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PDE FOR THE JOINT LAW OF THE PAIR OF A CONTINUOUS DIFFUSION AND ITS RUNNING MAXIMUM

LAURE COUTIN, MONIQUE PONTIER,* *IMT*

Abstract

Let X be a d -dimensional diffusion and M the running supremum of its first component. In this paper, we show that for any $t > 0$, the density (with respect to the $d + 1$ -dimensional Lebesgue measure) of the pair (M_t, X_t) is a weak solution of a Fokker-Planck partial differential equation on the closed set $\{(m, x) \in \mathbb{R}^{d+1}, m \geq x^1\}$, using an integral expansion of this density.

Keywords: Diffusion process, partial differential equation, running supremum process, joint law.

2020 Mathematics Subject Classification: Primary 60J60

Secondary 60G70,60H10

1. Introduction

The goal of this paper is to study the law of the pair (M_t, X_t) where X is a d -dimensional diffusion and M is the running maximum of the first component. In a previous work [9], using Malliavin calculus and specifically Nualart's seminal book [21], we have proved that, for any $t > 0$ the law of $V_t := (M_t, X_t)$ is absolutely continuous with respect to the Lebesgue measure with density $p_V(\cdot; t)$, and that the support of this density is included in the set $\{(m, x) \in \mathbb{R}^{d+1}, m \geq x^1\}$.

In the present work, we prove that the density p_V is a weak solution of a partial differential equation (PDE). Furthermore, we exhibit a boundary condition on the set $\{(m, x) \in \mathbb{R}^{d+1}, m = x^1\}$. This work extends the results given in [8] and in Ngom's

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thesis [20] obtained in the case where X is a Lévy process and where it is proved that the density is a weak solution to an integro-differential equation.

In the literature, there exist many studies on the law of V_t . When the process X is a Brownian motion, one can refer to [15, 17] where an explicit expression of p_V is given. When X is a one-dimensional linear diffusion, [11] provides an expression of p_V using the scale function, the speed measure and the density of the law of some hitting times. See also [1, 4] for the particular case of Ornstein-Uhlenbeck process. For some applications to the local score of a biologic sequence, the case of reflected Brownian motion is presented in [19]. The law of the maximum M_t is studied in [2] for general Gaussian processes. The case of a Lévy process X is deeply investigated in the literature, see for instance [12, 20]. Moreover Section 2.4 in Ngom's thesis [20] provides the existence and the regularity of the joint law density of the process (M_t, X_t) for a Lévy process X . In the case where X is a martingale (see e.g. [22, 13] or [10, 16]), the law of the running maximum is provided. Such studies concerning this running maximum are useful in financial area which involve hitting times, for instance for the pricing of barrier option. It is known that the law of hitting times is closely related to the one of the running maximum, see [6, 7, 23]. As an application of our work, think of a firm the activity of which is characterized by a set of processes (X^1, \dots, X^d) . But one of them, e.g., X^1 could be linked to an alarm, namely: when there exists $s \leq t$ such that X_s^1 exceeds a threshold a , that is equivalent to $M_t = \sup_{0 \leq s \leq t} X_s^1 \geq a$, some action is important to operate. So, the firm needs to know the law of such pair (M_t, X_t) ; more specifically the law of the stopping time $\tau_a = \inf\{u, X_u^1 \geq a\}$, is linked to the law of M as following: $\{\tau_a \leq t\} = \{M_t \geq a\}$. To know the probability of such an alert, the law of the pair (M_t, X_t) will be useful.

We provide an infinite expansion of the density of the law of the pair (M_t, X_t) which can leads to numerical approximation.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a d -dimensional Brownian. Let X be the diffusion process with values in \mathbb{R}^d solution of

$$dX_t = B(X_t)dt + A(X_t)dW_t, \quad t > 0 \quad (1)$$

where X_0 is a random variable independent of the Brownian motion W , with law μ_0 , and A (resp. B) is a map from \mathbb{R}^d to the set of $(d \times d)$ matrices (resp. to \mathbb{R}^d). Let

us denote $C_b^i(\mathbb{R}^d, \mathbb{R}^n)$ the set of functions on \mathbb{R}^d , which are i times differentiable, bounded, with bounded derivatives, taking their values in \mathbb{R}^n . Let $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ be the completed right-continuous filtration defined by $\mathcal{F}_t := \sigma\{X_0, W_s, s \leq t\} \vee \mathcal{N}$ where \mathcal{N} is the set of negligible sets of \mathcal{F} .

Under classical assumptions on A and B (cf.(4) and (5) below), then according to [9], for all $t > 0$, the law of $V_t = (\sup_{u \leq t} X_t^1, X_t)$ has a density with respect to the Lebesgue measure on \mathbb{R}^{d+1} .

The main results and notations are given in Section 2: in the d -dimensional case, under a quite natural assumption (meaning Hypothesis 2.1 below) on the regularity of p_V around the boundary of Δ , p_V is a weak solution of a Fokker-Planck PDE on the subset of \mathbb{R}^{d+1} defined by $\{(m, x), m \geq x^1\}$. When $A = I_d$, this assumption is satisfied, see Theorem 2.4. The main results are proved in Section 3 under Hypothesis 2.1. Section 4 is devoted to prove that Hypothesis 2.1 is satisfied when $A = I_d$. The main tool is an infinite expansion of p_V given in Proposition 3.2. In Section 5, one-dimensional case, a Lamperti transformation [18] allows to get the main result for any $A \in C_b^2(\mathbb{R}, \mathbb{R})$. Finally Appendix contains some technical tools useful for the proofs of main results.

2. Main results and some notations

In this section, we give our main results, the proofs will be given later on, as it is mentioned in the introduction.

2.1. Notations

Let Δ be the open set of \mathbb{R}^{D+1} given by $\Delta := \{(m, x), m \in \mathbb{R}, x \in \mathbb{R}^d, m > x^1, x = (x^1, \dots, x^d)\}$. From now on, we use Einstein's convention. The infinitesimal generator \mathcal{L} of the diffusion X defined in (1) is the partial differential operator on the space $C^2(\mathbb{R}^d, \mathbb{R})$ given by:

$$\mathcal{L} = B^i \partial_{x_i} + \frac{1}{2} (AA^t)^{ij} \partial_{x_i, x_j}^2. \quad (2)$$

where A^t denotes the transposed matrix.

Its adjoint operator is $\mathcal{L}^* f = \frac{1}{2} \Sigma^{ij} \partial_{ij}^2 f - [B^i - \partial_j (\Sigma^{ij})] \partial_i f - [\partial_i B^i - \frac{1}{2} \partial_{ij}^2 (\Sigma^{ij})] f$ where $\Sigma := AA^t$. In what follows, the operators \mathcal{L} and \mathcal{L}^* are extended to the space $C^2(\mathbb{R}^{d+1}, \mathbb{R})$,

for $\Phi \in C^2(\mathbb{R}^{d+1}, \mathbb{R})$ as

$$\mathcal{L}(\Phi)(m, x) = B^i(x)\partial_{x_i}\Phi(m, x) + \frac{1}{2}\Sigma^{ij}(x)\partial_{x_i x_j}^2\Phi(m, x),$$

and $\mathcal{L}^*(\Phi)(m, x) =$

$$\frac{1}{2}\Sigma^{ij}(x)\partial_{ij}^2\Phi(m, x) - [B^i - \partial_j(\Sigma^{ij})](x)\partial_{x_i}\Phi(m, x) + [\frac{1}{2}\partial_{x_i x_j}^2\Sigma^{ij} - \partial_{x_i}B^i](x)\Phi(m, x).$$

It can be stressed that these operators are degenerated since no derivative with respect to the variable m appears.

Let $A^1(x)$ be the d dimensional vector $A^1(x) = (A_j^1(x), j = 1, \dots, d) \in \mathbb{R}^d$ corresponding to the first column of $A(x)$, similarly $A_j(x)$ denotes its j th line.

Recall that M denotes the running maximum of the first component of X , meaning $M_t = \sup_{0 \leq s \leq t} \{X_s^1\}$ and V is the \mathbb{R}^{d+1} -valued process defined by $(V_t = (M_t, X_t), \forall t \geq 0)$. Finally, $\tilde{x} \in \mathbb{R}^{d-1}$ denotes the vector (x_2, \dots, x_d) .

In [9], under Assumptions (4) and (5) below, when the initial value is deterministic, $X_0 = x_0 \in \mathbb{R}^d$, the density of V_t exists and is denoted $p_V(\cdot; t, x_0)$. If μ_0 is the distribution of X_0 , the density of the law of V_t with respect to the Lebesgue measure on \mathbb{R}^{d+1} is

$$p_V(\cdot; t, \mu_0) := \int_{\mathbb{R}^d} p_V(\cdot; t, x_0) d\mu_0(x_0) \quad (3)$$

When there is no ambiguity, the dependency in μ_0 is omitted.

Since $M_t \geq X_t^1$, the support of $p_V(\cdot; t, \mu_0)$ is contained in $\bar{\Delta} := \{(m, x) \in \mathbb{R}^{d+1} | m \geq x^1\}$.

2.2. Mains results

The aim of this article is to show that the density p_V is a weak solution of a Fokker-Planck PDE. The coefficients B and A are assumed to satisfy

$$B \in C_b^1(\mathbb{R}^d, \mathbb{R}^d) \text{ and } A \in C_b^2(\mathbb{R}^d, \mathbb{R}^{d \times d}), \quad (4)$$

and that there exists a constant $c > 0$ such that the Euclidean norm of any vector v satisfies

$$c\|v\|^2 \leq v^t A(x) A^t(x) v, \quad \forall v, x \in \mathbb{R}^d. \quad (5)$$

Our first result will be established under the following hypothesis which is a quite natural assumption on the regularity of p_V in the neighbourhood of the boundary of Δ since the set of times where the process M increases is included in the set $\{t, M_t = X_t\}$:

Hypothesis 2.1. *The density of the law of $V_t = (M_t, X_t)$, denoted by p_V (3), satisfies*
 (i) *the map $(t, m, \tilde{x}) \mapsto \sup_{u>0} p_V(m, m - u, \tilde{x}; t)$ belongs to $L^1([0, T] \times \mathbb{R}^d, dt dm d\tilde{x})$.*
 (ii) *for all $t > 0$ almost surely in $(m, \tilde{x}) \in \mathbb{R}^d$, $\lim_{u \rightarrow 0^+} p_V(m, m - u, \tilde{x}; t)$ exists and is denoted by $p_V(m, m, \tilde{x}; t)$.*

Theorem 2.2. *Assume that A and B fulfil (4) and (5) and that (M, X) fulfils Hypothesis 2.1. Then, for all initial law μ_0 and $F \in C_b^2(\mathbb{R}^{d+1}, \mathbb{R})$:*

$$\begin{aligned} \mathbb{E}[F(M_t, X_t)] &= \mathbb{E}[F(X_0^1, X_0)] + \int_0^t \mathbb{E}[\mathcal{L}(F)(M_s, X_s)] ds \\ &\quad + \frac{1}{2} \int_0^t \mathbb{E} \left[\partial_m F(X_s^1, X_s) \|A^1(X_s)\|^2 \frac{p_V(X_s^1, X_s; s)}{p_X(X_s; s)} \right] ds. \end{aligned} \quad (6)$$

Actually p_X is the solution of the PDE $\partial_t p = \mathcal{L}^* p$, $p(\cdot; 0) = \mu_0$, where $\mathcal{L}^* f = \frac{1}{2} \Sigma^{ij} \partial_{ij}^2 f - [B^i - \partial_j(\Sigma^{ij})] \partial_i f - [\partial_i B^i - \frac{1}{2} \partial_{ij}^2(\Sigma^{ij})] f$. Let $a_{ij} := \Sigma^{ij}$, $a_i := [B^i - \partial_j(\Sigma^{ij})] \partial_i$, and $a_0 := \partial_i B^i - \frac{1}{2} \partial_{ij}^2(\Sigma^{ij})$. Under Assumptions (4) and (5), the operator \mathcal{L}^* satisfies all the assumptions of Theorem 3.5 [14] (see (3.2) (3.3) 3.4 page 177). As a consequence of Theorem 3.5 line 14 $p_X(x; s) > 0$.

Remark 2.1. (i) When A is the identity matrix of \mathbb{R}^d (denoted by I_d) and $B \in C_b^1(\mathbb{R}^d, \mathbb{R}^d)$, Hypothesis 2.1 is fulfilled, see Theorem 2.4 below. When $d = 1$, using a Lamperti transformation [18], one proves that Hypothesis 2.1 is always fulfilled, see Section 5.

(ii) This result is similar to Theorem 2.1 in [8] where the process X is a Lévy process. Proposition 4 in [8] gives a key of the last term in (6) with factor $\frac{1}{2}$. Firstly, roughly speaking, the local behaviour of $X_t^1 - X_s^1$ conditionally to \mathcal{F}_s is the one of $\|A^1(X_s)\|(W_t^1 - X_s^1)$. So, as in the Brownian case, the running maximum M of X^1 is increasing as soon as it is equal to X^1 and both M and X^1 are increasing; it is well known that the Brownian process W^1 is increasing with probability $\frac{1}{2}$, more specifically, we have $\mathbb{P}\{\lim_{t \rightarrow s^+} \frac{W_t^1 - W_s^1}{t-s} = -\infty\} = \mathbb{P}\{\lim_{t \rightarrow s^+} \frac{W_t^1 - W_s^1}{t-s} = +\infty\} = \frac{1}{2}$.

The starting point of the proof of Theorem 2.2 is the Itô's formula: let F belong

to $C_b^2(\mathbb{R}^{d+1}, \mathbb{R})$. The process M is increasing, hence $V = (M, X)$ is a semi-martingale. Applying Itô's formula to $F(V)$ and taking expectation of both members,

$$\mathbb{E}[F(V_t)] = \mathbb{E}[F(V_0)] + \int_0^t \mathbb{E}[\mathcal{L}(F)(V_s)] ds + \mathbb{E}\left[\int_0^t \partial_m F(V_s) dM_s\right].$$

The novelty comes from the third term of the right member of the previous equation. The following theorem proved in Section 3 achieves the proof of Theorem 2.2.

Theorem 2.3. *Assume that A and B fulfil (4) and (5) and that (M, X) fulfils Hypothesis 2.1. For all $\Psi \in C_b^1(\mathbb{R}^{d+1}, \mathbb{R})$, let F_Ψ be the map*

$F_\Psi : t \mapsto \mathbb{E}\left[\int_0^t \Psi(M_s, X_s) dM_s\right]$. Then F_Ψ is absolutely continuous with respect to the Lebesgue measure and its derivative is

$$\dot{F}_\Psi(t) = \frac{1}{2} \int_{\mathbb{R}^d} \Psi(m, m, \tilde{x}) \|A^1(m, \tilde{x})\|^2 p_V(m, m, \tilde{x}; t) dm d\tilde{x}.$$

Remark that, as it is expressed in Theorem 2.2, this derivative can be written

$$\frac{1}{2} \mathbb{E}\left[\Psi(X_t^1, X_t) \|A^1(X_t)\|^2 \frac{p_V(X_t^1, X_t; t)}{p_X(X_t; t)}\right].$$

Remark 2.2. The above proposition provides an explicit formulation of the derivative of the function F_Ψ . Note that the absolute continuity of F_Ψ could be established as a direct consequence of the existence of the density of the law of the hitting time $\tau_a = \inf\{s : X_s^1 \geq a\}$ when it exists, using the identity $\{\tau_a \leq t\} = \{M_t \geq a\}$.

Conversely, it could be proved that the absolute continuity of F_Ψ yields the existence of the density of the law of the hitting time τ_a , using a sequence of $C_b^2(\mathbb{R}, \mathbb{R})$ functions (F_n) approximating the indicator function $\mathbf{1}_{[a, \infty)}$, namely this density satisfies $f_{\tau_a}(t) = \frac{1}{2} \int_{\mathbb{R}^{d-1}} p_V(a, a, \tilde{x}; t) d\tilde{x}$.

Theorem 2.4. *Assume that $A = I_d$ and B satisfies Assumption (4) then, for all $t > 0$ the distribution of the pair (M_t, X_t) fulfils Hypotheses 2.1. As a consequence, for all $F \in C_b^2(\mathbb{R}^{d+1}, \mathbb{R})$*

$$\begin{aligned} \mathbb{E}[F(M_t, X_t)] &= \mathbb{E}[F(X_0^1, X_0)] + \int_0^t \mathbb{E}[\mathcal{L}(F)(M_s, X_s)] ds \\ &+ \frac{1}{2} \int_0^t \mathbb{E}\left[\partial_m F(X_s^1, X_s) \frac{p_V(X_s^1, X_s; s)}{p_X(X_s; s)}\right] ds. \end{aligned}$$

Proof. This theorem is a consequence of Theorem 2.2 and Proposition 4.1. \square

When $d = 1$ a Lamperti transformation leads to the following corollary:

Corollary 1. *Assume that $d = 1$, A and B satisfies (4) and (5), the density p_V satisfies Hypothesis 2.1 so*

$$\begin{aligned} \mathbb{E}[F(M_t, X_t)] &= \mathbb{E}[F(X_0, X_0)] + \int_0^t \mathbb{E}[\mathcal{L}(F)(M_s, X_s)] ds \\ &+ \frac{1}{2} \int_0^t \mathbb{E} \left[A^2(X_s) \partial_m F(X_s, X_s) \frac{p_V(X_s, X_s; s)}{p_X(X_s; s)} \right] ds. \end{aligned}$$

Remark 2.3. If p_V is regular enough, and if the initial law of X_0 satisfies $\mu_0(dx) = f_0(x)dx$, then Theorem 2.2 means that p_V is a weak solution in the set Δ of $\partial_t p = \mathcal{L}^* p$ where $\mathcal{L}^* f = \frac{1}{2} \Sigma^{ij} \partial_{ij}^2 f - [B^i - \partial_j(\Sigma^{ij})] \partial_i f - \partial_i B^i - \frac{1}{2} \partial_{ij}^2(\Sigma^{ij}) f$ with boundary condition

$$B^1(m, \tilde{x}) p_V(m, m, \tilde{x}; s) = \partial_{x_k}(\Sigma^{1,k} p_V)(m, m, \tilde{x}; s) + \frac{1}{2} \partial_m(\|A^1\|^2 p_V)(m, m, \tilde{x}; s). \quad (7)$$

This result is proved in Appendix A.3

This boundary condition also appears in Proposition 4 Equation (11) of [4] (Ornstein Uhlenbeck process). Finally, a similar PDE is studied in Chapter 1.2 of [14] where the authors have established the existence of a unique strong solution of this PDE, but in case of a non degenerate elliptic operator.

3. Proof of Theorem 2.3

We start this section with a road map of the proof of Theorem 2.3. Firstly we compute the right derivative of the application $F_\Psi : t \mapsto \mathbb{E}[\int_0^t \Psi(M_s, X_s) dM_s]$, namely $\lim_{h \rightarrow 0^+} T_{h,t}$ with $T_{h,t} = \frac{1}{h} \mathbb{E}_\mathbb{P}[\int_t^{t+h} \psi(V_s) dM_s]$. A first step is the decomposition

$$T_{h,t} = \frac{1}{h} \mathbb{E}_\mathbb{P} \left[\int_t^{t+h} (\psi(V_s) - \psi(V_t)) dM_s \right] + \frac{1}{h} \mathbb{E}_\mathbb{P}[\psi(V_t)(M_{t+h} - M_t)]. \quad (8)$$

Since $\psi \in C_b^1(\mathbb{R}^{d+1}, \mathbb{R})$ and the process M is increasing, the first term in (8), is dominated by:

$$\mathbb{E} \left[\int_t^{t+h} (\psi(V_s) - \psi(V_t)) dM_s \right] \leq \|\nabla \psi\|_\infty \mathbb{E} \left[\sup_{t \leq s \leq t+h} \|V_s - V_t\| (M_{t+h} - M_t) \right].$$

Lemma 3.1 states that $\sup_{t \leq s \leq t+h} \|X_s - X_t\|_p = O(\sqrt{h})$ and Lemma 3.2 yields $\|M_{t+h} - M_t\|_p = o(\sqrt{h})$ so that that this first term is an $o(h)$.

Concerning the second term in (8), $M_{t+h} - M_t$ can be written as $\sup_{0 \leq u \leq h} (X_{t+h}^1 - X_t^1 - M_t + X_t^1)_+$. In order to use the independence of the increments of Brownian motion we introduce a new process, independent of \mathcal{F}_t , which is an approximation of $X_{t+u}^1 - X_t^1$:

$$X_{t,u}^1 := A_k^1(X_t) \hat{W}_u^k \text{ where } \hat{W}_u^k := W_{t+u}^k - W_t^k; M_{t,h} := \sup_{0 \leq u \leq h} X_{t,u}^1. \quad (9)$$

Lemma 3.4 (ii) will set $\mathbb{E} [|M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+ |] = o(h)$, where $(x)_+ = \max(x, 0)$. Thus

$$\frac{1}{h} \mathbb{E} [\psi(V_t)(M_{t+h} - M_t)] = \mathbb{E} [\psi(V_t)(M_{t,h} - M_t + X_t^1)_+] + o(h) \quad (10)$$

Remark that the law of $M_{t,h}$ given \mathcal{F}_t is the law of $\|A^1(X_t)\| \sup_{0 \leq u \leq h} \hat{W}_u^1$, then using the function \mathcal{H} (13), a \mathcal{F}_t conditioning yields:

$$\frac{1}{h} \mathbb{E} [\psi(V_t)(M_{t+h} - M_t)] = \frac{2}{\sqrt{h}} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] + o(h). \quad (11)$$

Then $T_{h,t} = \frac{2}{\sqrt{h}} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] + o(h)$ as it appears in Proposition 3.1 (ii).

In Proposition 3.2, under Hypothesis 2.1, we compute $\lim_{h \rightarrow 0} T_{h,t}$.

Finally in Section 3.4 we prove $F_\psi : t \mapsto E[\int_0^t \psi(V_s) dM_s]$ is an absolutely continuous function with respect to Lebesgue measure, integral of its right derivative. Actually we prove that F_ψ is a continuous function belonging to the Sobolev space $W^{1,1}(I)$, $I = (0, T)$. This achieves the proof of Theorem 2.3.

The main propositions to prove are

Proposition 3.1. *Let B and A fulfil (4) and (5) and let $\Psi \in C_b^1(\mathbb{R}^{d+1}, \mathbb{R})$. Recall that A^1 is the vector $(A_j^1, j = 1, \dots, d)$, and $\|A^1(x)\|^2 = \sum_{j=1}^d (A_j^1(x))^2$.*

(i) *for all $T > 0$, there exists a constant $C > 0$, (depending on $\|A\|_\infty, \|B\|_\infty, \|\nabla A\|_\infty, \|\Psi\|_\infty, \|\nabla \Psi\|_\infty$ and T) such that $\forall t \in [0, T], h \in [0, 1]$,*

$$\left| \mathbb{E} \left[\int_t^{t+h} \Psi(V_s) dM_s - 2\sqrt{h} \left(\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right) \right] \right| \leq Ch \|\nabla \Psi\|_\infty, \quad (12)$$

(ii) *for all $t > 0, h \in [0, 1]$,*

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \left| \mathbb{E} \left[\int_t^{t+h} \Psi(V_s) dM_s \right] - 2\sqrt{h} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] \right| = 0,$$

where, denoting by Φ_G the standard Gaussian cumulative distribution function,

$$\mathcal{H}(\theta) := \int_\theta^\infty \frac{1}{\sqrt{2\pi}} (y - \theta) e^{-\frac{y^2}{2}} dy = \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} - \theta \Phi_G(-\theta). \quad (13)$$

The following remark will be useful:

Remark 3.1. The definition of \mathcal{H} in (13) implies that $\int_0^\infty \mathcal{H}(u)du = 1/4$. Moreover, $\mathcal{H}'(\theta) = -\Phi_G(-\theta) \leq 0$, in particular \mathcal{H} is non increasing.

Proposition 3.2. Assume that A and B fulfil (4) and (5) and that (M, X) fulfils Hypothesis 2.1, then for all $\Psi \in C_b^1(\mathbb{R}^{d+1}, \mathbb{R})$, for all $0 < T$ and for all $t \geq 0$:

$$\begin{aligned} (i) & t \mapsto \sup_{h>0} \frac{2\sqrt{h}}{h} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right) \right] \in L^1([0, T], \mathbb{R}), \\ (ii) & \lim_{h \rightarrow 0^+} \frac{2\sqrt{h}}{h} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right) \right] \\ &= \frac{1}{2} \int_{\mathbb{R}^d} \Psi(m, m, \tilde{x}) \|A^1(m, \tilde{x})\|^2 p_V(m, m, \tilde{x}; t) dm d\tilde{x} \end{aligned}$$

As a corollary, the function $t \rightarrow \frac{1}{2} \int_{\mathbb{R}^d} \Psi(m, m, \tilde{x}) \|A^1(m, \tilde{x})\|^2 p_V(m, m, \tilde{x}; t) dm d\tilde{x}$ belongs to $L^1([0, T], \mathbb{R})$.

The proof of Proposition 3.1 will be obtained with the lemmas in the following section.

3.1. Tools for proving Proposition 3.1

Here we provide some estimations of the expectations of the increments of the processes X and M . Assumptions (4) and (5) allow us to introduce a constant K which denotes either $\max(\|A\|_\infty, \|B\|_\infty)$ or $\max(\|A\|_\infty, \|B\|_\infty, \|\nabla A\|_\infty)$. Let C_p be the constant in the Burkholder-Davis-Gundy inequality (cf. Theorem B.36 in [3]).

Lemma 3.1. Let A and B be bounded. Then, for all $0 < h \leq 1$, for all $p \geq 1$ there exists a constant $C_{p,K}$ (depending only on p and K) such that:

$$\sup_{t>0} \mathbb{E} \left[\sup_{0 \leq s \leq h} \|X_{t+s} - X_t\|^p \right] \leq C_{p,K} h^{p/2}.$$

Proof. Using the fact that $(a+b)^p \leq 2^{p-1} [a^p + b^p]$, $a, b \geq 0$, one obtains:

$$0 \leq \sup_{s \leq h} \|X_{t+s} - X_t\|^p \leq 2^{p-1} \left[\sup_{u \leq h} \left(\left\| \int_t^{t+u} B(X_s) ds \right\| \right)^p + \sup_{u \leq h} \left(\left\| \int_t^{t+u} A_j(X_s) dW_s^j \right\| \right)^p \right].$$

Taking expectation of both members, the Burkholder-Davis-Gundy inequality implies

$$\mathbb{E}[\sup_{s \leq h} \|X_{t+s} - X_t\|^p] \leq 2^{p-1} (1 + C_p) \mathbb{E} \left[\left(\int_t^{t+h} \|B(X_s)\| ds \right)^p + \left(\int_t^{t+h} \|A(X_s)\|^2 ds \right)^{p/2} \right].$$

Assumption (4) on B and A yields $\mathbb{E}[\sup_{s \leq h} \|X_{t+s} - X_t\|^p] \leq 2^{p-1}(1 + C_p)(h^p K^p + h^{p/2} K^p)$. \square

Lemma 3.2. *Let B and A satisfy Assumptions (4) and (5). Then, for all $0 < h \leq 1$, for all $p \geq 1$ we get:*

$$\sup_{t > 0} \mathbb{E}[|M_{t+h} - M_t|^p] \leq C_{p,K} h^{p/2}; \quad \mathbb{E}[|M_{t+h} - M_t|^p] = o(h^{p/2}). \quad (14)$$

Proof. Recall $M_{t+h} - M_t = (\sup_{0 \leq u \leq h} (X_{t+u}^1 - X_t^1) + X_t^1 - M_t)_+$ recalling $(x)_+ = \max(x, 0)$. For any $a \geq 0$, one has $(x - a)_+ \leq |x| \mathbf{1}_{\{x > a\}}$, thus

$$0 \leq M_{t+h} - M_t \leq \left| \sup_{0 \leq u \leq h} (X_{t+u}^1 - X_t^1) \right| \mathbf{1}_{\{\sup_{0 \leq u \leq h} (X_{t+u}^1 - X_t^1) > M_t - X_t^1\}}.$$

Cauchy-Schwartz's inequality yields:

$$0 \leq \mathbb{E}[(M_{t+h} - M_t)^p] \leq \sqrt{\mathbb{E} \left[\left| \sup_{0 \leq u \leq h} (X_{t+u}^1 - X_t^1) \right|^{2p} \right] \mathbb{P}(\{ \sup_{0 \leq u \leq h} (X_{t+u}^1 - X_t^1) > M_t - X_t^1 \})}.$$

Replacing p by $2p$ in Lemma 3.1 leads to the inequality in (14) and the equality $\lim_{h \rightarrow 0} \sup_{0 \leq u \leq h} (X_{t+u}^1 - X_t^1) = 0$ holds almost surely. According to Theorem 1.1 in [9] extended to X_0 with law μ_0 on \mathbb{R}^d , the pair (M_t, X_t) admits a density, thus $\mathbb{P}\{M_t - X_t^1 = 0\} = 0$ holds almost surely. Therefore $E([M_{t+h} - M_t]^p)$ is bounded by the product of $h^{p/2}$ and a factor going to zero when h goes to 0, and this quantity is an $o(h^{p/2})$. \square

For any fixed t we recall the process $(X_{t,u}, u \in [0, h])$ and the running maximum of its first component as follows:

$$X_{t,u} := \sum_j A_j(X_t) \hat{W}_u^j, \quad M_{t,h} := \sup_{0 \leq u \leq h} X_{t,u}^1. \quad (15)$$

Lemma 3.3. *Under Assumptions (4) and (5), for all $p \geq 1$ there exists a constant $C_{p,K}$ such that for all $t \leq T$, for all $h \in [0, 1]$:*

$$\mathbb{E} \left[\sup_{s \leq h} |X_{s+t}^1 - X_t^1 - X_{t,s}^1|^p \right] \leq C_{p,K} h^p.$$

Proof. By definition, recalling $\hat{W}_u := W_{t+u} - W_t$, $u \geq 0$, we obtain

$$X_{s+t}^1 - X_t^1 - X_{t,s}^1 = \int_0^s B^1(X_{u+t}) du + \int_0^s [A^1(X_{u+t}) - A^1(X_t)] d\hat{W}_u.$$

Using once again $(a + b)^p \leq 2^{p-1}(a^p + b^p)$, $a, b \geq 0$, we get

$$\sup_{0 \leq s \leq h} |X_{s+t}^1 - X_t^1 - X_{t,s}^1|^p \leq 2^{p-1} \left[\left(\int_0^h \|B^1(X_{u+t})\| du \right)^p + \sup_{0 \leq s \leq h} \left\| \int_0^s [A^1(X_{u+t}) - A^1(X_t)] d\hat{W}_u \right\|^p \right].$$

Taking expectation of both sides and applying the Burkholder-Davis-Gundy inequality yield with $D_p = 2^{p-1}(1 + C_p)$:

$$\mathbb{E} \left[\sup_{0 \leq s \leq h} |X_{s+t}^1 - X_t^1 - X_{t,s}^1|^p \right] \leq D_p \left(\mathbb{E} \left[\int_0^h \|B^1(X_{u+t})\|^2 du \right]^{p/2} + \mathbb{E} \left[\int_0^h \|A^1(X_{u+t}) - A^1(X_t)\|^2 du \right]^{p/2} \right).$$

The first term above is bounded by $K^p h^p$ since B is bounded. The assumption that A belongs to $C_b^1(\mathbb{R}^d, \mathbb{R}^{d \times d})$ and Jensen's inequality imply that the second term is bounded by $K^p h^{p/2-1} \int_0^h \mathbb{E} \|X_{u+t} - X_t\|^p du$ thus

$$\mathbb{E} \left[\sup_{0 \leq s \leq h} |X_{s+t}^1 - X_t^1 - X_{t,s}^1|^p \right] \leq D_p K^p h^{p/2-1} \left(h^{p/2+1} + \int_0^h \mathbb{E} \|X_{u+t} - X_t\|^p du \right).$$

From Lemma 3.1 we obtain the uniform upper bound: $\mathbb{E}[\|X_{u+t} - X_t\|^p] \leq C_{p,K} u^{p/2}$ hence

$$\mathbb{E} \left[\sup_{s \leq h} |X_{s+t}^1 - X_t^1 - X_{t,s}^1|^p \right] \leq \frac{D_p K^p C_{p,K}}{\frac{p}{2} + 1} h^p.$$

□

Lemma 3.4. *Under Assumptions (4) and (5), one has*

- (i) $\exists C > 0 \quad \sup_{0 \leq t \leq T; 0 \leq h \leq 1} h^{-1} \mathbb{E} \left[\left| M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+ \right| \right] \leq C < \infty,$
- (ii) $\lim_{h \rightarrow 0^+} h^{-1} \mathbb{E} \left[\left| M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+ \right| \right] = 0.$

Proof. Fistly remark

$$\forall a \in \mathbb{R}, \quad |(x - a)_+ - (y - a)_+| \leq |x - y| [\mathbf{1}_{\{x > a\}} + \mathbf{1}_{\{y > a\}}], \quad (16)$$

and if f and g are functions on $[0, T]$, then

$$\forall s \in [0, T], \quad f(s) - \sup_{0 \leq u \leq T} g(u) \leq f(s) - g(s) \leq |f(s) - g(s)| \leq \sup_{v \leq T} |f(v) - g(v)|,$$

hence $\sup_{s \leq T} f(s) - \sup_{u \leq T} g(u) \leq \sup_{v \leq T} |f(v) - g(v)|$. Here the role of f and g is symmetrical so $\sup_{s \leq T} g(s) - \sup_{u \leq T} f(u) \leq \sup_{v \leq T} |f(v) - g(v)|$, and

$$\left| \sup_{s \leq T} g(s) - \sup_{u \leq T} f(u) \right| \leq \sup_{v \leq T} |f(v) - g(v)|. \quad (17)$$

We now consider $M_{t+h} - M_t = (\sup_{0 \leq u \leq h} (X_{u+t}^1 - X_t^1) - M_t + X_t^1)_+$, using (16)

$$\begin{aligned} & \left| M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+ \right| \leq \\ & \left| \sup_{0 \leq u \leq h} (X_{u+t}^1 - X_t^1) - M_{t,h} \right| \left[\mathbf{1}_{\{\sup_{0 \leq u \leq h} (X_{u+t}^1 - X_t^1) > M_t - X_t^1\}} + \mathbf{1}_{\{M_{t,h} > M_t - X_t^1\}} \right]. \end{aligned}$$

Then, for any t fixed, we apply inequality (17) to the maps $g : u \mapsto X_{u+t}^1 - X_t^1$ and $f : u \mapsto X_{t,u}^1$. Then $\left| M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+ \right| \leq$

$$\sup_{0 \leq u \leq h} |X_{u+t}^1 - X_t^1 - X_{t,u}^1| \left[\mathbf{1}_{\{\sup_{0 \leq u \leq h} (X_{u+t}^1 - X_t^1) > M_t - X_t^1\}} + \mathbf{1}_{\{M_{t,h} > M_t - X_t^1\}} \right].$$

From Cauchy-Schwartz's inequality and the fact that $(a+b)^2 \leq 2(a^2 + b^2)$, we get

$$\begin{aligned} & \mathbb{E} \left[\left| M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+ \right|^2 \right] \leq \\ & \sqrt{2 \mathbb{E} \left[\sup_{u \leq h} |X_{u+t}^1 - X_t^1 - X_{t,u}^1|^2 \right] \left(\mathbb{P} \left\{ \sup_{0 \leq u \leq h} (X_{u+t}^1 - X_t^1) > M_t - X_t^1 \right\} + \mathbb{P} \{ M_{t,h} > M_t - X_t^1 \} \right)}. \end{aligned}$$

Lemma 3.3 with $p = 2$ insures that the map $h \mapsto h^{-1} \sqrt{2 \mathbb{E} \left[\sup_{u \leq h} |X_{u+t}^1 - X_t^1 - X_{t,u}^1|^2 \right]}$ is uniformly bounded in t . Concerning the second factor,

- firstly the almost sure continuity with respect to h insures that the quantities $\lim_{h \rightarrow 0} \sup_{0 \leq u \leq h} (X_{u+t}^1 - X_t^1)$ and $\lim_{h \rightarrow 0} M_{t,h}$ are equal to 0;
- secondly the law of the pair (M_t, X_t) admits a density with respect to the Lebesgue measure on $\bar{\Delta}$ according to Theorem 1.1 [9] so $\mathbb{P}(\{0 = M_t - X_t^1\}) = 0$ and the limit of the second factor is equal to 0.

This concludes the proof of the lemma. \square

Recall Definition (15): $X_{t,h} = A_j(X_t)[W_{t+h}^j - W_t^j]$, $M_{t,h} = \sup_{0 \leq u \leq h} X_{t,u}^1$, $h \in [0, 1]$.

Lemma 3.5. *Under Assumptions (4) and (5), with \mathcal{H} defined in (13):*

$$\mathbb{E} \left[(M_{t,h} - M_t + X_t^1)_+ | \mathcal{F}_t \right] = 2 \|A^1(X_t)\| \sqrt{h} \mathcal{H} \left(\frac{M_t - X_t^1}{\|A^1(X_t)\| \sqrt{h}} \right).$$

Proof. For any t fixed, conditionally to \mathcal{F}_t the process $(X_{t,u}^1, u \in [0, h])$ (9) has the same law as $(\sqrt{h} \|A^1(X_t)\| \hat{W}_u, u \in [0, 1])$ where \hat{W} is a Brownian motion independent of \mathcal{F}_t , and for any h , the random variable $M_{t,h}$ has the same law as $\sqrt{h} \|A^1(X_t)\| \sup_{u \leq 1} \hat{W}_u$.

Following [17] Section 3.1.3, the random variable $\sup_{u \leq 1} \hat{W}_u$ has the same law

as $|G|$ where G is a standard Gaussian variable (independent of \mathcal{F}_t), with density $\frac{2}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\mathbf{1}_{[0,+\infty[}(z)$. Then using the function \mathcal{H} introduced in (13)

$$\begin{aligned}\mathbb{E}[(M_{t,h} - (M_t - X_t^1))_+ | \mathcal{F}_t] &= \int_0^\infty \left(\|A^1(X_t)\| \sqrt{h}z - (M_t - X_t^1) \right)_+ \frac{2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= 2\|A^1(X_t)\| \sqrt{h} \mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right).\end{aligned}$$

□

3.2. Proof of Proposition 3.1

Let $t > 0$. The key of this proof is to write the quantity

$$\mathbb{E} \left[\int_t^{t+h} \Psi(V_s) dM_s \right] - 2\sqrt{h} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right) \right]$$

as the sum of three terms,

$$\begin{aligned}\mathbb{E} \left[\int_t^{t+h} (\Psi(V_s) - \Psi(V_t)) dM_s \right] &+ \mathbb{E} \left[\Psi(V_t) \left((M_{t+h} - M_t) - \mathbb{E}[(M_{t,h} - M_t + X_t^1)_+ | \mathcal{F}_t] \right) \right] \\ &+ \mathbb{E} \left[\Psi(V_t) \mathbb{E}[(M_{t,h} - M_t + X_t^1)_+ | \mathcal{F}_t] - 2\sqrt{h} \Psi(V_t) \|A^1(X_t)\| \mathcal{H}\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right) \right].\end{aligned}\tag{18}$$

We now prove that each terms in sum (18) are both $o(h)$ and $O(h)$ uniformly in time.

(a) Using Lemma 3.5 the third term is null.

(b) Concerning the second term, using the fact that Ψ is bounded and Lemma 3.4 (i) for all $t \in [0, T]$

$$\begin{aligned}& \left| \mathbb{E} \left[\Psi(V_t) \left((M_{t+h} - M_t) - \mathbb{E}[(M_{t,h} - M_t + X_t^1)_+ | \mathcal{F}_t] \right) \right] \right| \leq \\ & \|\Psi\|_\infty \left| \mathbb{E} \left[M_{t+h} - M_t - \mathbb{E}[(M_{t,h} - M_t + X_t^1)_+ | \mathcal{F}_t] \right] \right| \leq Ch \|\Psi\|_\infty,\end{aligned}$$

as it is required in (12). Moreover using Lemma 3.4 (ii)

$$\lim_{h \rightarrow 0} \frac{1}{h} \left| \mathbb{E} \left[\Psi(V_t) \left((M_{t+h} - M_t) - \mathbb{E}[(M_{t,h} - M_t + X_t^1)_+ | \mathcal{F}_t] \right) \right] \right| = 0.$$

(c) Since $\nabla \Psi$ is bounded and the process M is increasing, the first term is bounded:

$$\mathbb{E} \left[\int_t^{t+h} [\Psi(V_s) - \Psi(V_t)] dM_s \right] \leq \|\nabla \Psi\|_\infty \mathbb{E} \left[\sup_{t \leq s \leq t+h} \|V_s - V_t\| (M_{t+h} - M_t) \right].$$

Using Cauchy-Schwarz's inequality

$$\mathbb{E} \left[\sup_{t \leq s \leq t+h} \|V_s - V_t\| (M_{t+h} - M_t) \right] \leq \sqrt{\mathbb{E} \left[\sup_{t \leq s \leq t+h} \|V_s - V_t\|^2 \right] \mathbb{E}[(M_{t+h} - M_t)^2]}.$$

Since $\|V_s - V_t\|^2 = (M_s - M_t)^2 + \|X_s - X_t\|^2$, we obtain $\sup_{t \leq s \leq t+h} \|V_s - V_t\|^2 \leq (M_{t+h} - M_t)^2 + \sup_{t \leq s \leq t+h} \|X_s - X_t\|^2$, hence

$$\mathbb{E} \left[\sup_{t \leq s \leq t+h} \|V_s - V_t\| (M_{t+h} - M_t) \right] \leq \sqrt{\mathbb{E}[(M_{t+h} - M_t)^2] + \mathbb{E} \left[\sup_{t \leq s \leq t+h} \|X_s - X_t\|^2 \right]} \sqrt{\mathbb{E}[(M_{t+h} - M_t)^2]}$$

Lemmas 3.1 and 3.2 ($p = 2$) yield the fact that the first factor is an $o(\sqrt{h})$ and the second is an $O(\sqrt{h})$ uniformly with respect to $t \geq 0$. Then $\mathbb{E}[\sup_{t \leq s \leq t+h} \|V_s - V_t\| (M_{t+h} - M_t)]$ is an $o(h)$ and an $O(h)$ uniformly with respect to $t \geq 0$. \square

3.3. Proof of Proposition 3.2

(i) Recall that A and B fulfil (4), (5) and (M, X) fulfils Hypothesis 2.1. Then, using the density p_V of the law of the pair (M_t, X_t) we have

$$\begin{aligned} & \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] \leq \\ & \|\Psi\|_\infty \|A\|_\infty \int_{\mathbb{R}^{d+1}} \mathcal{H} \left(\frac{m - x^1}{\sqrt{h} \|A^1(x^1, \tilde{x})\|} \right) p_V(m, x^1, \tilde{x}; t) dm dx^1 d\tilde{x}. \end{aligned}$$

The change of variable $x^1 = m - u\sqrt{h}$ yields

$$\begin{aligned} & \frac{\sqrt{h}}{h} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] \leq \\ & \|\Psi\|_\infty \|A^1\|_\infty \int_{\mathbb{R}^d \times [0, +\infty[} \mathcal{H} \left(\frac{u}{\|A^1(m - \sqrt{h}u, \tilde{x})\|} \right) p_V(m, m - \sqrt{h}u, \tilde{x}; t) dm d\tilde{x} du. \end{aligned} \tag{19}$$

Since \mathcal{H} is decreasing (Remark 3.1) and $0 \leq h \leq 1$, $\mathcal{H} \left(\frac{u}{\|A^1(m - \sqrt{h}u, \tilde{x})\|} \right) \leq \mathcal{H} \left(\frac{u}{\|A^1\|_\infty} \right)$:

$$\begin{aligned} & \left| \frac{\sqrt{h}}{h} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] \right| \leq \\ & \|\Psi\|_\infty \|A^1\|_\infty \int_{\mathbb{R}^d \times [0, +\infty[} \mathcal{H} \left(\frac{u}{\|A^1\|_\infty} \right) \sup_{r>0} p_V(m, m - r, \tilde{x}; t) dm d\tilde{x} du. \end{aligned}$$

Applying Tonelli's Theorem, computing the integral with respect to du in the right-hand with $\int_0^\infty \mathcal{H}(v) dv = 1/4$ (Remark 3.1), yield:

$$\sup_{h>0} \left| \frac{\sqrt{h}}{h} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] \right| \leq \frac{1}{4} \|\Psi\|_\infty \|A^1\|_\infty^2 \int_{\mathbb{R}^d} \sup_{r>0} p_V(m, m - r, \tilde{x}; t) dm d\tilde{x}.$$

Using Hypothesis 2.1 (i), we obtain that the map:

$$t \mapsto \sup_{h>0} \left| \frac{\sqrt{h}}{h} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] \right|$$

belongs to $L^1([0, T], \mathbb{R})$. Point (i) of Proposition 3.2 is proved.

(ii) Concerning the proof of point (ii), firstly note that

$$\mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] = \int_{\mathbb{R}^{d+1}} \Psi(m, x) \|A^1(x)\| \mathcal{H} \left(\frac{m - x^1}{\sqrt{h} \|A^1(x)\|} \right) p_V(m, x; t) dm dx.$$

After the change of variable $x^1 = m - u\sqrt{h}$, we obtain

$$\begin{aligned} \frac{\sqrt{h}}{h} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] &= \\ \int_{\mathbb{R}^d \times \mathbb{R}^+} \Psi(m, m - u\sqrt{h}, \tilde{x}) \|A^1(m - u\sqrt{h}, \tilde{x})\| \mathcal{H} \left(\frac{u}{\|A^1(m - \sqrt{h}u, \tilde{x})\|} \right) p_V(m, m - \sqrt{h}u, \tilde{x}; t) dm d\tilde{x} du. \end{aligned} \quad (20)$$

Using Lebesgue's dominated convergence Theorem, we let h go to 0 in (20) for $t > 0$, and using the fact that Ψ , A and \mathcal{H} are continuous and Hypothesis 2.1 (ii) we obtain

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sqrt{h}}{h} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] &= \\ \int_{\mathbb{R}^d \times [0, +\infty[} \Psi(m, m, \tilde{x}) \|A^1(m, \tilde{x})\| \mathcal{H} \left(\frac{u}{\|A^1(m, \tilde{x})\|} \right) p_V(m, m, \tilde{x}; t) dm d\tilde{x} du. \end{aligned}$$

Using the change of variable $z = \frac{u}{\|A^1(m, \tilde{x})\|}$, and Remark 3.1 $\int_0^\infty \mathcal{H}(z) dz = 1/4$, yields

$$\lim_{h \rightarrow 0} \frac{\sqrt{h}}{h} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] = \frac{1}{4} \int_{\mathbb{R}^d} \Psi(m, m, \tilde{x}) \|A^1(m, \tilde{x})\|^2 p_V(m, m, \tilde{x}; t) dm d\tilde{x}.$$

□

3.4. End of proof of Theorem 2.3

We recall Theorem 8.2 page 204 in Brezis [5]: let $f \in W^{1,1}(0, T)$, then f is almost surely equal to an absolutely continuous function. As a particular case, any $f \in W^{1,1}(0, T) \cap C(0, T)$ is absolutely continuous. Recall $F_\Psi : t \mapsto \mathbb{E} \left[\int_0^t \Psi(V_s) dM_s \right]$.

Lemma 3.6. *Assume that A and B fulfil (4) and (5) and that Ψ is a continuous bounded function. Then F_Ψ is a continuous function on \mathbb{R}^+ .*

Proof. Let $0 \leq s \leq t$. Since Ψ is bounded and M is non decreasing

$$|F_\Psi(t) - F_\Psi(s)| = \left| \mathbb{E} \left[\int_s^t \Psi(V_u) dM_u \right] \right| \leq \|\Psi\|_\infty \mathbb{E}[M_t - M_s].$$

The map $t \mapsto \mathbb{E}[M_t]$ being continuous, F_Ψ is a continuous function on \mathbb{R}^+ . □

Lemma 3.7. *Assume that A and B fulfil (4) and (5), (M, X) fulfils Hypothesis 2.1 and $\Psi \in C_b^1$. Then for all $T > 0$, the map F_ψ belongs to the Sobolev space $W^{1,1}([0, T])$ and its weak derivative is*

$$\dot{F}_\Psi(t) := \frac{1}{2} \int_{\mathbb{R}^d} \Psi(m, \tilde{x}) \|A^1(m, \tilde{x})\|^2 p_V(m, m, \tilde{x}; t) dm d\tilde{x}$$

Proof. Let $g : [0, T] \rightarrow \mathbb{R}$ be C^1 with compact support $[\alpha, \beta] \subset (0, T)$. This means both functions g and \dot{g} are continuous so bounded and that moreover $g(\alpha) = g(\beta) = 0$. Note that $\dot{g}(t) = \lim_{h \rightarrow 0} \frac{g(t) - g(t-h)}{h}$, $\forall t \in (0, T)$. Moreover, $\sup_{t \in [0, T]} \sup_{h \in [0, 1]} \left| \frac{g(t) - g(t-h)}{h} \right| \leq \|\dot{g}\|_\infty$. Observe that, since M is non decreasing and the coefficients A and B are bounded $|F_\psi(t)| \leq \|\Psi\|_\infty \mathbb{E}[M_T] < \infty$.

Then, using Lebesgue's dominated convergence Theorem

$$\int_0^T \dot{g}(s) F_\psi(s) ds = \int_0^T \lim_{h \rightarrow 0} \frac{g(s) - g(s-h)}{h} F_\psi(s) ds = \lim_{h \rightarrow 0} \int_0^T \frac{g(s) - g(s-h)}{h} F_\psi(s) ds.$$

Using the change of variable $u = s - h$ in the last integral

$$\begin{aligned} \int_0^T \frac{g(s) - g(s-h)}{h} F_\psi(s) ds &= h^{-1} \int_0^T g(s) F_\psi(s) ds - h^{-1} \int_{-h}^{T-h} g(u) F_\psi(u+h) du \\ &= \int_0^T g(s) \frac{F_\psi(s) - F_\psi(s+h)}{h} ds - h^{-1} \int_{-h}^0 g(s) F_\psi(s+h) ds + h^{-1} \int_{T-h}^T g(s) F_\psi(s+h) ds. \end{aligned}$$

Recalling $\text{supp}(g) = [\alpha, \beta] \subset (0, T)$, gF_ψ is bounded on $[0, T]$ extended by 0 on $[\alpha, \beta]^c$ so $\lim_{s \rightarrow 0} g(s) = \lim_{s \rightarrow T} g(s) = 0$ then $h^{-1} \int_{-h}^0 g(s) F_\psi(s+h) ds = h^{-1} \int_{T-h}^T g(s) F_\psi(s+h) ds = 0$ as soon as $0 < h \leq T - \beta$ thus $\lim_{h \rightarrow 0} \left[h^{-1} \int_{-h}^0 g(s) F_\psi(s+h) ds \right] = \lim_{h \rightarrow 0} \left[h^{-1} \int_{T-h}^T g(s) F_\psi(s+h) ds \right] = 0$ Applying Lebesgue's dominated convergence Theorem yields, F admits a weak derivative:

$$\int_0^T \dot{g}(s) F_\psi(s) ds = - \int_0^T g(s) \dot{F}_\Psi(s) ds.$$

Using Proposition 3.1 (ii)

$$\lim_{h \rightarrow 0^+} \left(- \frac{F_\Psi(t) - F_\Psi(t+h)}{h} - \frac{2}{\sqrt{h}} \mathbb{E} \left[\Psi(V_t) \|A^1(X_t)\| \mathcal{H} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \right) \right] \right) = 0.$$

Using Proposition 3.2 (ii):

$$-\dot{F}_\Psi(t+) := \lim_{h \rightarrow 0, h > 0} \frac{F_\Psi(t) - F_\Psi(t+h)}{h} = - \frac{1}{2} \int_{\mathbb{R}^d} \Psi(m, m, \tilde{x}) \|A^1(m, \tilde{x})\|^2 p_V(m, m, \tilde{x}; t) dm d\tilde{x},$$

and the points (i) of Propositions 3.1 and 3.2:

$$\sup_{h > 0} \left| \frac{F_\Psi(t) - F_\Psi(t+h)}{h} \right| \in L^1([0, T], dt),$$

so $\dot{F}_\Psi \in L^1([0, T], \mathbb{R})$.

According to [5] Chap 8 section 2 page 202, F_Ψ belongs to $W^{1,1}([0, T], \mathbb{R})$. \square

We now end the proof of Theorem 2.3: According to Theorem 8.2 page 204 of [5], F_ψ is equal almost surely to an absolutely continuous function. Since F_Ψ is continuous (Lemma 3.6), the equality holds everywhere. Then F_Ψ is an absolutely continuous function and its derivative is its right derivative. \square

4. Case $A = I_d$

In this rather technical section, we firstly prove that the density of the pair (M_t, X_t) fulfils Hypothesis 2.1: p_V (3) is continuous on the boundary of $\bar{\Delta}$ and is dominated by an integrable function:

Proposition 4.1. *Assume that B fulfils Assumption (4) and $A = I_d$, then (M, X) fulfils Hypothesis 2.1 meaning that for all probability measure μ_0 on \mathbb{R}^d*

$$(i) \quad \forall T > 0, \quad \sup_{(h,u) \in [0,1] \times \mathbb{R}_+} p_V(b, b - hu, \tilde{a}; t, \mu_0) \in L^1([0, T] \times \mathbb{R}^d, dt db d\tilde{a}).$$

$$(ii) \quad \text{Almost surely in } (m, \tilde{x}) \in \mathbb{R}^d, \forall t > 0, \quad \lim_{u \rightarrow 0, u > 0} p_V(m, m - u, \tilde{x}; t, \mu_0) = p_V(m, m, \tilde{x}; t, \mu_0).$$

As a by product using Theorems 2.2 and 2.3 this proposition achieves the proof of Theorem 2.4. The main tool for the proof of this proposition is an integral representation of the density:

Proposition 4.2. *For any probability measure μ_0 on \mathbb{R}^d , for all $t > 0$,*

$$p_V = p^0 - \sum_{k=m,1,\dots,d} (p^{k,\alpha} + p^{k,\beta}) \quad (21)$$

where the various p are defined by (∂_k is the derivative with respect to $k = m, x^1, \dots, x^d$ and $B^m = B^1$):

$$p^0(m, x; t) := \int_{\mathbb{R}^d} p_{W^{*1}, W}(m - x_0^1, x - x_0; t) \mu_0(dx_0),$$

$$p^{k,\alpha}(m, x; t) := \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{b < m} B^k(a) \partial_k p_{W^{*1}, W}(m - a^1, x - a; t - s) p_V(b, a; s) db da ds,$$

$$p^{k,\beta}(m, x; t) := \int_0^t \int_{\mathbb{R}^{(d+1)}} \mathbf{1}_{b < m} B^k(a) \partial_k p_{W^{*1}, W}(b - a^1, x - a; t - s) p_V(m, a; s) db da ds,$$

where $p_{W^{*1}, W}(\cdot, \cdot; t)$ is the density of the distribution of $(\sup_{s \leq t} W_s^1, W_t)$ for $t \geq 0$, see Appendix A.2

4.1. Integral representation of the density: proof of Proposition 4.2

Let $t > 0$ be fixed. Firstly, we assume that $\mu_0 = \delta_{x_0}$, x_0 being fixed in \mathbb{R}^d .

According to Lemma 4.2 below and using the fact that B is bounded, $\forall t \in [0, T]$, the functions $p^{k, \gamma} \in \infty L([0, T], L^1(\mathbb{R}^{d+1}))$ for $\gamma = \alpha, \beta$.

Let $F \in C_b^1(\mathbb{R}^{d+1}, \mathbb{R})$ with compact support. We will prove

$$\mathbb{E}_{\mathbb{P}}[F(M_t, X_t)] = \int_{\mathbb{R}^{d+1}} F(m, x) \left(p^0 - \sum_{k=m, x^1, \dots, x^d} (p^{k, \alpha} + p^{k, \beta})(m, x, t) \right) dm dx. \quad (22)$$

Using Malliavin calculus we obtain the following decomposition:

Lemma 4.1.

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[F(M_t, X_t)] &= \int_{\mathbb{R}^{d+1}} F(x_0^1 + b, x_0 + a) p_{W^{*1}, W}(b, a; t) db da \\ &\quad + \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} \partial_m F(X_s^1 + b, X_s + a) \mathbf{1}_{\{M_s < X_s^1 + b\}} B^1(X_s) p_{W^{*1}, W}(b, a; t-s) db da \right] ds \\ &\quad + \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} \partial_k F(\max(M_s, X_s^1 + b), X_s + a) B^k(X_s) p_{W^{*1}, W}(b, a; t-s) db da \right] ds. \end{aligned}$$

Proof. Let Z be the exponential martingale solution of

$$Z_t = 1 + \int_0^t Z_s B^k(x_0 + W_s) dW_s^k. \quad (23)$$

As previously Einstein's convention is used. Let $\mathbb{Q} = Z\mathbb{P}$, according to Girsanov's Theorem, using (23), $W_B := \left(W_t^k - \int_0^t B^k(x_0 + W_s) ds; k = 1, \dots, d \right)_{t \geq 0}$ is a \mathbb{Q} continuous martingale such that $\langle W_B \rangle_t = t$, $\forall t \geq 0$. That means that under \mathbb{Q} , W_B is a d -dimensional Brownian motion. Then the distribution of X (resp. (M, X)) under \mathbb{P} is the distribution of $W + x_0$ (resp. $(W^{1*} + x_0, W + x_0)$) under \mathbb{Q} and

$$\mathbb{E}_{\mathbb{P}}[F(M_t, X_t)] = \mathbb{E}_{\mathbb{Q}}[F(x_0^1 + W_t^{1*}, x_0 + W_t)] = \mathbb{E}_{\mathbb{P}}[F(x_0^1 + W_t^{1*}, x_0 + W_t) Z_t]. \quad (24)$$

Let $G := F(x_0^1 + W_t^{1*}, x_0 + W_t)$ and $u := ZB(x_0 + W)$, using (23):

$$\mathbb{E}_{\mathbb{P}}[F(M_t, X_t)] = \mathbb{E}_{\mathbb{P}}[F(x_0^1 + W_t^{1*}, x_0 + W_t)] + \mathbb{E}_{\mathbb{P}}[G\delta(u)] \quad (25)$$

As a first step we will apply (50) (Appendix) to the second term in (25). Thus we have to check that the pair $(G, u) \in \mathbb{D}^{1,2} \times \mathbb{L}^{1,2}$: F being bounded and smooth,

$G \in \mathbb{D}^{1,2}$; and according to Lemma A.1, the process u belongs to $\mathbb{L}^{1,2}$.

Using (53) ($\tau := \inf\{s, W_s^{1*} = W_t^{1*}\}$) the pair (W_t^{1*}, W_t) belongs to $\mathbb{D}^{1,2}$ with Malliavin gradient:

$$D_s W_t^{1*} = (\mathbf{1}_{[0,\tau]}(s), 0, \dots, 0), \quad D_s W_t^k = (\delta_{j=k}, \quad j = 1, \dots, d) \mathbf{1}_{[0,t]}(s), \quad k = 1, \dots, d.$$

Using the chain rule:

$$\begin{aligned} \langle DG, u \rangle_{\mathbb{H}} &= \int_0^t \partial_m F(x_0^1 + W_t^{*1}, x_0 + W_t) \mathbf{1}_{\{W_s^{1*} < W_t^{1*}\}} B^1(x_0 + W_s) Z_s ds \\ &\quad + \int_0^t \partial_k F(x_0^1 + W_t^{*1}, x_0 + W_t) B^k(x_0 + W_s) Z_s ds. \end{aligned}$$

We are now in position to apply (50) $E_{\mathbb{P}}[G\delta(u)] = E_{\mathbb{P}}[\langle DG, u \rangle_{\mathbb{H}}]$:

$$\begin{aligned} E_{\mathbb{P}}[G\delta(u)] &= E_{\mathbb{P}} \left[\int_0^t \partial_m F(x_0^1 + W_t^{*1}, x_0 + W_t) \mathbf{1}_{\{W_s^{1*} < W_t^{1*}\}} B^1(x_0 + W_s) Z_s ds \right] \\ &+ E_{\mathbb{P}} \left[\int_0^t \partial_k F(x_0^1 + W_t^{*1}, x_0 + W_t) B^k(x_0 + W_s) Z_s ds \right]. \end{aligned} \quad (26)$$

Plugging identity (26) into right hand of (25) and using Fubini Theorem to commute the integrals in ds and $d\mathbb{P}$, we obtain

$$\begin{aligned} E_{\mathbb{P}}[F(M_t, X_t)] &= E_{\mathbb{P}}[F(x_0^1 + W_t^{*1}, x_0 + W_t)] \quad (27) \\ &+ \int_0^t E_{\mathbb{P}} \left[\partial_m F(x_0^1 + W_t^{*1}, x_0 + W_t) \mathbf{1}_{\{W_s^{1*} < W_t^{1*}\}} Z_s B^1(x_0 + W_s) \right] ds \\ &+ \int_0^t E_{\mathbb{P}} \left[\partial_k F(x_0^1 + W_t^{*1}, x_0 + W_t) Z_s B^k(x_0 + W_s) \right] ds. \end{aligned}$$

As a second step we use the independence of the increments of the Brownian motion in order to make appear the density of (W_{t-s}^{1*}, W_{t-s}) . Recall (9): $\hat{W}_{t-s} := W_t - W_s$ and $(\hat{W}^1)_{t-s}^* = \max_{s \leq u \leq t} (W_u^1 - W_s^1)$. Then $W_t^{1*} = \max(W_s^{1*}, W_s^1 + \max_{s \leq u \leq t} (W_u^1 - W_s^1)) = \max(W_s^{1*}, W_s^1 + (\hat{W}^1)_{t-s}^*)$ so the expression (27) becomes

$$\begin{aligned} E_{\mathbb{P}}[F(M_t, X_t)] &= E_{\mathbb{P}}[F(x_0^1 + W_t^{*1}, x_0 + W_t)] + \\ &\int_0^t E_{\mathbb{P}} \left[\partial_m F \left(x_0^1 + W_s^1 + (\hat{W}^1)_{t-s}^*, x_0 + W_s + \hat{W}_{t-s} \right) \mathbf{1}_{\{W_s^{1*} < W_s^1 + (\hat{W}^1)_{t-s}^*\}} Z_s B^1(x_0 + W_s) \right] ds \\ &+ \int_0^t E_{\mathbb{P}} \left[\partial_k F \left(\max \left(x_0^1 + W_s^{1*}, x_0^1 + W_s^1 + (\hat{W}^1)_{t-s}^* \right), x_0 + W_s + \hat{W}_{t-s} \right) Z_s B^k(x_0 + W_s) \right] ds. \end{aligned}$$

The random vector $((\hat{W}^1)_{t-s}^*, \hat{W}_{t-s})$ is independent of the σ -field \mathcal{F}_s and has the same distribution as the pair (W_{t-s}^{1*}, W_{t-s}) . Let $p_{W^{*1}, W}(\cdot, \cdot; t-s)$ be the density of its

law, and express the expectation with this density:

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} [F(M_t, X_t)] &= \int_{\mathbb{R}^{d+1}} F(x_0^1 + b, x_0 + a) p_{W^{*1}, W}(b, a; t) dbda \\ &+ \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} \partial_m F(x_0^1 + W_s^1 + b, x_0 + W_s + a) \mathbf{1}_{\{W_s^{1*} < W_s^1 + b\}} Z_s B^1(x_0 + W_s) p_{W^{*1}, W}(b, a; t-s) dbda \right] ds \\ &+ \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} \partial_k F(x_0^1 + \max(W_s^{1*}, W_s^1 + b), x_0 + W_s + a) Z_s B^k(x_0 + W_s) p_{W^{*1}, W}(b, a; t-s) dbda \right] ds. \end{aligned}$$

Using Girsanov's Theorem for $Z \cdot \mathbb{P} = \mathbb{Q}$, since the law of (M, X) under \mathbb{P} is the law of $(x_0^1 + W^{1*}, x_0 + W)$, under \mathbb{Q} , using the equality (24):

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} [F(M_t, X_t)] &= \int_{\mathbb{R}^{d+1}} F(x_0^1 + b, x_0 + a) p_{W^{*1}, W}(b, a; t) dbda \\ &+ \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} \partial_m F(X_s^1 + b, X_s + a) \mathbf{1}_{\{M_s < X_s^1 + b\}} B^1(X_s) p_{W^{*1}, W}(b, a; t-s) dbda \right] ds \\ &+ \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} \partial_k F(\max(M_s, X_s^1 + b), X_s + a) B^k(X_s) p_{W^{*1}, W}(b, a; t-s) dbda \right] ds. \end{aligned}$$

□

We are now in position to achieve the proof of Proposition 4.2. Using some suitable translations of the variables (a, b) , $\mathbb{E}_{\mathbb{P}} [F(M_t, X_t)] = \sum_{k=0}^d I_k + I_m$. where

$$\begin{aligned} I_0 &= \int_{\mathbb{R}^{d+1}} F(b, a) p_{W^{*1}, W}(b - x_0^1, a - x_0; t) dbda, \tag{28} \\ I_m &= \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} \partial_m F(b, a) \mathbf{1}_{\{M_s < b\}} B^1(X_s) p_{W^{*1}, W}(b - X_s^1, a - X_s; t-s) dbda \right] ds \end{aligned}$$

and for $k = 1, \dots, d$,

$$I_k = \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} \partial_k F(\max(M_s, b), a) B^k(X_s) p_{W^{*1}, W}(b - X_s^1, a - X_s; t-s) dbda \right] ds.$$

Since B, F and its derivatives are bounded, all these integrals are finite. Using (54) in Appendix, the function $p_{W^{*1}, W}(\cdot, \cdot; t)$ is C^∞ on $\bar{\Delta} = \{(b, a), b \geq a_+^1, (a, b) \in \mathbb{R}^{d+1}\}$. The aim is now to identify the terms $p^0, p^{k,\alpha}, p^{k,\beta}$, $\mathbf{k} = m, 1, \dots, d$, defined in Proposition 4.2.

1. Firstly we identify $p^0(b, a; t)$ as the factor of $F(b, a)$ in the integrand of I_0 :

$$p^0(b, a; t) = p_{W^{*1}, W}(b - x_0^1, a - x_0; t).$$

2. We now deal with $I_k, k = 2, \dots, d$. Integrating by parts with respect to a^k between

$-\infty$ and ∞ in I_k for $k = 2, \dots, d$ yields

$$\begin{aligned} I_k &= - \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} F(\max(M_s, b), a) B^k(X_s) \partial_k p_{W^{*1}, W}(b - X_s^1, a - X_s; t - s) db da \right] ds \\ &= - \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} \mathbf{1}_{\{b > M_s\}} F(b, a) B^k(X_s) \partial_k p_{W^{*1}, W}(b - X_s^1, a - X_s; t - s) db da \right] ds \\ &\quad - \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} \mathbf{1}_{\{b < M_s\}} F(M_s, a) B^k(X_s) \partial_k p_{W^{*1}, W}(b - X_s^1, a - X_s; t - s) db da \right] ds \end{aligned}$$

We identify $-p^{k,\alpha}(b, a, t)$ inside the integral on the set $(b > M_s)$. Concerning the integral on the set $(b < M_s)$, we introduce the density of (M_s, X_s) and identify $-p^{k,\beta}(m, a; t)$ as factor of $F(m, a)$.

3. Finally, we identify the $p^{m,\gamma}$ and $p^{1,\gamma}$, $\gamma = \alpha, \beta$ which come from the sum of I_m and I_1 . Note that $p_{W^{*1}, W}(b - X_s^1, a - X_s; t - s) = 0$ on the set $\{b < a^1\}$. Integrating by parts with respect to b between $\max(a^1, M_s)$ and ∞ in I_m yields

$$\begin{aligned} I_m &= - \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^d} F(\max(a^1, M_s), a) B^1(X_s) p_{W^{*1}, W}(\max(a^1, M_s) - X_s^1, a - X_s; t - s) da \right] ds \\ &\quad - \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} \mathbf{1}_{M_s < b} F(b, a) B^1(X_s) \partial_m p_{W^{*1}, W}(b - X_s^1, a - X_s; t - s) db da \right] ds \end{aligned} \tag{29}$$

Integrating by parts with respect to a^1 between $-\infty$ and b in I_1 yields

$$\begin{aligned} I_1 &= \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^d} F(\max(M_s, b), b, \tilde{a}) B^1(X_s) p_{W^{*1}, W}(b - X_s^1, b - X_s^1, \tilde{a} - \tilde{X}_s; t - s) db d\tilde{a} \right] ds \\ &\quad - \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} F(\max(M_s, b), a) B^1(X_s) \partial_1 p_{W^{*1}, W}(b - X_s^1, a - X_s; t - s) db da \right] ds. \end{aligned} \tag{30}$$

(i) The term $p^{m,\beta}(b, a, t)$ comes from the second term in I_m (29) as the factor of $F(b, a)$:

$$- \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} \mathbf{1}_{M_s < b} F(b, a) B^1(X_s) \partial_m p_{W^{*1}, W}(b - X_s^1, a - X_s; t - s) db da \right] ds$$

(ii) The terms $-p^{1,\alpha}(b, a, t)$ and $-p^{1,\beta}(b, a; t)$ come from the second term in I_1 (30):

$$- \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} F(\max(b, M_s), a) B^1(X_s) \partial_1 p_{W^{*1}, W}(b - X_s^1, a - X_s; t - s) db da \right] ds.$$

Inside the integral on the set $(M_s < b)$ we identify $-p^{1,\alpha}(b, a, t)$ and inside the integral on the set $(M_s > b)$ we identify $-p^{1,\beta}(b, a; t)$ as the factor of $F(b, a)$, respectively as

the factor of $F(M_s, a)$.

(iii) The term $-p^{m,\alpha}(b, a, t)$ comes from the sum of first terms in I_1 (30) and I_m (29).

Now we replace the variable b by a_1 , $dbd\tilde{a}$ by da in the first terms of I_m and I_1 :

$$\begin{aligned} I_m^1 &= - \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^d} F(\max(a^1, M_s), a) B^1(X_s) p_{W^{*1}, W}(\max(a^1, M_s) - X_s^1, a - X_s; t - s) da \right] ds \\ I_1^1 &= \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^d} F(\max(M_s, a^1), a) B^1(X_s) p_{W^{*1}, W}(a^1 - X_s^1, a - X_s; t - s) da \right] ds. \end{aligned}$$

Note that

$$\begin{aligned} & - p_{W^{*1}, W}(\max(a^1, M_s) - X_s^1, a - X_s; t - s) + p_{W^{*1}, W}(a^1 - X_s^1, a - X_s; t - s) \\ &= [-p_{W^{*1}, W}(M_s - X_s^1, a - X_s; t - s) + p_{W^{*1}, W}(a^1 - X_s^1, a - X_s; t - s)] \mathbf{1}_{M_s > a^1} \\ &= - \int_{a^1}^{M_s} \partial_m p_{W^{*1}, W}(b - X_s^1, a - X_s, t - s) db \mathbf{1}_{M_s > a^1}. \end{aligned}$$

Then the sum of I_m^1 and I_1^1 is:

$$- \int_0^t \mathbb{E}_{\mathbb{P}} \left[\int_{\mathbb{R}^{d+1}} F(M_s, a) B^1(X_s) \partial_m p_{W^{*1}, W}(b - X_s^1, a - X_s; t - s) \mathbf{1}_{M_s > b > a^1} da db \right] ds.$$

We introduce the density of the law of the pair (M_s, X_s) and we identify $-p^{m,\alpha}(m, a; t)$ as the factor of $F(m, a)$.

These three steps achieve the proof of Proposition 4.2 when $\mu_0 = \delta_{x_0}$.

Finally when μ_0 is the law of X_0 , we have $p_V(m, w; t, \mu_0) = \int_{\mathbb{R}^d} p_V(m, x; t, \delta_{x_0}) \mu_0(dx_0)$.

Then integrating with respect to μ_0 the expression obtained in (21) for $p_V(m, x; t, \delta_{x_0})$ achieves the proof of Proposition 4.2 for any initial law μ_0 . \square

4.2. Proof of Proposition 4.1

Using some idea's used in Garroni section V.3.2 let us introduce the linear applications on $L^\infty([0, T], dt, L^1(\mathbb{R}^{d+1}, dmdx))$, $k = m, 1, \dots, d$:

$$\mathcal{I}^{k,\alpha}[p](m, x; t) := \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{b < m} B^k(a) \partial_k p_{W^{*1}, W}(m - a^1, x - a; t - s) p(b, a; s) db da ds, \quad (31)$$

$$\mathcal{I}^{k,\beta}[p](m, x; t) := \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{b < m} B^k(a) \partial_k p_{W^{*1}, W}(b - a^1, x - a; t - s) p(m, a; s) db da ds.$$

Let us introduce the functions, defined by induction:

$$p_0(m, x; t, \mu_0) = \int_{\mathbb{R}^d} p_{W^{*1}, W}(m - x_0^1, x - x_0; t) \mu_0(dx_0), \quad p_n = - \sum_{k=m, 1, \dots, d} (p_n^{k,\alpha} + p_n^{k,\beta}) \quad (32)$$

and for $k = m, 1, \dots, d$, $j = \alpha, \beta$ and $n \geq 1$, $p_{n+1}^{k,j}(m, x; t) := \mathcal{I}^{k,j}[p_n](m, x; t)$. Let us denote the operator

$$\mathcal{I} := - \sum_{j=\alpha, \beta; k=m, 1, \dots, d} \mathcal{I}^{k,j}. \quad (33)$$

Moreover one remarks that this means $p_{n+1} = \mathcal{I}(p_n)$ and Proposition 4.2 leads to $p_V = p_0 + \mathcal{I}(p_V)$. Let

$$P_n := \sum_{k=0}^n p_k, \quad n \geq 0 \quad (34)$$

Proposition 4.3. *Assume the vector B is bounded, then for all T the sequence $(P_n)_n$ converges in $L^\infty([0, T], L^1(\mathbb{R}^{d+1}, dx dm))$ to p_V . Moreover $p_V = \sum_{n=0}^\infty p_n$.*

The proof is a consequence of the two following lemmas.

Lemma 4.2. *Let $j = \alpha, \beta$, $k = m, 1, \dots, d$ and $T > 0$ the linear applications $\mathcal{I}^{k,j}$ are continuous on $L^\infty([0, T], dt, L^1(\mathbb{R}^{d+1}, dmdx))$: there exists a constant C such that for all $p \in L^\infty([0, T], dt, L^1(\mathbb{R}^{d+1}, dmdx))$:*

$$\sup_{s \in [0, t]} \|\mathcal{I}^{k,j}[p](\cdot, \cdot; s)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq C \int_0^t \frac{1}{\sqrt{t-s}} \sup_{u \in [0, s]} \|p(\cdot, \cdot; u)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} ds \quad (35)$$

As a consequence,

$$\sup_{s \in [0, t]} \|\mathcal{I}[p](\cdot, \cdot; s)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq 2(d+1)C \int_0^t \frac{1}{\sqrt{t-s}} \sup_{u \in [0, s]} \|p(\cdot, \cdot; u)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} ds. \quad (36)$$

Proof. Let $T > 0$, $p \in L^\infty([0, T] \times L^1(\mathbb{R}^{d+1}, dmdx))$ and $t \in [0, T]$ and let ϕ_{d+1} be the Gaussian law density restrained to the subset $\{b > a_+^1\}$ (up to a constant)

$$\phi_{d+1}(b, b - a^1, \tilde{a}; 2t) := \frac{1}{\sqrt{2\pi t}^{d+1}} \mathbf{1}_{b > a_+^1} e^{-\frac{b^2 + (b - a^1)^2 + \|\tilde{a}\|^2}{4t}}. \quad (37)$$

(i) Let $j = \alpha$ and $k = m, 1, \dots, d$, according to the definition of $\mathcal{I}^{k,\alpha}$ and the boundedness of B ,

$$|\mathcal{I}^{k,\alpha}[p](m, x; t)| \leq \|B\|_\infty \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{b < m} |\partial_k p_{W^*1, W}(m - a^1, x - a; t - s) p(b, a; s)| db d\tilde{a} ds.$$

Using Lemma A.2 there exists a constant D such that for $k = m, 1, \dots, d$:

$$|\partial_k p_{W^*1, W}(b, a; t)| \leq \frac{D}{\sqrt{t}} \phi_{d+1}(b, b - a^1, \tilde{a}; 2t). \quad (38)$$

So

$$|\mathcal{I}^{k,\alpha}[p](m, x; t)| \leq \|B\|_\infty \int_0^t \int_{\mathbb{R}^{d+1}} \frac{D}{\sqrt{t-s}} \phi_{d+1}(m - a^1, m - x^1, \tilde{x} - \tilde{a}; t-s) |p(b, a; s)| db da ds.$$

We operate an integration with respect to (m, x) using Tonelli's theorem and omitting the indicator functions. Since ϕ_{d+1} is the density of a Gaussian law, we get the following bound,

$$\begin{aligned} \|\mathcal{I}^{k,\alpha}[p](\cdot, \cdot; t)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} &\leq D \|B\|_\infty \int_0^t \int_{\mathbb{R}^{d+1}} \frac{1}{\sqrt{t-s}} |p(b, a; s)| db da ds \\ &\leq 2^{(d+1)/2} D \|B\|_\infty \int_0^t \frac{1}{\sqrt{t-s}} \sup_{u \leq s} \|p(\cdot, \cdot; u)\|_{L^1(\mathbb{R}^{d+1}, db da)} ds, \end{aligned}$$

meaning inequality (35) when $j = \alpha$.

(ii) Let $j = \beta$ and $k = m, 1, \dots, d$. According to the definition of $\mathcal{I}^{k,\beta}$ and the boundedness of B ,

$$|\mathcal{I}^{k,\beta}[p](m, x; t)| \leq \|B\|_\infty \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{b < m} |\partial_k p_{W^*1, W}(b - a^1, x - a; t-s)| p(m, a; s) db da ds.$$

Using (38) yields:

$$|\mathcal{I}^{k,\beta}[p](m, x; t)| \leq \|B\|_\infty \int_0^t \int_{\mathbb{R}^{d+1}} \frac{D}{\sqrt{t-s}} \phi_{d+1}(b - a^1, b - x^1, \tilde{x} - \tilde{a}; 2(t-s)) |p(m, a; s)| db da ds.$$

We operate an integration with respect to x then to b using Tonelli's theorem and omitting the indicator functions and using that ϕ is the density of a Gaussian law. So the bound with respect to a multiplicative constant:

$$\begin{aligned} \|\mathcal{I}^{k,\beta}[p](\cdot, \cdot; t)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} &\leq D \|B\|_\infty 2^{(d+1)/2} \int_0^t \int_{\mathbb{R}^{d+1}} \frac{1}{\sqrt{t-s}} |p(m, a; s)| dmda ds \\ &\leq D \|B\|_\infty 2^{(d+1)/2} \int_0^t \frac{1}{\sqrt{t-s}} \sup_{u \leq s} \|p(\cdot, \cdot; u)\|_{L^1(\mathbb{R}^{d+1}, dmda)} ds, \end{aligned}$$

meaning inequality (35) for $j = \beta$.

Finally, estimation (36) is obtained by adding estimations (35) for $j = \alpha, \beta$ and $k = m, 1, \dots, d$. •

The following lemma is a consequence of (36) in Lemma 4.2:

Lemma 4.3. For all n

$$\sup_{u \leq t} \|p_n(\cdot, \cdot; u)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq (2(d+1)C)^n t^{n/2} \frac{\Gamma(1/2)^n}{\Gamma(1+n/2)}, \quad (39)$$

$$\sup_{u \leq t} \|(p_V - P_n)(\cdot, \cdot; u)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq (2(d+1)C)^{n+1} t^{(n+1)/2} \frac{\Gamma(1/2)^{n+1}}{\Gamma((n+3)/2)}. \quad (40)$$

Proof. (i) For all $t > 0$, $p_0(\cdot; t)$ is a density of probability, so (39) is satisfied for $n = 0$. We now assume that (39) is satisfied for n . Using $p_{n+1} = \mathcal{I}[p_n]$, (36) and the induction e assumption:

$$\sup_{u \leq t} \|p_{n+1}(\cdot, \cdot; u)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq (2(d+1)C)^{n+1} \frac{\Gamma(1/2)^n}{\Gamma(1+n/2)} \int_0^t \frac{\sqrt{s^n}}{\sqrt{t-s}} ds.$$

We operate the change of variable $s = tu$ and use $\int_0^1 u^{a-1}(1-u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$:

$$\sup_{u \leq t} \|p_{n+1}(\cdot, \cdot; u)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq (2(d+1)C)^{n+1} t^{(n+1)/2} \frac{\Gamma(1/2)^n}{\Gamma(1+n/2)} \frac{\Gamma(1/2)\Gamma(1+n/2)}{\Gamma((n+3)/2)}$$

which proves (39) for all n .

(ii) Noting that $P_0 = p_0$ and $p_V - p_0 = \mathcal{I}[p_V]$ and applying (36) to p_V yield

$$\sup_{u \leq t} \|(p_V - P_0)(\cdot, \cdot; u)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq 2(d+1)Ct^{1/2}.$$

But $\Gamma(1/2)/\Gamma(3/2) = 2$ so (40) is satisfied for $n = 0$.

We now suppose that (40) is satisfied for n . Using $p_V - P_{n+1} = p_0 + \mathcal{I}(p_V) - (p_0 + \mathcal{I}(P_n)) = \mathcal{I}(p_V - P_n)$, the bound (36) and the induction assumption:

$$\sup_{u \leq t} \|[p_V - P_{n+1}](\cdot, \cdot; u)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq 2(d+1)C \int_0^t (2(d+1)C)^{n+1} \frac{\Gamma(1/2)^{n+1}}{\Gamma((3+n)/2)} \frac{\sqrt{s^{n+1}}}{\sqrt{t-s}} ds.$$

We now operate the change of variable $s = tu$ and $\int_0^1 u^{a-1}(1-u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

with $a = (n+3)/2, b = \frac{1}{2}$:

$$\sup_{u \leq t} \|[p_V - P_{n+1}](\cdot, \cdot; u)\|_{L^1(\mathbb{R}^{d+1}, dmdx)} \leq (2(d+1)C)^{n+2} t^{(n+2)/2} \frac{\Gamma(1/2)^{n+2}}{\Gamma((4+n)/2)}$$

which proves (40) for $n+1$ and thus for all n . \square

The series $\sum_n \frac{x^n}{\Gamma(n/2+1)}$ is convergent for any x , so Proposition 4.3 is a consequence of lemmas 4.2 and 4.3.

4.2.1. *Upper Bound of p_V meaning Hypothesis 2.1 (i).*

For all $T > 0$, $x_0 \in \mathbb{R}^d$, $p \in L^\infty([0, T], L^1(\mathbb{R}^{d+1}, dmdx))$ the support of which being included in $\{(m, x), m > x_0^1, m > x^1\}$ let us denote

$$N(p; t, x_0) := \sup_{(m, x) \in \mathbb{R}^{d+1}, m > x^1, m > x_0^1} \frac{|p(m, x; t)|}{\phi_{d+1}(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0; 2t)}. \quad (41)$$

Proposition 4.4. *For all $T > 0$ there exists a constant C_T and for all n there exists constants $C_n = \frac{[\|B\|_\infty D(2(d+1))2^{d/2}\Gamma(1/2)]^n}{\Gamma(1+n/2)}$ such that: for all $x_0 \in \mathbb{R}^d$, $0 < t \leq T$,*

$$(i) \quad |p_n(m, x; t, x_0)| \leq C_n t^{n/2} \phi_{d+1}(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0, 2t) \mathbf{1}_{m > \max(x^1, x_0^1)}$$

$$(ii) \quad |p_V(m, x; t, x_0)| \leq C_T \phi_{d+1}(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0, 2t) \mathbf{1}_{m > \max(x^1, x_0^1)}$$

(iii) *For all μ_0 initial probability measure on \mathbb{R}^d ,*

$$\sup_{u > 0} p_V(m, m - u, \tilde{x}, t; \mu_0) \in L^1([0, T] \times \mathbb{R}^d, dt dmd\tilde{x}).$$

Remark that, actually, this point (iii) is Hypothesis 2.1 (i).

Proof. Point (ii) is a consequence of point (i), since $p_V = \sum_{n=0}^{\infty} p_n$, and the series $\sum_n \frac{1}{\Gamma(1+n/2)} x^n$ admits an infinite radius of convergence (Proposition 4.3).

We prove point (i) by induction on n using point (ii) in Lemma A.2: $p_0(m, x; t, x_0) \leq$

$$\frac{e^{-\frac{(m-x^1)^2}{4t} - \frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t} - \frac{(m-x_0^1)^2}{4t}}}{\sqrt{(2\pi)^{d+1} t^{d+1}}} \mathbf{1}_{m > \max(x^1, x_0^1)} = \phi_{d+1}(m - x^1, m - x_0^1, \tilde{x} - \tilde{x}_0; 2t) \mathbf{1}_{m > \max(x^1, x_0^1)},$$

so $N(p_0; t, x_0) \leq 1$, which is (i) for $n = 0$, $C_0 = 1$.

We assume point (i) is true for p_n , meaning $N(p_n; t, x_0) \leq C_n t^{n/2}$. By definition $p_{n+1} = \mathcal{I}[p_n]$, Lemma 4.4 proved below yields:

$$N(p_{n+1}; t, x_0) = N(\mathcal{I}[p_n]; t, x_0) \leq 2(d+1)2^{d/2} \|B\|_\infty D C_n \int_0^t \frac{s^{n/2}}{\sqrt{2\pi(t-s)}} ds.$$

We operate the change of variable $s = tu$

$$N(p_{n+1}; t, x_0) \leq \frac{2(d+1)2^{d/2} \|B\|_\infty D}{\sqrt{2\pi}} C_n (\sqrt{t})^{n+1} \int_0^1 \frac{u^{n/2}}{\sqrt{1-u}} du$$

Using $\int_0^1 \frac{u^{n/2}}{\sqrt{1-u}} du = \frac{\Gamma((n+2)/2)\Gamma(1/2)}{\Gamma((n+3)/2)}$ and C_n definition:

$$N(p_{n+1}; t, x_0) \leq C_{n+1} (\sqrt{t})^{n+1},$$

this achieves the proof of point (i) in Proposition 4.4.

(iii) Then for all $x_0 \in \mathbb{R}^d$ and using $x^1 = m - u$,

$$\sup_{u>0} p_V(m, m - u, \tilde{x}; t) \leq C_T \phi_{d+1}(m - x_0^1, 0, \tilde{x} - \tilde{x}_0; 2t) \in L^1([0, T] \times \mathbb{R}^d, dt dmd\tilde{x}).$$

Since $p_V(m, x; t, \mu_0) = \int_{\mathbb{R}^d} p_V(m, x; t, x_0) \mu_0(dx_0)$ point (iii) is true. \square

Lemma 4.4. *Let $T > 0$, $x_0 \in \mathbb{R}^d$, $p \in L^\infty([0, T], dt, L^1(\mathbb{R}^{d+1}, dmdx))$ such that the support of $p(\cdot, \cdot; t)$ is included in $\{(m, x), m > x_0^1, m > x^1\}$ and for all $s \in]0, T]$ $N(p; s, x_0) < \infty$. Then for $j = \alpha, k = m, 1, \dots, d$, the support of function $\mathcal{I}^{j,k}[p](\cdot; t)$ is included in $\{(m, x), m > x_0^1, m > x^1\}$. Moreover for all $t \in [0, T]$ we have :*

$$N(\mathcal{I}[p]; t, x_0) \leq 2(d+1)2^{d/2} \|B\|_\infty D \int_0^t \frac{1}{\sqrt{2\pi(t-s)}} N(p; s, x_0) ds, \quad \forall t \in [0, T].$$

Proof. Let $T > 0$, $x_0 \in \mathbb{R}^d$, $p \in L^\infty([0, T], dt, L^1(\mathbb{R}^{d+1}, dmdx))$ such that for all $t > 0$ the support of $p(\cdot; t)$ is included in $\{(m, x), m > x_0^1, m > x^1\}$.

(i) For $j = \alpha, k = m, 1, \dots, d$, using the definition of $\mathcal{I}^{\alpha,k}$ yields:

$$\mathcal{I}^{k,\alpha}[p](m, x; t) := \int_0^t \int_{\mathbb{R}^{d+1}} B^k(a) \partial_k p_{W^{*1}, W}(m - a^1, x - a; t - s) \mathbf{1}_{x_0^1 < b < m, m > x^1} p(b, a; s) db da ds$$

So the support of $\mathcal{I}^{\alpha,k}[p](\cdot; t)$ is included in $\{(m, x) \in \mathbb{R}^{d+1}, m > \max(x_0^1, x^1)\}$. For now on, we only consider (m, x) such that $m > \max(x^1, x_0^1)$.

Let p be a function such that $\forall s \in]0, T]$ $N(p; x_0, s) < \infty$. The definition of $\mathcal{I}^{k,\alpha}$, the boundedness of B , the fact that $\partial_k p_{W^{*1}, W}$ satisfies (38) and the definition (41) of $N(p; t, x_0)$ imply

$$\begin{aligned} |\mathcal{I}^{k,\alpha}[p](m, x; t)| &\leq \|B\|_\infty \int_0^t \int_{\mathbb{R}^{d+1}} N(p; s, x_0) \frac{D}{\sqrt{(t-s)}} \mathbf{1}_{m > x^1} \mathbf{1}_{m > b > \max(a^1, x_0^1)} \\ &\quad \phi_{d+1}(m - a^1, m - x^1, \tilde{x} - \tilde{a}; t - s) \phi_{d+1}(b - x_0^1, b - a^1, \tilde{a} - \tilde{x}_0; s) db da ds. \end{aligned}$$

We integrate in \tilde{a} using Lemme A.3 (ii) with $u = \tilde{x}, v = \tilde{a}, w = \tilde{x}_0$ and the fact that ϕ_{d+1} is a Gaussian density of probability:

$$\begin{aligned} |\mathcal{I}^{\alpha,k}[p](m, x; t)| &\leq 2^{(d-1)/2} \|B\|_\infty D \tag{42} \\ \int_0^t \int_{\mathbb{R}^2} N(p; s, x_0) \mathbf{1}_{m > b > \max(a^1, x_0^1)} &\frac{e^{-\frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t}}}{\sqrt{(2\pi t)^{d-1}}} e^{-\frac{(m-a^1)^2}{4(t-s)} - \frac{(m-x^1)^2}{4(t-s)}} e^{-\frac{(b-x_0^1)^2}{4s} - \frac{(b-a^1)^2}{4s}} db da^1 ds. \end{aligned}$$

Using point (i') Lemma A.2 with $u = m$, $v = a^1$, $w = b$, $k = 1$, we integrate in da^1 up to b :

$$\int_{-\infty}^b \frac{e^{-\frac{(m-a^1)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)}} \frac{e^{-\frac{(b-a^1)^2}{4s}}}{\sqrt{(2\pi s)}} da^1 = \frac{e^{-\frac{(m-b)^2}{4t}}}{\sqrt{2\pi t}} \Phi_G \left(\sqrt{\frac{s}{2t(t-s)}} (b-m) \right)$$

where $\Phi_G(u) = \int_{-\infty}^u e^{-z^2/2} dz \leq \frac{1}{2} e^{-u^2/2}$ for $u = b-m < 0$ according to Lemma A.3

(iii). This yields the bound: $\frac{e^{-\frac{(m-b)^2}{4t}}}{\sqrt{2\pi t}} e^{-\frac{s(b-m)^2}{4t(t-s)}}$ and

$$2 \int_{-\infty}^b \frac{e^{-\frac{(m-a^1)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)}} \frac{e^{-\frac{(b-a^1)^2}{4s}}}{\sqrt{(2\pi s)}} da^1 \leq \frac{e^{-\frac{(m-b)^2}{4t}}}{\sqrt{2\pi t}} e^{-\frac{s(m-b)^2}{4t(t-s)}} = \frac{e^{-\frac{(m-b)^2}{4(t-s)}}}{\sqrt{2\pi t}}.$$

Plugging this inequality inside (42) yields with $C_{d,B} = 2^{(d+1)/2} \|B\|_{\infty} D$

$$\frac{|\mathcal{I}^{\alpha,k}[p](m, x; t)|}{C_{d,B}} \leq \int_0^t \int_{\mathbb{R}} N(p; s, x_0) \mathbf{1}_{m > b > x_0^1} \frac{e^{-\frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t}}}{\sqrt{(2\pi t)^d}} \frac{e^{-\frac{(m-b)^2}{4(t-s)} - \frac{(m-x^1)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)^2 s}} e^{-\frac{(b-x_0^1)^2}{4s}} db ds.$$

Omitting the indicator functions, Lemma A.3 (ii) with $u = m$, $v = b$, $w = x_0^1$, $k = 1$ implies

$$\int_{b < m} \frac{e^{-\frac{(m-b)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)2\pi s}} e^{-\frac{(b-x_0^1)^2}{4s}} db \leq \sqrt{\frac{2}{2\pi t}} e^{-\frac{(m-x_0^1)^2}{4t}}.$$

Inserting this result, we obtain

$$|\mathcal{I}^{\alpha,k}[p](m, x; t)| \leq \sqrt{2} C_{d,B} \int_0^t N(p; s, x_0) \frac{e^{-\frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t}}}{\sqrt{(2\pi t)^{d+1}}} \frac{e^{-\frac{(m-x_0^1)^2}{4t} - \frac{(m-x^1)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)}} ds.$$

For $0 < s < t$, $e^{-\frac{(m-x^1)^2}{4(t-s)}} \leq e^{-\frac{(m-x^1)^2}{4t}}$, so

$$|\mathcal{I}^{\alpha,k}[p](m, x; t)| \leq \sqrt{2} C_{d,B} \int_0^t N(p; s, x_0) \frac{e^{-\frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t}}}{\sqrt{(2\pi t)^{d+1}}} \frac{e^{-\frac{(m-x_0^1)^2}{4t} - \frac{(m-x^1)^2}{4t}}}{\sqrt{2\pi(t-s)}} ds.$$

Using the definition of ϕ_{d+1} we identify

$$|\mathcal{I}^{\alpha,k}[p](m, x; t)| \leq \sqrt{2} C_{d,B} \int_0^t N(p; s, x_0) \frac{\phi_{d+1}(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0; 2t)}{\sqrt{2\pi(t-s)}} ds$$

and with the definition of N , with respect to a multiplicative constant:

$$N(\mathcal{I}^{\alpha,k}[p], x_0, t) \leq \sqrt{2} C_{d,B} \int_0^t N(p; s, x_0) \frac{1}{\sqrt{2\pi(t-s)}} ds.$$

(ii) For $j = \beta$, $k = m, 1, \dots, d$ using the definition of $\mathcal{I}^{\beta,k}$ and the fact that the support of p is included in $\{(m, x), m > \max(x_0^1, x^1)\}$ yields $\mathcal{I}^{\beta,k}[p](m, x; t) =$

$$\int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{m > b > x^1, m > x_0^1, b > a^1} B^k(a) \partial_k p_{W^{1*}, W}(b - a^1, x - a, t - s) p(m, a, s) da db ds.$$

Thus the support of $\mathcal{I}^{\beta,k}[p](\cdot; t)$ is included in $\{(m, x), m > \max(x^1, x^0)\}$. For now on we only consider (m, x) satisfying $m > \max(x^1, x^0)$. Definition of $\mathcal{I}^{\beta,k}$, the boundedness of B , the inequality (38) satisfied by $\partial_k p_{W^*, W}$:

$$|\partial_k p_{W^*, W}(b - a^1, x - a, t - s)| \leq \frac{D}{\sqrt{t-s}} \phi_{d+1}(b - a^1, b - x^1, \tilde{x} - \tilde{a}, 2(t - s))$$

and the definition of $N(p; t, x_0)$ yield:

$$\begin{aligned} |\mathcal{I}^{\beta,k}[p](m, x; t)| &\leq \|B\|_\infty D \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{m > b > x^1, b > a^1} N(p; s, x_0) \\ &\frac{e^{-\frac{(b-a^1)^2}{4(t-s)} - \frac{(b-x^1)^2}{4(t-s)} - \frac{\|\tilde{x}-\tilde{a}\|^2}{4(t-s)}}}{\sqrt{(2\pi)^{d+1}(t-s)^{d+2}}} e^{-\frac{(m-x_0^1)^2}{4s} - \frac{(m-a^1)^2}{4s} - \frac{\|\tilde{x}_0-\tilde{a}\|^2}{4s}}}{\sqrt{(2\pi)^{d+1}s^{d+1}}} da^1 db ds. \end{aligned}$$

We integrate in \tilde{a} using Lemma A.3 (ii) with $u = \tilde{x}$, $v = \tilde{a}$ et $w = \tilde{x}_0$:

$$|\mathcal{I}^{\beta,k}[p](m, x; t)| \leq C_{d,B}. \quad (43)$$

$$\int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{m > b > x^1, b > a^1} \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{(2\pi t)^{d-1}}} N(p; s, x_0) \cdot \frac{e^{-\frac{(b-a^1)^2}{4(t-s)} - \frac{(b-x^1)^2}{4(t-s)}}}{\sqrt{(2\pi)^2(t-s)^3}} \frac{e^{-\frac{(m-x_0^1)^2}{4s} - \frac{(m-a^1)^2}{4s}}}{\sqrt{(2\pi)^2 s^2}} da^1 db ds.$$

Using Lemma A.3 (i') for $u = b$, $v = a^1$, $w = m$ $k = 1$

$$\begin{aligned} \int_{-\infty}^b \frac{e^{-\frac{(b-a^1)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)}} \frac{e^{-\frac{(m-a^1)^2}{4s}}}{\sqrt{2\pi s}} da^1 &= \sqrt{2} \frac{e^{-\frac{(b-m)^2}{4t}}}{\sqrt{2\pi t}} \Phi_G \left(\sqrt{\frac{t}{4s(t-s)}} \left[b - \left(\frac{s}{t} b + \frac{t-s}{t} m \right) \right] \right) \\ &= \frac{e^{-\frac{(b-m)^2}{4t}}}{\sqrt{2\pi t}} \Phi_G \left(\sqrt{\frac{t-s}{4st}} [b - m] \right) \leq \frac{e^{-\frac{(b-m)^2}{4t}}}{\sqrt{2\pi t}} e^{-\frac{t-s}{4st} [b-m]^2} = \frac{e^{-\frac{(b-m)^2}{4s}}}{\sqrt{2\pi t}} \end{aligned}$$

the last bound coming from Lemma A.3 (iii) since $b - m < 0$.

We plug this estimation in (43)

$$\begin{aligned} |\mathcal{I}^{\beta,k}[p](m, x; t)| &\leq \\ C_{d,B} \int_0^t \int_{\mathbb{R}} \mathbf{1}_{m > b > x^1} \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{(2\pi t)^d}} N(p; s, x_0) &\frac{e^{-\frac{(b-x^1)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)^2}} \frac{e^{-\frac{(m-x_0^1)^2}{4s} - \frac{(b-m)^2}{4s}}}{\sqrt{2\pi s}} db ds. \end{aligned}$$

We integrate with respect to b on \mathbb{R} and we use Lemma A.3 (ii) with $u = x^1$, $v = b$, $w = m$, $k = 1$:

$$|\mathcal{I}^{\beta,k}[p](m, x; t)| \leq \sqrt{2} C_{d,B} \int_0^t \frac{e^{-\frac{(m-x^1)^2}{4t} - \frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{(2\pi t)^{d+1}}} N(p; s, x_0) \frac{e^{-\frac{(m-x_0^1)^2}{4s}}}{\sqrt{2\pi(t-s)}} ds.$$

When $0 < s < t$, $e^{-\frac{(m-x_0^1)^2}{4s}} \leq e^{-\frac{(m-x_0^1)^2}{4t}}$ so:

$$|\mathcal{I}^{\beta,k}[p](m, x; t)| \leq \sqrt{2} C_{d,B} \int_0^t \frac{e^{-\frac{(m-x^1)^2}{4t} - \frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t} - \frac{(m-x_0^1)^2}{4t}}}{\sqrt{(2\pi t)^{d+1}}} N(p; s, x_0) \frac{1}{\sqrt{2\pi(t-s)}} ds.$$

Under the integral we identify the factor $\phi_{d+1}(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0; 2t)$ so

$$|\mathcal{I}^{\beta,k}[p](m, x; t)| \leq \sqrt{2}C_{d,B}\phi_{d+1}(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0; 2t) \int_0^t N(p; s, x_0) \frac{1}{\sqrt{2\pi(t-s)}} db ds.$$

Finally using the definition of N (41) we have proved

$$N(\mathcal{I}^{\beta,k}[p], x_0, t) \leq \sqrt{2}C_{d,B} \int_0^t N(p; s, x_0) \frac{1}{\sqrt{2\pi(t-s)}} ds$$

which achieves the proof of Lemma 4.4. \square

4.2.2. Proof of Hypothesis 2.1 (ii), case $A = I_d$

Proposition 4.5. *For any μ_0 probability measure on \mathbb{R}^d , for all $(m, \tilde{x}, t) \in \mathbb{R}^d \times]0, T]$, $u \mapsto p_V(m, m - u, \tilde{x}, t)$ admits a limit when u goes to 0, $u > 0$.*

Proof. The proof is a consequence of the three following lemmas.

Lemma 4.5. *Recall that $p^0(m, x; t) = \int_{\mathbb{R}^d} p_{W^*,W}(m - x_0^1, x - x_0; t) \mu_0(dx_0)$.*

$$\lim_{u \rightarrow 0, u > 0} p^0(b, b - u, \tilde{a}; t) = p^0(b, b, \tilde{a}; t), \quad \forall u > 0, \quad (b, \tilde{a}) \in \mathbb{R}^d, \forall t > 0.$$

Proof. We have $p^0(b, b - u, \tilde{x}; t) = \int_{\mathbb{R}^d} 2 \frac{b+u-x_0^1}{\sqrt{(2\pi)^{d+1}}} e^{-\frac{(b+u-x_0^1)^2}{2t} - \frac{\|\tilde{x}-\tilde{x}_0\|^2}{2t}} \mathbf{1}_{b \geq x_0^1}, \quad u \geq 0 \mu_0(dx_0)$. Then, since the integrand is dominated by $\frac{D}{\sqrt{(2\pi)^{d+1}t}}$ and μ_0 is a probability measure, using Lebesgue's dominated convergence Theorem yields:

$$\lim_{u \rightarrow 0, u > 0} p^0(b, b - u, \tilde{x}; t) = p^0(b, b, \tilde{x}; t), \quad \forall (b, \tilde{x}) \in \mathbb{R}^d, \forall t > 0.$$

\square

Lemma 4.6. *For $k = m, 1, \dots, d$ recall that*

$$p^{k,\alpha}(m, x; t) = \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{b < m} B^k(a) \partial_k p_{W^*,W}(m - a^1, x - a, t - s) p_V(b, a; s) db da ds.$$

The map $u \mapsto p^{k,\alpha}(m, m - u, \tilde{x}; t)$ converges to $p^{k,\alpha}(m, m, \tilde{x}; t)$ when u goes to 0^+ .

Proof. The proof will be a consequence of Lebesgue dominated theorem. First, the map $u \mapsto \mathbf{1}_{b < m} \partial_k p_{W^*,W}(m - a^1, m - u - a^1, \tilde{x} - \tilde{a}; t - s) p_V(b, a; s)$ converges to $\mathbf{1}_{b < m} \partial_k p_{W^*,W}(m - a^1, m - a^1, \tilde{x} - \tilde{a}; t - s) p_V(b, a; s)$ when u goes to 0^+ .

Second it is dominated by $q^{k,\alpha}(m, \tilde{x}, a, b; s, x_0) :=$

$$|B^k(a)| \mathbf{1}_{b < m} \sup_{u > 0} |\partial_k p_{W^*,W}(m - a^1, m - u - a^1, \tilde{x} - \tilde{a}; t - s) p_V(b, a; s, x_0)|$$

We seek to prove that

$$\int_0^T \int_{\mathbb{R}^{2d+1}} q^{k,\alpha}(m, \tilde{x}, a, b; s, x_0) ds db da \mu_0(dx_0) < +\infty. \quad (44)$$

According to estimation (38) of $\partial_k p_{W^{*1}, W}$ and estimation (ii) of Proposition 4.4, we obtain

$$q^{k,\alpha}(m, \tilde{x}, a, b; s, x_0) \leq \|B\|_\infty \mathbf{1}_{m>b>a^1} \frac{D}{\sqrt{t-s} \sqrt{2\pi(t-s)}^{d+1}} \exp\left[-\frac{(m-a^1)^2}{4(t-s)} - \frac{\|\tilde{x}-\tilde{a}\|^2}{4(t-s)}\right] \\ \frac{C_T}{\sqrt{2\pi s}^{d+1}} \exp\left[-\frac{(b-x_0^1)^2}{4s} - \frac{(b-a^1)^2}{4s} - \frac{\|\tilde{x}_0-\tilde{a}\|^2}{4s}\right].$$

We integrate with respect to \tilde{a} using Lemma A.3 (ii) for $k = d+1$, $u = \tilde{x}$, $v = \tilde{a}$ and $w = \tilde{x}_0$:

$$\int_{\mathbb{R}^{d-1}} q^{k,\alpha}(m, \tilde{x}, a^1, b; s, x_0^1) d\tilde{a} \leq \mathbf{1}_{m>b>a^1} \frac{\|B\|_\infty C_T D 2^{(d-1)/2}}{\sqrt{t-s} \sqrt{2\pi(t-s)}^2 \sqrt{2\pi s}^2} \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{2\pi t}^{d-1}} \\ \exp\left[-\frac{(m-a^1)^2}{4(t-s)} - \frac{(b-x_0^1)^2}{4s} - \frac{(b-a^1)^2}{4s}\right].$$

We integrate with respect to a_1 between $-\infty$ and b using Lemma A.3 (i') for $u = m$, $v = a^1$ and $w = b$

$$\int_{\mathbb{R}^d} \mathbf{1}_{a^1 < b} q^{k,\alpha}(m, \tilde{x}, b, a; s; x_0) da \leq \mathbf{1}_{b < m} \frac{\|B\|_\infty C_T D 2^{d/2}}{\sqrt{t-s} \sqrt{2\pi(t-s)} \sqrt{2\pi s}} \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}} - \frac{(b-m)^2}{4t}}{\sqrt{2\pi t}^d} \\ \exp\left[-\frac{(b-x_0^1)^2}{4s}\right] \Phi_G\left(\sqrt{\frac{t}{2s(t-s)}} \left(b - \left[\frac{s}{t}m + \frac{(t-s)}{t}b\right]\right)\right).$$

Note that $\sqrt{\frac{t}{2s(t-s)}} \left(b - \left[\frac{s}{t}m + \frac{(t-s)}{t}b\right]\right) = \sqrt{\frac{s}{2t(t-s)}} (b-m)$ and using Lemma A.3 (iii)

$$\int_{\mathbb{R}^d} q^{k,\alpha}(m, \tilde{x}, b, a; s, x_0) da \leq \mathbf{1}_{b < m} \frac{\|B\|_\infty C_T D 2^{d/2}}{\sqrt{t-s} \sqrt{2\pi(t-s)} \sqrt{2\pi s}} \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}} - \frac{(b-m)^2}{4t}}{\sqrt{2\pi t}^d} \\ \exp\left[-\frac{(b-x_0^1)^2}{4s}\right] \exp\left[-\frac{s}{t(t-s)} \frac{(b-m)^2}{4}\right].$$

We observe that $\frac{1}{t} + \frac{s}{t(t-s)} = \frac{1}{t-s}$ so that $\exp\left[-\frac{(b-m)^2}{4t}\right] \exp\left[-\frac{s}{t(t-s)} \frac{(b-m)^2}{4}\right] = \exp\left[-\frac{(b-m)^2}{4(t-s)}\right]$.

We integrate with respect to b (neglecting the indicator function) using Lemma A.3

(ii) for $u = m$, $v = b$ and $w = x_0^1$ and $\exp\left[-\frac{(m-x_0^1)^2}{4t}\right] \leq 1$:

$$\int_{\mathbb{R}^{d+1}} q^{k,\alpha}(m, \tilde{x}, b, a; s, x_0) da db \leq \mathbf{1}_{m > x_0^1} \frac{\|B\|_\infty C_T D 2^{(d+1)/2}}{\sqrt{t-s}} \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{t}^{d+1}}.$$

Since μ_0 is a probability measure then $\int_0^t \int_{\mathbb{R}^{2d+1}} q^{k,\alpha}(m, \tilde{x}, b, a; s, x_0) da db \mu_0(dx_0) ds < +\infty$.

This is (44) and achieves the proof of Lemma 4.6 \square

Lemma 4.7. For $k = m, 1, \dots, d$ recall that

$$p^{k,\beta}(m, x; t) = \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{b < m} B^k(a) \partial_k p_{W^{*1}, W}(b - a^1, x - a, t - s) p_V(m, a; s) db da ds.$$

The map $u \mapsto p^{k,\beta}(m, m - u, \tilde{x}; t)$ converges to 0 when u goes to 0^+ .

Proof. Using estimation (38) of $\partial_k p_{W^{*1}, W}$ and estimation (ii) of Proposition 4.4 concerning p_V , we dominate the integrand which defines $p^{k,\beta}(m, m - u, \tilde{x}; t)$ by :
 $q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s) :=$

$$\mathbf{1}_{m-u < b < m, a^1 < b} \frac{e^{-\frac{(b-a^1)^2}{4(t-s)} - \frac{(b-m+u)^2}{4(t-s)} - \frac{\|\tilde{x}-\tilde{a}\|^2}{4(t-s)} - \frac{(m-x_0^1)^2}{4s} - \frac{(m-a^1)^2}{4s} - \frac{\|\tilde{x}_0-\tilde{a}\|^2}{4s}}{\sqrt{t-s} \sqrt{2\pi(t-s)}^{d+1} \sqrt{2\pi s}^{d+1}}$$

up to a multiplicative constant. Meaning that

$$|p^{k,\beta}(m, m - u, \tilde{x}; t)| \leq \|B\|_\infty \int_0^t \int_{\mathbb{R}^{2d+1}} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s) db da ds \mu_0(dx_0). \quad (45)$$

We integrate with respect to \tilde{a} using Lemma A.3 (ii) with $u = \tilde{x}$, $v = \tilde{a}$ and $w = \tilde{x}_0$

$$\int_{\mathbb{R}^{d-1}} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s) d\tilde{a} \leq \sqrt{2}^{d-1} e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}} e^{-\frac{(b-a^1)^2}{4(t-s)} - \frac{(b-m+u)^2}{4(t-s)} - \frac{(m-x_0^1)^2}{4s} - \frac{(m-a^1)^2}{4s}} \frac{1}{\sqrt{2\pi t}^{d-1} \sqrt{t-s} \sqrt{2\pi(t-s)}^2 \sqrt{2\pi s}^2}$$

We integrate with respect to a^1 between $-\infty$ and b using Lemma A.3 (i') for $u = b$, $v = a^1$ and $w = m$:

$$\begin{aligned} \int_{\mathbb{R}^d} \mathbf{1}_{b > a^1} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s) da &\leq \\ \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}} e^{-\frac{(b-m)^2}{4t} - \frac{(b-m+u)^2}{4(t-s)} - \frac{(m-x_0^1)^2}{4s}}}{\sqrt{2\pi t}^d \sqrt{t-s} \sqrt{2\pi(t-s)} \sqrt{2\pi s}} \Phi_G \left(\sqrt{\frac{t}{s(t-s)}} \left(b - \frac{s}{t} b - \frac{t-s}{t} m \right) \right) \\ &= \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}} e^{-\frac{(b-m)^2}{4t} - \frac{(b-m+u)^2}{4(t-s)} - \frac{(m-x_0^1)^2}{4s}}}{\sqrt{2\pi t}^d \sqrt{t-s} \sqrt{2\pi(t-s)} \sqrt{2\pi s}} \Phi_G \left(\sqrt{\frac{t-s}{2st}} (b-m) \right). \end{aligned}$$

Since $b - m < 0$, using Lemma A.3 (iii)

$$\int_{\mathbb{R}^d} \mathbf{1}_{b > a^1} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s) da \leq \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}} e^{-\frac{(b-m)^2}{4t} - \frac{(b-m+u)^2}{4(t-s)} - \frac{(m-x_0^1)^2}{4s}}}{\sqrt{2\pi t}^d \sqrt{t-s} \sqrt{2\pi(t-s)} \sqrt{2\pi s}} e^{-\frac{t-s}{4st} (b-m)^2}.$$

Note that $e^{-\frac{(b-m)^2}{4t}} e^{-\frac{t-s}{4st}(b-m)^2} = e^{-\frac{(b-m)^2}{4s}}$. We integrate this last bound with respect to b between $m-u$ and m using Lemma A.3 (i') for the triplet $(m-u, b, m)$ and the fact that $\frac{s}{t}(m-u) + \frac{t-s}{t}m = \frac{s(m-u)+m(t-s)}{t}$

$$\int_{R^{d+1}} \mathbf{1}_{m-u < b < m, b < a^1} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s) dadb \leq \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{2\pi t}^{d+1}} \frac{e^{-\frac{(m-x_0^1)^2}{4s}}}{\sqrt{t-s}} \left[\Phi_G \left(\sqrt{\frac{t}{2s(t-s)}} \left(m - \frac{s(m-u)+m(t-s)}{t} \right) \right) - \Phi_G \left(\sqrt{\frac{t}{2s(t-s)}} \left(m - u - \frac{s(m-u)+m(t-s)}{t} \right) \right) \right].$$

Then,

$$\int_{R^{d+1}} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s) dadb \leq \frac{e^{-\frac{\|\tilde{x}-\tilde{x}_0\|^2}{4t}}}{\sqrt{2\pi t}^{d+1}} \frac{e^{-\frac{(m-x_0^1)^2}{4s}}}{\sqrt{t-s}} \left[\Phi_G \left(\sqrt{\frac{s}{2t(t-s)}} u \right) - \Phi_G \left(-\sqrt{\frac{t-s}{2t(t-s)}} u \right) \right].$$

Note that $\lim_{u \rightarrow 0} \Phi_G \left(\sqrt{\frac{s}{2t(t-s)}} u \right) - \Phi_G \left(-\sqrt{\frac{t-s}{2t(t-s)}} u \right) = 0$ and

$\left| \Phi_G \left(\sqrt{\frac{s}{2t(t-s)}} u \right) - \Phi_G \left(-\sqrt{\frac{t-s}{2t(t-s)}} u \right) \right| \leq 1$. Since μ is a probability measure, using Lebesgue dominated theorem

$$\lim_{u \rightarrow 0^+} \int_0^t \int_{R^{2d+1}} \mathbf{1}_{m-u < b < m, b < a^1} q^{k,\beta}(m, u, \tilde{x}, a, b, x_0, s) dadb \mu_0(dx_0) ds = 0.$$

Finally estimation (45) yields $\lim_{u \rightarrow 0^+} p^{k,\beta}(m, m-u, \tilde{x}; t) = 0$. \square

5. Cas $d = 1$

Proposition 5.1. *Let the real diffusion X given by $dX_t = B(X_t)dt + A(X_t)dW_t$ where A, B fulfil (4) and (5). Then the density of probability p_V satisfies Hypothesis 2.1, so for any initial law μ_0 and $F \in C_b^2(\mathbb{R}^2, \mathbb{R})$,*

$$\begin{aligned} \mathbb{E}[F(M_t, X_t)] &= \mathbb{E}[F(X_0, X_0)] + \int_0^t \mathbb{E}[\mathcal{L}(F)(M_s, X_s)] ds \\ &\quad + \frac{1}{2} \int_0^t \mathbb{E} \left[\partial_m F(X_s, X_s) \|A(X_s)\|^2 \frac{p_V(X_s, X_s; s)}{p_X(X_s; s)} \right] ds. \end{aligned} \quad (46)$$

Proof. We operate a Lamperti transformation [18]. Without loss of generality, A can be chosen positive. In case $d = 1$ Assumption (5): “ $\exists c > 0$ such that for any $x \in \mathbb{R}$, $A^2(x) \geq c$ ” could be expressed:

$$\exists c > 0 \text{ such that for any } x \in \mathbb{R}, A(x) \geq c. \quad (47)$$

Let φ such that $\varphi' = \frac{1}{A}$ and $\varphi(0) = 0$, so that φ' is uniformly bounded and $\varphi \in C^2(\mathbb{R})$, as is the function A . Moreover φ' being strictly positive, φ is strictly increasing hence invertible and we denote by φ^{-1} its inverse function. Under the initial condition $\varphi(0) = 0$, using Itô formula $Y = \varphi(X)$ satisfies

$$dY_t = \left[\frac{B}{A} \circ \varphi^{-1}(Y_t) - \frac{1}{2} A' \circ \varphi^{-1}(Y_t) \right] dt + dW_t, \quad Y_0 = \varphi(X_0). \quad (48)$$

Let $A_\varphi = 1$ and $B_\varphi := \frac{B}{A} \circ \varphi^{-1} - \frac{1}{2} A' \circ \varphi^{-1}$ which belongs to $C_b^1(\mathbb{R})$ as a consequence of $B \in C_b^1$, $A \in C_b^2$. Obviously, $\varphi' > 0$ implies that φ is increasing, $Y_t^* = \varphi(X_t^*) = \varphi(M_t)$.

Theorem 1.1 in [9] can be easily extended to the case where X admits an initial law μ_0 , thus the law of the pair (Y_t^*, Y_t) admits a density with respect to the Lebesgue measure. Moreover, Lemma 2.2 in [9] sets out $p_V(b, a; t) = \frac{p_{Y^*, Y}(\varphi(b), \varphi(a); t)}{A(b)A(a)}$. Now applying Theorem 2.4 to the pair $(B_\varphi, 1)$ the density $p_{Y^*, Y}$ satisfies Hypothesis 2.1. Since functions A and φ are continuous

$$\lim_{u \rightarrow 0^+} p_V(b, b - u; t) = \frac{p_{Y^*, Y}(\varphi(b), \varphi(b); t)}{A^2(b)}$$

that means p_V satisfies Hypothesis 2.1 (ii).

Using now (47)

$$\sup_{u > 0} p_V(b, b - u; t) \leq \frac{1}{c^2} \sup_{u > 0} p_{Y^*, Y}(\varphi(b), \varphi(b - u); t)$$

and since φ is increasing, if $u > 0$, $\varphi(b - u) < \varphi(b)$ and denoting $v = \varphi(b) - \varphi(b - u)$ it gets $v > 0$, and

$$\sup_{u > 0} p_V(b, b - u; t) \leq \frac{1}{c^2} \sup_{v > 0} p_{Y^*, Y}(\varphi(b), \varphi(b) - v; t).$$

After the change of variable $m = \varphi(b)$ so $db = A(b)dm$,

$$\int_0^T \int_{\mathbb{R}} \sup_{u > 0} p_{Y^*, Y}(\varphi(b), \varphi(b) - u; t) db dt = \int_0^T \int_{\mathbb{R}} A(\varphi^{-1}(m)) \sup_{u > 0} p_{Y^*, Y}(m, m - u; t) dm dt.$$

Since A is bounded and $p_{Y^*, Y}$ satisfies Hypothesis 2.1 (i),

$\int_0^T \int_{\mathbb{R}} A(\varphi^{-1}(m)) \sup_{u > 0} p_{Y^*, Y}(m, m - u; t) dm dt < \infty$ and p_V satisfies Hypothesis 2.1 (i) and (ii). \square

6. Conclusion

This paper establishes a PDE of which the density of the pair $[M_t, X_t]$ running maximum-diffusion process is a weak solution, under a quite natural assumption on the regularity of p_V around the boundary of Δ . This assumption is fulfilled when the matrix coefficient of diffusion A is the identity matrix or when the dimension $d = 1$. This PDE is degenerated then the classical results on uniqueness cannot be applied here. The case of non constant matrix A is an open problem. Such generalization could be useful in case of practical applications, as the management of barrier options, in models including stochastic volatility.

Appendix A. Tools

A.1. Malliavin calculus tools

The material of this subsection is taken from section 1.2 in [21].

Let $\mathbb{H} = L^2([0, T], \mathbb{R}^d)$ endowed with the usual scalar product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and the associated norm $\|\cdot\|_{\mathbb{H}}$. For all $h \in \mathbb{H}$, $W(h) := \int_0^T h(t) dW_t$ is a center Gaussian variable with variance equal to $\|h\|_{\mathbb{H}}^2$. If $(h_1, h_2) \in \mathbb{H}^2$, and $\langle h_1, h_2 \rangle_{\mathbb{H}} = 0$, then, the random variables $W(h_i)$, $i = 1, 2$, are independent.

Let \mathcal{S} denote the class of smooth random variables F defined by:

$$F = f(W(h_1), \dots, W(h_n)), \quad n \in \mathbb{N}, \quad h_1, \dots, h_n \in \mathbb{H}, \quad f \in C_b(\mathbb{R}^n). \quad (49)$$

Definition 1. The derivative of the smooth variable F defined in (49) is the \mathbb{H} valued random variable given by $DF := \sum_{i=1}^n \partial_i f(W(h_1), \dots, W(h_n)) h_i$.

We denote the domain of the operator D in $L^2(\Omega)$ by $\mathbb{D}^{1,2}$ meaning that $\mathbb{D}^{1,2}$ is the closure of the class of smooth random variables \mathcal{S} with respect to the norm

$$\|F\|_{1,2} = \{\mathbb{E}[|F|^2] + \mathbb{E}[\|DF\|_{\mathbb{H}}^2]\}^{1/2}.$$

Definition 2. $\mathbb{L}^{1,2}$ is the set of processes $(u_s, s \in [0, T])$ which satisfy

$u \in L^2(\Omega \times [0, T], \mathbb{R}^d)$ and for all $s \in [0, T]$, u_s belongs to $\mathbb{D}^{1,2}$ and

$$\|u\|_{\mathbb{L}^{1,2}}^2 = \|u\|_{L^2([0, T] \times \Omega)}^2 + \|Du\|_{L^2([0, T] \times \Omega)}^2 < \infty.$$

Definition 3. Let $u \in \mathbb{L}^{1,2}$, then the divergence $\delta(u)$ is the unique random variable of $L^2(\Omega)$ such that $\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u \rangle_{\mathbb{H}}]$, $\forall F \in \mathcal{S}$ smooth random variable.

We apply Definition 1.3.1 in [21] with $u \in \mathbb{L}^{1,2}$ and $G \in \mathbb{D}^{1,2}$:

$$\mathbb{E}[G\delta(u)] = \mathbb{E}[\langle DG, u \rangle_{\mathbb{H}}]. \quad (50)$$

Let $x_0 \in \mathbb{R}^d$. We introduce the exponential martingale

$$Z_t^{x_0} := \exp \left[\sum_{k=1}^d \left(\int_0^t B^k(x_0 + W_s) dW_s^k - \frac{1}{2} \int_0^t (B^k(x_0 + W_s))^2 ds \right) \right]. \quad (51)$$

When there is no ambiguity, we will omit the exponent x_0 .

Lemma A.1. *Let $B \in C_b^1(\mathbb{R}^d, \mathbb{R})$, then for all $x_0 \in \mathbb{R}^d$ the process $(B(x_0 + W_s)Z_s^{x_0}, s \in [0, T])$ belongs to $\mathbb{L}^{1,2}$.*

Proof. Let x_0 be fixed. In this proof we omit the exponent x_0 . Note that $Z_t^2 = \exp \left(2 \sum_{k=1}^d \int_0^t B^k(x_0 + W_s) dW_s^k - \frac{4}{2} \int_0^t \|B(x_0 + W_s)\|^2 ds + \frac{4-2}{2} \int_0^t \|B(x_0 + W_s)\|^2 ds \right) \leq e^{T\|B\|_\infty^2} \exp \left(2 \sum_{k=1}^d \int_0^t B^k(x_0 + W_s) dW_s^k - \frac{4}{2} \int_0^t \|B(x_0 + W_s)\|^2 ds \right).$

Then, Z_t belongs to $L^2(\Omega)$ for all $t \in [0, T]$ since

$$\sup_{t \in [0, T]} \mathbb{E}(Z_t^2) \leq e^{T\|B\|_\infty^2}. \quad (52)$$

Note that $Z_t = 1 + \sum_{k=1}^d \int_0^t B^k(x_0 + W_s) Z_s dW_s^k$, $t \in [0, T]$. Using Lemma 2.2.1, Theorem 2.2.1 of [21], and the definition of $\mathbb{L}^{1,2}$, applied to the \mathbb{R}^{d+1} -valued process $Y = (W, Z)$ with a null drift coefficient, the matrix Σ , $(d+1, d)$, defined by:

$$[\sigma^{j,k}(y), 1 \leq j, k \leq d] = I_d, \quad \sigma^{d+1,k}(y) = B^k(x_0^1 + y^1, \dots, x_0^d + y^d)^k z, \quad k = 1, \dots, d,$$

we obtain that Z belongs to $\mathbb{L}^{1,2}$. Since B is continuously differentiable with bounded derivatives, the process $(B(W_s + x_0)Z_s, s \in [0, T])$ belongs to $\mathbb{L}^{1,2}$. \square

The following remark will be often used: using line -15 page 135 of [9] or Exercise 1.2.11 p. 36 in [21]

$$D_s^1 W_t^{1*} = \mathbf{1}_{[0, \tau]}(s) \quad \text{where } \tau := \inf\{s, W_s^{1*} = W_t^{1*}\}. \quad (53)$$

A.2. Brownian motion case estimations

Let us recall the density of distribution of the pair $(W_t^{*,1}, W_t^1)$, where W^1 is a one-dimensional Brownian motion and $W^{*,1}$ its running maximum (see e.g., Section 3.2 in

[17] or [15]):

$$p_{W^{1*}, W^1}(b, a; t) = 2 \frac{2b - a}{\sqrt{2\pi t^3}} \exp - \frac{(2b - a)^2}{2t} \mathbf{1}_{b > \sup(a, 0)}.$$

Thus, using the independence of the components of the process W , the law of (W_t^{1*}, W_t) has a density with respect to the Lebesgue measure on \mathbb{R}^{d+1} denoted by $p_{W^{1*}, W}(\cdot; t)$:

$$p_{W^{1*}, W}(b, a; t) = 2 \frac{(2b - a^1)}{\sqrt{(2\pi)^d t^{d+2}}} e^{-\frac{(2b - a^1)^2}{2t} - \sum_{k=2}^d \frac{|a^k|^2}{2t}} \mathbf{1}_{b \geq 0, b \geq a^1}, \quad b \in \mathbb{R}, \quad a = (a^1, \dots, a^d) \in \mathbb{R}^d. \quad (54)$$

Lemma A.2. (i) For all $t > 0$, $p_{W^{1*}, W}(\cdot; t)$ is the restriction to $\bar{\Delta}$ of a $C^\infty(\mathbb{R}^{d+1})$ function and there exists a universal constant D such that for $x = b, a^1, a^2, \dots, a^d$,

$$|\partial_x p_{W^{1*}, W}(b, a; t)| \leq \frac{D}{\sqrt{(4\pi)^d t^{d+2}}} e^{-\frac{b^2 + (b - a^1)^2}{4t} - \sum_{k=2}^d \frac{(a^k)^2}{4t}} \mathbf{1}_{b > \max(a^1, 0)}. \quad (55)$$

As a consequence $\sum_{x=b, a^1, \dots, a^d} |\partial_x p_{W^{1*}, W}(b, a; t - s)| \in L^1([0, t] \times \mathbb{R}^{d+1}, db da ds)$.

$$\begin{aligned} (ii) \quad p_0(m, x; t, x_0) &\leq \frac{e^{-\frac{(m - x^1)^2}{4t} - \frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t} - \frac{(m - x_0^1)^2}{4t}}}{\sqrt{(2\pi)^{d+1} t^{d+1}}} \mathbf{1}_{m > \max(x^1, x_0^1)} \\ &= 2^{(d+1)/2} \phi_{d+1}(m - x^1, m - x_0^1, \tilde{x} - \tilde{x}_0; 2t) \mathbf{1}_{m > \max(x^1, x_0^1)}, \end{aligned}$$

Proof. (i) Let p_W be the density of a d dimensional Brownian motion, and the density of law of $W_t \forall t > 0 : p_W(\cdot; t) \in C^\infty(\mathbb{R}^d)$:

$$p_W(x; t) = \frac{1}{\sqrt{2^d \pi^d t^d}} e^{-\sum_{k=1}^d \frac{(x^k)^2}{2t}}, \quad t > 0, \quad x = (x^1, \dots, x^d) \in \mathbb{R}^d.$$

Its derivative with respect to x^1 is

$$\partial_{x^1} p_W(x; t) = -\frac{x^1}{\sqrt{2^d \pi^d t^{d+2}}} e^{-\sum_{k=1}^d \frac{(x^k)^2}{2t}}, \quad t > 0, \quad x = (x^1, \dots, x^d) \in \mathbb{R}^d.$$

Its second derivatives are

$$\begin{aligned} \partial_{x^1 x^k}^2 p_W(x; t) &= \frac{x^1 x^k}{\sqrt{2^d \pi^d t^{d+4}}} e^{-\sum_{k=1}^d \frac{(x^k)^2}{2t}}, \quad t > 0, \quad x = (x^1, \dots, x^d) \in \mathbb{R}^d, \quad k = 2, \dots, d. \\ \partial_{x^1 x^1}^2 p_W(x; t) &= \frac{(x^1)^2 - t}{\sqrt{2^d \pi^d t^{d+4}}} e^{-\sum_{k=1}^d \frac{(x^k)^2}{2t}}, \quad t > 0, \quad x = (x^1, \dots, x^d) \in \mathbb{R}^d. \end{aligned}$$

Using (2.1) page 106 of [14] we obtain the analogous of (2.2) page 107 of [14]: there exists a constant C such that

$$|\partial_{x^1 x^1}^2 p_W(x; t)| + |\partial_{x^1 x^k}^2 p_W(x; t)| \leq \frac{C}{t} p_W(x; 2t), \quad k = 1, \dots, d, \quad t > 0, \quad x \in \mathbb{R}^d. \quad (56)$$

Recall (54)

$$p_{W^{*1}, W}(b, a; t) = 2 \frac{2b - a^1}{\sqrt{(2\pi)^d t^{d+2}}} e^{-\frac{(2b - a^1)^2}{2t} - \sum_{k=2}^d \frac{(a^k)^2}{2t}} \mathbf{1}_{b \geq a^1_+}, \quad \forall (b, a) \in \mathbb{R}^{d+1}, \quad t > 0.$$

We observe

$$p_{W^{*1}, W}(b, a; t) = -2\partial_{x^1} p_W(2b - a^1, a^2, \dots, a^d; t) \mathbf{1}_{b \geq a^1_+}. \quad (57)$$

Then $p_{W^{*1}, W}(\cdot, \cdot; t)$ is the restriction to $\bar{\Delta}$ of a C^∞ function.

Moreover, using the chain rule, x being (b, a^1, \dots, a^d) :

$$|\partial_x p_{W^{*1}, W}(b, a; t)| \leq \frac{4C}{t} p_W(2b - a^1, a^2, \dots, a^d; 2t) \mathbf{1}_{b \geq a^1_+}. \quad (58)$$

On the set $\{(b, a), \quad b > \max(0, a^1)\}$ we have

$$(2b - a^1)^2 = (b + b - a^1)^2 \geq (b)^2 + (b - a^1)^2. \quad (59)$$

Plugging estimation (59) into (58) yields (55) with $D = 2^3 C$.

(ii) Recalling the definition

$$p_0(m, x; t, x_0) = p_{W^{*1}, W}(m - x_0^1, x - x_0; t) = 2 \frac{m - x^1 + m - x_0^1}{\sqrt{(2\pi)^d t^{d+2}}} e^{-\frac{(m - x^1 + m - x_0^1)^2}{2t} - \frac{\|\tilde{x} - \tilde{x}_0\|^2}{2t}} \mathbf{1}_{m \geq x^1 \vee x_0^1}$$

we deduce the standard bound which uses $xe^{-x^2} \leq e^{-x^2/2}$ and $(m - x^1 + m - x_0^1)^2 \geq (m - x^1)^2 + (m - x_0^1)^2$:

$$\begin{aligned} p_0(m, x; t, x_0) &\leq \frac{e^{-\frac{(m - x^1)^2}{4t} - \frac{\|\tilde{x} - \tilde{x}_0\|^2}{4t} - \frac{(m - x_0^1)^2}{4t}}}{\sqrt{(2\pi)^{d+1} t^{d+1}}} \mathbf{1}_{m > x^1 \vee x_0^1} \\ &= 2^{(d+1)/2} \phi_{d+1}(m - x^1, m - x_0^1, \tilde{x} - \tilde{x}_0; 2t) \mathbf{1}_{m > x^1 \vee x_0^1}, \end{aligned}$$

□

Lemma A.3. For all $0 < s < t$, $k \geq 1$ and all $u, v, w \in \mathbb{R}^k$

- (i) $\frac{\|u - v\|^2}{t - s} + \frac{\|v - w\|^2}{s} = \frac{t}{s(t - s)} \left\| v - \left(\frac{s}{t} u + \frac{t - s}{t} w \right) \right\|^2 + \frac{\|u - w\|^2}{t}$;
- (i') $k = 1, \quad \int_{-\infty}^b \frac{e^{-\frac{(u-v)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)}} \frac{e^{-\frac{(w-v)^2}{4s}}}{\sqrt{2\pi s}} dv = \sqrt{2} \frac{e^{-\frac{(u-w)^2}{4t}}}{\sqrt{2\pi t}} \Phi_G \left(\sqrt{\frac{t}{s(t-s)}} \left[b - \left(\frac{s}{t} u + \frac{t-s}{t} w \right) \right] \right)$
- (ii) $\int_{\mathbb{R}^k} \frac{e^{-\frac{\|u-v\|^2}{4(t-s)}}}{\sqrt{(2\pi(t-s))^k}} \frac{e^{-\frac{\|w-v\|^2}{4s}}}{\sqrt{(2\pi s)^k}} dv = 2^{k/2} \frac{e^{-\frac{\|u-w\|^2}{4t}}}{\sqrt{(2\pi t)^k}}$
- (iii) For $u > 0$, $1 - \Phi_G(u) := \int_u^{+\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz = \Phi_G(-u) \leq \frac{e^{-\frac{u^2}{2}}}{2}$.

Proof. Point (i) is proved by a development of both hands then an identification of the coefficients of the squared norms and scalar products: $\|u\|^2, \|v\|^2, \|w\|^2, u.v, u.w, v.w$. So we deduce (i') as the integral of

$$\frac{e^{-\frac{t}{4s(t-s)}\left(v - \left(\frac{s}{t}u + \frac{t-s}{t}w\right)\right)^2 - \frac{(u-w)^2}{4t}}}{\sqrt{2\pi(t-s)}\sqrt{2\pi s}}$$

with respect to v up to b .

(ii) is a consequence of point (i) then an integration on \mathbb{R}^k of the Gaussian density with respect to dv .

(iii) The function $u \mapsto \Phi_G(u) - \frac{e^{-\frac{u^2}{2}}}{2}$ is null at 0, has a null limit when u goes to $-\infty$ and its derivative is $u \mapsto -\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} + u\frac{e^{-\frac{u^2}{2}}}{2}$. Its derivative vanishes at $\sqrt{2/\pi}$ and is negative for $u \leq \sqrt{2/\pi}$ and positive after. Then, $u \mapsto \Phi_G(u) - \frac{e^{-\frac{u^2}{2}}}{2}$ is negative for $u \leq 0$.

□

A.3. Proof of Remark 2.3, boundary conditions of the PDE

Here we assume that p_V is regular enough. Let $\mu_0(dx) = f_0(x)dx$. Using Theorem 2.2, (6) means that: for all $F \in C_b^2(\mathbb{R}^{d+1}, \mathbb{R}^d)$

$$\int_{\bar{\Delta}} F(m, x) p_V(m, x; t) dm dx = \int_{\mathbb{R}^d} F(m, m, \tilde{x}) f_0(m, \tilde{x}) dm d\tilde{x} + \int_0^t \int_{\bar{\Delta}} \mathcal{L}F(m, x) p_V(m, x; s) dm dx ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \|A^1(m, \tilde{x})\|^2 \partial_m F(m, m, \tilde{x}) p_V(m, m, \tilde{x}; s) dm d\tilde{x} ds \quad (60)$$

recalling $\mathcal{L} = B^i \partial_{x_i} + \frac{1}{2} \Sigma^{ij} \partial_{x_i, x_j}^2$ where $\Sigma = AA^t$.

(i) Integrating by parts with respect to a convenient dx_k in $\int_0^t \int_{\bar{\Delta}} \mathcal{L}F(m, x) p_V(m, x; s) dm dx ds$ and noting that the support of $p_V(\cdot, \cdot; s)$ is $\bar{\Delta}$, the boundary terms uniquely concern the component x^1 :

$$\begin{aligned} & \int_0^t \int_{\bar{\Delta}} \mathcal{L}F(m, x) p_V(m, x; s) dm dx ds = - \int_0^t \int_{\bar{\Delta}} F(m, x) \partial_{x^k} (B^k p_V)(m, x; s) dm dx ds \\ & - \frac{1}{2} \int_0^t \int_{\bar{\Delta}} \partial_{x^l} F(m, x) \partial_{x^k} [\Sigma^{k,l}(m, x) p_V(m, x; s)] dm dx ds \\ & + \int_0^t \int_{\mathbb{R}^d} \left(F(m, m, \tilde{x}) B^1(m, \tilde{x}) + \frac{1}{2} \partial_{x^k} F(m, m, \tilde{x}) \Sigma^{1,k}(m, \tilde{x}) \right) p_V(m, m, \tilde{x}; s) dm d\tilde{x} ds. \end{aligned}$$

We again operate an integration by parts on the second term above on the right hand:

$$\begin{aligned} & - \frac{1}{2} \int_0^t \int_{\bar{\Delta}} \partial_{x^l} F(m, x) \partial_{x^k} \Sigma^{k,l}(m, x) p_V(m, x; s) dm dx ds = \\ & \frac{1}{2} \int_0^t \int_{\bar{\Delta}} F(m, x) \partial_{x^k, x^l}^2 [\Sigma^{k,l} p_V](m, x; s) dm dx ds - \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} F(m, m, \tilde{x}) \partial_{x^k} [\Sigma^{1,k} p_V](m, m, \tilde{x}; s) dm d\tilde{x} ds. \end{aligned}$$

Gathering these equalities yields

$$\begin{aligned}
& \int_0^t \int_{\bar{\Delta}} \mathcal{L}F(m, x) p_V(m, x; s) dm dx ds = \int_0^t \int_{\bar{\Delta}} F(m, x) \mathcal{L}^* p_V(m, x; s) dm dx ds \\
& - \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} F(m, m, \tilde{x}) \partial_{x^k} [\Sigma^{1,k} p_V](m, m, \tilde{x}; s) dm d\tilde{x} ds \\
& + \int_0^t \int_{\mathbb{R}^d} \left(F(m, m, \tilde{x}) B^1(m, \tilde{x}) p_V(m, m, \tilde{x}; s) + \frac{1}{2} [\partial_{x^k} F \Sigma^{1,k} p_V](m, m, \tilde{x}; s) \right) dm d\tilde{x} ds.
\end{aligned} \tag{61}$$

(ii) Using $F \in C_b^2(\mathbb{R}^{d+1}, \mathbb{R})$ with compact support in Δ (so $F(m, m, \tilde{x}) = 0$) we deduce the equality in Δ :

$$\partial_t p_V(m, x; s) = \mathcal{L}^* p_V(m, x; s), \quad \forall s > 0, \quad (m, x) \in \Delta. \tag{62}$$

We use (60), (61) and (62) applied to $F \in C_b^2(\mathbb{R}^{d+1}, \mathbb{R})$ with compact support in $\bar{\Delta}$:

$$\begin{aligned}
0 &= \int_{\mathbb{R}^d} F(m, m, \tilde{x}) f_0(m, \tilde{x}) dm d\tilde{x} - \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} F(m, m, \tilde{x}) \partial_{x^k} [\Sigma^{1,k} p_V](m, m, \tilde{x}; s) dm d\tilde{x} ds \\
&+ \int_0^t \int_{\mathbb{R}^d} \left(F(m, m, \tilde{x}) B^1(m, \tilde{x}) p_V(m, m, \tilde{x}; s) + \frac{1}{2} [\partial_{x^k} F \Sigma^{1,k} p_V](m, m, \tilde{x}; s) \right) dm d\tilde{x} ds \\
&+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \|A^1(m, \tilde{x})\|^2 \partial_m F(m, m, \tilde{x}) p(m, m, \tilde{x}; s) dm d\tilde{x} ds.
\end{aligned} \tag{63}$$

We now operate integration by parts on the last two terms:

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d} \left([\partial_{x^k} F \Sigma^{1,k} p_V](m, m, \tilde{x}; s) + \|A^1(m, \tilde{x})\|^2 \partial_m F(m, m, \tilde{x}) p_V(m, m, \tilde{x}; s) \right) dm d\tilde{x} ds = \\
& - \int_0^t \int_{\mathbb{R}^d} \left([F \partial_{x^k} (\Sigma^{1,k} p_V)](m, m, \tilde{x}; s) + \partial_m (\|A^1\|^2 p_V)(m, m, \tilde{x}; s) \right) dm d\tilde{x} ds
\end{aligned} \tag{64}$$

Plugging (64) into (63) yields the boundary condition, namely a PDE of which p_V is a solution in the weak sense:

$$\begin{aligned}
B^1(m, \tilde{x}) p_V(m, m, \tilde{x}; s) &= \frac{1}{2} \sum_{k \geq 1} \partial_{x^k} (\Sigma^{1,k} p_V)(m, m, \tilde{x}; s) + \\
& \frac{1}{2} \sum_{k \geq 1} \partial_{x^k} (\Sigma^{1,k} p_V)(m, m, \tilde{x}; s) + \frac{1}{2} \partial_m (\|A^1\|^2 p_V)(m, m, \tilde{x}; s)
\end{aligned}$$

simplified as

$$B^1(m, \tilde{x}) p_V(m, m, \tilde{x}; s) = \sum_{k \geq 1} \partial_{x^k} (\Sigma^{1,k} p_V)(m, m, \tilde{x}; s) + \frac{1}{2} \partial_m (\|A^1\|^2 p_V)(m, m, \tilde{x}; s)$$

with the initial condition $p_V(m, m, \tilde{x}; 0) = f_0(m, \tilde{x})$. \square

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