A Mayer optimal control problem on Wasserstein spaces over Riemannian manifolds

Frédéric Jean, Othmane Jerhaoui, Hasnaa Zidani

Available online at www.sciencedirect.com

ScienceDirect

IFAC Papers Online CONFERENCE PAPER ARCHIVE

IFAC PapersOnLine 55-16 (2022) 44–49

A Mayer optimal control problem on Wasserstein spaces over Riemannian manifolds

F. Jean * **O. Jerhaoui** ** **H. Zidani** ***

* *UMA, ENSTA Paris, Institut Polytechnique de Paris, 91120 Palaiseau, France (e-mail: frederic.jean@ensta-paris.fr).*
** *UMA, ENSTA Paris, Institut Polytechnique de Paris, 91120 Palaiseau, France (e-mail: othmane.jerhaoui@ensta-paris.fr).*
*** *INSA Rouen Normandie, 76800 Saint-Étienne-du-Rouvray, France (e-mail: hasnaa.zidani@insa-rouen.fr).*

Abstract: This paper concerns an optimal control problem on the space of probability measures over a compact Riemannian manifold. The motivation behind it is to model certain situations where the central planner of a deterministic controlled system has only a probabilistic knowledge of the initial condition. The lack of information here is very specific. In particular, we show that the value function verifies a dynamic programming principle and we prove that it is the unique viscosity solution to a suitable Hamilton Jacobi Bellman equation. The notion of viscosity is defined using test functions that are directionally differentiable in the the space of probability measures.

Keywords: Optimal Control, Viscosity solutions, Hamilton Jacobi Bellman equation, Wasserstein spaces, Multi-agent systems.

1. INTRODUCTION

The study of optimal control problems and viscosity theory in the space of probability measures has been an active area of research in the mathematical community in the last years, in particular because of its potential real-world applications in the modeling of multi-agent systems. The potential applications include crowd dynamics modeling Cristiani et al. (2014), opinion formation process modeling Bellomo et al. (2012), herd analysis Topaz and Bertozzi (2004), autonomous multi-vehicle navigation Ren and Beard (2008) and the modeling of uncertainties on the initial state of a deterministic controlled system Cardaliaguet and Quincampoix (2008).

At the individual level, the behavior of each agent is dictated not only by local interactions but also by the *non local* interactions that depend on the distribution of all agents. When the number of agents is assumed to be very large, the complexity of the system grows extremely fast. A suitable way to modelize this problem is through a macroscopic approach, where the discrete collection of agents is replaced by a spatial density that evolves in time. If we assume further that the total number of agents remains constant at all times during the evolution of the system, then one can normalize the density of the agents and assume that its total mass is equal to 1.

Hence, the evolution of the multi-agent system, seen as normalized spatial density in a given base space X (typically the Euclidean space or a Riemannian manifold), is described by a curve $t \mapsto \mu_t \in \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the space of Borel probability measures over X, and μ_t represents the spatial density of the multi-agent system at a given time $t \geq 0$. The conservation of the mass along the trajectory $t \mapsto \mu_t$ is described by the following continuity equation

$$\partial_t \mu_t + div(v_t \mu_t) = 0,$$

where $v_t(.)$ is a time-dependent Borel vector field, and the equation is understood in the sense of distributions.

In this paper, we take the base space $X = M$ to be a compact Riemannian manifold without boundary. We propose to study a simple model of multi-agent systems, where the non local interactions between the agents are not considered. This problem can be interpreted as a deterministic control system with imperfect information on the initial condition, i.e. the initial condition is not known precisely by the controller, but they only know that the initial condition follows a probability distribution $\mu_0 \in \mathcal{P}(M)$. More precisely, consider the following controlled equation:

$$\begin{cases} \dot{Y}(t) = f(Y(t), u(t)), & t \in [t_0, T], \\ Y(t_0) = x_0, & u(t) \in U, \end{cases} \tag{1}$$

where $f : M \times U \to TM$ is the dynamics, assumed to be Lipschitz with respect to the first variable and continuous with respect to the second variable, $x_0 \in M$ and $t_0 \in [0, T]$. The set U is the set of admissible control values which is assumed to be a compact subset of some metric space. The control function $u(.) \in U$ is a Borel measurable function $u : [t_0, T] \to U$. The main feature of this problem is that the initial position x_0 is not perfectly known, but rather distributed along the probability measure μ_0. Notice that, since $f(., u(t))$ is Lipschitz continuous and bounded, the evolution curve of the uncertainty, $t \mapsto \mu_t$ starting from μ_0, is the unique solution to the equation

10.1016/j.ifacol.2022.08.079

$$\begin{cases}\partial_t \mu_t + div(f(.,u(t))\mu_t) = 0,\ t \in (t_0,T),\\ \mu_{t_0} = \mu_0,\end{cases}$$

in the distributional sense. The measures μ_t are obtained by the pushforward of μ_0 by the flow at time t of the controlled equation (1).
The controller aims at minimizing the following final cost:

$$L(\mu) = \int l(y) d\mu(y),$$

where $l : M \to \mathbb{R}$ is a Lipschitz function. An immediate consequence of this assumption is that the function $L : \mathcal{P}(M) \to \mathbb{R}$ inherits the Lipschitz property from l as well. The quantity $L(\mu_T)$ represents the expectation of the deterministic final cost with respect to the measure μ_T.

To this optimal control problem, we associate the following value function:

$$\vartheta(t_0,\mu_0) = \inf_{u(.)\in U} L(\mu_T).$$

A similar problem was studied in Cardaliaguet and Quincampoix (2008) in the context of differential games and in Marigonda and Quincampoix (2018) for Mayer optimal control problems. We stress on the difference between the set of trajectories $t \mapsto \mu_t$ considered here and the set of trajectories $t \mapsto \mu_t$ considered in Marigonda and Quincampoix (2018). Indeed, in the latter case, the minimization is considered over a set of trajectories $t \mapsto \mu_t$ that appears to be larger. The trajectories $t \mapsto \mu_t$ are solutions to the following continuity equation

$$\begin{cases}\partial_t \mu_t + div(v_t\mu_t) = 0,\ t \in (t_0,T),\\ \mu_{t_0} = \mu_0,\end{cases}$$

where $v_t(.)$ is a vector field such that

$$v_t(.) \in \{f(.,u)\ :\ u \in U\}.$$

In this manuscript, we want to study the evolution of the lack of information on the initial condition in (1), modeled by a Borel probability measure μ_0. Hence, we only consider trajectories $t \mapsto \mu_t$ that are obtained by the pushforward of the initial uncertainty μ_0 by the flow at time t of the controlled equation (1).

The first main goal of this paper is to study the properties and the regularity of the value function. In particular we will show that the value function is Lipschitz continuous with respect to both variables and that it verifies the dynamic programming principle. The second goal of the paper is to prove that value function can be characterized as the unique viscosity solution of a suitable Hamilton Jacobi Bellman equation (HJB in short) of the form

$$\begin{cases}\partial_t v + H(\mu, D_\mu v) = 0, \quad (t,\mu) \in [0,T) \times \mathcal{P}_2(M),\\ v(T,\mu) = L(\mu),\end{cases}$$

in the space of probability measures $\mathcal{P}(M)$. In the literature, the study of HJB equations or Hamilton Jacobi equations in general in $\mathcal{P}(M)$, is heavily based on defining a suitable notion of sub/super-differentials to define the notion of viscosity (see for example Cardaliaguet and Quincampoix (2008); Jimenez et al. (2020); Gangbo et al. (2008); Gangbo and Tudorascu (2019)). Our approach is different. We aim at transposing the viscosity theory techniques that are used in the classical theory (Crandall et al. (1992)) to the space of Borel probability measures $\mathcal{P}_2(M)$. In particular, we define a suitable notion of viscosity using a class of real valued functions that admit *directional derivatives* at all points $\mu \in \mathcal{P}_2(M)$. We then prove a local comparison principle between any bounded upper semicontinuous subsolution and any lower semicontinuous supersolution. Finally, we prove that the value function is the unique viscosity solution to the above HJB equation by using the dynamic programming principle verified by the value function.
This paper is expository. All the results asserted in here are proven in Jean et al. (2022). The paper is structured as follows. In Section 2, we recall some classical notions of optimal transport theory and the geometry of the space of probability measures. In Section 3, we formulate the Mayer problem in the space of probability measures and we give the main properties of the value functions. Section 4 is devoted to the study of a suitable HJB equation that characterizes the value function. In particular, we define the Hamiltonian we are going to work with, then we define a notion of viscosity using a class of test functions that are directionally differentiable, we show the comparison principle and we show that the value function is the unique viscosity of the HJB equation.

2. PRELIMINARIES

In this section, we recall some facts about optimal transport and the geometry of Wasserstein spaces. Let $(M, \langle .,. \rangle)$ be a finite dimensional, compact and connected Riemannian manifold without boundary. We denote by $|\,.\,|$ the associated norm on the tangent bundle TM, and by $d(.,.)$ its Riemannian distance on M. The metric space (M,d), is a complete, separable and compact space and its topology is equivalent to the topology of the differentiable manifold M. The tangent bundle TM is itself a complete and separable Riemannian manifold when endowed with the Sasaki metric (Sasaki (1962)). We denote by $d_{TM}(.,.)$ the Riemannian distance on TM associated with the Sasaki metric.

2.1 The Wasserstein space $\mathcal{P}_2(M)$

We denote by $\mathcal{P}(M)$ the set of Borel probability measures over M and $\mathcal{P}_2(M)$ the set of Borel probability measures with bounded second moments:

$$\mathcal{P}_2(M) := \{\mu \in \mathcal{P}(M) : \int d^2(x,x_0) d\mu(x) < \infty,\ \forall x_0 \in M\}$$

Actually, since M is compact, we have $\mathcal{P}_2(M) = \mathcal{P}(M)$ but we will keep using the notation $\mathcal{P}_2(M)$. Recall that for any two topological spaces X and Y, any Borel probability measure μ on X and any Borel function $g : X \to Y$, the pushforward measure $g\sharp\mu$ on Y is defined by

$$g\sharp\mu(A) = \mu(g^{-1}(A)) \quad \forall A \subset Y,\ \text{a Borel set},$$

or equivalently, for all $h : Y \to \mathbb{R}$, a Borel measurable and bounded function, we have:

$$\int h\ dg\sharp\mu = \int h \circ g\ d\mu.$$

We define the Wasserstein distance $W_2(.,.)$ over $\mathcal{P}_2(M)$ by

$$W_2(\mu,\nu) := \sqrt{\inf\Big\{ \int d^2(x,y) d\gamma(x,y) \Big\}},$$

where the infimum is taken over all Borel probability measures of $M \times M$ that have marginals μ and ν, i.e. for all $A, B,$ Borel sets of M, we have

$$\gamma(A \times M) = \mu(A) \quad \text{and} \quad \gamma(M \times B) = \nu(B).$$

Such Borel probability measures are called admissible plans of μ and ν and the set of such plans is denoted $Adm(\mu,\nu)$. It is well known that W_2 verifies all the axioms of a distance and that the infinimum is always reached (Ambrosio and Gigli, 2013, Theorem 1.5). The admissible plans where the minimum is achieved are called optimal transport plans and the set of such plans is denoted $Opt(\mu,\nu) \subset Adm(\mu,\nu)$.
The metric space $(\mathcal{P}_2(M), W_2)$ is complete and separable. Furthermore, it is a geodesic space, i.e. any two points of $\mathcal{P}_2(M)$ can be joined by at least one geodesic. We recall that a curve $\alpha : [0,1] \to \mathcal{P}_2$ is a geodesic if

$$W_2(\alpha_t,\alpha_s) \leq |t-s| W_2(\alpha_0,\alpha_1),\ \forall t,s \in [0,1].$$

2.2 The tangent space $T_\mu(\mathcal{P}_2(M))$

In this subsection, we will adopt a purely metric perspective to define the tangent space of $(\mathcal{P}_2(M), W_2)$ at a given point μ. First, Let $\mathcal{P}(TM)$ be the set of Borel probability measures over TM. Since (TM, d_{TM}) is a complete geodesic space, we can define the Wasserstein space over TM as the set of all $\eta \in \mathcal{P}(TM)$ such that

$$\int d^2_{TM}\Big((x,v),(x_0,v_0)\Big) d\eta(x,v) < \infty,$$

for all $(x_0,v_0) \in TM$. We denote it by $\mathcal{P}_2(TM)$. We endow it with the usual Wasserstein distance for any $\eta,\gamma \in \mathcal{P}_2(TM)$:

$$W_2(\gamma,\eta) := \sqrt{\inf\Big\{ \int d^2_{TM}(x,y) d\beta(x,y) \Big\}},$$

the infinimum is taken over all admissible plans β with marginals γ and η. Notice that we kept the same notation for the Wasserstein distance. It would be clear from the context which base space is considered. We are now able to define the tangent space at a point $\mu \in \mathcal{P}_2(M)$.

Definition 1. (Tangent space). Let $\mu \in \mathcal{P}_2(M)$. The tangent space $T_\mu(\mathcal{P}_2(M)) \subset \mathcal{P}_2(TM)$, is the set of measures $\gamma \in \mathcal{P}_2(TM)$ such that $\pi^M \sharp \gamma = \mu$, where $\pi^M : TM \to M$ is the canonical projection onto M.

The tangent space at a point μ has a geometric meaning. In fact, it encodes all the information about geodesics emanating from μ as we describe hereafter.
Let $\exp : TM \to M$ be the exponential map of $(M, \langle .,. \rangle)$. The exponential $\mathbf{exp}_\mu(\gamma)$ of a measure $\gamma \in T_\mu(\mathcal{P}_2(M))$ is defined by

$$\mathbf{exp}_\mu(\gamma) := \exp \sharp \gamma \in \mathcal{P}_2(M).$$

We define the map $\mathbf{exp}_\mu^{-1} : \mathcal{P}_2(M) \to T_\mu(\mathcal{P}_2(M))$ by

$$\mathbf{exp}_\mu^{-1}(\nu) := \{\, \gamma \in T_\mu(\mathcal{P}_2(M)) \ :\ \mathbf{exp}_\mu(\gamma) = \nu \text{ and } \int |v|^2 \, \mathrm{d}\gamma(x,v) = (W(\mu,\nu))^2 \,\},$$

or in other words, the set of measures $\gamma \in \mathcal{P}_2(TM)$ such that $(\pi^M, \exp)\sharp\gamma$ is an optimal plan from μ to ν and $\int |v|^2 \, \mathrm{d}\gamma(x,v) = (W(\mu,\nu))^2$. We introduce the following notation

$$\Delta_t(x,v) = (x, tv), \quad \forall t \in \mathbb{R},\ (x,v) \in TM.$$

Remark 2. The map $\mathbf{exp}_\mu^{-1}$ is not really an inverse map to $\mathbf{exp}_\mu$ since only the measures $\gamma \in T_\mu(\mathcal{P}_2(M))$ such that $(\pi^M, \exp)\sharp\gamma$ are optimal plans in the inverse image of ν are considered. While this might seem confusing, the map $\mathbf{exp}_\mu^{-1}$ is defined this way so that for all $\gamma \in \mathbf{exp}_\mu^{-1}(\nu)$, the curve $t \to \exp(\Delta_t)\sharp\gamma$ is a geodesic connecting μ and ν, see the theorem below.

Theorem 3. (Gigli, 2011, Theorem 1.11) Let $\mu,\nu \in \mathcal{P}_2(M)$. A curve $(\mu_t) : [0,1] \to \mathcal{P}_2(M)$ is a geodesic connecting μ to ν if and only if there exists a measure $\gamma \in \mathbf{exp}_\mu^{-1}(\nu)$ such that

$$\mu_t := \exp \circ \Delta_t \sharp \gamma, \quad \forall\, t \in [0,1]. \tag{2}$$

The measure γ is uniquely identified by the geodesic. Moreover, for any $t \in (0,1)$ there exists a unique optimal plan from μ to μ_t. Finally, if there exist two different geodesics connecting μ to ν, they do not intersect in intermediate times (i.e. on $(0,1)$).

Introducing the following rescaling of a measure:

$$t \centerdot \gamma = \Delta_t \sharp \gamma, \quad \forall t \in \mathbb{R},\ \gamma \in \mathcal{P}_2(TM),$$

equation (2) can be rewritten in a more elegant way as

$$\mu_t = \exp \circ \Delta_t \sharp \gamma = \mathbf{exp}_\mu(t \centerdot \gamma), \quad \forall t \in [0,1].$$

With the characterization of geodesics in Theorem 3, notice that for any $\mu \in \mathcal{P}_2(M)$, all measures $\gamma \in T_\mu(\mathcal{P}_2(M))$ that produce geodesics, i.e. such that

$$t \mapsto \mathbf{exp}_\mu(t \centerdot \gamma), \tag{3}$$

is a geodesic defined in some right neighborhood of 0, say $[0,\varepsilon]$, can be seen as *initial velocities* of these geodesics. We mention that not all curves of this form are necessarily geodesics but all geodesics are of this form.
Moreover, using this characterization of geodesics, we can define a class of real valued functions $f : \mathcal{P}_2(M) \to \mathbb{R}$ that are *directionally differentiable* along all geodesics. In particular, the squared Wasserstein distance function

$$\mu \mapsto W^2(\mu,\sigma),$$

with $\sigma \in \mathcal{P}_2(M)$ fixed, is directionally differentiable along all geodesics. In fact, a much more general result holds: the squared Wasserstein distance is directionally differentiable along all curves of the form (3) even though they are not geodesics. For more details on this, we refer to Gigli (2011); Ohta (2009). We will only give the following result for the squared Wasserstein distance, which we will need in order to define test functions for viscosity notion in Section 4.

Theorem 4. (Gigli, 2011, Theorem 4.2) Let $\mu,\sigma \in \mathcal{P}_2(M)$, and $g : M \to TM$ be a squared integrable vector field with respect to μ. Let $\gamma = g\sharp\mu \in T_\mu(\mathcal{P}_2(M))$ and $t \mapsto \mathbf{exp}_\mu(t\centerdot\gamma)$ be a curve starting from μ, *not necessarily a geodesic*. Then it holds

$$\frac{d}{dt}\Big|_{t=0} W_2^2(\mathbf{exp}_\mu(t \centerdot \gamma), \sigma)^2 = -2 \sup \int \langle g(x), v \rangle d\zeta(x,v),$$

where the supremum is taken over all $\zeta \in \mathbf{exp}_\mu^{-1}(\sigma)$. We denote it by

$$D_\mu W_2^2(\mu,\sigma) \centerdot (g\sharp\mu) := \frac{d}{dt}\Big|_{t=0} W_2^2(\mathbf{exp}_\mu(t \centerdot \gamma), \sigma)^2,$$

and is understood as the *differential* of $W_2^2(.,\sigma)$ along $g\sharp\mu$.

3. OPTIMAL CONTROL PROBLEM IN $\mathcal{P}_2(M)$

Let $T > 0$ and U be a compact subset of a metric space. Consider the dynamics, defined for $T > t_0 \geq 0$ and $x_0 \in M$, as

$$\begin{cases}\dot{Y}(t) = f(Y(t), u(t)), \quad t \in [t_0, T], \\ Y(t_0) = x_0, u(t) \in U, \end{cases} \tag{4}$$

where $f : M \times U \to TM$ satisfies the following assumptions:

(H) $\begin{cases} f : M \times U \to TM \text{ is continuous and Lipschitz} \\ \text{ continuous with respect to the state, i.e.} \\ \exists k > 0 : d_{TM}(f(x,u), f(y,u)) \leq k\, d(x,y), \\ \forall u \in U,\ (x,y) \in M \times M. \end{cases}$

$\mathbf{(H)}_{co}$: for all $x \in M$, the set

$$f(x, U) := \{f(x,u) \ : \ u \in U\} \text{ is convex.}$$

We define the set of open-loop controls by

$$\mathcal{U} := \{u : [0,T] \to U \ : \ u(.) \text{ is measurable}\}.$$

Under the assumption **(H)**, classical results of ordinary differential equations hold. In particular, for any control $u(.) \in \mathcal{U}$ and $x_0 \in M$, there exists a unique Lipschitz trajectory $t \mapsto Y_t^{t_0,x_0,u}$ defined on all $[t_0, T]$. Moreover, we have the following estimates:

Proposition 5. There exist positive constants $C_1, C_2 > 0$ such that for all $x_0, z_0 \in M$, for all $t_0 \in [0,T]$, and $t \mapsto Y_t^{t_0,x_0,u}$, $t \mapsto Y_t^{t_0,z_0,u}$ be solutions of (4), it holds:

$$d(Y_T^{t_0,x_0,u}, Y_T^{t_0,z_0,u}) \leq C_1\, d(x_0, z_0),$$
$$d(Y_t^{t_0,x_0,u}, x_0) \leq C_2\, |t - t_0|, \quad t \in [t_0, T].$$

The control problem aims at minimizing the final cost

$$L(\mu_0) = \int l(Y_T^{t_0,x_0,u})\, d\mu_0(x_0),$$

over all trajectories that are solutions of the dynamics (4) with the initial condition $x_0 \in M$, distributed along the measure $\mu_0 \in \mathcal{P}_2(M)$. We consider the following assumption:

(H_l) $l : M \to \mathbb{R}$ is Lipschitz continuous with constant $Lip(l)$.

When μ_0 is equal to the Dirac mass δ_{x_0}, the resulting system corresponds to the classical case without uncertainties on the initial condition. This problem is thoroughly studied in the literature. When μ_0 is any probability measure of $\mathcal{P}_2(M)$, it is better to see this problem as an optimal control problem defined on the space of Borel probability measures $\mathcal{P}_2(M)$. First we rewrite the final cost the following way

$$L(\mu_0) = \int l(Y_T^{t_0,x_0,u})\, d\mu_0(x_0) = \int l(y)\, dY_T^{t_0,.,u}\sharp\mu_0\,(y),$$

and we minimize this cost over the set of trajectories $t \mapsto \mu_t^{\mu_0,u}$ of the space $\mathcal{P}_2(M)$ that verify

$$\begin{cases} \mu_t^{\mu_0,u} = Y_t^{t_0,.,u}\sharp\mu_0, \quad t \in [t_0, T], \text{ and } x \mapsto Y_t^{t_0,x,u} \\ \text{is the flow of (4)}, \\ \mu_{t_0} = \mu_0. \end{cases}$$

Since $x \mapsto Y_t^{t_0,x,u} \in M$ and $\mu_0 \in \mathcal{P}_2(M)$, then $t \mapsto Y_t^{t_0,.,u}\sharp\mu_0 \in \mathcal{P}_2(M)$ for all $t \in [t_0, T]$. It is a known fact, Ambrosio et al. (2008); Bernard (2008), that each trajectory $t \mapsto \mu_t^{\mu_0,u}$ is the unique solution to the following continuity equation

$$\begin{cases} \partial_t \mu_t^{\mu_0,u} + div(f(.,u(t))\mu_t^{\mu_0,u}) = 0,\ t \in (t_0, T) \\ \mu_{t_0}^{\mu_0,u} = \mu_0. \end{cases} \tag{5}$$

In the distributional sense. Hence the optimal control problem can be rewritten in the following way:

$$\begin{cases} \min\limits_{u(.)\in U} \int l(y) d\mu_T^{\mu_0,u}(y), \\ \text{such that } \begin{cases} \partial_t \mu_t^{\mu_0,u} + div(f(.,u(t))\mu_t^{\mu_0,u}) = 0, \\ \mu_{t_0}^{\mu_0,u} = \mu_0,\ t \in (t_0, T). \end{cases} \end{cases} \tag{6}$$

The associated *value function* to the above optimal control problem is defined as

$$\vartheta(t_0, \mu_0) := \inf_{u(.)\in\mathcal{U}} \int l(y)\, d\mu_T^{\mu_0,u}(y).$$

Under hypotheses **(H)** and **(H_l)**, we have the following two properties of the value function.

Proposition 6. (Jean et al. (2022)). Assume **(H)** and **(H_l)**. Then, the value function ϑ is Lipschitz continuous on $[0,T] \times \mathcal{P}_2(M)$. In particular, ϑ is bounded.

Theorem 7. (Jean et al. (2022)). Let $\mu \in \mathcal{P}_2(M)$, $t \in [0,T]$ and $h \in [t, T-t]$. Then it holds

$$\vartheta(t, \mu) = \inf_{u\in\mathcal{U}} \vartheta(t + h, \mu_{t+h}^{\mu,u}).$$

In the classical theory of viscosity solution, the value function is the unique viscosity solution of the Hamilton Jacobi Bellman equation. The goal of the next section is to show that the value function, in this setting, is also a viscosity solution to a Hamilton Jacobi Bellman equation of the form

$$\begin{cases} \partial_t v + H(\mu, D_\mu v) = 0, \quad (t, \mu) \in [0,T) \times \mathcal{P}_2(M), \\ v(T, \mu) = L(\mu), \end{cases}$$

where the Hamiltonian and test functions will be defined precisely.

4. HJB EQUATION IN $\mathcal{P}_2(M)$

We have defined all the elements we need to give a precise definition of the Hamiltonian, viscosity solutions and test functions. In this section, we prove that the value function is the unique viscosity solution to an HJB equation. The Hamiltonian we will work with has the following expression:

$$H(\mu, D_\mu v(t,\mu)) = \inf_{u\in U} D_\mu v(t,\mu) \cdot \Big(f(.,u)\sharp\mu\Big), \tag{7}$$

with $v : \mathbb{R} \times \mathcal{P}_2(M) \to \mathbb{R}$ is a real valued function that admits directional derivatives along the time variable and the measure variable. The function v represents a test function that has the following form:

$$\forall (t,\mu) \in [0,T) \times \mathcal{P}_2(M),\ v(t,\mu) = \psi(t) + a\, W_2^2(\mu, \sigma),$$

with $a \in \mathbb{R}$ and $\sigma \in \mathcal{P}_2(M)$ fixed and $\psi : [0,T) \to \mathbb{R}$ is a continuously differentiable function, $\mu \mapsto W_2^2(\mu, \sigma)$ is directionally differentiable in the sense of Theorem 4. We consider the following Hamilton Jacobi Bellman equation:

$$\begin{cases} \partial_t v + H(\mu, D_\mu v) = 0, \quad (t, \mu) \in [0,T) \times \mathcal{P}_2(M), \\ v(T, \mu) = L(\mu). \end{cases} \tag{8}$$

Next, we define the test functions that we are going to use to define the notion of viscosity solutions.

Definition 8. (Test functions).
Let $\mathcal{TEST}_1$ be the set defined as:

$$\mathcal{TEST}_1 := \{(t,\mu) \mapsto \psi(t) + a(\,(W_2^2(\mu,\sigma))\,) \ : \ a \in \mathbb{R}^+, \sigma \in \mathcal{P}_2(M)\},$$

where $\psi : [0,T) \to \mathbb{R}$ is a C^1 function.
We set $\mathcal{TEST}_2 = -\mathcal{TEST}_1 := \{-\phi \ : \ \phi \in \mathcal{TEST}_1\}$.

Definition 9. (Viscosity solutions).

- We say that a function $v : [0,T) \times \mathcal{P}_2(M) \to \mathbb{R}$ satisfies the inequality
$$\partial_t v + H(\mu, D_\mu v) \geq 0,$$
at $(t,\mu) \in [0,T) \times \mathcal{P}_2(M)$ in the viscosity sense if v is upper semicontinuous and for all $\mathcal{TEST}_1$ functions $\phi : [0,T) \times \mathcal{P}_2(M) \to \mathbb{R}$ such that $v - \phi$ attains a maximum at (t,μ), we have
$$\partial_t \phi + H(\mu, D_\mu \phi) \geq 0.$$
A function v satisfying $\partial_t v + H(\mu, D_\mu v) \geq 0$ on $[0,T) \times \mathcal{P}_2(M)$ in the viscosity sense is called a *viscosity subsolution* of (8).
- Similarly, we say that a function $v : [0,T) \times \mathcal{P}_2(M) \to \mathbb{R}$ satisfies the inequality
$$\partial_t v + H(\mu, D_\mu v) \leq 0,$$
at $(t,\mu) \in [0,T) \times \mathcal{P}_2(M)$ in the viscosity sense if v is lower semicontinuous and for all $\mathcal{TEST}_2$ functions $\phi : [0,T) \times \mathcal{P}_2(M) \to \mathbb{R}$ such that $v - \phi$ attains a minimum at (t,μ), then
$$\partial_t \phi + H(\mu, D_\mu \phi) \leq 0.$$
A function v satisfying $\partial_t v + H(\mu, D_\mu v) \leq 0$ on $[0,T) \times \mathcal{P}_2(M)$ in the viscosity sense is called a *viscosity supersolution* of (8).
- We say that a continuous function $v : [0,T] \times \mathcal{P}_2(M) \to \mathbb{R}$ is a *viscosity solution* of (8) if it is both a supersolution and a subsolution on $[0,T) \times \mathcal{P}_2(M)$ and verifies
$$v(T,\mu) = L(\mu).$$

In the literature, the study of viscosity solutions for HJB equations or general Hamilton Jacobi equations relies heavily on defining a suitable notion of sub/superdifferentials and in defining a notion of smooth functions in Wasserstein space in an extrinsic way using the so-called Lions calculus (see for example Cardaliaguet and Quincampoix (2008); Jimenez et al. (2020); Gangbo et al. (2008); Gangbo and Tudorascu (2019)). In all these approaches, the squared Wasserstein distance is not included in the set of test functions considered, which makes them unsuitable to extend the common viscosity techniques known in the classical theory. This is the reason why in here, we adopt a different approach. We define the viscosity notion using test functions that are directionally differentiable along both the time variable and the measure variable. Furthermore, using this approach of test functions, we can prove the comparison principle for equation (8) for any bounded upper semicontinuous subsolution and bounded lower semicontinuous supersolution. First, we need two key results.

Proposition 10. (Jean et al. (2022)). For all $\sigma, \mu \in \mathcal{P}_2(M)$ and $a > 0$, we have:
$$H(\mu, a\, D_\mu W_2^2(\mu,\sigma)) - H(\sigma, -a\, D_\sigma W_2^2(\mu,\sigma)) \leq 2a\, Lip(f)\, W_2^2(\mu,\sigma).$$

Remark 11. The above result is of fundamental importance to prove the comparison principle. Indeed, it allows us to use the variable doubling technique without having any regularity assumptions on the Hamiltonian. Furthermore, this result does not need any compactness assumptions on the Wasserstein space. The proof can also be adapted if for example the base space is the Euclidean space $\mathbb{R}^N$, rather than the compact manifold M. This is due to the fact that the squared Wasserstein distance in $\mathcal{P}_2(\mathbb{R}^N)$ is directionally differentiable and has an expression similar to the one given in Theorem 4 (see (Ambrosio et al., 2008, Theorem 7.3.2 and Proposition 7.3.6)).

Theorem 12. (Jean et al. (2022)). Assume $\boldsymbol{(H)}$ and $\boldsymbol{(H_l)}$. Let $u, v : [0,T] \times \mathcal{P}_2(M) \to \mathbb{R}$ be respectively bounded upper semicontinuous subsolution and lower semicontinuous supersolution on $[0,T] \times \mathcal{P}_2(M)$. Then it holds:
$$\sup_{[0,T]\times\mathcal{P}_2(M)} (v-w)_+ \leq \sup_{\{T\}\times\mathcal{P}_2(M)} (v-w)_+,$$
where $(a)_+ = max(a,0)$.

Theorem 13. (Jean et al. (2022)). Assume $\boldsymbol{(H)}$, $\boldsymbol{(H_l)}$ and $\boldsymbol{(H_{co})}$. Then the value function ϑ is the unique continuous viscosity solution to (8).

To prove that the value function is a viscosity solution to (8), we use the dynamic programming principle asserted in Theorem 7, whereas uniqueness comes from the comparison principle asserted in Theorem 12.

REFERENCES

Ambrosio, L. and Gigli, N. (2013). A user's guide to optimal transport. In *Modelling and optimisation of flows on networks*, 1–155. Springer.

Ambrosio, L., Gigli, N., and Savaré, G. (2008). *Gradient flows: in metric spaces and in the space of probability measures*. Springer Science & Business Media.

Bellomo, N., Herrero, M.A., and Tosin, A. (2012). On the dynamics of social conflicts: looking for the black swan. *arXiv preprint arXiv:1202.4554*.

Bernard, P. (2008). Young measures, superposition and transport. *Indiana University mathematics journal*, 247–275.

Cardaliaguet, P. and Quincampoix, M. (2008). Deterministic differential games under probability knowledge of initial condition. *International Game Theory Review*, 10(01), 1–16.

Crandall, M.G., Ishii, H., and Lions, P.L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American mathematical society*, 27(1), 1–67.

Cristiani, E., Piccoli, B., and Tosin, A. (2014). *Multiscale modeling of pedestrian dynamics*, volume 12. Springer.

Gangbo, W., Nguyen, T., Tudorascu, A., et al. (2008). Hamilton-jacobi equations in the wasserstein space. *Methods and Applications of Analysis*, 15(2), 155–184.

Gangbo, W. and Tudorascu, A. (2019). On differentiability in the wasserstein space and well-posedness for hamilton–jacobi equations. *Journal de Mathématiques Pures et Appliquées*, 125, 119–174.

Gigli, N. (2011). On the inverse implication of brenier-mccann theorems and the structure of (p 2 (m), w 2). *Methods and Applications of Analysis*, 18(2), 127–158.

Jean, F., Jerhaoui, O., and Zidani, H. (2022). Deterministic optimal control on riemannian manifolds under probability knowledge of the initial condition. *https://hal.archives-ouvertes.fr/hal-03564787/*.

Jimenez, C., Marigonda, A., and Quincampoix, M. (2020). Optimal control of multiagent systems in the wasserstein space. *Calculus of Variations and Partial Differential Equations*, 59(2), 1–45.

Marigonda, A. and Quincampoix, M. (2018). Mayer control problem with probabilistic uncertainty on initial positions. *Journal of Differential Equations*, 264(5), 3212–3252.

Ohta, S.i. (2009). Gradient flows on wasserstein spaces over compact alexandrov spaces. *American journal of mathematics*, 131(2), 475–516.

Ren, W. and Beard, R.W. (2008). *Distributed consensus in multi-vehicle cooperative control*, volume 27. Springer.

Sasaki, S. (1962). On the differential geometry of tangent bundles of riemannian manifolds ii. *Tohoku Mathematical Journal, Second Series*, 14(2), 146–155.

Topaz, C.M. and Bertozzi, A.L. (2004). Swarming patterns in a two-dimensional kinematic model for biological groups. *SIAM Journal on Applied Mathematics*, 65(1), 152–174.