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# MONOTONICITY OF THE PERIOD AND POSITIVE PERIODIC SOLUTIONS OF A QUASILINEAR EQUATION

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ABSTRACT. We investigate the monotonicity of the minimal period of the periodic solutions of some quasilinear differential equations and extend results for  $p = 2$  due to Chow and Wang, and Chicone, to the case of the  $p$ -Laplace operator. Our main result is the monotonicity of the period of optimal functions for a minimization problem related with a fundamental interpolation inequality. In particular we generalize to  $p \geq 2$  a recent proof of monotonicity due to Benguria, Depassier and Loss for the same optimality issue and  $p = 2$ .

## 1. INTRODUCTION

In this paper we study monotonicity properties of the *minimal period* of positive periodic solutions of

$$(\phi_p(w'))' + \mathcal{V}'(w) = 0, \quad (1)$$

where  $p \geq 2$ ,  $\phi_p(s) = |s|^{p-2}s$  and  $w \mapsto (\phi_p(w'))'$  is the  $p$ -Laplace operator. The potential function  $\mathcal{V}(w)$  is assumed to be non-negative for  $w \geq 0$ ,  $\mathcal{V}(0) > 0$ , it has a minimum at  $w = A > 0$  with  $\mathcal{V}(A) = 0 = \mathcal{V}'(A)$ , and satisfies additional conditions, which guarantee that (1) has positive periodic solutions enclosing the critical point  $(A, 0)$  in the phase plane  $(w, w')$ .

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The energy  $E = \frac{1}{p'} |w'|^p + \mathcal{V}(w)$  is conserved if  $w$  solves (1) and we are interested in the positive periodic solutions with energy less than  $E_* := \mathcal{V}(0)$  which are enclosed by the homoclinic orbit attached to  $(w, w') = (0, 0)$ . We shall further assume that  $\mathcal{V}$  is such that these solutions are uniquely determined, up to translations, by the energy level  $E$ , with minimal period  $T(E)$ .

Our purpose is to study under which conditions  $T$  is an increasing function of  $E$  in the range  $0 \leq E < E_*$  where  $E_*$  is the energy level of the homoclinic orbit. Furthermore we will consider the asymptotic behaviour of  $T(E)$  as  $E \rightarrow 0_+$  and as  $E \rightarrow (E_*)_-$ . Surprisingly enough, the limit of  $T(E)$  as  $E \rightarrow 0_+$  is different in the cases  $p = 2$  and  $p > 2$ . We shall assume that  $\mathcal{V}$  is a potential of class  $C^2$  defined on  $\mathbb{R}$  such that

$$\begin{aligned} &\mathcal{V} \text{ is a } C^2 \text{ function on } \mathbb{R}^+ \text{ and there are } A, B \in \mathbb{R} \text{ with } 0 < A < B \text{ such} \\ &\text{that } \mathcal{V}(0) = \mathcal{V}(B) = E_* > 0, \mathcal{V}'(0) = \mathcal{V}'(A) = \mathcal{V}''(A) \neq 0, \\ &\text{and } 0 < \mathcal{V}(w) < E_* \text{ for all } w \in (0, A) \cup (A, B). \end{aligned} \quad (\text{H1})$$

See Fig. 1. The potential  $\mathcal{V}(w)$  achieves its minimum on  $(0, B)$  at  $x = A$ . The point  $(w, w') = (A, 0)$  is a stationary solution of (1) giving rise to a *center* surrounded by closed periodic orbits with minimal period  $T(E)$ , such that these periodic orbits are enclosed by a homoclinic orbit attached to  $(0, 0)$ .

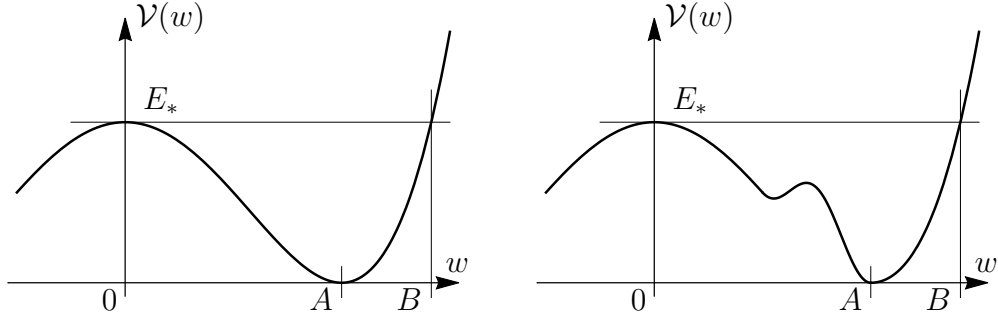


FIGURE 1. Two examples of  $\mathcal{V}$  satisfying assumption (H1). Our monotonicity results require stronger assumptions on  $\mathcal{V}$  which, typically, hold in the first case (left) but not in the second case (right).

Our first result is an extension to  $p > 2$  of a result of Chow and Wang [8, Theorem 2.1].

**Theorem 1.** *Let  $p > 2$ . Assume that  $\mathcal{V}$  satisfies (H1) and  $\mathcal{V}'' > 0$  on  $(0, B)$ . If  $w \mapsto |\mathcal{V}'(w)|^2 - p' \mathcal{V}(w) \mathcal{V}''(w)$  is positive, then  $E \mapsto T(E)$  is increasing on  $(0, E_*)$ .*

Notice that  $w \mapsto |\mathcal{V}'(w)|^2 - p' \mathcal{V}(w) \mathcal{V}''(w)$  is a positive function if and only if  $w \mapsto \mathcal{V}(w) |\mathcal{V}'(w)|^{-p'}$  is a monotonically increasing function.

Our second result is an extension to  $p > 2$  of the monotonicity result [7, Theorem A] under *Chicone's condition*, which is also a growth condition, but of higher order in the derivatives.

**Theorem 2.** *Let  $p > 2$ . Assume that  $\mathcal{V}$  is a  $C^3$  function on  $\mathbb{R}^+$  which satisfies (H1). If  $\mathcal{V}/(\mathcal{V}')^2$  is a convex function, then  $E \mapsto T(E)$  is increasing on  $(0, E_*)$ .*

A central motivation for this paper arises from the study of a minimization problem which is exposed in Appendix A (with additional references) and can be reduced to the study of all positive periodic solutions on  $\mathbb{R}$  of

$$(\phi_p(w'))' + \phi_q(w) - \phi_p(w) = 0. \quad (2)$$

Equation (2) enters in the framework of (1) with  $A = 1$  and potential

$$\mathcal{V}(w) = \frac{1}{q} |w|^q - |w|^p - \left(\frac{1}{q} - \frac{1}{p}\right), \quad (3)$$

so that  $E_* = 1/p - 1/q$ . A positive periodic solution exists only if the energy level satisfies the condition  $E < E_*$ , again we let  $T(E)$  be the minimal period of such a solution. Theorems 1 and 2 do not apply easily and we shall prove directly the following result, which is the main contribution of the paper.

**Theorem 3.** *Let  $p$  and  $q$  be two exponents such that  $2 < p < q$  and consider the positive periodic solutions of (2). Then the map  $E \mapsto T(E)$  is increasing on  $(0, E_*)$  with  $\lim_{E \rightarrow 0^+} T(E) = 0$  and  $\lim_{E \rightarrow (E_*)^-} T(E) = +\infty$ .*

If  $p' = p/(p-1)$  is the Hölder conjugate of the exponent  $p$  and

$$\mathcal{H}(u, v) := \mathcal{V}(u) + \frac{1}{p'} |v|^{p'},$$

Equation (1) can be rewritten as the Hamiltonian system of equations

$$u' = \frac{\partial \mathcal{H}}{\partial v} = \phi_{p'}(v) \quad \text{and} \quad v' = -\frac{\partial \mathcal{H}}{\partial u} = -\mathcal{V}'(u)$$

with  $w = u$  and  $w' = \phi_{p'}(v)$ . Although this Hamiltonian structure may superficially look similar to those of [22, Theorem 1], we have a definitely different set of assumptions. In [21], a very large set of Hamiltonian systems is considered but again our assumptions differ, for instance for the simple reason that the function  $\phi_{p'}$  is not of class  $C^2$ . Further references on the period function can be found in [24]. There are various other extensions of Chicone's result [7], see for instance [6], but they do not cover our assumptions. Also notice that there is a computation in [6, Section 4] which turns out to be equivalent to an argument used in the proof of our Theorem 4 (see below in Section 2), although it is stated neither in that form nor as in Theorem 1.

The monotonicity of the minimal period as a function of the energy level is a question of interest by itself and particularly in the model case of the potential  $\mathcal{V}$  given by (3), even in the case  $p = 2$ . For this last case we quote from [4] that: "It is somewhat surprising that, despite its ubiquity, the monotonicity of the period function for [this problem] in full generality was only established recently." In [20], Miyamoto and Yagasaki proved the monotonicity of the period function for  $p = 2$  and for  $q$  an integer. In [24], Yagasaki generalized the result to all values of  $q > 2$ . Both papers, [20, 24], rely on Chicone's criterion which is difficult to apply to non-integer values of  $q$ . Benguria, Depassier and Loss in [4] consider the positive periodic solutions of  $w'' + w^{q-1} - w = 0$  (*i.e.*, the case  $p = 2$  in our notation) and give a simplified proof of the monotonicity of the period established in [20]. Our approach provides yet another proof.

This paper is organized as follows. Section 2 is devoted to the proof of the  $p$ -Laplacian version of results which are classical when  $p = 2$  and are summarized in Theorems 1 and 2. We are not aware of such statements in the existing literature but they are natural extensions of the case  $p = 2$ . The result of Theorem 3 is by far more difficult. In Section 3 we start with problem (1) by making a change of variables and obtain an expression for the minimal period which goes along some of Chicone's ideas. We also prove some properties of the minimal period when the energy goes to zero and when it goes to the homoclinic level  $E_*$ . In Section 4 we establish a sufficient condition for the monotonicity of the minimal period extending, in particular, the results of [4] for  $p = 2$  to the more general case of the one-dimensional  $p$ -Laplace operator with  $p > 2$ . Our main result (Theorem 3) is proved in Section 5, with an entirely new strategy of proof based on the conditions found in Section 4.

## 2. A $p$ -LAPLACIAN VERSION OF SOME CLASSICAL RESULTS

This section is devoted to the proof of Theorems 1 and 2. We also provide a slightly more detailed statement of Theorem 1.

We begin by extending [8, Theorem 2.1] by Chow and Wang to the  $p$ -Laplacian case  $p \geq 2$ . We recall that  $p' = p/(p-1)$  denotes the Hölder conjugate of  $p$ . Equation (1) has a first integral given by

$$\frac{1}{p'} |w'|^p + \mathcal{V}(w) = E \quad (4)$$

for any energy level  $E \in (0, E_*)$  and the minimal period is given in terms of the energy by

$$T(E) = \frac{2}{p^{1/p}} \int_{w_1(E)}^{w_2(E)} \frac{dw}{(E - \mathcal{V}(w))^{1/p}} \quad (5)$$

where  $w_i(E)$ ,  $i = 1, 2$ , are two roots of  $\mathcal{V}(w) = E$  such that

$$0 < w_1(E) < A < w_2(E) < B \quad \text{and} \quad \mathcal{V}(w) < E \quad \forall w \in (w_1(E), w_2(E)).$$

At this point, let us notice that the map  $E \mapsto T(E)$  is a continuous function if we assume that  $(w - A) \mathcal{V}'(w) > 0$  for any  $w \in (0, A) \cup (A, B)$ , but that it is not the case if  $\mathcal{V}$  admits another local minimum than  $w = A$  in the interval  $(A, B)$ : see for instance Fig. 1. Let us define

$$\gamma(w, E) := p'(E - \mathcal{V}(w)), \quad \mathcal{R}(w) := \mathcal{V}'(w)^2 - p' \mathcal{V}(w) \mathcal{V}''(w)$$

and notice that

$$\frac{\partial \gamma}{\partial w} = -p' \mathcal{V}'(w) \quad \text{and} \quad \frac{\partial \gamma}{\partial E} = p'.$$

The following result is a detailed version of Theorem 1.

**Theorem 4.** *Let  $p \geq 2$  and consider Equation (1) where we assume that  $\mathcal{V}$  satisfies (H1). With the above notations, for any  $E \in (0, E_*)$ , it holds that*

$$\frac{dT}{dE}(E) = \frac{2}{p' E} \int_{w_1(E)}^{w_2(E)} \frac{\mathcal{R}(w)}{\gamma(w, E)^{1/p} \mathcal{V}'(w)^2} dw \quad (6)$$

if the integral in the right hand side is finite. Thus if  $\mathcal{R}$  is positive on  $(0, A) \cup (A, B)$ , then the minimal period is increasing.

Notice that from Assumption (H1), we know that  $\mathcal{V}(0) = E_* > 0$  and  $\mathcal{V}'(0) = 0$  so that  $\lim_{w \rightarrow 0_+} \mathcal{V}(w) |\mathcal{V}'(w)|^{-p'} = +\infty$  and

$$\left( \frac{\mathcal{V}}{|\mathcal{V}'|^{p'}} \right)' = \frac{\mathcal{R} \mathcal{V}'}{|\mathcal{V}'|^{p'+2}},$$

which is incompatible with  $\mathcal{R}$  being a negative valued function in a neighbourhood of  $w = 0_+$ . If we remove the assumption that  $\mathcal{V}'(0) = 0$ , then it makes sense to assume that  $\mathcal{R}$  is a negative function on  $(0, A) \cup (A, B)$ . In this case, the minimal period is decreasing.

*Proof.* The proof relies on the same strategy as for [8, Theorem 2.1]. We skip some details and emphasize only the changes needed to cover the case  $p > 2$ . Let us set

$$I(E) := \int_{w_1(E)}^{w_2(E)} \gamma(w, E)^{1/p'} dw \quad \text{and} \quad J(E) := \int_{w_1(E)}^{w_2(E)} (\gamma(w, E) - p' E) \gamma(w, E)^{1/p'} dw.$$

By differentiating with respect to  $E$ , we obtain

$$\frac{dI}{dE}(E) = \int_{w_1(E)}^{w_2(E)} \frac{dw}{\gamma(w, E)^{1/p}} = \frac{1}{2} T(E) \quad \text{and} \quad \frac{dJ}{dE}(E) = \int_{w_1(E)}^{w_2(E)} \frac{\gamma(w, E) - p' E}{\gamma(w, E)^{1/p}} dw,$$

which implies that

$$\frac{dJ}{dE}(E) = I(E) - p' E \frac{dI}{dE}(E).$$

Differentiating once more with respect to  $E$ , we get

$$\frac{d^2 J}{dE^2}(E) = (1 - p') \frac{dI}{dE}(E) - p' E \frac{d^2 I}{dE^2}(E). \quad (7)$$

On the other hand, by integrating by parts in

$$\int_{w_1}^{w_2} \gamma^{1+1/p'} \frac{\mathcal{V}'^2 - \mathcal{V} \mathcal{V}''}{\mathcal{V}'^2} dw = \int_{w_1}^{w_2} \gamma^{1+1/p'} \left( \frac{\mathcal{V}}{\mathcal{V}'} \right)' dw = -\frac{1+p'}{p'} \int_{w_1}^{w_2} \gamma^{1/p'} \frac{\mathcal{V}}{\mathcal{V}'} \frac{\partial \gamma}{\partial w} dw,$$

we obtain

$$J(E) = -\frac{p'}{p'+1} \int_{w_1(E)}^{w_2(E)} \gamma(w, E)^{1+1/p'} \frac{\mathcal{V}'(w)^2 - \mathcal{V}(w) \mathcal{V}''(w)}{\mathcal{V}'(w)^2} dw$$

by definition of  $J$  and  $\gamma$ . See [8] for further details in the case  $p = 2$ . By differentiating twice this expression of  $J(E)$  with respect to  $E$ , we obtain

$$\frac{d^2 J}{dE^2}(E) = -p' \int_{w_1(E)}^{w_2(E)} \frac{\mathcal{V}'(w)^2 - \mathcal{V}(w) \mathcal{V}''(w)}{\gamma(w, E)^{1/p} \mathcal{V}'(w)^2} dw.$$

Since  $T(E) = 2 \frac{dI}{dE}(E)$ , we learn from (7) that

$$\begin{aligned}
\frac{p' E}{2} \frac{dT}{dE}(E) &= p' E \frac{d^2I}{dE^2}(E) \\
&= (1 - p') \frac{dI}{dE}(E) - \frac{d^2J}{dE^2}(E) \\
&= (1 - p') \int_{w_1(E)}^{w_2(E)} \frac{dw}{\gamma(w, E)^{1/p}} + p' \int_{w_1(E)}^{w_2(E)} \frac{\mathcal{V}'(w)^2 - \mathcal{V}(w) \mathcal{V}''(w)}{\gamma(w, E)^{1/p} \mathcal{V}'(w)^2} dw \\
&= \int_{w_1(E)}^{w_2(E)} \frac{\mathcal{R}(w)}{\gamma(w, E)^{1/p} \mathcal{V}'(w)^2} dw.
\end{aligned}$$

This concludes the proof of (6).  $\square$

*Proof of Theorem 2.* Let us consider again Equation (1) with a potential  $\mathcal{V}$  which satisfies (H1). We adapt the proof of [7, Theorem A] to the case  $p > 2$ . Let us consider the function

$$\mathbf{h}(w) := \frac{w - A}{|w - A|} \sqrt{\mathcal{V}(w)}$$

for any  $w \in (0, A) \cup (A, B)$  and extend it by  $\mathbf{h}(A) = 0$  at  $w = A$ . With the notations of (5), we have  $\mathbf{h}(w_1(E)) = -\sqrt{E}$ ,  $\mathbf{h}(w_2(E)) = +\sqrt{E}$  and we can reparametrize the interval  $(w_1(E), w_2(E))$  with some  $\theta \in (-\pi/2, \pi/2)$  such that

$$\sqrt{E} \sin \theta = \mathbf{h}(w).$$

With this change of variables, the minimal period can be written as

$$T(E) = 2 \frac{E^{\frac{1}{2} - \frac{1}{p}}}{p'^{\frac{1}{p}}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\cos \theta)^{1 - \frac{2}{p}}}{(\mathbf{h}' \circ \mathbf{h}^{-1})(\sqrt{E} \sin \theta)} d\theta. \quad (8)$$

Its derivative with respect to  $E$  is given by

$$\frac{dT}{dE}(E) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{T(E)}{E} - (p' E)^{-\frac{1}{p}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathbf{h}''(w)}{\mathbf{h}'(w)^3} (\cos \theta)^{1 - \frac{2}{p}} \sin \theta d\theta$$

where we use the short-hand notation  $w = \mathbf{h}^{-1}(\sqrt{E} \sin \theta)$ . After an integration by parts and using  $w' = 1/\mathbf{h}'(w)$ , this expression becomes

$$\frac{dT}{dE}(E) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{T(E)}{E} + \frac{1}{2} (p')^{\frac{1}{p}} E^{\frac{1}{2} - \frac{1}{p}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3 \mathbf{h}''(w)^2 - \mathbf{h}'(w) \mathbf{h}'''(w)}{\mathbf{h}'(w)^5} (\cos \theta)^{3 - \frac{2}{p}} d\theta$$

and it results that

$$3 (\mathbf{h}'')^2 - \mathbf{h}' \mathbf{h}''' = \frac{|\mathcal{V}'|^4}{8 \mathcal{V}^2} \left( \frac{\mathcal{V}}{|\mathcal{V}'|^2} \right)''$$

is positive if and only if  $\mathcal{V}/(\mathcal{V}')^2$  is a convex function. This completes the proof of Theorem 2.  $\square$

## 3. ASYMPTOTIC RESULTS

As in Section 2, recall that (1) has a first integral given by (4) where  $E \geq 0$  is the energy level. In this short section, we shall assume that (H1) holds, define

$$\omega := \sqrt{\mathcal{V}''(A)} > 0 \quad (9)$$

and make the additional hypothesis

$$\liminf_{w \rightarrow 0_+} \frac{|\mathcal{V}'(w)|}{w^{p-1}} > 0. \quad (\text{H2})$$

This assumption is satisfied in case of (3) as soon as  $q > p > 2$  and in that case  $\omega = \sqrt{\mathcal{V}''(1)} = \sqrt{q-p}$ , but the following result holds for a much larger class of potentials.

**Lemma 5.** *Let  $p > 1$ . If  $\mathcal{V}$  is a potential such that (H1) holds, then we have*

$$T(E) \sim \frac{2\sqrt{2}\pi\Gamma(1-\frac{1}{p})}{(p')^{\frac{1}{p}}\omega\Gamma(\frac{3}{2}-\frac{1}{p})} E^{\frac{1}{2}-\frac{1}{p}} \quad \text{as } E \rightarrow 0_+$$

with  $\omega$  defined by (9). As a consequence, we obtain

$$\begin{aligned} \lim_{E \rightarrow 0_+} T(E) &= 0 && \text{if } p > 2, \\ \lim_{E \rightarrow 0_+} T(E) &= \frac{2\pi}{\omega} && \text{if } p = 2, \\ \lim_{E \rightarrow 0_+} T(E) &= +\infty && \text{if } p \in (1, 2). \end{aligned}$$

Additionally, if (H2) holds, then for any  $p > 1$  we have  $\lim_{E \rightarrow (E_*)_-} T(E) = +\infty$ .

*Proof.* In a neighbourhood of  $w = A$ , we can write  $\mathcal{V}(w) \sim \frac{1}{2}\omega^2(w-A)^2$ , use (5) and the change of variables  $w = A + \sqrt{2E}y/\omega$  to obtain

$$T(E) \sim \frac{2\sqrt{2}}{p^{1/p}\omega} E^{\frac{1}{2}-\frac{1}{p}} \int_{-1}^1 \frac{dy}{(1-y^2)^{1/p}} \quad \text{as } E \rightarrow 0_+.$$

We obtain the expression of the integral using the formulae [1, 6.2.1 & 6.2.2] for the Euler Beta function.

Now let us consider the limit as  $E \rightarrow (E_*)_-$ . We learn from (H2) that

$$E_* - \mathcal{V}(w) \geq \frac{\ell}{p} w^p$$

for some  $\ell > 0$  if  $w > 0$  is taken small enough. We deduce from (5) that  $T(E)$  diverges as  $E \rightarrow (E_*)_-$ .  $\square$



## 4. THE MONOTONICITY OF THE MINIMAL PERIOD

In this section, we develop a new sufficient condition so that the monotonicity of the period holds, see Lemma 6 below. This condition will be used in Section 5 to prove the monotonicity statement of Theorem 3. Applying directly the formulae of Section 2 to study the monotonicity of the minimal period for periodic solutions of (2), corresponding to the potential defined by (3), indeed involves very complicated expressions. For that reason, it is convenient to introduce a new change of variables as follows. Let us define

$$h(y) := \frac{y - A}{|y - A|} \sqrt{\mathcal{V}(y^{1/p})} \quad \forall y \in (0, A^p) \cup (A^p, B^p)$$

extend it by  $h(A^p) = 0$  at  $y = A^p$ . Let us make the simplifying assumption

$$(w - A) \mathcal{V}'(w) > 0 \quad \forall w \in (0, A) \cup (A, B). \quad (\text{H3})$$

Under this assumption,  $y_i(E)$ ,  $i = 1, 2$ , are the two roots in  $(0, B)$  of  $\mathcal{V}(y^{1/p}) = E$ . As in Theorem 4,  $\mathcal{V}(y^{1/p}) = E$  admits no other root in  $(0, B)$  for any  $E \in (0, E_*)$  and the map  $E \mapsto T(E)$  is continuous. Assumption (H1) does not imply the monotonicity of  $\mathcal{V}$  on either  $(0, A)$  nor  $(A, B)$ , while this monotonicity is granted under Assumption (H3): see Fig. 1. Also notice that

$$h'(y) > 0 \quad \forall y \in (y_1(E), A^p) \cup (A^p, y_2(E)).$$

Using the *change of variables*  $y \mapsto \theta \in (-\pi/2, \pi/2)$  such that

$$\sqrt{E} \sin \theta = h(y) \quad (10)$$

and (8) with  $\mathbf{h}(w(y)) := h(y)$  where  $w(y) := y^{1/p} - A$ , so that  $w'(y) = \frac{1}{p} y^{-1/p'}$ , the minimal period can now be computed as

$$T(E) = c_p E^{\frac{1}{2} - \frac{1}{p}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\cos \theta)^{1 - \frac{2}{p}}}{y^{\frac{1}{p'}} h'(y)} d\theta \quad \text{with} \quad c_p := \frac{2}{p p'^{\frac{1}{p}}}. \quad (11)$$

Let us define

$$J(E) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\cos \theta)^{1 - \frac{2}{p}}}{y^{\frac{1}{p'}} h'(y)} d\theta$$

and emphasize that  $J$  is a function of  $E$  as a consequence of the change of variables (10):  $y = y(E, \theta)$  is such that

$$\frac{\partial y}{\partial E} = \frac{\sin \theta}{2 \sqrt{E} h'(y)}.$$

By differentiating  $T(E)$  in (11) with respect to  $E$ , we find that

$$\frac{T'(E)}{T(E)} = \frac{p-2}{2p} \frac{1}{E} + \frac{J'(E)}{J(E)}$$

where

$$J'(E) = -\frac{1}{2 \sqrt{E}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K(y) (\cos \theta)^{1 - \frac{2}{p}} \sin \theta d\theta,$$

$y$  is given by (10) and

$$K(y) := -\frac{1}{h'(y)} \frac{d}{dy} \left( \frac{1}{y^{\frac{1}{p'}} h'(y)} \right) = \frac{y^2 h''(y) + \frac{1}{p'} y h'(y)}{y^{2+\frac{1}{p'}} (h'(y))^3}. \quad (12)$$

With  $p > 2$ ,  $E \mapsto T(E)$  is increasing if  $J'(E) > 0$ . Here is a sufficient condition on  $h$ , which is in fact an assumption on  $\mathcal{V}$ .

**Lemma 6.** *Assume that (H1) and (H3) hold. With the above notations, if the function  $K$  is decreasing on  $[A, B]$ , then  $J' > 0$  on  $(0, E_*)$  and the minimal period  $T(E)$  is a monotonically increasing function of  $E$ .*

*Proof.* With  $y(E, \theta)$  defined by (10), the result is a consequence of

$$J'(E) = -\frac{1}{2\sqrt{E}} \int_0^{\frac{\pi}{2}} \left( K(y(E, \theta)) - K(y(E, -\theta)) \right) (\cos \theta)^{1-\frac{2}{p}} \sin \theta d\theta$$

and  $y(E, -\theta) < y(E, \theta)$  if  $\theta \in (0, \pi/2)$ . □

In the next result, we give a sufficient condition on  $h$  so that the assumption on  $K$  in Lemma 6 is satisfied and hence its conclusion holds.

**Corollary 7.** *Assume that (H1) and (H3) hold. If  $h$  and  $1/h^2$  are convex functions, then the minimal period  $T(E)$  is a monotonically increasing function of  $E \in (0, E_*)$ .*

*Proof.* By convexity of  $1/h^2$ , we have that

$$0 < \frac{1}{2} \frac{d^2}{dy^2} \left( \frac{1}{h^2} \right) = -\frac{d}{dy} \left( \frac{h''}{h^3} \right)$$

and  $\frac{h''}{h^3}$  is a decreasing function. Next, according to (12), we have

$$K(y) = \frac{1}{y^{\frac{1}{p'}}} \frac{h''(y)}{(h'(y))^3} + \frac{1}{p'} \frac{1}{y^{2-\frac{1}{p'}}} \frac{1}{(h'(y))^2} \quad \text{if } y > 0$$

and observe that all the factors in the right hand side are positive decreasing functions, implying that  $K$  is a decreasing function on  $[A, B]$ . □

## 5. PROOF OF THE MAIN RESULT

In this section, we prove Theorem 3, corresponding to the case of a potential  $\mathcal{V}$  specifically given by (3), which clearly satisfies Assumptions (H1) and (H3). This is done by considering separately the cases  $q = 2p$ ,  $q > 2p$  and  $p < q < 2p$ , and by proving in each case that either Lemma 6 or Corollary 7 applies.

5.1. **Notations and strategy.** It is convenient to define

$$W(y) := y^m - m y + m - 1 \quad \text{and} \quad h(y) := \frac{y-1}{|y-1|} \sqrt{W(y)/q} \quad \forall y \in [0, \gamma_m]$$

where

$$m := \frac{q}{p} \quad \text{and} \quad \gamma_m := m^{\frac{1}{m-1}}.$$

The link with the notations of Section 1 and the framework of Theorem 3 goes as follows: if  $\mathcal{V}$  is defined by (3), then  $W(y) = q \mathcal{V}(y^{1/p})$ ,  $q E_* = m - 1$ ,  $\gamma_m = B = (q/p)^{p/(q-p)}$  and  $A = 1$ . With these notations, we have

$$0 = W(1) < W(y) < W(0) = W(\gamma_m) = m - 1 \quad \forall y \in (0, 1) \cup (1, \gamma_m).$$

The *change of variables*  $y \mapsto \theta \in (-\pi/2, \pi/2)$  defined by (10) amounts to

$$\sqrt{E} \sin \theta = h(y).$$

In this section, our goal is to prove Theorem 3. We consider the cases  $m = 2$ ,  $m > 2$  and  $1 < m < 2$  respectively in Sections 5.2, 5.3, and 5.4, which correspond respectively to  $q = 2p$ ,  $q > 2p$  and  $p < q < 2p$ . The main difficulty is to establish that  $K$  is monotonically decreasing if  $1 < m < 2$ .

5.2. **The case  $m = 2$ .** As a special case, note that  $W(y) = (y-1)^2$  and  $h(y) = (y-1)/\sqrt{q}$  if  $m = 2$ . In that case, the result of Theorem 3 is straightforward.

**Lemma 8.** *If  $m = 2$ , the minimal period  $T(E)$  is a monotonically increasing function of  $E \in (0, E_*)$  with  $E_* = \frac{1}{2p}$ .*

*Proof.* The function  $K$  defined by (12) is explicitly given by  $K(y) = \frac{q^2}{p'} y^{-1/p}$  hence monotonically decreasing and Lemma 6 applies.  $\square$

5.3. **The case  $m > 2$ .** We start with the following result.

**Lemma 9.** *If  $m > 2$ , then  $h$  and  $(h')^{-2}$  are convex.*

*Proof.* Let  $z = y^{m-1}$ . With  $0 \leq y \leq \gamma_m$ ,  $W(y) := y^m - m y + m - 1$  and the function  $h$  given by  $h(y) = \frac{y-1}{|y-1|} \sqrt{W(y)/q}$ , we find that the expression

$$4 W^{3/2} (\sqrt{W})'' = 2 W W'' - (W')^2$$

has the same sign as

$$F(y) := -m^2 + 2m(m-1)^2 y^{m-2} - 2m^2(m-2)y^{m-1} + m(m-2)y^{2m-2}.$$

- If  $y \geq 1$ , then  $y^{m-2} \geq 1$ ,  $-m^2 + 2m(m-1)^2 y^{m-2} \geq m(m-2)(2m-1)$  and

$$F(y) \geq m(m-2)(z-1)(z+1-2m) \geq 0.$$

- If  $y \leq 1$ , then  $y^{m-2} \leq 1$ ,  $y^{2m-2} \leq 1$ ,  $m(m-2)y^{2m-2} - m^2 \leq -2m y^{2m-2}$  and

$$-F(y) \geq 2m(z-1)(z+(m-1)^2) \geq 0.$$

In both cases, we conclude that  $h'' \geq 0$ .

The function  $((h')^{-2})''$  has the same sign as

$$G(y) := 2(m-1)(m-2) - m(2m-1)y + 2(m-1)(2m-1)y^{m-1} - 2(m-2)(2m-1)y^m + (m-2)y^{2m-1}.$$

Since  $G(1) = G'(1) = 0$  and

$$G''(y) = 2(m-1)(m-2)(2m-1)W(y) \geq 0,$$

we conclude that  $G \geq 0$  and  $((h')^{-2})'' \geq 0$ .  $\square$

Then, as a straightforward consequence of Lemmas 6 and 9, we have

**Lemma 10.** *If  $m > 2$ , then the minimal period  $T(E)$  is a monotonically increasing function of  $E \in (0, E_*)$ .*

*Proof.* It is a consequence of Lemma 9 and Corollary 7.  $\square$

5.4. **The case  $1 < m < 2$ .** We cannot apply Corollary 7 and we have to rely directly on Lemma 6. We recall that  $m = q/p$ . Let us start by computing  $K'$ .

**Lemma 11.** *The function  $y \mapsto -K'(y)$  has the same sign as  $p^2 y^2 f(a, m, y, z)$  where  $z = y^{m-1}$ , the parameters  $(a, m)$  are admissible in the sense that*

$$a = \frac{1}{p} \in \left(0, \frac{1}{2}\right), \quad m = \frac{q}{p} \in (1, 2),$$

and

$$\begin{aligned} f(a, m, y, z) = & -3my(z-1)^2(mz-1) \\ & + 2(m-1-my+yz)(2+(1-6m+m^2)z+2m^2z^2) \\ & + a\left(3my(z-1)^3 - 6(z-1)(mz-1)(m-1-my+yz)\right) \\ & + a^2\left(2(z-1)^2(m-1-my+yz)\right). \end{aligned}$$

*Proof.* We set  $y = x^p$  so that  $x = y^{1/p}$  and  $\frac{dx}{dy} = \frac{1}{p}y^{-1/p'}$ . Let

$$\Phi(x) := W(y) = x^{mp} - mx^p + m - 1 \quad \forall x \in [0, \gamma_m^{1/p}],$$

where  $W$  and  $h$  are as in Section 5.1, so that

$$|\sqrt{q}h'(y)|^2 = \left| \frac{\Phi'(x)}{2py^{1/p'}\sqrt{\Phi(x)}} \right|_{x=y^{1/p}}^2,$$

that is,  $4mp^3|y^{1/p'}h'(y)|^2 = (\Phi'(x))^2/\Phi(x)$  and  $K$  defined by (12) can be rewritten as

$$K(y) = -\frac{1}{2}y^{1/p'} \frac{d}{dy} \left( \frac{1}{y^{2/p'}h'(y)^2} \right) = -2mp^3 \frac{d}{dx} \left( \frac{\Phi(x)}{|\Phi'(x)|^2} \right).$$

Hence  $-K'$  has the same sign as  $\frac{d^2}{dx^2}(\Phi(x)|\Phi'(x)|^{-2})$ , *i.e.*, of  $6\Phi|\Phi''|^2 - 2\Phi\Phi'\Phi''' - 3|\Phi'|^2\Phi''$ . A detailed computation shows that

$$\frac{x^4}{q^2} |\Phi'(x)|^4 \frac{d^2}{dx^2} \left( \frac{\Phi(x)}{|\Phi'(x)|^2} \right) = p^2 y^2 f(a, m, y, z),$$

ending the proof of the lemma.  $\square$

**Lemma 12.** *With  $\mathcal{V}$  given by (3) and  $2 < p < q < 2p$ ,  $K$  defined by (12) is monotonically decreasing.*

*Proof.* Keeping the notations of Lemma 11, our goal is to prove that  $y \mapsto f(a, m, y, y^{m-1})$  is nonnegative for any  $y \in (0, \gamma_m)$  whenever the parameters  $(a, m)$  are *admissible*.

Let us start by considering its value at some remarkable points.

- At  $(y, z) = (0, 0)$ , we have  $f(a, m, 0, 0) = 2(1-a)(2-a)(m-1) > 0$ .
- At  $(y, z) = (1, 1)$ , we have  $f(a, m, 1, 1) = 0$  but a Taylor expansion shows that

$$f(a, m, y, y^{m-1}) = \frac{1}{12} (m-1)^3 c_{m,a} (y-1)^4 + O((y-1)^5) \quad \text{as } y \rightarrow 1 \quad (13)$$

for any  $a \in (0, 1/2)$ , where

$$c_{m,a} = 12ma(a-m-1) + m(2m^2 + 7m + 2) \geq c_{m,1/2} = m(m+1)(2m-1) > 0.$$

This proves that  $y \mapsto f(a, m, y, y^{m-1})$  is positive for any  $y \in (1-\varepsilon, 1) \cup (1, 1+\varepsilon)$  for some  $\varepsilon = \varepsilon(a, m) > 0$  whenever the parameters  $(a, m)$  are *admissible*.

- At  $(y, z) = (\gamma_m, m)$ , we have

$$f(a, m, \gamma_m, m) = (m-1)^3 c_{m,a}, \quad c_{m,a} = 2a^2 - 3a(2m+2-m\gamma_m) + 2(2m^2 + 5m + 2).$$

Using  $\inf_{m \in (1,2)} \frac{3}{4}(2m+2-m\gamma_m) = \lim_{m \rightarrow 1+} \frac{3}{4}(2m+2-m\gamma_m) = 3(1-e/4) > 1/2$ , we have

$$\begin{aligned} c_{m,a} &> c_{m,1/2} = (4-3\gamma_m)m^2 + (7-\frac{3}{2}\gamma_m)m + \frac{3}{2} \\ &> \lim_{m \rightarrow 1+} ((4-3\gamma_m)m^2 + (7-\frac{3}{2}\gamma_m)m + \frac{3}{2}) = \frac{1}{2}(25-9e) > 0. \end{aligned}$$

In the limit as  $m \rightarrow 2$ , we have  $y = z$  and

$$f(a, 2, y, z) = 2(1-a)(2-a)(z-1)^4. \quad (14)$$

Hence  $f(a, 2, y, y^{m-1})$  is positive unless  $y = 1$ . We are now going to take a given  $a \in (0, 1/2)$  and consider  $m \in (1, 2)$  as a parameter. Let us prove that for some  $m_* \in (1, 2)$ , we have  $f(a, m, y, y^{m-1}) \geq 0$  for any  $(m, y)$  such that  $m_* < m < 2$  and  $0 \leq y \leq \gamma_m$ . We assume by contradiction that there are two sequences  $(m_k)_{k \in \mathbb{N}}$  and  $(y_k)_{k \in \mathbb{N}}$  such that  $1 < m_k < 2$  for any  $k \in \mathbb{N}$ ,  $\lim_{k \rightarrow +\infty} m_k = 2$ ,  $0 \leq y_k \leq \gamma_{m_k}$  and  $f(a, m_k, y_k, y_k^{m_k-1}) < 0$  for any  $k \in \mathbb{N}$ . Up to the extraction of a subsequence,  $(y_k)_{k \in \mathbb{N}}$  converges to some limit  $y_\infty \in [0, 2]$  and by continuity of  $f$  we know that  $f(a, 2, y_\infty, y_\infty) \leq 0$ : the only possibility is  $y_\infty = 1$  by (14). Since  $f(a, m_k, y_k, y_k^{m_k-1}) < 0 = f(a, m_k, 1, 1)$ , we learn that  $y_k \neq 1$ . Since  $\lim_{k \rightarrow +\infty} y_k = 1$ , this contradicts (13) or, to be precise,  $|y_k - 1| \geq \varepsilon(a, m_k)$ , as the

reader is invited to check that  $\liminf_{k \rightarrow +\infty} \varepsilon(a, m_k) > 0$  because  $f$  is a smooth function of all of its arguments. If we redefine

$$m_*(a) := \inf \left\{ m \in (1, 2) : f(a, m, y, y^{m-1}) \geq 0 \forall y \in [0, \gamma_m] \right\},$$

then we know that for any  $a \in (0, 1/2)$ , we have  $m_*(a) < 2$ .

We want to prove that  $m_*(a) = 1$ . Again, let us argue by contradiction: if  $m_*(a) > 1$ , and assume that there are two sequences  $(m_k)_{k \in \mathbb{N}}$  and  $(y_k)_{k \in \mathbb{N}}$  such that  $1 < m_k < m_*(a)$  for any  $k \in \mathbb{N}$ ,  $\lim_{k \rightarrow +\infty} m_k = m_*(a)$ ,  $0 \leq y_k \leq \gamma_{m_k}$  and  $f(a, m_k, y_k, y_k^{m_k-1}) < 0$  for any  $k \in \mathbb{N}$ . Up to the extraction of a subsequence,  $(y_k)_{k \in \mathbb{N}}$  converges to some limit  $y_\infty \in [0, 2]$  and by continuity of  $f$  we know that  $f(a, m_*(a), y_\infty, y_\infty^{m_*(a)-1}) \leq 0$ . For the same reasons as above,  $y_\infty = 0$ ,  $y_\infty = 1$  and  $y_\infty = \gamma_{m_*(a)}$  are excluded. Altogether, we have proved that for

$$m = m_*(a),$$

we have  $f(a, m, y_\infty, y_\infty^{m-1}) = 0$  for some  $y_\infty \in (0, 1) \cup (1, \gamma_m)$  and we also have that  $f(a, m, y, y^{m-1}) \geq 0$  for any  $y \in (0, 1) \cup (1, \gamma_m)$ , so that  $y_\infty$  is a local minimizer of  $y \mapsto f(a, m, y, y^{m-1})$ . As a consequence, we have shown that for  $m = m_*(a) > 1$  and  $y = y_\infty \neq 1$ , we have

$$f(a, m, y, y^{m-1}) = 0 \quad \text{and} \quad \frac{\partial}{\partial y} f(a, m, y, y^{m-1}) = 0. \quad (15)$$

As we shall see below, this contradicts Lemma 13. Hence  $y \mapsto f(a, m, y, y^{m-1})$  takes nonnegative values for any *admissible* parameters  $(a, m)$  with  $1 < m < 2$ . By Lemma 11,  $K'(y) \leq 0$ , thus completing the proof.  $\square$

We still have to prove that (15) has no solution  $y \in (0, 1) \cup (1, \gamma_m)$ . Since

$$y \frac{\partial}{\partial y} f(a, m, y, y^{m-1}) = y \frac{\partial f}{\partial y}(a, m, y, z) + (m-1)z \frac{\partial f}{\partial z}(a, m, y, z),$$

we can relax the condition  $z = y^{m-1}$  and prove the slightly more general result.

**Lemma 13.** *With the notations of Lemma 11, assume that  $m > 1$ ,  $y \in (0, \gamma_m]$  and  $z \in (0, m]$ . For any admissible parameters  $(a, m)$ , if*

$$f(a, m, y, z) = 0, \quad (16a)$$

$$y \frac{\partial f}{\partial y}(a, m, y, z) + (m-1)z \frac{\partial f}{\partial z}(a, m, y, z) = 0. \quad (16b)$$

then  $z = 1$ .

*Proof.* Solving the system (16a)–(16b) is an *elimination problem* because the function  $f$ , as defined in Lemma 11, is a polynomial in the variables  $a$ ,  $y$  and  $z$ . Since (16a) is a first order equation in  $y$ , we can eliminate this variable and find that

$$y = \frac{n(a, m, z)}{d(a, m, z)}$$

with

$$\begin{aligned} \mathfrak{n}(a, m, z) &:= 2(m-1) \left( a^2(z-1)^2 - 3a(z-1)(mz-1) \right. \\ &\quad \left. + 2m^2z^2 + (m^2 - 6m + 1)z + 2 \right), \\ \mathfrak{d}(a, m, z) &:= m \left( 2a^2(z-1)^2 + 3a(z+1)^2(z-1) + 9z^2 + 8z + 1 \right) \\ &\quad - 2z \left( a^2(z-1)^2 + 3a(z-1) + z + 2 \right) \\ &\quad - m^2z \left( 6a(z-1) + z^2 + 8z + 9 \right) + 2m^3z(2z+1). \end{aligned}$$

After replacing, solving (16b) under the condition  $z \neq 1$  is reduced to a second order equation in  $z$ , whose discriminant is

$$\delta(a, m) := -3(a-1)^2(m-1)^2(a-m)^2(5a^2 - 10a(m+1) - 3m^2 + 14m - 3).$$

Since  $5a^2 - 10a(m+1) - 3m^2 + 14m - 3$  takes only positive values for *admissible*  $(a, m)$ , there are no other roots than  $z = 1$ . This is the desired contradiction, which completes the proof.  $\square$

Then, as a straightforward consequence of Lemmas 6 and 12, we have the

**Lemma 14.** *If  $1 < m < 2$ , then the minimal period  $T(E)$  is a monotonically increasing function of  $E \in (0, E_*)$ .*

Finally, we can conclude with the

*Proof of Theorem 3.* The monotonicity of the minimal period  $T$  of the solution of (2) follows from Lemmas 8, 10, and 14. The asymptotic behaviours of  $T$  as  $E \rightarrow 0_+$  and as  $E \rightarrow (E_*)_-$  are established in Lemma 5.  $\square$

## APPENDIX A. A VARIATIONAL PROBLEM

A central motivation for studying (2) arises from the minimization problem

$$\mu(\lambda) := \inf_{f \in W^{1,p}(\mathbb{S}^1) \setminus \{0\}} \frac{\|f'\|_{L^p(\mathbb{S}^1)}^2 + \lambda \|f\|_{L^p(\mathbb{S}^1)}^2}{\|f\|_{L^q(\mathbb{S}^1)}^2} \quad (17)$$

where  $q > p$  is an arbitrary exponent and  $\mathbb{S}^1$  is the unit circle. This problem can also be seen as the search for the optimal constant in the *interpolation inequality*

$$\|f'\|_{L^p(\mathbb{S}^1)}^2 + \lambda \|f\|_{L^p(\mathbb{S}^1)}^2 \geq \mu(\lambda) \|f\|_{L^q(\mathbb{S}^1)}^2 \quad \forall f \in W^{1,p}(\mathbb{S}^1).$$

Testing the inequality with constant functions shows that  $\mu(\lambda) \leq \bar{\mu}(\lambda) := \lambda |\mathbb{S}^1|^{\frac{2}{p} - \frac{2}{q}}$ . If  $p = 2$ , one can consider the interpolation inequality on  $\mathbb{S}^d$  for any integer  $d \geq 1$  and it is well known from the *carré du champ* method [2, 3] that equality holds if and only if  $\lambda \leq d/(q-2)$ . If  $\lambda > d/(q-2)$ , we have  $\mu(\lambda) < \bar{\mu}(\lambda)$  and optimal functions are non-constant, so that *symmetry breaking* occurs. This is a basic mechanism in *phase transition* theory that can be interpreted as a bifurcation of a branch of non-trivial functions from

a branch of constant functions. An important question is therefore to find the largest value of  $\lambda > 0$  such that  $\mu(\lambda) = \bar{\mu}(\lambda)$ . This is an open question for  $p > 2$ , even if  $d = 1$ .

In dimension  $d = 1$ , the bifurcation problem degenerates in the limit case  $p = 2$ , for which  $\lambda_1 = \lambda_2 = 1/(q - 2)$  according to [2]. We refer to [4, Section 1] for an introduction to the minimization problem (17) with  $p = 2$ , the issue of the branches and the monotonicity of the period problem. Proving that *symmetry breaking* occurs if and only if  $\lambda > 1/(q - 2)$  can be reduced to a proof of the *monotonicity of the minimal period* using Chicone's criterion [7, Theorem A], which provides an interesting alternative to the *carré du champ* method. The study of bifurcation problems using the period function goes back to [23] in case of equations with cubic non-linearities and was later extended to various classes of Hamiltonian systems in [22, 21, 10, 9, 19]. It is therefore natural to consider the case  $p > 2$ .

The minimization problem with  $p > 2$  was studied in [18]. There is an optimal function for (17) and the corresponding Euler-Lagrange equation turns out to be the nonlinear differential equation with nonlocal terms given by

$$- \|f'\|_{L^p(\mathbb{S}^1)}^{2-p} (\phi_p(f'))' + \lambda \|f\|_{L^p(\mathbb{S}^1)}^{2-p} \phi_p(f) = \mu(\lambda) \|f\|_{L^q(\mathbb{S}^1)}^{2-q} \phi_q(f), \quad (18)$$

where we look for positive solutions on  $W^{1,p}(\mathbb{S}^1) \setminus \{0\}$  or equivalently positive  $2\pi$ -periodic solutions on  $\mathbb{R}$ . So far, we do not know the precise value of  $\lambda$  for which there is symmetry breaking but according to [18] *rigidity* holds if  $0 < \lambda < \lambda_1$  for some  $\lambda_1 > 0$ , where rigidity means that any positive solution of (18) is a constant. In that range, we have  $\mu(\lambda) = \bar{\mu}(\lambda)$ . On the contrary, one can prove that *symmetry breaking* occurs if  $\lambda > \lambda_2$  for some  $\lambda_2 > \lambda_1$ , so that  $\mu(\lambda) < \bar{\mu}(\lambda)$  and (18) admits non-constant positive solutions for any  $\lambda > \lambda_2$ . As shown in [18], using homogeneity, scalings and a suitable change of variables, the study of (18) is reduced to the study of all positive periodic solutions on  $\mathbb{R}$  of (2): there are no non-local terms but the minimal period of periodic solutions is no more given. If  $p = 2$ , a precise description of the threshold value of  $\lambda$  is known in the framework of Markov processes if  $q$  is not too large (see [3] for an overview with historical references that go back to [2]) and from [5, 11, 14, 15, 16, 17, 13] using *entropy methods* applied to nonlinear elliptic and parabolic equations; also see [12] for an overview and extensions to various related variational problems. The results of [18] are also based on *entropy methods*. Almost nothing is known beyond [18] if  $p > 2$ , even for  $d = 1$ . The results of this paper are a contribution to a better understanding of the fundamental properties of the solutions of (2) in the simplest case, when  $p > 2$ .

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