



HAL
open science

Invariant Gibbs measures for 1D NLS in a trap

van Duong Dinh, Nicolas Rougerie

► **To cite this version:**

van Duong Dinh, Nicolas Rougerie. Invariant Gibbs measures for 1D NLS in a trap. 2023. hal-03926436v1

HAL Id: hal-03926436

<https://hal.science/hal-03926436v1>

Preprint submitted on 6 Jan 2023 (v1), last revised 30 Jan 2023 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

INVARIANT GIBBS MEASURES FOR 1D NLS IN A TRAP

VAN DUONG DINH AND NICOLAS ROUGERIE

ABSTRACT. We consider the one dimensional cubic nonlinear Schrödinger equation with trapping potential behaving like $|x|^s$ ($s > 1$) at infinity. We construct Gibbs measures associated to the equation and prove that the Cauchy problem is globally well-posed almost surely on their support. Consequently, the Gibbs measure is indeed invariant under the flow of the equation. We also address the construction and invariance of canonical Gibbs measures, conditioned on the L^2 mass.

CONTENTS

1. Introduction	2
2. Main results	3
2.1. Random data Cauchy theory	3
2.2. Grand-canonical measures	5
2.3. Canonical measures	7
2.4. Organization of the paper	8
3. Grand-canonical measures	8
3.1. Basic estimates and the defocusing case	8
3.2. Focusing measure, $s > 2$	13
3.3. Focusing measure, $s \leq 2$	15
3.4. Approximating measures	20
4. Cauchy problem and invariant measure, super-harmonic case	24
4.1. Deterministic local well-posedness	24
4.2. Measure invariance and global well-posedness	26
5. Cauchy problem and invariant measure, (sub)-harmonic case	32
5.1. Probabilistic local well-posedness	33
5.2. Measure invariance and global well-posedness	36
6. Canonical measures	42
6.1. Gaussian measure with a fixed renormalized mass	42
6.2. Gibbs measures conditioned on the mass	51
6.3. Invariance of Gibbs measures conditioned on mass	56
Appendix A. Bounds on the covariance operator	59
Appendix B. Basic functional-analytic estimates	60
Appendix C. L^p -boundedness of the smoothed spectral projectors	61
Appendix D. An estimate on the number of eigenvalues	62
References	65

Date: January, 2023.

2020 Mathematics Subject Classification. 35Q55.

Key words and phrases. Nonlinear Schrödinger Equation; Invariant Gibbs measure; Harmonic oscillator.

1. INTRODUCTION

We study the statistical mechanics of the trapped 1D cubic nonlinear Schrödinger equation

$$i \partial_t u = hu \pm |u|^2 u \quad (1.1)$$

on \mathbb{R} with the Schrödinger operator

$$h := -\partial_x^2 + V(x),$$

where $V(x) \xrightarrow{|x| \rightarrow \infty} +\infty$ is a trapping potential. Namely, we construct the associated Gibbs probability measures given *formally* by

$$d\mu_\nu^{\text{gc}}(u) = z_\nu^{-1} \exp\left(-\langle u, (h + \nu)u \rangle_{L^2} \mp \frac{1}{2} \int_{\mathbb{R}} |u|^4 du\right) du \quad (1.2)$$

and

$$d\mu_m^{\text{c}}(u) = z_m^{-1} \mathbf{1}_{\{\int_{\mathbb{R}} |u|^2 = m\}} \mu_0^{\text{gc}}(du) \quad (1.3)$$

and prove their invariance by the flow of (1.1). These measures correspond, for this nonlinear model, to the well known grand-canonical ensemble ((1.2) with chemical potential $\nu \in \mathbb{R}$) and canonical ensemble ((1.3) with particle number/mass $m > 0$) of statistical mechanics. Our main physical motivation comes from the mean-field approximation of Bose gases and Bose–Einstein condensates [36, 44, 48], whence our restriction to the cubic equation, corresponding to short-range pair interactions between particles. In this context, the measure (1.2) was rigorously derived from many-body quantum mechanics in [32, 33, 23, 24, 45].

The question of the invariance of such measures under the associated Hamiltonian flow has a rich history, initiated by [31], elements of which we recall below. As is well-known, the main issue is that, once rigorously defined (in particular, in the focusing case, a L^2 -mass cutoff is necessary in (1.2)), the above measures live on functional spaces of regularity/integrability levels at which it is challenging or impossible to construct the flow of (1.1). Instead of relying on such a deterministic flow, one often constructs a probabilistic Cauchy theory, taking advantage of the formal invariance of the Gibbs measures to substitute for conservation laws. The main datum of the problem, setting the regularity/integrability level, is in our context the growth at infinity of the potential V . We assume that it is polynomial, of order $s > 1$, which is the threshold for the measure (1.2) to make sense (see below).

Assumption 1.1. *Let $V \in C^\infty(\mathbb{R}, \mathbb{R}_+)$ satisfy for some $s > 1$:*

(i) *There exists $C \geq 1$ so that for all $|x| \geq 1$, $\frac{1}{C} \langle x \rangle^s \leq V(x) \leq C \langle x \rangle^s$.*

(ii) *For any $j \in \mathbb{N}$, there exists $C_j > 0$ so that $|\partial^j V(x)| \leq C_j \langle x \rangle^{s-j}$.*

The reader may think throughout that

$$V(x) = \left(1 + |x|^2\right)^{s/2}, \quad (1.4)$$

although we do not need such an exact formula. We construct a global-in-time Cauchy theory on the support of the measures (1.2) for any $s > 1$ in the defocusing case (+ sign in (1.1)) and any $s > 8/5$ in the focusing case (− sign in (1.1)). There is a noticeable dichotomy at $s = 2$ (the harmonic oscillator). Indeed, for $s > 2$, one can essentially construct the flow deterministically at the appropriate level of regularity (mostly based on tools from [60, 61, 62]), whereas for $s \leq 2$, we must rely on the randomization of initial data. We shall thus be particularly interested in the case $s \leq 2$, where the measures do not live on L^2 and thus in particular (1.3) must be interpreted in a renormalized sense (vaguely, $m = \infty - m'$ with $m' \in \mathbb{R}$).

Our results generalize known theorems and allow to simplify the proofs of some of them. Indeed, for the periodic NLS ($s = +\infty$ formally, restriction to a compact setting) the Cauchy theory was constructed in [1], and the invariance of (1.2) was deduced. The canonical measure (1.3) was then constructed and proved to be invariant in [42] (see also [6]). In [8] (see also [16]), the invariance of (1.2) was obtained for $s = 2$. In all other cases (in particular for the canonical measure (1.3) on \mathbb{R} with any kind of trap), our results seem new. In particular, they answer a question raised after [8, Theorem 1.1] concerning more general potentials than (1.4) for $s = 2$, having the same behavior at infinity.

One of our motivations for considering the more general case of $s \neq +\infty, 2$ is that all approaches of the topic at hand that we are aware of rely on the spectral problem for the linear operator $h = -\partial_x^2 + V(x)$ being exactly soluble. A detailed explicit knowledge of the eigenfunctions is used to construct the measure and the flow. This is certainly the case for $s = +\infty$ (plane waves [1, 42]) and $s = 2$ (link with Hermite polynomials [8, 18]) but also in other cases considered in the literature, like radial NLS on the disk or sphere (link with Bessel functions [4, 5, 56, 57]). All (non-radial) known results in higher dimension [2, 3, 19, 20, 21] seem to rely on plane waves.

In this paper, we propose a softer approach to the Cauchy theory (in particular, the probabilistic one, for $s \leq 2$), which relies less on exact formulae and allows the aforementioned generalizations to $s < 2$. A price to pay is that we do not prove multilinear estimates, and consequently our results are mostly restricted to the cubic nonlinearity. We could allow a slightly more general non-linearity (namely, behaving like $|u|^k u$ for more general, s -dependent, $k > 2$) but we refrain from stating this for simplicity and because the cubic nonlinearity is the most physically relevant one.

Another motivation to consider a general s is that, in experiments with cold alkali gases, the trapping potential can be quite general. The link between the first-principles, many-body, description of these experiments and the above formalism was made in [32, 33, 34, 23, 26, 25, 52] at the static level of equilibrium states and in [24, 45] for the dynamics. The Cauchy theory on the support of the (defocusing) measures we consider for $2 < s < +\infty$ was used as a working assumption in [24], whose results it would be interesting to generalize to $s \leq 2$ in view of ours and [8].

In the next section, we state our results and related remarks precisely. The rest of the paper is devoted to proofs.

Acknowledgments. This work was supported by the European Union's Horizon 2020 Research and Innovation Programme (Grant agreement CORFRONMAT No. 758620).

2. MAIN RESULTS

2.1. Random data Cauchy theory. In a finite dimensional setting, one may consider a system of ODEs

$$\begin{cases} \partial_t x_j &= \frac{\partial H}{\partial \xi_j}, \\ \partial_t \xi_j &= -\frac{\partial H}{\partial x_j}, \end{cases} \quad j = 1, \dots, n$$

with the Hamiltonian $H(x, \xi) = H(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$. The Gibbs measure associated to the system is given by

$$d\mu(x, \xi) = \frac{1}{Z} e^{-H(x, \xi)} dx d\xi,$$

where $Z > 0$ is a normalization constant. By the invariance of Lebesgue measure $dx d\xi = \prod_{j=1}^n dx_j d\xi_j$ (thanks to Liouville's theorem) and the conservation of the Hamiltonian H , the Gibbs measure μ is invariant under the time evolution of the system, i.e., for any measurable set $A \subset \mathbb{R}^{2n}$, $\mu(A) = \mu(\Phi(t)(A))$ for all $t \in \mathbb{R}$, where $\Phi(t)$ is the solution map.

Equation (1.1) also has a Hamiltonian structure, namely

$$\partial_t u = -i \frac{\partial H}{\partial \bar{u}},$$

where $H = H(u)$ is the Hamiltonian given by

$$H(u) = \langle u, hu \rangle_{L^2} \pm \frac{1}{2} \int_{\mathbb{R}} |u|^4. \quad (\text{Hamiltonian})$$

This Hamiltonian is conserved under the dynamics of (1.1) as well as the mass

$$M(u) = \int_{\mathbb{R}} |u|^2. \quad (\text{Mass})$$

Following the rationale of the finite dimensional case, one expects the Gibbs measure of the form

$$d\mu(u) = \frac{1}{Z} e^{-H(u)} du \quad (2.1)$$

to be invariant under the dynamics of (1.1). However, the above expression is formal since there is no infinite dimensional Lebesgue measure.

In the seminal work [31], Lebowitz, Rose, and Speer gave a rigorous justification, by means of techniques from constructive quantum field theory [49, 28], for the existence of Gibbs measures for 1D periodic NLS. More precisely, by rewriting (2.1) as

$$d\mu(u) = \frac{1}{Z} e^{\mp \frac{1}{2} \int_{\mathbb{R}} |u|^4} e^{-\langle u, hu \rangle_{L^2}} du, \quad (2.2)$$

they defined μ as an absolutely continuous probability measure with respect to the Gaussian measure

$$d\mu_0(u) = \frac{1}{Z_0} e^{-\langle u, hu \rangle_{L^2}} du.$$

Here h should be understood as $-\partial_x^2 + 1$ on \mathbb{T} . The above can be defined as a probability measure on $H^\theta(\mathbb{T})$ for any $\theta < \frac{1}{2}$, where $H^\theta(\mathbb{T})$ is the Sobolev space on the torus. Note that for the focusing nonlinearity, the Gibbs measure is constructed with a mass cutoff, i.e., $\mathbb{1}_{\{\int_{\mathbb{R}} |u|^2 < m\}}$ multiplying (2.2) for some positive number m . The result of [31] actually holds for a general power type nonlinearity and we refer to [31] for more details.

In [1], Bourgain pursued the study of Gibbs measures for 1D periodic NLS and proved the invariance of μ under the NLS flow. Here, by invariance, we mean that $\mu(A) = \mu(\Phi(t)(A))$ for any measurable set $A \subset H^\theta(\mathbb{T})$ with some $\theta < \frac{1}{2}$ and any $t \in \mathbb{R}$, where $\Phi(t)$ is the solution map. Moreover, there exists a set of full μ -measure such that the solution exists globally in time for all initial data belonging to this set. Such a result is usually referred to as almost sure global existence. In this context, the invariant Gibbs measure serves as a substitute for conserved quantities to control the growth in time of solutions. This allows to extend local in time solutions to global ones almost surely.

After these pioneering works, invariant Gibbs measure and almost-sure global existence on the support of Gibbs measure have been proved for many other dispersive equations (see e.g., [2] for 2D periodic NLS, [56, 57] for NLS on the disc, [10, 9, 17, 4] for nonlinear wave equations, [41, 40] for KdV-type systems, and [37, 55] for 1D derivative NLS,...).

In the above-mentioned works, invariant Gibbs measures were constructed in compact settings (torus or ball). There are few works addressing the invariant Gibbs measure on non-compact frameworks (see e.g., [8, 18] for NLS with harmonic potential, [11] for 1D NLS with cubic nonlinearity multiplied by a sufficiently smooth and integrable function, and also [54, 43] for almost sure well-posedness with (an)-harmonic potentials).

2.2. Grand-canonical measures. The first goal of this paper is to extend the result of Burq, Thomann, and Tzvetkov [8] to the 1D cubic NLS with potentials satisfying Assumption 1.1. We first recall the rigorous definition of the measure (1.2), setting $\nu = 0$ for simplicity of notation.

We start with the definition of the Gaussian measure. Under Assumption 1.1, the linear operator h is Hermitian and has compact resolvent. We write its' spectral decomposition as

$$h = \sum_{j \geq 1} \lambda_j |u_j\rangle \langle u_j| \quad (2.3)$$

with a non-decreasing sequence of positive eigenvalues $\lambda_j \rightarrow \infty$ and the associated eigenfunctions u_j . For $\theta \in \mathbb{R}$, we introduce the Sobolev space associated to h as

$$\mathcal{H}^\theta := \left\{ u = \sum_{j \geq 1} \alpha_j u_j : \|u\|_{\mathcal{H}^\theta} := \left(\sum_{j \geq 1} \lambda_j^\theta |\alpha_j|^2 \right)^{1/2} < \infty \right\} \quad (2.4)$$

with

$$\alpha_j = \langle u_j, u \rangle_{L^2}. \quad (2.5)$$

For $\beta \geq 0$ and $1 \leq p \leq \infty$, we define

$$\mathcal{W}^{\beta,p} := \left\{ u \in \mathcal{S}'(\mathbb{R}) : h^{\beta/2} u \in L^p(\mathbb{R}) \right\} \quad (2.6)$$

which is equipped with the norm

$$\|u\|_{\mathcal{W}^{\beta,p}} := \|h^{\beta/2} u\|_{L^p}.$$

Here $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions on \mathbb{R} . When $p = 2$, we actually have $\mathcal{W}^{\beta,2} \equiv \mathcal{H}^\beta$.

Definition 2.1 (Gaussian measure).

Let $\Lambda \geq \lambda_1$. On

$$E_{\leq \Lambda} := \text{span} \{u_j : \lambda_j \leq \Lambda\}, \quad (2.7)$$

we define the finite-dimensional Gaussian measure

$$d\mu_0^{\leq \Lambda}(u) := \prod_{\lambda_j \leq \Lambda} \frac{\lambda_j}{\pi} e^{-\lambda_j |\alpha_j|^2} d\alpha_j, \quad (2.8)$$

where α_j is as in (2.5) and $d\alpha_j = d\text{Re}(\alpha_j) d\text{Im}(\alpha_j)$ is the Lebesgue measure on \mathbb{C} .

The sequence of measures $\{\mu_0^{\leq \Lambda}\}_{\Lambda \geq \lambda_1}$ is tight in the Hilbert space \mathcal{H}^θ for any $\theta < \frac{1}{2} - \frac{1}{s}$ with $s > 1$ as in Assumption 1.1. Consequently, it defines a probability measure μ_0 on this space, having $\mu_0^{\leq \Lambda}$ as cylindrical projection on $E_{\leq \Lambda}$.

The tightness is proved in [32, Example 3.2]. We recall the main argument in Appendix A below for the convenience of the reader. The existence of the infinite-dimensional measure then follows from [51, Lemma 1].

For the interacting measures, we have the following result.

Proposition 2.2 (Grand-canonical measures).

Let $s > 1$, V satisfy Assumption 1.1, and μ_0 as defined above.

(1) **Defocusing case.** For any $s > 1$, the map $u \mapsto e^{-\frac{1}{2} \int_{\mathbb{R}} |u|^4}$ is in $L^1(d\mu_0)$. Consequently,

$$d\mu(u) := \frac{1}{Z} e^{-\frac{1}{2} \int_{\mathbb{R}} |u|^4} d\mu_0(u) \quad (2.9)$$

makes sense as a probability measure.

(2) **Focusing case, $s > 2$.** For any $s > 2$ and any $m > 0$, the map $u \mapsto e^{\frac{1}{2} \int_{\mathbb{R}} |u|^4} \mathbb{1}_{\{\int_{\mathbb{R}} |u|^2 \leq m\}}$ is in $L^1(d\mu_0)$. Consequently,

$$d\mu(u) := \frac{1}{Z} e^{\frac{1}{2} \int_{\mathbb{R}} |u|^4} \mathbb{1}_{\{\int_{\mathbb{R}} |u|^2 \leq m\}} d\mu_0(u) \quad (2.10)$$

makes sense as a probability measure.

(3) **Focusing case, $\frac{8}{5} < s \leq 2$.** For any $1 < s \leq 2$, the sequence $\{\mathcal{M}_{\leq \Lambda}(u)\}_{\Lambda \geq \lambda_1}$, with

$$\mathcal{M}_{\leq \Lambda}(u) := \sum_{\lambda_j \leq \Lambda} \left(|\alpha_j|^2 - \int |\alpha_j|^2 d\mu_0(u) \right),$$

is Cauchy in $L^2(d\mu_0)$. It has thus a limit (the renormalized mass), denoted by $\mathcal{M}(u)$. In addition, for any $\frac{8}{5} < s \leq 2$ and any $m > 0$, the map $u \mapsto e^{\frac{1}{2} \int_{\mathbb{R}} |u|^4} \mathbb{1}_{\{|\mathcal{M}(u)| \leq m\}}$ is in $L^1(d\mu_0)$, hence

$$d\mu(u) := \frac{1}{Z} e^{\frac{1}{2} \int_{\mathbb{R}} |u|^4} \mathbb{1}_{\{|\mathcal{M}(u)| \leq m\}} d\mu_0(u) \quad (2.11)$$

makes sense as a probability measure.

The defocusing case is dealt with in [32, Section 5] for $s > 2$ and in [33, Section 3] for $s > 1$, generalizing [8, Section 3] for $s = 2$. We recall the arguments below. The definition of the focusing measure for $s \neq 2, +\infty$ is new. For the harmonic potential $V(x) = |x|^2$, the construction of the measure was performed in [8, Section 3] with a continuous cut-off $\zeta(\mathcal{M}(u))$ instead of the rough one in (2.11). We vindicate that the above definitions make sense in Section 3 below.

We have the following result for the evolution of these measures under a suitably defined flow.

Theorem 2.3 (Invariance of grand-canonical measures).

Let $s > 1$ and V satisfy Assumption (1.1). Assume in addition that $s > \frac{8}{5}$ for the focusing nonlinearity. Then there exist $\theta < \frac{1}{2} - \frac{1}{s}$ and a set $\Sigma \subset \mathcal{H}^\theta$ such that:

- (1) $\mu(\Sigma) = 1$;
- (2) Equation (1.1) is globally well-posed for initial data $f \in \Sigma$ with flow $\Phi(t) : \Sigma \mapsto \Sigma$;
- (3) μ is invariant under the flow of (1.1), $\mu(\Phi(t)A) = \mu(A)$ for all measurable sets $A \subset \Sigma$;
- (4) There exists $C > 0$ such that for any $f \in \Sigma$,

$$\|\Phi(t)f\|_{\mathcal{H}^\theta} \leq C \left(\omega(f) + \log^{\frac{1}{2}}(1 + |t|) \right), \quad \forall t \in \mathbb{R}$$

for some constant $\omega(f)$ depending on f .

An ingredient of our proof is the following estimate (see e.g., [32, 33, 34]) on the k -particle density matrix associated to μ_0

$$\int |u^{\otimes k}\rangle \langle u^{\otimes k}| d\mu_0(u) \leq k!(h^{-1})^{\otimes k}, \quad \forall k \geq 1. \quad (2.12)$$

Another observation (see Lemma 3.3) is that for $0 \leq \beta < \frac{1}{2}$, the function

$$x \mapsto h^{\beta-1}(x, x) \in L^p(\mathbb{R}) \quad (2.13)$$

for all $\max\left\{1, \frac{2}{s(1-2\beta)}\right\} < p \leq \infty$, where

$$h^{-1}(x, y) = \sum_{j \geq 1} \lambda_j^{-1} u_j(x) \bar{u}_j(y) \quad (2.14)$$

is the integral kernel of h^{-1} . This property together with (2.12) enable us to prove (see Lemma 3.4) that the Gaussian measure μ_0 is supported on Sobolev spaces $\mathcal{W}^{\beta,p}$. Moreover,

there exist $C, c > 0$ such that

$$\int e^{c\|u\|_{\mathcal{W}^{\beta,p}}^2} d\mu_0(u) \leq C$$

provided that $0 \leq \beta < \frac{1}{2}$ and $p > \max\left\{2, \frac{4}{s(1-2\beta)}\right\}$ an even integer. As a result, we are able to define the defocusing Gibbs measure for all $s > 1$. Combining with a Gagliardo–Nirenberg inequality from [7], we define the focusing measure for $s > \frac{8}{5}$ (a restriction which is perhaps technical). Our proof of (2.13) uses Lieb–Thirring-type inequalities from [22] and standard inequalities for operators in Schatten ideals (Hölder and Kato–Seiler–Simon [50]) rather than L^p -bounds on individual eigenfunctions. This makes it applicable to general potentials. Since μ_0 is invariant under the linear Schrödinger flow associated with h , we also have that for any time t ,

$$\|e^{-ith}u\|_{\mathcal{W}^{\beta,p}} < \infty \quad \mu_0 \text{ almost surely.}$$

Combining with fractional chain rules, we directly recover estimates e.g., like

$$\|(e^{-ith}u)^2\|_{\mathcal{H}^\beta} < \infty \quad \mu_0 \text{ almost surely}$$

at $s = 2$, which was originally obtained as [8, Equation (1.2)] using bilinear estimates for Hermite functions. The above probabilistic Strichartz estimates are important tools in our construction of a local Cauchy theory.

For super-harmonic potentials, i.e., $s > 2$, since the Gibbs measure is supported on L^2 -based Sobolev spaces of positive indices, there is hope to obtain a deterministic local well-posedness on its' support. This is indeed feasible, using Strichartz estimates with a loss of derivatives proved by Yajima and Zhang [61]. For (sub)-harmonic potentials, i.e., $1 < s \leq 2$, the Gibbs measure lives on L^2 -based Sobolev spaces of negative indices, so it is difficult to obtain a satisfying deterministic local theory on its' support. In [8], such a deterministic result was proved by using a delicate multilinear estimate which relies heavily on L^p -bound of derivatives of eigenfunctions obtained in [30]. Here we give a softer argument and prove an almost sure local well-posedness for the equation.

2.3. Canonical measures. We next deal with the construction and invariance of canonical Gibbs measures (1.3) conditioned on the mass. This type of measures has so far been constructed only for the 1D periodic NLS [42, 13, 14] and for the 1D periodic derivative NLS [6]. To the best of our knowledge, there is no work constructing canonical Gibbs measures in non-compact settings.

To give rigorous meaning to (1.3), we follow the basic strategy of [42]. First, we construct a Gaussian measure conditioned on the L^2 mass. For $s \leq 2$ it is infinite almost surely, so that we need to use the renormalized mass as defined in Item (3) of Proposition 2.2. To unify the presentation we always condition the Gaussian measure on the renormalized mass, keeping in mind that when $s > 2$, the L^2 mass is finite almost surely, equal to the renormalized one plus its finite expectation with respect to the Gaussian measure.

More precisely, for $m \in \mathbb{R}$ and $m > -\text{Tr}[h^{-1}]$ if $s > 2$, we will take the limit $\varepsilon \rightarrow 0^+$ of the approximate canonical Gaussian measure

$$d\mu_0^{m,\varepsilon}(u) = \frac{1}{Z_0^{m,\varepsilon}} \mathbb{1}_{\{m-\varepsilon < \mathcal{M}(u) < m+\varepsilon\}} d\mu_0(u), \quad (2.15)$$

where $\mathcal{M}(u)$ is as in Proposition 2.2. Since we aim at conditioning on a zero-probability event, the normalization constant

$$Z_0^{m,\varepsilon} = \mu_0(m - \varepsilon < \mathcal{M}(u) < m + \varepsilon)$$

converges to zero as $\varepsilon \rightarrow 0^+$, hence taking the limit in (2.15) is not straightforward. We however prove that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon} Z_0^{m,\varepsilon} > 0$$

so that the definition

$$d\mu_0^m(u) := \lim_{\varepsilon \rightarrow 0^+} d\mu_0^{m,\varepsilon}(u)$$

indeed yields a probability measure. Then the interacting canonical Gibbs measure is defined by

$$d\mu^m(u) = \frac{1}{Z^m} e^{\mp \frac{1}{2} \int_{\mathbb{R}} |u|^4} d\mu_0^m(u).$$

Its invariance with respect to the NLS flow is a direct consequence of the invariance of the standard Gibbs measure in Theorem 2.3. We summarize this in the following theorem.

Theorem 2.4 (Canonical Gibbs measures).

Let $s > 1$, V satisfy Assumption 1.1, $m \in \mathbb{R}$, and assume $m > -\text{Tr}[h^{-1}]$ if $s > 2$. Assume in addition that $s > \frac{8}{5}$ for the focusing nonlinearity. Then μ^m makes sense as a probability measure. In addition, it is invariant under the flow of (1.1) (as defined in Theorem 2.3).

Canonical measures were constructed before in [42, 6]. In these references the normalization of the canonical Gibbs measure is proved by using a large deviation principle, a L^2 -based Sobolev embedding (for instance, $\mathcal{H}^{1/4} \subset L^4(\mathbb{R})$), and a sum over dyadic pieces. Adapting this argument to our context results in an unnecessary restriction to $s > 4$ because of the use of the L^2 -based Sobolev embedding. Our approach is different, based primarily on the almost sure L^p -regularity alluded to in the discussion below Theorem 2.3. This allows us to define the canonical Gibbs measure as soon as the grand-canonical one is defined.

2.4. Organization of the paper. Section 3 is devoted to the construction of Gibbs measures associated to (1.1). We also define and prove some properties of approximate measures which are needed for the proof of measure invariance. In Section 4, we consider the Cauchy problem in the case of super-harmonic potential, $s > 2$. The more difficult Cauchy problem for (sub)-harmonic potentials, $s \leq 2$, will be addressed in Section 5. The construction as well as the invariance of canonical measures conditioned on the mass are considered in Section 6. Some estimates and inequalities from the literature, used throughout the paper are recalled in appendices for the convenience of the reader.

3. GRAND-CANONICAL MEASURES

3.1. Basic estimates and the defocusing case. The definition of the Gaussian measure μ_0 was recalled in Definition 2.1. Several of our estimates will be based on the following observation. We quote it from [32] but it is probably well-known, see e.g., [23, 24] and references therein.

Lemma 3.1 (Density matrices of the Gaussian measure).

Let $s > 1$, V satisfy Assumption 1.1, and $\theta < \frac{1}{2} - \frac{1}{s}$. Then there exists a unique measure μ_0 living over \mathcal{H}^θ such that for every $\Lambda \geq \lambda_1$, $\mu_0^{\leq \Lambda}$ is the cylindrical projection of μ_0 on $E_{\leq \Lambda}$. Moreover, for every integer $k \geq 1$,

$$\int |u^{\otimes k}\rangle \langle u^{\otimes k}| d\mu_0(u) \leq k!(h^{-1})^{\otimes k} \quad (3.1)$$

as operators. In particular, for any self-adjoint operator A ,

$$\int |(Au)^{\otimes k}\rangle \langle (Au)^{\otimes k}| d\mu_0(u) \leq k!(Ah^{-1}A)^{\otimes k}. \quad (3.2)$$

As explained in [32, Lemma 3.3], Wick's theorem for Gaussian measures yields

$$\int |u^{\otimes k}\rangle \langle u^{\otimes k}| d\mu_0(u) = k! P_s^k (h^{-1})^{\otimes k} P_s^k,$$

where P_s^k is the orthogonal projection on k -symmetric functions, i.e.,

$$P_s^k v(x_1, \dots, x_k) = \frac{1}{k!} \sum_{\sigma \in \Pi(k)} v(x_{\sigma(1)}, \dots, x_{\sigma(k)})$$

with $\Pi(k)$ the group of all permutations of $\{1, \dots, k\}$. Since P_s^k commutes with $(h^{-1})^{\otimes k}$, we have

$$(h^{-1})^{\otimes k} = P_s^k (h^{-1})^{\otimes k} P_s^k + (1 - P_s^k) (h^{-1})^{\otimes k} (1 - P_s^k)$$

and (3.1) follows.

As a direct consequence of density matrices, we have the following decay of \mathcal{H}^θ -norms with respect to the Gaussian measure μ_0 .

Lemma 3.2 (*L^2 -Regularity on the support of the Gaussian measure*).

Let $s > 1$, V satisfy Assumption 1.1, and $\theta < \frac{1}{2} - \frac{1}{s}$. Then there exist $C, c > 0$ such that

$$\int e^{c\|u\|_{\mathcal{H}^\theta}^2} d\mu_0(u) \leq C. \quad (3.3)$$

In particular, for all $\lambda > 0$

$$\mu_0(u \in \mathcal{H}^\theta : \|u\|_{\mathcal{H}^\theta} > \lambda) \leq C e^{-c\lambda^2}. \quad (3.4)$$

Proof. It suffices to prove (3.3) since (3.4) follows from (3.3) and the Chebyshev inequality. We write

$$\int e^{c\|u\|_{\mathcal{H}^\theta}^2} d\mu_0(u) = \sum_{k \geq 0} \frac{c^k}{k!} \int \|u\|_{\mathcal{H}^\theta}^{2k} d\mu_0(u)$$

and observe that

$$\begin{aligned} \int \|u\|_{\mathcal{H}^\theta}^{2k} d\mu_0(u) &= \int \underbrace{\|u\|_{\mathcal{H}^\theta}^2 \cdots \|u\|_{\mathcal{H}^\theta}^2}_{k \text{ times}} d\mu_0(u) \\ &= \int \cdots \int \left(\int |h^{\theta/2} u(x_1)|^2 \cdots |h^{\theta/2} u(x_k)|^2 d\mu_0(u) \right) dx_1 \cdots dx_k. \end{aligned}$$

Denote $\delta_x^{(\eta)}$ a mollification of the Dirac delta function at x so that

$$\delta_x^{(\eta)} \xrightarrow{\eta \rightarrow 0} \delta_x$$

as measures. Identifying it with the associated multiplication operator we have

$$\begin{aligned} &\int |h^{\theta/2} u(x_1)|^2 \cdots |h^{\theta/2} u(x_k)|^2 d\mu_0(u) \\ &= \lim_{\eta \rightarrow 0} \text{Tr} \left[\delta_{x_1}^{(\eta)} \otimes \cdots \otimes \delta_{x_k}^{(\eta)} \left(\int |(h^{\theta/2} u)^{\otimes k} \rangle \langle (h^{\theta/2} u)^{\otimes k}| d\mu_0(u) \right) \delta_{x_k}^{(\eta)} \otimes \cdots \otimes \delta_{x_1}^{(\eta)} \right]. \end{aligned}$$

But (3.1) implies

$$\begin{aligned} &\text{Tr} \left[\delta_{x_1}^{(\eta)} \otimes \cdots \otimes \delta_{x_k}^{(\eta)} \left(\int |(h^{\theta/2} u)^{\otimes k} \rangle \langle (h^{\theta/2} u)^{\otimes k}| d\mu_0(u) \right) \delta_{x_k}^{(\eta)} \otimes \cdots \otimes \delta_{x_1}^{(\eta)} \right] \\ &\leq k! \text{Tr} \left[\delta_{x_1}^{(\eta)} \otimes \cdots \otimes \delta_{x_k}^{(\eta)} (h^{\theta-1})^{\otimes k} \delta_{x_k}^{(\eta)} \otimes \cdots \otimes \delta_{x_1}^{(\eta)} \right] \end{aligned}$$

and since

$$\text{Tr} \left[\delta_x^{(\eta)} h^{\theta-1} \delta_x^{(\eta)} \right] = \iint \delta_x^{(\eta)}(y_1) h^{\theta-1}(y_1, y_2) \delta_x^{(\eta)}(y_2) dy_1 dy_2$$

we may let $\eta \rightarrow 0$ to deduce

$$\int |h^{\theta/2} u(x_1)|^2 \cdots |h^{\theta/2} u(x_k)|^2 d\mu_0(u) \leq k! h^{\theta-1}(x_1, x_1) \cdots h^{\theta-1}(x_k, x_k).$$

This implies

$$\int \|u\|_{\mathcal{H}^\theta}^{2k} d\mu_0(u) \leq k! \left(\int_{\mathbb{R}} h^{\theta-1}(x, x) dx \right)^k = k! \left(\text{Tr}[h^{-(1-\theta)}] \right)^k.$$

In particular, we obtain

$$\int e^{c\|u\|_{\mathcal{H}^\theta}^2} d\mu_0(u) \leq \sum_{k \geq 0} \left(c \text{Tr}[h^{-(1-\theta)}] \right)^k \leq C$$

provided that $c \text{Tr}[h^{-(1-\theta)}] < 1$. This proves (3.3). Note that $\text{Tr}[h^{-(1-\theta)}] < \infty$ due to $\theta < \frac{1}{2} - \frac{1}{s}$ (see Lemma A.1). \square

Regularity properties of typical samples of the Gaussian measure are, as per (3.1), connected to properties of the covariance h^{-1} (Green function of the Schrödinger operator). When $s > 2$, it is relatively easy to construct the (defocusing, at least) interacting measures since h^{-1} is a trace-class operator, see [32, Example 5.2] and Appendix A. For $s \leq 2$, the following lemma plays a key role in our analysis. It generalizes estimates from [33, Section 3].

Lemma 3.3 (Integrability of the density).

Let $s > 1$, V satisfy Assumption 1.1, and $0 \leq \beta < \frac{1}{2}$. Then $x \mapsto h^{\beta-1}(x, x)$ is in $L^p(\mathbb{R})$ for all

$$\max \left\{ 1, \frac{2}{s(1-2\beta)} \right\} < p \leq \infty,$$

where $h^{\beta-1}(x, y)$ is the integral kernel of $h^{\beta-1}$ defined as in (2.14).

Proof. It suffices to prove that for any multiplication operator $\chi \geq 0$ satisfying $\chi^2 \in L^q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int_{\mathbb{R}} \chi^2(x) h^{\beta-1}(x, x) dx = \text{Tr}[\chi h^{\beta-1} \chi] = \|h^{(\beta-1)/2} \chi\|_{\mathfrak{S}^2}^2 \leq C \|\chi^2\|_{L^q(\mathbb{R})}, \quad (3.5)$$

where for any $1 \leq p \leq \infty$,

$$\mathfrak{S}^p := \left\{ A : \|A\|_{\mathfrak{S}^p} := \left(\text{Tr}[(A^*A)^{p/2}] \right)^{1/p} < \infty \right\}$$

is the p -th Schatten class of operators on a Hilbert space [47, 50] ($p = 1, 2, \infty$ correspond respectively to trace-class, Hilbert-Schmidt, and compact operators).

For $0 < \alpha < \frac{1-\beta}{2}$, we write

$$h^{(\beta-1)/2} \chi = h^{\alpha+(\beta-1)/2} \left(h^{-\alpha} (1 - \partial_x^2)^\alpha \right) \left((1 - \partial_x^2)^{-\alpha} \chi \right).$$

We have

$$h^{\alpha+(\beta-1)/2} \in \mathfrak{S}^{2p} \iff \text{Tr} \left[h^{-2p(\frac{1-\beta}{2}-\alpha)} \right] < \infty,$$

which holds provided that (see Lemma A.1)

$$2p \left(\frac{1-\beta}{2} - \alpha \right) > \frac{1}{2} + \frac{1}{s}.$$

Since $h \geq \frac{1}{2}(-\partial_x^2 + \lambda_1)$ with $\lambda_1 > 0$ the lowest eigenvalue of h , we infer that $h \geq C(1 - \partial_x^2)$ for some $C > 0$. Since $\alpha < \frac{1}{2}$ The operator monotonicity of $x \mapsto x^{2\alpha}$ (see [12, Theorem 2.6]) implies

$$h^{2\alpha} \geq C^{2\alpha} (1 - \partial_x^2)^{2\alpha}$$

or

$$h^{-\alpha} (1 - \partial_x^2)^{2\alpha} h^{-\alpha} \leq \frac{1}{C^{2\alpha}}$$

hence $h^{-\alpha} (1 - \partial_x^2)^\alpha$ is a bounded operator for $0 < \alpha < (1 - \beta)/2$.

By the Kato–Seiler–Simon inequality (see e.g., [50, Theorem 4.1]) for $1 \leq r < \infty$,

$$\|f(-i\nabla)g(x)\|_{\mathfrak{S}^r} \leq \|f\|_{L^r} \|g\|_{L^r},$$

we have

$$\|(1 - \partial_x^2)^{-\alpha} \chi\|_{\mathfrak{S}^{2q}} \leq \left(\int_{\mathbb{R}} \frac{d\xi}{(1 + |\xi|^2)^{2\alpha q}} \right)^{1/2q} \|\chi\|_{L^{2q}(\mathbb{R})} \leq C \|\chi\|_{L^{2q}(\mathbb{R})}$$

provided that $4\alpha q > 1$. Here we need $q < \infty$ hence $p > 1$.

Combining these estimates, the Hölder inequality in Schatten spaces (see [50, Theorem 2.8]) yields

$$\begin{aligned} \|h^{(\beta-1)/2} \chi\|_{\mathfrak{S}^2}^2 &\leq \|h^{\alpha+(\beta-1)/2}\|_{\mathfrak{S}^{2p}}^2 \|h^{-\alpha}(1 - \partial_x^2)^\alpha\|_{\mathfrak{S}^\infty}^2 \|(1 - \partial_x^2)^{-\alpha} \chi\|_{\mathfrak{S}^{2q}}^2 \\ &\leq C \|\chi\|_{L^{2q}(\mathbb{R})}^2 = C \|\chi^2\|_{L^q(\mathbb{R})}. \end{aligned} \quad (3.6)$$

This estimate holds true if the following conditions are fulfilled:

$$2p \left(\frac{1-\beta}{2} - \alpha \right) > \frac{1}{2} + \frac{1}{s}, \quad 4\alpha q > 1. \quad (3.7)$$

For $0 \leq \beta < 1/2$ and $\max \left\{ 1, \frac{2}{s(1-2\beta)} \right\} < p \leq \infty$, we pick $0 < \alpha < \frac{1-\beta}{2}$ such that

$$\frac{1}{4q} < \alpha < \frac{1-\beta}{2} - \frac{1}{2p} \left(\frac{1}{2} + \frac{1}{s} \right),$$

we see that (3.7) is satisfied and the result follows by inserting (3.6) into (3.5). \square

Applying the above, we will deduce that μ_0 is supported on Sobolev spaces $\mathcal{W}^{\beta,p}$ based on h as in (2.6). In fact, we have the following Fernique-type estimate giving the decay of such norms.

Lemma 3.4 (*L^p -Regularity on the support of the Gaussian measure*).

Let $s > 1$, V satisfy Assumption 1.1, $0 \leq \beta < \frac{1}{2}$, and $p > \max \left\{ 2, \frac{4}{s(1-2\beta)} \right\}$ be an even integer. Then there exist $C, c > 0$ such that

$$\int e^{c\|u\|_{\mathcal{W}^{\beta,p}}^2} d\mu_0(u) \leq C. \quad (3.8)$$

In particular, for $\theta < \frac{1}{2} - \frac{1}{s}$ and all $\lambda > 0$,

$$\mu_0 \left(u \in \mathcal{H}^\theta : \|u\|_{\mathcal{W}^{\beta,p}} > \lambda \right) \leq C e^{-c\lambda^2}. \quad (3.9)$$

Proof. It suffices to prove (3.8). To this end, we write

$$\int e^{c\|u\|_{\mathcal{W}^{\beta,p}}^2} d\mu_0(u) = \sum_{k \geq 0} \frac{c^k}{k!} \int \|u\|_{\mathcal{W}^{\beta,p}}^{2k} d\mu_0(u)$$

and estimate the summands.

For $2k = pm$ with $m \geq 0$ an integer, we have

$$\begin{aligned} \int \|u\|_{\mathcal{W}^{\beta,p}}^{2k} d\mu_0(u) &= \int \underbrace{\|u\|_{\mathcal{W}^{\beta,p}}^p \cdots \|u\|_{\mathcal{W}^{\beta,p}}^p}_{m \text{ times}} d\mu_0(u) \\ &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left(\int |h^{\beta/2} u(x_1)|^p \cdots |h^{\beta/2} u(x_m)|^p d\mu_0(u) \right) dx_1 \cdots dx_m. \end{aligned}$$

Using (3.2), we see that¹

$$\begin{aligned}
& \int |h^{\beta/2}u(x_1)|^p \cdots |h^{\beta/2}u(x_m)|^p d\mu_0(u) \\
&= \text{Tr} \left[(\delta_{x_1})^{\otimes n} \otimes \cdots \otimes (\delta_{x_m})^{\otimes n} \int |(h^{\beta/2}u)^{\otimes(mn)} \rangle \langle (h^{\beta/2}u)^{\otimes(mn)} | d\mu_0(u) (\delta_{x_1})^{\otimes n} \otimes \cdots \otimes (\delta_{x_m})^{\otimes n} \right] \\
&\leq (mn)! \text{Tr} \left[(\delta_{x_1})^{\otimes n} \otimes \cdots \otimes (\delta_{x_m})^{\otimes n} (h^{\beta-1})^{\otimes(mn)} (\delta_{x_1})^{\otimes n} \otimes \cdots \otimes (\delta_{x_m})^{\otimes n} \right] \\
&= (mn)! \text{Tr} \left[(\delta_{x_1})^{\otimes n} \otimes \cdots \otimes (\delta_{x_m})^{\otimes n} (h^{\beta-1} \otimes \cdots \otimes h^{\beta-1})^{\otimes n} (\delta_{x_1})^{\otimes n} \otimes \cdots \otimes (\delta_{x_m})^{\otimes n} \right] \\
&= (mn)! \text{Tr} \left[\left((\delta_{x_1} \otimes \cdots \otimes \delta_{x_m}) (h^{\beta-1} \otimes \cdots \otimes h^{\beta-1}) (\delta_{x_1} \otimes \cdots \otimes \delta_{x_m}) \right)^{\otimes n} \right] \\
&\leq (mn)! \left(\text{Tr} \left[(\delta_{x_1} \otimes \cdots \otimes \delta_{x_m}) (h^{\beta-1} \otimes \cdots \otimes h^{\beta-1}) (\delta_{x_1} \otimes \cdots \otimes \delta_{x_m}) \right] \right)^n \\
&\leq (mn)! \left(h^{\beta-1}(x_1, x_1) \cdots h^{\beta-1}(x_m, x_m) \right)^n,
\end{aligned}$$

where $p = 2n$. Thus we get

$$\int \|u\|_{\mathcal{W}^{\beta,p}}^{2k} d\mu_0(u) \leq k! B_{\beta,p}^k,$$

where

$$B_{\beta,p} := \left(\int_{\mathbb{R}} \left(h^{\beta-1}(x, x) \right)^{p/2} dx \right)^{2/p}. \quad (3.10)$$

Note that $B_{\beta,p}$ is finite thanks to Lemma 3.3 and the assumptions on p and β .

For $pm < 2k < p(m+1)$ with $m \geq 0$ an integer, we use Hölder's inequality to get

$$\begin{aligned}
& \int \|u\|_{\mathcal{W}^{\beta,p}}^{2k} d\mu_0(u) \int \left(\|u\|_{\mathcal{W}^{\beta,p}}^{pm} \right)^\delta \left(\|u\|_{\mathcal{W}^{\beta,p}}^{p(m+1)} \right)^{1-\delta} d\mu_0(u) \\
&\leq \left(\int \|u\|_{\mathcal{W}^{\beta,p}}^{pm} d\mu_0(u) \right)^\delta \left(\int \|u\|_{\mathcal{W}^{\beta,p}}^{p(m+1)} d\mu_0(u) \right)^{1-\delta} \\
&\leq \left(\left(\frac{pm}{2} \right)! B_{\beta,p}^{\frac{pm}{2}} \right)^\delta \left(\left(\frac{p(m+1)}{2} \right)! B_{\beta,p}^{\frac{p(m+1)}{2}} \right)^{1-\delta} \\
&= \left(\left(\frac{pm}{2} \right)! \right)^\delta \left(\left(\frac{p(m+1)}{2} \right)! \right)^{1-\delta} B_{\beta,p}^k,
\end{aligned}$$

where $\delta \in (0, 1)$ satisfying $2k = pm\delta + p(m+1)(1-\delta)$ or $\delta = m+1 - \frac{2k}{p}$. We claim that

$$\left(\left(\frac{pm}{2} \right)! \right)^\delta \left(\left(\frac{p(m+1)}{2} \right)! \right)^{1-\delta} \leq \left(\frac{p}{2} \right)! k! \quad (3.11)$$

Assuming this claim for the moment, we get

$$\int \|u\|_{\mathcal{W}^{\beta,p}}^{2k} d\mu_0(u) \leq \left(\frac{p}{2} \right)! k! B_{\beta,p}^k$$

hence

$$\int e^{c\|u\|_{\mathcal{W}^{\beta,p}}^2} d\mu_0(u) \leq \left(\frac{p}{2} \right)! \sum_{k \geq 0} (cB_{\beta,p})^k \leq C$$

provided that $c > 0$ is chosen such that $cB_{\beta,p} < 1$. This proves (3.8).

¹Strictly speaking the Dirac delta functions must be regularity as in the proof of Lemma 3.2.

It remains to prove the claim (3.11). We write $2k = pm + 2l$ with $1 \leq l \leq n - 1$. In particular, we have $\delta = 1 - \frac{l}{n}$ and $1 - \delta = \frac{l}{n}$. We also have

$$\begin{aligned} \left(\left(\frac{pm}{2} \right)! \right)^\delta \left(\left(\frac{p(m+1)}{2} \right)! \right)^{1-\delta} &= ((k-l)!)^\delta ((k-l+n)!)^{1-\delta} \\ &= \left(k! \frac{1}{k \cdots (k-l+1)} \right)^{1-\frac{l}{n}} (k!(k+1) \cdots (k-l+n))^{\frac{l}{n}} \\ &= k! \left(\frac{(k+1)^l \cdots (k-l+n)^l}{k^{n-l} \cdots (k-l+1)^{n-l}} \right)^{\frac{1}{n}} \\ &= k! \left(\underbrace{\frac{(k+1) \cdots (k-l+n)}{k^{n-l}} \cdots \frac{(k+1) \cdots (k-l+n)}{(k-l+1)^{n-l}}}_{l \text{ times}} \right)^{\frac{1}{n}}. \end{aligned}$$

We observe that each factor inside the bracket is of the form

$$\frac{(a+j) \cdots (a+j+n-l-1)}{a^{n-l}}$$

with $j = 1, \dots, l$ and $a \geq 1$. We can bound this term as

$$\left(1 + \frac{j}{a} \right) \cdots \left(1 + \frac{j+n-l-1}{a} \right) \leq (1+j) \cdots (j+n-l) \leq n!$$

for all $j = 1, \dots, l$. As a result, we obtain

$$\left(\left(\frac{pm}{2} \right)! \right)^\delta \left(\left(\frac{p(m+1)}{2} \right)! \right)^{1-\delta} \leq k!(n!)^{\frac{l}{n}} \leq n!k!$$

This proves (3.11). \square

Given the above estimates, the construction of the defocusing Gibbs measure is straightforward.

Proof of Proposition 2.2, Item (1). It suffices to prove that $Z \in (0, \infty)$. Since μ_0 is a probability measure, we obviously have $Z \leq 1$. As in the proof of Lemma 3.4, we have for any $s > 1$,

$$\int \|u\|_{L^4}^4 d\mu_0(u) \leq 2!B_{0,4}^2 < \infty,$$

where $B_{0,4}$ is as in (3.10). By Jensen's inequality

$$\int e^{-\frac{1}{2}\|u\|_{L^4}^4} d\mu_0(u) \geq \exp\left(-\frac{1}{2} \int \|u\|_{L^4}^4 d\mu_0(u)\right),$$

we infer that $Z > 0$. \square

3.2. Focusing measure, $s > 2$. We next tackle the more subtle definition of the focusing Gibbs measure. We first consider the case $s > 2$ where the mass is finite μ_0 -almost surely. Let us introduce some notation. For $u \in \mathcal{H}^\theta$ and $\Lambda \geq \lambda_1$, we denote

$$P_{\leq \Lambda} u = \sum_{\lambda_j \leq \Lambda} \alpha_j u_j, \quad P_{> \Lambda} u = \sum_{\lambda_j > \Lambda} \alpha_j u_j \quad (3.12)$$

the projections on the low and high frequencies respectively.

Lemma 3.5 (Decay estimates for the L^4 norm).

Let $s > 1$, V satisfy Assumption 1.1, and $0 \leq \rho < \frac{s-1}{2s}$. Then there exist $C, c > 0$ such that for $\theta < \frac{1}{2} - \frac{1}{s}$ and all $\Lambda, R > 0$,

$$\mu_0 \left(u \in \mathcal{H}^\theta : \|P_{>\Lambda}u\|_{L^4} > R \right) \leq Ce^{-c\Lambda^\rho R^2}. \quad (3.13)$$

Proof. Let $t > 0$ be a positive constant to be chosen later. We estimate

$$\begin{aligned} \mu_0 (\|P_{>\Lambda}u\|_{L^4} > R) &\leq e^{-tR^2} \int e^{t\|P_{>\Lambda}u\|_{L^4}^2} d\mu_0(u) \\ &= e^{-tR^2} \sum_{k \geq 0} \frac{t^k}{k!} \int \|P_{>\Lambda}u\|_{L^4}^{2k} d\mu_0(u). \end{aligned}$$

By the same argument as in the proof of Lemma 3.4, we have

$$\int \|P_{>\Lambda}u\|_{L^4}^{2k} d\mu_0(u) \leq 2!k!B_{\Lambda,0,4}^k, \quad \forall k \geq 0,$$

where

$$B_{\Lambda,0,4} := \left(\int_{\mathbb{R}} \left((P_{>\Lambda}h)^{-1}(x, x) \right)^2 dx \right)^{1/2} \quad (3.14)$$

with

$$(P_{>\Lambda}h)^{-1}(x, x) = \sum_{\lambda_j > \Lambda} \lambda_j^{-1} |u_j(x)|^2.$$

It follows that

$$\int e^{t\|P_{>\Lambda}u\|_{L^4}^2} d\mu_0(u) \leq 2! \sum_{k \geq 0} (tB_{\Lambda,0,4})^k.$$

For $0 \leq \rho < \frac{s-1}{2s}$, we have

$$\begin{aligned} (P_{>\Lambda}h)^{-1}(x, x) &= \sum_{\lambda_j > \Lambda} \lambda_j^{-1} |u_j(x)|^2 \\ &\leq \Lambda^{-\rho} \sum_{\lambda_j > \Lambda} \lambda_j^{\rho-1} |u_j(x)|^2 \\ &\leq \Lambda^{-\rho} h^{\rho-1}(x, x). \end{aligned}$$

Thanks to Lemma 3.3, we see that $x \mapsto h^{\rho-1}(x, x) \in L^2(\mathbb{R})$ for all $0 \leq \rho < \frac{s-1}{2s}$, hence $B_{\Lambda,0,4} \leq C\Lambda^{-\rho}$ for some constant $C > 0$. In particular, we have

$$\mu_0 (\|P_{>\Lambda}u\|_{L^4} > R) \leq e^{-tR^2} 2! \sum_{k \geq 0} (Ct\Lambda^{-\rho})^k.$$

Taking $t = \nu\Lambda^\rho$ with $\nu > 0$ sufficiently small so that $Ct\Lambda^{-\rho} = C\nu < 1$, we obtain (3.13). \square

We can now construct the focusing Gibbs measure in the super-harmonic case.

Proof of Proposition 2.2, Item (2). We have

$$Z \geq \int \mathbf{1}_{\{\|u\|_{L^2}^2 < m\}} d\mu_0(u) > \frac{1}{m} \int \|u\|_{L^2}^2 d\mu_0(u) = \frac{1}{m} \text{Tr}[h^{-1}] > 0.$$

It remains to show that $Z < \infty$. To this end, we use the layer cake representation to write

$$\int e^{\frac{1}{2}\|u\|_{L^4}^4} \mathbf{1}_{\{\|u\|_{L^2}^2 < m\}} d\mu_0(u) = \int_0^\infty \mu_0 \left(e^{\frac{1}{2}\|u\|_{L^4}^4} > \lambda, \|u\|_{L^2}^2 < m \right) d\lambda.$$

By a change of variable $\lambda \mapsto e^\lambda$, the problem is reduced to proving

$$\int_0^\infty \mu_0 \left(\|u\|_{L^4}^4 > 2\lambda, \|u\|_{L^2}^2 < m \right) e^\lambda d\lambda < \infty. \quad (3.15)$$

Let $\Lambda > 0$. By Sobolev embedding and the norm equivalence (B.1), we have for $\|u\|_{L^2}^2 < m$,

$$\begin{aligned} \|P_{\leq \Lambda} u\|_{L^4} &\leq C \|\langle \nabla \rangle^{\frac{1}{4}} P_{\leq \Lambda} u\|_{L^2} \\ &\leq C \|h^{\frac{1}{8}} P_{\leq \Lambda} u\|_{L^2} \\ &\leq C \Lambda^{\frac{1}{8}} \|P_{\leq \Lambda} u\|_{L^2} \\ &\leq C \Lambda^{\frac{1}{8}} \|u\|_{L^2} \\ &< C \Lambda^{\frac{1}{8}} m^{\frac{1}{2}}. \end{aligned}$$

For $\lambda > 0$, we can pick

$$\Lambda_0 = \frac{(2\lambda)^2}{2^8 C^8 m^4}. \quad (3.16)$$

so that

$$\|P_{\leq \Lambda_0} u\|_{L^4} < \frac{1}{2} (2\lambda)^{\frac{1}{4}}.$$

By the triangle inequality, we deduce that if $\|u\|_{L^4}^4 > 2\lambda$, then

$$\|P_{> \Lambda_0} u\|_{L^4} > \|u\|_{L^4} - \|P_{\leq \Lambda_0} u\|_{L^4} > \frac{1}{2} (2\lambda)^{\frac{1}{4}}.$$

Thus we have

$$\mu_0 \left(\|u\|_{L^4}^4 > 2\lambda, \|u\|_{L^2}^2 < m \right) \leq \mu_0 \left(\|P_{> \Lambda_0} u\|_{L^4} > \frac{1}{2} (2\lambda)^{\frac{1}{4}} \right).$$

Applying Lemma 3.5 and using (3.16), we arrive at

$$\mu_0 \left(\|P_{> \Lambda_0} u\|_{L^4} > \frac{1}{2} (2\lambda)^{\frac{1}{4}} \right) \leq C e^{-c\lambda^{2\rho + \frac{1}{2}}}$$

for any $0 \leq \rho < \frac{s-1}{2s}$. Choosing

$$\rho \in \left(\frac{1}{4}, \frac{s-1}{2s} \right)$$

and noting that $s > 2$, we prove (3.15). \square

3.3. Focusing measure, $s \leq 2$. We turn to the focusing case with $1 < s \leq 2$, Item (3) in Proposition 2.2. In this case, μ_0 lives over negative Sobolev spaces \mathcal{H}^θ with $\theta < \frac{1}{2} - \frac{1}{s}$, hence the mass is infinite μ_0 -almost surely. In this situation, we consider the renormalized mass as follows.

Lemma 3.6 (Renormalized mass).

Let $1 < s \leq 2$ and V satisfy Assumption 1.1. For every $\Lambda \geq \lambda_1$, we define the truncated renormalized mass

$$\mathcal{M}_{\leq \Lambda}(u) := \|P_{\leq \Lambda} u\|_{L^2}^2 - \int \|P_{\leq \Lambda} u\|_{L^2}^2 d\mu_0(u).$$

Then $\{\mathcal{M}_{\leq \Lambda}\}_{\Lambda \geq \lambda_1}$ converges strongly to a limit in $L^2(d\mu_0)$, i.e.,

$$\mathcal{M}(u) := \lim_{\Lambda \rightarrow \infty} \mathcal{M}_{\leq \Lambda}(u) \text{ in } L^2(d\mu_0).$$

In particular, we have

$$\int (\mathcal{M}(u))^2 d\mu_0(u) = \text{Tr}[h^{-2}] < \infty.$$

Proof. This is the same argument as in [34, Lemma 5.2]. We reproduce it for the reader's convenience. For $\Lambda \geq \lambda_1$, we have

$$\|P_{\leq \Lambda} u\|_{L^2}^2 = \sum_{\lambda_j \leq \Lambda} |\alpha_j|^2$$

and

$$\begin{aligned} \int \|P_{\leq \Lambda} u\|_{L^2}^2 d\mu_0(u) &= \sum_{\lambda_j \leq \Lambda} \int |\alpha_j|^2 d\mu_0(u) \\ &= \sum_{\lambda_j \leq \Lambda} \left(\int_{\mathbb{C}} |\alpha_j|^2 \frac{\lambda_j}{\pi} e^{-\lambda_j |\alpha_j|^2} d\alpha_j \right) \left(\prod_{\lambda_k \neq \lambda_j} \int_{\mathbb{C}} \frac{\lambda_k}{\pi} e^{-\lambda_k |\alpha_k|^2} d\alpha_k \right) \\ &= \sum_{\lambda_j \leq \Lambda} \frac{1}{\lambda_j}. \end{aligned}$$

Thus we get

$$\mathcal{M}_{\leq \Lambda}(u) = \sum_{\lambda_j \leq \Lambda} |\alpha_j|^2 - \lambda_j^{-1}. \quad (3.17)$$

Now for $\Theta \geq \Lambda$, we compute

$$\begin{aligned} \int |\mathcal{M}_{\leq \Theta}(u) - \mathcal{M}_{\leq \Lambda}(u)|^2 d\mu_0(u) &= \int \left(\sum_{\Lambda < \lambda_j \leq \Theta} |\alpha_j|^2 - \lambda_j^{-1} \right)^2 d\mu_0(u) \\ &= \sum_{\Lambda < \lambda_j, \lambda_k \leq \Theta} \int (|\alpha_j|^2 - \lambda_j^{-1})(|\alpha_k|^2 - \lambda_k^{-1}) d\mu_0(u) \\ &= \sum_{\Lambda < \lambda_j, \lambda_k \leq \Theta} \int (|\alpha_j|^2 |\alpha_k|^2 - \lambda_j^{-1} |\alpha_k|^2 - \lambda_k^{-1} |\alpha_j|^2 + \lambda_j^{-1} \lambda_k^{-1}) d\mu_0(u) \\ &= \sum_{\Lambda < \lambda_j, \lambda_k \leq \Theta} \int (|\alpha_j|^2 |\alpha_k|^2 - \lambda_j^{-1} \lambda_k^{-1}) d\mu_0(u) \\ &= \sum_{\Lambda < \lambda_j \leq \Theta} \int (|\alpha_j|^4 - \lambda_j^{-2}) d\mu_0(u) + \sum_{\substack{\Lambda < \lambda_j, \lambda_k \leq \Theta \\ \lambda_j \neq \lambda_k}} \int (|\alpha_j|^2 |\alpha_k|^2 - \lambda_j^{-1} \lambda_k^{-1}) d\mu_0(u) \\ &= \sum_{\Lambda < \lambda_j \leq \Theta} \lambda_j^{-2} \rightarrow 0 \text{ as } \Lambda, \Theta \rightarrow \infty \end{aligned}$$

due to $\text{Tr}[h^{-2}] = \sum_{j \geq 1} \lambda_j^{-2} < \infty$ since $2 > \frac{1}{2} + \frac{1}{s}$ for all $1 < s \leq 2$ (see Lemma A.1). This shows that $\{\mathcal{M}_{\leq \Lambda}(u)\}_{\Lambda \geq \lambda_1}$ is a Cauchy sequence in $L^2(d\mu_0)$. Thus there exists $\mathcal{M}(u) \in L^2(d\mu_0)$ such that $\mathcal{M}_{\leq \Lambda}(u) \rightarrow \mathcal{M}(u)$ strongly in $L^2(d\mu_0)$ as $\Lambda \rightarrow \infty$. Moreover, from the above computation, we have

$$\int (\mathcal{M}(u))^2 d\mu_0(u) = \lim_{\Lambda \rightarrow \infty} \int (\mathcal{M}_{\leq \Lambda}(u))^2 d\mu_0(u) = \lim_{\Lambda \rightarrow \infty} \sum_{\lambda_j \leq \Lambda} \lambda_j^{-2} = \text{Tr}[h^{-2}].$$

□

We have the following observation regarding the renormalized mass.

Lemma 3.7 (Decay estimates for the renormalized mass).

Let $1 < s \leq 2$, V satisfy Assumption 1.1, and $0 \leq \gamma < \frac{3s-2}{4s}$. Then there exist $C, c > 0$

such that for $\theta < \frac{1}{2} - \frac{1}{s}$ and all $\Lambda, R > 0$,

$$\mu_0(u \in \mathcal{H}^\theta : |\mathcal{M}_{>\Lambda}(u)| > R) \leq Ce^{-c\Lambda^\gamma R}, \quad (3.18)$$

where

$$\mathcal{M}_{>\Lambda}(u) := \sum_{\lambda_j > \Lambda} |\alpha_j|^2 - \lambda_j^{-1}. \quad (3.19)$$

Proof. We have

$$\mu_0(|\mathcal{M}_{>\Lambda}(u)| > R) \leq \mu_0(\mathcal{M}_{>\Lambda}(u) > R) + \mu_0(\mathcal{M}_{>\Lambda}(u) < -R) =: \text{(I)} + \text{(II)}.$$

For (I), we estimate for $0 < t < \frac{\Lambda}{2}$ to be chosen later,

$$\begin{aligned} \mu_0(\mathcal{M}_{>\Lambda}(u) > R) &\leq e^{-tR} \int e^{t\mathcal{M}_{>\Lambda}(u)} d\mu_0(u) \\ &= e^{-tR} \int e^{t\left(\sum_{\lambda_j > \Lambda} |\alpha_j|^2 - \lambda_j^{-1}\right)} d\mu_0(u) \\ &= e^{-tR} \int \prod_{\lambda_j > \Lambda} e^{-t\lambda_j^{-1}} e^{t|\alpha_j|^2} d\mu_0(u) \\ &= e^{-tR} \prod_{\lambda_j > \Lambda} e^{-t\lambda_j^{-1}} \left(\int_{\mathbb{C}} \frac{\lambda_j}{\pi} e^{-\lambda_j(1-t\lambda_j^{-1})|\alpha_j|^2} d\alpha_j \right) \left(\prod_{\substack{k \geq 1 \\ \lambda_k \neq \lambda_j}} \int_{\mathbb{C}} \frac{\lambda_k}{\pi} e^{-\lambda_k|\alpha_k|^2} d\alpha_k \right) \\ &= e^{-tR} \prod_{\lambda_j > \Lambda} e^{-t\lambda_j^{-1}} \frac{1}{1-t\lambda_j^{-1}}. \end{aligned}$$

Note that for $0 \leq x \leq \frac{1}{2}$, we have $\frac{1}{1-x} \leq Ce^{x+x^2}$ for some constant $C > 0$. Applying this to $x = t\lambda_j^{-1} \leq t\Lambda^{-1} \leq \frac{1}{2}$, we get

$$\mu_0(\mathcal{M}_{>\Lambda}(u) > R) \leq Ce^{-tR} \prod_{\lambda_j > \Lambda} e^{t^2\lambda_j^{-2}} = Ce^{-tR} e^{t^2 \sum_{\lambda_j > \Lambda} \lambda_j^{-2}}.$$

For $0 \leq \gamma < \frac{3s-2}{4s}$, we observe that

$$\sum_{\lambda_j > \Lambda} \lambda_j^{-2} \leq \Lambda^{-2\gamma} \sum_{\lambda_j > \Lambda} \lambda_j^{-2+2\gamma} \leq \Lambda^{-2\gamma} \text{Tr}[h^{-2+2\gamma}].$$

Here $\text{Tr}[h^{-2+2\gamma}] < \infty$ since $2 - 2\gamma > \frac{1}{2} + \frac{1}{s}$ for all $1 < s \leq 2$ (see Lemma A.1). Thus we get

$$\mu_0(\mathcal{M}_{>\Lambda}(u) > R) \leq Ce^{-tR+t^2\Lambda^{-2\gamma}\text{Tr}[h^{-2+2\gamma}]}.$$

Taking $t = \nu\Lambda^\gamma$ with $\nu > 0$ small, we obtain

$$\mu_0(\mathcal{M}_{>\Lambda}(u) > R) \leq Ce^{-c\Lambda^\gamma R}.$$

For (II), we have for $0 < t < \frac{\Lambda}{2}$,

$$\mu_0(\mathcal{M}_{>\Lambda}(u) < -R) \leq e^{-tR} \int e^{-t\mathcal{M}_{>\Lambda}(u)} d\mu_0(u) = e^{-tR} \prod_{\lambda_j > \Lambda} e^{t\lambda_j^{-1}} \frac{1}{1+t\lambda_j^{-1}}.$$

Note that for $0 \leq x \leq \frac{1}{2}$, we have $\frac{1}{1+x} \leq Ce^{-x+x^2}$ for some constant $C > 0$. Estimating as in (I), we prove as well that

$$\mu_0(\mathcal{M}_{>\Lambda}(u) < -R) \leq Ce^{-c\Lambda^\gamma R}.$$

Collecting both terms, we prove (3.18). \square

Our construction of the focusing Gibbs measure for $s \leq 2$ uses² the following interpolation inequality, due to Brézis and Mironescu [7]. It generalizes well-known results [38, 39] to fractional Sobolev spaces, which is handy in our case, for our measures can only afford at best $1/2$ derivative.

Lemma 3.8 (Fractional Gagliardo–Nirenberg inequality).

Let $1 < s \leq 2$, V satisfy Assumption 1.1, $0 < \beta < \frac{1}{2}$, $1 \leq p \leq \infty$, and

$$0 < \delta = \frac{1}{2 + 4\beta - \frac{4}{p}} < 1.$$

Then there exists $C > 0$ such that

$$\|u\|_{L^4} \leq C \|u\|_{L^2}^{1-\delta} \|u\|_{W^{\beta,p}}^\delta. \quad (3.20)$$

We have the inequality in usual Sobolev spaces $W^{\beta,p}(\mathbb{R})$ (see[7])

$$\|u\|_{L^4(\mathbb{R})} \leq C \|u\|_{L^2(\mathbb{R})}^{1-\delta} \|u\|_{W^{\beta,p}(\mathbb{R})}^\delta$$

provided that $1 \leq p \leq \infty$ and $\delta \in (0, 1)$ satisfy

$$\frac{1}{4} = \frac{1-\delta}{2} + \frac{\delta}{p} - \delta\beta$$

or

$$\delta = \frac{1}{2 + 4\beta - \frac{4}{p}}.$$

Hence (3.20) follows from the norm equivalence (B.1).

We now conclude the last part of Proposition 2.2.

Proof of Proposition 2.2, Item (3). We have

$$Z \geq \int \mathbf{1}_{\{|\mathcal{M}(u)| < m\}} d\mu_0(u) \geq \frac{1}{m^2} \int (\mathcal{M}(u))^2 d\mu_0(u) = \frac{1}{m^2} \text{Tr}[h^{-2}] > 0.$$

We next show that $Z < \infty$. We have

$$\begin{aligned} Z &= \int e^{\frac{1}{2}\|u\|_{L^4}^4} \mathbf{1}_{\{|\mathcal{M}(u)| < m\}} d\mu_0(u) \\ &= \int_0^\infty \mu_0 \left(e^{\frac{1}{2}\|u\|_{L^4}^4} > \lambda, |\mathcal{M}(u)| < m \right) d\lambda \\ &= \int_0^\infty \mu_0 \left(\|u\|_{L^4} > (2 \log \lambda)^{1/4}, |\mathcal{M}(u)| < m \right) d\lambda. \end{aligned}$$

Denote

$$\Lambda_0 := (\log \lambda)^l \quad (3.21)$$

for some $l > 0$ to be determined later. By the triangle inequality, we have

$$\begin{aligned} \mu_0 \left(\|u\|_{L^4} > (2 \log \lambda)^{1/4}, |\mathcal{M}(u)| < m \right) &\leq \mu_0 \left(\|P_{>\Lambda_0} u\|_{L^4} > \frac{1}{2} (2 \log \lambda)^{1/4}, |\mathcal{M}(u)| < m \right) \\ &\quad + \mu_0 \left(\|P_{\leq \Lambda_0} u\|_{L^4} > \frac{1}{2} (2 \log \lambda)^{1/4}, |\mathcal{M}(u)| < m \right) \\ &=: \text{(I)} + \text{(II)}. \end{aligned}$$

By Lemma 3.5, we have for any $0 \leq \rho < \frac{s-1}{2s}$,

$$\text{(I)} \leq \mu_0 \left(\|P_{>\Lambda_0} u\|_{L^4} > \frac{1}{2} (2 \log \lambda)^{1/4} \right) \leq C e^{-c\Lambda_0^\rho (\log \lambda)^{1/2}} = C e^{-c(\log \lambda)^{\rho l + 1/2}}. \quad (3.22)$$

²This is the origin of our technical restriction $s > 8/5$.

For (II), we denote

$$A_{\Lambda_0, \lambda} := \left\{ \|P_{\leq \Lambda_0} u\|_{L^4} > \frac{1}{2} (2 \log \lambda)^{1/4}, |\mathcal{M}(u)| < m \right\}.$$

We estimate

$$\begin{aligned} \|P_{\leq \Lambda_0} u\|_{L^2}^2 &= \sum_{\lambda_j \leq \Lambda_0} |\alpha_j|^2 \\ &= \sum_{\lambda_j \leq \Lambda_0} |\alpha_j|^2 - \lambda_j^{-1} + \sum_{\lambda_j \leq \Lambda_0} \lambda_j^{-1} \\ &= \mathcal{M}_{\leq \Lambda_0}(u) + \sum_{\lambda_j \leq \Lambda_0} \lambda_j^{-1} \\ &= \mathcal{M}(u) - \mathcal{M}_{> \Lambda_0}(u) + \sum_{\lambda_j \leq \Lambda_0} \lambda_j^{-1}, \end{aligned} \quad (3.23)$$

where $\mathcal{M}_{\leq \Lambda}(u)$ and $\mathcal{M}_{> \Lambda}(u)$ are defined in (3.17) and (3.19) respectively. Observe that

$$\sum_{\lambda_j \leq \Lambda_0} \lambda_j^{-1} \leq \Lambda_0^\nu \sum_{\lambda_j \leq \Lambda_0} \lambda_j^{-1-\nu} = \Lambda_0^\nu \text{Tr}[h^{-1-\nu}] \quad (3.24)$$

with $\text{Tr}[h^{-1-\nu}] < \infty$ provided that $\nu > \frac{1}{s} - \frac{1}{2}$. Define the set

$$\Omega_{\Lambda_0, \lambda} := \{|\mathcal{M}_{> \Lambda_0}(u)| \leq \Lambda_0^\nu\}.$$

We have

$$(II) = \mu_0(A_{\Lambda_0, \lambda}) \leq \mu_0(A_{\Lambda_0, \lambda} \cap \Omega_{\Lambda_0, \lambda}^c) + \mu_0(A_{\Lambda_0, \lambda} \cap \Omega_{\Lambda_0, \lambda}) =: (II_1) + (II_2).$$

By Lemma 3.7, we have for any $0 \leq \gamma < \frac{3s-2}{4s}$,

$$(II_1) \leq \mu_0(\Omega_{\Lambda_0, \lambda}^c) \leq C e^{-c\Lambda_0^{\gamma+\nu}} = C e^{-c(\log \lambda)^{(\gamma+\nu)l}}. \quad (3.25)$$

For $u \in A_{\Lambda_0, \lambda} \cap \Omega_{\Lambda_0, \lambda}$, we deduce from (3.23) and (3.24) that

$$\|P_{\leq \Lambda_0} u\|_{L^2} \leq \sqrt{m + C\Lambda_0^\nu} \leq C\Lambda_0^{\nu/2}.$$

Fix $0 < \beta < \frac{1}{2}$, we can find an even integer p sufficiently large such that $p > \frac{4}{s(1-2\beta)}$.

Moreover, using (3.20), we find that for any $u \in A_{\Lambda_0, \lambda} \cap \Omega_{\Lambda_0, \lambda}$,

$$\frac{1}{2} (2 \log \lambda)^{1/4} < \|P_{\leq \Lambda_0} u\|_{L^4} \leq C \|P_{\leq \Lambda_0} u\|_{L^2}^{1-\delta} \|P_{\leq \Lambda_0} u\|_{\mathcal{W}^{\beta, p}}^\delta \leq C \Lambda_0^{\nu(1-\delta)/2} \|P_{\leq \Lambda_0} u\|_{\mathcal{W}^{\beta, p}}^\delta$$

hence

$$\|P_{\leq \Lambda_0} u\|_{\mathcal{W}^{\beta, p}} > C \left(\frac{(\log \lambda)^{1/4}}{\Lambda_0^{\nu(1-\delta)/2}} \right)^{\frac{1}{\delta}} \sim (\log \lambda)^{\frac{1}{4\delta} - \frac{l\nu}{2}(\frac{1}{\delta}-1)} \sim (\log \lambda)^{\frac{1}{2} + \beta - \frac{1}{p} - \frac{l\nu}{2}(1+4\beta - \frac{4}{p})}.$$

Thus, by (3.9), we get

$$\begin{aligned} (II_2) &\leq \mu_0 \left(\|P_{\leq \Lambda_0} u\|_{\mathcal{W}^{\beta, p}} > C (\log \lambda)^{\frac{1}{2} + \beta - \frac{1}{p} - \frac{l\nu}{2}(1+4\beta - \frac{4}{p})} \right) \\ &\leq C e^{-c(\log \lambda)^{1+2\beta - \frac{2}{p} - l\nu(1+4\beta - \frac{4}{p})}}. \end{aligned} \quad (3.26)$$

Collecting (3.22), (3.25), and (3.26), we obtain

$$\begin{aligned} \mu_0 \left(\|u\|_{L^4} > (2 \log \lambda)^{1/4}, |\mathcal{M}(u)| < m \right) &\leq C e^{-c(\log \lambda)^{\rho l + 1/2}} + C e^{-c(\log \lambda)^{(\gamma+\nu)l}} \\ &\quad + C e^{-c(\log \lambda)^{1+2\beta - \frac{2}{p} - l\nu(1+4\beta - \frac{4}{p})}} \end{aligned}$$

for any $0 \leq \rho < \frac{s-1}{2s}$, any $0 \leq \gamma < \frac{3s-2}{4s}$, and any $\nu > \frac{2-s}{2s}$. Using the fact that for any $L > 0$, there exists $C_L > 0$ such that $e^{-c(\log \lambda)^{1+\varepsilon}} \leq C_L \lambda^{-L}$ provided that $\varepsilon > 0$, we have

$Z < \infty$ by choosing suitable values of l, ρ, γ , and ν so that the following conditions are satisfied:

$$\rho l + 1/2 > 1, \quad (\gamma + \nu)l > 1, \quad 1 + 2\beta - \frac{2}{p} - l\nu \left(1 + 4\beta - \frac{4}{p}\right) > 1.$$

By taking $\rho = \frac{s-1}{2s}-$, $\gamma = \frac{3s-2}{4s}-$, and $\nu = \frac{2-s}{2s}+$, the first two conditions imply

$$l > \max \left\{ \frac{s}{s-1}+, \frac{4s}{s+2} \right\}.$$

Taking $\beta = \frac{1}{2}-$ and p sufficiently large, the last condition yields

$$l < \frac{2s}{3(2-s)}.$$

Since $\frac{4s}{s+2} \leq \frac{s}{s-1} < \frac{2s}{3(2-s)}$ for $\frac{8}{5} < s \leq 2$, we can choose l satisfying the above conditions. This shows that for any $L > 0$,

$$\mu_0 \left(\|u\|_{L^4} > (2 \log \lambda)^{1/4}, |\mathcal{M}(u)| < m \right) \leq C_L \lambda^{-L}$$

which ensures $Z < \infty$. The proof is complete. \square

3.4. Approximating measures. We next define approximate measures for μ which are useful in proving the invariance of Gibbs measures under the flow of (1.1). Following [8], we introduce for $\Lambda \geq \lambda_1$,

$$Q_\Lambda u := \sum_{j \geq 1} \chi \left(\frac{\lambda_j}{\Lambda} \right) \alpha_j u_j = \chi(h/\Lambda)u, \quad (3.27)$$

where $\chi \in C_0^\infty(\mathbb{R})$ satisfies $\text{supp}(\chi) \subset [-1, 1]$, $\chi \in [0, 1]$, and $\chi = 1$ on $[-1/2, 1/2]$. It is known that

$$Q_\Lambda P_{\leq \Lambda} = P_{\leq \Lambda} Q_\Lambda = Q_\Lambda \quad (3.28)$$

and there exists $C > 0$ such that for all $\Lambda \geq \lambda_1$,

$$\|Q_\Lambda\|_{L^p \rightarrow L^p} \leq C, \quad 1 \leq p \leq \infty. \quad (3.29)$$

The latter follows from the L^p -boundedness of semi-classical pseudo-differential operators as we briefly recall in Appendix C.

For the defocusing nonlinearity, we define the approximate measure as

$$d\mu_\Lambda(u) := \frac{1}{Z^\Lambda} e^{-\frac{1}{2}\|Q_\Lambda u\|_{L^4}^4} d\mu_0(u) = d\mu^{\leq \Lambda}(u) \otimes d\mu_0^{> \Lambda}(u), \quad (3.30)$$

where

$$d\mu_0^{> \Lambda}(u) = \prod_{\lambda_j > \Lambda} \frac{\lambda_j}{\pi} e^{-\lambda_j |\alpha_j|^2} d\alpha_j \quad (3.31)$$

and

$$d\mu^{\leq \Lambda}(u) = \frac{1}{Z^\Lambda} e^{-\frac{1}{2}\|Q_\Lambda u\|_{L^4}^4} d\mu_0^{\leq \Lambda}(u)$$

with $\mu_0^{\leq \Lambda}$ as in (2.8) and

$$Z^\Lambda = \int e^{-\frac{1}{2}\|Q_\Lambda u\|_{L^4}^4} d\mu_0(u) = \int e^{-\frac{1}{2}\|Q_\Lambda u\|_{L^4}^4} d\mu_0^{\leq \Lambda}(u).$$

For the focusing nonlinearity, the definitions include the natural cut-offs:

$$d\mu_\Lambda(u) := d\mu^{\leq \Lambda}(u) \otimes d\mu_0^{> \Lambda}(u), \quad (3.32)$$

where

$$d\mu^{\leq \Lambda}(u) = \frac{1}{Z^\Lambda} e^{\frac{1}{2}\|Q_\Lambda u\|_{L^4}^4} \mathbb{1}_{\{\|P_{\leq \Lambda} u\|_{L^2}^2 < m\}} d\mu_0^{\leq \Lambda}(u)$$

if $s > 2$ and

$$d\mu^{\leq \Lambda}(u) = \frac{1}{Z^\Lambda} e^{\frac{1}{2}\|Q_\Lambda u\|_{L^4}^4} \mathbb{1}_{\{|\mathcal{M}_{\leq \Lambda}(u)| < m\}} d\mu_0^{\leq \Lambda}(u)$$

if $\frac{8}{5} < s \leq 2$. The partition functions Z^Λ turn these into probability measures, as usual.

Lemma 3.9 (Approximating measures).

Let $s > 1$, V satisfy Assumption 1.1, $\theta < \frac{1}{2} - \frac{1}{s}$, and $\Lambda \geq \lambda_1$. Assume in addition that $s > \frac{8}{5}$ for the focusing nonlinearity. Then the measures μ_Λ are well-defined and absolutely continuous with respect to the Gaussian measure μ_0 . Moreover, μ_Λ converge to μ in the sense that for any measurable set $A \subset \mathcal{H}^\theta$,

$$\lim_{\Lambda \rightarrow \infty} \mu_\Lambda(A) = \mu(A). \quad (3.33)$$

Proof. Thanks to (3.28) and (3.29), the well-definedness of approximating measures follows exactly as the case $\Lambda = \infty$ of the full measure considered above. In addition, the normalization constants Z^Λ are finite uniformly in Λ .

We first prove (3.33) in the defocusing case. Denote

$$G_\Lambda(u) := e^{-\frac{1}{2}\|Q_\Lambda u\|_{L^4}^4}, \quad G(u) := e^{-\frac{1}{2}\|u\|_{L^4}^4} \quad (3.34)$$

We first claim that $G_\Lambda(u) \rightarrow G(u)$ in measure with respect to μ_0 , i.e.,

$$\forall \varepsilon > 0, \quad \lim_{\Lambda \rightarrow \infty} \mu_0 \left(u \in \mathcal{H}^\theta : |G_\Lambda(u) - G(u)| > \varepsilon \right) = 0. \quad (3.35)$$

Since $\mu_0(\mathcal{H}^\theta) = 1$, the convergence in measure is preserved under composition and multiplication by continuous functions. It suffices to show that $\|Q_\Lambda u\|_{L^4} \rightarrow \|u\|_{L^4}$ in measure with respect to μ_0 . By Chebyshev's inequality, namely

$$\mu_0 \left(\left| \|Q_\Lambda u\|_{L^4} - \|u\|_{L^4} \right| > \varepsilon \right) \leq \frac{1}{\varepsilon^4} \int \left| \|Q_\Lambda u\|_{L^4} - \|u\|_{L^4} \right|^4 d\mu_0(u),$$

it suffices to show that

$$\int \|Q_\Lambda u - u\|_{L^4}^4 d\mu_0(u) \rightarrow 0 \text{ as } \Lambda \rightarrow \infty.$$

To see this, we denote $R_\Lambda := Q_\Lambda - \text{Id}$, hence

$$R_\Lambda u = \sum_{j \geq 1} R(\lambda_j) \alpha_j u_j, \quad R(\lambda_j) := \chi \left(\frac{\lambda_j}{\Lambda} \right) - 1.$$

Using (3.2), we have³

$$\begin{aligned} \int |R_\Lambda u(x)|^4 d\mu_0(u) &= \text{Tr} \left[(\delta_x)^{\otimes 2} \int |(R_\Lambda u)^{\otimes 2} \rangle \langle (R_\Lambda u)^{\otimes 2} | d\mu_0(u) (\delta_x)^{\otimes 2} \right] \\ &\leq 2! \text{Tr} \left[(\delta_x)^{\otimes 2} (R_\Lambda h^{-1} R_\Lambda)^{\otimes 2} (\delta_x)^{\otimes 2} \right] \\ &\leq 2! \left(\text{Tr} [\delta_x R_\Lambda h^{-1} R_\Lambda \delta_x] \right)^2 \\ &= 2! \left(\sum_{j \geq 1} |R(\lambda_j)|^2 \lambda_j^{-1} |u_j(x)|^2 \right)^2 \end{aligned}$$

By the choice of χ , we have $R(\lambda_j) = 0$ for $\lambda_j \leq \Lambda/2$ and $|R(\lambda_j)| \leq 2$ for all j , hence

$$\int \|R_\Lambda u\|_{L^4}^4 d\mu_0(u) \leq 32 \int \left(\sum_{\lambda_j > \Lambda/2} \lambda_j^{-1} |u_j(x)|^2 \right)^2 dx \rightarrow 0 \text{ as } \Lambda \rightarrow \infty.$$

Thus the claim follows.

³Again, regularizing the delta functions as in the proof of Lemma 3.2.

Now let A be a measurable set in \mathcal{H}^θ . We will show that

$$\lim_{\Lambda \rightarrow \infty} \int \mathbf{1}_A G_\Lambda(u) d\mu_0(u) = \int \mathbf{1}_A G(u) d\mu_0(u) \quad (3.36)$$

which is (3.33). Let $\varepsilon > 0$. We introduce the set

$$B_{\Lambda, \varepsilon} := \left\{ u \in \mathcal{H}^\theta : |G_\Lambda(u) - G(u)| \leq \varepsilon \right\}.$$

We have

$$\left| \int_{B_{\Lambda, \varepsilon}} \mathbf{1}_A (G_\Lambda(u) - G(u)) d\mu_0(u) \right| \leq \varepsilon.$$

On the other hand, since $G_\Lambda(u), G(u) \in L^2(d\mu_0)$ uniformly in Λ , we deduce from (3.35) that

$$\left| \int_{B_{\Lambda, \varepsilon}^c} \mathbf{1}_A (G_\Lambda(u) - G(u)) d\mu_0(u) \right| \leq \|G_\Lambda(u) - G(u)\|_{L^2(d\mu_0)} \sqrt{\mu_0(B_{\Lambda, \varepsilon}^c)} \rightarrow 0 \text{ as } \Lambda \rightarrow \infty.$$

Combining the above estimates, we prove (3.36).

For the focusing case, we denote

$$G_\Lambda(u) := e^{\frac{1}{2}\|Q_\Lambda u\|_{L^4}^4} \mathbf{1}_{\{\|P_{\leq \Lambda} u\|_{L^2}^2 < m\}}, \quad G(u) := e^{\frac{1}{2}\|u\|_{L^4}^4} \mathbf{1}_{\{\|u\|_{L^2}^2 < m\}} \quad (3.37)$$

for $s > 2$ and

$$G_\Lambda(u) := e^{\frac{1}{2}\|Q_\Lambda u\|_{L^4}^4} \mathbf{1}_{\{|\mathcal{M}_{\leq \Lambda}(u)| < m\}}, \quad G(u) := e^{\frac{1}{2}\|u\|_{L^4}^4} \mathbf{1}_{\{|\mathcal{M}(u)| < m\}} \quad (3.38)$$

for $\frac{8}{5} < s \leq 2$. Since $G_\Lambda(u), G(u) \in L^2(d\mu_0)$ uniformly in Λ , the same argument as in the defocusing case shows (3.33). \square

An immediate interest of the above measures is that they are easily shown to be invariant under the flow of the approximation NLS equation

$$\begin{cases} i \partial_t u_\Lambda - h u_\Lambda &= \pm Q_\Lambda (|Q_\Lambda u_\Lambda|^2 Q_\Lambda u_\Lambda), & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u_\Lambda|_{t=0} &= f. \end{cases} \quad (3.39)$$

As in e.g., [2, 3, 8], we shall use this fact extensively.

Lemma 3.10 (Approximate NLS dynamics).

Let $s > 1$, V satisfy Assumption 1.1, $\theta < \frac{1}{2} - \frac{1}{s}$, and $f \in \mathcal{H}^\theta$. Then for each $\Lambda \geq \lambda_1$, the solution to (3.39) exists globally in time. Moreover, the measures μ_Λ defined in (3.30) or (3.32) are invariant under the flow of (3.39).

Proof. Step 1, global existence. We write $u_\Lambda = u_\Lambda^{\text{lo}} + u_\Lambda^{\text{hi}}$ with $u_\Lambda^{\text{lo}} := P_{\leq \Lambda} u_\Lambda$ and $u_\Lambda^{\text{hi}} := P_{> \Lambda} u_\Lambda$. As $P_{> \Lambda} Q_\Lambda = 0$, the high frequency part satisfies

$$\begin{cases} i \partial_t u_\Lambda^{\text{hi}} - h u_\Lambda^{\text{hi}} &= 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u_\Lambda^{\text{hi}}|_{t=0} &= P_{> \Lambda} f. \end{cases} \quad (3.40)$$

If we write

$$u_\Lambda^{\text{hi}}(t, x) = \sum_{\lambda_j > \Lambda} \alpha_j^{\text{hi}}(t) u_j(x), \quad P_{> \Lambda} f = \sum_{\lambda_j > \Lambda} \alpha_{j0} u_j, \quad \alpha_{j0} = \langle f, u_j \rangle,$$

then we have

$$\begin{cases} i \partial_t \alpha_j^{\text{hi}} - \lambda_j \alpha_j^{\text{hi}} &= 0, \\ \alpha_j^{\text{hi}}|_{t=0} &= \alpha_{j0}, \end{cases}$$

hence $\alpha_j^{\text{hi}}(t) = e^{-it\lambda_j} \alpha_{j0}$ or

$$u_\Lambda^{\text{hi}}(t, x) = \sum_{\lambda_j > \Lambda} e^{-it\lambda_j} \alpha_{j0} u_j(x).$$

In particular, the high frequency part exists globally in time.

On the other hand, by applying $P_{\leq \Lambda}$ to both sides of (3.39) and using (3.28), we get

$$\begin{cases} i \partial_t u_\Lambda^{\text{lo}} - h u_\Lambda^{\text{lo}} &= \pm Q_\Lambda (|Q_\Lambda u_\Lambda^{\text{lo}}|^2 Q_\Lambda u_\Lambda^{\text{lo}}), \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u_\Lambda^{\text{lo}}|_{t=0} &= P_{\leq \Lambda} f. \end{cases} \quad (3.41)$$

If we write

$$u_\Lambda^{\text{lo}}(t, x) = \sum_{\lambda_j \leq \Lambda} \alpha_j^{\text{lo}}(t) u_j(x), \quad \alpha_j^{\text{lo}} = a_j^{\text{lo}} + i b_j^{\text{lo}},$$

then (3.41) is a Hamiltonian ODE of the form

$$\partial_t a_j^{\text{lo}} = \frac{\partial H}{\partial b_j^{\text{lo}}}, \quad \partial_t b_j^{\text{lo}} = -\frac{\partial H}{\partial a_j^{\text{lo}}}, \quad \lambda_j \leq \Lambda,$$

where

$$\begin{aligned} H(u_\Lambda^{\text{lo}}) &= \|h^{1/2} u_\Lambda^{\text{lo}}\|_{L^2}^2 \pm \frac{1}{2} \|Q_\Lambda u_\Lambda^{\text{lo}}\|_{L^4}^4 \\ &= \sum_{\lambda_j \leq \Lambda} \lambda_j |\alpha_j^{\text{lo}}|^2 \pm \frac{1}{2} \left\| Q_\Lambda \left(\sum_{\lambda_j \leq \Lambda} \alpha_j^{\text{lo}} u_j \right) \right\|_{L^4}^4 =: H(a_j^{\text{lo}}, b_j^{\text{lo}}) \end{aligned}$$

is conserved under the flow of (3.41). By the Cauchy-Lipschitz theorem, there exists a local solution to (3.41). In addition, thanks to the conservation of mass

$$M(u_\Lambda^{\text{lo}}) = \|u_\Lambda^{\text{lo}}\|_{L^2}^2 = \sum_{\lambda_j \leq \Lambda} |\alpha_j^{\text{lo}}|^2,$$

local solutions can be extended globally in time. Thus u_Λ^{lo} exists globally in time.

Step 2, measure invariance. We show that $\mu_0^{> \Lambda}$ and $\mu_0^{\leq \Lambda}$ are invariant under the flow of (3.40) and (3.41). To see this, we denote by $\Phi_\Lambda^{\text{hi}}(t)$ and $\Phi_\Lambda^{\text{lo}}(t)$ the solution maps of (3.40) and (3.41) separately.

Let A be a measurable set in \mathcal{H}^θ with $\theta < \frac{1}{2} - \frac{1}{s}$. We have

$$\begin{aligned} \mu_0^{> \Lambda}(\Phi_\Lambda^{\text{hi}}(t)(A)) &= \int_A d\mu_0^{> \Lambda}(u_\Lambda^{\text{hi}}(t)) \\ &= \int_A \prod_{\lambda_j > \Lambda} \frac{\lambda_j}{\pi} e^{-\lambda_j |\alpha_\Lambda^{\text{hi}}(t)|^2} d\alpha_j^{\text{hi}}(t) \\ &= \int_A \prod_{\lambda_j > \Lambda} \frac{\lambda_j}{\pi} e^{-\lambda_j |e^{-i\lambda_j} \alpha_{j0}|^2} d(e^{-it\lambda_j} \alpha_{j0}) \\ &= \int_A \prod_{\lambda_j > \Lambda} \frac{\lambda_j}{\pi} e^{-\lambda_j |\alpha_{j0}|^2} d\alpha_{j0} \\ &= \int_A d\mu_0^{> \Lambda}(u_0) = \mu_0^{> \Lambda}(A), \end{aligned}$$

where the fourth line follows from the invariance of the Lebesgue measure under rotations. This shows that $\mu_0^{> \Lambda}$ is invariant under the flow of (3.40).

Since $\mu_0^{\leq \Lambda}$ is finite dimensional, the invariance of $\mu_0^{\leq \Lambda}$ under the flow of (3.41) follows directly from the conservation of the Hamiltonian $H(u_\Lambda^{\text{lo}})$ and the Liouville theorem for $du_\Lambda^{\text{lo}} = \prod_{\lambda_j \leq \Lambda} da_j^{\text{lo}} db_j^{\text{lo}}$. We also use conservation of mass for the focusing measures. \square

4. CAUCHY PROBLEM AND INVARIANT MEASURE, SUPER-HARMONIC CASE

We start our analysis of the Cauchy problem with the case $s > 2$, where the Gibbs measure is supported on Sobolev spaces with positive indices, i.e., in \mathcal{H}^θ with $0 < \theta < \frac{1}{2} - \frac{1}{s}$. Thus there is a chance to expect a deterministic local well-posedness with initial data lying in the support of the Gibbs measure. This is indeed what we provide in Section 4.1. In Section 4.2, we globalize the flow by using the invariance of the approximate measures introduced above.

4.1. Deterministic local well-posedness. When $s > 2$, we may use tools from [60, 61, 62] to obtain the following deterministic local well-posedness result.

Proposition 4.1 (Deterministic local well-posedness, $s > 2$).

Let $s > 2$, V satisfy Assumption 1.1, and θ satisfy

$$\frac{1}{2} \left(\frac{1}{2} - \frac{1}{s} \right) < \theta < \frac{1}{2} - \frac{1}{s}. \quad (4.1)$$

Let (p, q) be an admissible pair as in Appendix B and $\gamma > 0$ satisfying

$$p > 4, \quad \frac{1}{2} - \frac{2}{p} < \gamma < \theta - \frac{2}{p} \left(\frac{1}{2} - \frac{1}{s} \right). \quad (4.2)$$

Then for any $f \in \mathcal{H}^\theta$, there exist $\delta > 0$ and a unique solution to (1.1) with initial datum $u|_{t=0} = f$ satisfying

$$u \in L^\infty([-\delta, \delta], \mathcal{H}^\theta) \cap L^p([-\delta, \delta], \mathcal{W}^{\gamma, q}).$$

In particular, if $\|f\|_{\mathcal{H}^\theta} \leq K$, then

$$\|u(t)\|_{\mathcal{H}^\theta} \leq 2CK \quad (4.3)$$

for all $|t| \leq \delta \sim K^{-\varrho}$ with $\varrho > 0$ and some universal constant $C > 0$. In addition, if $u_1(t)$ and $u_2(t)$ are respectively solutions to (1.1) with initial data $u_1|_{t=0} = f_1$, $u_2|_{t=0} = f_2 \in \mathcal{H}^\theta$ that satisfy $\|f_1\|_{\mathcal{H}^\theta}, \|f_2\|_{\mathcal{H}^\theta} \leq K$, then

$$\|u_1(t) - u_2(t)\|_{\mathcal{H}^\theta} \leq 2C\|f_1 - f_2\|_{\mathcal{H}^\theta} \quad (4.4)$$

for all $|t| \leq \delta \sim K^{-\varrho}$.

Let us comment briefly on the conditions (4.1) and (4.2). The second inequality in (4.1) ensures that the support of the Gibbs measure is contained in \mathcal{H}^θ . The first inequality in (4.1) guarantees the existence of an admissible pair (p, q) given in (4.2). The first condition in (4.2) allows us to use Hölder's inequality in time. The first inequality in the second condition in (4.2) coupled with the admissibility of (p, q) yields

$$\frac{1}{q} = \frac{1}{2} - \frac{2}{p} < \gamma$$

so that we have the Sobolev embedding $\mathcal{W}^{\gamma, q} \subset L^\infty(\mathbb{R})$. Finally, the second inequality in the second condition in (4.2) implies $\theta - \gamma > \frac{2}{p} \left(\frac{1}{2} - \frac{1}{s} \right)$ which is needed to use Strichartz estimates with a loss of derivatives

$$\|e^{-ith} f\|_{L^p([-\delta, \delta], \mathcal{W}^{\gamma, q})} \lesssim \|f\|_{\mathcal{H}^\theta}. \quad (4.5)$$

See Appendix B for more details.

Proof of Proposition 4.1. The proof is essentially given in [61]. For the reader's convenience, we recall some details. Let $\delta > 0$ be a small parameter to be chosen later. We denote

$$X_\delta^{\theta, \gamma} := C([-\delta, \delta], \mathcal{H}^\theta) \cap L^p([-\delta, \delta], \mathcal{W}^{\gamma, q})$$

with the norm

$$\|u\|_{X_\delta^{\theta, \gamma}} = \|u\|_{L^\infty([-\delta, \delta], \mathcal{H}^\theta)} + \|u\|_{L^p([-\delta, \delta], \mathcal{W}^{\gamma, q})}.$$

It suffices to show that the Duhamel functional

$$\Phi_f(u(t)) = e^{-i t h} f \mp i \int_0^t e^{-i(t-\tau)h} |u(\tau)|^2 u(\tau) d\tau$$

is a contraction on $(B_{X_\delta^{\theta,\gamma}}(L), d)$, where

$$B_{X_\delta^{\theta,\gamma}}(L) := \left\{ u \in X_\delta^{\theta,\gamma} : \|u\|_{X_\delta^{\theta,\gamma}} \leq L \right\}$$

is the ball in $X_\delta^{\theta,\gamma}$ centered at zero and of radius L and

$$d(u_1, u_2) := \|u_1 - u_2\|_{X_\delta^{\theta,\gamma}}.$$

By the unitarity of $e^{-i t h}$ on \mathcal{H}^θ and the fractional product rule (see Lemma B.3), we have

$$\begin{aligned} \sup_{t \in [-\delta, \delta]} \|\Phi_f(u)(t)\|_{\mathcal{H}^\theta} &\leq \|f\|_{\mathcal{H}^\theta} + \int_0^\delta \| |u(\tau)|^2 u(\tau) \|_{\mathcal{H}^\theta} d\tau \\ &\leq \|f\|_{\mathcal{H}^\theta} + C \int_0^\delta \|u(\tau)\|_{L^\infty}^2 \|u(\tau)\|_{\mathcal{H}^\theta} d\tau \\ &\leq \|f\|_{\mathcal{H}^\theta} + C \|u\|_{L^2([-\delta, \delta], L^\infty)}^2 \|u\|_{L^\infty([-\delta, \delta], \mathcal{H}^\theta)} \\ &\leq \|f\|_{\mathcal{H}^\theta} + C \delta^{1-\frac{2}{p}} \|u\|_{L^p([-\delta, \delta], \mathcal{W}^{\gamma,q})}^2 \|u\|_{L^\infty([-\delta, \delta], \mathcal{H}^\theta)}, \end{aligned}$$

where we have used the Hölder inequality in time and the Sobolev embedding $\mathcal{W}^{\gamma,q} \subset L^\infty(\mathbb{R})$ to get the last inequality. On the other hand, using (4.5), we have

$$\begin{aligned} \|\Phi_f(u)\|_{L^p([-\delta, \delta], \mathcal{W}^{\gamma,q})} &\leq \|e^{-i t h} f\|_{L^p([-\delta, \delta], \mathcal{W}^{\gamma,q})} + \left\| \int_0^t e^{-i(t-\tau)h} |u(\tau)|^2 u(\tau) d\tau \right\|_{L^p([-\delta, \delta], \mathcal{W}^{\gamma,q})} \\ &\leq C \|f\|_{\mathcal{H}^\theta} + \int_0^\delta \|\mathbb{1}_{\{\tau < t\}} e^{-i(t-\tau)h} |u(\tau)|^2 u(\tau)\|_{L^p([-\delta, \delta], \mathcal{W}^{\gamma,q})} d\tau \\ &\leq C \|f\|_{\mathcal{H}^\theta} + \int_0^\delta \|e^{-i t h} e^{i \tau h} |u(\tau)|^2 u(\tau)\|_{L^p([-\delta, \delta], \mathcal{W}^{\gamma,q})} d\tau \\ &\leq C \|f\|_{\mathcal{H}^\theta} + C \int_0^\delta \| |u(\tau)|^2 u(\tau) \|_{\mathcal{H}^\theta} d\tau \\ &\leq C \|f\|_{\mathcal{H}^\theta} + C \delta^{1-\frac{2}{p}} \|u\|_{L^p([-\delta, \delta], \mathcal{W}^{\gamma,q})}^2 \|u\|_{L^\infty([-\delta, \delta], \mathcal{H}^\theta)}. \end{aligned}$$

In particular, we have

$$\|\Phi_f(u)\|_{X_\delta^{\theta,\gamma}} \leq C \|f\|_{\mathcal{H}^\theta} + C \delta^{1-\frac{2}{p}} \|u\|_{X_\delta^{\theta,\gamma}}^3.$$

By writing

$$|u_1|^2 u_1 - |u_2|^2 u_2 = (u_1 - u_2)(|u_1|^2 + |u_2|^2) + (\bar{u}_1 - \bar{u}_2) u_1 u_2,$$

the same argument gives

$$\begin{aligned} \|\Phi_f(u_1) - \Phi_f(u_2)\|_{X_\delta^{\theta,\gamma}} &\leq \left\| \int_0^t e^{-i(t-\tau)h} (|u_1(\tau)|^2 u_1(\tau) - |u_2(\tau)|^2 u_2(\tau)) d\tau \right\|_{X_\delta^{\theta,\gamma}} \\ &\leq C \delta^{1-\frac{2}{p}} \left(\|u_1\|_{X_\delta^{\theta,\gamma}}^2 + \|u_2\|_{X_\delta^{\theta,\gamma}}^2 \right) \|u_1 - u_2\|_{X_\delta^{\theta,\gamma}}. \end{aligned}$$

Hence there exists $C > 0$ such that for each $u_1, u_2 \in B_{X_\delta^{\theta,\gamma}}(L)$,

$$\begin{aligned} \|\Phi_f(u)\|_{X_\delta^{\theta,\gamma}} &\leq C \|f\|_{\mathcal{H}^\theta} + C \delta^{1-\frac{2}{p}} L^3, \\ d(\Phi_f(u_1), \Phi_f(u_2)) &\leq C \delta^{1-\frac{2}{p}} L^2 d(u_1, u_2). \end{aligned}$$

Taking $L = 2C\|f\|_{\mathcal{H}^\theta}$ and choosing $\delta > 0$ such that $C\delta^{1-\frac{2}{p}}L^2 \leq \frac{1}{2}$, we see that Φ_f is a contraction mapping on $(B_{X_\delta^{\theta,\gamma}}(L), d)$. This shows the existence of a solution satisfying (4.3). Moreover, if $\|f\|_{\mathcal{H}^\theta} \leq K$, then we can take $L = 2CK$ and $\delta = \nu K^{-\varrho}$ with $\varrho = \frac{2}{1-\frac{2}{p}}$ and $\nu > 0$ sufficiently small so that

$$C\delta^{1-\frac{2}{p}}L^2 = 4C^3\nu^{1-\frac{2}{p}} \leq \frac{1}{2}.$$

Thus the solution satisfies $\|u(t)\|_{\mathcal{H}^\theta} \leq 2CK$ for all $|t| \leq \delta \sim K^{-\varrho}$.

To see (4.4), we estimate as before and get

$$\|u_1 - u_2\|_{X_\delta^{\theta,\gamma}} \leq C\|f_1 - f_2\|_{\mathcal{H}^\theta} + C\delta^{1-\frac{2}{p}} \left(\|u_1\|_{X_\delta^{\theta,\gamma}}^2 + \|u_2\|_{X_\delta^{\theta,\gamma}}^2 \right) \|u_1 - u_2\|_{X_\delta^{\theta,\gamma}}.$$

As $\|f_1\|_{\mathcal{H}^\theta}, \|f_2\|_{\mathcal{H}^\theta} \leq K$, we have

$$\|u_1\|_{X_\delta^{\theta,\gamma}}, \|u_2\|_{X_\delta^{\theta,\gamma}} \leq 2CK,$$

hence

$$\|u_1 - u_2\|_{X_\delta^{\theta,\gamma}} \leq C\|f_1 - f_2\|_{\mathcal{H}^\theta} + C\delta^{1-\frac{2}{p}}K^2\|u_1 - u_2\|_{X_\delta^{\theta,\gamma}}.$$

We choose $\delta > 0$ small enough to have $C\delta^{1-\frac{2}{p}}K^2 \leq \frac{1}{2}$. Then we can absorb the second term in the right hand side in the left hand side to obtain

$$\|u_1 - u_2\|_{X_\delta^{\theta,\gamma}} \leq 2C\|f_1 - f_2\|_{\mathcal{H}^\theta}$$

which yields (4.4). The proof is complete. \square

4.2. Measure invariance and global well-posedness. We next show that the flow can be globalized for initial data in a set of full Gibbs measure.

Theorem 4.2 (Almost sure global well-posedness, $s > 2$).

Let $s > 2$, V satisfy Assumption 1.1, and θ satisfy

$$\frac{1}{2} \left(\frac{1}{2} - \frac{1}{s} \right) < \theta < \frac{1}{2} - \frac{1}{s}.$$

Then there exists a set $\Sigma \subset \mathcal{H}^\theta$ satisfying $\mu(\Sigma) = 1$ such that for any $f \in \Sigma$, the corresponding solution to (1.1) with initial datum $u|_{t=0} = f$ exists globally in time and satisfies

$$\|u(t)\|_{\mathcal{H}^\theta} \leq C \left(\omega(f) + \log^{\frac{1}{2}}(1 + |t|) \right), \quad \forall t \in \mathbb{R} \quad (4.6)$$

for some constant $\omega(f) > 0$ depending on f and some universal constant $C > 0$. Moreover, the Gibbs measure μ is invariant under the flow of (1.1).

The proof is built on two main ingredients:

- The approximate NLS flow (3.39) is globalized with explicit bounds on a set of almost full measure.
- The local solution of the full flow (1.1) is shown to stay close to the approximate solution.

We start with the former ingredient, where we use crucially the invariance of μ_Λ under the approximate flow.

Lemma 4.3 (Uniform estimate for the approximate flow, $s > 2$).

Let s, θ be as in the previous statement and take θ_1 satisfying

$$\theta < \theta_1 < \frac{1}{2} - \frac{1}{s}.$$

Then for all $\Lambda \geq \lambda_1$ and $T, \varepsilon > 0$. There exist $\Sigma_{\Lambda, T, \varepsilon} \subset \mathcal{H}^{\theta_1}$ and $C > 0$ independent of Λ, T, ε such that:

- (1) $\mu_\Lambda(\Sigma_{\Lambda, T, \varepsilon}^c) \leq C\varepsilon$.

(2) For $f \in \Sigma_{\Lambda, T, \varepsilon}$, there exists a unique solution to (3.39) on $[-T, T]$ satisfying

$$\|u_\Lambda(t)\|_{\mathcal{H}^{\theta_1}} \leq C \left(\log \frac{T}{\varepsilon} \right)^{1/2}, \quad \forall |t| \leq T. \quad (4.7)$$

Proof. Let $K > 0$ and denote

$$B_K := \{u \in \mathcal{H}^{\theta_1} : \|u\|_{\mathcal{H}^{\theta_1}} \leq K\}. \quad (4.8)$$

Thanks to (3.29), the deterministic local well-posedness given in Proposition 4.1 applies mutatis mutandis to the approximate flow and implies that for $f \in B_K$, there exists a unique solution to (3.39) satisfying

$$\|u_\Lambda(t)\|_{\mathcal{H}^{\theta_1}} \leq 2CK, \quad \forall |t| \leq \delta \quad (4.9)$$

with

$$\delta = \nu(K+1)^{-\varrho}, \quad (4.10)$$

where $\varrho > 0$ is as in Proposition 4.1, $\nu > 0$ small, and $C > 0$ is independent of Λ, K . Here we choose δ slightly smaller than in the proof of Proposition 4.1, which will be convenient later.

Denote $J = \left\lfloor \frac{T}{\delta} \right\rfloor$ the integer part of $\frac{T}{\delta}$ and set

$$\Sigma_{\Lambda, T, K} := \bigcap_{j=-J}^J \Phi_\Lambda(-j\delta)(B_K), \quad (4.11)$$

where Φ_Λ is the solution map for (3.39). Since μ_Λ is invariant under the flow of (3.39), we have

$$\begin{aligned} \mu_\Lambda(\Sigma_{\Lambda, T, K}^c) &= \mu_\Lambda\left(\left(\bigcap_{j=-J}^J \Phi_\Lambda(-j\delta)(B_K)\right)^c\right) \\ &= \mu_\Lambda\left(\bigcup_{j=-J}^J (\Phi_\Lambda(-j\delta)(B_K))^c\right) \\ &= \mu_\Lambda\left(\bigcup_{j=-J}^J \Phi_\Lambda(-j\delta)(B_K^c)\right) \\ &\leq \sum_{j=-J}^J \mu_\Lambda(\Phi_\Lambda(-j\delta)(B_K^c)) \\ &\leq 2\frac{T}{\delta} \mu_\Lambda(B_K^c). \end{aligned}$$

Thanks to (3.4), we have

$$\begin{aligned} \mu_\Lambda(B_K^c) &= \int \mathbf{1}_{B_K^c} d\mu_\Lambda(u) \\ &= \int \mathbf{1}_{B_K^c} \frac{1}{Z^\Lambda} G_\Lambda(u) d\mu_0(u) \\ &\leq \frac{1}{Z^\Lambda} \|G_\Lambda(u)\|_{L^2(d\mu_0)} (\mu_0(B_K^c))^{1/2} \\ &\leq C e^{-cK^2}, \end{aligned}$$

where $G_\Lambda(u)$ is as in (3.34) for the defocusing case and in (3.37) for the focusing one. Here we have used the fact that $G_\Lambda(u) \in L^2(d\mu_0)$ and $Z^\Lambda \geq C > 0$ uniformly in Λ . In particular, we obtain

$$\mu_\Lambda(\Sigma_{\Lambda, T, K}^c) \leq \frac{2C}{\nu} T(K+1)^\varrho e^{-cK^2} \leq CT e^{-cK^2}$$

for some constants $C, c > 0$. Note that the constants C, c may change from line to line but are independent of Λ, T, K . By choosing

$$K = C \left(\log \frac{T}{\varepsilon} \right)^{1/2} \quad (4.12)$$

for a suitable constant $C > 0$, we obtain

$$\mu_\Lambda(\Sigma_{\Lambda, T, K}^c) \leq C\varepsilon.$$

Setting $\Sigma_{\Lambda, T, \varepsilon} = \Sigma_{\Lambda, T, K}$ with K as in (4.12), we have the first item. To see the second item, we observe that for $|t| \leq T$, there exist an integer j and $\delta_1 \in [-\delta, \delta]$ such that $t = j\delta + \delta_1$, hence

$$u_\Lambda(t) = \Phi_\Lambda(\delta_1)\Phi_\Lambda(j\delta)f.$$

Since $f \in \Phi_\Lambda(-j\delta)(B_K)$, we have $\Phi_\Lambda(j\delta)f \in B_K$ which, by (4.9), yields

$$\|u_\Lambda(t)\|_{\mathcal{H}^{\theta_1}} \leq 2CK, \quad \forall |t| \leq T. \quad (4.13)$$

Taking into account (4.12), we have the desired estimate. \square

We turn to the second main ingredient.

Lemma 4.4 (Comparison between approximate and exact flows, $s > 2$).

Let $\Sigma_{\Lambda, T, \varepsilon}$ be as in the previous lemma. Then for any $f \in \Sigma_{\Lambda, T, \varepsilon}$, there exists a unique solution to (1.1) with initial data $u|_{t=0} = f$ satisfying

$$\|u(t) - u_\Lambda(t)\|_{\mathcal{H}^\theta} \leq C(T, \varepsilon)\Lambda^{(\theta-\theta_1)/2}, \quad \forall |t| \leq T \quad (4.14)$$

for all Λ sufficiently large and some constant $C(T, \varepsilon) > 0$ independent of Λ . In particular, there exist $\Sigma_{T, \varepsilon} \subset \mathcal{H}^\theta$ and $C > 0$ independent of T, ε such that:

(1) $\mu(\Sigma_{T, \varepsilon}^c) \leq C\varepsilon$.

(2) For $f \in \Sigma_{T, \varepsilon}$, there exists a unique solution to (1.1) with initial data $u|_{t=0} = f$ on $[-T, T]$ satisfying

$$\|u(t)\|_{\mathcal{H}^\theta} \leq C \left(\log \frac{T}{\varepsilon} \right)^{1/2}, \quad \forall |t| \leq T. \quad (4.15)$$

Proof. Estimating the difference. Denote

$$v_\Lambda := Q_\Lambda u_\Lambda$$

with Q_Λ as in (3.27). We first study the difference $u - v_\Lambda$ on

$$X^{\theta, \gamma}([-\delta, \delta]) = L^\infty([-\delta, \delta], \mathcal{H}^\theta) \cap L^p([-\delta, \delta], \mathcal{W}^{\gamma, q}), \quad (4.16)$$

where δ is as in (4.10), (p, q) is an admissible pair (see Appendix B), and γ is such that

$$p > 4, \quad \frac{1}{2} - \frac{2}{p} < \gamma < \theta - \frac{2}{p} \left(\frac{1}{2} - \frac{2}{p} \right). \quad (4.17)$$

From (3.39), we have the following Duhamel formula

$$u(t) - v_\Lambda(t) = e^{-i t h} (f - Q_\Lambda f) \mp i \int_0^t e^{-i(t-\tau)h} \left(|u(\tau)|^2 u(\tau) - Q_\Lambda^2(|v_\Lambda(\tau)|^2 v_\Lambda(\tau)) \right) d\tau$$

which, by Strichartz estimates (see Appendix B), yields

$$\|u - v_\Lambda\|_{X^{\theta, \gamma}([-\delta, \delta])} \leq C \|f - Q_\Lambda f\|_{\mathcal{H}^\theta} + C \| |u|^2 u - Q_\Lambda^2(|v_\Lambda|^2 v_\Lambda) \|_{L^1([-\delta, \delta], \mathcal{H}^\theta)}.$$

For the linear term, we estimate (recall (3.27))

$$\|f - Q_\Lambda f\|_{\mathcal{H}^\theta} \leq C \Lambda^{(\theta-\theta_1)/2} \|f\|_{\mathcal{H}^{\theta_1}} \leq CK \Lambda^{(\theta-\theta_1)/2}.$$

For the nonlinear term, we write

$$|u|^2 u - Q_\Lambda^2(|v_\Lambda|^2 v_\Lambda) = |u|^2 u - |v_\Lambda|^2 v_\Lambda + (\text{Id} - Q_\Lambda^2)(|v_\Lambda|^2 v_\Lambda).$$

Since $f \in \Sigma_{\Lambda, T, \varepsilon}$ (see (4.11)), we have $\|f\|_{\mathcal{H}^{\theta_1}} = \|u_\Lambda(0)\|_{\mathcal{H}^{\theta_1}} \leq K$ and the local theory for (3.39) ensures the existence of a unique solution satisfying $\|u_\Lambda\|_{X^{\theta_1, \gamma}([- \delta, \delta])} \leq 2CK$ which, by (3.29), yields

$$\|v_\Lambda\|_{X^{\theta_1, \gamma}([- \delta, \delta])} \leq 2CK,$$

where $X^{\theta_1, \gamma}([- \delta, \delta])$ is as in (4.16) with p, q, γ as in (4.17). Estimating as in the proof of Proposition 4.1, we have

$$\begin{aligned} \|(\text{Id} - Q_\Lambda^2)(|v_\Lambda|^2 v_\Lambda)\|_{L^1([- \delta, \delta], \mathcal{H}^\theta)} &\leq C\Lambda^{(\theta - \theta_1)/2} \| |v_\Lambda|^2 v_\Lambda \|_{L^1([- \delta, \delta], \mathcal{H}^{\theta_1})} \\ &\leq C\delta^{1 - \frac{2}{p}} \Lambda^{(\theta - \theta_1)/2} \|v_\Lambda\|_{X^{\theta_1, \gamma}([- \delta, \delta])}^3 \\ &\leq C\delta^{1 - \frac{2}{p}} K^3 \Lambda^{(\theta - \theta_1)/2}. \end{aligned}$$

Since $\|f\|_{\mathcal{H}^\theta} \leq \|f\|_{\mathcal{H}^{\theta_1}} \leq K$, the local theory and (3.29) give

$$\|u\|_{X^{\theta, \gamma}([- \delta, \delta])}, \|v_\Lambda\|_{X^{\theta, \gamma}([- \delta, \delta])} \leq 2CK,$$

which implies

$$\begin{aligned} \| |u|^2 u - |v_\Lambda|^2 v_\Lambda \|_{L^1([- \delta, \delta], \mathcal{H}^\theta)} &\leq C\delta^{1 - \frac{2}{p}} \left(\|u\|_{X^{\theta, \gamma}([- \delta, \delta])}^2 + \|v_\Lambda\|_{X^{\theta, \gamma}([- \delta, \delta])}^2 \right) \|u - v_\Lambda\|_{X^{\theta, \gamma}([- \delta, \delta])} \\ &\leq C\delta^{1 - \frac{2}{p}} K^2 \|u - v_\Lambda\|_{X^{\theta, \gamma}([- \delta, \delta])}. \end{aligned}$$

By the choice of δ with some $\nu > 0$ small, we deduce

$$\|u - v_\Lambda\|_{X^{\theta, \gamma}([- \delta, \delta])} \leq CK\Lambda^{(\theta - \theta_1)/2} + \frac{1}{2}\|u - v_\Lambda\|_{X^{\theta, \gamma}([- \delta, \delta])}$$

hence

$$\|u - v_\Lambda\|_{X^{\theta, \gamma}([- \delta, \delta])} \leq 2CK\Lambda^{(\theta - \theta_1)/2}.$$

On the other hand, we infer from (4.13) that

$$\|u_\Lambda(t) - v_\Lambda(t)\|_{\mathcal{H}^\theta} \leq C\Lambda^{(\theta - \theta_1)/2} \|u_\Lambda(t)\|_{\mathcal{H}^{\theta_1}} \leq CK\Lambda^{(\theta - \theta_1)/2}$$

which gives

$$\begin{aligned} \|u(t) - u_\Lambda(t)\|_{\mathcal{H}^\theta} &\leq \|u(t) - v_\Lambda(t)\|_{\mathcal{H}^\theta} + \|u_\Lambda(t) - v_\Lambda(t)\|_{\mathcal{H}^\theta} \\ &\leq 3CK\Lambda^{(\theta - \theta_1)/2}, \quad \forall |t| \leq \delta. \end{aligned} \tag{4.18}$$

We can iterate the above argument $\lceil \frac{T}{\delta} \rceil$ many times. For instance, at the second iteration, we have

$$\|u - v_\Lambda\|_{X^{\theta, \gamma}([0, 2\delta])} \leq C\|u(\delta) - v_\Lambda(\delta)\|_{\mathcal{H}^\theta} + \text{nonlinear term}.$$

Note that

$$\|u(\delta) - v_\Lambda(\delta)\|_{\mathcal{H}^\theta} \leq \|u - v_\Lambda\|_{L^\infty([- \delta, \delta], \mathcal{H}^\theta)} \leq \|u - v_\Lambda\|_{X^{\theta, \gamma}([- \delta, \delta])} \leq 2CK\Lambda^{(\theta - \theta_1)/2}.$$

The nonlinear term can be handled as before by noting that $\|u_\Lambda(\delta)\|_{\mathcal{H}^{\theta_1}} = \|\Phi_\Lambda(\delta)\|_{\mathcal{H}^{\theta_1}} \leq K$ (see (4.11)) and

$$\|u(\delta)\|_{\mathcal{H}^\theta} \leq \|u_\Lambda(\delta)\|_{\mathcal{H}^{\theta_1}} + \|u(\delta) - u_\Lambda(\delta)\|_{\mathcal{H}^\theta} \leq K + 1$$

provided that Λ is taken sufficiently large. The above estimate is the reason why we take δ as in (4.10). Thus we get

$$\|u - v_\Lambda\|_{X^{\theta, \gamma}([0, 2\delta])} \leq (2C)^2 K\Lambda^{(\theta - \theta_1)/2}.$$

Arguing as in (4.18), we obtain

$$\|u(t) - u_\Lambda(t)\|_{\mathcal{H}^\theta} \leq (3C)^2 K\Lambda^{(\theta - \theta_1)/2}, \quad \forall t \in [0, 2\delta].$$

After $\left\lceil \frac{T}{\delta} \right\rceil$ iterations, we can sum over all sub-intervals to get

$$\|u - v_\Lambda\|_{X^{\theta,\gamma}([-T,T])} \leq C e^{c\frac{T}{\delta}} K \Lambda^{(\theta-\theta_1)/2} \leq C e^{cT(K+1)^e} K \Lambda^{(\theta-\theta_1)/2}.$$

By the same reasoning as in (4.18) and invoking (4.12), we prove (4.14).

Globalizing the flow. By (4.14), there exists Λ_1 sufficiently large such that for $f \in \Sigma_{\Lambda_1, T, \varepsilon}$, the corresponding solution to (1.1) with initial data $u|_{t=0}$ satisfies

$$\|u(t) - u_\Lambda(t)\|_{\mathcal{H}^\theta} \ll 1, \quad \forall |t| \leq T, \quad \forall \Lambda \geq \Lambda_1.$$

From this and (4.7), we have

$$\|u(t)\|_{\mathcal{H}^\theta} \leq C \left(\log \frac{T}{\varepsilon} \right)^{1/2}, \quad \forall |t| \leq T.$$

This proves (4.15) by setting $\Sigma_{T, \varepsilon} := \Sigma_{\Lambda_1, T, \varepsilon}$. It remains to prove the first item. We estimate

$$\mu(\Sigma_{\Lambda_1, T, \varepsilon}^c) = \int \mathbf{1}_{\Sigma_{\Lambda_1, T, \varepsilon}^c} d\mu(u) = \int_{\Sigma_{\Lambda_1, T, \varepsilon}^c} \frac{1}{Z} G(u) d\mu_0(u) \leq C \int_{\Sigma_{\Lambda_1, T, \varepsilon}^c} G(u) d\mu_0(u),$$

where $G(u)$ is as in (3.34) for the defocusing nonlinearity and in (3.37) for the focusing one. Here we have used the fact that $Z \geq C > 0$.

To estimate the last integral in terms of μ_{Λ_1} , we consider two cases. For the defocusing nonlinearity, we use the fact that $\|Q_\Lambda u\|_{L^4}^4 \leq C_1 \|u\|_{L^4}^4$ for some constant $C_1 > 0$. Without loss of generality, we may assume that $C_1 \geq 1$. By Hölder's inequality, we estimate

$$\begin{aligned} \int_{\Sigma_{\Lambda_1, T, \varepsilon}^c} e^{-\frac{1}{2}\|u\|_{L^4}^4} d\mu_0(u) &\leq \int_{\Sigma_{\Lambda_1, T, \varepsilon}^c} e^{-\frac{1}{2C_1}\|Q_{\Lambda_1} u\|_{L^4}^4} d\mu_0(u) \\ &\leq \left(\int_{\Sigma_{\Lambda_1, T, \varepsilon}^c} e^{-\frac{1}{2}\|Q_{\Lambda_1} u\|_{L^4}^4} d\mu_0(u) \right)^{1/C_1} \left(\int_{\Sigma_{\Lambda_1, T, \varepsilon}^c} d\mu_0(u) \right)^{1/C_1'} \\ &= \left(Z^{\Lambda_1} \int_{\Sigma_{\Lambda_1, T, \varepsilon}^c} d\mu_{\Lambda_1}(u) \right)^{1/C_1} \left(\int_{\Sigma_{\Lambda_1, T, \varepsilon}^c} d\mu_0(u) \right)^{1/C_1'} \\ &\leq C \left(\mu_{\Lambda_1}(\Sigma_{\Lambda_1, T, \varepsilon}^c) \right)^{1/C_1} \\ &< C \varepsilon^{1/C_1}, \end{aligned}$$

where we used that (C_1, C_1') is a Hölder-conjugate pair, μ_0 is a probability measure, and $Z^{\Lambda_1} \leq 1$. Here the last inequality follows from Lemma 4.3. In the focusing case, we have

$$\begin{aligned} \int_{\Sigma_{\Lambda_1, T, \varepsilon}^c} G(u) d\mu_0(u) &= \int_{\Sigma_{\Lambda_1, T, \varepsilon}^c} G(u) \mathbf{1}_{\{\|u\|_{L^2}^2 < m\}} d\mu_0(u) \\ &\leq \|G(u)\|_{L^2(d\mu_0)} \left(\int_{\Sigma_{\Lambda_1, T, \varepsilon}^c} \mathbf{1}_{\{\|u\|_{L^2}^2 < m\}} d\mu_0(u) \right)^{1/2} \\ &\leq C \left(\int_{\Sigma_{\Lambda_1, T, \varepsilon}^c} \mathbf{1}_{\{\|P_{\leq \Lambda_1} u\|_{L^2}^2 < m\}} d\mu_0(u) \right)^{1/2} \\ &\leq C \left(\int_{\Sigma_{\Lambda_1, T, \varepsilon}^c} e^{\frac{1}{2}\|Q_{\Lambda_1} u\|_{L^4}^4} \mathbf{1}_{\{\|P_{\leq \Lambda_1} u\|_{L^2}^2 < m\}} d\mu_0(u) \right)^{1/2} \\ &= C \left(\mu_{\Lambda_1}(\Sigma_{\Lambda_1, T, \varepsilon}^c) \right)^{1/2} \\ &< C \varepsilon^{1/2}. \end{aligned}$$

In both cases, we adjust ε slightly to get the desired result. The proof is complete. \square

Now we can conclude the proof of Theorem 4.2.

Proof of Theorem 4.2. Almost-sure flow, global in time. Fix $\varepsilon > 0$ and set $T_n = 2^n$, $\varepsilon_n = \varepsilon 2^{-n}$. Let $\Sigma_n = \Sigma_{T_n, \varepsilon_n}$ be as in Lemma 4.4 and set

$$\Sigma_\varepsilon = \bigcap_{n=1}^{\infty} \Sigma_n.$$

For $f \in \Sigma_\varepsilon$, we have $f \in \Sigma_n$ for all $n \geq 1$. By Lemma 4.4, the corresponding solution to (1.1) with initial data $u|_{t=0} = f$ exists globally in time since it exists on $[2^{-n}, 2^n]$ for all $n \geq 1$. In addition, we have

$$\mu(\Sigma_\varepsilon^c) = \mu\left(\bigcup_{n=1}^{\infty} \Sigma_n^c\right) \leq \sum_{n=1}^{\infty} \mu(\Sigma_n^c) \leq C \sum_{n=1}^{\infty} \varepsilon_n = C\varepsilon.$$

Define now

$$\Sigma = \bigcup_{\varepsilon > 0} \Sigma_\varepsilon \tag{4.19}$$

so that

$$\mu(\Sigma^c) = \mu\left(\bigcap_{\varepsilon > 0} \Sigma_\varepsilon^c\right) \leq \inf_{\varepsilon > 0} \mu(\Sigma_\varepsilon^c) = 0.$$

Hence we have indeed found the claimed set of full μ measure on which the flow is globally defined.

Growth in time. Pick $f \in \Sigma$ and $t \in \mathbb{R}$. It must be that $f \in \Sigma_\varepsilon$ for some $\varepsilon > 0$ and that

$$2^{n-1} \leq 1 + |t| \leq 2^n$$

for some integer $n \geq 1$. In particular, we have $n \leq 1 + \log_2(1 + |t|) \leq 1 + 2 \log(1 + |t|)$. For such n , we apply Lemma 4.4 with $f \in \Sigma_n$ to get

$$\|u(t)\|_{\mathcal{H}^\theta} \leq C \left(\log \frac{T_n}{\varepsilon_n}\right)^{1/2} = C \left(\log \frac{1}{\varepsilon} + n\right)^{1/2} \leq C \left(\log^{1/2} \frac{1}{\varepsilon} + \log^{1/2}(1 + |t|)\right)$$

for some constant $C > 0$. Since ε depends on f , we prove (4.6) with $\omega(f) = \log^{1/2} \frac{1}{\varepsilon}$.

Measure invariance. Let θ satisfy $\frac{1}{2} \left(\frac{1}{2} - \frac{1}{s}\right) < \theta < \frac{1}{2} - \frac{1}{s}$. By the time reversibility of the solution map, it suffices to show

$$\mu(A) \leq \mu(\Phi(t)(A)) \tag{4.20}$$

for all measurable sets $A \subset \Sigma$ and all $t \in \mathbb{R}$. By the inner regularity, there exists a sequence $\{F_n\}_n$ of closed set in \mathcal{H}^θ such that $F_n \subset A$ and $\mu(A) = \lim_{n \rightarrow \infty} \mu(F_n)$. We claim that it suffices to prove (4.20) for closed sets. Note that since $F_n \subset A$, the uniqueness of solutions implies that $\mu(\Phi(t)(F_n)) \leq \mu(\Phi(t)(A))$. If (4.20) is true for closed sets, then we have

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(F_n) \leq \limsup_{n \rightarrow \infty} \mu(\Phi(t)(F_n)) \leq \mu(\Phi(t)(A)),$$

hence (4.20) is true for all measurable sets. Now given a closed set $F \subset \mathcal{H}^\theta$. Take $\theta < \theta_1 < \frac{1}{2} - \frac{1}{s}$ and set

$$U_n := \{u \in F : \|u\|_{\mathcal{H}^{\theta_1}} \leq n\}.$$

From (3.9), we infer that

$$\begin{aligned}\mu(F \setminus U_n) &= \frac{1}{Z} \int_{F \setminus U_n} G(u) d\mu_0(u) \\ &\leq C \|G(u)\|_{L^2(d\mu_0)} (\mu_0(F \setminus U_n))^{1/2} \\ &\leq C (\mu_0(\|u\|_{\mathcal{H}^{\theta_1}} > n))^{1/2} \\ &\leq C e^{-cn^2} \rightarrow 0 \text{ as } n \rightarrow \infty.\end{aligned}$$

Thus we have

$$\mu(F) = \lim_{n \rightarrow \infty} \mu(U_n).$$

The same argument as above yields that it suffices to prove (4.20) for closed sets of \mathcal{H}^θ which are bounded in \mathcal{H}^{θ_1} .

Let U be such a set and fix $t > 0$ (the case $t < 0$ is similar). Since U is bounded in \mathcal{H}^{θ_1} , the local theory ensures the existence of $K > 0$ such that

$$\{\Phi(\tau)(U) : 0 \leq \tau \leq t\} \subset \{u \in \mathcal{H}^{\theta_1} : \|u\|_{\mathcal{H}^{\theta_1}} \leq K\} =: B_{\theta_1, K}.$$

Set

$$\delta = \nu K^{-\varrho},$$

where ϱ is as in Proposition 4.1 and $\nu > 0$ is a small constant independent of K . It suffices to prove

$$\mu(U) \leq \mu(\Phi(t)(U)), \quad \forall t \in [0, \delta]. \quad (4.21)$$

Indeed, we split $[0, t]$ into intervals of size δ and apply (4.21) on these intervals. Such an iteration is possible since by the continuity of the solution map, the image of each interval remains closed in \mathcal{H}^θ and included in $B_{\theta_1, K}$.

To prove (4.21), we take $\varepsilon > 0$ and denote by $B_{\theta, \varepsilon}$ the open ball in \mathcal{H}^θ centered at the origin and of radius ε . There exist $0 < c \ll 1$ and $\Lambda \geq \lambda_1$ sufficiently large such that

$$\begin{aligned}\Phi_\Lambda(t)(U + B_{\theta, c\varepsilon}) &\subset \Phi_\Lambda(t)(U) + B_{\theta, \varepsilon/2} && \text{(Lipschitz continuity (4.4))} \\ &\subset \Phi(t)(U) + B_{\theta, \varepsilon}. && \text{(Lemma 4.4)}\end{aligned}$$

Thus we have the chain of inequalities

$$\begin{aligned}\mu(U) &\leq \mu(U + B_{\theta, c\varepsilon}) \leq \liminf_{\Lambda \rightarrow \infty} \mu_\Lambda(U + B_{\theta, c\varepsilon}) && (\mu_\Lambda \rightharpoonup \mu \text{ weakly}) \\ &= \liminf_{\Lambda \rightarrow \infty} \mu_\Lambda(\Phi_\Lambda(t)(U + B_{\theta, c\varepsilon})) && \text{(invariance of } \mu_\Lambda) \\ &\leq \liminf_{\Lambda \rightarrow \infty} \mu_\Lambda(\Phi(t)(U) + B_{\theta, \varepsilon}) \\ &\leq \limsup_{\Lambda \rightarrow \infty} \mu_\Lambda(\Phi(t)(U) + \overline{B}_{\theta, \varepsilon}) \\ &\leq \mu(\Phi(t)(U) + \overline{B}_{\theta, \varepsilon}). && (\mu_\Lambda \rightharpoonup \mu \text{ weakly})\end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we obtain $\mu(U) \leq \mu(\Phi(t)(U))$ for all $t \in [0, \delta]$. The proof is complete. \square

5. CAUCHY PROBLEM AND INVARIANT MEASURE, (SUB)-HARMONIC CASE

We now consider the case $1 < s \leq 2$ for which the Gibbs measure is supported on L^2 -based Sobolev spaces of negative indices. It seems difficult to obtain deterministic local well-posedness for (1.1) in this case. In [8], a deterministic LWP was proved for $s = 2$ using some intricate multilinear estimates. The proof of these estimates relies heavily on a detailed understanding of spectral properties of the harmonic oscillator. We do not expect the proof of such multilinear estimates to carry through to the case $1 < s < 2$. In Section 5.1, we aim to prove an almost sure local well-posedness for (1.1). Section 5.2 is devoted to an almost sure global well-posedness and the measure invariance.

5.1. Probabilistic local well-posedness. Our proof of almost sure local well-posedness relies on the following probabilistic Strichartz estimates.

Lemma 5.1 (Probabilistic Strichartz estimates).

Let $1 < s \leq 2$, V satisfy Assumption 1.1, $0 \leq \beta < \frac{1}{2}$, and $p > \max\left\{2, \frac{4}{s(1-2\beta)}\right\}$ be an even integer. Then there exist $C, c > 0$ such that

$$\int e^{c\|e^{-ith}f\|_{\mathcal{W}^{\beta,p}}^2} d\mu_0(f) \leq C, \quad \forall t \in \mathbb{R}. \quad (5.1)$$

Furthermore, for $\theta < 1/2 - 1/s$ and all $\lambda > 0$,

$$\mu_0\left(f \in \mathcal{H}^\theta : \|e^{-ith}f\|_{L^\infty(\mathbb{R}, \mathcal{W}^{\beta,p})} > \lambda\right) \leq Ce^{-c\lambda^2} \quad (5.2)$$

and for all $T > 0$, all $q \geq 1$, and all $\lambda > 0$,

$$\mu_0\left(f \in \mathcal{H}^\theta : \|e^{-ith}f\|_{L^q([-T,T], \mathcal{W}^{\beta,p})} > \lambda\right) \leq Ce^{-c\frac{\lambda^2}{T^{2/q}}}. \quad (5.3)$$

Proof. Denote $g_f(t) := e^{-ith}f$. We write

$$f = \sum_{j \geq 1} \alpha_j u_j, \quad \alpha_j = \langle f, u_j \rangle,$$

hence

$$g_f(t) = \sum_{j \geq 1} \beta_j(t) u_j, \quad \beta_j(t) = \alpha_j e^{-it\lambda_j}.$$

Observe that

$$\begin{aligned} d\mu_0(g_f(t)) &= \prod_{j \geq 1} \frac{\lambda_j}{\pi} e^{-\lambda_j |\beta_j(t)|^2} d\beta_j(t) \\ &= \prod_{j \geq 1} \frac{\lambda_j}{\pi} e^{-\lambda_j |\alpha_j|^2} d\alpha_j = d\mu_0(f), \end{aligned}$$

where we have used the invariance of the Lebesgue measure under rotations. Thus we get

$$\int e^{c\|e^{-ith}f\|_{\mathcal{W}^{\beta,p}}^2} d\mu_0(f) = \int e^{c\|g_f(t)\|_{\mathcal{W}^{\beta,p}}^2} d\mu_0(g_f(t)), \quad \forall t \in \mathbb{R}.$$

By Lemma 3.4, we prove (5.1). This immediately yields (5.2). To see (5.3), we use Hölder's inequality in time to get

$$\|e^{-ith}f\|_{L^q([-T,T], \mathcal{W}^{\beta,p})} \leq CT^{1/q} \|e^{-ith}f\|_{L^\infty([-T,T], \mathcal{W}^{\beta,p})}.$$

Thus

$$\mu_0\left(f \in \mathcal{H}^\theta : \|e^{-ith}f\|_{L^q([-T,T], \mathcal{W}^{\beta,p})} > \lambda\right) \leq \mu_0\left(f \in \mathcal{H}^\theta : \|e^{-ith}f\|_{L^\infty([-T,T], \mathcal{W}^{\beta,p})} > C\frac{\lambda}{T^{1/q}}\right)$$

and (5.3) follows from (5.2). \square

We are now able to state the almost sure LWP for (1.1).

Proposition 5.2 (Probabilistic local well-posedness, $1 < s \leq 2$).

Let $1 < s \leq 2$, V satisfy Assumption 1.1, $0 \leq \beta < \frac{s-1}{2s}$, and $\theta < \frac{1}{2} - \frac{1}{s}$. For any $K > 0$, there exist $\Sigma(K) \subset \mathcal{H}^\theta$ satisfying $\mu_0(\Sigma^c(K)) \leq Ce^{-cK^2}$ for some constants $C, c > 0$ and $\delta \sim K^{-4} > 0$ such that for all $f \in \Sigma(K)$, there exists a unique solution to (1.1) with initial data $u|_{t=0} = f$ satisfying

$$u(t) - e^{-ith}f \in C([- \delta, \delta], \mathcal{H}^\beta) \cap L^8([- \delta, \delta], \mathcal{W}^{\beta,4}).$$

In addition, for all $f \in \Sigma(K)$, we have $\|f\|_{\mathcal{H}^\theta} \leq K$ and the corresponding solution to (1.1) with initial data $u|_{t=0} = f$ satisfies

$$\|u(t)\|_{\mathcal{H}^\theta} \leq 2K, \quad \forall |t| \leq \delta. \quad (5.4)$$

Furthermore, if $u_1(t)$ and $u_2(t)$ are respectively solutions to (1.1) with initial data $u_1|_{t=0} = f_1$, $u_2|_{t=0} = f_2$ with $f_1, f_2 \in \Sigma(K)$, then

$$\|u_1(t) - u_2(t)\|_{\mathcal{H}^\theta} \leq \|f_1 - f_2\|_{\mathcal{H}^\theta} + \|e^{-ith}f_1 - e^{-ith}f_2\|_{L^8([-\delta, \delta], \mathcal{W}^{\beta, 4})} \quad (5.5)$$

for all $|t| \leq \delta$.

Note that since the pair (8, 4) is Strichartz-admissible and $\beta > \theta$, (5.5) implies an estimate with a loss of derivatives

$$\|u_1(t) - u_2(t)\|_{\mathcal{H}^\theta} \leq C\|f_1 - f_2\|_{\mathcal{H}^\beta}. \quad (5.6)$$

As preparation for the proof of Proposition 5.2, denote

$$g_f(t) := e^{-ith}f, \quad v(t) := u(t) - g_f(t).$$

We seek for a solution v to the equation

$$i\partial_t v - hv = \pm |g_f + v|^2(g_f + v), \quad v|_{t=0} = 0.$$

To this end, we define for $0 < \delta \leq 1$,

$$X_\delta^\beta := C([-\delta, \delta], \mathcal{H}^\beta) \cap L^8([-\delta, \delta], \mathcal{W}^{\beta, 4}).$$

Our purpose is to prove that the functional

$$\Phi_f(v(t)) := \mp i \int_0^t e^{-i(t-\tau)h} (|g_f + v|^2(g_f + v))(\tau) d\tau$$

is a contraction mapping on the ball

$$B_{X_\delta^\beta}(L) := \left\{ v \in X_\delta^\beta : \|v\|_{X_\delta^\beta} \leq L \right\}$$

equipped with the distance

$$d(v_1, v_2) := \|v_1 - v_2\|_{X_\delta^\beta},$$

with $L > 0$ to be determined later.

Let us start with the following nonlinear estimates.

Lemma 5.3 (Nonlinear estimates).

There exists $C > 0$ such that for any $0 < \delta \leq 1$, any $f \in \mathcal{H}^\theta$ satisfying

$$\|gf\|_{L^8([-1, 1], \mathcal{W}^{\beta, 4})} \leq K$$

with some $K > 0$, and any $v, v_1, v_2 \in X_\delta^\beta$,

$$\begin{aligned} \|\Phi_f(v)\|_{X_\delta^\beta} &\leq C\delta^{1/2} \left(K^3 + \|v\|_{X_\delta^\beta}^3 \right), \\ \|\Phi_f(v_1) - \Phi_f(v_2)\|_{X_\delta^\beta} &\leq C\delta^{1/2} \left(K^2 + \|v_1\|_{X_\delta^\beta}^2 + \|v_2\|_{X_\delta^\beta}^2 \right) \|v_1 - v_2\|_{X_\delta^\beta}. \end{aligned}$$

Proof. By Strichartz estimates (see Appendix B), we have

$$\|\Phi_f(v)\|_{X_\delta^\beta} \leq C \left\| h^{\beta/2} (|g_f + v|^2(g_f + v)) \right\|_{L^{8/7}([-\delta, \delta], L^{4/3})}$$

By the product rule (see Lemma B.3), we have, using Hölder in time,

$$\begin{aligned} \|\Phi_f(v)\|_{X_\delta^\beta} &\leq C \|g_f + v\|_{L^{24/7}([-\delta, \delta], L^4)}^2 \|h^{\beta/2}(g_f + v)\|_{L^{24/7}([-\delta, \delta], L^4)} \\ &\leq C\delta^{1/2} \|g_f + v\|_{L^8([-\delta, \delta], L^4)}^2 \|h^{\beta/2}(g_f + v)\|_{L^8([-\delta, \delta], L^4)}. \end{aligned}$$

By the norm equivalence (B.1), we infer that

$$\begin{aligned} \|\Phi_f(v)\|_{X_\delta^\beta} &\leq C\delta^{1/2}\|h^{\beta/2}(g_f+v)\|_{L^8([-\delta,\delta],L^4)}^3 \\ &\leq C\delta^{1/2}\left(\|g_f\|_{L^8([-\delta,\delta],\mathcal{W}^{\beta,4})}^3 + \|v\|_{L^8([-\delta,\delta],\mathcal{W}^{\beta,4})}^3\right) \\ &\leq C\delta^{1/2}\left(\|g_f\|_{L^8([-\delta,\delta],\mathcal{W}^{\beta,4})}^3 + \|v\|_{X_\delta^\beta}^3\right). \end{aligned}$$

On the other hand, we have

$$\|\Phi_f(v_1) - \Phi_f(v_2)\|_{X_\delta^\beta} \leq C \left\| h^{\beta/2} \left(|g_f + v_1|^2(g_f + v_1) - |g_f + v_2|^2(g_f + v_2) \right) \right\|_{L^{8/7}([-\delta,\delta],L^{4/3})}.$$

By writing

$$\begin{aligned} |g_f + v_1|^2(g_f + v_1) - |g_f + v_2|^2(g_f + v_2) \\ = |g_f + v_1|^2(v_1 - v_2) + |g_f + v_2|^2(v_1 - v_2) + (g_f + v_1)(g_f + v_2)(\bar{v}_1 - \bar{v}_2), \end{aligned}$$

and estimating as above, we have

$$\begin{aligned} \|\Phi_f(v_1) - \Phi_f(v_2)\|_{X_\delta^\beta} &\leq C \|h^{\beta/2}(v_1 - v_2)\|_{L^{24/7}([-\delta,\delta],L^4)} \\ &\quad \times \left(\|h^{\beta/2}(g_f + v_1)\|_{L^{24/7}([-\delta,\delta],L^4)}^2 + \|h^{\beta/2}(g_f + v_2)\|_{L^{24/7}([-\delta,\delta],L^4)}^2 \right) \\ &\leq C\delta^{1/2} \|h^{\beta/2}(v_1 - v_2)\|_{L^8([-\delta,\delta],L^4)} \\ &\quad \times \left(\|h^{\beta/2}(g_f + v_1)\|_{L^8([-\delta,\delta],L^4)}^2 + \|h^{\beta/2}(g_f + v_2)\|_{L^8([-\delta,\delta],L^4)}^2 \right) \\ &\leq C\delta^{1/2} \|v_1 - v_2\|_{X_\delta^\beta} \left(\|g_f\|_{L^8([-\delta,\delta],\mathcal{W}^{\beta,4})}^2 + \|v_1\|_{X_\delta^\beta}^2 + \|v_2\|_{X_\delta^\beta}^2 \right). \end{aligned}$$

Thus we get the desired estimates. \square

Now we may complete the

Proof of Proposition 5.2. Denote

$$\Sigma(K) := \left\{ f \in \mathcal{H}^\theta : \|f\|_{\mathcal{H}^\theta} \leq K, \|g_f\|_{L^8([-1,1],\mathcal{W}^{\beta,4})} \leq K \right\},$$

where $g_f(t) = e^{-ith}f$. By Lemma 3.2 and Lemma 5.1, we have

$$\begin{aligned} \mu_0(\Sigma^c(K)) &\leq \mu_0(f \in \mathcal{H}^\theta : \|f\|_{\mathcal{H}^\theta} > K) + \mu_0(f \in \mathcal{H}^\theta : \|g_f\|_{L^8([-1,1],\mathcal{W}^{\beta,4})} > K) \\ &\leq Ce^{-cK^2}. \end{aligned} \tag{5.7}$$

By Lemma 5.3, there exists $C > 0$ such that for all $0 < \delta \leq 1$, all $f \in \Sigma(K)$, and all $v, v_1, v_2 \in B_{X_\delta^\beta}(L)$,

$$\begin{aligned} \|\Phi_f(v)\|_{X_\delta^\beta} &\leq C\delta^{1/2}(K^3 + L^3), \\ d(\Phi_f(v_1), \Phi_f(v_2)) &\leq C\delta^{1/2}(K^2 + L^2)d(v_1, v_2). \end{aligned}$$

Pick $L = K$ and $\delta = \nu K^{-4}$ with $\nu > 0$ sufficiently small so that $0 < \delta \leq 1$ and

$$2C\delta^{1/2}K^2 = 2C\sqrt{\nu} \leq \frac{1}{2}.$$

We get

$$\begin{aligned} \|\Phi_f(v)\|_{X_\delta^\beta} &\leq K, \\ d(\Phi_f(v_1), \Phi_f(v_2)) &\leq \frac{1}{2}d(v_1, v_2), \end{aligned}$$

hence Φ_f is a contraction $\left(B_{X_\delta^\beta}(K), d\right)$. In particular, for each $f \in \Sigma(K)$, there exists a unique solution to (1.1) with initial datum $u|_{t=0} = f$ satisfying

$$u(t) - e^{-ith}f \in C([- \delta, \delta], \mathcal{H}^\beta) \cap L^8([- \delta, \delta], \mathcal{W}^{\beta,4}).$$

Moreover, for all $f \in \Sigma(K)$, we have $\|f\|_{\mathcal{H}^\theta} \leq K$ and the corresponding solution to (1.1) satisfies

$$\begin{aligned} \|u(t)\|_{\mathcal{H}^\theta} &\leq \|e^{-ith}f\|_{\mathcal{H}^\theta} + \|u(t) - e^{-ith}f\|_{\mathcal{H}^\theta} \\ &\leq \|f\|_{\mathcal{H}^\theta} + \|u(t) - e^{-ith}f\|_{\mathcal{H}^\beta} \\ &\leq 2K, \quad \forall |t| \leq \delta \end{aligned}$$

which is (5.4).

Let us prove (5.5). We have

$$\begin{aligned} \|u_1(t) - u_2(t)\|_{\mathcal{H}^\theta} &\leq \|e^{-ith}f_1 - e^{-ith}f_2\|_{\mathcal{H}^\theta} + \|v_1(t) - v_2(t)\|_{\mathcal{H}^\theta} \\ &\leq \|f_1 - f_2\|_{\mathcal{H}^\theta} + \|v_1(t) - v_2(t)\|_{\mathcal{H}^\beta}, \end{aligned} \quad (5.8)$$

where

$$v_1(t) := u_1(t) - e^{-ith}f_1, \quad v_2(t) := u_2(t) - e^{-ith}f_2.$$

Using Duhamel's formula, we estimate as above to get

$$\begin{aligned} \|v_1 - v_2\|_{X_\delta^\beta} &\leq \| |g_{f_1} + v_1|^2(g_{f_1} + v_1) - |g_{f_2} + v_2|^2(g_{f_2} + v_2) \|_{L^{8/7}([- \delta, \delta], \mathcal{W}^{\beta,4/3})} \\ &\leq C\delta^{1/2} \left(\|g_{f_1}\|_{L^8([- \delta, \delta], \mathcal{W}^{\beta,4})}^2 + \|g_{f_2}\|_{L^8([- \delta, \delta], \mathcal{W}^{\beta,4})}^2 + \|v_1\|_{X_\delta^\beta}^2 + \|v_2\|_{X_\delta^\beta}^2 \right) \\ &\quad \times \left(\|g_{f_1} - g_{f_2}\|_{L^8([- \delta, \delta], \mathcal{W}^{\beta,4})} + \|v_1 - v_2\|_{X_\delta^\beta} \right). \end{aligned}$$

Since $f_1, f_2 \in \Sigma(K)$, we have

$$\|g_{f_1}\|_{L^8([-1,1], \mathcal{W}^{\beta,4})}, \|g_{f_2}\|_{L^8([-1,1], \mathcal{W}^{\beta,4})}, \|v_1\|_{X_\delta^\beta}, \|v_2\|_{X_\delta^\beta} \leq K,$$

hence

$$\begin{aligned} \|v_1 - v_2\|_{X_\delta^\beta} &\leq \| |g_{f_1} + v_1|^2(g_{f_1} + v_1) - |g_{f_2} + v_2|^2(g_{f_2} + v_2) \|_{L^{8/7}([- \delta, \delta], \mathcal{W}^{\beta,4/3})} \\ &\leq C\delta^{1/2}K^2 \left(\|g_{f_1} - g_{f_2}\|_{L^8([- \delta, \delta], \mathcal{W}^{\beta,4})} + \|v_1 - v_2\|_{X_\delta^\beta} \right). \end{aligned}$$

By choosing $\delta > 0$ sufficiently small so that $C\delta^{1/2}K^2 \leq \frac{1}{2}$, we get

$$\|v_1 - v_2\|_{X_\delta^\beta} \leq \|g_{f_1} - g_{f_2}\|_{L^8([- \delta, \delta], \mathcal{W}^{\beta,4})}.$$

This together with (5.8) imply (5.5). The proof is complete. \square

5.2. Measure invariance and global well-posedness.

Theorem 5.4 (Almost sure global well-posedness, $1 < s \leq 2$).

Let $1 < s \leq 2$, V satisfy Assumption 1.1, and $\theta < \frac{1}{2} - \frac{1}{s}$. Assume in addition that $s > \frac{8}{5}$ for the focusing nonlinearity. Then there exists a set $\Sigma \subset \mathcal{H}^\theta$ satisfying $\mu(\Sigma) = 1$ such that for any $f \in \Sigma$, the corresponding solution to (1.1) with initial data $u|_{t=0} = f$ exists globally in time and satisfies

$$\|u(t)\|_{\mathcal{H}^\theta} \leq C \left(\omega(f) + \log^{\frac{1}{2}}(1 + |t|) \right), \quad \forall t \in \mathbb{R}$$

for some constant $\omega(f) > 0$ depending on f and some universal constant $C > 0$. Moreover, the Gibbs measure μ is invariant under the flow of (1.1).

The proof of this theorem follows by the same line of reasoning as in Theorem 4.2. However, due to the lack of deterministic local well-posedness, we need to proceed differently.

Lemma 5.5 (Uniform estimate for the approximate flow, $1 < s \leq 2$).

Let s, θ be as in the previous statement and take θ_1 satisfying

$$\theta < \theta_1 < \frac{1}{2} - \frac{1}{s}.$$

Then for all $\Lambda \geq \lambda_1$ and $T, \varepsilon > 0$. There exist $\Sigma_{\Lambda, T, \varepsilon} \subset \mathcal{H}^{\theta_1}$ and $C > 0$ independent of Λ, T, ε such that:

(1) $\mu_{\Lambda}(\Sigma_{\Lambda, T, \varepsilon}^c) \leq C\varepsilon$.

(2) For $f \in \Sigma_{\Lambda, T, \varepsilon}$, there exists a unique solution to (3.39) on $[-T, T]$ satisfying

$$\|u_{\Lambda}(t)\|_{\mathcal{H}^{\theta_1}} \leq C \left(\log \frac{T}{\varepsilon} \right)^{1/2}, \quad \forall |t| \leq T. \quad (5.9)$$

Proof. Pick $0 \leq \beta < \beta_1 < \frac{s-1}{2s}$ so that

$$\theta_1 - \theta = \beta_1 - \beta = \eta > 0. \quad (5.10)$$

For $K > 0$, we denote

$$\Sigma_{\theta_1, \beta_1}(K) := \left\{ f \in \mathcal{H}^{\theta_1} : \|f\|_{\mathcal{H}^{\theta_1}} \leq K, \|e^{-i t h} f\|_{L^s([-1, 1], \mathcal{W}^{\beta_1, 4})} \leq K \right\}.$$

Using (3.29), the probabilistic local well-posedness given in Proposition 5.2 applies to (3.39) and we have that for $f \in \Sigma_{\theta_1, \beta_1}(K)$, there exists a unique solution to (3.39) satisfying

$$\|u_{\Lambda}(t)\|_{\mathcal{H}^{\theta_1}} \leq 2K, \quad \forall |t| \leq \delta \quad (5.11)$$

with

$$\delta = \nu(K+1)^{-4}, \quad (5.12)$$

where $\nu > 0$ is a small constant independent of Λ, K . As for the super-harmonic case, here we choose δ a bit smaller than νK^{-4} as in Proposition 5.2 which is useful for the iteration process (see Lemma 5.6).

Denote $J = \left\lceil \frac{T}{\delta} \right\rceil$ and set

$$\Sigma_{\Lambda, K, \theta_1, \beta_1} := \bigcap_{j=-J}^J \Phi_{\Lambda}(-j\delta)(\Sigma_{\theta_1, \beta_1}(K)), \quad (5.13)$$

where Φ_{Λ} is the solution map of (3.39). Note that the set $\Sigma_{\Lambda, K, \theta_1, \beta_1}$ contains all initial data f such that the corresponding solutions $u_{\Lambda}(t)$ of (3.39) satisfy

$$\|u_{\Lambda}(t)\|_{\mathcal{H}^{\theta_1}} \leq 2K, \quad \forall |t| \leq T. \quad (5.14)$$

In fact, for $|t| \leq T$, there exist an integer j and $\delta_1 \in (-\delta, \delta)$ such that $t = j\delta + \delta_1$. Thus

$$u_{\Lambda}(t) = \Phi_{\Lambda}(\delta_1)\Phi_{\Lambda}(j\delta)f.$$

Since $f \in \Phi_{\Lambda}(-j\delta)(\Sigma_{\theta_1, \beta_1}(K))$, we have $\Phi_{\Lambda}(j\delta)f \in \Sigma_{\theta_1, \beta_1}(K)$ and the observation follows from (5.11).

Since μ_{Λ} is invariant under the flow of (3.39), we have

$$\begin{aligned} \mu_{\Lambda}(\Sigma_{\Lambda, K, \theta_1, \beta_1}^c) &= \mu_{\Lambda}\left(\left(\bigcap_{j=-J}^J \Phi_{\Lambda}(-j\delta)(\Sigma_{\theta_1, \beta_1}(K))\right)^c\right) \\ &\leq \sum_{j=-J}^J \mu_{\Lambda}(\Phi_{\Lambda}(-j\delta)(\Sigma_{\theta_1, \beta_1}^c(K))) \\ &= \sum_{j=-J}^J \mu_{\Lambda}(\Sigma_{\theta_1, \beta_1}^c(K)) \\ &\leq 2\frac{T}{\delta} \mu_{\Lambda}(\Sigma_{\theta_1, \beta_1}^c(K)). \end{aligned}$$

Thanks to Proposition 5.2 we have

$$\begin{aligned}\mu_\Lambda(\Sigma_{\theta_1, \beta_1}^c(K)) &= \int \mathbf{1}_{\Sigma_{\theta_1, \beta_1}^c(K)} d\mu_\Lambda(u) \\ &= \int \mathbf{1}_{\Sigma_{\theta_1, \beta_1}^c(K)} \frac{1}{Z^\Lambda} G_\Lambda(u) d\mu_0(u) \\ &\leq \frac{1}{Z^\Lambda} \|G_\Lambda(u)\|_{L^2(d\mu_0)} \left(\mu_0(\Sigma_{\theta_1, \beta_1}^c(K))\right)^{1/2} \\ &\leq C e^{-cK^2},\end{aligned}$$

where we have used $G_\Lambda(u) \in L^2(d\mu_0)$ and $Z^\Lambda \geq C > 0$ uniformly in Λ and (5.7). In particular, we obtain

$$\mu_\Lambda(\Sigma_{\Lambda, K, \theta_1, \beta_1}^c) \leq \frac{2C}{\nu} T(K+1)^4 e^{-cK^2} \leq CT e^{-cK^2}$$

for some constants $C, c > 0$. By choosing K as in (4.12) and setting $\Sigma_{\Lambda, T, \varepsilon} = \Sigma_{\Lambda, K, \theta_1, \beta_1}$, we get the desired estimates. \square

Lemma 5.6 (Comparison between approximate and exact flows, $1 < s \leq 2$).

Let $\Sigma_{\Lambda, T, \varepsilon}$ be as in the previous lemma. Then for any $f \in \Sigma_{\Lambda, T, \varepsilon}$, there exists a unique solution to (1.1) with initial data $u|_{t=0} = f$ satisfying

$$\|u(t) - u_\Lambda(t)\|_{\mathcal{H}^\theta} \leq C(T, \varepsilon) \Lambda^{-\eta/2}, \quad \forall |t| \leq T \quad (5.15)$$

for all Λ sufficiently large and some constant $C(T, \varepsilon) > 0$ independent of Λ , where η is as in (5.10). In particular, there exist $\Sigma_{T, \varepsilon} \subset \mathcal{H}^\theta$ and $C > 0$ independent of T, ε such that:

- (1) $\mu(\Sigma_{T, \varepsilon}^c) \leq C\varepsilon$.
- (2) For $f \in \Sigma_{T, \varepsilon}$, there exists a unique solution to (1.1) with initial data $u|_{t=0} = f$ on $[-T, T]$ satisfying

$$\|u(t)\|_{\mathcal{H}^\theta} \leq C \left(\log \frac{T}{\varepsilon}\right)^{1/2}, \quad \forall |t| \leq T. \quad (5.16)$$

Proof. Estimating the difference. We first study the difference between u and u_Λ on $I_0 = [-\delta, \delta]$, where δ is as in (5.12). To do this, we denote $g_f(t) = e^{-ith} f$ and $v_\Lambda := Q_\Lambda u_\Lambda$. By Duhamel's formula, we write for $t \in I_0$,

$$u(t) = g_f(t) + v(t), \quad v(t) := \mp i \int_0^t e^{-i(t-\tau)h} (|g_f + v|^2(g_f + v))(\tau) d\tau$$

and

$$v_\Lambda(t) = Q_\Lambda g_f(t) + w_\Lambda(t), \quad w_\Lambda(t) := \mp i \int_0^t e^{-i(t-\tau)h} (Q_\Lambda^2 (|Q_\Lambda g_f + w_\Lambda|^2(Q_\Lambda g_f + w_\Lambda)))(\tau) d\tau.$$

Denote

$$X^\beta(I_0) := C(I_0, \mathcal{H}^\beta) \cap L^8(I_0, \mathcal{W}^{\beta, 4}).$$

By Strichartz estimates (cf Appendix B), we have

$$\|v - w_\Lambda\|_{X^\beta(I_0)} \lesssim \| |g_f + v|^2(g_f + v) - Q_\Lambda^2 (|Q_\Lambda g_f + w_\Lambda|^2(Q_\Lambda g_f + w_\Lambda)) \|_{L^{8/7}(I_0, \mathcal{W}^{\beta, 4/3})}.$$

We write

$$\begin{aligned}&|g_f + v|^2(g_f + v) - Q_\Lambda^2 (|Q_\Lambda g_f + w_\Lambda|^2(Q_\Lambda g_f + w_\Lambda)) \\ &= |g_f + v|^2(g_f + v) - |Q_\Lambda g_f + w_\Lambda|^2(Q_\Lambda g_f + w_\Lambda) \\ &\quad + (\text{Id} - Q_\Lambda^2) (|Q_\Lambda g_f + w_\Lambda|^2(Q_\Lambda g_f + w_\Lambda)).\end{aligned}$$

Since $f \in \Sigma_{\theta_1, \beta_1}(K)$, by Proposition 5.2 and (3.29), we have

$$\|w_\Lambda\|_{X^{\beta_1}(I_0)} \leq CK. \quad (5.17)$$

Estimating as in the proof of Lemma 5.3 yields

$$\begin{aligned}
& \|(\text{Id} - Q_\Lambda^2)(|Q_\Lambda g_f + w_\Lambda|^2(Q_\Lambda g_f + w_\Lambda))\|_{L^{8/7}(I_0, \mathcal{W}^{\beta, 4/3})} \\
& \leq C\Lambda^{(\beta-\beta_1)/2} \| |Q_\Lambda g_f + w_\Lambda|^2(Q_\Lambda g_f + w_\Lambda) \|_{L^{8/7}(I_0, \mathcal{W}^{\beta_1, 4/3})} \\
& \leq C\delta^{1/2}\Lambda^{-\eta/2} \left(\|Q_\Lambda g_f\|_{L^8(I_0, \mathcal{W}^{\beta_1, 4})}^3 + \|w_\Lambda\|_{L^8(I_0, \mathcal{W}^{\beta_1, 4})}^3 \right) \\
& \leq C\delta^{1/2}\Lambda^{-\eta/2} \left(\|g_f\|_{L^8(I_0, \mathcal{W}^{\beta_1, 4})}^3 + \|w_\Lambda\|_{X^{\beta_1}(I_0)}^3 \right) \\
& \leq C\delta^{1/2}K^3\Lambda^{-\eta/2}.
\end{aligned}$$

Since $\|f\|_{\mathcal{H}^\theta} \leq \|f\|_{\mathcal{H}^{\theta_1}} \leq K$ and $\|g_f\|_{L^8([-1,1], \mathcal{W}^{\beta, 4})} \leq \|g_f\|_{L^8([-1,1], \mathcal{W}^{\beta_1, 4})} \leq K$ due to $f \in \Sigma_{\theta_1, \beta_1}(K)$, Proposition 5.2 and (3.29) give

$$\|v\|_{X^\beta(I_0)}, \|w_\Lambda\|_{X^\beta(I_0)} \leq K.$$

By writing

$$\begin{aligned}
& |g_f + v|^2(g_f + v) - |Q_\Lambda g_f + w_\Lambda|^2(Q_\Lambda g_f + w_\Lambda) \\
& = (g_f - Q_\Lambda g_f + v - w_\Lambda)R_2(g_f, Q_\Lambda g_f, v, w_\Lambda),
\end{aligned}$$

where R_2 is a homogeneous polynomial of degree 2 (omitting the complex conjugates for simplicity) and estimating as in Lemma 5.3, we obtain

$$\begin{aligned}
& \|Q_\Lambda^2(|g_f + v|^2(g_f + v) - |Q_\Lambda g_f + w_\Lambda|^2(Q_\Lambda g_f + w_\Lambda))\|_{L^{8/7}(I_0, \mathcal{W}^{\beta, 4/3})} \\
& \leq C\delta^{1/2} \left(\|g_f - Q_\Lambda g_f\|_{L^8(I_0, \mathcal{W}^{\beta, 4})} + \|v - w_\Lambda\|_{L^8(I_0, \mathcal{W}^{\beta, 4})} \right) \\
& \quad \times \left(\|g_f\|_{L^8(I_0, \mathcal{W}^{\beta, 4})}^2 + \|Q_\Lambda g_f\|_{L^8(I_0, \mathcal{W}^{\beta, 4})}^2 + \|v\|_{L^8(I_0, \mathcal{W}^{\beta, 4})}^2 + \|w_\Lambda\|_{L^8(I_0, \mathcal{W}^{\beta, 4})}^2 \right) \\
& \leq C\delta^{1/2} \left(\Lambda^{(\beta-\beta_1)/2} \|g_f\|_{L^8(I_0, \mathcal{W}^{\beta_1, 4})} + \|v - w_\Lambda\|_{X^\beta(I_0)} \right) \\
& \quad \times \left(\|g_f\|_{L^8(I_0, \mathcal{W}^{\beta, 4})}^2 + \|v\|_{X^\beta(I_0)}^2 + \|w_\Lambda\|_{X^\beta(I_0)}^2 \right) \\
& \leq C\delta^{1/2}K^2 \left(K\Lambda^{-\eta/2} + \|v - w_\Lambda\|_{X^\beta(I_0)} \right).
\end{aligned}$$

Thus we obtain

$$\|v - w_\Lambda\|_{X^\beta(I_0)} \leq C\delta^{1/2}K^3\Lambda^{-\eta/2} + C\delta^{1/2}K^2\|v - w_\Lambda\|_{X^\beta(I_0)}.$$

By the choice of δ (see (4.12)), we get

$$\|v - w_\Lambda\|_{X^\beta(I_0)} \leq K\Lambda^{-\eta/2}.$$

From this and (5.14), we deduce

$$\begin{aligned}
\|u - u_\Lambda\|_{L^\infty(I_0, \mathcal{H}^\theta)} & \leq \|u - v_\Lambda\|_{L^\infty(I_0, \mathcal{H}^\theta)} + \|v_\Lambda - u_\Lambda\|_{L^\infty(I_0, \mathcal{H}^\theta)} \\
& \leq \|g_f - Q_\Lambda g_f\|_{L^\infty(I_0, \mathcal{H}^\theta)} + \|v - w_\Lambda\|_{L^\infty(I_0, \mathcal{H}^\theta)} + \|Q_\Lambda u_\Lambda - u_\Lambda\|_{L^\infty(I_0, \mathcal{H}^\theta)} \\
& \leq \|f - Q_\Lambda f\|_{\mathcal{H}^\theta} + \|v - w_\Lambda\|_{L^\infty(I_0, \mathcal{H}^\beta)} + \|Q_\Lambda u_\Lambda - u_\Lambda\|_{L^\infty(I_0, \mathcal{H}^\theta)} \\
& \leq C\Lambda^{(\theta-\theta_1)/2} \|f\|_{\mathcal{H}^{\theta_1}} + \|v - w_\Lambda\|_{X^\beta(I_0)} + C\Lambda^{(\theta-\theta_1)/2} \|u_\Lambda\|_{L^\infty(I_0, \mathcal{H}^{\theta_1})} \\
& \leq CK\Lambda^{-\eta/2}.
\end{aligned} \tag{5.18}$$

Iterating in time. We now iterate this argument to other sub-intervals of $[-T, T]$. The next iteration is on $I_1 := [0, 2\delta]$. To this end, we claim that, taking Λ sufficiently large,

$$\|u(\delta)\|_{\mathcal{H}^\theta} \leq K + 1, \quad \|e^{-i\text{th}}u(\delta)\|_{L^8([-1,1], \mathcal{W}^{\beta, 4})} \leq K + 1. \tag{5.19}$$

In fact, the first observation follows directly from (5.18) and

$$\|u_\Lambda(\delta)\|_{\mathcal{H}^\theta} = \|\Phi_\Lambda(\delta)f\|_{\mathcal{H}^\theta} \leq \|\Phi_\Lambda(\delta)f\|_{\mathcal{H}^{\theta_1}} \leq K$$

since $u_\Lambda(\delta) = \Phi_\Lambda(\delta)f \in \Sigma_{\theta_1, \beta_1}(K)$ (see (5.13)). We turn to the second inequality in (5.19), using Strichartz estimates and (5.10)

$$\begin{aligned} \|e^{-ith}u(\delta)\|_{L^8([-1,1], \mathcal{W}^{\beta_1, 4})} &\leq \|e^{-ith}u_\Lambda(\delta)\|_{L^8([-1,1], \mathcal{W}^{\beta_1, 4})} + \|e^{-ith}(u(\delta) - u_\Lambda(\delta))\|_{L^8([-1,1], \mathcal{W}^{\beta_1, 4})} \\ &\leq \|e^{-ith}u_\Lambda(\delta)\|_{L^8([-1,1], \mathcal{W}^{\beta_1, 4})} + C\|u(\delta) - u_\Lambda(\delta)\|_{\mathcal{H}^\beta} \\ &\leq K + C\Lambda^{-\eta/2}\|u - u_\Lambda\|_{L^\infty(I_0, \mathcal{H}^{\beta_1})}. \end{aligned} \quad (5.20)$$

Next, using Duhamel's formulas

$$u(t) = e^{-ith}f \mp i \int_0^t e^{-i(t-\tau)h}(|u|^2u)(\tau)d\tau$$

and

$$u_\Lambda(t) = e^{-ith}f \mp i \int_0^t e^{-i(t-\tau)h}(Q_\Lambda(|Q_\Lambda u_\Lambda|^2 Q_\Lambda u_\Lambda))(\tau)d\tau,$$

we have

$$\begin{aligned} \|u - u_\Lambda\|_{L^\infty(I_0, \mathcal{H}^{\beta_1})} &\leq C\| |u|^2u - Q_\Lambda(|Q_\Lambda u_\Lambda|^2 Q_\Lambda u_\Lambda) \|_{L^{8/7}(I_0, \mathcal{W}^{\beta_1, 4/3})} \\ &\leq C\delta^{1/2} \left(\|u\|_{L^8(I_0, \mathcal{W}^{\beta_1, 4})}^3 + \|u_\Lambda\|_{L^8(I_0, \mathcal{W}^{\beta_1, 4})}^3 \right) \end{aligned} \quad (5.21)$$

$$\begin{aligned} &\leq C\delta^{1/2}K^3 \\ &\leq K \end{aligned} \quad (5.22)$$

by the choice of δ . Here we have used the fact that

$$\begin{aligned} \|u\|_{L^8(I_0, \mathcal{W}^{\beta_1, 4})} &\leq \|gf\|_{L^8(I_0, \mathcal{W}^{\beta_1, 4})} + \|u - gf\|_{L^8(I_0, \mathcal{W}^{\beta_1, 4})} \leq 2K, \\ \|u_\Lambda\|_{L^8(I_0, \mathcal{W}^{\beta_1, 4})} &\leq \|gf\|_{L^8(I_0, \mathcal{W}^{\beta_1, 4})} + \|u_\Lambda - gf\|_{L^8(I_0, \mathcal{W}^{\beta_1, 4})} \leq 2K, \end{aligned}$$

where the second line follows from the probabilistic local well-posedness. Combining (5.20) with (5.21) yields the second inequality in (5.19). Note that (5.19) is the reason why we choose δ as in (5.12).

Thanks to (5.19) and $u_\Lambda(\delta) = \Phi_\Lambda(\delta)f \in \Sigma_{\theta_1, \beta_1}(K)$, we can repeat the above argument using the following Duhamel formulas for $t \in [0, 2\delta]$,

$$\begin{aligned} u(t) &= g_\delta(t) + v(t), \quad v(t) := \mp i \int_\delta^t e^{-i(t-\tau)h}(|g_\delta + v|^2(g_\delta + v))(\tau)d\tau \\ v_\Lambda(t) &= Q_\Lambda g_{\Lambda, \delta}(t) + w_\Lambda(t), \\ w_\Lambda(t) &:= \mp i \int_\delta^t e^{-i(t-\tau)h}(Q_\Lambda^2(|Q_\Lambda g_{\Lambda, \delta} + w_\Lambda|^2(Q_\Lambda g_{\Lambda, \delta} + w_\Lambda)))(\tau)d\tau \end{aligned}$$

with $g_\delta(t) := e^{-i(t-\delta)h}u(\delta)$ and $g_{\Lambda, \delta}(t) := e^{-i(t-\delta)h}u_\Lambda(\delta)$ and get

$$\|v - w_\Lambda\|_{X^\beta(I_1)} \leq K\Lambda^{-\eta/2}.$$

The same reasoning as in (5.18) yields

$$\|u - u_\Lambda\|_{L^\infty(I_1, \mathcal{H}^\theta)} \lesssim CK\Lambda^{-\eta/2}.$$

Iterating this bound $\left\lceil \frac{T}{\delta} \right\rceil$ many times and taking into account the choice of K (see (4.12)), we prove (5.15).

Globalizing the flow. Finally, (5.16) follows from (5.14) and (5.15) exactly as in the proof of Lemma 4.3. We omit the details. \square

We are now able to prove Theorem 5.4.

Proof of Theorem 5.4. The almost sure global well-posedness and the growth in time are proved by the same argument as in the proof of Theorem 4.2. Since the continuity of the solution flow does not depend solely on \mathcal{H}^θ norm of initial data (see (5.5) and (5.6)), the argument given in the super-harmonic case should be modified by choosing a suitable ball in higher Sobolev spaces. We present here another approach using an equivalent characterization of measure invariance (see e.g., [37, Theorem 6.5]), i.e., for all $F \in L^1(\mathcal{H}^\theta, d\mu)$,

$$\int F(\Phi(t)u)d\mu(u) = \int F(u)d\mu(u), \quad \forall t \in \mathbb{R}. \quad (5.23)$$

By iteration, it suffices to show (5.23) for $t > 0$ small. In addition, by a density argument, the problem is reduced to show (5.23) for F bounded and continuous.

Now fix $t > 0$ small and let $\varepsilon > 0$. We write for $\Lambda \geq \lambda_1$,

$$\begin{aligned} \left| \int F(\Phi(t)u)d\mu(u) - \int F(u)d\mu(u) \right| &\leq \left| \int F(\Phi(t)u)d\mu(u) - \int F(\Phi(t)u)d\mu_\Lambda(u) \right| \\ &\quad + \left| \int F(\Phi(t)u)d\mu_\Lambda(u) - \int F(\Phi_\Lambda(t)u)d\mu_\Lambda(u) \right| \\ &\quad + \left| \int F(\Phi_\Lambda(t)u)d\mu_\Lambda(u) - \int F(u)d\mu_\Lambda(u) \right| \\ &\quad + \left| \int F(u)d\mu_\Lambda(u) - \int F(u)d\mu(u) \right| \\ &=: \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)}. \end{aligned}$$

Since $\mu_\Lambda \rightharpoonup \mu$ weakly as $\Lambda \rightarrow \infty$, the boundedness of F implies

$$\text{(I)} + \text{(IV)} \rightarrow 0 \text{ as } \Lambda \rightarrow \infty.$$

Since μ_Λ is invariant under the flow map $\Phi_\Lambda(t)$, an equivalent characterization of invariance as in (5.23) yields $\text{(III)} = 0$. It remains to estimate (II) . Pick $\theta < \theta_1 < \frac{1}{2} - \frac{1}{s}$ and $0 \leq \beta < \beta_1 < \frac{s-1}{2s}$ such that $\theta_1 - \theta = \beta_1 - \beta = \eta > 0$. Denote

$$\Sigma_{\theta_1, \beta_1}(K) := \left\{ f \in \mathcal{H}^{\theta_1} : \|f\|_{\mathcal{H}^{\theta_1}} \leq K, \|e^{-ith}f\|_{L^8([-1,1], \mathcal{W}^{\beta_1, 4})} \leq K \right\}.$$

We estimate

$$\begin{aligned} \text{(II)} &\leq \left| \int_{\Sigma_{\theta_1, \beta_1}(K)} F(\Phi(t)u) - F(\Phi_\Lambda(t)u)d\mu_\Lambda(u) \right| + \left| \int_{\Sigma_{\theta_1, \beta_1}^c(K)} F(\Phi(t)u) - F(\Phi_\Lambda(t)u)d\mu_\Lambda(u) \right| \\ &=: \text{(II)}_1 + \text{(II)}_2. \end{aligned}$$

Since F is bounded, we have

$$\text{(II)}_2 \leq 2\|F\|_{L^\infty} \mu_\Lambda(\Sigma_{\theta_1, \beta_1}^c(K)) \leq C\|F\|_{L^\infty} \left(\mu_0(\Sigma_{\theta_1, \beta_1}^c(K)) \right)^{1/2} \leq Ce^{-cK^2} < \frac{\varepsilon}{2}$$

provided that K is taken sufficiently large. For such a K , the same argument as in the proof of Lemma 5.5 yields

$$\|\Phi(t)u - \Phi_\Lambda(t)u\|_{\mathcal{H}^\theta} \leq CK\Lambda^{-\eta/2}$$

for all $u \in \Sigma_{\theta_1, \beta_1}(K)$. Note that $t > 0$ is taken small here. Since F is continuous, we have

$$\|F(\Phi(t)u) - F(\Phi_\Lambda(t)u)\|_{\mathcal{H}^\theta} < \varepsilon/2$$

for all $u \in \Sigma_{\theta_1, \beta_1}(K)$ provided that Λ is chosen large enough. Since μ_Λ is a probability measure, we have $\text{(II)}_1 \leq \varepsilon/2$, hence

$$\text{(II)} \rightarrow 0 \text{ as } \Lambda \rightarrow \infty.$$

Collecting the above estimates, we prove the invariance. \square

Proof of Theorem 2.3. It follows from Theorems 4.2 and 5.4. \square

6. CANONICAL MEASURES

We aim at constructing, and proving the invariance of, Gibbs measures conditioned on mass. Towards this purpose, we first define the Gaussian measure with a fixed renormalized mass in Section 6.1. We then define the Gibbs measures conditioned on mass in Section 6.2. Finally, in Section 6.3, we prove that these measures are invariant under the dynamics of (1.1).

6.1. Gaussian measure with a fixed renormalized mass. Our purpose in this section is to give a rigorous definition of Gaussian measure conditioned on mass. There is a difference between the cases $s > 2$ and $1 < s \leq 2$ in that the latter the mass is infinite almost surely. We however treat the two cases on the same footing by conditioning on the renormalized mass, although this is slightly redundant for $s > 2$ (where the mass is finite almost surely). To do this, we shall define the Gaussian measure with a fixed renormalized mass which is formally given by

$$d\mu_0^m(u) \text{ “ = ” } \frac{\mathbb{1}_{\{\mathcal{M}(u)=m\}}}{\mu_0(\mathcal{M}(u) = m)} d\mu_0(u), \quad (6.1)$$

where $m \in \mathbb{R}$, $\mathcal{M}(u) = \|u\|_{L^2}^2 - \langle \|u\|_{L^2}^2 \rangle_{\mu_0}$ is the renormalized mass (see Lemma 3.6), and μ_0 is the Gaussian measure. When $s > 2$, since the expectation of the mass with respect to the Gaussian measure is finite,

$$\langle \|u\|_{L^2}^2 \rangle_{\mu_0} = \text{Tr}[h^{-1}] < \infty,$$

we obtain a Gaussian measure with a fixed mass $m + \text{Tr}[h^{-1}] > 0$.

The formulation (6.1) is purely formal because we essentially have that

$$\mu_0(\mathcal{M}(u) = m) = 0.$$

Inspired by an idea of Oh and Quastel [42], we will define the above measure as a limit, as $\varepsilon \rightarrow 0^+$, of the constrained measure

$$d\mu_0^{m,\varepsilon}(u) = \frac{1}{Z_0^{m,\varepsilon}} \mathbb{1}_{\{m-\varepsilon < \mathcal{M}(u) < m+\varepsilon\}} d\mu_0(u),$$

where

$$Z_0^{m,\varepsilon} = \mu_0(m - \varepsilon < \mathcal{M}(u) < m + \varepsilon).$$

Note that for each $\varepsilon > 0$, the measure $\mu_0^{m,\varepsilon}$ is well-defined a priori, however, since the normalized constant $Z_0^{m,\varepsilon}$ converges to zero as $\varepsilon \rightarrow 0^+$, some care is needed to justify the meaning of this limit.

The construction of the canonical Gaussian measure consists of two steps:

- We first find a sequence of measures on the finite dimensional spaces

$$E_{\leq \Lambda} = \mathbb{1}_{h \leq \Lambda} L^2(\mathbb{R})$$

that will define the cylindrical projections of the target measure.

- We then show that the above sequence of measures is tight on a suitable Hilbert space, so that the target measure can be defined as its limit.

This is summarized in the following statement, whose proof occupies this whole subsection.

Proposition 6.1 (Gaussian measure with a fixed renormalized mass).

Let $s > 1$, V satisfy Assumption 1.1, $\theta < \frac{1}{2} - \frac{1}{s}$, $m \in \mathbb{R}$, and assume $m > -\text{Tr}[h^{-1}]$ if $s > 2$.

(1) For any borelian set A of $E_{\leq \Lambda}$, the limit

$$\lim_{\varepsilon \rightarrow 0^+} \mu_0^{m,\varepsilon}(A) = \lim_{\varepsilon \rightarrow 0^+} \frac{\mu_0(A \cap \{m - \varepsilon < \mathcal{M}(u) < m + \varepsilon\})}{\mu_0(m - \varepsilon < \mathcal{M}(u) < m + \varepsilon)} \quad (6.2)$$

exists, and thus we may define a probability measure on $E_{\leq \Lambda}$ by setting

$$\mu_0^{m,\leq \Lambda}(A) := \lim_{\varepsilon \rightarrow 0^+} \mu_0^{m,\varepsilon}(A).$$

(2) There exists a unique measure μ_0^m supported in $\mathcal{H}^\theta \cap \{\mathcal{M}(u) = m\}$ such that for all $\Lambda \geq \lambda_1$, $\mu_0^{m,\leq \Lambda}$ is the cylindrical projection of μ_0^m on $E_{\leq \Lambda}$. Moreover, we have for any measurable set $A \subset \mathcal{H}^\theta$

$$\mu_0^m(A) = \lim_{\varepsilon \rightarrow 0^+} \mu_0^{m,\varepsilon}(A).$$

We start by defining the finite dimensional measures as follows.

Lemma 6.2 (Cylindrical projections I).

The projection of $\mu_0^{m,\varepsilon}$ on $E_{\leq \Lambda}$ is given by

$$\begin{aligned} d\mu_0^{m,\varepsilon}|_{E_{\leq \Lambda}} &= \frac{1}{Z_0^{m,\varepsilon}} \mu_0^{> \Lambda}(m - \varepsilon - \vartheta_\Lambda < \mathcal{M}_{> \Lambda}(u) < m + \varepsilon - \vartheta_\Lambda) d\mu_0^{\leq \Lambda}(u) \\ &=: d\mu_0^{m,\varepsilon,\leq \Lambda}(u), \end{aligned} \quad (6.3)$$

where

$$\vartheta_\Lambda := \mathcal{M}_{\leq \Lambda}(u) = \sum_{\lambda_j \leq \Lambda} |\alpha_j|^2 - \lambda_j^{-1}$$

and $\mathcal{M}_{> \Lambda}(u)$ is as in (3.19). The measure $\mu_0^{m,\varepsilon,\leq \Lambda}$ is the cylindrical projection of $\mu_0^{m,\varepsilon}$ on $E_{\leq \Lambda}$ in the sense that for $\Theta \geq \Lambda$,

$$\mu_0^{m,\varepsilon,\leq \Theta}|_{E_{\leq \Lambda}} = \mu_0^{m,\varepsilon,\leq \Lambda}. \quad (6.4)$$

Proof. For any borelian set A of $E_{\leq \Lambda}$, we have

$$\begin{aligned} \mu_0^{m,\varepsilon}(A) &= \frac{1}{Z_0^{m,\varepsilon}} \int \mathbb{1}_{A \cap \{m - \varepsilon < \mathcal{M}(u) < m + \varepsilon\}} d\mu_0(u) \\ &= \frac{1}{Z_0^{m,\varepsilon}} \int_A \left(\int \mathbb{1}_{\{m - \varepsilon - \vartheta_\Lambda < \mathcal{M}_{> \Lambda}(u) < m + \varepsilon - \vartheta_\Lambda\}} d\mu_0^{> \Lambda}(u) \right) d\mu_0^{\leq \Lambda}(u) \\ &= \frac{1}{Z_0^{m,\varepsilon}} \int_A \mu_0^{> \Lambda}(m - \varepsilon - \vartheta_\Lambda < \mathcal{M}_{> \Lambda}(u) < m + \varepsilon - \vartheta_\Lambda) d\mu_0^{\leq \Lambda}(u), \end{aligned}$$

where $d\mu_0^{\leq \Lambda}(u)$ and $d\mu_0^{> \Lambda}(u)$ are defined as in (2.8) and (3.31) respectively. This shows (6.3). To see (6.4), we observe that $A \times \mathbb{C}^N$ is a borelian set of $E_{\leq \Theta}$ with $N = \#\{\lambda_j : \Lambda <$

$\lambda_j \leq \Theta\}$, hence

$$\begin{aligned}
\mu_0^{m,\varepsilon,\leq\Theta} \Big|_{E_{\leq\Lambda}}(A) &= \int_{A \times \mathbb{C}^N} d\mu_0^{m,\varepsilon,\leq\Theta}(u) \\
&= \frac{1}{Z_0^{m,\varepsilon}} \int_{A \times \mathbb{C}^N} \mu_0^{>\Theta}(m - \varepsilon - \vartheta_\Theta < \mathcal{M}_{>\Theta}(u) < m + \varepsilon - \vartheta_\Theta) d\mu_0^{\leq\Theta}(u) \\
&= \frac{1}{Z_0^{m,\varepsilon}} \int_{A \times \mathbb{C}^N} \left(\int \mathbb{1}_{\{m - \varepsilon - \vartheta_\Theta < \mathcal{M}_{>\Theta}(u) < m + \varepsilon - \vartheta_\Theta\}} d\mu_0^{>\Theta}(u) \right) d\mu_0^{\leq\Theta}(u) \\
&= \frac{1}{Z_0^{m,\varepsilon}} \int_A \left(\int_{\mathbb{C}^N} \left(\int \mathbb{1}_{\{m - \varepsilon - \vartheta_\Theta < \mathcal{M}_{>\Theta}(u) < m + \varepsilon - \vartheta_\Theta\}} d\mu_0^{>\Theta}(u) \right) \prod_{\Lambda < \lambda_j \leq \Theta} \frac{\lambda_j}{\pi} e^{-\lambda_j |\alpha_j|^2} d\alpha_j \right) d\mu_0^{\leq\Lambda}(u) \\
&= \frac{1}{Z_0^{m,\varepsilon}} \int_A \left(\int \mathbb{1}_{\{m - \varepsilon - \vartheta_\Lambda < \mathcal{M}_{>\Lambda}(u) < m + \varepsilon - \vartheta_\Lambda\}} d\mu_0^{>\Lambda}(u) \right) d\mu_0^{\leq\Lambda}(u) \\
&= \frac{1}{Z_0^{m,\varepsilon}} \int_A \mu_0^{>\Lambda}(m - \varepsilon - \vartheta_\Lambda < \mathcal{M}_{>\Lambda}(u) < m + \varepsilon - \vartheta_\Lambda) d\mu_0^{\leq\Lambda}(u) \\
&= \int_A d\mu_0^{m,\varepsilon,\leq\Lambda}(u) = \mu_0^{m,\varepsilon,\leq\Lambda}(A) \tag{6.5}
\end{aligned}$$

which proves (6.4). \square

To prove that the limit in (6.2) exists, we denote f_Λ the density function of $\mathcal{M}_{>\Lambda}(u)$ with respect to $\mu_0^{>\Lambda}$. In particular, we have

$$\mu_0^{>\Lambda}(m - \varepsilon - \vartheta_\Lambda < \mathcal{M}_{>\Lambda}(u) < m + \varepsilon - \vartheta_\Lambda) = \int_{m - \varepsilon - \vartheta_\Lambda}^{m + \varepsilon - \vartheta_\Lambda} f_\Lambda(x) dx$$

and

$$Z_0^{m,\varepsilon} = \mu_0(m - \varepsilon < \mathcal{M}(u) < m + \varepsilon) = \int_{m - \varepsilon}^{m + \varepsilon} f_0(x) dx,$$

hence

$$\mu_0^{m,\varepsilon}(A) = \int_A \left(\int_{m - \varepsilon - \vartheta_\Lambda}^{m + \varepsilon - \vartheta_\Lambda} f_\Lambda(x) dx \right) \left(\int_{m - \varepsilon}^{m + \varepsilon} f_0(x) dx \right)^{-1} d\mu_0^{\leq\Lambda}(u). \tag{6.6}$$

Lemma 6.3 (Uniform continuity of the density function).

For any $\Lambda \geq 0$, f_Λ is bounded and uniformly continuous on $(m_0, +\infty)$, where $m_0 = -\text{Tr}[h^{-1}]$ if $s > 2$ and $m_0 = -\infty$ if $1 < s \leq 2$. In addition, for $\Lambda > 0$ sufficiently large, there exists $C > 0$ such that $\|f_\Lambda\|_{L^\infty((m_0, +\infty))} \leq C\Lambda$.

Proof. Denote ϕ_Λ the characteristic function of $\mathcal{M}_{>\Lambda}(u)$ with respect to $\mu_0^{>\Lambda}$. We have

$$\begin{aligned}
\phi_\Lambda(s) &= \mathbb{E}_{\mu_0^{>\Lambda}}[e^{is\mathcal{M}_{>\Lambda}(u)}] \\
&= \int e^{is \sum_{\lambda_j > \Lambda} |\alpha_j|^2 - \lambda_j^{-1}} d\mu_0^{>\Lambda}(u) \\
&= \int \prod_{\lambda_j > \Lambda} e^{is(|\alpha_j|^2 - \lambda_j^{-1})} \prod_{\lambda_k > \Lambda} \frac{\lambda_k}{\pi} e^{-\lambda_k |\alpha_k|^2} d\alpha_k \\
&= \prod_{\lambda_j > \Lambda} \left(\int_{\mathbb{C}} e^{-is\lambda_j^{-1}} \frac{\lambda_j}{\pi} e^{-(1-is\lambda_j^{-1})\lambda_j |\alpha_j|^2} d\alpha_j \right) \left(\prod_{\substack{\lambda_k \neq \lambda_j \\ \lambda_k > \Lambda}} \int_{\mathbb{C}} \frac{\lambda_k}{\pi} e^{-\lambda_k |\alpha_k|^2} d\alpha_k \right) \\
&= \prod_{\lambda_j > \Lambda} \frac{e^{-is\lambda_j^{-1}}}{1 - is\lambda_j^{-1}}.
\end{aligned}$$

Each factor of this product has complex norm smaller than or equal to 1, thus the norm of this product is bounded by the norm of a product of any two terms. In particular, we have

$$|\phi_\Lambda(s)| \leq \left| \frac{e^{-is\lambda_{j_1}^{-1}}}{1 - is\lambda_{j_1}^{-1}} \frac{e^{-is\lambda_{j_2}^{-1}}}{1 - is\lambda_{j_2}^{-1}} \right| = \frac{1}{\sqrt{1 + s^2\lambda_{j_1}^{-2}}} \frac{1}{\sqrt{1 + s^2\lambda_{j_2}^{-2}}} \leq \frac{1}{1 + s^2\lambda_{j_2}^{-2}},$$

where j_1 and j_2 are the first two indices such that $\lambda_{j_2} \geq \lambda_{j_1} > \Lambda$. We deduce that $\phi_\Lambda \in L^1(\mathbb{R})$ with

$$\|\phi_\Lambda\|_{L^1(\mathbb{R})} \leq C\lambda_{j_2}.$$

We have (see Lemma D.1) that the number $N(\Lambda)$ of eigenvalues of h below Λ satisfies

$$c\Lambda^{\frac{1}{2} + \frac{1}{s}} \leq N(\Lambda) \leq C\Lambda^{\frac{1}{2} + \frac{1}{s}}$$

for positive constants $c, C > 0$. Hence, we may pick some $k > 0$ large enough but independent of Λ to ensure that

$$N(k\Lambda) - N(\Lambda) \geq (ck^{1/2+1/s} - C)\Lambda^{\frac{1}{2} + \frac{1}{s}} > 1$$

for Λ sufficiently large. Hence λ_{j_2} can be chosen of order Λ in the above argument and we deduce that

$$\|\phi_\Lambda\|_{L^1(\mathbb{R})} \leq C\Lambda$$

for Λ sufficiently large.

Using the following relation between density and characteristic functions

$$f_\Lambda(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-isx} \phi_\Lambda(s) ds,$$

we infer that f_Λ is bounded and uniformly continuous on $(m_0, +\infty)$. Note that when $s > 2$, we have $\mathcal{M}_{>\Lambda}(u) = \|P_{>\Lambda}u\|_{L^2}^2 - \langle \|P_{>\Lambda}u\|_{L^2}^2 \rangle_{\mu_0} > -\text{Tr}[h^{-1}]$, hence the density function f_Λ is defined on $(-\text{Tr}[h^{-1}], +\infty)$. \square

Lemma 6.4 (Positivity of the density function).

Let $s > 1$, $m \in \mathbb{R}$, and assume $m > -\text{Tr}[h^{-1}]$ if $s > 2$. We have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon} \int_{m-\varepsilon}^{m+\varepsilon} f_0(x) dx = f_0(m) > 0.$$

Proof. Since f_0 is uniformly continuous on \mathbb{R} , we have

$$\frac{1}{2\varepsilon} \int_{m-\varepsilon}^{m+\varepsilon} f_0(x) dx \xrightarrow{\varepsilon \rightarrow 0^+} f_0(m).$$

It remains to prove that $f_0(m) > 0$. Since the first eigenvalue of h is simple (see e.g., [35, Theorem 11.8]), we have

$$\begin{aligned} & \frac{1}{2\varepsilon} \int_{m-\varepsilon}^{m+\varepsilon} f_0(x) dx \\ &= \frac{1}{2\varepsilon} \mu_0(m - \varepsilon < \mathcal{M}(u) < m + \varepsilon) \\ &= \frac{1}{2\varepsilon} \mu_0(\mathcal{M}_{>\lambda_1}(u) \in (m_0, +\infty), m - \varepsilon - \mathcal{M}_{>\lambda_1}(u) < |\alpha_1|^2 - \lambda_1^{-1} < m + \varepsilon - \mathcal{M}_{>\lambda_1}(u)) \\ &= \frac{1}{2\varepsilon} \mu_0(\mathcal{M}_{>\lambda_1}(u) \in (m_0, +\infty), \lambda_1^{-1} + m - \varepsilon - \mathcal{M}_{>\lambda_1}(u) < |\alpha_1|^2 < \lambda_1^{-1} + m + \varepsilon - \mathcal{M}_{>\lambda_1}(u)) \\ &= \frac{1}{2\varepsilon} \int_{m_0}^{+\infty} f_{\lambda_1}(x) \left(\int_{\max\{0, \lambda_1^{-1} + m - \varepsilon - x\} < |\alpha_1|^2 < \lambda_1^{-1} + m + \varepsilon - x} \frac{\lambda_1}{\pi} e^{-\lambda_1|\alpha_1|^2} d\alpha_1 \right) dx. \end{aligned}$$

We estimate it further as

$$\begin{aligned}
& \frac{1}{2\varepsilon} \int_{m-\varepsilon}^{m+\varepsilon} f_0(x) dx \\
&= \frac{1}{2\varepsilon} \int_{m_0}^{\lambda_1^{-1}+m-\varepsilon} f_{\lambda_1}(x) \left(\int_{\lambda_1^{-1}+m-\varepsilon-x < |\alpha_1|^2 < \lambda_1^{-1}+m+\varepsilon-x} \frac{\lambda_1}{\pi} e^{-\lambda_1|\alpha_1|^2} d\alpha_1 \right) dx \\
&+ \frac{1}{2\varepsilon} \int_{\lambda_1^{-1}+m-\varepsilon}^{\lambda_1^{-1}+m+\varepsilon} f_{\lambda_1}(x) \left(\int_{0 < |\alpha_1|^2 < \lambda_1^{-1}+m+\varepsilon-x} \frac{\lambda_1}{\pi} e^{-\lambda_1|\alpha_1|^2} d\alpha_1 \right) dx \\
&= \frac{1}{2\varepsilon} \int_{m_0}^{\lambda_1^{-1}+m-\varepsilon} f_{\lambda_1}(x) \left(\int_{1+\lambda_1(m-\varepsilon-x)}^{1+\lambda_1(m+\varepsilon-x)} e^{-\lambda} d\lambda \right) dx \\
&+ \frac{1}{2\varepsilon} \int_{\lambda_1^{-1}+m-\varepsilon}^{\lambda_1^{-1}+m+\varepsilon} f_{\lambda_1}(x) \left(\int_0^{1+\lambda_1(m+\varepsilon-x)} e^{-\lambda} d\lambda \right) dx \\
&\geq \frac{1}{2\varepsilon} \int_{m_0}^{\lambda_1^{-1}+m-\varepsilon} f_{\lambda_1}(x) e^{-1-\lambda_1 m + \lambda_1 x} (e^{\lambda_1 \varepsilon} - e^{-\lambda_1 \varepsilon}) dx \\
&= \frac{1}{2\varepsilon} \int_{m_0}^{\lambda_1^{-1}+m} f_{\lambda_1}(x) e^{-1-\lambda_1 m + \lambda_1 x} (e^{\lambda_1 \varepsilon} - e^{-\lambda_1 \varepsilon}) dx \\
&- \frac{1}{2\varepsilon} \int_{\lambda_1^{-1}+m-\varepsilon}^{\lambda_1^{-1}+m} f_{\lambda_1}(x) e^{-1-\lambda_1 m + \lambda_1 x} (e^{\lambda_1 \varepsilon} - e^{-\lambda_1 \varepsilon}) dx.
\end{aligned}$$

The last term converges to zero as $\varepsilon \rightarrow 0^+$ due to the dominated convergence theorem. Thus letting $\varepsilon \rightarrow 0^+$, we get

$$\begin{aligned}
f_0(m) &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon} \int_{m-\varepsilon}^{m+\varepsilon} f_0(x) dx \geq \lim_{\varepsilon \rightarrow 0^+} \int_{m_0}^{\lambda_1^{-1}+m} f_{\lambda_1}(x) e^{-1-\lambda_1 m + \lambda_1 x} \frac{e^{\lambda_1 \varepsilon} - e^{-\lambda_1 \varepsilon}}{2\varepsilon} dx \\
&= \int_{m_0}^{\lambda_1^{-1}+m} f_{\lambda_1}(x) e^{-1-\lambda_1 m + \lambda_1 x} \lambda_1 dx.
\end{aligned}$$

Assume for contradiction that $f_0(m) = 0$. Since $f_{\lambda_1}(x) \geq 0$ for all $x \in (m_0, +\infty)$, we infer that $f_{\lambda_1}(x) = 0$ for all $x \in (m_0, \lambda_1^{-1} + m)$. In particular, we have

$$\begin{aligned}
0 &= \int_{m_0}^{\lambda_1^{-1}+m} f_{\lambda_1}(x) dx \\
&= \mu_0^{>\lambda_1}(m_0 < \mathcal{M}_{>\lambda_1}(u) < \lambda_1^{-1} + m) \\
&= \mu_0^{>\lambda_1}(\mathcal{M}_{>\lambda_2}(u) \in (m_0, +\infty), m_0 - \mathcal{M}_{>\lambda_2}(u) < \sum_{\lambda_j=\lambda_2} |\alpha_j|^2 - \lambda_j^{-1} < \lambda_1^{-1} + m - \mathcal{M}_{>\lambda_2}(u)).
\end{aligned}$$

To proceed further, we denote

$$\alpha = (\alpha_j)_{\lambda_j=\lambda_2} \in \mathbb{C}^N,$$

with N the multiplicity of λ_2 , hence $d\alpha = \prod_{\lambda_j=\lambda_2} d\alpha_j$ and $|\alpha|^2 = \sum_{\lambda_j=\lambda_2} |\alpha_j|^2$. The above measure becomes

$$\begin{aligned} & \int_{m_0}^{N\lambda_2^{-1}+\lambda_1^{-1}+m} f_{\lambda_2}(x) \left(\int_{\max\{0, m_0+N\lambda_2^{-1}-x\}}^{<|\alpha|^2 < N\lambda_2^{-1}+\lambda_1^{-1}+m-x} \left(\frac{\lambda_2}{\pi}\right)^N e^{-\lambda_2|\alpha|^2} d\alpha \right) dx \\ &= \int_{m_0}^{N\lambda_2^{-1}+m_0} f_{\lambda_2}(x) \left(\int_{m_0+N\lambda_2^{-1}-x}^{<|\alpha|^2 < N\lambda_2^{-1}+\lambda_1^{-1}+m-x} \left(\frac{\lambda_2}{\pi}\right)^N e^{-\lambda_2|\alpha|^2} d\alpha \right) dx \\ &+ \int_{N\lambda_2^{-1}+m_0}^{N\lambda_2^{-1}+\lambda_1^{-1}+m} f_{\lambda_2}(x) \left(\int_{0}^{<|\alpha|^2 < N\lambda_2^{-1}+\lambda_1^{-1}+m-x} \left(\frac{\lambda_2}{\pi}\right)^N e^{-\lambda_2|\alpha|^2} d\alpha \right) dx \end{aligned}$$

Using polar coordinates, this becomes (up to a factor $\sigma(\mathbb{S}^{2N-1})$)

$$\begin{aligned} & \int_{m_0}^{N\lambda_2^{-1}+m_0} f_{\lambda_2}(x) \left(\int_{\sqrt{m_0+N\lambda_2^{-1}-x}}^{\sqrt{N\lambda_2^{-1}+\lambda_1^{-1}+m-x}} \left(\frac{\lambda_2}{\pi}\right)^N e^{-\lambda_2 r^2} r^{2N-1} dr \right) dx \\ &+ \int_{N\lambda_2^{-1}+m_0}^{N\lambda_2^{-1}+\lambda_1^{-1}+m-x} f_{\lambda_2}(x) \left(\int_0^{\sqrt{N\lambda_2^{-1}+\lambda_1^{-1}+m-x}} \left(\frac{\lambda_2}{\pi}\right)^N e^{-\lambda_2 r^2} r^{2N-1} dr \right) dx. \end{aligned}$$

By a change of variable $\lambda = \lambda_2 r^2$, the above quantity is (up to a factor $\frac{1}{2\pi^N}$)

$$\begin{aligned} & \int_{m_0}^{N\lambda_2^{-1}+m_0} f_{\lambda_2}(x) \left(\int_{\lambda_2(m_0+N\lambda_2^{-1}-x)}^{\lambda_2(N\lambda_2^{-1}+\lambda_1^{-1}+m-x)} e^{-\lambda} \lambda^{N-1} d\lambda \right) dx \\ &+ \int_{N\lambda_2^{-1}+m_0}^{N\lambda_2^{-1}+\lambda_1^{-1}+m} f_{\lambda_2}(x) \left(\int_0^{\lambda_2(N\lambda_2^{-1}+\lambda_1^{-1}+m-x)} e^{-\lambda} \lambda^{N-1} d\lambda \right) dx. \end{aligned}$$

Since this sum is equal to zero and both terms are non-negative, we infer that $f_{\lambda_2}(x) = 0$ for all $x \in (m_0, N\lambda_2^{-1} + \lambda_1^{-1} + m)$. Arguing in a similar manner, we can prove that $f_{\Lambda}(x) = 0$ for all $x \in (m_0, \text{Tr}[(P_{\leq \Lambda} h)^{-1}] + m)$ and all $\Lambda \geq 0$. Thus

$$1 = \int_{m_0}^{+\infty} f_{\Lambda}(x) dx = \int_{\text{Tr}[(P_{\leq \Lambda} h)^{-1}] + m}^{+\infty} f_{\Lambda}(x) dx, \quad \forall \Lambda \geq 0.$$

This contradicts the fact that

$$\begin{aligned} \int_{\text{Tr}[(P_{\leq \Lambda} h)^{-1}] + m}^{+\infty} f_{\Lambda}(x) dx &= \mu_0^{>\Lambda} \left(\mathcal{M}_{>\Lambda}(u) > \text{Tr}[(P_{\leq \Lambda} h)^{-1}] + m \right) \\ &\leq \mu_0^{>\Lambda} \left((\mathcal{M}_{>\Lambda}(u))^2 > (\text{Tr}[(P_{\leq \Lambda} h)^{-1}] + m)^2 \right) \\ &\leq \frac{1}{(\text{Tr}[(P_{\leq \Lambda} h)^{-1}] + m)^2} \int (\mathcal{M}_{>\Lambda}(u))^2 d\mu_0^{>\Lambda}(u) \\ &= \frac{1}{(\text{Tr}[(P_{\leq \Lambda} h)^{-1}] + m)^2} \sum_{\lambda_j > \Lambda} \lambda_j^{-2} \rightarrow 0 \text{ as } \Lambda \rightarrow \infty \end{aligned}$$

due to $\text{Tr}[h^{-2}] = \sum \lambda_j^{-2} < \infty$. Note that for m fixed, we have

$$\text{Tr}[(P_{\leq \Lambda} h)^{-1}] + m \xrightarrow{\Lambda \rightarrow \infty} \begin{cases} m + \text{Tr}[h^{-1}] > 0 & \text{if } s > 2, \\ +\infty & \text{if } 1 < s \leq 2. \end{cases}$$

The proof is complete. \square

Lemma 6.5 (Cylindrical projections II).

Let $s > 1$, $m \in \mathbb{R}$, and assume $m > -\text{Tr}[h^{-1}]$ if $s > 2$. Then for any $\Lambda \geq \lambda_1$ and any

borelian set A of $E_{\leq \Lambda}$, the limit $\lim_{\varepsilon \rightarrow 0^+} \mu_0^{m, \varepsilon}(A)$ exists and

$$\lim_{\varepsilon \rightarrow 0^+} \mu_0^{m, \varepsilon}(A) = \int_A \frac{f_\Lambda(m - \vartheta_\Lambda)}{f_0(m)} d\mu_0^{\leq \Lambda}(u) =: \mu_0^{m, \leq \Lambda}(A). \quad (6.7)$$

In particular,

$$d\mu_0^{m, \leq \Lambda}(u) = \frac{f_\Lambda(m - \vartheta_\Lambda)}{f_0(m)} d\mu_0^{\leq \Lambda}(u)$$

satisfies for any $\Theta \geq \Lambda$,

$$\mu_0^{m, \leq \Theta} \Big|_{E_{\leq \Lambda}} = \mu_0^{m, \leq \Lambda}. \quad (6.8)$$

Proof. On the one hand, by the uniform continuity of f_Λ , we have

$$\left(\frac{1}{2\varepsilon} \int_{m-\varepsilon-\vartheta_\Lambda}^{m+\varepsilon-\vartheta_\Lambda} f_\Lambda(x) dx \right) \left(\frac{1}{2\varepsilon} \int_{m-\varepsilon}^{m+\varepsilon} f_0(x) dx \right)^{-1} \xrightarrow{\varepsilon \rightarrow 0^+} \frac{f_\Lambda(m - \vartheta_\Lambda)}{f_0(m)}.$$

On the other hand, the boundedness of f_Λ gives

$$\frac{1}{2\varepsilon} \int_{m-\varepsilon-\vartheta_\Lambda}^{m+\varepsilon-\vartheta_\Lambda} f_\Lambda(x) dx \leq \|f_\Lambda\|_{L^\infty}.$$

Since $f_0(m) > 0$, we have for $\varepsilon > 0$ sufficiently small,

$$\frac{1}{2\varepsilon} \int_{m-\varepsilon}^{m+\varepsilon} f_0(x) dx \geq \frac{f_0(m)}{2}.$$

In particular, we get for $\varepsilon > 0$ sufficiently small,

$$\left(\int_{m-\varepsilon-\vartheta_\Lambda}^{m+\varepsilon-\vartheta_\Lambda} f_\Lambda(x) dx \right) \left(\int_{m-\varepsilon}^{m+\varepsilon} f_0(x) dx \right)^{-1} d\mu_0^{\leq \Lambda}(u) \leq \frac{2\|f_\Lambda\|_{L^\infty}}{f_0(m)} d\mu_0^{\leq \Lambda}(u)$$

which is integrable on A . From (6.6), the dominated convergence theorem yields (6.7).

To see (6.8), we use (6.5) to have for any borelian set A of $E_{\leq \Lambda}$,

$$\begin{aligned} \mu_0^{m, \leq \Theta} \Big|_{E_{\leq \Lambda}}(A) &= \mu_0^{m, \leq \Theta}(A \times \mathbb{C}^N) \\ &= \lim_{\varepsilon \rightarrow 0^+} \mu_0^{m, \varepsilon, \leq \Theta}(A \times \mathbb{C}^N) \\ &= \lim_{\varepsilon \rightarrow 0^+} \mu_0^{m, \varepsilon, \leq \Lambda}(A) \\ &= \mu_0^{m, \leq \Lambda}(A), \end{aligned}$$

where $N = \#\{\lambda_j : \Lambda < \lambda_j \leq \Theta\}$. \square

We have constructed a sequence of measures $\{\mu_0^{m, \leq \Lambda}\}_{\Lambda \geq \lambda_1}$ on the finite dimensional spaces $\{E_{\leq \Lambda}\}_{\Lambda \geq \lambda_1}$ that satisfies the cylindrical property (6.8). Thus we have completed the proof of Item (1) of Proposition 6.1.

To prove Item (2), we will show that there exists a unique measure μ_0^m on an infinite dimensional Hilbert space satisfying

$$\mu_0^m \Big|_{E_{\leq \Lambda}} = \mu_0^{m, \leq \Lambda}.$$

By Skorokhod's criterion (see [51, Lemma 1]), it suffices to check that the sequence $\{\mu_0^{m, \leq \Lambda}\}_{\Lambda \geq \lambda_1}$ is tight in the sense that

$$\lim_{R \rightarrow \infty} \sup_{\Lambda \geq \lambda_1} \mu_0^{m, \leq \Lambda}(\{u \in E_{\leq \Lambda} : \|u\|_{\mathcal{H}^\theta} \geq R\}) = 0$$

for $\theta < \frac{1}{2} - \frac{1}{s}$. This clearly follows from the following lemma, whose proof will thus complete that of Proposition 6.1.

Lemma 6.6 (Tightness of the canonical Gaussian measure).

Let $s > 1$, $m \in \mathbb{R}$, and assume $m > -\text{Tr}[h^{-1}]$ if $s > 2$. Then for any $\theta < \frac{1}{2} - \frac{1}{s}$, there exists $C(m, \theta) > 0$ such that

$$\int_{E_{\leq \Lambda}} \|u\|_{\mathcal{H}^\theta}^2 d\mu_0^{m, \leq \Lambda}(u) \leq C(m, \theta), \quad \forall \Lambda \geq \lambda_1.$$

Proof. It suffices to prove that

$$\int \lambda_j |\alpha_j|^2 d\mu_0^{m, \leq \Lambda}(u) \leq C(m), \quad \forall \Lambda \geq \lambda_1, \quad \forall \lambda_1 \leq \lambda_j \leq \Lambda \quad (6.9)$$

for some constant $C(m)$ depending only on m . Indeed, we have

$$\begin{aligned} \int_{E_{\leq \Lambda}} \|u\|_{\mathcal{H}^\theta}^2 d\mu_0^{m, \leq \Lambda}(u) &= \int \sum_{\lambda_j \leq \Lambda} \lambda_j^\theta |\alpha_j|^2 d\mu_0^{m, \leq \Lambda}(u) \\ &= \sum_{\lambda_j \leq \Lambda} \lambda_j^{\theta-1} \int \lambda_j |\alpha_j|^2 d\mu_0^{m, \leq \Lambda}(u) \\ &\leq C(m) \sum_{\lambda_j \leq \Lambda} \lambda_j^{-1+\theta} \\ &\leq C(m) \text{Tr}[h^{-1+\theta}] < \infty, \end{aligned}$$

where we have used the fact that $1 - \theta > \frac{1}{2} + \frac{1}{s}$ to get the finiteness of $\text{Tr}[h^{-1+\theta}]$ (see Lemma A.1).

We are thus reduced to proving (6.9), which we shall deduce from the fact that for all $\Lambda \geq \lambda_1$, all $\lambda_1 \leq \lambda_j \leq \Lambda$, and all $\varepsilon > 0$ sufficiently small,

$$\int \lambda_j |\alpha_j|^2 d\mu_0^{m, \varepsilon, \leq \Lambda}(u) \leq C(m). \quad (6.10)$$

We have

$$\begin{aligned} &\int \lambda_j |\alpha_j|^2 d\mu_0^{m, \varepsilon, \leq \Lambda}(u) \\ &= \frac{1}{Z_0^{m, \varepsilon}} \int \lambda_j |\alpha_j|^2 \mu_0^{> \Lambda}(m - \varepsilon - \vartheta_\Lambda < \mathcal{M}_{> \Lambda}(u) < m + \varepsilon - \vartheta_\Lambda) d\mu_0^{\leq \Lambda}(u) \\ &= \frac{1}{Z_0^{m, \varepsilon}} \int \lambda_j |\alpha_j|^2 \left(\int \mathbb{1}_{\{m - \varepsilon - \vartheta_\Lambda < \mathcal{M}_{> \Lambda}(u) < m + \varepsilon - \vartheta_\Lambda\}} d\mu_0^{> \Lambda}(u) \right) d\mu_0^{\leq \Lambda}(u) \\ &= \frac{1}{Z_0^{m, \varepsilon}} \int \lambda_j |\alpha_j|^2 \left(\int \mathbb{1}_{\{m - \varepsilon - |\alpha_j|^2 + \lambda_j^{-1} < \mathcal{M}_{\neq j}(u) < m + \varepsilon - |\alpha_j|^2 + \lambda_j^{-1}\}} d\mu_0^{\neq j}(u) \right) \frac{\lambda_j}{\pi} e^{-\lambda_j |\alpha_j|^2} d\alpha_j \\ &= \frac{1}{Z_0^{m, \varepsilon}} \int \lambda_j |\alpha_j|^2 \mu_0^{\neq j}(m - \varepsilon - |\alpha_j|^2 + \lambda_j^{-1} < \mathcal{M}_{\neq j}(u) < m + \varepsilon - |\alpha_j|^2 + \lambda_j^{-1}) \frac{\lambda_j}{\pi} e^{-\lambda_j |\alpha_j|^2} d\alpha_j, \end{aligned}$$

where

$$\mathcal{M}_{\neq j}(u) = \sum_{k \neq j} |\alpha_k|^2 - \lambda_k^{-1}, \quad d\mu_0^{\neq j}(u) = \prod_{k \neq j} \frac{\lambda_k}{\pi} e^{-\lambda_k |\alpha_k|^2} d\alpha_k.$$

Denote F_j the density function of $\mathcal{M}_{\neq j}(u)$ with respect to $\mu_0^{\neq j}$. We rewrite

$$\int (\lambda_j |\alpha_j|^2)^q d\mu_0^{m, \varepsilon, \leq \Lambda}(u) = \frac{1}{Z_0^{m, \varepsilon}} \int \lambda_j |\alpha_j|^2 \left(\int_{m - \varepsilon - |\alpha_j|^2 + \lambda_j^{-1}}^{m + \varepsilon - |\alpha_j|^2 + \lambda_j^{-1}} F_j(x) dx \right) \frac{\lambda_j}{\pi} e^{-\lambda_j |\alpha_j|^2} d\alpha_j.$$

To proceed further, we denote by Φ_j the characteristic function of $\mathcal{M}_{\neq j}(u)$ with respect to $\mu_0^{\neq j}$. As in the proof of Lemma 6.3, we have

$$\Phi_j(s) = \mathbb{E}_{\mu_0^{\neq j}} [e^{is \mathcal{M}_{\neq j}(u)}] = \prod_{k \neq j} \frac{e^{-is \lambda_k^{-1}}}{1 - is \lambda_k^{-1}}.$$

Since each factor of this product has complex norm smaller than or equal to 1, we bound the norm of this product by the norm of a product of any two terms. Taking the first two terms, we obtain that for all $\lambda_j \geq \lambda_1$,

$$|\Phi_j(s)| \leq \frac{1}{1 + s^2 \lambda_3^{-2}}.$$

In particular, $\|\Phi_j\|_{L^1(\mathbb{R})} \leq C\lambda_3$ for all $\lambda_j \geq \lambda_1$. Thus F_j is bounded (uniformly in j) and uniformly continuous for all $\lambda_j \geq \lambda_1$.

By Lemma 6.4, we have for $\varepsilon > 0$ sufficiently small,

$$\frac{1}{2\varepsilon} Z_0^{m,\varepsilon} \geq \frac{f_0(m)}{2}.$$

We also have

$$\frac{1}{2\varepsilon} \int_{m-\varepsilon-|\alpha_j|^2+\lambda_j^{-1}}^{m+\varepsilon-|\alpha_j|^2+\lambda_j^{-1}} F_j(x) dx \leq \|F_j\|_{L^\infty} \leq C\|\Phi_j\|_{L^1(\mathbb{R})} \leq C\lambda_3, \quad \forall \lambda_j \geq \lambda_1.$$

It follows that

$$\begin{aligned} \int \lambda_j |\alpha_j|^2 d\mu_0^{m,\varepsilon,\leq\Lambda}(u) &\leq \frac{2C\lambda_3}{f_0(m)} \int \lambda_j |\alpha_j|^2 \frac{\lambda_j}{\pi} e^{-\lambda_j |\alpha_j|^2} d\alpha_j \\ &= \frac{2C\lambda_3}{f_0(m)} \int_0^\infty \lambda e^{-\lambda} d\lambda \\ &= C(m) \end{aligned}$$

for all $\lambda_j \geq \lambda_1$ and all $\varepsilon > 0$ sufficiently small. This proves (6.10).

We now prove (6.9). Using the layer cake representation, the problem is reduced to showing that

$$\int_0^\infty \mu_0^{m,\leq\Lambda}(\lambda_j |\alpha_j|^2 > \lambda) d\lambda = \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty \mu_0^{m,\varepsilon,\leq\Lambda}(\lambda_j |\alpha_j|^2 > \lambda) d\lambda.$$

Thanks to the weak convergence $\mu_0^{m,\varepsilon,\leq\Lambda} \rightarrow \mu_0^{m,\leq\Lambda}$, (6.9) follows from the dominated convergence theorem and the fact that

$$\begin{aligned}
& \mu_0^{m,\varepsilon,\leq\Lambda}(\lambda_j|\alpha_j|^2 > \lambda) \\
&= \int \mathbb{1}_{\{\lambda_j|\alpha_j|^2 > \lambda\}} d\mu_0^{m,\varepsilon,\leq\Lambda}(u) \\
&= \frac{1}{Z_0^{m,\varepsilon}} \int \mathbb{1}_{\{\lambda_j|\alpha_j|^2 > \lambda\}} \mu_0^{>\Lambda}(m - \varepsilon - \vartheta_\Lambda < \mathcal{M}_{>\Lambda}(u) < m + \varepsilon - \vartheta_\Lambda) d\mu_0^{\leq\Lambda}(u) \\
&= \frac{1}{Z_0^{m,\varepsilon}} \int \mathbb{1}_{\{\lambda_j|\alpha_j|^2 > \lambda\}} \left(\int \mathbb{1}_{\{m - \varepsilon - \vartheta_\Lambda < \mathcal{M}_{>\Lambda}(u) < m + \varepsilon - \vartheta_\Lambda\}} d\mu_0^{>\Lambda}(u) \right) d\mu_0^{\leq\Lambda}(u) \\
&= \frac{1}{Z_0^{m,\varepsilon}} \int \mathbb{1}_{\{\lambda_j|\alpha_j|^2 > \lambda\} \cap \{|\alpha_j|^2 - \lambda_j^{-1} \in \mathbb{R}\}} \\
&\quad \times \left(\int \mathbb{1}_{\{m - \varepsilon - |\alpha_j|^2 + \lambda_j^{-1} < \mathcal{M}_{\neq j}(u) < m + \varepsilon - |\alpha_j|^2 + \lambda_j^{-1}\}} d\mu_0^{\neq j}(u) \right) \frac{\lambda_j}{\pi} e^{-\lambda_j|\alpha_j|^2} d\alpha_j \\
&= \frac{1}{Z_0^{m,\varepsilon}} \int \mathbb{1}_{\{\lambda_j|\alpha_j|^2 > \lambda\} \cap \{|\alpha_j|^2 - \lambda_j^{-1} \in \mathbb{R}\}} \left(\int_{m - \varepsilon - |\alpha_j|^2 + \lambda_j^{-1}}^{m + \varepsilon - |\alpha_j|^2 + \lambda_j^{-1}} F_j(x) dx \right) \frac{\lambda_j}{\pi} e^{-\lambda_j|\alpha_j|^2} d\alpha_j \\
&\leq \frac{2C\lambda_3}{f_0(m)} \int \mathbb{1}_{\{\lambda_j|\alpha_j|^2 > \lambda, |\alpha_j|^2 - \lambda_j^{-1} \in \mathbb{R}\}} \frac{\lambda_j}{\pi} e^{-\lambda_j|\alpha_j|^2} d\alpha_j \\
&\leq \frac{2C\lambda_3}{f_0(m)} \int_\lambda^\infty e^{-\tau} d\tau \\
&= \frac{2C\lambda_3}{f_0(m)} e^{-\lambda}
\end{aligned}$$

which is integrable on $(0, \infty)$. \square

6.2. Gibbs measures conditioned on the mass. Once the Gaussian measure conditioned on mass is constructed, our next task is to define the Gibbs measure with a fixed renormalized mass as follows:

$$d\mu^m(u) = \frac{1}{Z^m} e^{\mp \frac{1}{2} \|u\|_{L^4}^4} d\mu_0^m(u),$$

where

$$Z^m := \int e^{\mp \frac{1}{2} \|u\|_{L^4}^4} d\mu_0^m(u)$$

is the normalization constant. Here the minus sign stands for the defocusing nonlinearity, and the plus sign is for the focusing one.

Proposition 6.7 (Gibbs measures conditioned on mass).

Let $s > 1$, V satisfy Assumption 1.1, $m \in \mathbb{R}$, and assume $m > -\text{Tr}[h^{-1}]$ if $s > 2$. Assume in addition that $s > \frac{8}{5}$ for the focusing nonlinearity. Then μ^m makes sense as a probability measure.

We need the following observation.

Lemma 6.8 (Decay of the renormalized mass).

Let $s > 1$, V satisfy Assumption 1.1, $0 \leq \gamma < \frac{3s-2}{4s}$, $m \in \mathbb{R}$, and assume $m > -\text{Tr}[h^{-1}]$ if $s > 2$. Then there exist $C, c > 0$ such that for $\theta < \frac{1}{2} - \frac{1}{s}$, all $\Lambda \geq \lambda_2$, all $R > 0$, and all $\varepsilon > 0$ sufficiently small,

$$\mu_0^{m,\varepsilon} \left(u \in \mathcal{H}^\theta : |\mathcal{M}_{>\Lambda}(u)| > R \right) \leq C e^{-c\Lambda^\gamma R}. \quad (6.11)$$

In particular, we have

$$\mu_0^m \left(u \in \mathcal{H}^\theta : |\mathcal{M}_{>\Lambda}(u)| > R \right) \leq C e^{-c\Lambda^\gamma R}. \quad (6.12)$$

Proof. Since $\mu_0^{m,\varepsilon} \rightarrow \mu_0^m$ as $\varepsilon \rightarrow 0^+$ (see Proposition 6.1), it is enough to prove (6.11). We estimate

$$\mu_0^{m,\varepsilon}(|\mathcal{M}_{>\Lambda}(u)| > R) \leq \mu_0^{m,\varepsilon}(\mathcal{M}_{>\Lambda}(u) > R) + \mu_0^{m,\varepsilon}(\mathcal{M}_{>\Lambda}(u) < -R) = \text{(I)} + \text{(II)}.$$

For (I), we have for $0 < t < \frac{\Lambda}{2}$ to be chosen shortly,

$$\mu_0^{m,\varepsilon}(\mathcal{M}_{>\Lambda}(u) > R) \leq e^{-tR} \int e^{t\mathcal{M}_{>\Lambda}(u)} d\mu_0^{m,\varepsilon}(u).$$

By the definition of $\mu_0^{m,\varepsilon}$,

$$\begin{aligned} \int e^{t\mathcal{M}_{>\Lambda}(u)} d\mu_0^{m,\varepsilon}(u) &= \frac{1}{Z_0^{m,\varepsilon}} \int e^{t\mathcal{M}_{>\Lambda}(u)} \mathbf{1}_{\{m-\varepsilon < \mathcal{M}(u) < m+\varepsilon\}} d\mu_0(u) \\ &= \frac{1}{Z_0^{m,\varepsilon}} \int e^{t\mathcal{M}_{>\Lambda}(u)} \left(\int \mathbf{1}_{\{m-\varepsilon-\varrho_\Lambda < \mathcal{M}_{\leq\Lambda}(u) < m+\varepsilon-\varrho_\Lambda\}} d\mu_0^{\leq\Lambda}(u) \right) d\mu_0^{>\Lambda}(u), \end{aligned}$$

where

$$\varrho_\Lambda = \mathcal{M}_{>\Lambda}(u).$$

Denote g_Λ the density function of $\mathcal{M}_{\leq\Lambda}(u)$ with respect to $\mu_0^{\leq\Lambda}$. We rewrite the above quantity as

$$\frac{1}{Z_0^{m,\varepsilon}} \int e^{t\mathcal{M}_{>\Lambda}(u)} \left(\int_{m-\varepsilon-\varrho_\Lambda}^{m+\varepsilon-\varrho_\Lambda} g_\Lambda(x) dx \right) d\mu_0^{>\Lambda}(u).$$

Let ψ_Λ be the characteristic function of $\mathcal{M}_{\leq\Lambda}(u)$ with respect to $\mu_0^{\leq\Lambda}$. We compute

$$\begin{aligned} \psi_\Lambda(s) &= \mathbb{E}_{\mu_0^{\leq\Lambda}}[e^{is\mathcal{M}_{\leq\Lambda}(u)}] \\ &= \int e^{is\mathcal{M}_{\leq\Lambda}(u)} d\mu_0^{\leq\Lambda}(u) \\ &= \int e^{is \sum_{\lambda_j \leq \Lambda} |\alpha_j|^2 - \lambda_j^{-1}} \prod_{\lambda_k \leq \Lambda} \frac{\lambda_k}{\pi} e^{-\lambda_k |\alpha_k|^2} d\alpha_k \\ &= \prod_{\lambda_j \leq \Lambda} \frac{e^{-is\lambda_j^{-1}}}{1 - is\lambda_j^{-1}}. \end{aligned}$$

Since $\Lambda \geq \lambda_2$, we see that

$$|\psi_\Lambda(s)| \leq \frac{1}{1 + s^2 \lambda_2^{-2}},$$

hence $\|\psi_\Lambda\|_{L^1(\mathbb{R})} \leq C\lambda_2$. Thus we deduce that g_Λ is uniformly bounded (in Λ) and uniformly continuous. In particular, we have

$$\frac{1}{2\varepsilon} \int_{m-\varepsilon-\varrho_\Lambda}^{m+\varepsilon-\varrho_\Lambda} g_\Lambda(x) dx \leq \|g_\Lambda\|_{L^\infty} \leq C\|\psi_\Lambda\|_{L^1(\mathbb{R})} \leq C\lambda_2.$$

On the other hand, Lemma 6.4 implies for $\varepsilon > 0$ sufficiently small,

$$\frac{1}{2\varepsilon} Z_0^{m,\varepsilon} \geq \frac{f_0(m)}{2}.$$

Thus for $\varepsilon > 0$ small enough, we get

$$\frac{1}{Z_0^{m,\varepsilon}} \int_{m-\varepsilon-\varrho_\Lambda}^{m+\varepsilon-\varrho_\Lambda} g_\Lambda(x) dx \leq \frac{2C\lambda_2}{f_0(m)},$$

hence

$$\int e^{t\mathcal{M}_{>\Lambda}(u)} d\mu_0^{m,\varepsilon}(u) \leq \frac{2C\lambda_2}{f_0(m)} \int e^{t\mathcal{M}_{>\Lambda}(u)} d\mu_0^{>\Lambda}(u).$$

By the same argument as in the proof of Lemma 3.7, we have

$$\int e^{t\mathcal{M}_{>\Lambda}(u)} d\mu_0^{>\Lambda}(u) = \prod_{\lambda_j > \Lambda} e^{-t\lambda_j^{-1}} \frac{1}{1-t\lambda_j^{-1}} \leq C \prod_{\lambda_j > \Lambda} e^{t^2\lambda_j^{-2}} = Ce^{t^2 \sum_{\lambda_j > \Lambda} \lambda_j^{-2}}.$$

For $0 \leq \gamma < \frac{3s-2}{4s}$, we have

$$\sum_{\lambda_j > \Lambda} \lambda_j^{-2} \leq \Lambda^{-2\gamma} \text{Tr}[h^{-2+2\gamma}].$$

In particular, we obtain

$$\mu_0^{m,\varepsilon}(\mathcal{M}_{>\Lambda}(u) > R) \leq Ce^{-tR+t^2\Lambda^{-2}\text{Tr}[h^{-2+2\gamma}]}.$$

Taking $t = \nu\Lambda^\gamma$ with $\nu > 0$ small yields

$$(I) \leq Ce^{-c\Lambda^\gamma R}$$

for any $0 \leq \gamma < \frac{3s-2}{4s}$. The term (II) is treated in a similar manner and we prove (6.11). \square

Lemma 6.9 (Decay of the L^4 -norm).

Let $s > 1$, V satisfy Assumption 1.1, $0 \leq \gamma < \frac{3s-2}{4s}$, $m \in \mathbb{R}$, and assume $m > -\text{Tr}[h^{-1}]$ if $s > 2$. Then there exist $C, c > 0$ such that for $\theta < \frac{1}{2} - \frac{1}{s}$, all $\Lambda \geq \lambda_2$, all $R > 0$ and all $\varepsilon > 0$ sufficiently small,

$$\mu_0^{m,\varepsilon}(u \in \mathcal{H}^\theta : \|P_{>\Lambda}u\|_{L^4} > R) \leq Ce^{-c\Lambda^\rho R^2}. \quad (6.13)$$

In particular, we have

$$\mu_0^m(u \in \mathcal{H}^\theta : \|P_{>\Lambda}u\|_{L^4} > R) \leq Ce^{-c\Lambda^\rho R^2}. \quad (6.14)$$

Proof. Since $\mu_0^{m,\varepsilon} \rightarrow \mu_0^m$ as $\varepsilon \rightarrow 0^+$, it suffices to prove (6.13). Let $t > 0$ be a positive constant to be determined later. We estimate

$$\begin{aligned} \mu_0^{m,\varepsilon}(\|P_{>\Lambda}u\|_{L^4} > R) &\leq e^{-tR^2} \int e^{t\|P_{>\Lambda}u\|_{L^4}^2} d\mu_0^{m,\varepsilon}(u) \\ &= e^{-tR^2} \sum_{k \geq 0} \frac{t^k}{k!} \int \|P_{>\Lambda}u\|_{L^4}^{2k} d\mu_0^{m,\varepsilon}(u). \end{aligned}$$

We have

$$\begin{aligned} &\int \|P_{>\Lambda}u\|_{L^4}^{2k} d\mu_0^{m,\varepsilon}(u) \\ &= \frac{1}{Z_0^{m,\varepsilon}} \int \|P_{>\Lambda}u\|_{L^4}^{2k} \mathbf{1}_{\{m-\varepsilon < \mathcal{M}(u) < m+\varepsilon\}} d\mu_0(u) \\ &= \frac{1}{Z_0^{m,\varepsilon}} \int \|P_{>\Lambda}u\|_{L^4}^{2k} \left(\int \mathbf{1}_{\{m-\varepsilon-\varrho_\Lambda < \mathcal{M}_{\leq \Lambda}(u) < m+\varepsilon-\varrho_\Lambda\}} d\mu_0^{\leq \Lambda}(u) \right) d\mu_0^{>\Lambda}(u) \\ &= \frac{1}{Z_0^{m,\varepsilon}} \int \|P_{>\Lambda}u\|_{L^4}^{2k} \left(\int_{m-\varepsilon-\varrho_\Lambda}^{m+\varepsilon-\varrho_\Lambda} g_\Lambda(x) dx \right) d\mu_0^{>\Lambda}(u), \end{aligned}$$

where g_Λ is the density function of $\mathcal{M}_{\leq \Lambda}(u)$ with respect to $\mu_0^{\leq \Lambda}$ and $\varrho_\Lambda = \mathcal{M}_{>\Lambda}(u)$. As in the proof of Lemma 6.8, we have for $\varepsilon > 0$ sufficiently small,

$$\frac{1}{Z_0^{m,\varepsilon}} \int_{m-\varepsilon-\varrho_\Lambda}^{m+\varepsilon-\varrho_\Lambda} g_\Lambda(x) dx \leq \frac{2C\lambda_2}{f_0(m)}$$

hence

$$\int \|P_{>\Lambda}u\|_{L^4}^{2k} d\mu_0^{m,\varepsilon}(u) \leq \frac{2C\lambda_2}{f_0(m)} \int \|P_{>\Lambda}u\|_{L^4}^{2k} d\mu_0^{>\Lambda}(u).$$

By the same argument as in the proof of Lemma 3.4, we have

$$\int \|P_{>\Lambda}u\|_{L^4}^{2k} d\mu_0^{>\Lambda}(u) \leq 2!k!B_{\Lambda,0,4}^k, \quad \forall k \geq 0,$$

where $B_{\Lambda,0,4}$ is as in (3.14). In particular, we have for $\varepsilon > 0$ sufficiently small,

$$\int \|P_{>\Lambda}u\|_{L^4}^{2k} d\mu_0^{m,\varepsilon}(u) \leq Ck!B_{\Lambda,0,4}^k, \quad \forall k \geq 0,$$

hence

$$\mu_0^{m,\varepsilon}(\|P_{>\Lambda}u\|_{L^4} > R) \leq Ce^{-tR^2} \sum_{k \geq 0} (tB_{\Lambda,0,4})^k.$$

Arguing exactly as in the proof of Lemma 3.5, we obtain (6.13). \square

Proof of Proposition 6.7, the super-harmonic case $s > 2$. Let us consider the focusing non-linearity. It suffices to prove

$$Z^m = \int e^{\frac{1}{2}\|u\|_{L^4}^4} d\mu_0^m(u) < \infty \quad (6.15)$$

since it is clear that $Z^m \geq 1$. We write

$$\begin{aligned} Z^m &= \int e^{\frac{1}{2}\|u\|_{L^4}^4} d\mu_0^m(u) = \int_0^\infty \mu_0^m(\|u\|_{L^4}^4 > 2\lambda) e^\lambda d\lambda \\ &= C(\lambda_0) + \int_{\lambda_0}^\infty \mu_0^m(\|u\|_{L^4}^4 > 2\lambda) e^\lambda d\lambda \end{aligned}$$

for some $\lambda_0 > 0$ to be chosen shortly. On the support of μ_0^m , we have

$$\|u\|_{L^2}^2 = \mathcal{M}(u) + \text{Tr}[h^{-1}] = m + \text{Tr}[h^{-1}].$$

Repeating the same reasoning as in the proof of Item 2 of Proposition 2.2, we have for $\lambda > \lambda_0$,

$$\mu_0^m(\|u\|_{L^4}^4 > 2\lambda) \leq \mu_0^m\left(\|P_{>\Lambda_0}u\|_{L^4} > \frac{1}{2}(2\lambda)^{\frac{1}{4}}\right),$$

where $\Lambda_0 \sim \lambda^2$ is defined in (3.16). Taking λ_0 sufficiently large so that $\Lambda_0 \gtrsim \lambda_0^2 \geq \lambda_2$, we apply Lemma 6.9 to get

$$\mu_0^m\left(\|P_{>\Lambda_0}u\|_{L^4} > \frac{1}{2}(2\lambda)^{\frac{1}{4}}\right) \leq Ce^{-c\Lambda_0^\rho \lambda^{1/2}} = Ce^{-c\lambda^{2\rho+1/2}}$$

for any $0 \leq \rho < \frac{s-1}{2s}$. Choosing $\rho \in \left(\frac{1}{4}, \frac{s-1}{2s}\right)$, we prove that $Z^m < \infty$.

For the defocusing nonlinearity, it suffices to show $Z^m > 0$. By Jensen's inequality, it is enough to prove that

$$\int \|u\|_{L^4}^4 d\mu_0^m(u) < \infty$$

which is a consequence of (6.15). \square

Before giving the proof of Proposition 6.7 in the (sub)-harmonic case $s \leq 2$, we have the following observation.

Lemma 6.10 (Decay of $\mathcal{W}^{\beta,p}$ -norm).

Let $s > 1$, V satisfy Assumption 1.1, $0 \leq \beta < \frac{1}{2}$, $p > \max\left\{2, \frac{4}{s(1-2\beta)}\right\}$ be an even integer, $m \in \mathbb{R}$, and assume $m > -\text{Tr}[h^{-1}]$ if $s > 2$. Then there exists $c > 0$ such that for $\theta < \frac{1}{2} - \frac{1}{s}$, all $\Lambda \geq \lambda_1$ sufficiently large, all $R > 0$, and all $\varepsilon > 0$ sufficiently small,

$$\mu_0^{m,\varepsilon}\left(u \in \mathcal{H}^\theta : \|P_{\leq \Lambda}u\|_{\mathcal{W}^{\beta,p}} > R\right) \leq C\Lambda e^{-cR^2}. \quad (6.16)$$

In particular, we have

$$\mu_0^m\left(u \in \mathcal{H}^\theta : \|P_{\leq \Lambda}u\|_{\mathcal{W}^{\beta,p}} > R\right) \leq C\Lambda e^{-cR^2}. \quad (6.17)$$

Proof. We only need to prove (6.16). We estimate

$$\begin{aligned} \mu_0^{m,\varepsilon}(\|P_{\leq\Lambda}u\|_{\mathcal{W}^{\beta,p}} > R) &\leq e^{-tR^2} \int e^{t\|P_{\leq\Lambda}u\|_{\mathcal{W}^{\beta,p}}^2} d\mu_0^{m,\varepsilon}(u) \\ &= e^{-tR^2} \sum_{k \geq 0} \frac{t^k}{k!} \int \|P_{\leq\Lambda}u\|_{\mathcal{W}^{\beta,p}}^{2k} d\mu_0^{m,\varepsilon}(u). \end{aligned}$$

We have

$$\begin{aligned} &\int \|P_{\leq\Lambda}u\|_{\mathcal{W}^{\beta,p}}^{2k} d\mu_0^{m,\varepsilon}(u) \\ &= \frac{1}{Z_0^{m,\varepsilon}} \int \|P_{\leq\Lambda}u\|_{\mathcal{W}^{\beta,p}}^{2k} \mathbf{1}_{\{m-\varepsilon < \mathcal{M}(u) < m+\varepsilon\}} d\mu_0(u) \\ &= \frac{1}{Z_0^{m,\varepsilon}} \int \|P_{\leq\Lambda}u\|_{\mathcal{W}^{\beta,p}}^{2k} \left(\int \mathbf{1}_{\{m-\varepsilon-\vartheta_\Lambda < \mathcal{M}_{>\Lambda}(u) < m+\varepsilon-\vartheta_\Lambda\}} d\mu_0^{>\Lambda}(u) \right) d\mu_0^{\leq\Lambda}(u) \\ &= \frac{1}{Z_0^{m,\varepsilon}} \int \|P_{\leq\Lambda}u\|_{\mathcal{W}^{\beta,p}}^{2k} \left(\int_{m-\varepsilon-\vartheta_\Lambda}^{m+\varepsilon-\vartheta_\Lambda} f_\Lambda(x) dx \right) d\mu_0^{\leq\Lambda}(u), \end{aligned}$$

with f_Λ the density function of $\mathcal{M}_{>\Lambda}(u)$ with respect to $\mu_0^{>\Lambda}$ and $\vartheta_\Lambda = \mathcal{M}_{\leq\Lambda}(u)$. From Lemmas 6.3 and 6.4, we infer that for $\varepsilon > 0$ sufficiently small,

$$\frac{1}{Z_0^{m,\varepsilon}} \int_{m-\varepsilon-\vartheta_\Lambda}^{m+\varepsilon-\vartheta_\Lambda} f_\Lambda(x) dx \leq \frac{2\|f_\Lambda\|_{L^\infty}}{f_0(m)} \leq C\Lambda.$$

We deduce that for $\varepsilon > 0$ small,

$$\int \|P_{\leq\Lambda}u\|_{\mathcal{W}^{\beta,p}}^{2k} d\mu_0^{m,\varepsilon}(u) \leq C\Lambda \int \|P_{\leq\Lambda}u\|_{\mathcal{W}^{\beta,q}}^{2k} d\mu_0^{\leq\Lambda}(u).$$

The integral in the right hand side is estimated exactly as in Lemma 3.4 to get

$$\int \|P_{\leq\Lambda}u\|_{\mathcal{W}^{\beta,q}}^{2k} d\mu_0^{\leq\Lambda}(u) \leq \left(\frac{p}{2}\right)! k! B_{\beta,p}^k, \quad \forall k \geq 0.$$

In particular, we obtain

$$\mu_0^{m,\varepsilon}(\|P_{\leq\Lambda}u\|_{\mathcal{W}^{\beta,p}} > R) \leq C\Lambda e^{-tR^2} \sum_{k \geq 0} (tB_{\beta,p})^k \leq C\Lambda e^{-cR^2}$$

provided that $t > 0$ is taken small enough. \square

Remark 6.1. From the argument presented in the proofs of Lemmas 6.9 and 6.10 (see also the proof of Lemma 3.2), we can show that: for $s > 1$, $\theta < \frac{1}{2} - \frac{1}{s}$, and $m > 0$,

$$\mu_0^m(u \in \mathcal{H}^\theta : \|u\|_{\mathcal{H}^\theta} > \lambda) \leq C e^{-c\lambda^2} \quad (6.18)$$

for some constants $C, c > 0$.

Proof of Proposition 6.7, the (sub)-harmonic case $s \leq 2$. We only consider the harder case of the focusing nonlinearity. It is enough to prove that

$$Z^m = \int e^{\frac{1}{2}\|u\|_{L^4}^4} d\mu_0^m(u) < \infty.$$

We have

$$Z^m = \int_0^\infty \mu_0^m(e^{\frac{1}{2}\|u\|_{L^4}^4} > \lambda) d\lambda = C(\lambda_0) + \int_{\lambda_0}^\infty \mu_0^m(e^{\frac{1}{2}\|u\|_{L^4}^4} > \lambda) d\lambda$$

for some $\lambda_0 > 0$ to be fixed later. On the support of μ_0^m , we have $\mathcal{M}(u) = m$. Therefore, thanks to (6.12), (6.14), and (6.17), we can repeat the same argument as in the proof of Item 3 of Proposition 2.2 to conclude $Z^m < \infty$. The only different point is the estimation of the term (II₂) where there is an additional factor $C\Lambda_0$. However, this term is just

$C(\log \lambda)^l$ (see (3.21)) and it can be absorbed by the exponential decay (see (3.26)). The constant λ_0 is chosen so that Λ_0 is sufficiently large as needed to apply Lemmas 6.8, 6.9, and 6.10. \square

6.3. Invariance of Gibbs measures conditioned on mass. Thanks to the invariance of the standard Gibbs measure μ (see Theorem 2.3), we can deduce the invariance of the Gibbs measures conditioned on mass.

Proposition 6.11 (Invariance of canonical Gibbs measures).

Let $s > 1$, V satisfy Assumption 1.1, $m \in \mathbb{R}$, and assume $m > -\text{Tr}[h^{-1}]$ if $s > 2$. Assume in addition that $s > \frac{8}{5}$ for the focusing nonlinearity. Then μ^m is invariant under the flow of (1.1).

The proof is based on the following lemma.

Lemma 6.12 (Approximate canonical measures).

Let $s > 1$, V satisfy Assumption 1.1, $m \in \mathbb{R}$, and assume $m > -\text{Tr}[h^{-1}]$ if $s > 2$. Assume in addition that $s > \frac{8}{5}$ for the focusing nonlinearity. Denote

$$d\mu^{m,\varepsilon}(u) = \frac{1}{Z^{m,\varepsilon}} e^{\mp \frac{1}{2}\|u\|_{L^4}^4} d\mu_0^{m,\varepsilon}(u), \quad Z^{m,\varepsilon} = \int e^{\mp \frac{1}{2}\|u\|_{L^4}^4} d\mu_0^{m,\varepsilon}(u).$$

Then $\mu^{m,\varepsilon}$ converges weakly to μ^m as $\varepsilon \rightarrow 0^+$ in the sense that

$$\lim_{\varepsilon \rightarrow 0^+} \int F(u) d\mu^{m,\varepsilon}(u) = \int F(u) d\mu^m(u)$$

for all bounded continuous functions $F : \mathcal{H}^\theta \rightarrow \mathbb{R}$.

Proof. We first show that $Z^{m,\varepsilon} \rightarrow Z^m$ as $\varepsilon \rightarrow 0^+$. To simplify the notation, we denote

$$G(u) = e^{\mp \frac{1}{2}\|u\|_{L^4}^4}$$

and

$$w_\Lambda := P_{\leq \Lambda} u.$$

For $\Lambda \geq \lambda_1$ sufficiently large and $\varepsilon > 0$, we estimate

$$\begin{aligned} |Z^{m,\varepsilon} - Z^m| &\leq \left| \int G(u) - G(w_\Lambda) d\mu_0^{m,\varepsilon}(u) \right| + \left| \int G(w_\Lambda) d\mu_0^{m,\varepsilon}(u) - \int G(w_\Lambda) d\mu_0^m(u) \right| \\ &\quad + \left| \int G(u) - G(w_\Lambda) d\mu_0^m(u) \right| =: \text{(I)} + \text{(II)} + \text{(III)}. \end{aligned}$$

Regarding (II), we write

$$\int G(w_\Lambda) d\mu_0^{m,\varepsilon}(u) = \int_0^\infty \mu_0^{m,\varepsilon}(G(w_\Lambda) > \lambda) d\lambda.$$

For each $\Lambda \geq \lambda_1$, we have $\mu_0^{m,\varepsilon}(G(w_\Lambda) > \lambda) \rightarrow \mu^m(G(w_\Lambda) > \lambda)$ as $\varepsilon \rightarrow 0^+$. In addition, we have for $\varepsilon > 0$ small,

$$\begin{aligned} \mu_0^{m,\varepsilon}(G(w_\Lambda) > \lambda) &\leq \frac{1}{\lambda^2} \int G^2(w_\Lambda) d\mu_0^{m,\varepsilon}(u) \\ &= \frac{1}{\lambda^2} \int e^{\mp \|P_{\leq \Lambda} u\|_{L^4}^4} d\mu_0^{m,\varepsilon}(u) \\ &\leq \frac{C\Lambda}{\lambda^2}, \end{aligned}$$

for Λ sufficiently large. The dominated convergence theorem implies that for $\Lambda \geq \lambda_1$ sufficiently large, (II) $\rightarrow 0$ as $\varepsilon \rightarrow 0^+$.

To estimate (I), we let $\delta > 0$. By continuity, there exists $\nu > 0$ such that if $\|u - w_\Lambda\|_{L^4} < \nu$, then $|G(u) - G(w_\Lambda)| < \frac{1}{2}\delta$. We write

$$(I) \leq \left| \int_{\|u - w_\Lambda\|_{L^4} \geq \nu} G(u) - G(w_\Lambda) d\mu_0^{m,\varepsilon}(u) \right| + \left| \int_{\|u - w_\Lambda\|_{L^4} < \nu} G(u) - G(w_\Lambda) d\mu_0^{m,\varepsilon}(u) \right|.$$

Since $G(u), G(w_\Lambda) \in L^2(d\mu_0^{m,\varepsilon})$ uniformly in Λ for $\varepsilon > 0$ sufficiently small, we have

$$(I) \leq \|G(u) - G(w_\Lambda)\|_{L^2(d\mu_0^{m,\varepsilon})} (\mu_0^{m,\varepsilon}(\|u - w_\Lambda\|_{L^4} \geq \nu))^{1/2} + \frac{\delta}{2}.$$

By Lemma 6.9, we also have for $\varepsilon > 0$ small,

$$\mu_0^{m,\varepsilon}(\|u - w_\Lambda\|_{L^4} \geq \nu) = \mu_0^{m,\varepsilon}(\|P_{>\Lambda} u\|_{L^4} > \nu) \leq C e^{-c\Lambda^\rho \nu^2}$$

for any $0 \leq \rho < \frac{s-1}{2s}$. Fix $0 < \rho < \frac{s-1}{2s}$, we deduce, for Λ large enough, that

$$\|G(u) - G(w_\Lambda)\|_{L^2(d\mu_0^{m,\varepsilon})} (\mu_0^{m,\varepsilon}(\|u - w_\Lambda\|_{L^4} \geq \nu))^{1/2} < \frac{\delta}{2}$$

for all $\varepsilon > 0$ sufficiently small. This shows that (I) $\rightarrow 0$ as $\Lambda \rightarrow \infty$ provided that $\varepsilon > 0$ is taken sufficiently small. A similar argument goes for (III) and we prove $Z^{m,\varepsilon} \rightarrow Z^m$.

Now we write, for F a test function,

$$\begin{aligned} \int F(u) d\mu^{m,\varepsilon}(u) &= \frac{1}{Z^{m,\varepsilon}} \int F(u) G(u) d\mu_0^{m,\varepsilon}(u) \\ &= \left(\frac{1}{Z^{m,\varepsilon}} - \frac{1}{Z^m} \right) \int F(u) G(u) d\mu_0^{m,\varepsilon}(u) + \frac{1}{Z^m} \int F(u) G(u) d\mu_0^m(u). \end{aligned}$$

The first term in the right hand side goes to zero as $\varepsilon \rightarrow 0^+$ because $Z^{m,\varepsilon} \rightarrow Z^m$ as $\varepsilon \rightarrow 0^+$ and

$$\int F(u) G(u) d\mu_0^{m,\varepsilon}(u) \leq \|F\|_{L^\infty} \|G(u)\|_{L^1(d\mu_0^{m,\varepsilon})} \leq C$$

for some constant $C > 0$ independent of $\varepsilon > 0$ small.

For the second term, we estimate

$$\begin{aligned} &\left| \int F(u) G(u) d\mu_0^{m,\varepsilon}(u) - \int F(u) G(u) d\mu_0^m(u) \right| \\ &= \left| \int F(u) (G(u) - G(w_\Lambda)) + G(w_\Lambda) (F(u) - F(w_\Lambda)) d\mu_0^{m,\varepsilon}(u) \right| \\ &\quad + \left| \int F(w_\Lambda) G(w_\Lambda) d\mu_0^{m,\varepsilon}(u) - \int F(w_\Lambda) G(w_\Lambda) d\mu_0^m(u) \right| \\ &\quad + \left| \int F(u) (G(u) - G(w_\Lambda)) + G(w_\Lambda) (F(u) - F(w_\Lambda)) d\mu_0^m(u) \right|. \end{aligned}$$

Arguing as above, we deduce that

$$\int F(u) G(u) d\mu_0^{m,\varepsilon}(u) \rightarrow \int F(u) G(u) d\mu_0^m(u) \text{ as } \varepsilon \rightarrow 0^+.$$

The proof is complete. \square

Proof of Proposition 6.11. Invariance of the approximate canonical measure. We first show that $\mu^{m,\varepsilon}$ is invariant under the flow of (1.1). In fact, we can rewrite $\mu^{m,\varepsilon}$ as

$$d\mu^{m,\varepsilon}(u) = \frac{1}{Z^{m,\varepsilon}} \mathbb{1}_{\{m-\varepsilon < \mathcal{M}(u) < m+\varepsilon\}} d\mu(u).$$

Here μ is the standard Gibbs measure which is invariant under the flow of (1.1). Now let A be a measurable set in \mathcal{H}^θ with $\theta < \frac{1}{2} - \frac{1}{s}$ as in Theorem 2.3. We have

$$\begin{aligned}
\mu^{m,\varepsilon}(A) &= \frac{1}{Z^{m,\varepsilon}} \int_A \mathbb{1}_{\{m-\varepsilon < \mathcal{M}(u) < m+\varepsilon\}} d\mu(u) \\
&= \frac{1}{Z^{m,\varepsilon}} \int \mathbb{1}_{\{m-\varepsilon < \mathcal{M}(u) < m+\varepsilon\} \cap A} d\mu(u) \\
&= \frac{1}{Z^{m,\varepsilon}} \mu(\{m-\varepsilon < \mathcal{M}(u) < m+\varepsilon\} \cap A) \\
&= \frac{1}{Z^{m,\varepsilon}} \mu(\Phi(t)(\{m-\varepsilon < \mathcal{M}(u) < m+\varepsilon\} \cap A)) && \text{(invariance of } \mu) \\
&= \frac{1}{Z^{m,\varepsilon}} \mu(\{m-\varepsilon < \mathcal{M}(u) < m+\varepsilon\} \cap \Phi(t)(A)) && \text{(mass conservation)} \\
&= \frac{1}{Z^{m,\varepsilon}} \int_{\Phi(t)(A)} \mathbb{1}_{\{m-\varepsilon < \mathcal{M}(u) < m+\varepsilon\}} d\mu(u) \\
&= \mu^{m,\varepsilon}(\Phi(t)(A)),
\end{aligned}$$

where $\Phi(t)$ is the solution map of (1.1). This shows that $\mu^{m,\varepsilon}$ is invariant under the flow of (1.1).

Invariance of the canonical Gibbs measure. We now prove the invariance of μ^m under the flow of (1.1). By the same reasoning as in the proof of Theorem 4.2 using (6.18), it suffices to show

$$\mu^m(U) \leq \mu^m(\Phi(t)(U))$$

for any closed set U of \mathcal{H}^θ which is bounded in \mathcal{H}^{θ_1} with $\theta < \theta_1 < \frac{1}{2} - \frac{1}{s}$ and for all $t \in \mathbb{R}$. The problem is further reduced to show

$$\mu^m(U) \leq \mu^m(\Phi(t)(U)), \quad \forall t \in [0, \delta] \quad (6.19)$$

with $\delta > 0$ sufficiently small. By the local theory, for each $\varepsilon > 0$, there exists $0 < c \ll 1$ such that

$$\Phi(t)(U + B_{\gamma,c\varepsilon}) \subset \Phi(t)(U) + B_{\theta,\varepsilon}, \quad (6.20)$$

where $B_{\theta,\varepsilon}$ is the ball in \mathcal{H}^θ centered at zero and of radius ε . Here $\gamma = \theta$ for $s > 2$ (see (4.4)) and $\gamma = \beta$ for $1 < s \leq 2$ (see (5.6)). Since $\mu^{m,\varepsilon} \rightharpoonup \mu^m$ weakly as $\varepsilon \rightarrow 0^+$, we have

$$\begin{aligned}
\mu^m(U) &\leq \mu^m(U + B_{\gamma,c\varepsilon}) \\
&\leq \liminf_{\varepsilon \rightarrow 0^+} \mu^{m,\varepsilon}(U + B_{\gamma,c\varepsilon}) && (\mu^{m,\varepsilon} \rightharpoonup \mu^m \text{ weakly}) \\
&= \liminf_{\varepsilon \rightarrow 0^+} \mu^{m,\varepsilon}(\Phi(t)(U + B_{\gamma,c\varepsilon})) && \text{(invariance of } \mu^{m,\varepsilon}) \\
&\leq \liminf_{\varepsilon \rightarrow 0^+} \mu^{m,\varepsilon}(\Phi(t)(U) + B_{\theta,\varepsilon}) && \text{(due to (6.20))} \\
&\leq \limsup_{\varepsilon \rightarrow 0^+} \mu^{m,\varepsilon}(\Phi(t)(U) + B_{\theta,\varepsilon}) \\
&\leq \limsup_{\varepsilon \rightarrow 0^+} \mu^{m,\varepsilon}\left(\left(\Phi(t)(U) + \overline{B_{\theta,\varepsilon}}\right)\right) \\
&\leq \mu^m(\Phi(t)(U) + \overline{B_{\theta,\varepsilon}}). && (\mu^{m,\varepsilon} \rightharpoonup \mu^m \text{ weakly})
\end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we get (6.19). This proves the invariance of μ^m under the flow of (1.1). \square

Proof of Theorem 2.4. This follows by combining Propositions 6.7 and 6.11. \square

APPENDIX A. BOUNDS ON THE COVARIANCE OPERATOR

Lemma A.1 (Schatten norm of the Green function).

Let $s > 0$ and $p > \frac{1}{2} + \frac{1}{s}$. Then we have

$$\mathrm{Tr}[h^{-p}] = \sum_{j \geq 1} \lambda_j^{-p} < \infty. \quad (\text{A.1})$$

Proof. We follow an argument of [32, Example 3.2]. Let $\lambda_1 > 0$ be the first eigenvalue of h . We have

$$h + \lambda_1 \leq 2h,$$

hence

$$\mathrm{Tr}[h^{-p}] \leq 2^p \mathrm{Tr}[(h + \lambda_1)^{-p}].$$

Thanks to a version of Lieb–Thirring’s inequality [22, Theorem 1], we have

$$\mathrm{Tr}[(h + \lambda_1)^{-p}] \leq \frac{1}{2\pi} \iint_{\mathbb{R} \times \mathbb{R}} \frac{dx d\xi}{(|\xi|^2 + V(x) + \lambda_1)^p}.$$

Using Assumption 1.1 and the layer-cake representation, one finds that

$$\iint_{\mathbb{R} \times \mathbb{R}} \frac{dx d\xi}{(|\xi|^2 + V(x) + \lambda_1)^p} = \int_{\mathbb{R}} (V(x) + \lambda_1)^{1/2-p} dx < +\infty$$

under our stated assumptions. Here is another short proof. We have

$$\begin{aligned} \iint_{\mathbb{R} \times \mathbb{R}} \frac{dx d\xi}{(|\xi|^2 + V(x) + \lambda_1)^p} &= \int_{\mathbb{R}} \int_{|x| \leq 1} \frac{dx d\xi}{(|\xi|^2 + V(x) + \lambda_1)^p} + \int_{\mathbb{R}} \int_{|x| \geq 1} \frac{dx d\xi}{(|\xi|^2 + V(x) + \lambda_1)^p} \\ &\leq \int_{\mathbb{R}} \int_{|x| \leq 1} \frac{dx d\xi}{(|\xi|^2 + \lambda_1)^p} + C \int_{\mathbb{R}} \int_{|x| \geq 1} \frac{dx d\xi}{(|\xi|^2 + |x|^s + 1)^p} \\ &= (\text{I}) + (\text{II}), \end{aligned}$$

where $V(x) \geq 0$ for all $x \in \mathbb{R}$ and $V(x) \geq C|x|^s$ for $|x| \geq 1$. Since $p > \frac{1}{2} + \frac{1}{s}$, it is clear that (I) $< \infty$. To see that (II) $< \infty$, we take

$$\theta = \frac{4}{2+s}$$

so that

$$\frac{1}{2-\theta} = \frac{2}{\theta s} = \frac{1}{2} + \frac{1}{s}.$$

By Young’s inequality, we have

$$\begin{aligned} |\xi|^2 + |x|^s + 1 &\gtrsim (1 + |\xi|^2) + (1 + |x|^s) \\ &\gtrsim (1 + |\xi|)^2 + (1 + |x|)^s \\ &\gtrsim (1 + |\xi|)^{2-\theta} (1 + |x|)^{\frac{\theta s}{2}}. \end{aligned}$$

It follows that

$$\int_{\mathbb{R}} \int_{|x| \geq 1} \frac{dx d\xi}{(|\xi|^2 + |x|^s + 1)^p} \lesssim \left(\int_{\mathbb{R}} \frac{d\xi}{(1 + |\xi|)^{(2-\theta)p}} \right) \left(\int_{\mathbb{R}} \frac{dx}{(1 + |x|)^{\frac{\theta s p}{2}}} \right) < \infty.$$

Note that $(2 - \theta)p = \frac{\theta s p}{2} = \frac{p}{\frac{1}{2} + \frac{1}{s}} > 1$. □

APPENDIX B. BASIC FUNCTIONAL-ANALYTIC ESTIMATES

We collect here known functional inequalities used repeatedly in the paper. We start with the following norm equivalence due to [60].

Lemma B.1 (Norm equivalence).

Let $s > 1$, V satisfy Assumption 1.1, $\beta > 0$, and $1 < p < \infty$. Then we have

$$\|h^{\beta/2}u\|_{L^p} \sim \|\langle D \rangle^\beta u\|_{L^p} + \|\langle x \rangle^{\beta s/2} u\|_{L^p}, \quad (\text{B.1})$$

where $\langle D \rangle := \sqrt{1 - \partial_x^2}$.

Proof. In [60, Lemma 2.4], the above norm equivalence was proved for $s > 2$ using pseudo-differential calculus. However, the same proof applies to $1 < s \leq 2$ as well. \square

Lemma B.2 (Sobolev embedding).

Let $s > 1$ and V satisfy Assumption 1.1.

(i) If $1 < r < \bar{r} < \infty$ and $\beta > 0$ satisfy $\frac{1}{\bar{r}} \geq \frac{1}{r} - \beta$, then $\mathcal{W}^{\beta,r} \subset L^{\bar{r}}(\mathbb{R})$.

(ii) If $\beta > \frac{1}{r}$, then $\mathcal{W}^{\beta,r} \subset L^\infty(\mathbb{R})$.

Proof. Direct consequence of embeddings for standard Sobolev spaces on \mathbb{R} and the norm equivalence (B.1). \square

Lemma B.3 (Fractional product rule).

Let $s > 1$, V satisfy Assumption 1.1, $\beta > 0$, and $1 < p < \infty$. Then

$$\|h^{\beta/2}(fg)\|_{L^2} \leq C\|f\|_{L^{q_1}}\|h^{\beta/2}g\|_{L^{q_2}} + C\|g\|_{L^{r_1}}\|h^{\beta/2}f\|_{L^{r_2}} \quad (\text{B.2})$$

provided that

$$\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2}, \quad q_2, r_2 \in (1, \infty), \quad q_1, r_1 \in (1, \infty].$$

Proof. Direct consequence of (B.1) and the following product rule (see e.g., [53, Proposition 1.1 of Chapter 2]):

$$\|\langle D \rangle^\beta (fg)\|_{L^2} \leq C\|f\|_{L^{q_1}}\|\langle D \rangle^\beta g\|_{L^{q_2}} + C\|g\|_{L^{r_1}}\|\langle D \rangle^\beta f\|_{L^{r_2}}.$$

\square

We next recall some Strichartz estimates whose proofs can be found in [27, 15] for the case $s \leq 2$ and in [61] for $s > 2$.

Definition B.4 (Admissible pairs).

A pair (p, q) is called admissible if

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2}, \quad q \in [2, \infty].$$

Proposition B.5 (Strichartz estimates).

(i) [27, 15] Let $1 < s \leq 2$ and (p, q) be an admissible pair. Then there exists $C > 0$ such that

$$\|e^{-ith}f\|_{L^p((-1,1),L^q(\mathbb{R}))} \leq C\|f\|_{L^2(\mathbb{R})}. \quad (\text{B.3})$$

Moreover, for any admissible pairs (p, q) and (a, b) , there exists $C > 0$ such that

$$\left\| \int_0^t e^{-i(t-\tau)h} F(\tau) d\tau \right\|_{L^p((-1,1),L^q(\mathbb{R}))} \leq C\|F\|_{L^{a'}((-1,1),L^{b'}(\mathbb{R}))}. \quad (\text{B.4})$$

(ii) Let $s > 2$, (p, q) be an admissible pair, and $\sigma > \frac{2}{p} \left(\frac{1}{2} - \frac{1}{s} \right)$. Then there exists $C > 0$ such that

$$\|e^{-ith}f\|_{L^p((-1,1),L^q(\mathbb{R}))} \leq C\|f\|_{\mathcal{H}^\sigma}. \quad (\text{B.5})$$

In particular, we have

$$\left\| \int_0^t e^{-i(t-\tau)h} F(\tau) d\tau \right\|_{L^p((-1,1), L^q(\mathbb{R}))} \leq C \|F\|_{L^1((-1,1), \mathcal{H}^\sigma)}. \quad (\text{B.6})$$

For potentials that grow at most quadratically at infinity, i.e., $|\partial^\alpha V(x)| \leq C_\alpha$ for $|\alpha| \geq 2$, it is known (see [27]) that the following dispersive estimate holds

$$\|e^{-ith} f\|_{L^\infty} \leq C |t|^{-1/2} \|f\|_{L^1}, \quad \forall |t| \leq \delta \quad (\text{B.7})$$

with some small constant $\delta > 0$. Using this and the unitary property, we obtain Strichartz estimates (see e.g., [29]) on the time interval $[-\delta, \delta]$. After dividing $(-1, 1)$ into intervals of size 2δ and applying Strichartz estimates for these intervals, we can sum over all sub-intervals to get (B.3) and (B.4).

For super-quadratic potentials, there is a loss of derivatives in (B.5). Moreover, dispersive estimates as in (B.7) are no longer available due to the unboundedness and lack of smoothness of the kernel of e^{-ith} (see [59]). Thus the above inhomogeneous Strichartz estimates (B.6) may not hold true for $(a, b) \neq (\infty, 2)$.

APPENDIX C. L^p -BOUNDEDNESS OF THE SMOOTHENED SPECTRAL PROJECTORS

We here discuss the boundedness of the smoothened spectral projector Q_Λ (defined in (3.27)) as an operator on $L^p(\mathbb{R})$. This comes from known facts that we collect in the following lemma.

Lemma C.1 (L^p -boundedness of smooth spectral projections).

Let $1 < p < \infty$. Then there exists $C > 0$ such that for all $\Lambda \geq \lambda_1$,

$$\|Q_\Lambda\|_{L^p \rightarrow L^p} \leq C.$$

Proof. The proof follows from the L^p -boundedness of pseudo-differential operators with symbol in $S(1, g)$ (see e.g., [58, Theorem 22.3]). Here

$$g = \frac{dx^2}{\langle x \rangle^2} + \frac{d\xi^2}{\langle \xi \rangle^2}$$

is a metric and $S(1, g)$ is the Hörmander symbol class consisting of functions $a \in C^\infty(\mathbb{R}^2)$ such that

$$|\partial_x^j \partial_\xi^k a(x, \xi)| \leq C_{jk} \langle x \rangle^{-j} \langle \xi \rangle^{-k}, \quad \forall (x, \xi) \in \mathbb{R}^2.$$

We write the symbol of Q_Λ as

$$p_\Lambda(x, \xi) = \chi(q_\Lambda(x, \xi))$$

with

$$q_\Lambda(x, \xi) = \Lambda^{-1}(\xi^2 + V(x)),$$

and we will show that $p_\Lambda \in S(1, g)$ uniformly in $\Lambda \geq \lambda_1$. To see this, we observe that for $j, k \geq 0$, there exist $C_j, C_k > 0$ independent of Λ such that

$$|\partial_x^j q_\Lambda(x, \xi)| \leq C_j \langle x \rangle^{-j}, \quad |\partial_\xi^k q_\Lambda(x, \xi)| \leq C_k \langle \xi \rangle^{-k} \quad (\text{C.1})$$

for all (x, ξ) in the support of p_Λ . Note that the mixed derivatives

$$\partial_x^j \partial_\xi^k q_\Lambda(x, \xi) = 0$$

for $j, k \neq 0$. In fact, for the x -derivative, it is straightforward when $j = 0$. For $j \geq 1$, we have $\partial_x^j q_\Lambda(x, \xi) = \Lambda^{-1} \partial_x^j V(x)$. By Assumption 1.1, we have

$$|\partial_x^j q_\Lambda(x, \xi)| \leq C_j \Lambda^{-1} \langle x \rangle^{s-j} \leq C_j \langle x \rangle^{-j}.$$

Actually, for $|x| \leq 1$, we have $\langle x \rangle^s \leq 2^{s/2}$ and $\Lambda^{-1} \leq \lambda_1^{-1}$, hence $\Lambda^{-1} \langle x \rangle^s \leq C$. On the other hand, for $|x| \geq 1$, since $V(x) \leq \Lambda$ on the support of p_Λ , the assumption on V gives $\frac{1}{C} \langle x \rangle^s \leq V(x) \leq \Lambda$, hence $\Lambda^{-1} \langle x \rangle^s \leq C$. For the ξ -derivative, it is obvious when $k = 0$

and $k \geq 3$ since $\partial_\xi^k q_\Lambda(x, \xi) = 0$ for $k \geq 3$. For $k = 1, 2$, we have $\partial_\xi^k q_\Lambda(x, \xi) = 2\Lambda^{-1}\xi^{2-k}$. On the support of p_Λ , we have $|\xi| \leq \Lambda^{\frac{1}{2}}$, hence

$$|\partial_\xi^k q_\Lambda(x, \xi)| \leq C\Lambda^{-\frac{k}{2}} \leq C\langle \xi \rangle^{-k}.$$

Here, to obtain the second estimate, we have used the fact that

$$\frac{\lambda_1}{1 + \lambda_1} \Lambda^{-1} \leq (1 + \Lambda)^{-1} \leq \langle \xi \rangle^{-2}$$

as $\Lambda \geq \lambda_1$ and $|\xi|^2 \leq \Lambda$.

The result now follows from (C.1) and the Faà di Bruno formula saying that $\partial_x^j g(f(x))$ is a linear combination of

$$g^{(n)}(f(x)) (f'(x))^{n_1} (f''(x))^{n_2} \cdots (f^{(n)}(x))^{n_j},$$

where $n = n_1 + \cdots + n_j$ and the sum is over all partitions of j , i.e., all j -tuples (n_1, \cdots, n_j) satisfying $n_1 + 2n_2 + \cdots + jn_j = j$. \square

APPENDIX D. AN ESTIMATE ON THE NUMBER OF EIGENVALUES

We are interested in the number of eigenvalues of the Schrödinger operator $h = -\partial_x^2 + V(x)$ below a certain energy threshold which is needed in the construction of Gaussian measure conditioned on mass (see Lemma 6.3).

Lemma D.1 (Cwikel-Lieb-Rozenbljum law). *Let $s > 1$ and V satisfy Assumption 1.1. Then, for $\Lambda > 0$ sufficiently large, we have that*

$$c\Lambda^{\frac{1}{2} + \frac{1}{s}} \leq \#\{\lambda_j : \lambda_j \leq \Lambda\} \leq C\Lambda^{\frac{1}{2} + \frac{1}{s}}$$

for fixed constants $c, C > 0$.

Proof. To see this, we recall the following result due to Rozenbljum [46]. Let $V(x) \geq 1$ and $V(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$. Assume that the following conditions hold:

(V1) $\sigma(2\Lambda) \leq C\sigma(\Lambda)$, where $\sigma(\Lambda) := |\{x \in \mathbb{R} : V(x) < \Lambda\}|$.

(V2) $V(x) \leq CV(y)$ almost everywhere when $|x - y| < 1$.

(V3) There exist a continuous function $\eta(t) \geq 0$, $0 \leq t < 1$, $\eta(0) = 0$, and an index $\beta \in [0, 1/2)$ such that

$$\int_{\substack{|x-y| \leq 1 \\ |x+z-y| \leq 1}} |V(x+z) - V(x)| dx < \eta(|z|) |z|^{2\beta} (V(y))^{1+\beta}$$

for any $y, z \in \mathbb{R}$ and $|z| < 1$. Then, for Λ sufficiently large, we have that

$$\#\{\lambda_j : \lambda_j \leq \Lambda\} \propto \int_{\mathbb{R}} (\Lambda - V(x))_+^{1/2} dx, \quad (\text{D.1})$$

where $(f(x))_+ := \max\{f(x), 0\}$.

We will check that the above conditions are fulfilled for V as in Assumption 1.1. First, we observe that by using the change of variable $u(t, x) \mapsto e^{-iat}u(t, x)$ for (1.1) with some constant $a > 0$, we can assume (in addition to Assumption 1.1) that $V(x) \geq 1$ for all $x \in \mathbb{R}$. From this, we infer that there exists $C \geq 1$ such that

$$\frac{1}{C} \langle x \rangle^s \leq V(x) \leq C \langle x \rangle^s, \quad \forall x \in \mathbb{R}. \quad (\text{D.2})$$

Let us now check the conditions (V1)-(V3).

- Checking (V1): From (D.2), $V(x) < \Lambda$ implies $|x| < \langle x \rangle < (C\Lambda)^{1/s}$, hence

$$\sigma(\Lambda) \leq |\{x \in \mathbb{R} : |x| < (C\Lambda)^{1/s}\}|. \quad (\text{D.3})$$

On the other hand, for $|x| < \left(\frac{1}{C2^{s/2}}\Lambda\right)^{1/s}$, we have for Λ large,

$$\langle x \rangle = \sqrt{1 + |x|^2} < \sqrt{1 + \left(\frac{1}{C2^{s/2}}\Lambda\right)^{2/s}} \leq \left(\frac{1}{C2^{s/2}}\Lambda\right)^{1/s} 2^{1/2}$$

hence, by (D.2),

$$V(x) \leq C\langle x \rangle^s < C\left(\frac{1}{C}\Lambda\right) = \Lambda.$$

This shows that

$$\left| \left\{ x \in \mathbb{R} : |x| < \left(\frac{1}{C2^{s/2}}\Lambda\right)^{1/s} \right\} \right| \leq \sigma(\Lambda). \quad (\text{D.4})$$

From (D.3) and (D.4), we get (V1).

- Checking (V2): We have

$$V(x) \sim \langle x \rangle^s \leq \langle y \rangle^s \langle x - y \rangle^s \leq C\langle y \rangle^s \sim CV(y)$$

for all $|x - y| \leq 1$.

- Checking (V3): By Taylor's formula, we have

$$|V(x+z) - V(x)| \leq |z| \int_0^1 |V'(x+tz)| dt,$$

which, by Assumption 1.1, yields

$$|V(x+z) - V(x)| \leq C|z| \int_0^1 \langle x+tz \rangle^{s-1} dt.$$

As $s > 1$, we infer that

$$|V(x+z) - V(x)| \leq C|z|\langle x \rangle^{s-1}\langle z \rangle^{s-1}.$$

Now denote

$$\Omega := \{x \in \mathbb{R} : |x - y| \leq 1, |x + z - y| \leq 1\}.$$

For $x \in \Omega$, we have $|x| \leq 1 + |y| < 2\langle y \rangle$. In particular,

$$|V(x+z) - V(x)| \leq C|z|\langle y \rangle^{s-1}\langle z \rangle^{s-1}, \quad \forall x \in \Omega$$

and

$$|\Omega| \leq |\{x \in \mathbb{R} : |x| < 2\langle y \rangle\}| \sim \langle y \rangle.$$

Therefore, we obtain

$$\int_{\substack{|x-y| \leq 1 \\ |x+z-y| \leq 1}} |V(x+z) - V(x)| dx < C|z|\langle y \rangle^s \langle z \rangle^{s-1} \leq C|z|V(y)$$

for all $y, z \in \mathbb{R}^d$ with $|z| < 1$. This is (V3) with $\eta(t) = t$ and $\beta = 0$.

We have so far proved that (D.1) holds for V as in Assumption 1.1. The claim of the lemma follows, as we now explain. For an upper bound, we use (D.3) to get

$$\int_{\mathbb{R}} (\Lambda - V(x))_+^{1/2} dx = \int_{V(x) \leq \Lambda} (\Lambda - V(x))^{1/2} dx \leq (2\Lambda)^{1/2} \sigma(\Lambda) \leq C\Lambda^{1/2+1/s}.$$

For a lower bound, we estimate for a constant $\nu > 0$ small to be chosen later,

$$\int_{\mathbb{R}} (\Lambda - V(x))_+^{1/2} dx \geq \int_{\nu\Lambda < V(x) \leq \Lambda/2} (\Lambda - V(x))^{1/2} dx \geq \left(\frac{\Lambda}{2}\right)^{1/2} (\sigma(\Lambda/2) - \sigma(\nu\Lambda)).$$

Thanks to (D.3) and (D.4), we deduce

$$\begin{aligned}
\int_{\mathbb{R}} (\Lambda - V(x))_+^{1/2} dx &\geq \left(\frac{\Lambda}{2}\right)^{1/2} (\sigma(\Lambda/2) - \sigma(\nu\Lambda)) \\
&\geq \left(\frac{\Lambda}{2}\right)^{1/2} \left(\left| B\left(0, \left(\frac{1}{C2^{s/2}\frac{\Lambda}{2}}\right)^{1/s}\right) \right| - |B(0, C\nu\Lambda)^{1/s}| \right) \\
&\geq C\Lambda^{1/2+1/s}
\end{aligned}$$

provided that $\nu > 0$ is taken sufficiently small, where $B(0, R)$ is the segment $[-R, R]$. \square

REFERENCES

- [1] BOURGAIN, J. Periodic nonlinear Schrödinger equation and invariant measures. *Comm. Math. Phys.* 166, 1 (1994), 1–26. [3](#), [4](#)
- [2] BOURGAIN, J. Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. *Comm. Math. Phys.* 176 (1996), 421–445. [3](#), [4](#), [22](#)
- [3] BOURGAIN, J. Invariant measures for the Gross-Pitaevskii equation. *J. Math. Pures Appl.* 76, 8 (1997), 649–702. [3](#), [22](#)
- [4] BOURGAIN, J., AND BULUT, A. Almost sure global well posedness for the radial nonlinear Schrödinger equation on the unit ball I: the 2D case. *Annales I. H. Poincaré (C)* 31, 6 (2014), 1267–1288. [3](#), [4](#)
- [5] BOURGAIN, J., AND BULUT, A. Almost sure global well posedness for the radial nonlinear Schrödinger equation on the unit ball II: the 3D case. *J. Eur. Math. Soc.* 16 (2014), 1289–1325. [3](#)
- [6] BRERETON, J. T. Invariant measure construction at a fixed mass. *Nonlinearity* 32 (2019), 496–558. [3](#), [7](#), [8](#)
- [7] BRÉZIS, H., AND MIRONESCU, P. Where Sobolev interacts with Gagliardo-Nirenberg. *J. Funct. Anal.* 277 (2019), 2839–2864. [7](#), [18](#)
- [8] BURQ, N., THOMANN, L., AND TZVETKOV, N. Long time dynamics for the one dimensional non linear Schrödinger equation. *Ann. Inst. Fourier (Grenoble)* 63, 6 (2013), 2137–2198. [3](#), [4](#), [5](#), [6](#), [7](#), [20](#), [22](#), [32](#)
- [9] BURQ, N., AND TZVETKOV, N. Invariant measure for a three dimensional nonlinear wave equation. *Int. Math. Res. Not.* 2007 (2007), art. ID rnm108. [4](#)
- [10] BURQ, N., AND TZVETKOV, N. Random data Cauchy theory for supercritical wave equations. II. A global existence result. *Invent. Math.* 173, 3 (2008), 477–496. [4](#)
- [11] CACCIAFESTA, F., AND DE SUZZONI, A.-S. Invariant measure for the Schrödinger equation on the real line. *J. Funct. Anal.* 269, 1 (2015), 271–324. [4](#)
- [12] CARLEN, E. Trace inequalities and quantum entropy: An introductory course. In *Entropy and the Quantum* (2010), R. Sims and D. Ueltschi, Eds., vol. 529 of *Contemporary Mathematics*, American Mathematical Society, pp. 73–140. Arizona School of Analysis with Applications, March 16-20, 2009, University of Arizona. [10](#)
- [13] CARLEN, E., FRÖHLICH, J., AND LEBOWITZ, J. Exponential relaxation to equilibrium for a one-dimensional focusing non-linear Schrödinger equation with noise. *Comm. Math. Phys.* 342, 1 (2016), 303–332. [7](#)
- [14] CARLEN, E., FRÖHLICH, J., LEBOWITZ, J., AND WANG, W.-M. Quantitative bounds on the rate of approach to equilibrium for some one-dimensional stochastic nonlinear schrödinger equations. *Nonlinearity* 32, 4 (2019), 1352–1374. [7](#)
- [15] CARLES, R. Linear vs nonlinear effects for nonlinear Schrödinger equations with potential. *Commun. Contemp. Math.* 7 (2005), 483–508. [60](#)
- [16] DE BOUARD, A., DEBUSSCHE, A., AND FUKUIZUMI, R. Long time behavior of Gross-Pitaevskii equation at positive temperature. *SIAM J. Math. Anal.* 50, 6 (2018). [3](#)
- [17] DE SUZZONI, A.-S. Invariant measure for the cubic wave equation on the unit ball of \mathbb{R}^3 . *Dyn. Partial Differ. Equ.* 8, 2 (2011), 127–147. [4](#)
- [18] DENG, Y. Two-dimensional nonlinear Schrödinger equation with random radial data. *Anal. PDE* 5, 5 (2012), 913–960. [3](#), [4](#)
- [19] DENG, Y., NAHMOD, A. R., AND YUE, H. Invariant Gibbs measures and global strong solutions for nonlinear Schrödinger equations in dimension two. arXiv:1910.08492, 2019. [3](#)
- [20] DENG, Y., NAHMOD, A. R., AND YUE, H. Invariant Gibbs measure and global strong solutions for the Hartree NLS equation in dimension three. *J. Math. Phys.* 62 (2021), 031514. [3](#)
- [21] DENG, Y., NAHMOD, A. R., AND YUE, H. Random tensors, propagation of randomness, and nonlinear dispersive equations. *Invent. Math.* 228, 2 (2022), 539–686. [3](#)
- [22] DOLBEAULT, J., FELMER, P., LOSS, M., AND PATUREL, E. Lieb-Thirring type inequalities and Gagliardo-Nirenberg inequalities for systems. *J. Funct. Anal.* 238, 1 (2006), 193–220. [7](#), [59](#)
- [23] FRÖHLICH, J., KNOWLES, A., SCHLEIN, B., AND SOHINGER, V. Gibbs measures of nonlinear Schrödinger equations as limits of quantum many-body states in dimensions $d \leq 3$. *Comm. Math. Phys.* 356 (2017), 883–980. [2](#), [3](#), [8](#)
- [24] FRÖHLICH, J., KNOWLES, A., SCHLEIN, B., AND SOHINGER, V. A microscopic derivation of time-dependent correlation functions of the 1D cubic nonlinear Schrödinger equation. *Adv. Math.* 353 (2019), 67–115. [2](#), [3](#), [8](#)
- [25] FRÖHLICH, J., KNOWLES, A., SCHLEIN, B., AND SOHINGER, V. The Euclidean φ_2^4 theory as a limit of an interacting bose gas. arXiv:2201.07632, 2022. [3](#)
- [26] FRÖHLICH, J., KNOWLES, A., SCHLEIN, B., AND SOHINGER, V. The mean-field limit of quantum Bose gases at positive temperature. *J. Amer. Math. Soc.* 35 (2022), 995–1030. [3](#)

- [27] FUJIWARA, D. A construction of the fundamental solution for the Schrödinger equation. *J. Analyse Math.* 35 (1979), 41–96. [60](#), [61](#)
- [28] GLIMM, J., AND JAFFE, A. *Quantum Physics: A Functional Integral Point of View*. Springer-Verlag, 1987. [4](#)
- [29] KEEL, M., AND TAO, T. Endpoint Strichartz estimates. *Amer. J. Math.* 120, 5 (1998), 955–980. [61](#)
- [30] KOCH, H., AND TATARU, D. L^p eigenfunction bounds for the Hermite operator. *Duke Math. J.* 128, 2 (2005), 369–392. [7](#)
- [31] LEBOWITZ, J. L., ROSE, H. A., AND SPEER, E. R. Statistical mechanics of the nonlinear Schrödinger equation. *J. Stat. Phys.* 50, 3-4 (1988), 657–687. [2](#), [4](#)
- [32] LEWIN, M., NAM, P., AND ROUGERIE, N. Derivation of nonlinear Gibbs measures from many-body quantum mechanics. *J. Éc. Polytech. Math.* 2, 1 (2016), 553–606. [2](#), [3](#), [5](#), [6](#), [8](#), [10](#), [59](#)
- [33] LEWIN, M., NAM, P., AND ROUGERIE, N. Gibbs measures based on 1D (an)harmonic oscillators as mean-field limits. *J. Math. Phys.* 59, 4 (2018). [2](#), [3](#), [6](#), [10](#)
- [34] LEWIN, M., NAM, P., AND ROUGERIE, N. Classical field theory limit of many-body quantum Gibbs states in 2D and 3D. *Invent. Math.* 224 (2021), 315–444. [3](#), [6](#), [16](#)
- [35] LIEB, E. H., AND LOSS, M. *Analysis*, 2nd ed., vol. 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. [45](#)
- [36] LIEB, E. H., SEIRINGER, R., SOLOVEJ, J. P., AND YNGVASON, J. *The mathematics of the Bose gas and its condensation*. Oberwolfach Seminars. Birkhäuser, 2005. [2](#)
- [37] NAHMUD, A. R., OH, T., REY-BELLET, L., AND STAFFILANI, G. Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS. *J. Eur. Math. Soc.* 14 (2011), 1275–1330. [4](#), [41](#)
- [38] NIRENBERG, L. On elliptic partial differential equations. *Ann. Scu. Norm. Sup. di Pisa* 13, 2 (1959), 115–162. [18](#)
- [39] NIRENBERG, L. An extended interpolation inequality. *Ann. Scu. Norm. Sup. di Pisa* 20, 4 (1966), 733–737. [18](#)
- [40] OH, T. Invariance of the Gibbs measure for the Schrödinger-Benjamin-Ono system. *SIAM J. Math. Anal.* 41 (2009), 2207–2225. [4](#)
- [41] OH, T. Invariant Gibbs measures and a.s. global well-posedness for coupled KdV systems. *Differential and Integral Equations* 22 (2009), 637–668. [4](#)
- [42] OH, T., AND QUASTEL, J. On invariant Gibbs measures conditioned on mass and momentum. *J. Math. Soc. Japan* 65, 1 (2013), 13–35. [3](#), [7](#), [8](#), [42](#)
- [43] POIRET, A., ROBERT, D., AND THOMANN, L. Probabilistic global well-posedness for the supercritical nonlinear harmonic oscillator. *Anal. PDE* 7, 4 (2014), 997–1026. [4](#)
- [44] ROUGERIE, N. Scaling limits of bosonic ground states, from many-body to nonlinear Schrödinger. *EMS Surveys in Mathematical Sciences* 7, 2 (2020), 253–408. [2](#)
- [45] ROUT, A., AND SOHINGER, V. A microscopic derivation of Gibbs measures for the 1D focusing cubic nonlinear Schrödinger equation. arXiv:2206.03392, 2022. [2](#), [3](#)
- [46] ROZENBLJUM, G. V. Asymptotics of the eigenvalues of the Schrödinger operator. *Math. USSR Sbornik* 22 (1974), 349–371. [62](#)
- [47] SCHATTEN, R. *Norm Ideals of Completely Continuous Operators*, vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Folge, 1960. [10](#)
- [48] SCHLEIN, B. Derivation of effective evolution equations from microscopic quantum dynamics. *arXiv eprints* (2008). Lecture Notes for a course at ETH Zurich. [2](#)
- [49] SIMON, B. *The $P(\Phi)_2$ Euclidean (quantum) field theory*. Princeton University Press, Princeton, N.J., 1974. Princeton Series in Physics. [4](#)
- [50] SIMON, B. *Trace ideals and their applications*, vol. 35 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1979. [7](#), [10](#), [11](#)
- [51] SKOROKHOD, A. *Integration in Hilbert space*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1974. [5](#), [48](#)
- [52] SOHINGER, V. A microscopic derivation of Gibbs measures for nonlinear Schrödinger equations with unbounded interaction potentials. *Int. Math. Res. Not.* 180 (2022), 14964–15063. [3](#)
- [53] TAYLOR, M. E. *Tools for PDEs: Pseudodifferential operators, Paradifferential operators, and Layer potentials*, vol. 80 of *Mathematical Surveys and Monographs*. AMS, 2000. [60](#)
- [54] THOMANN, L. Random data Cauchy problem for supercritical Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26, 6 (2009), 2385–2402. [4](#)
- [55] THOMANN, L., AND TZVETKOV, N. Gibbs measure for the periodic derivative nonlinear Schrödinger equation. *Nonlinearity* 23, 11 (2010), 2771. [4](#)
- [56] TZVETKOV, N. Invariant measures for the nonlinear Schrödinger equation on the disc. *Dyn. Partial Differ. Equ.* 3, 2 (2006), 111–160. [3](#), [4](#)

- [57] TZVETKOV, N. Invariant measures for the defocusing nonlinear Schrödinger equation. *Ann. Inst. Fourier (Grenoble)* 58, 7 (2008), 2543–2604. [3](#), [4](#)
- [58] WONG, M. W. *An Introduction to Pseudo-differential Operators, 3rd edition*, vol. 6 of *Series on Analysis, Applications and Computation*. World Scientific, York University, Canada, 2014. [61](#)
- [59] YAJIMA, K. Smoothness and non-smoothness of the fundamental solution of time dependent Schrödinger equations. *Comm. Math. Phys.* 181 (1996), 605–629. [61](#)
- [60] YAJIMA, K., AND ZHANG, G. Smoothing property for Schrödinger equations with potential superquadratic at infinity. *Comm. Math. Phys.* 221, 3 (2001), 537–590. [2](#), [24](#), [60](#)
- [61] YAJIMA, K., AND ZHANG, G. Local smoothing property and Strichartz inequality for Schrödinger equations with potential superquadratic at infinity. *J. Differential Equations* 202 (2004), 81–110. [2](#), [7](#), [24](#), [60](#)
- [62] ZHANG, G., YAJIMA, K., AND LIU, F. H^s solutions for nonlinear Schrödinger equations with potentials superquadratic at infinity. *Phys. Lett. A* 356, 2 (2006), 95–98. [2](#), [24](#)

(V. D. Dinh) ECOLE NORMALE SUPÉRIEURE DE LYON & CNRS, UMPA (UMR 5669), LYON, FRANCE
Email address: contact@duondinh.com

(N. Rougerie) ECOLE NORMALE SUPÉRIEURE DE LYON & CNRS, UMPA (UMR 5669), LYON, FRANCE
Email address: nicolas.rougerie@ens-lyon.fr