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# MONOTONICITY OF THE PERIOD AND POSITIVE PERIODIC SOLUTIONS OF A QUASILINEAR EQUATION

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ABSTRACT. We investigate the monotonicity of the minimal period of the periodic solutions of some quasilinear differential equations and extend results for  $p = 2$  due to Chow and Wang, and to Chicone, to the case of the  $p$ -Laplace operator. Our main result is the monotonicity of the period for positive solutions of a nonlinear Euler-Lagrange equation for a minimization problem related with a fundamental interpolation inequality. In particular we generalize to  $p$  greater than 2 recent results of Benguria, Depassier and Loss.

## 1. INTRODUCTION

In this paper we study monotonicity properties of the *minimal period* of positive periodic solutions of

$$(\phi_p(w'))' + \mathcal{V}'(w) = 0, \quad (1)$$

where  $p \geq 2$ ,  $\phi_p(s) = |s|^{p-2}s$ , and  $\mathcal{V} : \mathbb{R} \rightarrow \mathbb{R}$  is smooth. The potential function  $\mathcal{V}(w)$  is assumed to be non-negative for  $w \geq 0$ ,  $\mathcal{V}(0) > 0$ , it has a minimum at  $w = A > 0$  with  $\mathcal{V}(A) = 0 = \mathcal{V}'(A)$ , and satisfies some additional conditions listed in Section 3, which guarantee that (1) has positive periodic solutions enclosing the critical point  $(A, 0)$  in the phase plane  $(w, w')$ .

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The energy  $E = \frac{1}{p} |w'|^p + \mathcal{V}(w)$  is conserved if  $w$  solves (1) and we are interested in the positive periodic solutions with energy less than  $E_* := \mathcal{V}(0)$  which are enclosed by the homoclinic orbit attached to  $(w, w') = (0, 0)$ . We further assume that  $\mathcal{V}$  is such that these solutions are uniquely determined, up to translations, by the energy level  $E$ , with minimal period  $T(E)$ .

The purpose of this paper is to study under which conditions  $T$  is an increasing function of  $E$  in the range  $0 \leq E \leq E_*$  where  $E_*$  is the energy level of the homoclinic orbit. Furthermore we will consider the asymptotic behaviour of  $T(E)$  as  $E \rightarrow 0_+$  and as  $E \rightarrow (E_*)_-$ . Surprisingly enough, the cases  $p = 2$  and  $p > 2$  differ as  $E \rightarrow 0_+$ .

Our first result is an extension to  $p > 2$  of a result of Chow and Wang [8, Theorem 2.1].

**Theorem 1.** *Let  $p > 2$  and assume that  $\mathcal{V}$  is a  $C^2$  function on  $\mathbb{R}^+$  such that  $\mathcal{V}(A) = 0 = \mathcal{V}'(A)$  and  $\mathcal{V}'' > 0$  on  $(0, B)$  with  $B := \min\{w > A : \mathcal{V}(w) \geq \mathcal{V}(0)\}$ . If  $w \mapsto |\mathcal{V}'(w)|^2 - p' \mathcal{V}(w) \mathcal{V}''(w)$  is a positive function, then  $E \mapsto T(E)$  is increasing on  $(0, E_*)$ .*

Notice that  $w \mapsto |\mathcal{V}'(w)|^2 - p' \mathcal{V}(w) \mathcal{V}''(w)$  is a positive function if and only if  $w \mapsto \mathcal{V}(w) |\mathcal{V}'(w)|^{-p'}$  is a monotone increasing function.

Our second result is also an extension to  $p > 2$  of the monotonicity result in [7, Theorem A] under *Chicone's condition*, which is also a growth condition, but of higher order in the derivatives.

**Theorem 2.** *Let  $p > 2$  and assume that  $\mathcal{V}$  is a  $C^3$  function on  $\mathbb{R}^+$  such that  $\mathcal{V}(A) = 0 = \mathcal{V}'(A)$  and let  $B := \min\{w > A : \mathcal{V}(w) \geq \mathcal{V}(0)\}$ . If  $\mathcal{V}/(\mathcal{V}')^2$  is a convex function, then  $E \mapsto T(E)$  is increasing on  $(0, E_*)$ .*

A central motivation for this paper arises from the study of the minimization problem

$$\mu(\lambda) := \inf_{f \in W^{1,p}(\mathbb{S}^1) \setminus \{0\}} \frac{\|f'\|_{L^p(\mathbb{S}^1)}^2 + \lambda \|f\|_{L^p(\mathbb{S}^1)}^2}{\|f\|_{L^q(\mathbb{S}^1)}^2} \quad (2)$$

where  $q > p$  is an arbitrary exponent and  $\mathbb{S}^1$  is the unit circle. The problem can also be seen as the search for the optimal constant in the interpolation inequality

$$\|f'\|_{L^p(\mathbb{S}^1)}^2 + \lambda \|f\|_{L^p(\mathbb{S}^1)}^2 \geq \mu(\lambda) \|f\|_{L^q(\mathbb{S}^1)}^2 \quad \forall f \in W^{1,p}(\mathbb{S}^1).$$

Testing the inequality with constant functions shows that  $\mu(\lambda) \leq \bar{\mu}(\lambda) := \lambda |\mathbb{S}^1|^{\frac{2}{p} - \frac{2}{q}}$ . If  $p = 2$ , it is well known from the *carré du champ* method [2, 3] that equality holds if and only if  $\lambda \leq d/(q-2)$ . If  $\lambda > d/(q-2)$ , we have  $\mu(\lambda) < \bar{\mu}(\lambda)$  and optimal functions are non constant, so that *symmetry breaking* occurs. The minimization problem with  $p > 2$  was studied in [18]. There is an optimal function for (2) and the corresponding Euler-Lagrange equation turns out to be the nonlinear differential equation with nonlocal terms given by

$$-\|f'\|_{L^p(\mathbb{S}^1)}^{2-p} (\phi_p(f'))' + \lambda \|f\|_{L^p(\mathbb{S}^1)}^{2-p} \phi_p(f) = \mu(\lambda) \|f\|_{L^q(\mathbb{S}^1)}^{2-q} \phi_q(f), \quad (3)$$

where we look for positive solutions on  $W^{1,p}(\mathbb{S}^1) \setminus \{0\}$  or equivalently positive  $2\pi$ -periodic solutions on  $\mathbb{R}$ . So far, we do not know the precise value of  $\lambda$  for which there is symmetry breaking but according to [18] *rigidity* holds if  $0 < \lambda < \lambda_1$  for some explicit  $\lambda_1 > 0$ , where rigidity means that any positive solution of (3) is a constant. In that range, we have  $\mu(\lambda) = \bar{\mu}(\lambda)$ . On the contrary, one can prove that *symmetry breaking* occurs if  $\lambda > \lambda_2$  for some  $\lambda_2 > \lambda_1$ , so that  $\mu(\lambda) < \bar{\mu}(\lambda)$  and (3) admits non-constant positive solutions for any  $\lambda > \lambda_2$ . Using homogeneity, scalings and a suitable change of variables, the study of (3) is reduced in [18] to the study of positive periodic solutions on  $\mathbb{R}$  of

$$(\phi_p(w'))' + \phi_q(w) - \phi_p(w) = 0. \quad (\star)$$

In this equation, there are no non-local terms but the minimal period of periodic solutions is no more given. Equation  $(\star)$  enters in the framework of (1) with  $A = 1$  and potential

$$\mathcal{V}(w) = \frac{1}{q} |w|^q - \frac{1}{p} |w|^p - \left(\frac{1}{q} - \frac{1}{p}\right), \quad (4)$$

so that  $E_* = 1/p - 1/q$ . Positive periodic solutions exist only if the energy level satisfies the condition  $E < E_*$ . Again, let  $T(E)$  be the minimal period of such a solution. Theorems 1 and 2 do not apply easily and we shall prove directly the following result, which is the main contribution of this paper.

**Theorem 3.** *Let  $p$  and  $q$  be two exponents such that  $2 < p < q$  and consider the positive periodic solutions of  $(\star)$ . Then the map  $E \mapsto T(E)$  is increasing on  $(0, E_*)$  with  $\lim_{E \rightarrow 0^+} T(E) = 0$  and  $\lim_{E \rightarrow (E_*)^-} T(E) = +\infty$ .*

The study of (3) is motivated by rigidity and symmetry breaking results associated with interpolation inequalities on the unit sphere  $\mathbb{S}^d$  in one and higher dimensions, that is,  $d \geq 1$ . If  $p = 2$ , a precise description of the threshold value of  $\lambda$  is known in the framework of Markov processes if  $q$  is not too large (see [3] for an overview with historical references that go back to [2]) and from [5, 11, 14, 15, 16, 17, 13] using *entropy methods* applied to nonlinear elliptic and parabolic equations; also see [12] for an overview and extensions to various related variational problems.

Almost nothing is known beyond [18] if  $p > 2$ , even for  $d = 1$ . Our results are a contribution to a better understanding of the fundamental properties of the solutions of (1) in the simplest of the cases when  $p > 2$ . Without the Assumption that  $\mathcal{V}'(A) = 0$  in Theorems 1 and 2 (which is also satisfied in Theorem 3), it is easy to give similar results so that  $E \mapsto T(E)$  is decreasing, but in phase plane the solutions are not described anymore by orbits enclosed by a homoclinic orbit. Some comments on this issue can be found in Section 2.

In dimension  $d = 1$ , the bifurcation problem (3) degenerates in the limit case  $p = 2$ , for which  $\lambda_1 = \lambda_2 = 1/(q - 2)$  according to [2]. We refer to [4, Section 1] for an introduction to the minimization problem (2) with  $p = 2$ , the issue of the branches and the monotonicity of the period problem. Proving that *symmetry breaking* occurs if and only if  $\lambda > 1/(q - 2)$  can be reduced to a proof of the *monotonicity of the minimal period* using Chicone's criterion [7, Theorem A]. The study of bifurcation problems using the

period function goes back to [23] in case of equations with cubic non-linearities and was later extended to various classes of Hamiltonian systems in [22, 21, 10, 9, 19].

If  $p' = p/(p - 1)$  is the Hölder conjugate of the exponent  $p$  and

$$\mathcal{H}(u, v) := \mathcal{V}(u) + \frac{1}{p'} |v|^{p'},$$

Equation (1) can be rewritten as the Hamiltonian system of equations

$$u' = \frac{\partial \mathcal{H}}{\partial v} = \phi_{p'}(v) \quad \text{and} \quad v' = -\frac{\partial \mathcal{H}}{\partial u} = -\mathcal{V}'(u)$$

with  $w = u$  and  $w' = \phi_{p'}(v)$ . Although this Hamiltonian structure may superficially look similar to the conditions of [22, Theorem 1], we have a definitely different set of assumptions. In [21], a much larger set of Hamiltonian systems is considered but again our assumptions differ, for instance for the simple reason that the function  $\phi_{p'}$  is not of class  $C^2$ . Further references on the period function can be found in [24]. There are various other extensions of Chicone's result [7], see for instance [6]. Also notice that there is a computation in [6, Section 4] which turns out to be equivalent to an argument used in the proof of our Theorem 4 (see below in Section 2), although it is stated neither in that form nor as in Theorem 1. The Hamiltonian version of the method has interesting applications to Lotka-Volterra systems.

The monotonicity of the minimal period as a function of the energy level is a question of interest by itself and particularly in the model case of the potential  $\mathcal{V}$  as in (4), even in the case  $p = 2$ . We quote from [4] that: "It is somewhat surprising that, despite its ubiquity, the monotonicity of the period function for [this problem] in full generality was only established recently." In [20], Miyamoto and Yagasaki proved the monotonicity of the period function for  $p = 2$  and for  $q$  an integer. In [24], Yagasaki generalized the result to all values of  $q > 2$ . Both papers, [20, 24], rely on Chicone's criterion which is difficult to apply to non-integer values of  $q$ . The purpose of Benguria, Depassier and Loss in [4] was to give a simplified proof of the monotonicity of the period of the positive solutions of  $w'' + w^{q-1} - w = 0$  (corresponding to  $p = 2$  in our notations).

We point out that in many situations in the paper we will consider the equation

$$(\phi_p(w'))' + V'(w) = 0 \tag{5}$$

where  $V$  is a potential of class  $C^2$  defined on  $\mathbb{R}$  such that

$$\begin{aligned} & \text{There are } a, b \in \mathbb{R} \text{ with } a < 0 < b \text{ such that } V(a) = V(b) = E_* > 0, \\ & V'(a) = V'(b) = V'(0) = 0, V''(0) \neq 0, \\ & \text{and } 0 < V(w) < E_* \text{ for all } w \in (a, 0) \cup (0, b). \end{aligned} \tag{H1}$$

The potential  $V(w)$  achieves its minimum on  $(a, b)$  at  $x = 0$ . The relationship of  $V$  with  $\mathcal{V}$  is given by  $V(w) = \mathcal{V}(w + A)$ ,  $a = -A$  and  $b = B - A$ . The origin  $w = 0$ ,  $w' = 0$  is a stationary point of (5) giving rise to a *center* surrounded by closed periodic orbits with minimal period  $T(E)$ , such that these periodic orbits are enclosed by a homoclinic orbit attached to  $(a, 0)$ .

This paper is organized as follows. Section 2 is devoted to the proof of the  $p$ -Laplacian version of results which are classical when  $p = 2$  and are summarized in Theorems 1 and 2. We are not aware of such statements in the existing literature but they are natural extensions of the case  $p = 2$  and might already be known, so we do not claim any deep originality. The result of Theorem 3 is by far more difficult. In Section 3 we start with problem (1) by making a change of variables and obtain an expression for the minimal period following Chicone's ideas. We also prove some properties of the minimal period when the energy goes to zero and when it goes to the homoclinic level  $E_*$ . In Section 4 we prove the monotonicity of the minimal period extending, in particular, the results of [4] for  $p = 2$  to the more general case of the one-dimensional  $p$ -Laplacian operator  $w \mapsto (\phi_p(w'))'$ , with  $p > 2$ . Our main result (Theorem 3) is proved in Section 5, the proof is highly non-trivial.

## 2. A $p$ -LAPLACIAN VERSION OF SOME CLASSICAL RESULTS

This section is devoted to the proof of Theorems 1 and 2. We also provide a slightly more detailed statement of Theorem 1.

We begin by extending [8, Theorem 2.1] by Chow and Wang to the  $p$ -Laplacian situation when  $p \geq 2$ .

We recall that  $p' = p/(p - 1)$  denotes the Hölder conjugate of  $p$ . Equation (5) has a first integral given by

$$\frac{1}{p'} |w'|^p + V(w) = E \quad (6)$$

for any energy level  $E \in (0, E_*)$  and the minimal period is given in terms of the energy by

$$T(E) = \frac{2}{p'^{1/p}} \int_{w_1(E)}^{w_2(E)} \frac{dw}{(E - V(w))^{1/p}} \quad (7)$$

where  $w_i(E)$ ,  $i = 1, 2$ , are two roots of  $V(w) = E$  such that

$$a < w_1(E) < 0 < w_2(E) < b \quad \text{and} \quad V(w) < E \quad \forall w \in (w_1(E), w_2(E)).$$

At this point, let us notice that the map  $E \mapsto T(E)$  is a continuous function if we assume that  $w V'(w) > 0$  for any  $w \in (a, 0) \cup (0, b)$ , but that it is not the case if  $V$  admits another local minimum than  $w = 0$  in the interval  $(a, b)$ . Let us define

$$\gamma(w, E) := p' (E - V(w)), \quad R(w) := V'(w)^2 - p' V(w) V''(w)$$

and notice that

$$\frac{\partial \gamma}{\partial w} = -p' V'(w) \quad \text{and} \quad \frac{\partial \gamma}{\partial E} = p'.$$

The following result is a detailed version of Theorem 1.

**Theorem 4.** *Let  $p \geq 2$  and consider Equation (5) where we assume that  $V$  satisfies (H1). With the above notations, for any  $E \in (0, E_*)$ , it holds that*

$$\frac{dT}{dE}(E) = \frac{2}{p' E} \int_{w_1(E)}^{w_2(E)} \frac{R(w)}{\gamma(w, E)^{1/p} V'(w)^2} dw \quad (8)$$

*if the integral in the right-hand side is finite. Thus if  $R$  is positive on  $(a, 0) \cup (0, b)$ , then the minimal period is increasing.*

Notice that from Assumption (H1), we know that  $V(a) = E_* > 0$  and  $V'(a) = 0$  so that  $\lim_{w \rightarrow a_+} V(w) |V'(w)|^{-p'} = +\infty$  and

$$\left( \frac{V}{|V'|^{p'}} \right)' = \frac{R V'}{|V'|^{p'+2}}$$

which is incompatible with  $R$  being a negative valued function in a neighbourhood of  $w = a_+$ . If we remove the assumption that  $V'(a) = 0$ , then it makes sense to assume that  $R$  is a negative function on  $(a, 0) \cup (0, b)$ . In this case, the minimal period is decreasing.

*Proof.* The proof relies on the same strategy as for [8, Theorem 2.1]. We skip some details and emphasize only the changes needed to cover the case  $p > 2$ . Let us set

$$I(E) := \int_{w_1(E)}^{w_2(E)} \gamma(w, E)^{1/p'} dw \quad \text{and} \quad J(E) := \int_{w_1(E)}^{w_2(E)} (\gamma(w, E) - p' E) \gamma(w, E)^{1/p'} dw.$$

By differentiating with respect to  $E$ , we obtain

$$\frac{dI}{dE}(E) = \int_{w_1(E)}^{w_2(E)} \frac{dw}{\gamma(w, E)^{1/p}} = \frac{1}{2} T(E) \quad \text{and} \quad \frac{dJ}{dE}(E) = \int_{w_1(E)}^{w_2(E)} \frac{\gamma(w, E) - p' E}{\gamma(w, E)^{1/p}} dw,$$

which implies that

$$\frac{dJ}{dE}(E) = I(E) - p' E \frac{dI}{dE}(E). \quad (9)$$

Differentiating once more with respect to  $E$ , we get

$$\frac{d^2 J}{dE^2}(E) = (1 - p') \frac{dI}{dE}(E) - p' E \frac{d^2 I}{dE^2}(E). \quad (10)$$

On the other hand, by integrating by parts in

$$\int_{w_1}^{w_2} \gamma^{\frac{p'+1}{p'}} \frac{V'^2 - V V''}{V'^2} dw = \int_{w_1}^{w_2} \gamma^{\frac{p'+1}{p'}} \left( \frac{V}{V'} \right)' dw = -\frac{p'+1}{p'} \int_{w_1}^{w_2} \gamma^{\frac{1}{p'}} \frac{V}{V'} \frac{\partial \gamma}{\partial w} dw,$$

we obtain

$$J(E) = -\frac{p'}{p'+1} \int_{w_1(E)}^{w_2(E)} \gamma(w, E)^{\frac{p'+1}{p'}} \frac{V'(w)^2 - V(w) V''(w)}{V'(w)^2} dw$$

by definition of  $J$  and  $\gamma$ . See [8] for further details in the case  $p = 2$ . By differentiating twice this expression of  $J(E)$  with respect to  $E$ , we obtain

$$\frac{d^2 J}{dE^2}(E) = -p' \int_{w_1(E)}^{w_2(E)} \frac{V'(w)^2 - V(w)V''(w)}{\gamma(w, E)^{1/p} V'(w)^2} dw.$$

Since  $T(E) = 2 \frac{dI}{dE}(E)$ , we learn from (10) that

$$\begin{aligned} \frac{p' E}{2} \frac{dT}{dE}(E) &= p' E \frac{d^2 I}{dE^2}(E) \\ &= (1 - p') \frac{dI}{dE}(E) - \frac{d^2 J}{dE^2}(E) \\ &= (1 - p') \int_{w_1(E)}^{w_2(E)} \frac{dw}{\gamma(w, E)^{1/p}} + p' \int_{w_1(E)}^{w_2(E)} \frac{V'(w)^2 - V(w)V''(w)}{\gamma(w, E)^{1/p} V'(w)^2} dw \\ &= \int_{w_1(E)}^{w_2(E)} \frac{R(w)}{\gamma(w, E)^{1/p} V'(w)^2} dw. \end{aligned}$$

This concludes the proof of (8).  $\square$

*Proof of Theorem 2.* Let us consider again Equation (5) with a potential  $V$  which satisfies (H1). We adapt the proof of [7, Theorem A] to the case  $p > 2$ . Let us consider the function

$$\mathbf{h}(w) := \frac{w}{|w|} \sqrt{V(w)} \quad (11)$$

for any  $w \in (a, 0) \cup (0, b)$  and extend it by  $\mathbf{h}(0) = 0$  at  $w = 0$ . With the notations of (7), we have  $\mathbf{h}(w_1(E)) = -\sqrt{E}$ ,  $\mathbf{h}(w_2(E)) = +\sqrt{E}$  and we can reparametrize the interval  $(w_1(E), w_2(E))$  with some  $\theta \in (-\pi/2, \pi/2)$  such that

$$\sqrt{E} \sin \theta = \mathbf{h}(w). \quad (12)$$

With this change of variables, the minimal period can be written as

$$T(E) = 2 \frac{E^{\frac{1}{2} - \frac{1}{p}}}{p'^{\frac{1}{p}}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos \theta^{1 - \frac{2}{p}}}{(\mathbf{h}' \circ \mathbf{h}^{-1})(\sqrt{E} \sin \theta)} d\theta. \quad (13)$$

Its derivative with respect to  $E$  is given by

$$\frac{dT}{dE}(E) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{T(E)}{E} - (p' E)^{-\frac{1}{p}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathbf{h}''(w)}{\mathbf{h}'(w)^3} \cos \theta^{1 - \frac{2}{p}} \sin \theta d\theta$$

where we use the short-hand notation  $w = \mathbf{h}^{-1}(\sqrt{E} \sin \theta)$ . After an integration by parts, this expression becomes

$$\frac{dT}{dE}(E) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{T(E)}{E} + \frac{1}{2} (p')^{\frac{1}{p}} E^{\frac{1}{2} - \frac{1}{p}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3 \mathbf{h}''(w)^2 - \mathbf{h}'(w) \mathbf{h}'''(w)}{\mathbf{h}'(w)^5} \cos \theta^{3 - \frac{2}{p}} d\theta$$

and one can show that

$$3 (\mathbf{h}'')^2 - \mathbf{h}' \mathbf{h}''' = \frac{|V'|^4}{8 V^2} \left( \frac{V}{|V'|^2} \right)''$$



is positive if and only if  $V/(V')^2$  is a convex function. This completes the proof of Theorem 2.  $\square$

### 3. ASYMPTOTIC RESULTS

As in Section 2, let  $V(w) = \mathcal{V}(w + A)$  and recall that (5) has a first integral given by (6) where  $E \geq 0$  is the energy level. In this short section, we shall assume that (H1) holds with  $a = -A$ , define

$$\omega := \sqrt{V''(0)} = \sqrt{\mathcal{V}''(A)} > 0 \quad (14)$$

and make the additional hypothesis

$$\liminf_{w \rightarrow 0_+} \frac{|V'(w + a)|}{w^{p-1}} = \liminf_{w \rightarrow 0_+} \frac{|\mathcal{V}'(w)|}{w^{p-1}} > 0. \quad (\text{H2})$$

This assumption is satisfied in case of (4) as soon as  $q > p > 2$  and in that case  $\omega = \sqrt{\mathcal{V}''(1)} = \sqrt{q - p}$ , but the following result holds for a much larger class of potentials.

**Lemma 5.** *Let  $p > 1$ . If  $V$  is a potential such that (H1) holds, then we have*

$$T(E) \sim \frac{2\sqrt{2}\pi\Gamma(1 - \frac{1}{p})}{p^{1/p}\omega\Gamma(\frac{3}{2} - \frac{1}{p})} E^{\frac{1}{2} - \frac{1}{p}} \quad \text{as } E \rightarrow 0_+$$

with  $\omega$  defined by (14). As a consequence, we obtain

$$\begin{aligned} \lim_{E \rightarrow 0_+} T(E) &= 0 && \text{if } p > 2, \\ \lim_{E \rightarrow 0_+} T(E) &= \frac{2\pi}{\omega} && \text{if } p = 2, \\ \lim_{E \rightarrow 0_+} T(E) &= +\infty && \text{if } p \in (1, 2). \end{aligned}$$

Additionally, if (H2) holds, then for any  $p > 1$  we have  $\lim_{E \rightarrow (E_*)_-} T(E) = +\infty$ .

*Proof.* In a neighbourhood of  $w = 0$ , we can write  $V(w) \sim \frac{1}{2}\omega^2 w^2$ , use (7) and the change of variables  $w = \sqrt{2E}y/\omega$  to obtain

$$T(E) \sim \frac{2\sqrt{2}}{p^{1/p}\omega} E^{\frac{1}{2} - \frac{1}{p}} \int_{-1}^1 \frac{dy}{(1 - y^2)^{1/p}} \quad \text{as } E \rightarrow 0_+.$$

We obtain the expression of the integral using the formulae [1, 6.2.1 & 6.2.2] for the Euler Beta function.

Now let us consider the limit as  $E \rightarrow (E_*)_-$ . We learn from (H2) that

$$E_* - v(w) \geq \frac{\ell}{p} (w - a)^p$$

for some  $\ell > 0$  if  $w - a > 0$  is taken small enough. We deduce from (7) that  $T(E)$  diverges as  $E \rightarrow (E_*)_-$ .  $\square$

## 4. THE MONOTONICITY OF THE MINIMAL PERIOD

Applying the formulae of Section 2 to study the monotonicity of the minimal period for periodic solutions of  $(\star)$  leads later to very complicated expressions for our problem with potential (4). For that reason, it is convenient to introduce a new change of variables as follows. Let  $A = -a > 0$  and define

$$h(y) := \mathfrak{h}(w) = \frac{w}{|w|} \sqrt{V(w)} \quad \text{with} \quad y = (w + A)^p \quad (15)$$

for any  $w \in (a, 0) \cup (0, b)$  and extend it by  $h(0) = 0$  at  $w = 0$ . Here  $\mathfrak{h}$  is defined as in Section 2 (proof of Theorem 1, Eq. (12)) while  $h$  is such that

$$h(y) = \mathfrak{h}\left(y^{\frac{1}{p}} - A\right) \quad \forall y \in (0, B)$$

with  $B = (A + b)^p$ . We have that

$$\mathfrak{h}'(w) = p y^{\frac{1}{p'}} h'(y).$$

Let us make the simplifying assumption

$$w V'(w) > 0 \quad \forall w \in (a, 0) \cup (0, b). \quad (\text{H3})$$

Under this assumption,  $w_i(E)$ ,  $i = 1, 2$ , are the two roots in  $(a, b)$  of  $V(w) = E$ , as in Theorem 4,  $V(w) = E$  admits no other root in  $(a, b)$  for any  $E \in (0, E_*)$  and the map  $E \mapsto T(E)$  is continuous. Also notice that

$$h'(y) > 0 \quad \forall y \in (y_1(E), A^p) \cup (A^p, y_2(E))$$

where  $y_1(E) := (A - |w_1(E)|)^p$  and  $y_2(E) := (A + w_2(E))^p$ .

By the above definition of  $h$  and (13), the minimal period can now be computed as

$$T(E) = c_p E^{\frac{1}{2} - \frac{1}{p}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos \theta^{1 - \frac{2}{p}}}{y^{\frac{1}{p'}} h'(y)} d\theta \quad \text{with} \quad c_p := \frac{2}{p p^{\frac{1}{p}}} \quad (16)$$

using the *change of variables*  $y \mapsto \theta \in (-\pi/2, \pi/2)$  such that

$$\sqrt{E} \sin \theta = h(y) = \frac{y - A}{|y - A|} \sqrt{V(y^{1/p} - A)}. \quad (17)$$

Let us define

$$J := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos \theta^{1 - \frac{2}{p}}}{y^{\frac{1}{p'}} h'(y)} d\theta$$

and notice that  $J$  is a function of  $E$  as a consequence of the change of variables (17):  $y = y(E, \theta)$  is such that

$$\frac{\partial y}{\partial E} = \frac{\sin \theta}{2 \sqrt{E} h'(y)}.$$

By differentiating  $T(E)$  in (16) with respect to  $E$ , we find that

$$\frac{T'(E)}{T(E)} = \frac{p-2}{2p} \frac{1}{E} + \frac{J'(E)}{J(E)}$$

where

$$J'(E) = -\frac{1}{2\sqrt{E}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K(y) \cos \theta^{1-\frac{2}{p}} \sin \theta \, d\theta,$$

$y$  is given by (17) and

$$K(y) := -\frac{1}{h'(y)} \frac{d}{dy} \left( \frac{1}{y^{\frac{1}{p'}} h'(y)} \right) = \frac{y^2 h''(y) + \frac{1}{p'} y h'(y)}{y^{2+\frac{1}{p'}} (h'(y))^3}. \quad (18)$$

With  $p > 2$ ,  $E \mapsto T(E)$  is increasing if  $J'(E) > 0$ . Here is a sufficient condition on  $h$ , which is in fact an assumption on  $V$ .

**Lemma 6.** *Assume that (H1) and (H3) hold. With the above notations, if the function  $K$  is decreasing on  $[A, B]$ , then  $J' > 0$  on  $(0, E_*)$  and the minimal period  $T(E)$  is a monotone increasing function of  $E$ .*

*Proof.* With  $y(E, \theta)$  defined by (17), the result is a consequence of

$$J'(E) = -\frac{1}{2\sqrt{E}} \int_0^{\frac{\pi}{2}} \left( K(y(E, \theta)) - K(y(E, -\theta)) \right) \cos \theta^{1-\frac{2}{p}} \sin \theta \, d\theta$$

and  $y(E, -\theta) < y(E, \theta)$  if  $\theta \in (0, \pi/2)$ .  $\square$

We deduce from Lemma 6 a sufficient condition on  $h$  to obtain that the minimal period is monotone increasing.

**Corollary 7.** *Assume that (H1) and (H3) hold. If  $h$  and  $1/h^2$  are convex functions, then the minimal period  $T(E)$  is a monotone increasing function of  $E \in (0, E_*)$ .*

*Proof.* By convexity of  $1/h^2$ , we have that

$$0 < \frac{1}{2} \frac{d^2}{dy^2} \left( \frac{1}{h^2} \right) = -\frac{d}{dy} \left( \frac{h''}{h^3} \right)$$

and  $\frac{h''}{h^3}$  is a decreasing function. Next, from (18) written as

$$K(y) = \frac{1}{y^{\frac{1}{p'}}} \frac{h''(y)}{(h'(y))^3} + \frac{1}{p'} \frac{1}{y^{2-\frac{1}{p'}}} \frac{1}{(h'(y))^2} \quad \text{if } y > 0, \quad (19)$$

we observe that all the factors on the right hand of this expression are positive decreasing functions, implying that  $K$  is a decreasing function on  $[A, B]$ .  $\square$

## 5. PROOF OF THE MAIN RESULT

By applying Lemma 6 and Corollary 7, we prove Theorem 3. The main difficulty is to establish that  $K$  is monotone decreasing if  $1 < m < 2$ , which is done in Section 5.3.

5.1. **Notations.** Let us consider (1) with  $\mathcal{V}$  given by (4) and  $q > p \geq 2$ , hence  $\mathcal{V}'(w) = \phi_q(w) - \phi_p(w)$ , and (1) is reduced to

$$(\phi_p(w'))' + \phi_q(w) - \phi_p(w) = 0. \quad (\star)$$

In particular  $w = 1$  is a trivial solution of this equation. All conditions of Section 1 for  $\mathcal{V}$  are satisfied,  $\mathcal{V}$  (resp.  $V$ ) reaches a minimum at  $w = 1$  (resp.  $w = 0$ ) and

$$E_* = \frac{q-p}{pq} = \mathcal{V}(B) = \mathcal{V}(0) \quad \text{where} \quad B := \left(\frac{q}{p}\right)^{\frac{1}{q-p}}. \quad (20)$$

In the discussion, we shall consider the three cases:  $m = 2$ ,  $m > 2$  and  $1 < m < 2$ , where

$$m = \frac{q}{p}.$$

We have that  $V(w) = \mathcal{V}(w + A)$  with  $A = 1$ , *i.e.*,

$$V(w) = \frac{1}{q} |w + 1|^q - \frac{1}{p} |w + 1|^p - \left(\frac{1}{q} - \frac{1}{p}\right),$$

With the definitions of (15), we find that

$$qV(w) = y^m - my - (1 - m) \quad \text{with} \quad w = y^{\frac{1}{p}} - 1$$

and the *change of variables*  $y \mapsto \theta \in (-\pi/2, \pi/2)$  defined by (17) amounts to

$$\sqrt{E} \sin \theta = h(y) = \frac{y-1}{|y-1|} \sqrt{\frac{1}{q} (y^m - my + m - 1)}.$$

It is convenient to define

$$\gamma_m := m^{\frac{1}{m-1}} = B = \left(\frac{q}{p}\right)^{\frac{p}{q-p}} \quad \text{and} \quad W(y) := y^m - my + m - 1 \quad \forall y \in [0, \gamma_m].$$

With these notations, we have

$$0 = W(1) < W(y) < W(0) = W(\gamma_m) = m - 1 \quad \forall y \in (0, 1) \cup (1, \gamma_m).$$

As a special case, note that  $W(y) = (y-1)^2$  and  $h(y) = (y-1)/\sqrt{q}$  if  $m = 2$ . In that case, the result of Theorem 3 is straightforward.

**Lemma 8.** *If  $m = 2$  and  $\mathcal{V}$  given by (4), the minimal period  $T(E)$  is a monotone increasing function of  $E \in (0, E_*)$  with  $E_* = \frac{1}{2p}$ .*

*Proof.* The function  $K$  defined by (18) is explicitly given by  $K(y) = \frac{q^2}{p'} y^{-1/p}$  hence monotone decreasing and Lemma 6 applies.  $\square$

5.2. **The case  $m > 2$ .** We obtain the following result.

**Lemma 9.** *If  $m > 2$ ,  $h$  and  $(h')^{-2}$  are convex.*

*Proof.* Let  $z = y^{m-1}$ . With  $0 \leq y \leq \gamma_m$ ,  $W(y) := y^m - my + m - 1$  and the function  $h$  given by  $h(y) = \frac{y-1}{|y-1|} \sqrt{W(y)/q}$ , we find that the expression

$$4W^{3/2} (\sqrt{W})'' = 2WW'' - (W')^2$$

has its sign given by

$$F(y) := -m^2 + 2m(m-1)^2 y^{m-2} - 2m^2(m-2)y^{m-1} + m(m-2)y^{2m-2}.$$

- If  $y \geq 1$ , then  $y^{m-2} \geq 1$ ,  $-m^2 + 2m(m-1)^2 y^{m-2} \geq m(m-2)(2m-1)$  and

$$F(y) \geq m(m-2)(z-1)(z+1-2m) \geq 0.$$

- If  $y \leq 1$ , then  $y^{m-2} \leq 1$ ,  $y^{2m-2} \leq 1$ ,  $m(m-2)y^{2m-2} - m^2 \leq -2my^{2m-2}$  and

$$-F(y) \geq 2m(z-1)(z+(m-1)^2) \geq 0.$$

In both cases, we conclude that  $h'' \geq 0$ .

The function  $((h')^{-2})''$  has the sign of

$$G(y) := 2(m-1)(m-2) - m(2m-1)y + 2(m-1)(2m-1)y^{m-1} - 2(m-2)(2m-1)y^m + (m-2)y^{2m-1}.$$

Since  $G(1) = G'(1) = 0$  and

$$G''(y) = 2(m-1)(m-2)(2m-1)W(y) \geq 0,$$

we conclude that  $g \geq 0$  and  $((h')^{-2})'' \geq 0$ .  $\square$

This proves Theorem 3 as a consequence of Corollary 7 and Lemma 9 if  $m > 2$ .

**5.3. The case  $1 < m < 2$ .** We cannot apply Corollary 7 and we have to directly rely on Lemma 6. We recall that  $m = q/p$ . Let us start by computing  $K'$ .

**Lemma 10.** *The function  $y \mapsto -K'(y)$  has the sign of  $p^2 y^2 f(a, m, y, z)$  where  $z = y^{m-1}$ , the parameters  $(a, m)$  are admissible in the sense that*

$$a = \frac{1}{p} \in \left(0, \frac{1}{2}\right), \quad m = \frac{q}{p} \in (1, 2),$$

and

$$\begin{aligned} f(a, m, y, z) = & -3my(z-1)^2(mz-1) \\ & + 2(m-1-my+yz)(2+(1-6m+m^2)z+2m^2z^2) \\ & + a\left(3my(z-1)^3 - 6(z-1)(mz-1)(m-1-my+yz)\right) \\ & + a^2\left(2(z-1)^2(m-1-my+yz)\right). \end{aligned}$$

*Proof.* We set  $y = x^p$  so that  $x = y^{1/p}$  and  $\frac{dx}{dy} = \frac{1}{p}y^{-1/p'}$ . Let

$$\Phi(x) := W(y) = x^{mp} - mx^p + m - 1 \quad \forall x \in [0, \gamma_m^{1/p}],$$

where  $W$  and  $h$  are as in Section 5.1, so that

$$|\sqrt{q}h'(y)|^2 = \left| \frac{\Phi'(x)}{2py^{1/p'}\sqrt{\Phi(x)}} \right|_{x=y^{1/p}}^2,$$

that is,  $4 m p^3 |y^{1/p'} h'(y)|^2 = (\Phi'(x))^2/\Phi(x)$  and  $K$  as in (18) can be rewritten as

$$K(y) = -\frac{1}{2} y^{1/p'} \frac{d}{dy} \left( \frac{1}{y^{2/p'} h'(y)^2} \right) = -2 m p^3 \frac{d}{dx} \left( \frac{\Phi(x)}{|\Phi'(x)|^2} \right).$$

Hence  $-K'$  has the sign of  $\frac{d^2}{dx^2} (\Phi(x) |\Phi'(x)|^{-2})$ , i.e., of  $6 \Phi |\Phi''|^2 - 2 \Phi \Phi' \Phi''' - 3 |\Phi'|^2 \Phi''$  and the detailed computation shows that

$$\frac{x^4}{q^2} |\Phi'(x)|^4 \frac{d^2}{dx^2} \left( \frac{\Phi(x)}{|\Phi'(x)|^2} \right) = p^2 y^2 f(a, m, y, z),$$

ending the proof of the lemma.  $\square$

**Lemma 11.** *With  $\mathcal{V}$  given by (4) and  $2 < p < q < 2p$ ,  $K$  defined by (18) is monotone decreasing.*

*Proof.* Keeping the notations of Lemma 10, our goal is to prove that  $y \mapsto f(a, m, y, y^{m-1})$  is nonnegative for any  $y \in (0, \gamma_m)$  whenever the parameters  $(a, m)$  are *admissible*.

Let us start by considering its value at some remarkable points.

- At  $(y, z) = (0, 0)$ , we have  $f(a, m, 0, 0) = 2(1-a)(2-a)(m-1) > 0$ .
- At  $(y, z) = (1, 1)$ , we have  $f(a, m, 1, 1) = 0$  but a Taylor expansion shows that

$$f(a, m, y, y^{m-1}) = \frac{1}{12} (m-1)^3 c_{m,a} (y-1)^4 + O((y-1)^5) \quad \text{as } y \rightarrow 1 \quad (21)$$

for any  $a \in (0, 1/2)$ , where

$$c_{m,a} = 12 m a (a - m - 1) + m (2 m^2 + 7 m + 2) \geq c_{m,1/2} = m (m + 1) (2 m - 1) > 0.$$

This proves that  $y \mapsto f(a, m, y, y^{m-1})$  is positive for any  $y \in (1 - \varepsilon, 1) \cup (1, 1 + \varepsilon)$  for some  $\varepsilon = \varepsilon(a, m) > 0$  whenever the parameters  $(a, m)$  are *admissible*.

- At  $(y, z) = (\gamma_m, m)$ , we have

$$f(a, m, \gamma_m, m) = (m-1)^3 c_{m,a}, \quad c_{m,a} = 2 a^2 - 3 a (2 m + 2 - m \gamma_m) + 2 (2 m^2 + 5 m + 2).$$

Using  $\inf_{m \in (1,2)} \frac{3}{4} (2 m + 2 - m \gamma_m) = \lim_{m \rightarrow 1+} \frac{3}{4} (2 m + 2 - m \gamma_m) = 3(1 - e/4) > 1/2$ , we have

$$\begin{aligned} c_{m,a} &> c_{m,1/2} = (4 - 3 \gamma_m) m^2 + (7 - \frac{3}{2} \gamma_m) m + \frac{3}{2} \\ &> \lim_{m \rightarrow 1+} ((4 - 3 \gamma_m) m^2 + (7 - \frac{3}{2} \gamma_m) m + \frac{3}{2}) = \frac{1}{2} (25 - 9 e) > 0. \end{aligned}$$

In the limit as  $m \rightarrow 2$ , we have  $y = z$  and

$$f(a, 2, y, z) = 2(1-a)(2-a)(z-1)^4. \quad (22)$$

Hence  $f(a, 2, y, y^{m-1})$  is positive unless  $y = 1$ . We are now going to take a given  $a \in (0, 1/2)$  and consider  $m \in (1, 2)$  as a parameter. Let us prove that for some  $m_* \in (1, 2)$ , we have  $f(a, m, y, y^{m-1}) \geq 0$  for any  $(m, y)$  such that  $m_* < m < 2$  and  $0 \leq y \leq \gamma_m$ . We assume by contradiction that there are two sequences  $(m_k)_{k \in \mathbb{N}}$  and  $(y_k)_{k \in \mathbb{N}}$  such that  $1 < m_k < 2$  for any  $k \in \mathbb{N}$ ,  $\lim_{k \rightarrow +\infty} m_k = 2$ ,  $0 \leq y_k \leq \gamma_{m_k}$  and  $f(a, m_k, y_k, y_k^{m_k-1}) < 0$

for any  $k \in \mathbb{N}$ . Up to the extraction of a subsequence,  $(y_k)_{k \in \mathbb{N}}$  converges to some limit  $y_\infty \in [0, 2]$  and by continuity of  $f$  we know that  $f(a, 2, y_\infty, y_\infty) \leq 0$ : the only possibility is  $y_\infty = 1$  by (22). Since  $f(a, m_k, y_k, y_k^{m_k-1}) < 0 = f(a, m_k, 1, 1)$ , we learn that  $y_k \neq 1$ . Since  $\lim_{k \rightarrow +\infty} y_k = 1$ , this contradicts (21) or, to be precise,  $|y_k - 1| \geq \varepsilon(a, m_k)$ , as the reader is invited to check that  $\liminf_{k \rightarrow +\infty} \varepsilon(a, m_k) > 0$  because  $f$  is a smooth function of all of its arguments. If we redefine

$$m_*(a) := \inf \left\{ m \in (1, 2) : f(a, m, y, y^{m-1}) \geq 0 \forall y \in [0, \gamma_m] \right\},$$

then we know that for any  $a \in (0, 1/2)$ , we have  $m_*(a) < 2$ .

We want to prove that  $m_*(a) = 1$ . Again, let us argue by contradiction: if  $m_*(a) > 1$ , and assume that there are two sequences  $(m_k)_{k \in \mathbb{N}}$  and  $(y_k)_{k \in \mathbb{N}}$  such that  $1 < m_k < m_*(a)$  for any  $k \in \mathbb{N}$ ,  $\lim_{k \rightarrow +\infty} m_k = m_*(a)$ ,  $0 \leq y_k \leq \gamma_{m_k}$  and  $f(a, m_k, y_k, y_k^{m_k-1}) < 0$  for any  $k \in \mathbb{N}$ . Up to the extraction of a subsequence,  $(y_k)_{k \in \mathbb{N}}$  converges to some limit  $y_\infty \in [0, 2]$  and by continuity of  $f$  we know that  $f(a, m_*(a), y_\infty, y_\infty^{m_*(a)-1}) \leq 0$ . For the same reasons as above,  $y_\infty = 0$ ,  $y_\infty = 1$  and  $y_\infty = \gamma_{m_*(a)}$  are excluded. Altogether, we have proved that for

$$m = m_*(a),$$

we have  $f(a, m, y_\infty, y_\infty^{m-1}) = 0$  for some  $y_\infty \in (0, 1) \cup (1, \gamma_m)$  and we also have that  $f(a, m, y, y^{m-1}) \geq 0$  for any  $y \in (0, 1) \cup (1, \gamma_m)$ , so that  $y_\infty$  is a local minimizer of  $y \mapsto f(a, m, y, y^{m-1})$ . As a consequence, we have shown that for  $m = m_*(a) > 1$  and  $y = y_\infty \neq 1$ , we have

$$f(a, m, y, y^{m-1}) = 0 \quad \text{and} \quad \frac{\partial}{\partial y} f(a, m, y, y^{m-1}) = 0. \quad (23)$$

As we shall see below, this contradicts Lemma 12. Hence  $y \mapsto f(a, m, y, y^{m-1})$  takes nonnegative values for any *admissible* parameters  $(a, m)$  with  $1 < m < 2$ . By Lemma 10,  $K'(y) \leq 0$ , thus completing the proof.  $\square$

We still have to prove that (23) has no solution  $y \in (0, 1) \cup (1, \gamma_m)$ . Since

$$y \frac{\partial}{\partial y} f(a, m, y, y^{m-1}) = y \frac{\partial f}{\partial y}(a, m, y, z) + (m-1)z \frac{\partial f}{\partial z}(a, m, y, z),$$

we can relax the condition  $z = y^{m-1}$  and prove the slightly more general result.

**Lemma 12.** *With the notations of Lemma 10, assume that  $m > 1$ ,  $y \in (0, \gamma_m]$  and  $z \in (0, m]$ . For any admissible parameters  $(a, m)$ , if*

$$f(a, m, y, z) = 0, \quad (24a)$$

$$y \frac{\partial f}{\partial y}(a, m, y, z) + (m-1)z \frac{\partial f}{\partial z}(a, m, y, z) = 0. \quad (24b)$$

then  $z = 1$ .

*Proof.* Solving the system (24a)–(24b) is an *elimination problem* because the function  $f$ , as defined in Lemma 10, is a polynomial in the variables  $a$ ,  $y$  and  $z$ . Since (24a) is a first order equation in  $y$ , we can eliminate this variable and find that

$$y = \frac{n(a, m, z)}{d(a, m, z)}$$

with

$$\begin{aligned} n(a, m, z) &:= 2(m-1)(a^2(z-1)^2 - 3a(z-1)(mz-1) \\ &\quad + 2m^2z^2 + (m^2 - 6m + 1)z + 2), \\ d(a, m, z) &:= m(2a^2(z-1)^2 + 3a(z+1)^2(z-1) + 9z^2 + 8z + 1) \\ &\quad - 2z(a^2(z-1)^2 + 3a(z-1) + z + 2) \\ &\quad - m^2z(6a(z-1) + z^2 + 8z + 9) + 2m^3z(2z + 1). \end{aligned}$$

After replacing, solving (24b) under the condition  $z \neq 1$  is reduced to a second order equation in  $z$ , whose discriminant is

$$\delta(a, m) := -3(a-1)^2(m-1)^2(a-m)^2(5a^2 - 10a(m+1) - 3m^2 + 14m - 3).$$

Since  $5a^2 - 10a(m+1) - 3m^2 + 14m - 3$  takes only positive values for *admissible*  $(a, m)$ , there are no other roots than  $z = 1$ . This is the desired contradiction, which completes the proof thanks to Lemma 6.  $\square$

*Proof of Theorem 3.* It is a straightforward consequence of Lemmas 6 and 11.  $\square$

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