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Mixed subdivisions suitable for the greedy Canny-Emiris formula

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Abstract The Canny-Emiris formula [4] gives the sparse resultant as a ratio between the determinant of a Sylvester-type matrix and a minor of it, by a subdivision algorithm. The most complete proof of the formula was given by D’Andrea, Sombra and Jerónimo in [11] under general conditions on the underlying mixed subdivision. Well before this proof, Canny and Pedersen had proposed [6] a greedy algorithm which provides smaller matrices, in general. The goal of this paper is to give an explicit class of mixed subdivisions for the greedy approach such that the formula holds, and the dimensions of the matrices are reduced compared to the subdivision algorithm. We measure this reduction for the case when the Newton polytopes are zonotopes generated by n line segments (where n is the rank of the underlying lattice), and for the case of multihomogeneous systems. This article comes with a *Julia* implementation of the treated cases.

Keywords Combinatorics, resultant theory, mixed subdivision, zonotopes, tropical geometry

1 Introduction

Sparse resultants offer a standard and efficient way of studying algebraic systems while exploiting their structure. They have applications in elimination and implicitization theory and many other areas of algebraic geometry. We examine matrix-based methods for expressing and computing this resultant. This paper contains the the results of [7] which was published in the proceedings of ISSAC’22 and provides a bigger family of examples. We also add a new result in Section 2 with which we give a broader view of the subdivision refinement in Theorem 2 through tropical geometry.

The Canny-Emiris formula was conjectured in [4] as a rational formula for the sparse resultant that generalizes Macaulay’s classic formula in [17]. It gives a combinatorial construction of a Sylvester-type matrix $\mathcal{H}_{\mathcal{A},\rho}$ depending on the family of supports $\mathcal{A} = (\mathcal{A}_0, \dots, \mathcal{A}_n)$ in a lattice M of rank n , and a mixed subdivision $S(\rho)$ defined from a lifting function ρ on the Minkowski sum Δ of the Newton polytopes $\Delta_i = \text{conv}(\mathcal{A}_i)$. Each row of this matrix corresponds to a lattice point $b \in M$ contained in a translation δ of the polytope Δ . Moreover, to each lattice point we can associate a type vector $t_b = (t_{b,0}, \dots, t_{b,n})$ corresponding to the dimensions of the components $D_i \subset \Delta_i$ of the cell $D \in S(\rho)$ in which b is lying. Imposing that

$$\sum_{i=0}^n t_{b,i} = n \quad \forall b \in (\Delta + \delta) \cap M,$$

it is possible to build such matrix $\mathcal{H}_{\mathcal{A},\rho}$ and a principal submatrix $\mathcal{E}_{\mathcal{A},\rho}$ of $\mathcal{H}_{\mathcal{A},\rho}$ so that the sparse resultant can be expressed in the form:

$$\text{Res}_{\mathcal{A}} = \frac{\det(\mathcal{H}_{\mathcal{A},\rho})}{\det(\mathcal{E}_{\mathcal{A},\rho})}.$$

D’Andrea, Jerónimo, and Sombra proved this formula in [11] under the assumption that $S(\rho)$ admits an incremental chain of mixed subdivisions:

$$S(\theta_0) \preceq \dots \preceq S(\theta_n) \preceq S(\rho)$$

satisfying some combinatorial properties, where \preceq denotes that $S(\theta_i)$ refines $S(\theta_{i-1})$, namely each cell $D \in S(\theta_i)$ is contained in a cell of $S(\theta_{i-1})$. This proof extended the first proof given by D'Andrea in [9] for generalized unmixed systems.

On the other hand, Canny and Pedersen gave in [6] another approach for the construction of the matrix $\mathcal{H}_{\mathcal{A},\rho}$. Starting at a lattice point $b \in M$, one can construct the matrix by only adding the rows corresponding to the columns that have a nonzero entry in a previously considered row. This is a greedy way of understanding these matrices: their construction only considers the strictly necessary rows and columns given the mixed subdivision $S(\rho)$.

This paper aims to give a family of lifting functions ρ that correspond to mixed subdivisions for which:

- i) the proof of the formula in [11] holds and
- ii) the size of the matrices is reduced.

The family that we propose is associated with a vector $v \in \text{Hom}(M_{\mathbb{R}}, \mathbb{R})$ outside the hyperplane arrangement associated with the polytope Δ .

We expect this family of lifting functions to reduce the size of the Canny-Emiris matrices obtained by the greedy algorithm for a general sparse system. We measure this reduction only for the case where the Newton polytopes are zonotopes generated by n independent line segments. Namely, we simplify the computations to the case where the supports $\mathcal{A}_0, \dots, \mathcal{A}_n$ are:

$$\mathcal{A}_i = \{(b_j)_{j=1, \dots, n} \in \mathbb{Z}^n \mid 0 \leq b_j \leq a_{ij}\} \quad i = 0, \dots, n$$

assuming that $0 < a_{0j} \leq \dots \leq a_{n-1j}$ for all $j = 1, \dots, n$. The main results of this paper are Theorem 5 and Theorem 6 and show that the greedy algorithm will end by reaching only those lattice points with type vector t_b satisfying:

$$\sum_{i=0}^I t_{b,i} \leq I + 1 \quad \forall I < n.$$

To find all the lattice points in cells with a given type vector, we introduce the type functions:

$$\varphi_b : \{1, \dots, n\} \rightarrow \{0, \dots, n\} \quad t_{b,i} = |\varphi_b^{-1}(i)|.$$

These combinatorial objects contain all the information of the cells of the mixed subdivision and can help us construct the matrices. In Corollary 4, we give a combinatorial measure of the number of rows of the matrix $\mathcal{H}_{\mathcal{G}}$ given by the greedy algorithm as:

$$\sum_{\varphi_b: \{1, \dots, n\} \rightarrow \{0, \dots, n\}} \prod_{j=1}^n a_{\varphi_b(j)j}$$

where φ_b satisfies $|\varphi_b^{-1}(\{0, \dots, I\})| \leq I + 1$. This result is not optimal amongst all the possible mixed subdivisions of an n -zonotopesystem, but it might be amongst the ones given by affine lifting functions; see Examples 1, 6.

We show that some multihomogeneous resultant matrices can be seen as an instance of the previous case by embedding their Newton polytopes and the mixed subdivisions of their Minkowski sum into an n -zonotope. Moreover, we add restrictions to the type functions so that they also count the size of these matrices; see Corollary 5. Despite the existence of many exact determinantal formulas for some of these cases; see [3, 2, 14, 22, 5], we expect our approach to have an easier generalization to general sparse systems through the use of the type functions and the underlying combinatorics.

The paper is organized as follows. In Section 1, we summarize the proof of the Canny-Emiris formula in [11] and we explain the greedy approach in [6]. In Section 2, a concrete family of mixed subdivisions is given by considering a hyperplane arrangement associated with the Newton polytopes and it is proved that the Canny-Emiris formula holds in this case. Section 3 is devoted to combinatorially finding the size of the Canny-Emiris matrices when the Newton polytopes are zonotopes generated by n line segments. In Section 4, the multihomogeneous resultant is seen as an instance of the previous case. Moreover, we count the number of lattice points that the greedy approach gives on these mixed subdivisions in a combinatorial way. In the conclusions, we present a list of possible ways to generalize our measure of the

Canny-Emiris matrices to other sparse systems.

This article comes with a *JULIA* implementation of the treated cases (n -zonotopes and multihomogeneous systems); see <https://github.com/carleschecanualart/CannyEmiris>. More than improving the existing formulas (which give, in general, smaller Sylvester matrices), the goal of this implementation is to introduce the type functions in the construction of the matrices.

1.1 The Canny-Emiris formula

Let M be a lattice of rank n and $M_{\mathbb{R}} = M \otimes \mathbb{R}$ the corresponding real vector space. Let $N = \text{Hom}(M, \mathbb{Z})$ be its dual and $\mathbb{T}_N = N \otimes \mathbb{C}^\times$ the underlying torus. Let $\mathcal{A}_0, \dots, \mathcal{A}_n \subset M$ be a family of supports corresponding to the polynomials:

$$F_i = \sum_{a \in \mathcal{A}_i} u_{i,a} \chi^a \in \mathbb{Z}[u_{i,a}][M] \quad a \in \mathcal{A}_i \quad i = 0, \dots, n$$

where χ^a are the characters in \mathbb{T}_N of the lattice points $a \in \mathcal{A}_i$. Let $\Delta_i = \text{conv}(\mathcal{A}_i) \subset M_{\mathbb{R}}$ for $i = 0, \dots, n$ be the convex hulls of the supports, also known as Newton polytopes, and Δ their Minkowski sum in $M_{\mathbb{R}}$.

The incidence variety $Z(\mathbf{F})$ is defined as the zero set in $\mathbb{T}_N \times \prod_{i=0}^n \mathbb{P}^{\mathcal{A}_i}$ of the polynomials $\mathbf{F} = (F_0, \dots, F_n)$. Denote by $\pi : \mathbb{T}_N \times \prod_{i=0}^n \mathbb{P}^{\mathcal{A}_i} \rightarrow \prod_{i=0}^n \mathbb{P}^{\mathcal{A}_i}$ the projection onto the second factor and let $\pi_*(Z(\mathbf{F}))$ be the direct image of the zero set of F_0, \dots, F_n .

Definition 1 The sparse resultant, denoted as $\text{Res}_{\mathcal{A}}$, is any primitive polynomial in $\mathbb{Z}[u_{i,a}]$ defining the direct image $\pi_*(Z(\mathbf{F}))$.

There are some lattice operations that can help us simplify the computation of these objects.

Lemma 1 [11, Proposition 3.2] Let $\phi : M \rightarrow M'$ be a monomorphism of lattices of rank n . Then, $\text{Res}_{\phi(\mathcal{A})} = \text{Res}_{\mathcal{A}}^{[M':\phi(M)]}$.

Remark 1 Moreover, the sparse resultant is invariant under translations. Therefore, we can always assume $0 \in \mathcal{A}_i$ for all $i = 0, \dots, n$.

Definition 2 A mixed subdivision of Δ is a decomposition of this polytope in a union of cells $\Delta = \cup D$ such that:

- i) the intersection of two cells is either a cell or empty,
- ii) every face of a cell is also a cell of the subdivision and,
- iii) every cell D has a component structure $D = D_0 + \dots + D_n$ where D_i is a cell of the subdivision in Δ_i .

The usual way to construct mixed subdivisions is by considering piecewise affine convex lifting functions $\rho_i : \Delta_i \rightarrow \mathbb{R}$ as explained in [16]. A global lifting function $\rho : \Delta \rightarrow \mathbb{R}$ is obtained after taking the inf-convolution of the previous functions, as explained in [11, Sec. 2].

Definition 3 A mixed subdivision of Δ is tight if, for every n -cell D , its components satisfy:

$$\sum_{i=0}^n \dim D_i = n.$$

In the case of $n + 1$ polynomials and n variables, this property guarantees that every n -cell has a component that is 0-dimensional. The cells that have a single 0-dimensional component are called mixed (i -mixed if it is the i -th component). The rest of the cells are called non-mixed.

Let δ be a generic vector such that the lattice points in the interior of $\Delta + \delta$ lie in n -cells. Then, consider:

$$\mathcal{B} = (\Delta + \delta) \cap M.$$

Each element $b \in \mathcal{B}$ lies in one of these translated cells $D + \delta$ and let D_i be the components of this cell. As the subdivision is tight, there is at least one i such that $\dim D_i = 0$.

Following the language of [19], we call $t_b = (t_{b,0}, \dots, t_{b,n})$ the type vector associated with b , defined as $t_{b,i} = \dim D_i$ for $b \in D + \delta$.

Definition 4 The row content is a function

$$\text{rc} : \mathcal{B} \rightarrow \cup_{i=0}^n \{i\} \times \mathcal{A}_i$$

where, for $b \in \mathcal{B}$ lying in an n -cell D , $\text{rc}(b)$ is a pair $(i(b), a(b))$ with $i(b) = \max\{i \in \{0, \dots, n\} \mid t_{b,i} = 0\}$ and $a(b) = D_{i(b)}$.

This provides a partition of \mathcal{B} into subsets:

$$\mathcal{B}_i = \{b \in \mathcal{B} \mid i(b) = i\}.$$

Finally, we construct the Canny-Emiris matrices $\mathcal{H}_{\mathcal{A},\rho}$ whose rows correspond to the coefficients of the polynomials $\chi^{b-a(b)} F_{i(b)}$ for each of the $b \in \mathcal{B}$. In particular, the entry corresponding to a pair $b, b' \in \mathcal{B}$ is:

$$\mathcal{H}_{\mathcal{A},\rho}[b, b'] = \begin{cases} u_{i(b), b'-b+a(b)} & b' - b + a(b) \in \mathcal{A}_i \\ 0 & \text{otherwise} \end{cases}$$

Remark 2 Each entry contains, at most, a single coefficient $u_{i,a}$. In particular, the row content allows us to choose a maximal submatrix of $\mathcal{H}_{\mathcal{A},\rho}$ from the matrix of a map sending a tuple of polynomials (G_0, \dots, G_n) to $G_0 F_0 + \dots + G_n F_n$. These class of matrices are called Sylvester-type matrices.

Let $\mathcal{C} \subset \mathcal{B}$ be a subset of the supports in translated cells. The matrix $\mathcal{H}_{\mathcal{A},\rho,\mathcal{C}}$ is defined by considering the submatrix of the corresponding rows and columns associated with elements in \mathcal{C} . In particular, we look at the set of lattice points lying in translated non-mixed cells and consider:

$$\mathcal{B}^\circ = \{b \in \mathcal{B} \mid b \text{ lies in a translated non-mixed cell}\}.$$

With this, we form the principal submatrix:

$$\mathcal{E}_{\mathcal{A},\rho} = \mathcal{H}_{\mathcal{A},\rho,\mathcal{B}^\circ}$$

The Canny-Emiris conjecture states that the sparse resultant is the quotient of the determinants of these two matrices:

$$\text{Res}_{\mathcal{A}} = \frac{\det(\mathcal{H}_{\mathcal{A},\rho})}{\det(\mathcal{E}_{\mathcal{A},\rho})}.$$

This result was conjectured by Canny and Emiris and proved by D'Andrea, Jerónimo, and Sombra; see [4, 11], under the restriction that the mixed subdivision $S(\rho)$ given by the lifting ρ satisfies a certain condition. This condition is given on a chain of mixed subdivisions.

Definition 5 Let $S(\phi), S(\psi)$ be two mixed subdivisions of $\Delta = \sum_{i=0}^n \Delta_i$. We say that $S(\psi)$ refines $S(\phi)$ and write $S(\phi) \preceq S(\psi)$ if for every cell $C \in S(\psi)$ there is a cell $D \in S(\phi)$ such that $C \subset D$. An incremental chain of mixed subdivisions $S(\theta_0) \preceq \dots \preceq S(\theta_n)$ is a chain of mixed subdivisions of Δ refining each other.

Remark 3 In [11, Definition 2.4], a common lifting function $\omega \in \prod_{i=0}^n \mathbb{R}^{\mathcal{A}_i}$ is considered and the $S(\theta_i)$ are given by the lifting functions $\omega^{<i} = (\omega_0, \dots, \omega_{i-1}, 0)$ as long as $S(\theta_i) \preceq S(\theta_{i+1})$. The last zero represents the lifting on $(\Delta_i, \dots, \Delta_n)$. The resulting mixed subdivision is the same as if we considered the zero lifting in $\sum_{j=i}^n \Delta_j$.

Definition 6 The mixed volume of n polytopes $P_1, \dots, P_n \subset M_{\mathbb{R}}$, denoted as $\text{MV}_M(P_1, \dots, P_n)$, is the coefficient of $\prod_{i=1}^n \lambda_i$ in:

$$\text{Vol}_n(\lambda_1 P_1 + \dots + \lambda_n P_n)$$

which is a polynomial in $\lambda_1, \dots, \lambda_n$ [8, Theorem 6.7].

Proposition 1 [15, Theorem 3.4] Let $S(\rho)$ be a tight mixed subdivision of $\Delta = (\Delta_0, \dots, \Delta_n)$. For $i = 0, \dots, n$, the mixed volume of all the polytopes except Δ_i equals the volume of the i -mixed cells.

$$\text{MV}(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n) = \sum_{D \text{ } i\text{-mixed}} \text{Vol}_n D$$

In particular, $MV(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n)$ equals the degree of the sparse resultant in the coefficients of F_i ; see [8, Chapter 7, Theorem 6.3]. Each of the rows of $\mathcal{H}_{\mathcal{A}, \rho}$ will correspond to a lattice point b and each entry on that row will have degree 1 with respect to the coefficients of $F_{i(b)}$ and zero with respect to the coefficients of the rest of polynomials. Therefore, if we add the lattice points in i -mixed cells, the degree of $\mathcal{H}_{\mathcal{A}, \rho}$ with respect to the coefficients of F_i will be at least the degree of the resultant with respect to the same coefficients.

Definition 7 [11, Definition 3.4] The fundamental subfamily of \mathcal{A} is the minimal family of supports $\mathcal{A}_I = (\mathcal{A}_i)_{i \in I}$ such that the resultant has positive degree with respect to the coefficients of F_i for $i \in I$. This definition can be given in other equivalent terms as shown in [21, Corollary 1.1].

Remark 4 Using Proposition 1, we can see that if the fundamental subfamily is empty, then the resultant is equal to 1 while if the fundamental subfamily is $\{i\}$ then \mathcal{A}_i is given by a single point $\{a\}$ and the resultant is $u_{i,a}^{m_i}$ for $m_i = MV(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n)$. The Canny-Emiris formula holds naturally [11, Proposition 4.26] in both cases.

Definition 8 An incremental chain $S(\theta_0) \preceq \dots \preceq S(\theta_n)$ is admissible if for each $i = 0, \dots, n$, each n -cell D of the subdivision $S(\theta_i)$ satisfies either of the following two conditions

- i) the fundamental subfamily of \mathcal{A}_D contains at most one support or
- ii) $\mathcal{B}_{D,i}$ is contained in the union of the translated i -mixed cells of $S(\rho_D)$.

A mixed subdivision $S(\rho)$ is called admissible if it admits an admissible incremental chain $S(\theta_0) \preceq \dots \preceq S(\theta_n) \preceq S(\rho)$ refining it.

With all these properties, together with the use of the product formulas, see [21, 12, 11], one can reproduce the proof of the Canny-Emiris formula given in [11, Theorem 4.27] under the conditions of admissibility in $S(\rho)$.

1.2 The greedy algorithm

Using the previous notation, we state the greedy algorithm in [6] for the construction of the matrix. Let $b \in \mathcal{B}$ be a lattice point in a translated cell. The first step of the algorithm is to add the row of the matrix corresponding to b , and then continue by considering the lattice points corresponding to the columns that have a nonzero entry in this row. These lattice points are:

$$b - a(b) + \mathcal{A}_{i(b)}.$$

All these lattice points will have to be added as rows of the matrix. If we add the lattice point b' at some point of the algorithm after having added another lattice point b , we say that we *reach* b' from b . The algorithm terminates when there are no more lattice points to add and it might give a square matrix $\mathcal{H}_{\mathcal{G}}$ which has less rows and columns than $\mathcal{H}_{\mathcal{A}, \rho}$, which was constructed using all the lattice points in \mathcal{B} . The rows and columns associated to lattice points in non-mixed cells also provide a minor $\mathcal{E}_{\mathcal{G}}$ of $\mathcal{H}_{\mathcal{G}}$.

It was not proved by Canny and Pedersen whether this approach would always include all the lattice points in mixed cells as rows of the matrix, independently of the starting point. As these points are necessary to achieve the degree of the resultant, see Proposition 1, we consider them to be the starting points of the algorithm.

Remark 5 We know that the entry corresponding to the diagonal of the matrix $\mathcal{H}_{\mathcal{A}, \rho, \mathcal{C}}$ will be $\prod_{b \in \mathcal{C}} u_{i(b), a(b)}$ for any subset $\mathcal{C} \subset \mathcal{B}$. This term can be used in order to deduce that these matrices have non-zero determinant; see [11, Proposition 4.13].

Theorem 1 *If the Canny-Emiris formula holds for a mixed subdivision $S(\rho)$ and the greedy algorithm provides matrices $\mathcal{H}_{\mathcal{G}}$ and $\mathcal{E}_{\mathcal{G}}$ by starting at the lattice points in mixed cells, then:*

$$\text{Res}_{\mathcal{A}} = \frac{\det(\mathcal{H}_{\mathcal{G}})}{\det(\mathcal{E}_{\mathcal{G}})}.$$

Proof In general, there is a subset $\mathcal{G} \subset \mathcal{B}$ corresponding to the rows and columns of $\mathcal{H}_{\mathcal{G}}$. We are assuming that \mathcal{G} contains all the lattice points in translated mixed cells. Let $\mathcal{H}_{\mathcal{A},\rho}$ be the matrix containing all lattice points in translated cells of $S(\rho)$. Without loss of generality, we can assume that the matrix takes the following form:

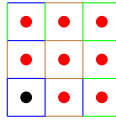
$$\mathcal{H}_{\mathcal{A},\rho} = \begin{pmatrix} \mathcal{H}_{\mathcal{G}} & 0 \\ \bullet & \mathcal{H}_{\mathcal{B}-\mathcal{G}} \end{pmatrix}$$

where $\mathcal{H}_{\mathcal{G}}$ is the minor corresponding to the lattice points in \mathcal{G} and $\mathcal{H}_{\mathcal{B}-\mathcal{G}}$ is the minor corresponding to the lattice points not in \mathcal{G} . The zeros appear due to the fact that there is no pair $b \notin \mathcal{G}, b' \in \mathcal{G}$ such that $b \in b' - a(b') + \mathcal{A}_{i(b)}$. The same block-triangular structure also appears in the principal submatrix $\mathcal{E}_{\mathcal{A},\rho}$ and all the lattice points that are not in \mathcal{G} must be non-mixed, implying that $\mathcal{E}_{\mathcal{B}-\mathcal{G}} = \mathcal{H}_{\mathcal{B}-\mathcal{G}}$.

Finally, using the fact that the determinant of a block-triangular matrix is the product of the determinants of the diagonal blocks, we can prove the resultant formula:

$$\text{Res}_{\mathcal{A}} = \frac{\det(\mathcal{H}_{\mathcal{A},\rho})}{\det(\mathcal{E}_{\mathcal{A},\rho})} = \frac{\det(\mathcal{H}_{\mathcal{G}}) \det(\mathcal{H}_{\mathcal{B}-\mathcal{G}})}{\det(\mathcal{E}_{\mathcal{G}}) \det(\mathcal{H}_{\mathcal{B}-\mathcal{G}})} = \frac{\det(\mathcal{H}_{\mathcal{G}})}{\det(\mathcal{E}_{\mathcal{G}})}.$$

Example 1 Let f_0, f_1, f_2 be three bilinear equations corresponding to the supports $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. A possible mixed subdivision $S(\rho)$ is the following:



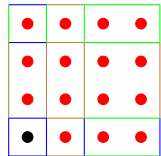
where the dots indicate the lattice points in translated mixed cells. The number of lattice points in translated cells is 9. However, if we construct the matrix greedily starting from the lattice points in translated mixed cells, we have an 8×8 matrix.

Example 2 Let f_0, f_1, f_2 be three bihomogeneous equations with supports

$$\mathcal{A}_0 = \{(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1)\},$$

$$\mathcal{A}_1 = \{(0, 0), (1, 0), (0, 1), (1, 1), (0, 2), (1, 2)\}, \quad \mathcal{A}_2 = \{(0, 0), (1, 0), (0, 1), (0, 1)\}.$$

The expected number of supports lying in translated cells is 16. Let $\rho_0 = (0, 3, 6, 3, 6, 9)$, $\rho_1 = (0, 2, 2, 4, 4, 6)$ and $\rho_2 = (0, 1, 1, 2)$ be the lifting functions and $\delta = (-\frac{1}{2}, \frac{1}{2})$ give the following mixed subdivision:

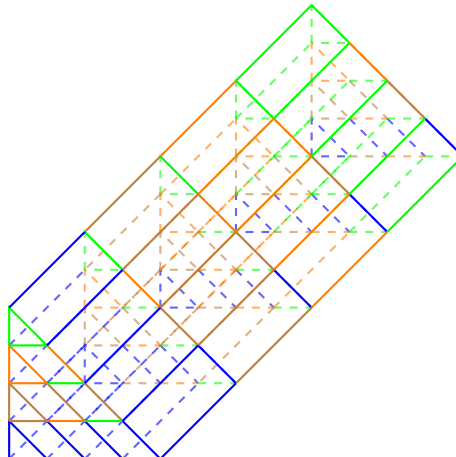


If we take the greedy approach the resulting matrix is 15×15 , whereas otherwise it would be 16×16 .

Example 3 Let f_0, f_1, f_2, f_3 be four polynomials with

$$\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1)\}$$

and $\rho_0 = (0, 3, 6, 3, 6, 9)$, $\rho_1 = (0, 2, 4, 2, 4, 6)$, $\rho_2 = (0, 1, 2, 1, 2, 3)$ and $\rho_3 = (0, 0, 0, 0, 0, 0)$ gives the mixed subdivision:



If we take the traslation $\delta = (-2/3, -2/3, -1/2)$ the number of points in traslated mixed cells is 24, but the degree of the resultant is $3 + 3 + 3 + 3 = 12$. We want to show that we can construct a smaller matrix using the greedy algorithm. If we start at the point $(0, 0, 0)$, we get the matrix and we achieve a matrix of size 20, with the principal submatrix marked in green.

2 A family of mixed subdivisions with tropical refinement

In this section, we give a family of lifting functions associated to the polytopes $\Delta_0, \dots, \Delta_n$ and we prove that the Canny-Emiris formula holds for the corresponding mixed subdivisions.

Definition 9 We can define a hyperplane arrangement $\mathcal{H} \subset N_{\mathbb{R}}$ by considering the span of the $(n - 1)$ -dimensional cones of the normal fan of Δ ; see [23] for more on polytopes and hyperplane arrangements.

Example 4 A polytope Δ (green), together with its normal fan (blue) and the hyperplane arrangement \mathbb{H}_{Δ} (red).



Definition 10 Let \mathcal{H} be the hyperplane arrangement associated to Δ and take a vector $v \in N_{\mathbb{R}}$ which does not lie in \mathcal{H} . We consider lifting functions $\omega_i : \mathcal{A}_i \rightarrow \mathbb{R}$ defined as:

$$\omega_i(x) = \lambda_i \langle v, x \rangle \quad i = 0, \dots, n \quad x \in \Delta_i$$

for $\lambda_0, \dots, \lambda_n \in \mathbb{R}$ satisfying $\lambda_0 > \dots > \lambda_n \geq 0$ and small enough. Let $\rho = (\omega_0, \dots, \omega_n)$ be a lifting giving a mixed subdivision $S(\rho)$.

Remark 6 This choice of the lifting function can also be seen as a case of the approach of [9], in a first proof of the rational formula for generalized unmixed systems. In particular, it is possible to think of the choice of the row content $a(b)$ associated to each lattice point as trying to solve the simplex method with the lifting function as objective. This family guarantees that we are always choosing this point in the same direction.

Theorem 2 $S(\rho)$ is an admissible mixed subdivision.

Proving that $S(\rho)$ is an admissible mixed subdivision consists on both proving that it has an incremental chain satisfying $S(\theta_0) \preceq \dots \preceq S(\theta_n) \preceq S(\rho)$ and that this incremental chain satisfies the conditions in Definition 8.

The easiest way to prove the chain condition would be to use [11, Proposition 2.11], which claims that for each $i = 0, \dots, n$, there is an open neighborhood of $0 \in U \subset \mathbb{R}^{\mathcal{A}_i}$ such that for $\omega_i \in U$ we have $S(\theta_i) \preceq S(\theta_{i+1})$. In this case, for λ_{i+1} small enough satisfying $\lambda_i > \lambda_{i+1} > 0$, ω_i lies in U . Therefore, the $S(\theta_i)$ form an incremental chain.

However, we can drop the restriction that λ_{i+1} is small enough by proving a more general result on refinement of mixed subdivisions.

2.1 Tropical refinement

This section constitutes an independent section inside this article in which we describe, in much broader generality than we need, the refinement of mixed dsubdivisions. In particular, we draw the full picture of when a coherent mixed subdivision refines another one, by only changing the lifting function in Δ_i . In terms of the previous notation, we would like to know whether $S(\theta_i) \preceq S(\theta_{i+1})$ for some $i = 0, \dots, n$. Instead of studying a given mixed subdivision, we define a dual of such object by introducing tropical geometry. After proving such result using tropical geometry, the family of lifting functions given in Section 2 will satisfy the refinement.

Remark 7 As in this paper we are mainly interested in affine lifting functions, we restrict to such case. However, the following results could be reproduced for any piecewise affine lifting function.

The general context of tropical geometry consists of working over rings of polynomials over \mathbb{R} with the tropical operations:

$$x \oplus y = \min(x, y) \quad x \otimes y = x + y$$

Definition 11 A tropical polynomial is the expression:

$$\text{trop}(f)(x) = \bigoplus_{a \in \mathcal{A}} \omega_a x^{\otimes a} = \min_{a \in \mathcal{A}} (\omega_a + ax)$$

for $x \in \mathbb{R}^n$ where \mathcal{A} is the support of f . A tropical hypersurface $V(\text{trop}(f))$ in \mathbb{R}^n is the set of points where the previous minimum is attained, at least, twice.

Remark 8 We can consider the coefficients ω_a to be the values of a lifting function. If the lifting is affine, we have $\omega_a = \langle v, a \rangle$ for some vector $v \in N_{\mathbb{R}}$. Therefore, the tropical polynomial with coefficients ω_a would be:

$$\min_{a \in \mathcal{A}} (a(x + v))$$

Definition 12 A tropical system \mathcal{T}_r is formed by $r + 1$ tropical polynomials with supports $P_0, \dots, P_r \subset M$:

$$\text{trop}(f_i^{\omega_i})(x) = \bigoplus_{a_i \in P_i} \omega_{i,a} \otimes x^{\otimes a} = \min_{a \in P_i} (\omega_{i,a} + a \cdot x)$$

where the coefficients of the system are given by some lifting function of the P_i . In some references like [18], it is important to specify a valuation in the field but here we can suppose it to be trivial.

In our context, we have a family of tropical systems \mathcal{T}_i for $i = 0, \dots, n$ of the supports:

$$\mathcal{A}_0, \dots, \mathcal{A}_{i-1}, \sum_{j=i}^n \mathcal{A}_j \subset M$$

The last tropical polynomial is formed by imposing 0 coefficients, therefore, it is defined by:

$$\min_{a \in \sum_{j=r}^n \Delta_j} \langle a, x \rangle$$

which corresponds to the normal fan of $\sum_{j=r}^n \Delta_j$. This coincides with the assumptions for $S(\theta_i)$ in Remark 3.

Proposition 2 *The expression $\min_{a \in \mathcal{A}} \langle a, x \rangle$ is achieved twice in the $(n - 1)$ -dimensional cones of the normal fan of $\Delta = \text{conv}(\mathcal{A})$.*

Proof A j -th dimensional cone $\mathcal{N}_F \subset M_{\mathbb{R}}$ is a of the normal fan of Δ corresponds to a $n - j$ -dimensional face of Δ . Take $v \in \mathcal{N}_F$, then $\min_{a \in \Delta} \langle a, x \rangle$ is the same for all $a \in F$, which is a face. Therefore, it is achieved, at least twice. On the other hand, if the minimum is achieved at least twice at v , then consider the convex hull

$$\text{conv}\{a_i \in \Delta \mid \min \langle a_i, v \rangle \text{ is achieved}\}$$

and it is a positive dimensional face F of Δ , therefore v is in the a cone of dimension at most $(n - 1)$ in \mathcal{F} .

Proposition 3 *The expression $\min_{a \in \mathcal{A}} \langle a, x + v \rangle$ is achieved twice in the $(n - 1)$ -skeleton of the normal fan of Δ translated after $v \in N_{\mathbb{R}}$.*

Proof The same proof as the previous works after translating by v .

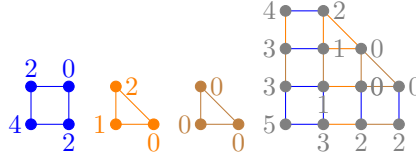
In this context, we can see the tropical system \mathcal{T}_i as the superposition in $N_{\mathbb{R}}$ of the normal fans $\mathcal{F}_0, \dots, \mathcal{F}_n$ centered at different points $v_i \in N_{\mathbb{R}}$ which correspond to each of the lifting functions $\omega_i : \Delta_i \rightarrow \mathbb{R}$.

Definition 13 A polyhedral complex \mathcal{P} is a union of cells (bounded or unbounded) in $N_{\mathbb{R}}$ such that:

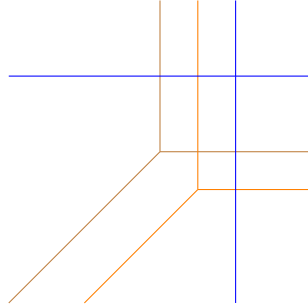
- Every face of a cell in \mathcal{P} is also in \mathcal{P} .
- The (possibly empty) intersection of two cells in \mathcal{P} is also in \mathcal{P} .

Fans are a good example of polyhedral complexes. Therefore, a tropical system defines a polyhedral complex.

Now, we claim that moving the orange fan close enough, we will be refining the mixed subdivision. In particular, moving the orange fan to each of the adjacent cells on the hyperplane arrangement corresponds to all the possible ways to refine the previous mixed subdivision. For instance, if we take the traslation given by the vector $(1, -1)$, which would be the normal vector to the affine lifting $c - x + y$ with $c = 1$. The mixed subdivision looks like:



and the tropical system after the traslation vector $(1, -1)$, corresponds to:



Let's do a recap of all the notation we have so far. Let $\omega_i : \mathcal{A}_i \rightarrow \mathbb{R}$ be the lifting function. As in Theorem 2, $S(\theta_i)$ be the mixed subdivisions of the candidate incremental chain given by the lifting functions $(\omega_0, \dots, \omega_{i-1}, 0, \dots, 0)$. Let \mathcal{T}_i be the tropical systems dual to each of the mixed subdivisions $S(\theta_i)$ for $i = 0, \dots, n$. Let \mathbb{H}_i be the hyperplane arrangement associated to each of the tropical systems.

At this point, we have all the ingredients to state and prove the tropical refinement result.

Theorem 3 (Tropical refinement) *Let $i = 1, \dots, n$. The mixed subdivision $S(\theta_i)$ refines $S(\theta_{i-1})$, if and only if, the normal vector to the lifting function $\omega_{i-1} : \mathcal{A} \rightarrow \mathbb{R}$ lives in a chamber of \mathbb{H}_i adjacent to $0 \in \mathbb{R}^n$.*

Let's construct the tools we need for proving this result.

Definition 14 We say that a ray r of the normal fan \mathcal{F}_i preserves adjacencies if it is adjacent to the same cells in \mathcal{T}_i and \mathcal{T}_{i-1}

Lemma 2 *Let $S(\theta_i)$ be a mixed subdivision of $\Delta_0, \dots, \Delta_{i-1}, \sum_{j=i}^n \Delta_j$ for $i = 0, \dots, n$. The lifting of Δ_i will give $S(\theta_i) \preceq S(\theta_{i+1})$, if and only if, each ray of \mathcal{F}_i preserves the adjacencies after the translation.*

Proof Suppose there is a ray r that doesn't preserve an adjacencies. Then, take the 0-dimensional cell of the corresponding polyhedral complex where this adjacency fails and it must correspond to an n -cell of $S(\theta_{i+1})$ that is not contained in the cell of $S(\theta_i)$ corresponding to such adjacency.

On the other hand, take a cell C of $S(\theta_{i+1})$ that is not contained in any of the cells of $S(\theta_i)$ and, as we only lifted the polytope Δ_i , the corresponding dual cell on the polyhedral complex has to fail to be adjacent to the same rays.

Proof (of the Theorem 3) Consider p as a point (0-dimensional cell) in the polyhedral complex that is dual to an n -cell D of $S(\theta_{i-1})$. Let v be the normal vector to the lifting function $\omega_i : \mathcal{A}_i \rightarrow \mathbb{R}$. We have to prove that v lies in an adjacent cell to 0 in \mathbb{H}_i , if and only if, D is contained in a cell D' of $S(\theta_k)$.

Firstly, suppose there was not such cell D' . This would mean that the adjacencies would not be preserved and we can find a ray r in \mathcal{F}_i where this property is failing. Consider the ray of a fan \mathcal{F}_k for $k = 0, \dots, i - 1$ where this adjacency has changed and this means that we have crossed a hyperplane containing such ray in the previous fan.

On the other hand, if there is such cell D' , then the lifting of Δ_i preserves adjacencies. However, if we had moved v to a non-adjacent cell to 0, we would have crossed a hyperplane therefore, we would be able to find rays in such hyperplane where the adjacencies are not preserved.

This result extends the proposition 2.11 on [11, Proposition 2.11] and gives a full picture of refinement of mixed subdivisions. Therefore, we naturally understand all the ways to refine a given mixed subdivision $S(\theta_i)$ with affine lifting functions on Δ_i .

Corollary 1 *The chambers of the hyperplane arrangement \mathbb{H}_i are in one to one correspondence to all the possible ways to refine $S(\theta_i)$. In particular, if $S(\theta_i)$ is tight, the chambers of \mathbb{H}_i correspond to tight mixed subdivisions.*

In the context of Theorem 2, in the direction of $v \notin \mathbb{H}_\Delta$, the function $\langle \lambda_i v, x \rangle$ will reach the hyperplane arrangement \mathbb{H}_i when $\lambda_i = \lambda_{i-1}$. Therefore, for any $0 < \lambda_i < \lambda_{i-1}$, the subdivision $S(\theta_{i+1})$ will refine $S(\theta_i)$ for $i = 0, \dots, n$.

2.2 The proof of Theorem 2.1

We close here the parentheses of tropical refinement and go back to the proof of Theorem 2.

Theorem 4 *The mixed subdivision $S(\rho)$ in Definition 8 is admissible.*

Proof All the lattice points with row content 0 are 0-mixed. Therefore, $S(\theta_0)$ satisfies *ii*) in Definition 8. Let D be an n -cell of $S(\theta_i)$. If $\dim D_i = 0$, then the fundamental subfamily of \mathcal{A}_D is at most $\{i\}$ as shown in Remark 4. We show that, for our choice of the lifting function, the rest of cells D satisfy *ii*) in Definition 8.

Let $D \in S(\theta_i)$ such that $\dim D_i > 0$. Suppose that this cell contains a lattice point $b \in \mathcal{B}$ that has row content i but is not i -mixed. Therefore, this lattice point b will be in a cell of $S(\rho)$ with a 0-dimensional j -th component for some $j < i$. Take $C \supset D$ in $S(\theta_j)$ containing the previous lattice point b . If $\dim C_j > 0$, then the lifting function $\omega_j = \lambda_j \langle v, x \rangle$ takes the same value in all the points of C_j . Therefore, the vector v is normal to C_j and has to be contained in the hyperplane arrangement associated to Δ . As this is not the case, $\dim C_j = 0$ and consequently $\dim D_j = 0$, contradicting the initial hypothesis.

This proves that the family of lifting functions that we have defined, always provides an admissible mixed subdivision.

3 The case of n -zonotopes

For simplicity, we suppose that the lattice is $M = \mathbb{Z}^n$.

Definition 15 A zonotope is a polytope given as a sum of line segments. An n -zonotope is generated by n line segments, which span a lattice of dimension n .

Consider linearly independent $v_1, \dots, v_n \in \mathbb{Z}^n$ and the line segments $\overline{0v_1}, \dots, \overline{0v_n} \subset \mathbb{R}^n$ forming an n -zonotope $Z \subset \mathbb{R}^n$. If the Newton polytopes are n -zonotopes whose defining line segments are integer multiples of the $\overline{0v_j}$, we can write the supports of the system as:

$$\mathcal{A}'_i = \left\{ \sum_{j=1}^n \lambda_j v_j \in \mathbb{Z}^n \mid \lambda_j \in \mathbb{Z}, \quad 0 \leq \lambda_j \leq a_{ij} \right\}.$$

for some $a_{ij} \in \mathbb{Z}_{>0}$. Let V be the nonsingular matrix whose columns are the coordinates of the v_j in the canonical basis of \mathbb{Z}^n for $j = 1, \dots, n$ and consider it as a monomorphism of lattices $V : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ of rank n . Let e_1, \dots, e_n be the canonical basis of \mathbb{Z}^n .

Corollary 2 *Let $\mathcal{A}'_0, \dots, \mathcal{A}'_n$ be the previous family of supports, then $\text{Res}_{\mathcal{A}'} = \text{Res}_{\mathcal{A}}^{|\det(V)|}$, where:*

$$\mathcal{A}_i = \left\{ (b_j)_{j=1, \dots, n} \in \mathbb{Z}^n \mid 0 \leq b_j \leq a_{ij} \right\} \quad i = 0, \dots, n$$

Proof Using Lemma 1, we can view the map $V : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ as a monomorphism of lattices sending the canonical basis e_i to v_i for $i = 1, \dots, n$. The absolute value of the determinant $|\det(V)|$ is the index of the image. This last result follows from the reduction of V to its Smith normal form [20, Theorem 2.3].

Remark 9 The normal vectors of the n -zonotope are given by n pairs $(\eta_j, -\eta_j)_{j=1, \dots, n}$ in N . The results that follow in this section could be proved without using Corollary 2, after changing b_j by $\langle b, \eta_j \rangle$ for $j = 1, \dots, n$, choosing η_j to be the element in the pair such that $0 \leq \langle b, \eta_j \rangle \leq a_{ij}$.

In order to prove our results, we assume that the a_{ij} are ordered, meaning that $0 < a_{0j} \leq a_{1j} \leq \dots \leq a_{n-1j}$ and $j = 1, \dots, n$, where we exclude \mathcal{A}_n from this assumption; notice that Example 6 wouldn't satisfy this property without excluding \mathcal{A}_n . Consider a translation $\delta \in \mathbb{R}^n$ which is negative in each component and small enough. Then, the lattice points in translated cells of a mixed subdivision of the previous system are:

$$\mathcal{B} = \{(b_j)_{j=1, \dots, n} \in \mathbb{Z}^n \mid 0 \leq b_j < \sum_{i=0}^n a_{ij}\}.$$

Let $v \notin \cup_{i=1}^n \{x_j = 0\}$ define the mixed subdivision $S(\rho)$ as in the previous section. We assume $v_j < 0$ for $j = 1, \dots, n$ and get the following result.

Proposition 5 *Let $b \in \mathcal{B}$ and $i \in \{0, \dots, n\}$. Then:*

$$t_{b,i} = |\{j \in \{1, \dots, n\} \mid \sum_{k=0}^{i-1} a_{kj} \leq b_j < \sum_{k=0}^i a_{kj}\}|$$

and the row content $i(b)$ is the maximum index in $\{0, \dots, n\}$ such that:

$$\exists j \in \{1, \dots, n\} : \sum_{k=0}^{i(b)-1} a_{kj} \leq b_j < \sum_{k=0}^{i(b)} a_{kj}$$

with the support $a(b) \in \mathcal{A}_{i(b)}$ satisfying:

$$a(b)_j = \begin{cases} 0 & b_j < \sum_{k=0}^{i(b)-1} a_{kj}, \\ a_{i(b)j} & b_j \geq \sum_{k=0}^{i(b)} a_{kj}. \end{cases}$$

Proof It can be deduced from Definition 10 that the mixed subdivision $S(\theta_i)$ is given by the lifting function $\theta_i = \sum_{j=1}^n \theta_{i,j}$ where the lifting in the component x_j only depends on $\theta_{i,j}$. As we said in Definition 2, each of the $\theta_{i,j}$ are given as the inf-convolution of the lifting function in each of the polytopes. The first non-trivial lifting function is $S(\theta_1)$: it is piecewise linear and by Remark 3 depends only on the lifting given in Δ_0 . Therefore, after taking the inf-convolution, we get:

$$\theta_{0j} = \begin{cases} 0 & \{x_j = 0\} \\ \lambda_0 v_j & \{x_j = a_{0j}\} \\ \lambda_0 v_j & \{x_j = \sum_{i=0}^n a_{ij}\} \end{cases}$$

This structure is guaranteed by the fact that $\lambda_0 v_j < 0$. One can see that $S(\theta_1)$ has two types of cells D with respect to the j -th coordinates: those in which $\overline{0a_{j,0}e_j} \subset D_0$ (which have to be in $x_j \leq a_{0j}$ for $x \in D$) and those with $\overline{0a_{j,0}e_j} \not\subset D_0$ (in $x_j \geq a_{0j}$). In terms of the lattice points, as we are considering $\delta_j < 0$, those $b \in \mathcal{B}$ lying in a cell D with $\overline{0a_{j,0}e_j} \subset D_0$ satisfy that $0 \leq b_j < a_{0j}$ and the rest satisfy $b_j \geq a_{0j}$. As a consequence, $t_{b,0}$ must be the number of $j \in \{1, \dots, n\}$ such that $0 \leq b_j < a_{0j}$.

Consider now the structure of the mixed subdivision $S(\theta_{i+1})$ with respect to the j -th coordinate. Using the inf-convolution, we get:

$$\theta_{ij} = \begin{cases} 0 & \{x_j = 0\} \\ \lambda_0 v_j & \{x_j = a_{0j}\} \\ (\lambda_0 + \lambda_1) v_j & \{x_j = a_{0j} + a_{1j}\} \\ \dots & \\ (\lambda_0 + \dots + \lambda_{i-1}) v_j & \{x_j = \sum_{i=0}^{i-1} a_{ij}\} \\ (\lambda_0 + \dots + \lambda_{i-1}) v_j & \{x_j = \sum_{i=0}^n a_{ij}\} \end{cases}$$

Again, this structure is guaranteed by $\lambda_i v_j < 0$ and by $\lambda_{i-1} > \lambda_i$. With the same argument as before, the cells in which $\overline{0a_{ij}e_j} \subset D_i$ are precisely those that satisfy $\sum_{k=0}^{i-1} a_{kj} \leq b_j < \sum_{k=0}^i a_{kj}$ for $b \in D$. As before, this also proves that $t_{b,i}$ is the number of $j \in \{1, \dots, n\}$ such that $\sum_{k=0}^{i-1} a_{kj} \leq b_j < \sum_{k=0}^i a_{kj}$. An explicit example of how this construction looks like can be found in Figure 1.

The second claim follows from the definition of row content with respect to the type vector t_b . Let $b \in \mathcal{B}$ and let $i(b)$ be its row content. For $j = 1, \dots, n$, we either have $b_j < \sum_{k=0}^{i(b)-1} a_{kj}$ or $b_j \geq \sum_{k=0}^{i(b)} a_{kj}$. In the first case, the vertex associated to the row content, will be in the face of $\Delta_{i(b)}$ defined by the equality $\{x_j = 0\}$ and in the second case, the one defined by the equality $\{x_j = a_{i(b)j}\}$.


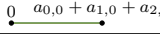
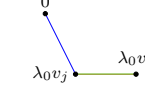
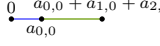
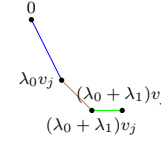
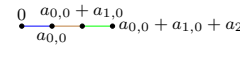
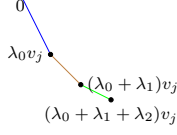
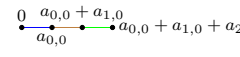
Step	Lifting	Subdivision
$S(\theta_0)$		
$S(\theta_1)$		
$S(\theta_2)$		
$S(\rho)$		

Fig. 1 This table explains how the process of passing from the proposed lifting on $\Delta_0, \Delta_1, \Delta_2$ to the mixed subdivision works in the j -th coordinate for $v_j < 0$ for any of the two components of Example 1. One clearly sees that, for instance, $0a_{0,0}e_0 \subset D_0$, if and only if, $x_0 \leq a_{0,0}$ for $x \in D$. The product of two subdivisions of this form gives the mixed subdivision in the figure of Example 1.

Remark 10 If $v_j > 0$, we would change the inequalities by $\sum_{k=i}^n a_{kj} \leq b_j < \sum_{k=i-1}^i a_{kj}$, but the results that follow would not change. Any other mixed subdivisions of this particular system can also be formed this way.

Definition 16 The type function $\varphi_b : \{1, \dots, n\} \rightarrow \{0, \dots, n\}$ associated to each lattice point $b \in \mathcal{B}$ is defined as the vector of indices satisfying:

$$\sum_{k=0}^{\varphi_b(j)-1} a_{kj} \leq b_j < \sum_{k=0}^{\varphi_b(j)} a_{kj}$$

Following Proposition 5, it satisfies that $t_{b,i} = |\varphi_b^{-1}(i)|$.

From the components of $a(b)$ in Proposition 5, we deduce that the range of values for $(b - a(b) + \mathcal{A}_{i(b)})_j$ is:

$$\begin{cases} [b_j, b_j + a_{i(b)j}] & b_j < \sum_{k=0}^{i(b)-1} a_{kj} \\ [b_j - a_{i(b)j}, b_j] & b_j \geq \sum_{k=0}^{i(b)} a_{kj} \end{cases}$$

Corollary 3 The range of possible type functions for $b' \in b - a(b) + \mathcal{A}_{i(b)}$ are:

$$\varphi_{b'}(j) \in \begin{cases} \{\varphi_b(j) - 1, \varphi_b(j)\} & i(b) < \varphi_b(j) \\ \{\varphi_b(j), \dots, i(b)\} & i(b) > \varphi_b(j) \end{cases}$$

Proof Take I to be the index such that $\sum_{k=0}^{I-1} a_{kj} \leq b_j < \sum_{k=0}^I a_{kj}$ and we get the inequalities:

$$\begin{cases} b_j - a_{i(b)j} \geq \sum_{k=0}^{I-1} a_{kj} - a_{i(b)j} \geq \sum_{k=0}^{I-2} a_{kj} & i(b) < I \\ b_j + a_{i(b)j} < \sum_{k=0}^I a_{kj} + a_{i(b)j} \leq \sum_{k=0}^{i(b)} a_{kj} & i(b) > I \end{cases}$$

In the first row, we used that $a_{i(b)j} \leq a_{I-1j}$.

Definition 17 We define the greedy subset $\mathcal{G} \subset \mathcal{B}$ to be formed by all the lattice points $b \in \mathcal{B}$ such that:

$$\sum_{i=0}^I t_{b,i} \leq I + 1 \quad \forall I < n.$$

Theorem 5 Let $b \in \mathcal{G}$ and $b' \notin \mathcal{G}$. Then, $b' \notin b - a(b) + \mathcal{A}_{i(b)}$

Proof Let I be the greatest index such that $\sum_{i=0}^I t_{b',i} > I + 1$. As it is the greatest, we must have $t_{b',I+1} = 0$ and $\sum_{i=I+2}^n t_{b',i} < n - I - 1$.

On the other hand, $\sum_{i=0}^I t_{b,i} \leq I + 1$. Using Corollary 3, the previous sum cannot grow in $b - a(b) + \mathcal{A}_{i(b)}$ when $i(b) > I$. If $\sum_{i=0}^I t_{b,i} = I + 1$, then $\sum_{i=I+1}^n t_{b,i} < n - I$ which implies that there is $i > I$ with $t_{b,i} = 0$ and $i(b) > I$.

Suppose $\sum_{i=0}^I t_{b,i} < I + 1$ and $i(b) < I$. Using Corollary 3, we have:

$$\sum_{i=I+1}^n t_{b,i} \geq n - I \quad \text{and} \quad \sum_{i=I+1}^n t_{\bar{b},i} \geq n - I - 1$$

for $\bar{b} \in b - a(b) + \mathcal{A}_{i(b)}$. Therefore,

$$\sum_{i=I+2}^n t_{b',i} < n - I - 1 \leq \sum_{i=I+2}^n t_{\bar{b},i}$$

meaning that it is not possible that b' has a type function on the range of $b - a(b) + \mathcal{A}_{i(b)}$.

Definition 18 Let $I_b \in \{0, \dots, n\}$ be the index satisfying:

$$I_b = \begin{cases} \max\{i \in \{0, \dots, n\} \mid t_{b,i} \geq 2\} & b \text{ lies in a non-mixed cell} \\ 0 & b \text{ lies in a mixed cell} \end{cases}$$

Let $g_b = |\{i < I_b \mid t_{i,b} = 0\}|$ be the number of zeros that t_b has before I_b .

Lemma 3 Let $b \in \mathcal{G}$ and suppose that $g_b = 0$. Then, b lies in a mixed cell.

Proof Suppose that b lies in a non-mixed cell. This would mean that there is no zero before I_b implying that $\sum_{i=0}^{I_b} t_{b,i} = \sum_{i=0}^{I_b-1} t_{b,i} + t_{b,I_b} \geq I_b + 2$.

Lemma 4 If $t_{b,I} = 0$ and $b \in \mathcal{G}$, $\sum_{i=0}^I t_{b,i} < I + 1$.

Proof Otherwise, $\sum_{i=0}^{I-1} t_{b,i} \geq I + 1$.

Theorem 6 Let \mathcal{G} be the greedy subset and $b \in \mathcal{G}$ such that $g_b = K$ for $K > 0$. Then, there is $b' \in \mathcal{G}$ with $g_{b'} = K - 1$ such that for some $\bar{b} \in b' - a(b') + \mathcal{A}_{i(b)}$, $\varphi_{\bar{b}} = \varphi_b$. As a consequence, we reach b from b' .

Proof Consider t_b to be the type vector of b and suppose that t_b has two or more zeros after I_b . Then,

$$\sum_{i=I_b+1}^n t_{b,i} \leq n - I_b - 2$$

implying that $\sum_{i=0}^{I_b} t_{b,i} \geq I_b + 2$, and $b \notin \mathcal{G}$.

If t_b has one zero after I_b , it implies that $i(b) > I_b$. If $g_b > 0$, it needs to have at least one zero before I_b . Therefore, the type vector contains a sequence of the form

$$(\dots, \overbrace{0}^{I'}, 1, \dots, 1, \overbrace{t_{b,I}}^I, \dots)$$

for some $I' < I \leq I_b$ with $t_{b,I} \geq 2$. Consider the type function:

$$\varphi_{b'}(j) = \begin{cases} \varphi_b(j) - 1 & I' < \varphi_b(j) \leq I \\ \varphi_b(j) & \text{otherwise} \end{cases}$$

The corresponding type vector $t_{b'}$ contains a sequence:

$$(\dots, \overbrace{1}^{I'}, 1, \dots, 1, \overbrace{t_{b,I} - 1}^I, \dots).$$

Using Lemma 4, $\sum_{i=0}^{I'} t_{b,i} < I' + 1$, therefore we will have:

$$\sum_{i=0}^{I'} t_{b',i} \leq I' + 1.$$

The same will hold for all the partial sums from I' to I_b implying there is $b' \in \mathcal{G}$ with type function $\varphi_{b'}$. Using Corollary 3, φ_b is in the range of type functions in $b' - a(b') + \mathcal{A}_{i(b')}$. As long as $i(b') < n$, we can find $b' \in \mathcal{G}$ such that:

$$(b - b' + a(b'))_j \leq a_{\varphi_{b'}(j)} \leq a_{j,i(b')}$$

so $b \in b' - a(b') + \mathcal{A}_{i(b')}$.

If $i(b) = i(b') = n$, we must have $a(b) = a(b') = 0$, so we reach a point $\bar{b} \in b' - a(b') + \mathcal{A}_{i(b')}$ in the same cell as b such that:

$$(b - \bar{b})_j < (b - b')_j \quad \forall I' < \varphi_b(j) \leq I$$

As $i(\bar{b})$ is always the same, after a finite number of steps, we have $b \in \bar{b} - a(\bar{b}) + \mathcal{A}_{i(\bar{b})}$.

If t_b does not have any zero after I_b , then $i(b) < I_b$. The vector contains a sequence of the form

$$(\dots, \overbrace{0}^{i(b)}, t_{b,i(b)+1}, \dots, t_{b,I_b}, \dots)$$

for $t_{b,I} \geq 1$ with $i(b) < I \leq I_b$. In this case, consider the type function:

$$\varphi_{b'}(j) = \begin{cases} \varphi_b(j) - 1 & i(b) < \varphi_b(j) \leq I \\ \varphi_b(j) & \text{otherwise} \end{cases}$$

with type vector $(\dots, \overbrace{t_{b,i(b)+1}}^{i(b)}, \dots, \overbrace{t_{b,I_b}}^{I_b-1}, \overbrace{0}^{I_b}, \dots)$.

$$\sum_{i=i(b')+1}^n t_{b',i} \geq n - i(b) + \sum_{\substack{t_{b,i} \geq 2 \\ i > i(b')}} (t_{b',i} - 1) \implies$$

$$\sum_{i=0}^{i(b)} t_{b',i} \leq i(b) - \sum_{\substack{t_{b,i} \geq 2 \\ i > i(b')}} (t_{b',i} - 1) \leq i(b) + 1$$

which implies that

$$\sum_{i=0}^{i(b)} t_{b',i} + t_{b',i(b')+1} \leq i(b) - \sum_{\substack{t_{b,i} \geq 2 \\ i > i(b')}} (t_{b',i-1}) + t_{b',i(b')+1} \leq i(b') + 1$$

This argument holds for bounding the partial sums for $I > i(b)$ so there is $b' \in \mathcal{G}$ with type function $\varphi_{b'}$ and φ_b is in the range of type functions in $b' - a(b') + \mathcal{A}_{i(b')}$. In this case, it is not possible that $i(b') = n$ so the same argument on the previous case holds in order to say that $b \in b' - a(b') + \mathcal{A}_{i(b')}$.

Theorem 5 and Theorem 6 imply that if we start the greedy algorithm from the lattice points in mixed cells, we will reach exactly the lattice points in \mathcal{G} . This actually reduces the size of the Canny-Emiris matrices.

Corollary 4 *The size of the matrix $\mathcal{H}_{\mathcal{G}}$ is:*

$$\sum_{\varphi_b: \{1, \dots, n\} \rightarrow \{0, \dots, n\}} \prod_{j=1}^n a_{\varphi_b(j)} j$$

where the sum is over the functions that satisfy $\varphi_b^{-1}(\{0, \dots, I\}) \leq I + 1 \quad \forall I < n$.

Dimension	Canny-Emiris	Greedy	Resultant degree
2	9	8	6
3	64	50	24
4	625	432	360
5	7776	4802	3720

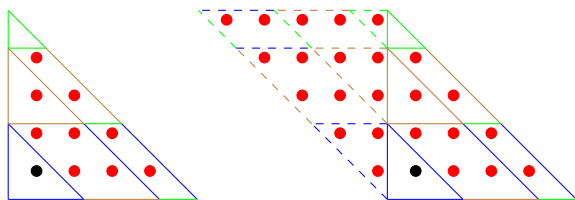
Fig. 2 This table represents the size of the matrices we achieve for zonotopes of dimensions from 2 to 5 with $a_{ij} = 1$ using the greedy approach versus the original Canny-Emiris formula. We also compare to the degree of the resultant.

Proof Each type function φ_b corresponds to a cell $D \in S(\rho)$. The lattice points $b \in D$ satisfy Definition 16. Therefore, for each j , there are $a_{\varphi_b(j)j}$ possible values of b_j . The product over all of them gives the desired count.

We could not yet prove whether this is minimal with respect to the application of the greedy approach to any other admissible mixed subdivision using affine liftings, but the many of the examples using that we have verify it. However, some other known results of elimination theory suggest that using other piecewise affine liftings might lead to more exact results; namely the relevance of degree reverse lexicographical order when using Grobner basis[1]. We should remark that this combinatorial formula (and the one in Corollary 4) should be compared with the same sum over all the type functions without the restriction. For practical purposes, we show an example of our computations of the matrix dimensions in Figure 2.

4 Multihomogeneous formulas

Example 6 Let f_0, f_1, f_2 be three homogeneous polynomials of degrees 2, 2, 1 respectively. We choose $v = (-1, -2)$ and $\delta = (-3/4, -3/4)$ and define an admissible mixed subdivision $S(\rho)$ in the Minkowski sum Δ of their Newton polytopes Δ_i . Let \mathcal{B} be the set of lattice points in $\Delta + \delta$. Consider a system of polynomials whose Newton polytopes are n -zonotopes generated by the vectors $w_1 = (1, 0)$ and $w_2 = (-1, 1)$ and let $a_{0,1} = a_{0,2} = a_{1,1} = a_{1,2} = 2$ and $a_{2,1} = a_{2,2} = 1$ be the bounds of the supports as in Section 3. Let $S(\bar{\rho})$ be the mixed subdivision in the Minkowski sum $\bar{\Delta}$ of $\bar{\Delta}_i$ of this system given by the same v, δ as the previous, and let $\bar{\mathcal{B}}$ be the set of lattice points in $\bar{\Delta} + \delta$.



It turns out that the mixed subdivision $S(\rho)$ embeds, into $S(\bar{\rho})$ implying that $\mathcal{B} = \bar{\mathcal{B}} \cap \Delta$. As the greedy reduction applies to the second system, it must apply to the first as well. Finally, we get a 9×9 matrix \mathcal{H}_G for the homogeneous system, excluding the black lattice point in the figure.

Similar to Example 6, let's now consider multihomogeneous polynomial systems and embed them into n -zonotopes. Let $n_1, \dots, n_s \in \mathbb{N}_{>0}$ be natural numbers and let $M = \bigoplus_{l=1}^s \mathbb{Z}^{n_l}$ be our lattice. Each multihomogeneous polynomial system can be written as:

$$F_i = \sum_{a \in \mathcal{A}_i} u_{i,a} \chi^a, \quad i = 0, \dots, n$$

where the supports are:

$$\mathcal{A}_i = \left\{ (b_{jl})_{l=1, \dots, s}^{j=1, \dots, n_l} \in \bigoplus_{l=1}^s \mathbb{Z}^{n_l} \mid b_{jl} \geq 0, \sum_{j=0}^{n_l} b_{jl} \leq d_{i,l} \right\}$$

where $\mathbf{d}_i = (d_{i,1}, \dots, d_{i,s})$ is the multidegree of F_i . Each of these supports can be embedded into the following sets of supports:

$$\bar{\mathcal{A}}_i = \left\{ (b_{jl}) \in \bigoplus_{j=1}^s \mathbb{Z}^{n_j} \mid 0 \leq \sum_{J=j}^{n_l} b_{Jl} \leq d_{i,l} \right\} \quad l = 1, \dots, s \quad j = 1, \dots, n_l.$$

Let $\Delta_i, \bar{\Delta}_i$ be the Newton polytopes of each of the systems and $\Delta, \bar{\Delta}$ be their respective Minkowski sums.

Lemma 5 *The Newton polytopes $\overline{\Delta}_i$ of the system of polynomials with supports in $\overline{\mathcal{A}}_i$ are n -zonotopes whose line segments $(w_{j,l})_{l=1,\dots,s}^{j=1,\dots,n_l}$ are given by the columns of the matrix:*

$$W = \begin{bmatrix} W_1 & \dots & 0 \\ \vdots & \ddots & \\ 0 & \dots & W_s \end{bmatrix}, \quad W_l = \begin{pmatrix} 1 & -1 & 0 & \dots & \dots \\ 0 & 1 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & -1 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

where the square blocks W_l are of size n_l for $l = 1, \dots, s$. Moreover, $\overline{\mathcal{H}} = \cup_{l=1}^s \cup_{j=1}^{n_l} \{\langle x, w_{j,l} \rangle = 0\} \subset \prod_{l=1}^s \mathbb{R}^{n_l}$ is the hyperplane arrangement associated to $\overline{\Delta}$.

Proof Let $b \in \oplus_{j=1}^s \mathbb{Z}^{n_j}$ be a lattice point. As the columns of W form a basis of the lattice, we can write $b = \sum_{l=1}^s \sum_{j=1}^{n_l} \lambda_{j,l} w_{j,l}$ and these coefficients are precisely $\lambda_{j,l} = \sum_{J=j}^{n_l} b_{J,l}$. Then,

$$b \in \overline{\mathcal{A}}_i \iff 0 \leq \lambda_{j,l} \leq d_{i,l} \quad l = 1, \dots, s \quad j = 1, \dots, n_l.$$

The normal vectors to the faces of $\overline{\Delta}$ are given by the columns $(\eta_{j,l})_{l=1,\dots,s}^{j=1,\dots,n_l}$ of the matrix:

$$H = \begin{bmatrix} H_1 & \dots & 0 \\ \vdots & \ddots & \\ 0 & \dots & H_s \end{bmatrix}, \quad H_l = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots \\ 1 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & 1 & 1 & 0 \\ 1 & \dots & 1 & 1 & 1 \end{pmatrix}.$$

One can check that $\langle w_{j,l}, \eta_{j',l'} \rangle \neq 0$, if and only if, $l = l'$ and $j = j'$. Therefore, $v \in \overline{\mathcal{H}}$, if and only if, it belongs to the span of $\sum_{l=1}^s n_l - 1$ columns of H , and this will only happen if $\langle v, w_{j,l} \rangle = 0$ for some pair j, l .

Remark 11 As a consequence of Lemma 5, we can apply the results of Sec. 3 to the system with supports $\overline{\mathcal{A}}_i$. The matrix H gives the normals to our polytopes, so we can use it in the sense of Remark 9.

Let $v \notin \overline{\mathcal{H}}$ and suppose that we take $\langle v, w_{j,l} \rangle < 0$, for $l = 1, \dots, s$ and $j = 1, \dots, n_l$. Consider $S(\overline{\rho})$ to be the admissible mixed subdivision of $\overline{\Delta}$ given by v as in Sec. 2. Let $S(\rho)$ be the mixed subdivision given by the same vector in Δ . Using $\langle v, w_{j,l} \rangle < 0$, one can check that this mixed subdivision is also admissible as v does not belong to the hyperplane arrangement \mathcal{H} associated to Δ . Let $\mathcal{B}, \overline{\mathcal{B}}$ be the sets of lattice points in translated cells of Δ and $\overline{\Delta}$, respectively. We can see the polytopes Δ_i as a product of simplices $\Delta_{i,1} \times \dots \times \Delta_{i,s}$ in each of the factors of $M_{\mathbb{R}} = \prod_{l=1}^s \mathbb{R}^{n_l}$.

Theorem 7 *The mixed subdivision $S(\rho)$ coincides with $S(\overline{\rho}) \cap \Delta$.*

Proof The vector $v \in \prod_{l=1}^s \mathbb{R}^{n_l}$ has to satisfy that:

$$v_{1,l} < 0 \quad v_{j+1,l} - v_{j,l} < 0 \quad j = 1, \dots, n_l - 1, \quad l = 1, \dots, s$$

In other words $v_{n_l,l} < v_{n_l-1,l} < \dots < v_{1,l} < 0$. This means that the mixed subdivision $S(\theta_1)$ lifts the vertices of $\Delta_{0,l}$ in the order $0, d_{0,l}w_{1,l}, \dots, d_{0,l}w_{n_l,l}$ from higher to lower in the same lines of Proposition 5. This means that the cells $D \in S(\rho)$ such that $d_{0,l}w_{j,l} \subset D_i$ are contained between the hyperplanes $\{\langle x, w_{j,l} \rangle = \sum_{j=0}^{i-1} d_{j,l}\}$ and $\{\langle x, w_{j,l} \rangle = \sum_{j=0}^i d_{j,l}\}$ for $l = 1, \dots, s \quad j = 1, \dots, n_l$. $S(\overline{\rho})$ has the same structure. Therefore, $S(\rho)$ coincides with $S(\overline{\rho}) \cap \Delta$.

Therefore, if we apply the greedy algorithm to the multihomogeneous system with supports in the \mathcal{A}_i , we will obtain the same greedy subset $\mathcal{G} \subset \mathcal{B}$, with the restriction on the type vectors given in Definition 17. In particular, the domain of the type functions will now be a multiset in each group of variables:

$$\varphi_b : \{\{1, \dots, n_1\}, \dots, \{1, \dots, n_s\}\} \rightarrow \{0, \dots, n\}.$$

The following proposition gives conditions to guarantee that the type function φ_b corresponds to a lattice point $b \in \mathcal{B}$.

Proposition 6 *A lattice point $b \in \overline{\mathcal{B}}$ belongs to \mathcal{B} iff its type function satisfies:*

$$\varphi_b(j) \leq \varphi_b(j'), \quad \forall j < j' \quad j, j' = 1, \dots, n_l, \quad l = 1, \dots, s.$$

Proof Suppose that there is φ_b for $b \in \mathcal{B}$ such that for some $l \in \{1, \dots, s\}$ and some pair $j < j'$ in $\{1, \dots, n_l\}$ the function satisfies $\varphi_b(j) > \varphi_b(j')$. Using the definition of the type functions and the matrix H , one sees that:

$$\sum_{k=0}^{\varphi_b(j)-1} d_{k,l} \leq \sum_{\bar{j}=1}^j b_{\bar{j},l} \quad \sum_{k=0}^{\varphi_b(j')} d_{k,l} > \sum_{\bar{j}=1}^{j'} b_{\bar{j},l} \implies \sum_{\bar{j}=j}^{j'} b_{\bar{j},l} < 0.$$

Therefore, there must be $\bar{j} \in \{j, \dots, j'\}$ such that $b_{\bar{j},l} < 0$ and $b \notin \mathcal{B}$. On the other hand, if we find $b_{j,l} < 0$, we can use the same argument to say that the type function φ_b cannot satisfy the previous restriction.

Corollary 5 *The size of the matrix $\mathcal{H}_{\mathcal{G}}$ for multihomogeneous systems is:*

$$\sum_{\varphi_b} \prod_{l=1}^s \prod_{k=0}^n \binom{d_{k,l}}{\bar{n}_{k,l,\varphi_b}}$$

where $\bar{n}_{l,k,\varphi_b} = |\{j \in \{1, \dots, n_l\} \mid \varphi_b(j) = k\}|$ and φ_b satisfies the restrictions of Corollary 4 and Proposition 6.

Proof Let $D \in S(\rho)$ be the cell associated to a type function φ_b for $b \in \mathcal{G}$. We can consider that this cell has a decomposition:

$$D = \sum_{k=0}^n \sum_{l=1}^s D_{k,l}$$

where $D_{k,l}$ is a cell of $\Delta_{k,l}$. The number of lattice points in D corresponds to the product over the number of lattice points in each of the $D_{k,l}$. As a face of $\Delta_{k,l}$, $D_{k,l}$ is a simplex of degree $d_{k,l}$ and dimension the number of $j \in \{1, \dots, n_l\}$ such that $\varphi_b(j) = k$.

The count follows by noticing that the lattice points in a translated simplex of degree d and dimension n of size length are contained in a simplex of degree $d - n$ and same dimension. Therefore, there are $\binom{d}{n}$ of them.

There exist exact determinantal resultant formulas for some multihomogeneous cases, obtained by using the Weyman complex and other techniques [3, 2, 13, 14, 22]. Our approach does not improve those cases, but the use of type functions might be easier to generalize to a general case. Let's give an example of the size of these matrices with respect to some of the existing formulas.

Example 7 For the polynomial system of Example 1, there are exact formulas of Sylvester type [13] which give a matrix of size 6, smaller than that of size 8.

We could also exploit the incremental algorithm for constructing the Canny-Emiris formula [5], but we would be losing the combinatorial properties. Therefore, we would not have a proof of the formula for such matrices or we wouldn't be able to guarantee that they have a non-zero determinant as in Remark 5. Moreover, such implementation requires the precomputation of mixed volumes.

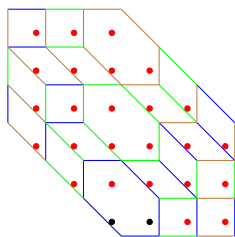
5 Conclusions

The main contribution of this paper is a first approach to estimating of the possible reduction of the size of the Canny-Emiris matrices produced by the greedy algorithm in [6]. Apart from the treated cases, we could consider other systems for which the mixed subdivision can be embedded in an n -zonotope and impose restrictions on the type functions accordingly. We could also try to drop the hypothesis that $a_{0j} \leq \dots \leq a_{n-1j}$: the examples show that, for that case, the reduction in the cells that are not in \mathcal{G} is lower. We also expect to measure when the Newton polytopes are m -zonotopes for $m > n$. In such cases, the examples show that there will still be some reduction.

Example 8 Here an example for

$$\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \{(0, 0), (1, 0), (-1, 1), (1, 1), (-1, 2), (0, 2)\} \subset \mathbb{Z}^2$$

and our choice of the mixed subdivision would give:



In this case, there is a reduction on the lattice points of the cells not in \mathcal{G} (lattice points in black), but not all the lattice points can be excluded.

This article includes an implementation of the Canny-Emiris formula in *JULIA* for the case of n -zonotopes and multihomogeneous systems which does not depend on the choice of a lifting function whose code and explanation can be found at [this link](#). The practical use of the construction of symbolic resultant matrices can be found in [10] and others.

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