



**HAL**  
open science

## Simple visibility design in network games

Gabrielle Demange

► **To cite this version:**

Gabrielle Demange. Simple visibility design in network games. David Gale, Mathematical Economist: Essays in Appreciation on his 100th Birthday, In press. hal-03925344

**HAL Id: hal-03925344**

**<https://hal.science/hal-03925344>**

Submitted on 5 Jan 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Simple visibility design in network games

Gabrielle Demange  
January 5, 2023

**Abstract** Individuals interact through social media networks by posting contributions, comments and the like. These actions entail complementarities and generate spillovers. A social media carefully designs its platform, often with the objective of increasing users' activity. Here I study visibility strategies, which promote some individuals by making their contributions more visible than others'. Under some specifications, optimal strategies are simple, making visible the individuals whose total impact (total number of followers for example) is maximal. Comparisons with other targeting strategies are discussed.

*Journal of Economic Literature Classification Number:* C72, D85, C69.

*Keywords:* networks, bilateral interactions, strategic complements, targeting.

**Acknowledgements** This is written for the volume "David Gale, Mathematical Economist: Essays in Appreciation on his 100th Birthday" edited by Ali Khan, Nobusumi Sagara and Alexander Zaslavski. I am grateful to them for this initiative and to the referee for useful comments. I thank the participants of the workshop "Digital Social Networks" at the Fondation des Treilles in Tourtour (France) where I presented part of this work.

## 1 Introduction

The tremendous growth in the social networks industry has triggered a variety of research. Statisticians and empiricists develop tools to analyze large networks and exploit the access to a huge amount of data, game theorists model how individuals form links and interact on a social network, marketing and computer scientists design targeting strategies taking advantage of detailed information on users. The growth of social networks is mainly conducted by companies, called social media platforms,

---

Gabrielle Demange  
Paris School of Economics-EHESS, 48 boulevard Jourdan, 75014 Paris, France. e-mail: demange@pse.ens.fr.

say, Facebook, Twitter or Instagram to name a few. Companies not only facilitate interactions but also shape them through the platform's design encapsulated in recommendation algorithms. The design matters for users, as illustrated by Instagram project in May 2022 to change its algorithm. In the current setup, the users see mostly pictures of friends or people they follow; the proposed algorithm would have recommended ads and videos from creators they do not follow. Given the users' protests,<sup>1</sup> the proposal is not yet implemented (as of December 2022). The design chosen by a platform, whose objective is to maximize its revenues, may create tensions with the users and lower their incentives to participate. I propose a simple analysis of one aspect of this design, namely *visibility*, by considering strategies that promote some individuals' actions.<sup>2</sup>

The analysis is based on a simple game in which individuals embedded in a social network interact directly with their neighbors. For example, in a friendship network, an individual exerts an effort level to help his/her friends and benefits from their effort to help; on a social media network, a user posts contributions accessible to his/her followers and reads the contributions of those he/she follows (his/her influencers). Individuals' payoffs positively depend on a weighted combination of their neighbors' actions, called exposure. As a result, the game exhibits strategic complementarities, meaning that the incentives for an individual to increase one's action rise with increases in the others' actions. Nash equilibria describe the full extent of these incentives, in particular they embody not only the influence of neighbors but also that of neighbors of neighbors, and, by extension, the indirect influence channelled through the network of relationships. This paper analyzes the equilibria for specific payoffs for which the best responses are linear in exposure. Equilibria are unique and easy to compute when they exist. Uniqueness allows for an analysis of design issues.

On an Internet platform, a variety of tools can be used for discriminating actions. Contributions can be presented in a biased order or displayed during different periods of time, making them more or less visible. How does a platform design visibility when its objective is to maximize aggregate activity (the sum of the actions)? I examine this question by modeling visibility in a simple way by *weights*, interpreted for example as the relative time to display contributions. Importantly, the platform, which knows the network structure, correctly anticipates how its strategies affect equilibria.

Empirical studies conducted by computer scientists provide a guide for modeling social networks, especially social media networks. Their topology is analyzed by computing various statistics such as the distributions of in- and out-degrees, clustering, the proportion of the directed links. For example, Kwak, Lee, Park and Moon (2010) analyze the topology of the follower graph on Twitter. They show important differences with other social networks; in particular the graph is directed due to the lack of reciprocity of links. An important insight of these studies is that influence -measured by the number of retweets- is related but not aligned with the number of followers. Furthermore, statistics such as centrality indices are rather bad predictors

---

<sup>1</sup> *Stop trying to be TikTok: how video-centric Instagram sparked a revolt* (Guardian 31 July 2022).

<sup>2</sup> To avoid confusion, I use the word "action" for the individuals and "strategy" for the designer.

of behaviors and real interactions. This justifies a strategic analysis that distinguishes the network structure from the behaviors of agents embedded in the network.

The problem of a designer (planner, firm) trying to influence individuals' actions in situations with complementarities has been investigated in the economics literature. Studies differ along several dimensions, in particular in the interaction model, the designer's objective and the tools of intervention. A first class of interaction models builds on Ballester, Calvo Armengol and Zenou (2006) who introduced games on a network with a linear best responses as in this paper. They consider the problem of removing a well-chosen agent -the key player- so as to minimize aggregate activity. It has been followed by an extensive literature studying interventions aiming at modifying aggregate activity, as referred to in Demange (2017), where I study an extension of the game with increasing but non-linear best responses in exposure. Galeotti, Golub and Goyal (2020) consider the maximization of welfare. These interventions modify the marginal incentives of individuals to contribute but do not change the importance of the links. In contrast, a visibility strategy as considered here (partially) changes the network structure; in particular, all the links toward an agent whose visibility weight is null are withdrawn. The problem of changing the network structure is challenging and results are limited (Ballester, Calvo Armengol and Zenou (2006) already cited and Belhaj, Bervoets, Deroian 2016). In a second class of interaction models, actions are discrete, most often binary. They describe, for instance, the choice of a location, the adoption of a new technology, or the health status, see, e.g., Schelling (1969) in a behavioral model, Morris (2003) in a strategic framework, or Domingos and Richardson (2001) in a marketing context. Analytical results on equilibria and optimal interventions are difficult to obtain in these models; simulations suggest that some characteristics of the behavioral process (say the probability of adoption as a function of neighbors and history) play an important role (Dodds and Watts (2004, Kempe, Kleinberg and Tardos 2003).

The paper relies on linear analysis (especially on positive matrices) and networks, domains to which David Gale made important contributions. He wrote a famous book on linear models (1960), proved an important result on the feasibility of flows in networks (1957). Gale and Politof (1981) extend to general networks Shapley's results on maximal network flow (1961) and complements or substitutes in assignment problems (1962).

The plan is as follows. Section 2 introduces the model and describes equilibria. Section 3 examines aggregate activity and visibility strategies.

## 2 The basic interaction model

There is a set of  $n$  agents,  $N = \{1, \dots, n\}$ , who are in a situation of interaction. Interactions are bilateral, specified by a weighted network. Sect. 2.1 describes the game and provides two illustrations. Sect. 2.2 defines and solves the equilibria of the game.

## 2.1 The game

Each agent chooses an action represented by a nonnegative real number,  $x_i$  for  $i$ . The payoff to  $i$ , denoted  $u_i$ , depends not only on  $i$ 's own action but also on others' actions. Specifically, denoting  $\mathbf{x} = (x_i)_{i \in N}$  the vector of all actions<sup>3</sup>  $u_i$  is given by:

$$u_i(\mathbf{x}) = x_i(\underline{x}_i + \sum_j X_{ij}x_j) - \frac{x_i^2}{2} \text{ for each } i \quad (1)$$

where  $\underline{x}_i$  is positive,  $X_{ij}$  is nonnegative and  $X_{ii} = 0$ .  $u_i$  is made of benefit (the first term) and cost (the second term). Cost is quadratic in the action level.  $i$ 's benefit is linear in  $i$ 's action with a return that depends on  $j$ 's action if  $X_{ij} > 0$ .  $X_{ij}$  measures the strength of this dependence and is called the *impact* of  $j$  on  $i$ . Given  $\mathbf{x}$ ,  $i$ 's *exposure* is defined as the linear combination of others' actions weighted by their impact,  $\sum_j X_{ij}x_j$ . The return to  $x_i$  is thus equal to  $\underline{x}_i$  augmented of  $i$ 's exposure.  $\underline{x}_i$  is the action that  $i$  would take if  $i$ 's exposure was null. Call  $\underline{x}_i$   $i$ 's *standalone action*. An agent  $i$  who has no interaction with others, that is  $X_{ij}$  and  $X_{ji}$  are null for each  $j$ , optimally plays the standalone action whatever other players are doing.

The game is thus characterized by the  $n$ -vector of positive standalone actions  $\underline{\mathbf{x}} = (\underline{x}_i)$  and the  $n \times n$  impact matrix  $\mathbf{X} = (X_{ij})$  with nonnegative elements and null diagonal.

### Examples

In the following two examples, the impacts are derived from an observed network of links between agents. The network has  $N$  as the set of nodes and is described by its adjacency matrix  $\mathbf{G}$ :  $G_{ij} = 1$  if  $(i, j)$  is a link and  $G_{ij} = 0$  if  $(i, j)$  is not a link.  $\mathbf{G}$  is not necessarily symmetric (Example 1) and, even if symmetric, the impact matrix  $\mathbf{X}$  derived from  $\mathbf{G}$  is not necessarily symmetric (Example 2).

**Example 1: unweighted network** Let  $\mathbf{G}$  represent the follower graph of a platform such as Twitter. Let  $G_{ij} = 1$  if  $i$  "follows"  $j$ , meaning that  $i$  receives the tweets posted by  $j$ .  $j$  thus influences or impacts  $i$ . But  $j$  does not have to reciprocate and follow  $i$ : as emphasized by Kwak et al. (2010), the Twitter follower graph is not symmetric, in contrast with social media such as Facebook where links are reciprocated. Individuals post contributions and  $x_i$  represents the number or quality of  $i$ 's contributions. Suppose that there is no congestion between followers nor influencers in the access to contributions, meaning that the impact of  $j$  on  $i$  is independent of the number of  $j$ 's followers and the number of  $i$ 's influencers. In

<sup>3</sup> In what follows, vectors and matrices are denoted in bold letters.  $\mathbf{1}$  denotes the  $n$ -vector of ones and  $\mathbb{I}_n$  the  $n$ -identity matrix. Given two  $n$ -vectors  $\mathbf{v}$  and  $\mathbf{v}'$ ,  $\mathbf{v} \geq \mathbf{v}'$  if  $v_i \geq v'_i$  for each  $i$ . Similarly, for two matrices  $\mathbf{A}$  and  $\mathbf{A}'$  of the same dimension,  $\mathbf{A} \geq \mathbf{A}'$  if  $A_{ij} \geq A'_{ij}$  for each  $(i, j)$ .  $\bar{\mathbf{v}}$  denotes the transpose of vector  $\mathbf{v}$  and similarly for a matrix. Finally, given vector  $\mathbf{v}$ ,  $\mathbf{dg}(\mathbf{v})$  denotes the diagonal matrix with  $v_j$  as the  $j$ -th element on the diagonal.

that case,  $i$ 's exposure is equal to  $\delta \sum_j G_{ij} x_j$ , the sum of the contributions of  $i$ 's influencers up to a discount factor  $\delta$  that measures the effect of contributions relative to the standalone actions and cost. Hence  $\mathbf{X} = \delta \mathbf{G}$ .

**Example 2** Let  $\mathbf{G}$  represent a network between relatives.  $G_{ij} = 1$  if  $j$  cares for  $i$ . In a friendship network,  $\mathbf{G}$  is symmetric. Let  $x_j$  represent the total effort (care or overall time) spent by  $j$  with the relatives he/she cares of. Assuming effort to be split equally among them,  $j$ 's effective effort on each relative is equal to  $\frac{x_j}{G_{+j}}$  where  $G_{+j} = \sum_i G_{ij}$  is  $j$ 's in-degree. Define  $i$ 's exposure as the sum of the effective attention  $i$  receives, up to a discount factor  $\delta$ :  $\delta \sum_j G_{ij} \frac{x_j}{G_{+j}}$ . This defines the impact matrix  $\mathbf{X}$  by  $X_{ij} = \frac{G_{ij}}{G_{+j}}$ . Defining  $j$ 's impact total by  $X_{+j} = \sum_i X_{ij}$ , total impacts are equal across individuals.

## 2.2 Equilibrium

The analysis of interactions relies on the notion of (Nash) equilibrium, which requires that no player observing others' actions could increase his/her payoff by changing his/her own action. Formally:

**Definition 1** Let  $\mathbf{x}_{-i} = (x_j)_{j \neq i}$  denote the actions of agents other than  $i$ .

$i$ 's best responses to  $\mathbf{x}_{-i}$  are  $i$ 's actions that maximize  $i$ 's payoff taking  $(\mathbf{x}_{-i})$  as given, i.e. maximize:  $x_i \rightarrow u_i(x_i, \mathbf{x}_{-i})$ .

$\mathbf{x}$  is an *equilibrium* if for each  $i$ ,  $x_i$  is a best response to others' actions  $\mathbf{x}_{-i}$ .

In our game, payoffs are concave and playing a null action is never optimal because the standalone actions are assumed to be positive. As a result, best responses are surely positive, characterized by first order conditions. Specifically,  $i$ 's best response to  $\mathbf{x}_{-i}$  is  $x_i = \underline{x}_i + \sum_j X_{ij} x_j$ , which is a linear and increasing function of  $i$ 's exposure. The game therefore exhibits *strategic complementarities*, meaning that players have incentives to (weakly) increase their own action when others' actions increase. In Example 1, strategic complementarities mean that the more numerous the influencers' contributions, the more a follower wants to contribute. In Example 2, strategic complementarities mean that the larger the relatives' effective effort toward an individual, the more effort the individual is ready to exert.

The problems of existence and computation of equilibria are not easy to solve in general since equilibria are fixed points of the collection of best responses. However, in our game, the problems are very easy to solve due to the linearity of best responses, as stated in Property 1:

*Property 1* Let  $\rho$  denote the dominant eigenvalue of the nonnegative matrix  $\mathbf{X}$ . If  $\rho < 1$ , then an equilibrium exists and is unique given by

$$\mathbf{x} = (\mathbb{I}_n - \mathbf{X})^{-1} \underline{\mathbf{x}}. \quad (2)$$

If  $\rho \geq 1$  there is no equilibrium.

Formula (2) is standard and first appears in Ballester et al. (2006), based on an explicit computation of the fixed points of the best responses. I give the proof for completeness. I also provide an alternative proof based on strategic complementarities, which allows for a better understanding of the condition for existence; furthermore it extends to games with non-linear (but increasing) best responses without invoking a topological fixed point theorem.<sup>4</sup> As shown by Topkis (1979), equilibria (if they exist) can be obtained by iterating best responses. Here, starting from standalone actions, iterated responses reinforce each other and increase. As a result, they converge to an equilibrium, except if the spillover effects due to complementarities are strong enough to make the sequence of best responses unbounded. The dominant eigenvalue of the impact matrix measures the strength of these spillover effects: when the eigenvalue is equal or greater than 1, iterated best responses become infinitely large and no equilibrium exists.

**Proof** To prove (2), recall that  $\mathbf{x}$  is an equilibrium if each  $i$  plays a best response to others' actions:  $x_i = \underline{x}_i + \sum_j X_{ij}x_j$  or in matrix form  $(\mathbb{I}_n - \mathbf{X})\mathbf{x} = \underline{\mathbf{x}}$ . According to Perron Frobenius theorem on positive matrices (see footnote 5),  $\mathbb{I}_n - \mathbf{X}$  is invertible and its inverse is positive if the dominant eigenvalue of  $\mathbf{X}$  is less than 1: (2) follows.

Consider now the alternative proof based on Topkis process of iterated best responses. Start by assuming that players exert their standalone actions. Denote  $\mathbf{x}(0) = \underline{\mathbf{x}}$ . Standalone actions are optimal against a null exposure, so, are not optimal against  $\mathbf{x}(0)$  for players who follow someone. Their best responses, denoted by  $\mathbf{x}(1)$ , are larger:  $x_i(1) = x_i(0) + \sum_j X_{ij}x_j(0)$  for each  $i$ , or  $\mathbf{x}(1) = \mathbf{x}(0) + \mathbf{X}\mathbf{x}(0)$ . But now,  $x_i(1)$  is not optimal for  $i$  against  $\mathbf{x}(1)$ ; let players adjust  $\mathbf{x}(1)$  to their best responses  $\mathbf{x}(2) = \mathbf{x}(0) + \mathbf{X}\mathbf{x}(1)$ . Iterating best responses, actions  $\mathbf{x}(t)$  at step  $t$  follow the process  $\mathbf{x}(t) = \mathbf{x}(0) + \mathbf{X}\mathbf{x}(t-1)$ . Successive adjustments thus satisfy  $\mathbf{x}(t) - \mathbf{x}(t-1) = \mathbf{X}(\mathbf{x}(t-1) - \mathbf{x}(t-2))$  for  $t \geq 1$ , which implies  $\mathbf{x}(t) - \mathbf{x}(t-1) = \mathbf{X}^{t-1}(\mathbf{x}(1) - \mathbf{x}(0))$ . Since  $\mathbf{x}(1) - \mathbf{x}(0) = \mathbf{X}\mathbf{x}(0)$ , we obtain that the adjustment at  $t$  is equal to  $\mathbf{X}^t\mathbf{x}(0)$ . Writing  $\mathbf{x}(t)$  as the sum of the successive adjustments starting from  $\mathbf{x}(0) = \underline{\mathbf{x}}$ , we obtain:

$$\mathbf{x}(t) = [\mathbb{I}_n + \mathbf{X} + \mathbf{X}^2 + \dots + \mathbf{X}^t]\underline{\mathbf{x}}. \quad (3)$$

When  $t$  increases, there are two cases.

Case 1:  $\rho < 1$ . Perron Frobenius theorem<sup>5</sup> implies that the sums  $\sum_{k=0}^t \mathbf{X}^k$  converge and their limit is the inverse of  $\mathbb{I}_n - \mathbf{X}$ :

$$(\mathbb{I}_n - \mathbf{X})^{-1} = \sum_{k=0}^{\infty} \mathbf{X}^k \quad (4)$$

Taking the limit in (3) proves (2).

<sup>4</sup> In Demange (2017), I consider best responses that are increasing in exposure. When they are not linear, the conditions for the existence of an equilibrium differ from those for their uniqueness.

<sup>5</sup> According to the theorem,  $\rho$  is at least as large as the absolute value of any other eigenvalue. This implies that for  $\rho < 1$ , the sums  $\sum_{k=0}^t \mathbf{X}^k$  converge when  $t$  increases; their limit is the inverse of  $\mathbb{I}_n - \mathbf{X}$  by taking the limit in the identity  $(\mathbb{I}_n - \mathbf{X})(\sum_{k=0}^t \mathbf{X}^k) = \mathbb{I}_n - \mathbf{X}^{t+1}$ .

Case 2:  $\rho \geq 1$ . The iterated best responses  $\mathbf{x}(t)$  become increasingly large because  $\mathbf{X}^t \underline{\mathbf{x}}$  does not tend to zero when  $t$  increases. To show this, consider the transpose of  $\mathbf{X}$ , denoted by  $\tilde{\mathbf{X}}$ . Matrix  $\tilde{\mathbf{X}}$  is a nonnegative matrix with the same eigenvalues as  $\mathbf{X}$ . Hence,  $\tilde{\mathbf{X}}$  has a nonnegative dominant eigenvector  $\tilde{\mathbf{y}}$  associated to  $\rho$ :  $\tilde{\mathbf{y}}\mathbf{X} = \rho\tilde{\mathbf{y}}$ . Iterating implies  $\tilde{\mathbf{y}}\mathbf{X}^t \underline{\mathbf{x}} = \rho^t \tilde{\mathbf{y}}\underline{\mathbf{x}}$ . Since  $\underline{\mathbf{x}}$  is positive,  $\tilde{\mathbf{y}}\underline{\mathbf{x}}$  is positive. This implies that  $\tilde{\mathbf{y}}\mathbf{X}^t \underline{\mathbf{x}}$  either stays constant and positive (for  $\rho = 1$ ) or increases indefinitely (for  $\rho > 1$ ).<sup>6</sup> Whatever case,  $\mathbf{X}^t \underline{\mathbf{x}}$  does not go to zero and iterated best responses become increasingly large when  $t$  increases.  $\square$

**Remark** For general games, an equilibrium is quite demanding in terms of computational point of view (as fixed points are) and players' rationality. As the argument is only based on best responses, the analysis applies also to a behavioral setting, in which agents' behaviors are described by exogenous reaction functions that are linear and increasing in others' actions.

### 3 Aggregate activity and visibility

Aggregate activity in our model is measured by the sum of the actions  $\sum_i x_i$ . Enhancing aggregate activity<sup>7</sup> is the objective for a social media platform, which gives free access to users and draws its revenues from ads. These revenues are linked to the time users spend on the platform, proxied by aggregate activity. Maximizing revenues is thus achieved by maximizing aggregate activity. This section assumes that the designer can change the visibility of users' actions. Subsection 3.1 introduces how visibility is modeled and presents a result that illustrates the spillover effects it generates (Proposition 1). Subsection 3.2 considers the maximization problem of aggregate activity by a designer who can change the visibility of agents' actions and who anticipates the effect of its interventions on the equilibria.

#### 3.1 Visibility strategy and supermodularity of aggregate activity

A visibility strategy is described by individuals' *multipliers* that multiply their impact. Proposition 1 shows that the increase in aggregate action from increasing the multiplier on a particular individual is larger the larger the others' multipliers are, a property called supermodularity.

Formally, a visibility strategy is described by a nonnegative vector  $\mathbf{v} = (v_i)$ ;  $v_j$  is called  $j$ 's multiplier as it has the same effect as multiplying  $j$ 's impact on each other player by  $v_j$ , i.e. each  $x_{ij}$  is modified into  $x_{ij}v_j$ . Equivalently, for each  $i$ ,  $i$ 's payoffs

<sup>6</sup> The result holds even if the sequence  $\mathbf{X}^t/\rho^t$  does not converge, which may happen when  $\mathbf{X}$  is not regular.

<sup>7</sup> Enhancing aggregate activity is an objective for a designer in other contexts. For example, a health agency aims to maximize individuals' precautionary efforts, possibly building on peer networks.



are:

$$u_i(\mathbf{x}, \mathbf{v}) = x_i(x_i + \sum_j X_{ij}v_jx_j) - \frac{x_i^2}{2}.$$

Setting a null multiplier on  $j$  eliminates  $j$ 's impact on anyone but  $j$  is still impacted by influencers' actions. In an unweighted network (Example 1), a null multiplier on  $j$  is akin to delete all links to  $j$  but not from  $j$ .

Consider equilibria associated to the payoffs transformed by  $\mathbf{v}$ . Since  $\mathbf{v}$  has the same effect as changing the matrix  $\mathbf{X}$  into  $\mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v})$ , Property 1 applies and yields the following results. Denote by  $\rho(\mathbf{v})$  the dominant eigenvalue of  $\mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v})$ . An equilibrium exists if (and only if)  $\rho(\mathbf{v}) < 1$ , in which case the equilibrium is  $\mathbf{x}(\mathbf{v}) = (\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}))^{-1}\underline{\mathbf{x}}$ . Let  $V$  measure aggregate activity given  $\mathbf{v}$ :  $V(\mathbf{v}) = \sum_i x_i(\mathbf{v})$ . Observe that lowering the multipliers, the matrix  $\mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v})$  is (elementwise) decreased hence its dominant eigenvalue is decreased as well. Thus  $\rho(\mathbf{v}) < 1$  implies that, for any  $\mathbf{v}' \leq \mathbf{v}$ , an equilibrium exists and  $V$  is well defined. The next proposition states the supermodularity of  $V$ .

**Proposition 1** *Let the dominant eigenvalue  $\rho(\mathbf{v})$  of  $\mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v})$  be strictly less than 1.  $\mathbf{v} \geq \mathbf{v}'$  implies for each  $i$*

$$V(\mathbf{v}_{-i}, v_i) - V(\mathbf{v}_{-i}, v'_i) \geq V(\mathbf{v}'_{-i}, v_i) - V(\mathbf{v}'_{-i}, v'_i). \quad (5)$$

**Proof** Let us first prove the following expression for the incremental value.

$$\begin{aligned} & V(\mathbf{v}_{-i}, v_i) - V(\mathbf{v}_{-i}, v'_i) = \\ & \tilde{\mathbf{I}}(\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v_i))^{-1}\mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{0}_{-i}, v_i - v'_i)(\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v'_i))^{-1}\underline{\mathbf{x}}. \end{aligned} \quad (6)$$

where  $\mathbf{0}_{-i}$  is the null ob  $N - i$ . Writing  $V(\mathbf{v}) = \tilde{\mathbf{I}}(\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}))^{-1}\underline{\mathbf{x}}$ , we have

$$\begin{aligned} & V(\mathbf{v}_{-i}, v_i) - V(\mathbf{v}_{-i}, v'_i) = \\ & \tilde{\mathbf{I}}[(\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v_i))^{-1} - (\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v'_i))^{-1}]\underline{\mathbf{x}}. \end{aligned} \quad (7)$$

Rewrite the matrix inside the square brackets by using the following computation. Letting  $\mathbf{A} = \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v_i)$  and  $\mathbf{A}' = \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v'_i)$ :

$$\begin{aligned} (\mathbb{I}_n - \mathbf{A})^{-1} - (\mathbb{I}_n - \mathbf{A}')^{-1} &= (\mathbb{I}_n - \mathbf{A})^{-1}[(\mathbb{I}_n - \mathbf{A}') - (\mathbb{I}_n - \mathbf{A})](\mathbb{I}_n - \mathbf{A}')^{-1} \\ &= (\mathbb{I}_n - \mathbf{A})^{-1}(\mathbf{A} - \mathbf{A}')(\mathbb{I}_n - \mathbf{A}')^{-1}. \end{aligned}$$

We obtain:

$$\begin{aligned} & (\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v_i))^{-1} - (\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v'_i))^{-1} \\ &= (\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v_i))^{-1}\mathbf{X}(\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v_i) - \mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v'_i))(\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v'_i))^{-1} \end{aligned}$$

Now  $\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v_i) - \mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v'_i)$  is the diagonal matrix whose elements are all null except the  $i$ -th diagonal term which is positive equal to  $v_i - v'_i$ . Using (7), this proves (6).

To end the proof, observe that the incremental value  $V(\mathbf{v}_{-i}, v_i) - V(\mathbf{v}_{-i}, v'_i)$  is obtained by multiplying the product of the three matrices in (6) by the positive vectors  $\mathbf{1}$  (on the left) and  $\underline{\mathbf{x}}$  (on the right). For  $v_i \geq v'_i$  the matrix  $\mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{0}, v_i - v'_i)$  has all its elements nonnegative. The two matrices on the left and the right in (6) are nonnegative. To prove the inequality (5), it thus suffices to show that the two positive matrices on the left and the right in (6) are (weakly) decreased elementwise when  $\mathbf{v}_{-i}$  is decreased to  $\mathbf{v}'_{-i}$ . This holds true since

$(\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v_i))^{-1} \geq (\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}'_{-i}, v_i))^{-1}$ : Matrix  $\mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v_i)$  is obtained by multiplying the elements in  $j$ 's column of  $\mathbf{X}$  by  $v_j$ . Hence  $\mathbf{v}_{-i} \geq \mathbf{v}'_{-i}$  implies  $\mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v_i) \geq \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}'_{-i}, v_i)$ . Formula (4) implies the desired inequality.

$(\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}_{-i}, v'_i))^{-1} \geq (\mathbb{I}_n - \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{v}'_{-i}, v'_i))^{-1}$ : same proof.  $\square$

### 3.2 Manager's visibility strategies

This section investigates strategies that modify the visibility of individuals' actions with the objective of maximizing aggregate activity. Of course, without constraints, activity can be made infinitely large. I assume here a constraint on the overall time to display actions, as I describe now. For that purpose, it is more convenient to normalize the multipliers and introduce a display factor.

A *visibility* strategy is of the form  $(\mathbf{w}, \tau)$  where  $\mathbf{w} = (w_i)$  are nonnegative and sum to  $n$  and  $\tau$  is a positive scalar. Call  $\mathbf{w}$  the *weights* and  $\tau$  the *display factor*.  $\mathbf{w}$  determines the relative visibility of individuals' actions:  $i$ 's actions are presented  $w_i/w_j$  times more than  $j$ 's ones. Given the display factor  $\tau$ ,  $j$ 's actions have the same impact on others as if there were multiplied by  $\tau w_j$ . Setting  $v_i = \tau w_i$  for each  $i$ , strategy  $(\mathbf{w}, \tau)$  is equivalent to the multipliers  $\mathbf{v}$  as defined in the previous section. The chosen representation makes the interpretation easier.

The overall time to display actions must be unchanged by the strategies. Specifically, let one unit of action be displayed per unit of time (a normalization). The overall time to display actions  $\mathbf{x}$  is  $\sum_i x_i$ . Since strategy  $(\mathbf{w}, \tau)$  transforms this overall time into  $\tau \sum_i w_i x_i$ , the time constraint writes

$$\tau \sum_i w_i x_i \leq \sum_i x_i. \quad (8)$$

Basically, given the weights, the overall time to display  $\mathbf{x}$  is reallocated across individuals.

As explained in the previous section, strategy  $(\mathbf{w}, \tau)$  transforms the impact matrix  $\mathbf{X}$  into  $\tau \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{w})$ , which has dominant eigenvalue  $\tau \rho(\mathbf{w})$ . An equilibrium exists if (and only if)  $\tau \rho(\mathbf{w}) < 1$ , in which case the equilibrium is unique given by  $\mathbf{x}(\mathbf{w}, \tau) = (\mathbb{I}_n - \tau \mathbf{X}\mathbf{d}\mathbf{g}(\mathbf{w}))^{-1} \underline{\mathbf{x}}$ .

The designer objective is to find a visibility strategy that maximizes aggregate activity over all strategies that satisfy the time constraint. The designer has full information on the game (i.e. on  $\mathbf{X}$  and  $\mathbf{x}$ ) and anticipates how interventions affect the equilibria. This leads to the following definitions of feasible and optimal strategies.

**Definition 2** A visibility strategy  $(\mathbf{w}, \tau)$  is said to be *feasible* if an equilibrium exists, i.e. if  $\tau\rho(\mathbf{w}) < 1$ , and the equilibrium  $\mathbf{x}(\mathbf{w}, \tau)$  satisfies the following constraint

$$\tau \sum_i w_i x_i(\mathbf{w}, \tau) \leq \sum_i x_i(\mathbf{w}, \tau). \quad (9)$$

The feasible strategy  $(\mathbf{w}, \tau)$  is *optimal* if it maximizes aggregate activity  $\sum_i x_i(\mathbf{w}, \tau)$  over all the feasible strategies.

Strategy  $(\mathbf{1}, 1)$  does not modify the impacts. It is therefore feasible if  $\rho(\mathbf{X}) < 1$  but typically not optimal, as the next proposition shows.  $j$ 's impact total is defined as the sum of  $j$ 's impact values:  $X_{+j} = \sum_i X_{ij}$  (it is equal to in-degree in a graph)

**Proposition 2** Let  $\sigma$  denote the maximal impact total over  $N$ :  $\sigma = \max_j X_{+j}$ .

If  $\sigma < 1$ , then optimal visibility strategies exist. An optimal strategy  $(\mathbf{w}, \tau)$  sets positive weights on the individuals whose impact total is maximal, i.e.  $w_j > 0$  only if  $X_{+j} = \sigma$ , and sets the factor  $\tau$  to the maximal level so that constraint (9) is binding. The aggregate activity is equal to  $\frac{\sum_i x_i}{1-\sigma}$ .

If  $\sigma \geq 1$ , then aggregate activity can be made arbitrarily large by feasible visibility strategies.

**Proof** Let us first prove three claims.

**Claim 1** Let  $\mathbf{w}$  be a weight vector. Assume  $\rho(\mathbf{w}) > 0$ . Then

$$\lim_{\tau \xrightarrow{\leftarrow} \frac{1}{\rho(\mathbf{w})}} \sum_j w_j x_j(\mathbf{w}, \tau) = \infty. \quad (10)$$

**Proof** To prove (10), let  $\mathbf{y}$  be a nonnegative left eigenvector of  $\mathbf{X}d\mathbf{g}(\mathbf{w})$ :  $\tilde{\mathbf{y}}\mathbf{X}d\mathbf{g}(\mathbf{w}) = \rho(\mathbf{w})\tilde{\mathbf{y}}$ . Applying (4) and multiplying by  $\tilde{\mathbf{y}}$  yields  $\tilde{\mathbf{y}}(\mathbb{I}_n - \tau\mathbf{X}d\mathbf{g}(\mathbf{w}))^{-1} = \frac{1}{1-\tau\rho(\mathbf{w})}\tilde{\mathbf{y}}$ . Multiplying by  $\mathbf{x}$  yields:  $\tilde{\mathbf{y}}\mathbf{x}(\mathbf{w}, \tau) = \frac{1}{1-\tau\rho(\mathbf{w})}\tilde{\mathbf{y}}\mathbf{x}$ . As  $\tilde{\mathbf{y}}\mathbf{x} > 0$ , this implies  $\lim_{\tau \xrightarrow{\leftarrow} \frac{1}{\rho(\mathbf{w})}} \sum_j y_j x_j(\mathbf{w}, \tau) = \infty$ . Since  $\tilde{\mathbf{y}}\mathbf{X}d\mathbf{g}(\mathbf{w}) = \rho(\mathbf{w})\tilde{\mathbf{y}}$ ,  $\mathbf{y}$  has all its components null outside the support of  $\mathbf{w}$ . Thus  $\mathbf{y} \leq k\mathbf{w}$  for some positive constant  $k$ . (10) follows.  $\square$

(10) is not necessarily true if  $\rho(\mathbf{w}) = 0$ . For example, set all the weights null except for one agent, say agent 1, the dominant eigenvalue is null and 1's action is equal to the standalone value whatever  $\tau$  so that (10) fails. Claim 1 extends as follows.

**Claim 2:** Let  $\mathbf{w}$  be a weight vector. By convention set  $1/\rho(\mathbf{w}) = \infty$  if  $\rho(\mathbf{w}) = 0$ . Then

$$\lim_{\tau \xrightarrow{\leftarrow} \frac{1}{\rho(\mathbf{w})}} \tau \sum_j w_j x_j(\mathbf{w}, \tau) = \infty. \quad (11)$$

**Proof** If  $\rho(\mathbf{w}) > 0$ , (11) follows directly from Claim 1. If  $\rho(\mathbf{w}) = 0$ , it suffices to observe that  $x_j(\mathbf{w}, \tau) \geq \underline{x}_j$  for each  $j$  and standalone actions are positive.  $\square$

**Claim 3** Let  $\mathbf{w}$  be a weight vector. The following inequality holds at equilibrium:

$$\sum_i x_i(\mathbf{w}, \tau) \leq \sum_i \underline{x}_i + \tau \sigma \sum_j w_j x_j(\mathbf{w}, \tau) \quad (12)$$

with an equality for  $\mathbf{w}$  with support included in  $I^{\max}$ .

**Proof** At equilibrium,  $\mathbf{x}(\mathbf{w}, \tau) - \tau \mathbf{X} \mathbf{d} \mathbf{g}(\mathbf{w}) \mathbf{x}(\mathbf{w}, \tau) = \underline{\mathbf{x}}$  implies:

$$\begin{aligned} \sum_i x_i(\mathbf{w}, \tau) &= \sum_i \underline{x}_i + \tau \sum_i (\sum_j X_{ij} w_j x_j) \\ &= \sum_i \underline{x}_i + \tau \sum_j [\sum_i X_{ij}] w_j x_j \end{aligned}$$

By definition  $\sum_i X_{ij} \leq \sigma$  for any  $j$  with an equality for  $j$  in  $I^{\max}$ : this proves (12).  $\square$

To proceed we distinguish two cases according to the value of  $\sigma$ .

**Case 1:**  $\sigma \geq 1$ . Let us show that aggregate activity is unbounded at feasible strategies. Consider  $\mathbf{w}$  with support in  $I^{\max}$  and values of  $\tau$  in  $[0, 1/\rho(\mathbf{w})[$ . (12) is satisfied as an equality. Thus  $\sum_i x_i(\mathbf{w}, \tau) \geq \tau \sigma (\sum_j w_j x_j(\mathbf{w}, \tau))$ . Since  $\sigma \geq 1$ , this implies that strategy  $(\mathbf{w}, \tau)$  is feasible for any  $\tau$  in  $[0, 1/\rho(\mathbf{w})[$ . As  $\lim_{\tau \rightarrow \infty} \tau \sum_j w_j x_j(\mathbf{w}, \tau) = \infty$  from Claim 2, it follows that aggregate activity is unbounded at feasible strategies.

**Case 2:**  $\sigma < 1$ . Let us first show that given  $\mathbf{w}$  there is a maximal value of  $\tau$  for which the strategy  $(\mathbf{w}, \tau)$  is feasible. The feasibility constraint writes as  $\phi(\tau) \geq 0$  where the function  $\phi$  is defined for  $\tau$  on  $[0, 1/\rho(\mathbf{w})[$  by:  $\phi(\tau) = \sum_i x_i(\mathbf{w}, \tau) - \tau \sum w_i x_i(\mathbf{w}, \tau)$ . When  $\tau$  is null, the equilibrium is equal to  $\underline{\mathbf{x}}$  and  $\phi(0) = \sum_i \underline{x}_i$  is positive. Let us show  $\phi(\tau) < 0$  for large enough  $\tau$ . By (12),  $\phi(\tau) \leq \sum_i \underline{x}_i + (\sigma - 1)\tau \sum_j w_j x_j(\mathbf{w}, \tau)$ . By Claim 2,  $\lim_{\tau \xrightarrow{\leq} \frac{1}{\rho(\mathbf{w})}} \tau \sum_j w_j x_j(\mathbf{w}, \tau) = \infty$ . Hence  $\sigma < 1$  implies  $\phi(\tau) < 0$  for  $\tau$  large enough in  $[0, 1/\rho(\mathbf{w})[$ . Since  $\phi(0) > 0$  and  $\phi$  is continuous, this proves that the set of  $\tau$  in  $[0, 1/\rho(\mathbf{w})[$  for which  $\phi(\tau) \geq 0$  has a maximum  $\tau^*$  and that  $\phi(\tau^*) = 0$ , i.e., the feasibility constraint (9) is binding at  $\tau^*$ .

To conclude the proof, let us show that aggregate activity is maximized at a feasible strategy with support on  $I^{\max}$ . At a feasible strategy, the display constraint (9) together with (12) imply

$$\sum_i x_i(\mathbf{w}, \tau) \leq \sum_i \underline{x}_i + \sigma \left( \sum_j x_j(\mathbf{w}, \tau) \right). \quad (13)$$

with an equality only if both (9) and (12) are satisfied as equality. It follows that aggregate activity is at most equal to  $\frac{\sum_i \underline{x}_i}{1-\sigma}$  and strictly less than this upper bound for  $\mathbf{w}$  with support not included in  $I^{\max}$ . For  $\mathbf{w}$  with support on  $I^{\max}$ , choosing the value  $\tau^*$  for which (9) is satisfied as an equality, (13) is satisfied as an equality: at  $(\mathbf{w}, \tau^*)$ , aggregate activity is equal to its upper bound  $\frac{\sum_i \underline{x}_i}{1-\sigma}$ .  $\square$

Interestingly, optimal visibility strategies are simple and easy to compute. Information on the standalone actions are not necessary. In an unweighted network (Example 1), it suffices to observe the in-degrees. An optimal strategy makes visible only the actions of individuals with maximal impact total. In Example 2 where each individual shares effort among those impacted, impact totals are all equal, hence interventions have all the same effect on aggregate activity and are useless.<sup>8</sup>

In general  $\sigma$  is at least as large as the dominant eigenvalue  $\rho$ . But the optimality of a strategy is not based on raising the dominant eigenvalue of the transformed matrix to  $\sigma$ . In fact, the dominant eigenvalue is even null for a strategy making a single agent in  $I^{max}$  visible, as is surely the case if  $I^{max}$  is a singleton. Let this agent be 1. Then  $w_1 = n$  and all other weights are null. All the eigenvalues of matrix  $\mathbf{X}d\mathbf{g}(\mathbf{w})$  are null since  $(\mathbf{X}d\mathbf{g}(\mathbf{w}))^2$  is null (recall that  $X_{11}$  is null). The equilibrium is easy to compute given  $\tau$ : player 1 is not impacted by anyone and plays 1's standalone action  $x_1 = \underline{x}_1$ . The other players are impacted by 1 only:  $x_i = \underline{x}_i + n\tau X_{i1}\underline{x}_1$ . The maximal value of  $\tau$  satisfies  $\tau n\underline{x}_1 = \sum_i \underline{x}_i + n\sigma\underline{x}_1$ . The increase in aggregate activity is due to the display time, not to the spillovers between actions since all the cycles are cut. If there are several agents in  $I^{max}$ , other strategies are optimal, making several agents visible and generating spillover between them (if their mutual impacts are positive). The equilibrium follows a simple 'hierarchical' structure: compute the equilibrium between agents in  $I^{max}$  and derive others' actions by their exposure to these equilibrium actions. The additional activity generated by making several agents interacting between each other and impacting other because of their visibility is factored out by lowering the weights of each and the display unit time.

### 3.3 Discussion

Different visibility strategies could be considered. For example, let the designer change the relative individual impacts without display time consideration. A strategy is simply described by weights  $\mathbf{w}$ , as  $\tau$  is fixed equal to 1. Optimal strategies are difficult to characterize in that setting. The difficulty comes from the fact that aggregate activity is neither concave nor convex in the weights. The supermodularity of  $V$  (Proposition 1) implies its convexity in each individual weight but the overall convexity may not hold due to cross-derivatives. We cannot use a first order approach nor conclude that a single agent is made visible (as is the case for a convex  $V$ ).

Interventions that change standalone actions are much easier to analyze than visibility strategies. The reason is that aggregate activity is linear in  $\underline{\mathbf{x}}$ , given by  $\tilde{\mathbf{I}}(\mathbb{I}_n - \mathbf{X})^{-1}\underline{\mathbf{x}}$ . In Demange (2017), I consider a designer who is endowed with an amount that can be allocated to agents to increase their standalone actions, i.e. their marginal return to actions. An optimal strategy is characterized by centrality indices computed in the direction of influence, i.e. on  $\tilde{\mathbf{X}}$ , the transpose of  $\mathbf{X}$ . Specifically,

---

<sup>8</sup> With equal total sums,  $\sigma = \rho$  and  $\tilde{\mathbf{I}}\mathbf{X} = \sigma\mathbf{1}$ . It follows that aggregate activity without intervention is already equal to  $\frac{\sum_i \underline{x}_i}{1-\sigma}$ .

consider the power/prestige indices introduced in sociology by Katz (1953) and Bonacich (1987) defined by  $(\mathbb{I}_n - \tilde{\mathbf{X}})^{-1} \mathbf{1}$ .<sup>9</sup> The amount is optimally allocated to the agents whose index is maximal. Optimal strategies are thus very easy to describe and depend only of the positions of the agents in the network. When best responses are not linear in exposure, however, optimal strategies are not so easy to describe as aggregate activity is no longer linear in standalone actions. In that case, optimal strategies depend on the concavity/convexity of the best response functions and the amount to be allocated.

## References

- Ballester, C., Calvo Armengol, A., & Zenou, Y. (2006). Who's who in networks. wanted: the key player. *Econometrica*, 74(5), 1403-1417.
- Belhaj, M., Bervoets, S., & Deroian, F. (2016). Efficient networks in games with local complementarities. *Theoretical Economics*, 11(1), 357-380.
- Bonacich P. (1987) "Power and centrality: a family of measures", *American Journal of Sociology*, 92(5), 1170-1182.
- Demange, G. (2017). Optimal targeting strategies in a network under complementarities. *Games and Economic Behavior*, 105, 84-103.
- Dodds, P. S., & Watts, D. J. (2004). Universal behavior in a generalized model of contagion. *Physical review letters*, 92(21), 218701.
- Domingos, P., & Richardson, M. (2001). Mining the network value of customers. In *Proceedings of the seventh ACM SIGKDD international conference on Knowledge discovery and data mining* 57-66.
- Gale, D. (1957). A theorem on flows in networks. *Pacific J. Math*, 7(2), 1073-1082.
- Gale, D. (1960). *The theory of linear economic models*. first edition McGraw-Hill Book Company, London 1960. Reprinted University of Chicago press, 1989.
- Gale, D., & Politof, T. (1981). Substitutes and complements in network flow problems. *Discrete Applied Mathematics*, 3(3), 175-186.
- Eckles, D., Kizilcec, R. F., & Bakshy, E. (2016). Estimating peer effects in networks with peer encouragement designs. *Proceedings of the National Academy of Sciences*, 113(27), 7316-7322.
- Galeotti, A., Golub, B., & Goyal, S. (2020). Targeting interventions in networks. *Econometrica*, 88(6), 2445-2471.
- Halberstam, Y., & Knight, B. (2016). Homophily, group size, and the diffusion of political information in social networks: Evidence from Twitter. *Journal of public economics*, 143, 73-88.
- Kwak, H., Lee, C., Park, H., & Moon, S. (2010, April). What is Twitter, a social network or a news media?. In *Proceedings of the 19th international conference on World wide web* (pp. 591-600).
- Katz L. (1953). A New Status Index Derived from Sociometric Analysis, *Psychometrika*, 18(1), 39-43.
- Kempe, D., Kleinberg, J., & Tardos, E. (2003). Maximizing the spread of influence through a social network. In *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining* 137-146.
- Morris, S. (2000). Contagion. *The Review of Economic Studies*, 67(1), 57-78.
- Richardson, M., & Domingos, P. (2002). Mining knowledge-sharing sites for viral marketing. In *Proceedings of the eighth ACM SIGKDD international conference on Knowledge discovery and data mining* 61-70.

---

<sup>9</sup> When standalone actions are equal across individuals, actions are proportional to similar indices but computed on  $\mathbf{X}$ , i.e. in the direction of following

- Schelling, T. C. (1969). Models of segregation. *The American Economic Review*, 59(2), 488-493.
- Shapley, L. S. (1961). On network flow functions. *Naval Research Logistics Quarterly*, 8(2), 151-158.
- Shapley, L. S. (1962). Complements and substitutes in the optimal assignment problem. *Naval Research Logistics Quarterly*, 9(1), 45-48.
- Topkis, D. M. (1979). Equilibrium points in nonzero-sum  $n$ -person submodular games. *SIAM Journal on Control and Optimization*, 17(6), 773-787.