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# A comparison principle for Hamilton-Jacobi equation with moving in time boundary

N. Forcadel<sup>1</sup>, M. Zaydan<sup>1</sup>

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## Abstract

In this paper we consider an Hamilton-Jacobi equation on a moving in time domain. The boundary is described by a  $C^1$  function. We show how we derive this equation from the work of [26]. We only prove a comparison principle since the proof of other theoretical results can be found in [20]. At the end of the paper, we consider a short homogenization result in order to reinforce the traffic flow interpretation of the equation.

## 1 Introduction

In this paper, we consider an Hamilton-Jacobi equation posed on a moving in time domain. More precisely, the equation is posed in several interval of the real axis whose boundary (called "*junction points*") move in time. The junction points are denoted by  $b_i(t) \in \mathbb{R}$  at time  $t$  and we set for  $i \in \{1, \dots, N+1\}$ ,

$$B_i = \{(t, x) \in (0, T) \times \mathbb{R}, \text{ s.t. } b_{i-1}(t) < x < b_i(t)\}.$$

We will show in Section 2 that the considered equation can be obtained by a first order bus-vehicles interaction model, introduced in [26], where authors assumed that buses represent a moving capacity restriction, i.e. the density of vehicles is reduced near the buses zones. In order to simplify the notations, let us first introduce the *flux limiting function*, (see [20]). For  $i \in \{1, \dots, N\}$ ,  $t \in \mathbb{R}^+$  and  $p = (p_1, p_2) \in \mathbb{R}^2$

$$F_{A_i}(t, p_1, p_2) = \max(A_i(t), H_{i,i}^+(t, p_1), H_{i+1,i}^-(t, p_2))$$

where  $A_i$  is a locally Lipschitz function and  $H_{i,i}^+$  (resp.  $H_{i+1,i}^-$ ) is the nondecreasing (resp. nonincreasing) part of the Hamiltonian  $H_{i,i}$  (resp.  $H_{i+1,i}$ ) whose definition is given later. The equation is given by

$$\begin{cases} u_t + H_i(u_x) = 0 & \text{if } (t, x) \in B_i, i = 1, \dots, N+1 \\ \frac{d}{dt}(u(t, b_i(t))) + F_{A_i}(t, u_x^{i,-}(t, x), u_x^{i,+}(t, x)) = 0 & \text{if } x = b_i(t), i = 1, \dots, N \\ u(0, x) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where  $u_t = \frac{\partial u}{\partial t}$  and  $u_x = \frac{\partial u}{\partial x}$  denotes respectively the time and the space derivative. Moreover, we denote by

$$\begin{aligned} u_x^{i,+}(t, b_i(t)) &= \lim_{\substack{(t,x) \rightarrow (t,b_i(t)) \\ x > b_i(t)}} u_x(t, x) \\ u_x^{i,-}(t, b_i(t)) &= \lim_{\substack{(t,x) \rightarrow (t,b_i(t)) \\ x < b_i(t)}} u_x(t, x). \end{aligned}$$

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<sup>1</sup>Normandie Univ, INSA de Rouen, LMI (EA 3226 - FR CNRS 3335), 76000 Rouen, France, 685 Avenue de l'Université, 76801 St Etienne du Rouvray cedex. France

Equation (1.1) is quite similar to the one introduced by the Imbert and Monneau in [20]. The difference here is that we consider a junction which moves in time. Stability, existence of solution and even the reduction of the class of test functions for (1.1) can be easily obtained adapting the proofs of these results in [20]. In this paper, we prove a comparison principle for equation (1.1). We borrow the idea introduced in [4] and we use a localization procedure in order to insert the "good" test function in the next step of the proof. Let us now clarify the notations used in (1.1).

**Assumptions and Notations (A).**

- (A1) The functions  $b_1, \dots, b_N$  are time dependent derivable functions such that  $b_{i+1} > b_i$ . We denote also by  $b_0 = -\infty$  and  $b_{N+1} = +\infty$ . Moreover, we assume that for all  $j \in \{1, \dots, N\}$ ,  $b'_j$  is a locally Lipschitz function.
- (A2) The Hamiltonians  $H_1, \dots, H_{N+1} : \mathbb{R} \rightarrow \mathbb{R}$  satisfy the following assumptions: for all  $i \in \{1, \dots, N + 1\}$ ,

$$\left\{ \begin{array}{l} H_i \text{ is continuous,} \\ H_i \text{ is superlinear i.e. } \lim_{|p| \rightarrow +\infty} \frac{H_i(p)}{|p|} = +\infty. \end{array} \right.$$

- (A3) For  $i \in \{1, \dots, N + 1\}$  and for  $k = i, i + 1$ ,  $H_{k,i}(t, p) = H_k(p) - b'_i(t)p$ . Moreover, we assume that for all  $i \in \{1, \dots, N\}$ ,  $k = i, i + 1$  and for all  $t \in \mathbb{R}^+$ , the Hamiltonian  $H_{k,i}(t, \cdot)$  is quasi-convex. We denote by  $H_{k,i}^+(t, \cdot)$  and  $H_{k,i}^-(t, \cdot)$  respectively the non-decreasing and the non-increasing part of  $H_{k,i}(t, \cdot)$ .
- (A4) For all  $i \in \{1, \dots, N\}$ , the flux limiter  $A_i : [0, T] \rightarrow \mathbb{R}$  is a locally Lipschitz function.

**Main results.** Our main result is the proof of a comparison principle for equation (1.1). In [20, 4, 21], a proof of comparison principle for (1.1) in the case where  $b_i = \text{constant}$  is done. In fact, they prove this result in a more general domain (such a network, junction or two half spaces in  $\mathbb{R}^N$ ) and more general Hamiltonians (depending on  $x$  and  $t$ ). In [20], they prove a comparison principle by replacing the classical penalization term  $\frac{(x - y)^2}{2\varepsilon}$  by the new term  $\varepsilon G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$  where  $G$  is a vertex test function which allows to compare the Hamiltonians in different branches of the domain. As we mentioned above, our proof uses the idea introduced in [4] which keep the classical term  $\frac{(x - y)^2}{2\varepsilon}$  and uses the fact that  $H^+$  and  $H^-$  are respectively increasing and decreasing functions. Let us mention also the work [28] where the authors consider a Kirchoff-type Neumann condition at the junction and proved that its solution satisfy a comparison principle and then they proved that the flux-limited solutions reduce to Kirchoff-type viscosity solutions. Finally, concerning comparison principle for Hamilton-Jacobi equations with boundary conditions of Neumann type, let us cite [27, 1, 5, 19, 2, 13, 23]. Combaining the comparison principle for (1.1) with Perron method, we obtain the following main result

**Theorem 1.1.** *Assume (A) and that the initial datum  $u_0$  is Lipschitz continuous function. Then there exists a unique continuous viscosity solution  $u$  of (1.1) such that for all  $T > 0$ , there exists a constant  $C_T > 0$  such that for all  $(t, x) \in [0, T] \times \mathbb{R}$ ,*

$$|u(t, x) - u_0(x)| \leq C_T.$$

The second main result of this paper is an homogenization result. We consider a macroscopic model describing the presence of a bus (or a large truck) and prove that the solution of the Hamilton-Jacobi formulation of this model converges towards the unique solution of equation (1.1) with one Hamiltonian and one boundary function. As previous works [15, 14, 16], the proof of convergence relies on the construction of suitable correctors. The difference here is that we don't consider a microscopic model since to our knowledge, no microscopic model considering the bus as a moving capacity constraint exist.

## 2 Traffic flow motivation and derivation of a Hamilton-Jacobi equation

### 2.1 A first order bus-vehicles interaction model

In this section, we show how we can obtain equation (1.1). To simplify the work and since the idea remains the same, we consider the case of one Hamiltonian  $H$  and one function  $b$  describing the bus trajectory. Before starting, we mention that our model was introduced in [26] in order to study the interaction between buses and the surrounding traffic flow. Several papers about modeling the effect of buses on the traffic flow exist, see [24, 11, 12, 9, 17].

The idea is to consider the traffic flow on a single road where a bus is moving. In this model, we assume that the fundamental physical parameters of the model, i.e. the maximum density and the maximum mean velocity, don't depend on the position  $x$  if  $x \neq b(t)$ , i.e. the characteristics of the infrastructure don't change with the position far from the bus. The traffic flow is assumed to be described by a first order macroscopic model of the LWR type if the space variable  $x \neq b(t)$ . Bus should be considered as a moving capacity restriction from other drivers point of view. Authors in [26] extended the notion of demand and supply introduced in [25] to the moving frame using the change of variables  $\zeta = x - b(t)$ . The model is given by

$$\begin{cases} \rho_t + (f(\rho))_x = 0 & \text{if } x \neq b(t) \\ \tilde{f}(t, \rho(t, x^-)) = \min(B(t), \tilde{f}_D(t, \rho(t, x^-)), \tilde{f}_S(t, \rho(t, x^+))) & \text{if } x = b(t) \end{cases} \quad (2.1)$$

where  $\rho$  is the density of vehicles at time  $t$  and position  $x$ ,  $f$  is a strictly concave function (as Greenshield model [18]), reaching its unique maximum at a critical density  $\rho_c$ , describing the flow and  $\tilde{f}(t, p) = f(p) - b'(t) \cdot p$ . The function  $B$  is the limiter of the passing flux through the bus at time  $t$ . The definition of  $\tilde{f}$  yields that for all  $t$ , the function  $\tilde{f}(t, \cdot)$  reaches a unique maximum at a point denoted  $\tilde{\rho}_c(t)$ . The functions  $\tilde{f}_D$  and  $\tilde{f}_S$  are respectively the Demand and Supply functions defined as follows

$$\tilde{f}_D(t, p) = \begin{cases} \tilde{f}(t, \tilde{\rho}_c(t)) & \text{if } p \geq \tilde{\rho}_c(t) \\ \tilde{f}(t, p) & \text{if } p < \tilde{\rho}_c(t) \end{cases}$$

and

$$\tilde{f}_S(t, p) = \begin{cases} \tilde{f}(t, p) & \text{if } p \geq \tilde{\rho}_c(t) \\ \tilde{f}(t, \tilde{\rho}_c(t)) & \text{if } p < \tilde{\rho}_c(t). \end{cases}$$

Before passing to the Hamilton-Jacobi formulation, let us present the two point below in order to clarify the model.

- The trajectory of the bus can be approximated by assuming that  $b' = 0$  (bus-stops) or that  $b'$  is equal to the desired bus-speed  $V_b$  (if the bus enjoys special lanes) or is the minimum between the desired bus speed  $V_b$  and the local traffic speed, i.e.

$$b'(t) = \min\left(V_b, V\left(\rho\left(t, b(t)^+\right)\right)\right).$$

In this paper, we will only consider the second case i.e. when the velocity of  $b$  is  $V_b$  (see section 4). The case where  $b' = 0$  reduces to the work [20]. In the case where the velocity of the bus depends on the density of vehicles, we will obtain a strongly coupled PDE-ODE system and we will have to introduce a good notion of solution for the system. In this case, we were not able to get a uniqueness result. Note that several paper like [24, 7, 6, 8, 12] considered the case where  $b$  depends on the density of vehicles but considered a different macroscopic model as this paper.

- The second equation in (2.1) means that the passing flux through  $x = b(t)$  is equal to the minimum between the upstream Demand, the downstream Supply and the flux limiter  $B(t)$ . Note that the flux at time  $t$  is limited only if  $B(t) < \tilde{f}(t, \tilde{\rho}_c) = \max \tilde{f}(t, \cdot)$ . The Demand function at the point  $x = b(t)^-$  is the greatest possible outflow at that point and the Supply function at the point  $x = b(t)^+$  is the greatest possible inflow at that point. Note that the passing flux through the bus is  $\tilde{f}$  and not  $f$ . In fact,  $f$  describes the flux at a fixed point  $x$  while the "real" passing flux through the bus is equal to the flux assuming that the bus is a fixed point minus the non-passing flux due to the variation of the position of  $b$ .

## 2.2 The Hamilton-Jacobi formulation

In order to derive the Hamilton-Jacobi equation, we proceed as in [22] considering the continuous analogue of the discrete vehicles label defined by

$$\begin{cases} U^1(t, x) = g(t) - \int_x^{b(t)} \rho(t, y) dy & \text{if } x < b(t) \\ U^2(t, x) = g(t) + \int_{b(t)}^x \rho(t, y) dy & \text{if } x > b(t) \end{cases}$$

with

$$g(t) = - \int_0^t f(\rho(s, b(s)^-)) - b'(s) \rho(s, b(s)^-) ds.$$

Formally, we have the following equalities

$$\begin{aligned} U_t^1 &= g'(t) - \int_x^{b(t)} \rho_t(t, y) dy - b'(t) \rho(t, b(t)^-) \\ &= g'(t) + \int_x^{b(t)} (f(\rho(t, y)))_y dy - b'(t) \rho(t, b(t)^-) \\ &= g'(t) - f(\rho(t, x)) + f(\rho(t, b(t)^-)) - b'(t) \rho(t, b(t)^-). \end{aligned}$$

Recalling the definition of  $g$ , we deduce that  $U_t^1 + f(U_x^1) = 0$  if  $x < b(t)$ . Similarly, we have  $U_t^2 + f(U_x^2) = 0$  if  $x > b(t)$ . In fact, the last equality is true because  $-g'(t)$  represents the passing flux at  $b(t)$  which is equal to the outgoing flux at  $b(t)$ , i.e.

$$g'(t) = -f(\rho(t, b(t)^+)) + b'(t) \rho(t, b(t)^+).$$

We now set

$$u(t, x) = \begin{cases} -U^1(t, x) & \text{if } x < b(t) \\ -U^2(t, x) & \text{if } x > b(t) \end{cases}$$

and we define the Hamiltonian  $H(p) = -f(-p)$ . Then we deduce that we have

$$u_t + H(u_x) = 0 \quad \text{if } x \neq b(t).$$

**The junction condition.** Recalling the definition of  $U^1$  and  $U^2$ , we have that

$$\frac{d}{dt}(u(t, b(t))) = -g'(t) = \min \left( B(t), \tilde{f}_D \left( t, \rho \left( t, b(t)^- \right) \right), \tilde{f}_S \left( t, \rho \left( t, b(t)^+ \right) \right) \right).$$

Let  $\tilde{H}(t, p) = H(p) - b'(t)p$  and  $A(t) = -B(t)$ . Denoting  $\tilde{H}^+(t, \cdot)$  and  $\tilde{H}^-(t, \cdot)$  respectively the non-decreasing part and the non-increasing part of  $\tilde{H}(t, \cdot)$ , we deduce the following junction condition

$$\frac{d}{dt}u(t, b(t)) + \max(A(t), \tilde{H}^+(t, u_x^-(t, b(t))), \tilde{H}^-(t, u_x^+(t, b(t)))) = 0.$$

### 3 Comparison principle for (1.1)

In this section we present the main result of this paper which is the comparison principle for (1.1). We give first the definition of viscosity solutions. As usual, we begin by introducing the class of test functions. For  $T > 0$ , set  $B = (0, T) \times \mathbb{R}$ .

**Test functions.** We denote by  $\mathcal{C}^1(B)$  the class of test functions. If  $\varphi \in \mathcal{C}^1(B)$ , then

- $\varphi$  is continuous.
- The restriction of  $\varphi$  on each  $B_i$  is  $\mathcal{C}^1$ .
- For all  $i = 1, \dots, N$ , the time dependent function  $\varphi(t, b_i(t))$  is  $\mathcal{C}^1$  in time. Moreover,

$$\begin{aligned} \frac{d}{dt}\varphi(t, b_i(t)) &= \varphi_t^+(t, b_i(t)) + b_i'(t)\varphi_x^+(t, b_i(t)) \\ &= \varphi_t^-(t, b_i(t)) + b_i'(t)\varphi_x^-(t, b_i(t)). \end{aligned}$$

We recall the definition of the upper and lower semi-continuous envelopes  $u^*$  and  $u_*$  of a locally bounded function  $u$  on  $B$ ,

$$u^*(t, x) = \limsup_{(s, y) \rightarrow (t, x)} u(s, y) \text{ and } u_*(t, x) = \liminf_{(s, y) \rightarrow (t, x)} u(s, y).$$

**Definition 3.1.** Assume (A) and let  $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ .

- i) We say that  $u$  is a sub-solution (resp. super-solution) of (1.1) in  $[0, T] \times \mathbb{R}$  if  $u^*(0, x) \leq u_0(x)$  (resp.  $u_*(0, x) \geq u_0(x)$ ) and if for all test function  $\varphi \in \mathcal{C}^1(B)$  touching  $u^*$  from above (resp. touching  $u_*$  from below) at  $(t_0, x_0) \in B$ , we have

$$\begin{aligned} \varphi_t + H_i(u_x) &\leq 0 \text{ (resp. } \geq 0) \text{ at } (t_0, x_0) \text{ if } (t_0, x_0) \in B_i \\ \frac{d}{dt}\varphi(t_0, b_i(t_0)) + F_{A_i}(t_0, u_x^{i,-}(t_0, x_0), u_x^{i,+}(t_0, x_0)) &\leq 0 \text{ (resp. } \geq 0) \text{ if } x_0 = b_i(t_0). \end{aligned}$$

- ii) We say that  $u$  is a viscosity solution of (1.1) if  $u$  is a sub-solution and a super-solution of (1.1).

**Theorem 3.2** (Reduction of test functions). Assume (A). We fix  $i \in \{1, \dots, N\}$  and assume that

$$A_i(t) \geq A_i^0(t) = \max\left(\min_{\mathbb{R}} H_{i,i}(t, \cdot), \min_{\mathbb{R}} H_{i+1,i}(t, \cdot)\right).$$

Let  $t_0 \in (0, T)$  and let  $p_{i,i}^{A_i(t_0)}$  and  $q_{i+1,i}^{A_i(t_0)}$  two constant satisfying

$$\begin{cases} H_{i,i}\left(t_0, p_{i,i}^{A_i(t_0)}\right) = H_{i,i}^-\left(t_0, p_{i,i}^{A_i(t_0)}\right) = A_i(t_0) \\ H_{i+1,i}\left(t_0, q_{i+1,i}^{A_i(t_0)}\right) = H_{i+1,i}^+\left(t_0, q_{i+1,i}^{A_i(t_0)}\right) = A_i(t_0). \end{cases}$$

We consider the following Hamilton-Jacobi equation

$$u_t + H_k(u_x) = 0 \text{ for } (t, x) \in B_k, k = i, i + 1. \quad (3.1)$$

- Let  $u : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$  an upper semi-continuous sub-solution of (3.1) and satisfying

$$u(t, b_i(t)) = \limsup_{(s, y) \rightarrow (t, b_i(t)), y > b_i(s)} u(s, y) = \limsup_{(s, y) \rightarrow (t, b_i(t)), y < b_i(s)} u(s, y). \quad (3.2)$$

If for any test function  $\varphi$  touching  $u$  from above at  $(t_0, b_i(t_0))$  with

$$\varphi(t, x) = g(t) + q_{i+1, i}^{A_i(t_0)}(x - b_i(t)) 1_{\{x - b_i(t) > 0\}} + p_{i, i}^{A_i(t_0)}(x - b_i(t)) 1_{\{x - b_i(t) < 0\}} \quad (3.3)$$

for some  $g \in C^1(0, +\infty)$ , we have

$$\frac{d}{dt} \varphi(t_0, b_i(t_0)) + F_{A_i}(t_0, \varphi_x^{i,-}(t_0, b_i(t_0)), \varphi_x^{i,+}(t_0, b_i(t_0))) \leq 0$$

then  $u$  is a sub-solution of

$$\frac{d}{dt} u(t, b_i(t)) + F_{A_i}(t, u_x^{i,-}(t, b_i(t)), u_x^{i,+}(t, b_i(t))) = 0 \quad \text{at } t_0.$$

- Let  $u : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$  a lower semi-continuous super-solution of (3.1). If for any test function  $\varphi$  touching  $u$  from below at  $(t_0, b_i(t_0))$  with  $\varphi$  is defined as in (3.3), we have

$$\frac{d}{dt} \varphi(t_0, b_i(t_0)) + F_{A_i}(t_0, \varphi_x^{i,-}(t_0, b_i(t_0)), \varphi_x^{i,+}(t_0, b_i(t_0))) \geq 0$$

then  $u$  is a super-solution of

$$\frac{d}{dt} u(t, b_i(t)) + F_{A_i}(t, u_x^{i,-}(t, b_i(t)), u_x^{i,+}(t, b_i(t))) = 0 \quad \text{at } t_0.$$

The proof of this theorem is similar to the proof of Theorem 2.7 in [20].

The next proposition is concerned with the supremum of sub-solutions. Such a result is used in the Perron process to construct solutions.

**Proposition 3.3.** *Assume (A). Let  $\mathcal{A}$  be a nonempty set and let  $(u_a)_{a \in \mathbb{A}}$  be a family of sub-solutions of (1.1) on  $(0, T) \times \mathbb{R}$  and satisfying (3.2) for all  $i \in \{1, \dots, N\}$ . Let us assume that*

$$u = \sup_{a \in \mathcal{A}} u_a$$

*is locally bounded on  $(0, T) \times \mathbb{R}$ . Then  $u$  is a sub-solution of (1.1) on  $(0, T) \times \mathbb{R}$ .*

The proof is standard. The only new idea is to prove that  $u^*$  satisfies (3.2) in order to use the result of the preceding theorem. By Perron method, and the last proposition, we easily obtain the following result.

**Theorem 3.4.** *Assume (A) and that the initial datum  $u_0$  is Lipschitz continuous. Then there exists a viscosity solution  $u$  of (1.1) in  $[0, T) \times \mathbb{R}$  and a constant  $C_T > 0$  such that*

$$|u(t, x) - u_0(x)| \leq C_T.$$

**Theorem 3.5** (Comparison principle). *Let  $T > 0$ . Assume that  $u_0$  is a Lipschitz continuous function. Let  $u$  be an upper semi-continuous sub solution and  $v$  be a lower semi-continuous super solution of (1.1), s.t. there exists a constant  $K > 0$ , s.t. for all  $t \in [0, T]$ , we have  $u(t, x) \leq u_0(x) + Kt$  and  $v(t, x) \geq u_0(x) - Kt$ , then we have*

$$u(t, x) \leq v(t, x) \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}.$$

As we mentioned before, we will adapt the idea introduced in [4]. The main difference here is the localization procedure in order to choose the good test function. Before starting the proof, we state the following useful remarks.

**Remark 3.6.** We recall that for all  $t > 0$ , and for all  $i \in \{1, \dots, N+1\}$ ,  $j \in \{1, \dots, N\}$ , the Hamiltonian  $H_{i,j}(t, \cdot)$  is superlinear (see (A2)). Therefore, there exists a constant  $C_t > 0$ , such that for all  $p \in \mathbb{R}$ , we have  $|p| \leq \max(C_t, H_{i,j}(t, p))$ . We will denote by  $C_T$  the upper bound of  $C_t$  for  $t \in [0, T]$ .

**Lemma 3.7.** There exists a constant  $B_T > 0$  and a modulus of continuity  $w_T$  such that for all  $t \in [0, T]$ ,  $p \in \mathbb{R}$  and for all  $i \in \{1, \dots, N+1\}$  and for  $k = i, i+1$ , we have

$$\begin{cases} |H_{k,i}(t, p) - H_{k,i}(s, p)| \leq B_T |t - s| \cdot |p| \\ |H_{k,i}^+(t, p) - H_{k,i}^+(s, p)| \leq B_T \max(|t - s| \cdot |p|, w_T(|t - s|)) \\ |H_{k,i}^-(t, p) - H_{k,i}^-(s, p)| \leq B_T \max(|t - s| \cdot |p|, w_T(|t - s|)). \end{cases}$$

*Proof.* The proof of these inequalities is very simple. We get the first line by the definition of the Hamiltonian  $H_{k,i}$ . To prove the second and the third lines, we simply use the continuity of the functions for  $k = i, i+1$

$$\begin{cases} t \rightarrow \min_{\mathbb{R}} H_{k,i}(t, \cdot) \\ t \rightarrow p_0^{k,i}(t) = \max\{p \text{ s.t. } H_{k,i}(t, p) = \min_{\mathbb{R}} H_{k,i}(t, \cdot)\} \\ t \rightarrow q_0^{k,i}(t) = \min\{p \text{ s.t. } H_{k,i}(t, p) = \min_{\mathbb{R}} H_{k,i}(t, \cdot)\}. \end{cases}$$

□

*Proof of Theorem 3.5.* We introduce

$$M = \sup_{(t,x) \in [0,T] \times \mathbb{R}} \{u(t, x) - v(t, x)\}.$$

We want to prove that  $M \leq 0$ . We argue by contradiction and assume that  $M > 0$ . Let  $L$  and  $R$  two constant such that  $L < \min_{t \in [0,T]} b_1(t)$  and  $R > \max_{t \in [0,T]} b_N(t)$ . Let  $\eta > 0$ , we introduce

$$M_\eta = \sup_{\substack{t \in [0,T] \\ L \leq x \leq R}} \left\{ u(t, x) - v(t, x) - \frac{\eta}{T-t} \right\}. \quad (3.4)$$

Since we consider the maximum of an upper-semi continuous function on a compact set, we deduce that the maximum is reached at a point that we denote  $(t_\eta, x_\eta)$ .

**Case 1:**  $M_\eta \leq 0$ . Then we consider the following supremum

$$M_{\varepsilon, \alpha} = \sup_{\substack{t, s \in [0, T] \\ x, y \in \mathbb{R}}} \left\{ u(t, x) - v(s, y) - \frac{\eta}{T-t} - \frac{(x-y)^2}{2\varepsilon} - \frac{(t-s)^2}{2\varepsilon} - \alpha x^2 - \alpha y^2 \right\}$$

Classically,  $M_{\varepsilon, \alpha} \geq M/2 > 0$  for  $\eta$  and  $\alpha$  small enough. Moreover, the maximum is reached at  $(t, s, x, y)$  and  $\alpha x \rightarrow 0$ ,  $\alpha y \rightarrow 0$  as  $\alpha \rightarrow 0$ . We denote by  $\bar{x}$  the common limit of  $x$  and  $y$  as  $\varepsilon$  goes to zero and by  $\bar{t}$  the common limit of  $t$  and  $s$  as  $\varepsilon$  goes to zero. It's clear that  $\bar{t} \neq 0$  since  $u_0$  is Lipschitz. Moreover, taking  $\varepsilon$  to zero and using the upper-semi continuity property, we obtain that  $u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{x}) - \frac{\eta}{T-\bar{t}} \geq M/2$ , which implies that  $\bar{x} \notin [L, R]$  because  $M_\eta \leq 0$ . We deduce that whether  $x < b_1(t)$  and  $y < b_1(s)$  or  $x > b_N(t)$  and  $y > b_N(s)$ . Using the fact that  $u$  is a sub-solution and  $v$  is a super-solution, we obtain for  $j = 1$  or  $j = N+1$

$$\begin{cases} \frac{\eta}{(T-t)^2} + \frac{t-s}{\varepsilon} + H_j \left( \frac{x-y}{\varepsilon} + 2\alpha x \right) \leq 0 \\ \frac{t-s}{\varepsilon} + H_j \left( \frac{x-y}{\varepsilon} - 2\alpha y \right) \geq 0. \end{cases}$$

Subtracting the two inequalities and taking  $\alpha$  to zero, we obtain a contradiction.



**Case 2:**  $M_\eta > 0$  and  $x_\eta \neq b_i(t_\eta)$  for all  $i \in \{1, \dots, N\}$ . In this case, we consider

$$M_{\varepsilon, \alpha} = \sup_{\substack{t, s \in [0, T] \\ x, y \in \mathbb{R}}} \left\{ \begin{array}{l} u(t, x) - v(s, y) - \frac{\eta}{T-t} - \frac{(x-y)^2}{2\varepsilon} - \frac{(t-s)^2}{2\varepsilon} \\ -\alpha \left( (x-x_\eta)^2 + (t-t_\eta)^2 \right) \end{array} \right\}$$

Classically,  $M_{\varepsilon, \alpha} \geq M_\eta > 0$ . Moreover, the maximum is reached at  $(t, s, x, y)$  and we denote by  $\bar{x}$  and  $\bar{t}$  respectively the common limit of  $x$  and  $y$  and the common limit of  $t$  and  $s$  as  $\varepsilon$  goes to zero. Moreover, taking  $\varepsilon$  to zero, and using the upper-semi continuity, we obtain that

$$u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{x}) - \frac{\eta}{T-\bar{t}} - \alpha \left( (\bar{x} - x_\eta)^2 + (\bar{t} - t_\eta)^2 \right) \geq M_\eta. \quad (3.5)$$

If  $\bar{x} \notin [L, R]$ , we proceed as the case where  $M_\eta \leq 0$ . If not, then (3.5) and the definition of  $M_\eta$  implies that

$$M_\eta - \alpha \left( (\bar{x} - x_\eta)^2 + (\bar{t} - t_\eta)^2 \right) \geq M_\eta$$

which yields that  $\bar{t} = t_\eta$  and  $\bar{x} = x_\eta$ . Writting the viscosity inequalities, we obtain that

$$\begin{cases} \frac{\eta}{(T-t)^2} + \frac{t-s}{\varepsilon} + 2\alpha(t-t_\eta) + H_j \left( \frac{x-y}{\varepsilon} + 2\alpha(x-x_\eta) \right) \leq 0 \\ \frac{t-s}{\varepsilon} + H_j \left( \frac{x-y}{\varepsilon} + 2\alpha(y-x_\eta) \right) \geq 0 \end{cases}$$

where  $j$  is the index such that  $b_{j-1}(t_\eta) < x_\eta < b_j(t_\eta)$ . Sending  $\alpha$  to zero, we obtain a contradiction.

**Case 3:**  $M_\eta > 0$  and there exists  $i_0 \in \{1, \dots, N\}$  s.t.  $x_\eta = b_{i_0}(t_\eta)$ . We first introduce

$$M_{\nu, \alpha} = \sup_{\substack{t, s \in [0, T] \\ L \leq x \leq R}} \left\{ \begin{array}{l} u(t, x) - v(s, x + b_{i_0}(s) - b_{i_0}(t)) - \frac{\eta}{T-t} - \frac{(t-s)^2}{2\nu} \\ -\alpha (x - b_{i_0}(t))^2 - (t - t_\eta)^2 \end{array} \right\}$$

Classically, we have that

$$\begin{cases} M_{\nu, \alpha} \geq M_\eta \text{ and the maximum is reached at a point that we denote by } (t_\nu, s_\nu, x_\nu), \\ (t_\nu, s_\nu, x_\nu) \xrightarrow{\nu \rightarrow 0} (t_\eta, t_\eta, x_\eta), \\ \alpha (x - b_{i_0}(t)) \xrightarrow{\alpha \rightarrow 0} 0. \end{cases}$$

The second point implies that for  $\nu$  small enough,  $x_\nu \neq b_i(t_\nu)$  for all  $i \neq i_0$ .

We need the following lemma.

**Lemma 3.8.** *Let  $(\hat{t}, \hat{s}, \hat{x})$  be the limit (up to a subsequence) of  $(t_\nu, s_\nu, x_\nu)$  as  $\alpha$  goes to zero. We have that*

$$\limsup_{\nu \rightarrow 0} \left( \frac{(\hat{t} - \hat{s})^2}{2\nu} \right) = 0. \quad (3.6)$$

*Proof.* The proof is very simple and relies only on the upper-semi continuity property of the function. Since  $M_{\nu, \alpha} \geq M_\eta$ , taking  $\alpha$  to zero, we obtain

$$u(\hat{t}, \hat{x}) - v(\hat{s}, \hat{x} + b_{i_0}(\hat{s}) - b_{i_0}(\hat{t})) - \frac{\eta}{T-\hat{t}} - \frac{(\hat{t} - \hat{s})^2}{2\nu} - (\hat{t} - t_\eta)^2 \geq M_\eta > 0.$$

Then, taking  $\nu$  to zero and recalling that  $\lim_{\nu \rightarrow 0} |\hat{t} - \hat{s}| = 0$  implies that

$$\begin{aligned} M_\eta &\geq \limsup_{\nu \rightarrow 0} \left( u(\hat{t}, \hat{x}) - v(\hat{s}, \hat{x} + b_{i_0}(\hat{s}) - b_{i_0}(\hat{t})) - \frac{\eta}{T - \hat{t}} \right) \\ &\geq \limsup_{\nu \rightarrow 0} \left( u(\hat{t}, \hat{x}) - v(\hat{s}, \hat{x} + b_{i_0}(\hat{s}) - b_{i_0}(\hat{t})) - \frac{\eta}{T - \hat{t}} - \frac{(\hat{t} - \hat{s})^2}{2\nu} - (\hat{t} - t_\eta)^2 \right) \\ &\geq M_\eta. \end{aligned}$$

The last inequality implies that

$$\limsup_{\nu \rightarrow 0} \left( \frac{(\hat{t} - \hat{s})^2}{2\nu} + (\hat{t} - t_\eta)^2 \right) = 0$$

and in particular (3.6) is true.  $\square$

We now continue the proof. We have to distinguish two different cases:

**Subcase**  $x_\nu \neq b_{i_0}(t_\nu)$ . We define the new supremum,

$$M_{\nu, \alpha, \varepsilon} = \sup_{\substack{t, s \in [0, T] \\ L \leq x, y \leq R}} \left\{ \begin{array}{l} u(t, x) - v(s, y) - \frac{\eta}{T - t} - \frac{(t - s)^2}{2\nu} - \alpha(x - b_{i_0}(t))^2 - (t - t_\eta)^2 \\ -G(t, s, x, y) - \psi(t, s, x) \end{array} \right\}$$

with

$$\begin{cases} G(t, s, x, y) = \frac{(x + b_{i_0}(s) - b_{i_0}(t) - y)^2}{2\varepsilon} \\ \psi(t, s, x) = (t - t_\nu)^2 + (s - s_\nu)^2 + (x - b_{i_0}(t) - x_\nu + b_{i_0}(t_\nu))^2. \end{cases} \quad (3.7)$$

The maximum is reached at  $(t, s, x, y)$  and the fact that  $u_0$  is Lipschitz continuous, that  $b_{i_0}$  is a continuous function and the definition of  $G$ , yields that

$$(t, s, x, y) \xrightarrow{\varepsilon \rightarrow 0} (t_\nu, s_\nu, x_\nu, x_\nu + b_{i_0}(s_\nu) - b_{i_0}(t_\nu)). \quad (3.8)$$

Equation (3.8) implies that for  $\varepsilon$  small enough,  $x \neq b_{i_0}(t)$  and  $y \neq b_{i_0}(s)$ . We now write the viscosity inequalities assuming that  $x_\nu < b_{i_0}(t_\nu)$ . The case where  $x_\nu > b_{i_0}(t_\nu)$  is similar only replacing  $H_{i_0}$  by  $H_{i_0+1}$ . In order to simplify the notations, we will use the following notations:

$$\begin{cases} p_{\varepsilon, \nu, \alpha} = 2\alpha(x - b_{i_0}(t)) + 2(x - b_{i_0}(t) + b_{i_0}(t_\nu) - x_\nu) + \frac{x + b_{i_0}(s) - b_{i_0}(t) - y}{\varepsilon} \\ p_{\varepsilon, \nu} = \frac{x + b_{i_0}(s) - b_{i_0}(t) - y}{\varepsilon} \end{cases} \quad (3.9)$$

Using the fact that  $u$  is a sub solution of (1.1), and the definition of  $H_{i_0, i_0}$ , we deduce that

$$\frac{\eta}{(T - t)^2} + \frac{t - s}{\nu} + 2(t - t_\eta) + 2(t - t_\nu) + H_{i_0, i_0}(t, p_{\varepsilon, \nu, \alpha}) \leq 0. \quad (3.10)$$

Using the fact that  $v$  is a supersolution of (1.1), we obtain

$$\frac{t - s}{\nu} + 2(s_\nu - s) + H_{i_0, i_0}(s, p_{\varepsilon, \nu}) \geq 0. \quad (3.11)$$

Combining (3.10) and (3.11), we obtain

$$\frac{\eta}{(T - t)^2} + 2(t - t_\eta) + 2(t - t_\nu) + 2(s - s_\nu) \leq H_{i_0, i_0}(s, p_{\varepsilon, \nu}) - H_{i_0, i_0}(t, p_{\varepsilon, \nu, \alpha}). \quad (3.12)$$

The goal is to take first  $\varepsilon$ , then  $\alpha$  and finally  $\nu$  to zero. Using (3.10) and Remark 3.6, we deduce that there exists a constant  $C_T > 0$  such that

$$|p_{\varepsilon,\nu,\alpha}| \leq \max \left( C_T, \frac{s-t}{\nu} + 2(t_\eta - t) + 2(t_\nu - t) \right) = C_{\nu,T} \quad (3.13)$$

which implies that

$$|p_{\varepsilon,\nu}| \leq C_{\nu,T} + o(\alpha) + o(\varepsilon). \quad (3.14)$$

Estimates (3.13) and (3.14) implies that  $p_{\varepsilon,\nu,\alpha}$  and  $p_{\varepsilon,\nu}$  converge as  $\varepsilon$  goes to zero (up to subsequence). Denoting by  $\bar{p}_{\nu,\alpha} = \lim_{\varepsilon \rightarrow 0} p_{\varepsilon,\nu,\alpha}$  and by  $\bar{p}_\nu = \lim_{\varepsilon \rightarrow 0} p_{\varepsilon,\nu}$  and taking  $\varepsilon$  to zero in (3.12), we obtain

$$\begin{aligned} \frac{\eta}{(T-t_\nu)^2} + 2(t_\nu - t_\eta) &\leq H_{i_0,i_0}(s_\nu, \bar{p}_\nu) - H_{i_0,i_0}(t_\nu, \bar{p}_{\nu,\alpha}) \\ &= H_{i_0,i_0}(s_\nu, \bar{p}_\nu) - H_{i_0,i_0}(s_\nu, \bar{p}_{\nu,\alpha}) \\ &\quad + H_{i_0,i_0}(s_\nu, \bar{p}_{\nu,\alpha}) - H_{i_0,i_0}(t_\nu, \bar{p}_{\nu,\alpha}). \end{aligned}$$

Recalling Remark 3.7 and using (3.13), (more precisely, we use (3.13) after taking  $\varepsilon$  to 0), we deduce that

$$H_{i_0,i_0}(s_\nu, \bar{p}_{\nu,\alpha}) - H_{i_0,i_0}(t_\nu, \bar{p}_{\nu,\alpha}) \leq B_T |t_\nu - s_\nu| \tilde{C}_{\nu,T}.$$

with  $\tilde{C}_{\nu,T} = \max \left( C_T, \frac{s_\nu - t_\nu}{\nu} + 2(t_\eta - t_\nu) \right)$ . Therefore, we obtain

$$\frac{\eta}{(T-t_\nu)^2} + 2(t_\nu - t_\eta) \leq H_{i_0,i_0}(s_\nu, \bar{p}_\nu) - H_{i_0,i_0}(s_\nu, \bar{p}_{\nu,\alpha}) + B_T |t_\nu - s_\nu| \tilde{C}_{\nu,T}.$$

First, we send  $\alpha$  to zero to get that the limit of  $H_{i_0,i_0}(s_\nu, \bar{p}_\nu) - H_{i_0,i_0}(s_\nu, \bar{p}_{\nu,\alpha}) = 0$  and then, recalling Lemma 3.8 and the definition of  $\tilde{C}_{\nu,T}$ , we send  $\nu$  to zero to obtain a contradiction.

**Subcase  $x_\nu = b_{i_0}(t_\nu)$ .** In this case, we will use the following lemma

**Lemma 3.9.** *We have the following inequality*

$$-\frac{\eta}{(T-t_\nu)^2} + \frac{s_\nu - t_\nu}{\nu} + 2(t_\eta - t_\nu) \geq \max \left( \min_{\mathbb{R}} H_{i_0+1,i_0}(t_\nu, \cdot), \min_{\mathbb{R}} H_{i_0,i_0}(t_\nu, \cdot) \right).$$

*Proof.* We can assume that the maximum  $M_{\nu,\alpha}$  is strict, (if not we add the term  $-(t-t_\nu)^2 - (s-s_\nu)^2 - (x-x_\nu)^2$ ).

We introduce the function  $\psi : [0, T] \rightarrow \mathbb{R}$  defined by

$$\psi(t) = u(t, b_{i_0}(t)) - v(s_\nu, b_{i_0}(s_\nu)) - \frac{\eta}{T-t} - \frac{(t-s_\nu)^2}{2\nu} - (t-t_\eta)^2$$

This function reaches its strict maximum at  $t_\nu$ . Let  $\phi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  defined as follows

$$\begin{aligned} \phi(t, x) = u(t, x) - v(s_\nu, b_{i_0}(s_\nu)) - \frac{\eta}{T-t} - \frac{(t-s_\nu)^2}{2\nu} - \alpha(x - b_{i_0}(t))^2 - (t-t_\eta)^2 \\ - L|x - b_{i_0}(t)| \end{aligned}$$

with  $L > 0$  a constant such that for all  $i \in \{1, \dots, N+1\}$

$$\begin{cases} H_i(L) - 3K_T L > \frac{T}{\nu} + 2T \\ H_i(-L) - 3K_T L > \frac{T}{\nu} + 2T \end{cases} \quad (3.15)$$

with  $K_T$  an upper-bound of  $|b'_j|$  on  $[0, T]$  for all  $j \in \{1, \dots, N\}$ . The constant  $L$  is well defined due to the superlinearity property of  $H_i$  (see (A2)).

The maximum of this function is reached at a point  $(t, x)$  with  $t$  close to  $s_\nu$  (which implies that  $t \neq 0$  and  $t \neq T$ ). If  $x \neq b_{i_0}(t)$ , then writing the subsolution inequality, we obtain a contradiction using (3.15). We deduce that  $x = b_{i_0}(t)$ . Moreover, using that the strict maximum of  $\psi$  is reached at  $t_\nu$ , we deduce that  $t = t_\nu$  and  $x = b_{i_0}(t_\nu)$ . Writing the subsolution inequality, we obtain

$$\frac{\eta}{(T - t_\nu)^2} + \frac{t_\nu - s_\nu}{\nu} + 2(t_\nu - t_\eta) + \max(A_{i_0}(t_\nu), H_{i_0, i_0}^+(t_\nu, -L), H_{i_0+1, i_0}^-(t_\nu, L)) \leq 0.$$

The inequality above implies directly the desired result.  $\square$

In order to introduce the new supremum  $M'_{\nu, \alpha, \varepsilon}$ , we will define two constant  $\lambda_1$  and  $\lambda_2$  whose existence is due to the preceding lemma and the properties of  $H_{k, i}(t, \cdot)$  for  $k = i, i + 1$ .

**Lemma 3.10.** *Let  $p_0^{i_0+1, i_0}(t)$  and  $q_0^{i_0, i_0}(t)$  the two functions defined in Remark 3.7. Let  $\nu$  small enough such that  $2(t_\eta - t_\nu) < \frac{\eta}{2T^2}$ . We define  $\lambda_1$  and  $\lambda_2$  such that  $\lambda_1 > p_0^{i_0+1, i_0}(t_\nu)$ ,  $\lambda_2 < q_0^{i_0, i_0}(t_\nu)$  and*

$$\begin{cases} -\frac{\eta}{(T - t_\nu)^2} + \frac{s_\nu - t_\nu}{\nu} + 2(t_\eta - t_\nu) < H_{i_0+1, i_0}^+(t_\nu, \lambda_1) < \frac{s_\nu - t_\nu}{\nu} - \frac{\eta}{2T^2} \\ -\frac{\eta}{(T - t_\nu)^2} + \frac{s_\nu - t_\nu}{\nu} + 2(t_\eta - t_\nu) < H_{i_0, i_0}^-(t_\nu, \lambda_2) < \frac{s_\nu - t_\nu}{\nu} - \frac{\eta}{2T^2}. \end{cases}$$

The existence of  $\lambda_1$  and  $\lambda_2$  is due to the quasi-convexity property of  $H_{i_0+1, i_0}(t, \cdot)$  and  $H_{i_0, i_0}(t, \cdot)$ . We also have that

$$H_{i_0+1, i_0}^+(s_\nu, \lambda_1) < \frac{s_\nu - t_\nu}{\nu}.$$

In fact, using that  $\lim_{p \rightarrow +\infty} \frac{H_{i_0+1, i_0}^+(t_\nu, p)}{p} = +\infty$ , we deduce that there exists  $C_T > 0$  such that

$$\lambda_1 \leq \max\left(C_T, \frac{s_\nu - t_\nu}{\nu} - \frac{\eta}{2T^2}\right)$$

and in particular

$$|\lambda_1| \leq \max\left(C_T, \frac{s_\nu - t_\nu}{\nu} - \frac{\eta}{2T^2}, p_0^{i_0+1, i_0}(t_\nu)\right).$$

Using the fact that the continuous function  $p_0^{i_0+1, i_0}$  is bounded on  $[0, T]$ , we deduce using Remark 3.7 that

$$\lim_{\nu \rightarrow 0} (H_{i_0+1, i_0}^+(s_\nu, \lambda_1) - H_{i_0+1, i_0}^+(t_\nu, \lambda_1)) = 0$$

and that for  $\nu$  small enough,

$$H_{i_0+1, i_0}^+(s_\nu, \lambda_1) < \frac{s_\nu - t_\nu}{\nu}.$$

Similarly, we have also  $H_{i_0, i_0}^-(s_\nu, \lambda_2) < \frac{s_\nu - t_\nu}{\nu}$ .

Before defining  $M'_{\nu, \alpha, \varepsilon}$ , we recall the definition of function  $G$ , see (3.7) and the notations used above, see (3.9). We set

$$M'_{\nu, \alpha, \varepsilon} = \sup_{\substack{t, s \in [0, T] \\ L \leq x, y \leq R}} \left\{ \begin{array}{l} u(t, x) - v(s, y) - \frac{\eta}{T-t} - \frac{(t-s)^2}{2\nu} - \alpha(x - b_{i_0}(t))^2 - (t - t_\eta)^2 \\ -G(t, s, x, y) - \psi(t, s, x) - \varphi(x - b_{i_0}(t)) + \varphi(y - b_{i_0}(s)) \end{array} \right\}$$

with

$$\varphi(x) = \begin{cases} \lambda_1 x & \text{if } x \geq 0 \\ \lambda_2 x & \text{if } x < 0. \end{cases}$$

The maximum is reached at a point  $(t, s, x, y)$  and we have that

$$(t, s, x, y) \xrightarrow{\varepsilon \rightarrow 0} (t_\nu, s_\nu, b_{i_0}(t_\nu), b_{i_0}(s_\nu)).$$

We distinguish three cases depending on the sign of  $x - b_{i_0}(t)$ .

**If  $x > b_{i_0}(t)$ .** If  $y > b_{i_0}(s)$ , we obtain the contradiction proceeding as in the case where  $x_\nu \neq b_{i_0}(t_\nu)$ . If  $y \leq b_{i_0}(s)$ , then using the fact that  $u$  is a subsolution, we obtain

$$\frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + 2(t-t_\eta) + 2(t-t_\nu) + H_{i_0+1, i_0}(t, p_{\varepsilon, \nu, \alpha} + \lambda_1) \leq 0. \quad (3.16)$$

Using that  $H_{i_0+1, i_0}(t, p) \geq H_{i_0+1, i_0}^+(t, p)$  and the fact that  $p_{\varepsilon, \nu, \alpha} > 0$ , and using (3.16), we deduce that

$$\frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + 2(t-t_\eta) + 2(t-t_\nu) + H_{i_0+1, i_0}^+(t, \lambda_1) \leq 0.$$

Sending  $\varepsilon$  to zero, we obtain a contradiction with the definition of  $\lambda_1$ .

**If  $x < b_{i_0}(t)$ .** We proceed as in the case where  $x > b_{i_0}(t)$  using that  $H_{i_0, i_0}(t, p) \geq H_{i_0, i_0}^-(t, p)$ , that  $p_{\varepsilon, \nu, \alpha} < 0$  and the definition of  $\lambda_2$ .

**If  $x = b_{i_0}(t)$ .** Using the fact that  $u$  is a subsolution, we obtain that

$$\frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + 2(t-t_\eta) + 2(t-t_\nu) + F_{A_{i_0}}(t, p_{\varepsilon, \nu} + \lambda_2, p_{\varepsilon, \nu} + \lambda_1) \leq 0. \quad (3.17)$$

This time, we distinguish three cases depending on the sign of  $y - b_{i_0}(s)$ .

**If  $y > b_{i_0}(s)$ .** Note first that using (3.17), we have that

$$\frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + 2(t-t_\eta) + 2(t-t_\nu) + H_{i_0+1, i_0}^-(t, p_{\varepsilon, \nu} + \lambda_1) \leq 0. \quad (3.18)$$

Using the fact that  $v$  is a super-solution, we have that

$$\frac{t-s}{\nu} + 2(s_\nu - s) + H_{i_0+1, i_0}(s, p_{\varepsilon, \nu} + \lambda_1) \geq 0. \quad (3.19)$$

We claim that

$$\frac{t-s}{\nu} + 2(s_\nu - s) + H_{i_0+1, i_0}^-(s, p_{\varepsilon, \nu} + \lambda_1) \geq 0. \quad (3.20)$$

In order to obtain this inequality, we will prove that

$$\frac{t-s}{\nu} + 2(s_\nu - s) + H_{i_0+1, i_0}^+(s, p_{\varepsilon, \nu} + \lambda_1) < 0. \quad (3.21)$$

If (3.21) is true, then combining it with (3.19), (3.20) will remain true. For  $\varepsilon$  small enough and using the fact that  $p_{\varepsilon, \nu} < 0$ , we have that

$$\frac{t-s}{\nu} + 2(s_\nu - s) + H_{i_0+1, i_0}^+(s, p_{\varepsilon, \nu} + \lambda_1) \leq \frac{t-s}{\nu} + 2(s_\nu - s) + H_{i_0+1, i_0}^+(s, \lambda_1) < 0.$$

In fact, the above inequality is true for  $\varepsilon$  small enough using the definition of  $\lambda_1$ . Finally, combining (3.18) and (3.20), we deduce that

$$\begin{aligned} \frac{\eta}{(T-t)^2} + 2(t-t_\eta) + 2(t-t_\nu) + 2(s-s_\nu) &\leq H_{i_0+1,i_0}^-(s, p_{\varepsilon,\nu} + \lambda_1) \\ &\quad - H_{i_0+1,i_0}^-(t, p_{\varepsilon,\nu} + \lambda_1). \end{aligned} \quad (3.22)$$

Recalling that  $\lim_{p \rightarrow +\infty} \frac{H_{i+1,i}^+(t,p)}{p} = +\infty$  and  $\lim_{p \rightarrow -\infty} \frac{H_{i+1,i}^-(t,p)}{p} = -\infty$ , we deduce, using (3.21) and (3.18) that there exists a constant  $C_T > 0$  such that

$$\begin{aligned} |p_{\varepsilon,\nu} + \lambda_1| &\leq \max(H_{i_0+1,i_0}^+(s, p_{\varepsilon,\nu} + \lambda_1), H_{i_0+1,i_0}^-(t, p_{\varepsilon,\nu} + \lambda_1)) \\ &\leq \max\left(C_T, \frac{s-t}{\nu} + 2(s-s_\nu), \frac{s-t}{\nu} + 2(t_\eta-t) + 2(t_\nu-t)\right). \end{aligned} \quad (3.23)$$

As in the case where  $x_\nu \neq b_{i_0}(t_\nu)$ , we take first  $\varepsilon$  to zero in (3.22), and then taking  $\nu$  to zero, thanks to Remark 3.7 and Lemma 3.8, we obtain a contradiction.

**If  $y < b_{i_0}(s)$ .** Note first that using (3.17), we have that

$$\frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + 2(t-t_\eta) + 2(t-t_\nu) + H_{i_0,i_0}^+(t, p_{\varepsilon,\nu} + \lambda_2) \leq 0.$$

As above, we can prove that

$$\frac{t-s}{\nu} + 2(s_\nu - s) + H_{i_0,i_0}^+(s, p_{\varepsilon,\nu} + \lambda_2) \geq 0.$$

and then we obtain the contradiction.

**If  $y = b_{i_0}(s)$ .** In this case, we have

$$\frac{t-s}{\nu} + 2(s_\nu - s) + F_{A_{i_0}}(s, \lambda_2, \lambda_1) \geq 0.$$

As above, we use the sub-solution inequality and the locally Lipschitz property for  $A_{i_0}$  then we send first  $\varepsilon$  to zero and then  $\nu$  to zero to obtain the contradiction. □

## 4 A homogenization problem

The goal of this section is to prove that after rescaling, the solution of the Hamilton-Jacobi equation formulation of (4.1) below converges towards the unique solution of (1.1) including only one Hamiltonian and one function  $b$ . Most of the results are presented without much details since they can be found in previous works [15, 14].

### 4.1 Presentation of the model

We consider the following model which modelize a moving capacity restriction (like a bus or more generally called "moving bottleneck") of the density of the vehicles, for  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ ,

$$\rho_t + (f(\rho)\phi(x-b(t)))_x + (g(\rho)(1-\phi(x-b(t))))_x = 0 \quad (4.1)$$

where  $\rho$  is the density of vehicles,  $b$  represents the position of the bottleneck,  $f$  is the flux function outside the bottleneck region,  $g$  is the flux function in the bottleneck region and  $\phi$  is a transition function. We make the following assumptions.

**Assumptions (B).**

- (B1) The flux function  $f$  is the Greenshields fundamental digram [18] given by

$$f(\rho) = \rho V_{max} \left( 1 - \frac{\rho}{\rho_{max}} \right)$$

where  $V_{max}$  represents the maximal mean velocity of vehicles and  $\rho_{max}$  is the maximal density far from the bus.

- (B2) The flux function around the bus  $g$  is given by

$$g(\rho) = \rho V_{max} \left( 1 - \frac{\rho}{\sigma_{max}} \right)$$

where  $\sigma_{max}$  is the maximal density around the bus. Moreover,  $\sigma_{max} < \rho_{max}$ .

- (B3)  $b$  is a linear function describing the trajectory of the bus and is defined by

$$b(t) = V_b t \quad \text{and we assume that } 0 < V_b < V_{max}.$$

- (B4) The function  $\phi$  is a  $C^1$  transition function and is given by

$$\phi(t) = \begin{cases} 0 & \text{if } x \in [-r, r] \\ 1 & \text{if } x < -r - 1 \text{ ou } x > r + 1. \end{cases}$$

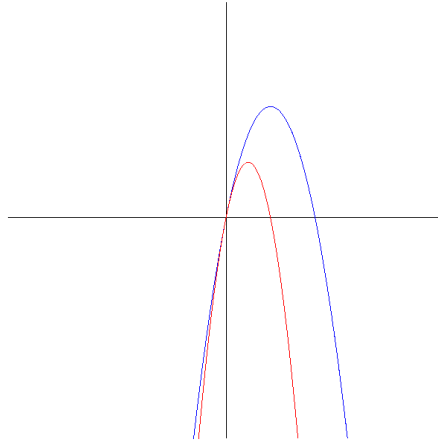


Figure 1: Schematic representation of  $f$  (blue) and  $g$  (red) .

We assume that the initial density satisfies

$$0 \leq \rho(0, x) \leq \begin{cases} \rho_{max} & \text{if } |x| > r + 1 \\ \sigma_{max} & \text{if } |x| \leq r + 1. \end{cases}$$

## 4.2 Main result

Like in subsection 2.2, we will derive the Hamilton-Jacobi equation from model (4.1) by defining the analogue of the discrete vehicles label,

$$u(t, x) = h(t) - \int_0^x \rho(t, y) dy$$

where

$$h(t) = \int_0^t (f(\rho(s, 0)) \phi(-b(s)) + g(\rho(s, 0)) (1 - \phi(-b(s)))) ds.$$

A simple computations yields to

$$u_t - f(-u_x) \phi(x - b(t)) - g(-u_x) (1 - \phi(x - b(t))) = 0.$$

Setting  $H(p) = -f(-p)$  and  $F(p) = -g(-p)$  and recalling the definition of the function  $b$  (see assumption (B3)), we obtain the following Hamilton-Jacobi equation

$$u_t + H(u_x) \phi(x - V_b t) + F(u_x) (1 - \phi(x - V_b t)) = 0 \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

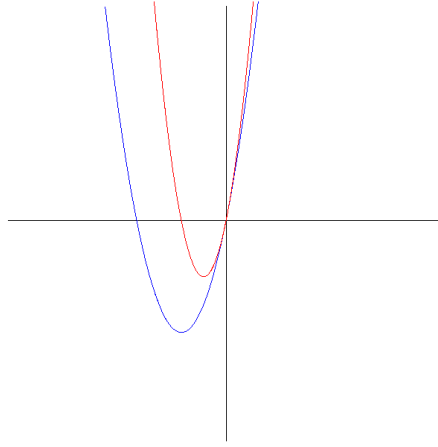


Figure 2: Schematic representation of  $H$  (blue) and  $F$  (red) .

In order to introduce the convergence result, let us define the new Hamiltonians  $\tilde{H}$  and  $\tilde{F}$  defined as

$$\begin{cases} \tilde{H}(p) = H(p) - b'(t)p = H(p) - V_b p \\ \tilde{F}(p) = F(p) - b'(t)p = F(p) - V_b p. \end{cases}$$

Clearly,  $\tilde{F} > \tilde{H}$  and we will use the following notations

$$\begin{cases} \tilde{H}_0 = \min_{\mathbb{R}} \tilde{H} \\ \tilde{F}_0 = \min_{\mathbb{R}} \tilde{F}. \end{cases}$$

The main result of this section is the following theorem. Let  $u^\varepsilon$  be the unique solution of

$$\begin{cases} u_t^\varepsilon + H(u_x^\varepsilon) \phi\left(\frac{x - V_b t}{\varepsilon}\right) + F(u_x^\varepsilon) \left(1 - \phi\left(\frac{x - V_b t}{\varepsilon}\right)\right) = 0 & (t, x) \in \mathbb{R}^+ \times \mathbb{R} \\ u^\varepsilon(0, x) = u_0(x) & x \in \mathbb{R}. \end{cases} \quad (4.2)$$

We assume that the initial condition  $u_0$  is a Lipshtiz function satisfying

$$(A0) \begin{cases} -\rho_{max} \leq (u_0)_x \leq 0 & \text{if } |x| > r + 1 \\ -\sigma_{max} \leq (u_0)_x \leq 0. & \text{if } |x| \leq r + 1 \end{cases} \quad (4.3)$$



**Theorem 4.1** (Junction condition by homogenization). *Assume (B) and (A0). For  $\varepsilon > 0$ , let  $u^\varepsilon$  be the unique solution of (4.2). Then there exists  $A \in [\tilde{F}_0, 0]$  such that  $u^\varepsilon$  converges locally uniformly to the unique viscosity solution  $u^0$  of the following equation*

$$\begin{cases} u_t + H(u_x) = 0 & \text{if } x \neq V_b t \\ \frac{d}{dt} u(t, V_b t) + \max(A, \tilde{H}^+(u_x^-(t, V_b t)), \tilde{H}^-(u_x^+(t, V_b t))) = 0 & \text{if } x = V_b t \\ u(0, x) = u_0(x). \end{cases} \quad (4.4)$$

### 4.3 Viscosity solutions

In this subsection, we give the definition of viscosity solutions of equation (4.2) for  $\varepsilon = 1$ . We then study the space and time oscillations of the solution. The considered equation is given by

$$\begin{cases} u_t + H(u_x) \phi(x - V_b t) + F(u_x) (1 - \phi(x - V_b t)) = 0 & (t, x) \in \mathbb{R}^+ \times \mathbb{R} \\ u(0, x) = u_0(x) & x \in \mathbb{R}. \end{cases} \quad (4.5)$$

#### 4.3.1 Definition

We will introduce now the standard notion of viscosity solutions of equation (4.5).

**Definition 4.2** (Viscosity solutions for (4.5)). *Let  $T > 0$ . An upper semi-continuous function (resp. lower semi-continuous)  $u : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is a viscosity sub-solution (resp. super-solution) of ((4.5)) on  $[0, T] \times \mathbb{R}$ , if  $u(0, x) \leq u_0(x)$  (resp.  $u(0, x) \geq u_0(x)$ ) and for all  $(t, x) \in (0, T) \times \mathbb{R}$  and for all  $\varphi \in C^1([0, T] \times \mathbb{R})$  such that  $u - \varphi$  reaches a maximum (resp. a minimum) at the point  $(t, x)$ , we have*

$$\varphi_t + H(\varphi_x) \phi(x - V_b t) + F(u_x) (1 - \phi(x - V_b t)) \leq 0 \quad (\text{resp } \geq 0).$$

*We say that a function  $u$  is a viscosity solution of ((4.5)) if  $u^*$  and  $u_*$  are respectively a sub-solution and a super-solution of ((4.5)).*

### 4.4 Results for viscosity solutions of (4.5)

We begin by stating the comparison principle for (4.5) whose proof is standard [3, 10].

**Proposition 4.3** (Comparison principle for (4.5)). *Let  $u$  be a sub-solution of (4.5) and  $v$  be a super-solution of (4.5). Let us also assume that there exists a constant  $K > 0$  such that for all  $(t, x) \in [0, T] \times \mathbb{R}$ ,*

$$u(t, x) \leq u_0(x - V_b t) + Kt \quad \text{and} \quad -v(t, x) \leq -u_0(x - V_b t) + Kt. \quad (4.6)$$

*Then we have  $u(t, x) \leq v(t, x)$  for all  $(t, x) \in [0, T] \times \mathbb{R}$ .*

**Theorem 4.4.** *Let  $C_1 = (|H_0| + |F_0|)$  and  $C_2 = \rho_{\max} V_b$ . There exists a unique viscosity solution of (4.5) such that*

$$u_0(x - V_b t) - C_2 t \leq u(t, x) \leq u_0(x - V_b t) + C_1 t.$$

*Moreover, for all  $x, y \in \mathbb{R}$  such that  $x \geq y$  and for all  $t, s \in [0, T]$  such that  $t \geq s$ , we have*

$$\begin{aligned} -C_2(t - s) &\leq u(t, x) - u(s, x) \leq (C_1 + C_2)(t - s) \quad \text{and} \\ -\rho_{\max}(x - y) &\leq u(t, x) - u(t, y) \leq 0. \end{aligned}$$

In order to prove Theorem 4.4, we will study the following simpler equation since it's invariant by time translation.

$$\begin{cases} w_t - V_b w_x + H(w_x) \phi(x) + F(w_x) (1 - \phi(x)) = 0 \\ w(0, x) = u_0(x). \end{cases} \quad (4.7)$$

The unique solution  $u$  of (4.5) is given by

$$u(t, x) = w(t, x - V_b t)$$

where  $w$  is the unique viscosity solution of (4.7).

**Remark 4.5.** *The definition of viscosity solution of equation (4.7) is the same as Definition 4.2 i.e replacing  $w_x$  and  $w_t$  respectively by  $\varphi_x$  and  $\varphi_t$ . Moreover, a comparison principle exists for (4.7).*

**Lemma 4.6** (Existence of barriers for (4.7)). *The functions*

$$w^+(t, x) = u_0(x) + C_1 t \quad \text{and} \quad w^-(t, x) = u_0(x) - C_2 t$$

*are respectively super and sub-solutions of (4.7).*

Applying Perron's method joint to the comparison principle, we obtain the following result.

**Theorem 4.7** (Existence and uniqueness of viscosity solutions for (4.7)). *There exists a unique continuous solution  $w$  of (4.7) which satisfies*

$$u_0(x) - C_2 t \leq w(t, x) \leq u_0(x) + C_1 t.$$

## 4.5 Control of the oscillations for (4.7)

**Proposition 4.8** (Control of the oscillations). *Let  $T > 0$ . The unique solution  $w$  of (4.7) satisfies the following: for all  $x, y \in \mathbb{R}$ ,  $x \geq y$  and for all  $t, s \in [0, T]$ ,  $t \geq s$ , we have*

$$-C_2 t \leq w(t, x) - w(s, x) \leq C_1(t - s) \quad \text{and} \quad (4.8)$$

$$-\rho_{max}(x - y) \leq w(t, x) - w(t, y) \leq 0. \quad (4.9)$$

*Proof.* We begin by proving inequality (4.8). Let  $h > 0$ . We define  $v(t, x) = w(t + h, x)$  and the goal is to prove that

$$w(t, x) - C_2 h \leq v(t, x) \leq w(t, x) + C_1 h. \quad (4.10)$$

All members of inequality (4.10) are viscosity solutions on  $(0, +\infty)$  of (4.7) since equation (4.7) is invariant by time translation and by addition of constants. Using Lemma 4.6, we have that

$$w(0, x) - C_2 h \leq v(0, x) \leq w(0, x) + C_1 h.$$

The comparison principle for equation (4.7) implies directly that (4.10) is true.

We now turn to the proof of (4.9). In the rest of the proof we will use the following notation:

$$\Omega = \{(t, x, y) \in [0, T] \times \mathbb{R}^2 \text{ s.t. } x \geq y\}.$$

**Proof of the upper inequality for the control of the space oscillations.** We introduce,

$$M = \sup_{(t, x, y) \in \Omega} \{w(t, x) - w(t, y)\}.$$

We want to prove that  $M \leq 0$ . We argue by contradiction and assume that  $M > 0$ .

**Step 1: the test function.** For  $\eta, \alpha > 0$ , small parameters, we define

$$\varphi(t, x, y) = w(t, x) - w(t, y) - \frac{\eta}{T-t} - \alpha x^2 - \alpha y^2.$$

Classically,  $\varphi$  reaches a maximum at a point that we denote by  $(\bar{t}, \bar{x}, \bar{y}) \in \Omega$  and for  $\eta$  and  $\alpha$  small enough, we have that

$$\begin{cases} 0 < \frac{M}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}), \\ \alpha|\bar{x}|, \alpha|\bar{y}| \rightarrow 0 \text{ as } \alpha \rightarrow 0 \\ T > \bar{t} > 0 \\ \bar{x} > \bar{y}. \end{cases}$$

**Step 2: utilisation of the equation.** By doubling the time variable and passing to the limit in this duplication parameter, we get that

$$\begin{aligned} \frac{\eta}{(T-\bar{t})^2} &\leq 2\alpha\bar{x}V_b - H(2\alpha\bar{x})\phi(\bar{x}) - F(2\alpha\bar{x})(1-\phi(\bar{x})) \\ &\quad + 2\alpha\bar{y}V_b + H(-2\alpha\bar{y})\phi(\bar{y}) + F(-2\alpha\bar{y})(1-\phi(\bar{y})). \end{aligned}$$

Passing to the limit as  $\alpha$  goes to 0 and using the fact that  $H(0) = F(0) = 0$ , we obtain a contradiction.

**Proof of the lower inequality for the control of the space oscillations** We introduce

$$M = \sup_{(t,x,y) \in \Omega} \{w(t, y) - w(t, x) - \rho_{max}(x - y)\}.$$

We want to prove that  $M \leq 0$ . We argue by contradiction and assume that  $M > 0$ .

**Step 1: the test function.** For  $\eta, \alpha, \nu > 0$  small parameters, we define

$$\varphi(t, s, x, y) = w(t, y) - w(s, x) - \rho_{max}(x - y) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t} - \alpha x^2 - \alpha y^2.$$

The maximum of  $\varphi$  for  $(t, s, x, y) \in [0, T]^2 \times \mathbb{R}^2$  such that  $x \geq y$  reaches a maximum at a point that we denote by  $(\bar{t}, \bar{s}, \bar{x}, \bar{y})$  and for  $\eta, \alpha$  and  $\nu$  small enough, we have that

$$\begin{cases} 0 < \frac{M}{2} \leq \varphi(\bar{t}, \bar{s}, \bar{x}, \bar{y}), \\ \alpha|\bar{x}|, \alpha|\bar{y}| \rightarrow 0 \text{ as } \alpha \rightarrow 0 \\ T > \bar{t}, \bar{s} > 0 \\ \bar{x} > \bar{y}. \end{cases}$$

**Step 2: Utilisation of the equation** Let  $\psi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  defined as follows

$$\psi(t, y) = w(\bar{s}, \bar{x}) + \rho_{max}(\bar{x} - y) + \frac{(t-\bar{s})^2}{2\nu} + \frac{\eta}{T-t} + \alpha\bar{x}^2 + \alpha y^2.$$

Since  $w - \psi$  reaches a maximum at  $(\bar{t}, \bar{y})$ , we deduce using the control of the time oscillations of  $w$  (estimate (4.8)) that  $\psi_{\bar{t}}(\bar{t}, \bar{y}) \geq -\rho_{max}V_b$ . Denoting  $p_\alpha = (-\rho_{max} + 2\alpha\bar{y})$  and recalling that  $w$  is a sub-solution of (4.7), we obtain that

$$-\rho_{max}V_b \leq \frac{\eta}{(T-\bar{t})^2} + \frac{\bar{t}-\bar{s}}{\nu} \leq V_b p_\alpha - H(p_\alpha)\phi(\bar{y}) - F(p_\alpha)(1-\phi(\bar{y})).$$

Sending  $\alpha$  to zero and recalling that  $H(-\rho_{max}) = 0$  and  $F(-\rho_{max}) > 0$ , we obtain that  $-\rho_{max}V_b < -\rho_{max}V_b$  which yields to a contradiction. We deduce that  $M \leq 0$  and the proof is complete.  $\square$

## 5 Proof of convergence

The proof of convergence is based on the construction of correctors. Let  $\lambda$  be a constant greater than  $\tilde{H}_0$ . The definition of  $\tilde{H}$  ensures the existence of two constants  $\tilde{p}_+^\lambda$  and  $\tilde{p}_-^\lambda$  such that

$$\begin{cases} \tilde{H}(\tilde{p}_+^\lambda) = \tilde{H}^+(\tilde{p}_+^\lambda) = \lambda \\ \tilde{H}(\tilde{p}_-^\lambda) = \tilde{H}^-(\tilde{p}_-^\lambda) = \lambda \end{cases}$$

where  $\tilde{H}^+$  and  $\tilde{H}^-$  are respectively the non-decreasing and the non-increasing part of  $\tilde{H}$ . For every  $\lambda \geq \tilde{H}_0$ , we define the following function

$$W^\lambda(t, x) = \tilde{p}_+^\lambda(x - V_b t) \mathbf{1}_{\{x - V_b t > 0\}} + \tilde{p}_-^\lambda(x - V_b t) \mathbf{1}_{\{x - V_b t < 0\}}.$$

**Theorem 5.1.** *There exists a unique constant  $A \in [\tilde{F}_0, 0]$  such that there exists  $w$  solution of the following equation*

$$w_t + H(w_x) \phi(x - V_b t) + F(w_x) (1 - \phi(x - V_b t)) = A$$

and such that  $w^\varepsilon(t, x) = \varepsilon w\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$  converges locally uniformly towards the function  $W^A$ .

*Proof.* We will not go into details because the proof is very similar to the proof of [15, 14]. The idea is to construct a corrector on a truncated domain. We consider  $l \gg r$  and we want to find  $\lambda_l \in \mathbb{R}$  such that there exists  $w^l$  solution of

$$\begin{cases} -V_b v_x^l + H(v_x^l) \phi(x) + F(v_x^l) (1 - \phi(x)) = \lambda_l & \text{if } x \in (-l, l) \\ \tilde{H}^+(v_x^l) = \lambda_l & \text{if } x = l \\ \tilde{H}^-(v_x^l) = \lambda_l & \text{if } x = -l. \end{cases} \quad (5.1)$$

To do this, we consider the following approximated problem

$$\begin{cases} \delta v^{\delta, l} - V_b v_x^{\delta, l} + H(v_x^{\delta, l}) \phi(x) + F(v_x^{\delta, l}) (1 - \phi(x)) = 0 & \text{if } x \in (-l, l) \\ \delta v^{\delta, l} + \tilde{H}^+(v_x^{\delta, l}) = 0 & \text{if } x = l \\ \delta v^{\delta, l} + \tilde{H}^-(v_x^{\delta, l}) = 0 & \text{if } x = -l. \end{cases} \quad (5.2)$$

We construct a unique solution  $v^{\delta, l}$  of problem (5.2) such that

$$0 \leq v^{\delta, l} \leq \frac{|\tilde{H}_0|}{\delta}.$$

In particular, we remark also that  $\delta v^{\delta, l}(0) \leq |\tilde{F}_0|$ . Then, as in the proof 4.5, we prove for all  $x, y \in [-l, l]$  such that  $x \geq y$

$$-\rho_{max}(x - y) \leq v^{\delta, l}(x) - v^{\delta, l}(y) \leq 0. \quad (5.3)$$

We can prove (5.3) only considering the sub-solution inequality using that

$$\begin{cases} \tilde{H}^+(0) = H(0) = F(0) = 0, \\ \tilde{H}^-(\rho_{max}), F(\rho_{max}) > H(\rho_{max}) = 0. \end{cases}$$

Considering the function  $v^{\delta, l}(x) - v^{\delta, l}(0)$  and passing to the limit as  $\delta$  goes to zero (due to Arzelà-Ascoli Theorem), we obtain a solution of problem (5.1) where  $\lambda_l = \lim_{\delta \rightarrow 0} -\delta v^{\delta, l}(0)$ .

The rest of the proof is the same as in [15], and even simpler since the constructed solution of problem (5.1) is Lipschitz so we don't need to consider lim sup, lim inf and the function  $m$  (see [15, 14]). Finally we obtain a unique constant  $A$  and a function  $v$  solution of

$$-V_b v_x + H(v_x) \phi(x) + F(v_x)(1 - \phi(x)) = A, \quad x \in \mathbb{R}$$

such that  $v^\varepsilon(x) = \varepsilon v\left(\frac{x}{\varepsilon}\right)$  converges locally uniformly towards the function  $\tilde{p}_+^A 1_{\{x>0\}} + \tilde{p}_-^A 1_{\{x<0\}}$ . The function  $w(t, x) = v(x - V_b t)$  is the desired function of Theorem 5.1.  $\square$

The following lemma is a direct result of Theorem 4.4 .

**Lemma 5.2** (Uniform gradient bound). *Assume (A0) and (B). Then the solution  $u^\varepsilon$  of (4.2) satisfies for all  $t > 0$ , for all  $x, y \in \mathbb{R}$ ,  $x \geq y$ ,*

$$-\rho_{max}(x - y) \leq u^\varepsilon(t, x) - u^\varepsilon(t, y) \leq 0.$$

We now turn to the proof of Theorem 4.1 .

*Proof of Theorem 4.1.* We introduce

$$\bar{u}(t, x) = \limsup_{\varepsilon \rightarrow 0}^* u^\varepsilon \quad \text{and} \quad \underline{u}(t, x) = \liminf_{\varepsilon \rightarrow 0}^* u^\varepsilon.$$

We want to prove that  $\bar{u}$  and  $\underline{u}$  are respectively a sub-solution and a super-solution of (4.4). In this case, the comparison principle will imply that  $\bar{u} \leq \underline{u}$ . But, by construction, we have  $\underline{u} \leq \bar{u}$ , hence we will get  $\underline{u} = \bar{u} = u^0$ , the unique solution of (4.4).

Let us prove that  $\bar{u}$  is a sub-solution of (4.4) (the proof for  $\underline{u}$  is similar and we skip it). We argue by contradiction and assume that there exists a test function  $\varphi \in \mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R})$  and a point  $(\bar{t}, \bar{x}) \in (0, +\infty) \times \mathbb{R}$  such that for  $\bar{r}, \eta > 0$  and  $\theta > 0$

$$\begin{cases} \bar{u}(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x}) \\ \bar{u} \leq \varphi \\ \bar{u} \leq \varphi - 2\eta \\ \varphi_t(\bar{t}, \bar{x}) + H(\varphi_x(\bar{t}, \bar{x})) = \theta \\ \frac{d}{dt} \varphi(\bar{t}, V_b \bar{t}) + \max(A, \tilde{H}^+(\varphi_x^-(\bar{t}, \bar{x})), \tilde{H}^-(\varphi_x^+(\bar{t}, \bar{x}))) = \theta \end{cases} \begin{array}{l} \text{on } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}) \\ \text{outside } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}) \\ \text{if } \bar{x} \neq V_b \bar{t} \\ \text{if } \bar{x} = V_b \bar{t} \end{array} \quad (5.4)$$

Lemma 5.2 implies that the function  $\bar{u}$  satisfies for all  $t > 0$  and  $x, y \in \mathbb{R}$ ,  $x \geq y$ ,

$$-\rho_{max}(x - y) \leq \bar{u}(t, x) - \bar{u}(t, y) \leq 0. \quad (5.5)$$

**First case:**  $\bar{x} \neq V_b \bar{t}$ . We choose  $r$  small enough such that  $x \neq V_b t$  for all  $(t, x) \in \mathcal{Q}_{r, r}(\bar{t}, \bar{x})$  and then we prove that  $\varphi$  is a super-solution of (4.2) on  $\mathcal{Q}_{r, r}(\bar{t}, \bar{x})$  using the last inequality of (5.4), inequality (5.5) and the fact that

$$\phi\left(\frac{x - V_b t}{\varepsilon}\right) = 1.$$

**Getting a contradiction.** We have for  $\varepsilon$  small enough,

$$u^\varepsilon \leq \varphi - \eta \quad \text{outside } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

Using the comparison principle on bounded subsets we get

$$u^\varepsilon \leq \varphi - \eta \quad \text{on } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

Passing to the limit as  $\varepsilon \rightarrow 0$ , we get  $\bar{u} \leq \varphi - \eta$  on  $\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$  and this contradicts the fact that  $\bar{u}(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x})$ .

**Second case:**  $\bar{x} = V_b \bar{t}$ . In this case, using Theorem 3.2, the definition of the test function  $\varphi$  is given by

$$\varphi(t, x) = g(t) + \tilde{p}_+^A(x - V_b t) 1_{\{x - V_b t > 0\}} + \tilde{p}_-^A(x - V_b t) 1_{\{x - V_b t < 0\}}$$

with  $g \in C^1(\mathbb{R}^+)$  and the last line in (5.4) becomes

$$g(\bar{t}) + A = \theta. \quad (5.6)$$

We define the perturbed test function  $\varphi^\varepsilon$  as

$$\varphi^\varepsilon(t, x) = \begin{cases} g(t) + w^\varepsilon(t, x) & \text{on } Q_{2\bar{r}, 2\bar{r}}(\bar{r}, V_b \bar{t}) \\ \varphi(t, x) & \text{outside } Q_{2\bar{r}, 2\bar{r}}(\bar{t}, V_b \bar{t}). \end{cases}$$

where  $w^\varepsilon$  is defined in Theorem 5.1. Using (5.6) and the definition of  $w$ , we prove that  $\varphi^\varepsilon$  satisfies in the viscosity sense

$$\varphi_t^\varepsilon + H(\varphi_x^\varepsilon) \phi\left(\frac{x - V_b t}{\varepsilon}\right) + F(\varphi_x^\varepsilon) \left(1 - \phi\left(\frac{x - V_b t}{\varepsilon}\right)\right) \geq \frac{\theta}{2} \quad \text{on } Q_{\bar{r}, \bar{r}}(\bar{t}, V_b \bar{t}).$$

**Getting the contradiction.** We have that for  $\varepsilon$  small enough

$$u^\varepsilon + \eta \leq \varphi = g(t) + W^A(t, x) \quad \text{on } Q_{2\bar{r}, 2\bar{r}}(\bar{t}, V_b \bar{t}) \setminus Q_{\bar{r}, \bar{r}}(\bar{t}, V_b \bar{t}).$$

Using the fact that  $w^\varepsilon \rightarrow W^A$ , we have for  $\varepsilon$  small enough

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{on } Q_{2\bar{r}, 2\bar{r}}(\bar{t}, V_b \bar{t}) \setminus Q_{\bar{r}, \bar{r}}(\bar{t}, V_b \bar{t}).$$

By the comparison principle on bounded subsets, the previous inequality holds in  $Q_{\bar{r}, \bar{r}}(\bar{t}, V_b \bar{t})$ . Passing to the limit as  $\varepsilon \rightarrow 0$  and evaluating the inequality in  $(\bar{t}, V_b \bar{t})$ , we obtain

$$\bar{u}(\bar{t}, V_b \bar{t}) + \frac{\eta}{2} \leq \varphi(\bar{t}, V_b \bar{t}) = \bar{u}(\bar{t}, V_b \bar{t}),$$

which is a contradiction. □

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