

# Limit theorems for iid products of positive matrices

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## Abstract

We study stochastic properties of the norm cocycle associated with iid products of positive matrices. We obtain the almost sure invariance principle (ASIP) with rate  $o(n^{1/p})$  under the optimal condition of a moment of order  $p > 2$  and the Berry-Esseen theorem with rate  $O(1/\sqrt{n})$  under the optimal condition of a moment of order 3. The results are also valid for the matrix norm. For the matrix coefficients, we also have the ASIP but we obtain only partial results for the Berry-Esseen theorem. The proofs make use of coupling coefficients that surprisingly decay exponentially fast to 0 while there is only a polynomial decay in the case of invertible matrices. All the results are actually valid in the context of iid products of matrices leaving invariant a suitable cone.

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## 1 Introduction

Let  $d \geq 2$  be an integer. Let  $G$  be the semi-group of  $d$ -dimensional positive allowable matrices: by positive, we mean that all entries are greater than or equal to 0, by allowable, we mean that any row and any column admits a strictly positive element. We endow  $\mathbb{R}^d$  with the  $\ell^1$  norm

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and  $G$  with the corresponding operator norm. We denote both norms by  $\|\cdot\|$ . Recall that  $\|g\| = \sup_{\|x\|=1} \|gx\|$ . Define also

$$S^+ := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : \|x\| = 1 \text{ and } x_i \geq 0, \forall i \in \{1, \dots, d\}\}. \quad (1.1)$$

Let  $\mu$  be a probability on the Borel sets of  $G$ . Let  $(Y_n)_{n \in \mathbb{N}}$  be independent and identically distributed (iid) random variables with law  $\mu$  living on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . For every  $n \in \mathbb{N}$ , set  $A_n := Y_n \cdots Y_1$ . We wish to study the asymptotic behaviour of the sequences  $(\log \|A_n\|)_{n \in \mathbb{N}}$  and  $(\log \|A_n x\|)_{n \in \mathbb{N}}$ ,  $x \in S^+$ . Other sequences of interest are  $(\log \langle A_n x, y \rangle)_{n \in \mathbb{N}}$  for  $x, y \in S^+$ , where  $\langle \cdot, \cdot \rangle$  stands for the standard inner product on  $\mathbb{R}^d$ ;  $(\log \kappa(A_n))_{n \in \mathbb{N}}$ , with  $\kappa$  the spectral radius or  $(\log(\inf_{x \in S^+} \|A_n x\|))_{n \in \mathbb{N}}$ .

In a series of paper [7], [9], [10], [12] and [13] we studied the stochastic properties of the norm cocycle (i.e.  $(\log \|A_n x\|)_{n \in \mathbb{N}}$ ) associated with the left random walk on  $GL_d(\mathbb{R})$  under optimal or close to optimal moment conditions. The moment conditions required in these works are in particular optimal in case of the almost sure invariance principle (ASIP) with rate, and close to optimal in the case of the Berry-Esseen theorem. We also obtained results for the matrix norm, the matrix coefficients and the spectral radius. A key ingredient to get these result is to obtain a suitable control of some coupling coefficients introduced in [7], under appropriate moment conditions for  $\mu$ . In the context of positive matrices these coefficients are defined in Section 3 and can be written as follows: for  $p \geq 1$ ,

$$\tilde{\delta}_{p,\infty}(n) := \mathbb{E} \left( \sup_{x,y \in S^+} |\log \|A_n x\| - \log \|A_n y\||^p \right).$$

As we shall see, in the context of positive matrices, these couplings coefficients decrease exponentially fast even if  $\mu$  has only polynomial moments, in contrast with the case of invertible matrices where the decay is only arithmetical. More precisely we shall prove in Proposition 3.2 below that, when  $\mu$  is strictly contracting and almost admits a moment of order  $p \geq 1$  (see Section 2 for a definition of these notions), there exists  $a \in ]0, 1[$  such that  $\tilde{\delta}_{p,\infty}(n) = O(a^n)$ .

As we already mentioned, a suitable control of this kind of coefficients (together with the Markovian structure of the random walk) is one of the main arguments used in [10] and [12] for obtaining rates in the ASIP, as well as Berry-Esseen type bounds in the case of the left random walk on  $GL_d(\mathbb{R})$ . We follow this strategy in Section 6, where we obtain rates of order  $o(n^{1/p})$  in the ASIP when  $\mu$  has a moment of order  $p > 2$ , and in Section 7 where we obtain rates of order  $O(n^{1-p/2})$  for Berry-Esseen type bounds (for the quantities  $\log \|A_n\|$  and  $\log \|A_n x\|$ ) when  $\mu$  has a moment of order  $p \in ]2, 3]$ .

Let us mention that the study of iid products of positive matrices benefited from a lot of works. Let us cite, among others, Hennion [15], Hennion and Hervé [16], Buraczewski et al. [4], Buraczewski and Mentemeier [5] or Xiao, Grama and Liu [24], [25], [26] and [27].

Hennion [15] obtained the strong law of large numbers and the central limit theorem (CLT) under optimal moment conditions in the more general situation of products of dependent positive random matrices satisfying some mixing conditions. All the other above mentioned papers, except [16] and [27], assume exponential moment for  $\mu$  which allows to use in a natural way the Guivarc'h-Nagaev method based on perturbation of operators.

In fact, in the context of products of positive random matrices, Hennion and Hervé [16] and Xiao, Grama and Liu [27] recently observed that the Guivarc'h-Nagaev method applies under polynomial moment conditions. In particular, Hennion and Hervé obtained the Berry-Esseen theorem with rate  $O(1/\sqrt{n})$ , under a moment of order  $p > 4$  while Xiao, Grama and Liu obtained the same rate (as well as a first order Edgeworth expansion) under a moment of order 3, but assuming an additional technical condition (see their condition A2, that we shall discuss at the end of Section 7.2).

The paper is organised as follows. In Section 2, we introduce some notations and definitions and we also recall several key properties in the study of positive matrices. In Section 3, we establish the existence of a unique invariant probability and we estimate our coupling coefficients (see Proposition 3.2). In Section 4, we recall the strong law of large numbers of Hennion [15] and provide some complementary results that can be obtained via Proposition 3.2. In Section 5, we recall the CLT and provide several identifications of the asymptotic variance  $s^2$ . Moreover, we show that the known aperiodicity condition (see Definition 5.1) is sufficient for  $s^2 > 0$ , under a moment of order 2. In Section 6, we obtain the ASIP for the norm cocycle, the matrix norm, the spectral radius and the matrix coefficients under optimal polynomial moment conditions. In Section 7, we obtain the Berry-Esseen theorem for all the above mentioned quantities. The obtained rates are optimal (in terms of moment conditions) in the case of the norm cocycle and the matrix norm, but we have a loss in the case of the spectral radius and the matrix coefficients. In Section 8 we study the regularity of the invariant measure and in Section 9, we provide some deviation inequalities for the norm cocycle and the matrix coefficients. In Section 10, we explain how to generalise our results to matrices leaving invariant a suitable cone (notice that the positive matrices of size  $d$  may be seen as the matrices leaving invariant the cone  $(\mathbb{R}^+)^d$ ). Finally, in Section 11, we provide technical results related to the previous section.

In all the paper we denote  $\mathbb{N} := \{1, 2, \dots\}$ .

## 2 Norm cocycle and matrix norm

We put on  $G$  the topology inherited from (the distance associated with) the norm. Then,  $G$  becomes a locally compact space. Let  $G^+$  be the sub-semi-group of  $G$  whose entries are all strictly positive. Actually,  $G^+$  is the interior of  $G$ .

Notice that for  $g \in G$ , we actually have  $\|g\| = \sup_{x \in S^+} \|gx\|$  and that, if  $g = (g_{ij})_{1 \leq i, j \leq d}$ ,

$$\|g\| = \max_{1 \leq j \leq d} \sum_{i=1}^d g_{ij}. \quad (2.1)$$

For every  $g \in G$ , set  $v(g) = \inf_{x \in S^+} \|gx\|$ . If  $g = (g_{ij})_{1 \leq i, j \leq d}$ , we have

$$v(g) = \min_{1 \leq j \leq d} \sum_{i=1}^d g_{ij}. \quad (2.2)$$

By definition of  $G$ ,  $v(g) > 0$  for every  $g \in G$ .

We then define  $N(g) := \max(\|g\|, 1/v(g))$  and  $L(g) = \frac{\|g\|}{v(g)}$ . Notice that  $N(g)^2 \geq L(g) \geq 1$  for every  $g \in G$ .

We endow  $S^+$  with the following metric (see Proposition 10.1 for a proof that it is indeed a metric). For every  $x, y \in S^+$ ,

$$d(x, y) = \varphi(m(x, y)m(y, x)),$$

where

$$\varphi(s) = \frac{1-s}{1+s} \quad \forall s \in [0, 1], \quad (2.3)$$

and

$$m(u, v) = \inf \left\{ \frac{u_i}{v_i} : i \in \{1, \dots, d\}, v_i > 0 \right\}.$$

Notice that the diameter of  $S^+$  is 1 and that  $d(x, y) = 1$  if and only if there exists  $i_0 \in \{1, \dots, d\}$  such that  $x_{i_0} = 0$  and  $y_{i_0} > 0$  or  $x_{i_0} > 0$  and  $y_{i_0} = 0$ .

Using that for  $u, v \in S^+$ ,  $\max_{1 \leq i \leq d} u_i \leq 1$  and  $\max_{1 \leq i \leq d} v_i \geq 1/d$ , we see that  $m(u, v) \leq d$ .

The semi-group  $G$  is acting on  $S^+$  as follows.

$$g \cdot x = \frac{gx}{\|gx\|} \quad \forall (g, x) \in G \times S^+.$$

We then define a cocycle by setting  $\sigma(g, x) = \log(\|gx\|)$  for every  $(g, x) \in G \times S^+$ . The cocycle property reads

$$\sigma(gg', x) = \sigma(g, g' \cdot x) + \sigma(g', x). \quad (2.4)$$

Following Hennion [15, Lemma 10.6], for every  $g \in G$  we define  $c(g) := \sup_{x, y \in S^+} d(gx, gy)$ .

Let us recall some properties that one may find in Hennion [15], see his Lemmas 5.2, 5.3 and 10.6 and his Proposition 3.1.

**Proposition 2.1.** *For every  $(g, g', x, y) \in G^2 \times (S^+)^2$  we have*

- (i)  $|\sigma(g, x)| \leq \log N(g);$
- (ii)  $\|x - y\| \leq 2d(x, y);$
- (iii)  $|\sigma(g, x) - \sigma(g, y)| \leq 2L(g)d(x, y);$
- (iv)  $|\sigma(g, x) - \sigma(g, y)| \leq 2 \ln(1/(1 - d(x, y))) ;$
- (v)  $c(gg') \leq c(g)c(g');$
- (vi)  $c(g) \leq 1$  and  $c(g) < 1$  iff  $g \in G^+;$
- (vii)  $d(g \cdot x, g \cdot y) \leq c(g)d(x, y).$

Let us also mention a closed-form expression for  $c(g)$  obtained in Lemma 10.7 of [15] (see also Proposition 10.3 below). For every  $g = (g_{ij})_{1 \leq i, j \leq d}$  we have

$$c(g) = \max_{1 \leq i, j, k, \ell \leq d} \frac{|g_{ij}g_{k\ell} - g_{i\ell}g_{kj}|}{g_{ij}g_{k\ell} + g_{i\ell}g_{kj}}. \quad (2.5)$$

Notice that  $(g, x) \rightarrow gx$  is continuous on  $G \times S^+$  (for the distance on  $G$  induced by the operator norm and the distance on  $S^+$  induced by  $\|\cdot\|$ ) and does not vanish. Hence, it follows from item (ii) that  $(g, x) \rightarrow g \cdot x$  is continuous on  $G \times S^+$  (for the distance on  $G$  induced by the operator norm and the distance  $d$  on  $S^+$ ).

Let us give some more properties that will be useful in the sequel. Set  $e = \{1/d, \dots, 1/d\} \in S^+$ . For  $g \in G$ , we denote by  $g^t$  the adjoint matrix of  $g$ .

**Lemma 2.2.** *For every  $(g, x, y) \in G \times (S^+)^2$ ,*

- (i)  $|\sigma(g, x) - \sigma(g, y)| \leq \log L(g);$
- (ii)  $\|ge\| \leq \|g\| \leq d\|ge\|;$
- (iii)  $\|g\| \leq d\|g^t\|;$
- (iv)  $|\sigma(g, x) - \sigma(g, y)| \leq 2(2 + \log L(g))d(x, y).$

**Remark.** The inequality in item (iv) of Lemma 2.2 is much better than the one in item (iii) of Proposition 2.1.

**Proof.** Items (i) and (ii) are obvious. Item (iii) is an easy consequence of (2.1). Let us prove item (iv). Let  $x, y \in S^+$ . Assume that  $d(x, y) \leq 1/2$ . Notice that for every  $t \in [0, 1/2]$ ,  $\ln(1/(1 - t)) \leq 2t$ . Hence, using item (iv) of Proposition 2.1, we see that  $|\sigma(g, x) - \sigma(g, y)| \leq 4d(x, y)$ . If  $2d(x, y) \geq 1$ , then the desired conclusion follows from item (i) of Lemma 2.2.  $\square$

**Proposition 2.3.**  $(S^+, d)$  is complete and  $S^{++}$  is closed where

$$S^{++} := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : \|x\| = 1 \text{ and } x_i > 0, \forall i \in \{1, \dots, d\}\}. \quad (2.6)$$

**Remark.** This proposition is probably well known. We did not find a reference for it. However, a hint of proof of completeness is given after Theorem 4.1 of Bushell [6], for Hilbert's metric given by  $d_H(x, y) = -\ln(m(x, y)m(y, x))$ . See Proposition 10.1 for a proof in a more general situation.

Let us state some of the assumptions used throughout the paper.

**Definition 2.1.** Let  $\mu$  be a Borel probability on  $G$  and  $p \geq 1$ . We say that  $\mu$  admits a moment of order  $p$  if

$$\int_G (\log(N(g)))^p d\mu(g) < \infty.$$

We say that  $\mu$  almost admits a moment of order  $p$  if

$$\int_G (\log(L(g)))^p d\mu(g) < \infty.$$

**Remark.** Clearly, since  $L(g) \leq N(g)^2$ , if  $\mu$  admits a moment of order  $p \geq 1$ , it almost admits a moment of order  $p \geq 1$ , but the converse is not true in general, see the example in Section 6. Assume now that  $\mu$  almost admits a moment of order  $p \geq 1$ . Then,  $\mu$  admits a moment of order  $p$  iff  $\int_G |\log \|g\||^p d\mu(g) < \infty$  iff  $\int_G |\log v(g)|^p d\mu(g) < \infty$ .

Similarly, we say that  $\mu$  admits or almost admits an exponential moment of order  $\gamma > 0$ , if there exists  $\delta > 0$  such that, respectively,

$$\int_G e^{\delta(\log N(g))^\gamma} d\mu(g) < \infty,$$

or

$$\int_G e^{\delta(\log L(g))^\gamma} d\mu(g) < \infty.$$

**Definition 2.2.** We say that  $\mu$  is *strictly contracting* if there exists  $r \in \mathbb{N}$ , such that  $\mu^{*r}(G^+) > 0$ .

Equivalently, the closed semi-group  $\Gamma_\mu$  generated by the support of  $\mu$  has non empty intersection with  $G^+$ .

### 3 Invariant measure and coupling coefficients

Recall that a Borel (with respect to the distance  $d(\cdot, \cdot)$ ) probability  $\nu$  on  $S^+$  is said to be  $\mu$ -invariant if for every Borel non negative function  $\varphi$  on  $S^+$ ,  $\int_{G \times S^+} \varphi(g \cdot x) d\mu(g) d\nu(x) = \int_{S^+} \varphi(x) d\nu(x)$ . It is well known and easy to prove (recall that  $(g, x) \rightarrow g \cdot x$  is continuous on  $G \times S^+$ ) that the support of a  $\mu$ -invariant measure  $\nu$  is  $\Gamma_\mu$ -invariant, i.e. satisfies  $\Gamma_\mu \cdot \text{supp } \nu \subset \text{supp } \nu$ .

As recalled below, when  $\mu$  is strictly contracting, it admits a unique  $\mu$ -invariant probability on  $S^+$ . We need some further notations to describe its support.

Let  $g \in G^+$ . By the Perron-Frobenius theorem (see Theorem 1.1.1 of [21]), there exists a unique  $x \in S^{++}$  such that  $gx = \kappa(g)x$ , where  $\kappa(g)$  is the spectral radius of  $g$ . We denote that vector by  $u_g$ . Notice the following bound that will be useful in the sequel,

$$\kappa(g) \geq v(g) \quad \forall g \in G. \quad (3.1)$$

Following [4] (see (2.4) there) we define

$$\Lambda_\mu = \overline{\{u_g : g \in \Gamma_\mu \cap G^+\}},$$

where the closure is taken with respect to  $d$ . By Proposition 2.3,  $\Lambda_\mu \subset S^{++}$ .

It follows from Lemma 4.2 of [4] that  $\Lambda_\mu$  is  $\Gamma_\mu$ -invariant (i.e.  $\Gamma_\mu \cdot \Lambda_\mu \subset \Lambda_\mu$ ).

We recall the following result of Hennion and Hervé [17].

**Proposition 3.1.** *Assume that  $\mu$  is strictly contracting. Then, there exists a unique  $\mu$ -invariant probability  $\nu$  on  $S^+$ . Moreover  $\text{supp } \nu = \Lambda_\mu$ .*

The existence and uniqueness of an invariant probability for strictly contracting  $\mu$  is proved in Theorem 2.1 of [17] and the characterization of the support of the invariant measure follows from Lemma 4.3 of [4]. Since there is no explicit proof of the latter fact in [4], let us give an argument.

For every  $n \in \mathbb{N}$ , set  $B_n := Y_1 \cdots Y_n$  (with  $(Y_n)_{n \in \mathbb{N}}$  iid with law  $\mu$ ). It follows from the proof of Theorem 2.1 of [17] that,  $\mathbb{P}$ -almost surely, for every  $x \in S^+$ ,  $(B_n \cdot x)_{n \in \mathbb{N}}$  converges to some random variable  $Z$  whose law  $\nu$  is  $\mu$ -invariant. Then  $\text{supp } \nu$  is  $\Gamma_\mu$ -invariant and  $\Lambda_\mu \subset \text{supp } \nu$  by Lemma 4.2 of [4]. Now, since  $\Gamma_\mu \cdot \Lambda_\mu \subset \Lambda_\mu$ , for every  $x \in \Lambda_\mu$ ,  $B_n \cdot x \in \Lambda_\mu$   $\mathbb{P}$ -almost surely, for every  $n \in \mathbb{N}$ . Hence  $Z \in \Lambda_\mu$   $\mathbb{P}$ -almost surely (recall that  $\Lambda_\mu$  is closed for  $d$ ), which implies that  $\nu(\Lambda_\mu) = 1$ , hence that  $\text{supp } \nu \subset \Lambda_\mu$ .

Recall that  $(Y_n)_{n \in \mathbb{N}}$  is a sequence of iid random variables taking values in  $G$ , with law  $\mu$  and living on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and that  $A_n := Y_n \cdots Y_1$ .

For every  $p \geq 1$  and every  $n \in \mathbb{N}$  define

$$\delta_{p,\infty}(n) := \sup_{x,y \in S^+} \mathbb{E}(|\sigma(Y_n, A_{n-1} \cdot x) - \sigma(Y_n, A_{n-1} \cdot y)|^p).$$

Those coefficients have been introduced in [7], in the setting of products of iid matrices in  $GL_d(\mathbb{R})$ , and proved to be very useful in [10] and [12], see also [9].

We shall see that those coefficients decrease exponentially fast to 0, as soon as  $\mu$  (almost) admits a moment of order 1, while we obtained only a polynomial speed of convergence in the case of  $GL_d(\mathbb{R})$ .

Actually, we will prove the result for the stronger coefficients

$$\tilde{\delta}_{p,\infty}(n) := \mathbb{E}\left(\sup_{x,y \in S^+} |\sigma(Y_n, A_{n-1} \cdot x) - \sigma(Y_n, A_{n-1} \cdot y)|^p\right).$$

**Proposition 3.2.** *Assume that  $\mu$  is strictly contracting and almost admits a moment of order  $p \geq 1$ . Then, there exists  $0 < a < 1$  such that*

$$\delta_{p,\infty}(n) \leq \tilde{\delta}_{p,\infty}(n) = O(a^n), \quad (3.2)$$

and

$$\sup_{x,y \in S^+} \sup_{n \in \mathbb{N}} |\sigma(A_n, x) - \sigma(A_n, y)| \in L^p. \quad (3.3)$$

In particular,

$$\sup_{n \in \mathbb{N}} |\log \|A_n\| - \log v(A_n)| \in L^p. \quad (3.4)$$

**Proof.** Let  $n \in \mathbb{N}$ . By item (iv) of Lemma 2.2 and item (vii) of Proposition 2.1, for every  $x, y \in S^+$ , we have

$$|\sigma(Y_n, A_{n-1} \cdot x) - \sigma(Y_n, A_{n-1} \cdot y)| \leq (4 + 2 \log L(Y_n)) d(A_{n-1} \cdot x, A_{n-1} \cdot y) \leq (4 + 2 \log L(Y_n)) c(A_{n-1}).$$

Let  $r \in \mathbb{N}$  be as in Definition 2.2. Then, by item (vi) of Proposition 2.1, there exists  $\varepsilon > 0$  such that

$$\mu^{*r}(c(g) \leq 1 - \varepsilon) =: \gamma > 0. \quad (3.5)$$

Hence, if  $m = \lfloor (n-1)/r \rfloor$ ,

$$\mathbb{E}[(c(A_{n-1}))^p] \leq \prod_{k=1}^m \mathbb{E}[(c(Y_{kr} \cdots Y_{(k-1)r+1}))^p] \leq (\gamma(1 - \varepsilon)^p + 1 - \gamma)^m.$$



This proves the desired exponential convergence of  $(\tilde{\delta}_{p,\infty}(n))_{n \in \mathbb{N}}$ .

Using the cocycle property, we see that for every  $n \in \mathbb{N}$  and every  $x, y \in S^+$ ,

$$\begin{aligned} |\sigma(A_n, x) - \sigma(A_n, y)| &\leq \sum_{k=1}^n |\sigma(Y_k, A_{k-1} \cdot x) - \sigma(Y_k, A_{k-1} \cdot y)| \\ &\leq \sum_{k=1}^{\infty} |\sigma(Y_k, A_{k-1} \cdot x) - \sigma(Y_k, A_{k-1} \cdot y)|. \end{aligned}$$

Using the triangle inequality in  $L^p$ , we infer that

$$\begin{aligned} \mathbb{E} \left[ \sup_{x, y \in S^+} \sup_{n \in \mathbb{N}} |\sigma(A_n, x) - \sigma(A_n, y)|^p \right] &\leq \left( \sum_{k=1}^{\infty} (\tilde{\delta}_{p,\infty}(k))^{1/p} \right)^p \\ &\leq r^p \mathbb{E} \left[ (2(2 + \log L(Y_1)))^p \right] \left( \sum_{m \geq 0} (\gamma(1 - \varepsilon)^p + 1 - \gamma)^{m/p} \right)^p \\ &\leq \frac{2^p r^p \mathbb{E} \left[ (2 + \log L(Y_1))^p \right]}{\left( 1 - (\gamma(1 - \varepsilon)^p + 1 - \gamma)^{1/p} \right)^p}, \end{aligned}$$

proving (3.3). To prove (3.4), note first that for every  $g \in G$ ,

$$\begin{aligned} |\log \|g\| - \log v(g)| &= \sup_{x \in S^+} \log \|gx\| - \inf_{y \in S^+} \log \|gy\| = \sup_{x \in S^+} \log \|gx\| + \sup_{y \in S^+} (-\log \|gy\|) \\ &= \sup_{x, y \in S^+} (\log \|gx\| - \log \|gy\|) \leq \sup_{x, y \in S^+} |\sigma(g, x) - \sigma(g, y)|. \end{aligned}$$

Therefore (3.4) follows from (3.3). □

## 4 The strong law of large numbers

Except the  $L^1$ -convergences, the results of that section are essentially contained in Hennion's paper [15] (where a more general situation is considered), see his Theorem 2 and its proof. Notice that Hennion assumed a moment of order 1 for  $\tilde{\mu}$  (the pushforward measure of  $\mu$  by the application  $g \mapsto g^t$ ) while we also consider the case where  $\mu$  itself admits a moment of order 1.

We first recall the version of Kingman's subadditive ergodic theorem relevant to our setting (see [19, Theorems 1 and 2]). The fact that  $\lambda_\mu$  in the next proposition is constant follows from Kolmogorov's 0 – 1 law.

**Proposition 4.1** (Kingman). *Assume that  $\int_G |\log \|g\|| d\mu(g) < \infty$ . Then,  $(\frac{1}{n} \log \|A_n\|)_{n \geq 1}$  converges  $\mathbb{P}$ -a.s. and in  $L^1$  to some constant  $\lambda_\mu \in \mathbb{R}$ .*

**Remark.** Using that  $\|g\| \geq v(g)$  for every  $g \in G^+$ , we see that  $\log^- \|g\| \leq \log^- v(g)$ , where  $\log^-(x) = \max(-\log x, 0)$  for every  $x > 0$ . In particular, if  $\mu$  or  $\tilde{\mu}$  admits a moment of order 1, then,  $\int_G |\log \|g\|| d\mu(g) < \infty$ .

The proposition implies in particular that  $\lambda_\mu = \lim_{n \rightarrow +\infty} \mathbb{E}(\log \|A_n\|)/n$ .

Notice that  $\int_G |\log \|g\|| d\mu(g) < \infty$  if and only if  $\int_G |\log \|g\|| d\tilde{\mu}(g) < \infty$ . Hence, applying the proposition to  $\tilde{\mu}$ , using item (iii) of Lemma 2.2 and the fact that  $Y_1^t \cdots Y_n^t$  has same law as  $Y_n^t \cdots Y_1^t$ , we infer that  $\lambda_\mu = \lambda_{\tilde{\mu}}$ .

We then provide the strong law of large numbers for various quantities related to  $(A_n)_{n \in \mathbb{N}}$  and identify the limit under a stronger assumption. In the sequel  $\|\cdot\|_1$  stands for the  $L^1$ -norm on our underlying probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

**Theorem 4.2.** *Assume that  $\mu$  is strictly contracting and that  $\mu$  admits a moment of order 1. Then, for every  $x \in S^+$ ,*

$$\lim_{n \rightarrow +\infty} \frac{\sigma(A_n, x)}{n} = \lim_{n \rightarrow +\infty} \frac{\log v(A_n)}{n} = \lim_{n \rightarrow +\infty} \frac{\log \kappa(A_n)}{n} = \lambda_\mu \quad \mathbb{P}\text{-a.s.}, \quad (4.1)$$

where  $\lambda_\mu = \int_{G \times S^+} \sigma(g, x) d\mu(g) d\nu(x)$ . Moreover, the convergences also hold in  $L^1$  and, we even have

$$\left\| \sup_{x \in S^+} \left| \frac{\sigma(A_n, x)}{n} - \lambda_\mu \right| \right\|_1 \xrightarrow{n \rightarrow +\infty} 0 \text{ and } \sup_{x \in S^+} \left| \frac{\sigma(A_n, x)}{n} - \lambda_\mu \right| \xrightarrow{n \rightarrow +\infty} 0 \quad \mathbb{P}\text{-a.s.}$$

**Remark.** The  $\mathbb{P}$ -a.s. and  $L^1$  convergence of  $(\frac{1}{n} \log v(A_n))_{n \in \mathbb{N}}$  when  $\int_G |\log v(g)| d\mu(g) < \infty$  (which holds if  $\mu$  admits a moment of order 1) follow from Kingman's subadditive ergodic Theorem applied to  $(-\log v(A_n))_{n \in \mathbb{N}}$ . The formula for  $\lambda_\mu$  may be derived from the formula in the middle of page 1568 of [15].

**Proof.** By Proposition 4.1 and the remark after it, we have the  $\mathbb{P}$ -a.s. and  $L^1$  convergence of  $((\log \|A_n\|)/n)_{n \in \mathbb{N}}$  to  $\lambda_\mu$ .

By (3.4), we infer the  $L^1$  convergence for  $((\log v(A_n))/n)_{n \in \mathbb{N}}$ . To prove the almost sure convergence, define first  $Z := \sup_{n \in \mathbb{N}} |\log \|A_n\| - \log v(A_n)|$ . By (3.4),  $Z \in L^1$  and, for every  $\varepsilon > 0$ , by Fubini's theorem,

$$\sum_{n \in \mathbb{N}} \mathbb{P}(|\log \|A_n\| - \log v(A_n)| \geq \varepsilon n) \leq \frac{\mathbb{E}(Z)}{\varepsilon} < \infty.$$

The  $\mathbb{P}$ -a.s. convergence for  $((\log v(A_n))/n)_{n \in \mathbb{N}}$  then follows from the one for  $((\log \|A_n\|)/n)_{n \in \mathbb{N}}$  and the Borel-Cantelli lemma.

The convergences for  $((\log \kappa(A_n))/n)_{n \in \mathbb{N}}$  follow from the bounds  $v(A_n) \leq \kappa(A_n) \leq \|A_n\|$  (see (3.1) for the first bound).

Finally, notice that for every  $n \in \mathbb{N}$ ,

$$\sup_{x \in S^+} |\sigma(A_n, x) - n\lambda_\mu| \leq \max(|\log \|A_n\| - n\lambda_\mu|, |\log v(A_n) - n\lambda_\mu|),$$

which proves the remaining convergences.

Hence, it remains to identify  $\lambda_\mu$ . From the above, using the  $\mu$ -invariance of  $\nu$ , we infer that

$$\begin{aligned} \int_{G \times S^+} \sigma(g, x) d\mu(g) d\nu(x) &= \frac{1}{n} \int_{S^+} \mathbb{E} \left( \sum_{k=1}^n \sigma(Y_k, A_{k-1} \cdot x) \right) d\nu(x) \\ &= \frac{1}{n} \int_{S^+} \mathbb{E}(\sigma(A_n, x)) d\nu(x) \xrightarrow{n \rightarrow +\infty} \lambda_\mu. \end{aligned}$$

□

We shall now consider the case of matrix coefficients. The proof will rely on Lemma 4.3 below, which is essentially contained in Lemma 2.1 of [17] (see also Lemma 6.3 of [4] for (4.4)). We need also some further notations. As in [17], set

$$T := \inf\{n \in \mathbb{N} : Y_n \cdots Y_1 \in G^+\}. \quad (4.2)$$

From item (i) of [17, Lemma 2.1], note that  $\mu$  is strictly contracting if and only if  $\mathbb{P}(T < \infty) = 1$ .

**Lemma 4.3.** *Assume that  $\mu$  is strictly contracting. With the above notations,*

$$\inf_{n \geq T} \inf_{x, y \in S^+} \frac{\langle y, A_n x \rangle}{\|Y_1^t \cdots Y_n^t y\|} > 0 \quad \mathbb{P}\text{-a.s.} \quad (4.3)$$

and

$$\inf_{n \in \mathbb{N}} \inf_{x \in S^+} \frac{\|A_n x\|}{\|A_n\|} = \inf_{n \in \mathbb{N}} \frac{v(A_n)}{\|A_n\|} > 0 \quad \mathbb{P}\text{-a.s.} \quad (4.4)$$

Inequality (4.3) is just a reformulation of item (iii) of Lemma 2.1 from [17], and (4.4) follows from (4.3) and the fact that for every  $n \in \mathbb{N}$  and every  $x \in S^+$ , using items (ii) and (iii) of Lemma 2.2, we get

$$\frac{\|A_n x\|}{\|A_n\|} \geq \frac{\langle e, A_n x \rangle}{d^2 \|A_n^t e\|}.$$

Recall that  $\tilde{\mu}$  stands for the pushforward measure of  $\mu$  by the map  $g \rightarrow g^t$ .

**Theorem 4.4.** Assume that  $\mu$  is strictly contracting and that  $\tilde{\mu}$  admits a moment of order 1. Then,

$$\left( \sup_{x, y \in S^+} \left| \frac{\log \langle y, A_n x \rangle}{n} - \lambda_\mu \right| \right)_{n \in \mathbb{N}} \xrightarrow{n \rightarrow +\infty} 0 \quad \mathbb{P}\text{-a.s.},$$

where  $\lambda_\mu = \int_{G \times S^+} \sigma(g, x) d\tilde{\mu}(g) d\tilde{\nu}(x)$  ( $\tilde{\nu}$  being the only  $\tilde{\mu}$ -invariant probability on  $S^+$ ). In particular,

$$\left( \left| \frac{\inf_{x, y \in S^+} \log \langle y, A_n x \rangle}{n} - \lambda_\mu \right| \right)_{n \in \mathbb{N}} \xrightarrow{n \rightarrow +\infty} 0 \quad \mathbb{P}\text{-a.s.}$$

Moreover,  $((\log \|A_n\| - n\lambda_\mu)/n)_{n \in \mathbb{N}}$  and  $((\log \kappa(A_n) - n\lambda_\mu)/n)_{n \in \mathbb{N}}$  converge  $\mathbb{P}$ -a.s. and in  $L^1$  to 0; and  $((\log v(A_n) - n\lambda_\mu)/n)_{n \in \mathbb{N}}$  converges  $\mathbb{P}$ -a.s. to 0.

**Proof.** First notice that Proposition 4.1 applies, which yields the  $\mathbb{P}$ -a.s. and  $L^1$  convergence for  $((\log \|A_n\|)/n)_{n \in \mathbb{N}}$  and for  $((\log \|A_n^t\|)/n)_{n \in \mathbb{N}}$  by item (iii) of Lemma 2.2.

By Lemma 4.3, there exists a random variable  $W > 0$  such that, for every  $x, y \in S^+$  and every  $n \in \mathbb{N}$ , on the set  $\{T \leq n\}$  (recall that  $T$  is defined in (4.2)),

$$0 \leq \log \|A_n\| - \log \langle y, A_n x \rangle \leq \log W + \log \|A_n\| - \log \|Y_1^t \cdots Y_n^t y\|. \quad (4.5)$$

Let  $\varepsilon > 0$ . Using that  $(Y_1, \dots, Y_n)$  and  $(Y_n, \dots, Y_1)$  have the same law, we get

$$\begin{aligned} \sum_{n \geq 1} \mathbb{P}(\sup_{y \in S^+} |\log \|Y_1^t \cdots Y_n^t y\| - \log \|Y_1^t \cdots Y_n^t e\|| \geq \varepsilon n) \\ \leq \sum_{n \geq 1} \mathbb{P}(\sup_{y \in S^+} \sup_{m \in \mathbb{N}} |\log \|Y_m^t \cdots Y_1^t y\| - \log \|Y_m^t \cdots Y_1^t e\|| \geq \varepsilon n) < \infty, \end{aligned}$$

where we used Proposition 3.2 for  $\tilde{\mu}$ .

By the Borel-Cantelli lemma, using item (ii) of Lemma 2.2, we infer that

$$\frac{\sup_{y \in S^+} |\log \|Y_1^t \cdots Y_n^t y\| - \log \|A_n^t\||}{n} \xrightarrow{n \rightarrow +\infty} 0 \quad \mathbb{P}\text{-a.s.} \quad (4.6)$$

Combining this with (4.5) (recall that  $\mathbb{P}(T < \infty) = 1$  and that  $\|g\| \leq d\|g^t\|$  for every  $g \in G$ ) we obtain that

$$\sup_{x, y \in S^+} \frac{|\log \|A_n\| - \log \langle y, A_n x \rangle|}{n} \xrightarrow{n \rightarrow +\infty} 0 \quad \mathbb{P}\text{-a.s.}$$

This gives the desired convergence for the coefficients. The  $\mathbb{P}$ -a.s. convergences for  $(\log \kappa(A_n)/n)_{n \in \mathbb{N}}$  and  $(\log v(A_n)/n)_{n \in \mathbb{N}}$  follow from the inequalities

$$\frac{\inf_{x, y \in S^+} \log \langle y, A_n x \rangle}{n} \leq \frac{\log v(A_n)}{n} \leq \frac{\log \kappa(A_n)}{n} \leq \frac{\log \|A_n\|}{n}.$$

The  $L^1$  convergence for  $(\log \kappa(A_n)/n)_{n \in \mathbb{N}}$ , follows from Theorem 4.2 applied to  $\tilde{\mu}$ , using item (iii) of Lemma 2.2 and noticing that  $(Y_1, \dots, Y_n)$  has the same law as  $(Y_n, \dots, Y_1)$ .  $\square$

Under our assumptions, one cannot expect the  $L^1$  convergence in Theorem 4.4 for  $v(A_n)$ .

For instance take  $\mu$  such that for every  $k \in \mathbb{N}$ ,  $\mu(\{g_k\}) = \frac{1}{3k(k+1)}$  and  $\mu(\{h\}) = \mu(\{\text{Id}\}) = 1/3$ , with  $g_k = \begin{pmatrix} 2^{-k} & 1/2 \\ 0 & 1/2 \end{pmatrix}$  and  $h = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ . Then, for every  $g \in \text{supp } \mu$ ,  $\|g\| \leq 1$ , which implies that for every  $g \in \Gamma_\mu$  (the closed semi-group generated by the support of  $\mu$ ),  $v(g) \leq \|g\| \leq 1$ . Moreover, using (2.2),  $v(g_k) = 2^{-k}$  and  $v(g_k^t) = 1/2$ . In particular,  $\tilde{\mu}$  admits a moment of order 1 while  $\mu$  does not, since  $\mathbb{E}(\log v(Y_1)) \leq \sum_{k \in \mathbb{N}} \frac{-k \log 2}{3k(k+1)} = -\infty$ .

For every integer  $n \geq 2$ , set  $\Lambda_n := \{Y_2 = \dots = Y_n = \text{Id}\}$ . Then,  $\mathbb{E}(\log v(A_n)) \leq \mathbb{E}(\log v(Y_1)\mathbf{1}_{\Lambda_n}) = 3^{-(n-1)}\mathbb{E}(\log v(Y_1)) = -\infty$ .

Similarly, even if  $\mu$  and  $\tilde{\mu}$  are strictly contracting and admit a moment of order 1, we may not have  $L^1$  convergence for the coefficients. For instance, let  $\mu$  be such that  $\mu(\{\text{Id}\}) = \mu(\{h\}) = 1/2$ . Then,  $\mu^{*n}(\{\text{Id}\}) \geq 2^{-n}$  and, with  $\{e_1, e_2\}$  the canonical basis of  $\mathbb{R}^2$ ,  $\mu^{*n}(\{g \in G : \langle e_1, ge_2 \rangle = 0\}) > 0$ , so that  $\mathbb{E}(\log \langle e_1, A_n e_2 \rangle) = -\infty$ .

## 5 The CLT and the asymptotic variance

In this section, we state and prove various CLTs. Those CLTs are proved in Hennion [15] by a slightly different approach (also based on a martingale-coboundary decomposition). Again, Hennion only proved result under moment conditions on  $\tilde{\mu}$  but, since the CLT is a result about convergence in law, it is easy to derive results under  $\mu$  from the ones under  $\tilde{\mu}$  and vice versa, using for instance item (iii) of Lemma 2.2 and the fact that  $Y_n \cdots Y_1$  has same law as  $Y_1 \cdots Y_n$ .

Our proof allows us to identify the asymptotic variance  $s^2$  in several ways and to characterize the fact that  $s^2 > 0$ . The obtained characterization is the same as in [4] and [5] but its proof does not require exponential moments as in those works.

We start by proving a martingale-coboundary decomposition. In the case of invertible matrices, such a decomposition was only available for  $p \geq 2$  while here it holds as soon as  $p \geq 1$ .

**Proposition 5.1.** *Assume that  $\mu$  is strictly contracting and admits a moment of order  $p \geq 1$ . There exists a continuous and bounded function  $\psi$  on  $X$  such that  $(\sigma(Y_n, A_{n-1} \cdot x) - \lambda_\mu + \psi(A_n \cdot x) - \psi(A_{n-1} \cdot x))_{n \in \mathbb{N}}$  is a sequence of martingale differences in  $L^p$ . If moreover  $W_0$  is a random variable with law  $\nu$ , independent of  $(Y_n)_{n \in \mathbb{N}}$ , then  $(\sigma(Y_n, A_{n-1} \cdot W_0) - \lambda_\mu + \psi(A_n \cdot W_0) - \psi(A_{n-1} \cdot W_0))_{n \in \mathbb{N}}$  is a stationary and ergodic sequence of martingale differences in  $L^p$ .*

**Remark.** The function  $\psi$  in the theorem is given by

$$\psi(x) := \sum_{n \geq 1} \left( \int_{G \times G} \sigma(g, g' \cdot x) d\mu(g) d\mu^{*(n-1)}(g') - \lambda_\mu \right). \quad (5.1)$$

**Proof.** Let  $\psi$  be given by (5.1). The fact that  $\psi$  is well-defined and continuous follows from Proposition 3.2.

Then, notice that

$$\sigma(g, x) - \lambda_\mu = \sigma(g, x) - \int_G \sigma(g', x) d\mu(g') + \int_G \sigma(g', x) d\mu(g') - \lambda_\mu$$

and, using the definition of  $\psi$ ,

$$\int_G \sigma(g, x) d\mu(g) - \lambda_\mu + \int_G \psi(g \cdot x) d\mu(g) = \psi(x).$$

Now,  $\left( \sigma(Y_n, A_{n-1} \cdot x) - \int_G \sigma(g, A_{n-1} \cdot x) d\mu(g) \right)_{n \in \mathbb{N}}$  is a sequence of martingale differences in  $L^p$  (notice that  $x \mapsto \int_G \sigma(g, x) d\mu(g)$  is bounded). Moreover,

$$\int_G \sigma(g, A_{n-1} \cdot x) d\mu(g) - \lambda_\mu + \psi(A_n \cdot x) - \psi(A_{n-1} \cdot x) = \psi(A_n \cdot x) - \int_G \psi(g A_{n-1} \cdot x) d\mu(g),$$

and the right-hand side defines a sequence of bounded martingale differences.

The final statement follows from the fact that  $((Y_n, A_{n-1} \cdot W_0))_{n \in \mathbb{N}}$  is a stationary and (uniquely) ergodic Markov chain.  $\square$

**Definition 5.1.** We say that a probability  $\mu$  on  $G$  is aperiodic if the group generated by  $\{\log \kappa(g) : g \in \Gamma_\mu\}$  is dense in  $\mathbb{R}$ .

**Proposition 5.2.** Assume that  $\mu$  is strictly contracting and that  $\mu$  admits a moment of order 2. Then, there exists  $s^2 \geq 0$  such that, with  $W_0$  as in Proposition 5.1,

$$\frac{1}{n} \mathbb{E}[(\sigma(A_n, W_0) - n\lambda_\mu)^2] \xrightarrow{n \rightarrow +\infty} s^2 \quad (5.2)$$

and  $\frac{1}{\sqrt{n}}(\sigma(A_n, W_0) - n\lambda_\mu) \Rightarrow \mathcal{N}(0, s^2)$ . In addition, if there do not exist  $m \in \mathbb{N}$  and  $\psi_m$  continuous on  $S^+$  such that

$$\sigma(g, x) - m\lambda_\mu = \psi_m(x) - \psi_m(g \cdot x) \quad \text{for } \mu^{\otimes m} \otimes \nu\text{-almost every } (g, x) \in G \times S^+, \quad (5.3)$$

then  $s^2 > 0$ . In particular, if  $\mu$  is aperiodic, then  $s^2 > 0$ .

**Remark.** Under the assumptions of the proposition we actually have the functional central limit theorem. Moreover, it is well known that the variance is given by

$$\begin{aligned} s^2 &= \mathbb{E}(\sigma(A_1, W_0)^2) + 2 \sum_{n \geq 2} \mathbb{E}(\sigma(A_1, W_0)\sigma(A_n, W_0)) \\ &= \int_{G \times S^+} \sigma^2(g, x) d\mu(g) d\nu(x) + 2 \sum_{n \geq 2} \int_{G^2 \times S^+} \sigma(g, x) \sigma(g'g, x) d\mu^{*(n-1)}(g') d\mu(g) d\nu(x). \end{aligned}$$

**Proof.** For every  $n \in \mathbb{N}$ , set  $D_n := \sigma(Y_n, A_{n-1} \cdot W_0) - \lambda_\mu + \psi(A_n \cdot W_0) - \psi(A_{n-1} \cdot W_0)$ . By Proposition 5.1,  $(D_n)_{n \in \mathbb{N}}$  is a stationary and ergodic sequence of martingale differences in  $L^2$ . In particular,  $(D_1 + \dots + D_n)/\sqrt{n} \Rightarrow \mathcal{N}(0, s^2)$ , with  $s^2 = \mathbb{E}(D_1^2) = \mathbb{E}((D_1 + \dots + D_n)^2)/n$ . Hence, the CLT with the description of the variance follows from the following reformulation of Proposition 5.1:

$$\sigma(A_n, W_0) - n\lambda_\mu = (D_1 + \dots + D_n) + \psi(W_0) - \psi(A_n \cdot W_0). \quad (5.4)$$

Assume now that  $s^2 = 0$ . Then

$$\int_G (\sigma(g, x) - \lambda_\mu - \psi(x) + \psi(g \cdot x))^2 d\mu(g) d\nu(x) = 0.$$

Hence, (5.3) holds with  $m = 1$  and  $\psi_1 = \psi$ . Let  $m > 1$ . Notice that  $\mu^{*m}$  is strictly contracting and admits a moment of order  $p$  and that the unique  $\mu^{*m}$ -invariant measure is the unique  $\mu$ -invariant measure. Notice also that  $\lambda_{\mu^{*m}} = m\lambda_\mu$ . Applying the above argument to  $\mu^{*m}$ , we infer that there exists a continuous  $\psi_m$  satisfying (5.3).

Using that  $\psi_m$  is continuous, we see that (5.3) holds for every  $g \in \text{supp } \mu^{*m}$  and every  $x \in \text{supp } \nu$ . Let  $g \in \text{supp } \mu^{*m} \subset \Gamma_\mu$ . Then,  $u_g \in \Lambda_\mu \subset \text{supp } \nu$  (recall that  $u_g$  has been defined before (3.1)). Since  $g \cdot u_g = u_g$  and  $\sigma(g, u_g) = \log \kappa(g)$ , we infer that  $\psi_m(g \cdot u_g) = \psi_m(u_g)$  and that  $\log \kappa(g) = m\lambda_\mu$ . Hence,  $\log \kappa(\Gamma_\mu) \subset \lambda_\mu \mathbb{N}$  and  $\mu$  cannot be aperiodic.  $\square$

Let us now give the CLT for  $\sigma(A_n, x)$ ,  $\log \|A_n\|$ ,  $\log v(A_n)$  and  $\log \kappa(A_n)$ . Below and in the rest of the section, we shall use the notation:  $\phi_s(t) = \mathbb{P}(sZ \leq t)$  with  $Z$  a standard normal variable.

**Proposition 5.3.** *Assume that  $\mu$  is strictly contracting and admits a moment of order 2. Then, the following limit exists*

$$s^2 := \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E}((\log \|A_n\| - n\lambda_\mu)^2), \quad (5.5)$$

and we even have

$$\begin{aligned}
s^2 &= \lim_{n \rightarrow +\infty} \mathbb{E}[(\sigma(A_n, W_0) - n\lambda_\mu)^2] \\
&= \lim_{n \rightarrow +\infty} \frac{1}{n} \sup_{x \in S^+} \mathbb{E}[(\sigma(A_n, x) - n\lambda_\mu)^2] \\
&= \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E}[(\log v(A_n) - n\lambda_\mu)^2],
\end{aligned}$$

and

$$s^2 = \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E}[(\log \kappa(A_n) - n\lambda_\mu)^2]. \quad (5.6)$$

Moreover the CLT in Proposition 5.2 also holds if we replace  $\sigma(A_n, W_0)$  with  $\sigma(A_n, x)$ ,  $\log \|A_n\|$ ,  $\log v(A_n)$  or  $\log \kappa(A_n)$  and we also have

$$\sup_{x \in S^+} \sup_{t \in \mathbb{R}} \left| \mathbb{P}(\sigma(A_n, x) - n\lambda_\mu \leq t\sqrt{n}) - \phi_s(t) \right| \xrightarrow{n \rightarrow +\infty} 0.$$

If we assume that  $\tilde{\mu}$  is strictly contracting and admits a moment of order 2 then the CLTs for  $(\log \|A_n\|)_{n \in \mathbb{N}}$  and  $(\log \kappa(A_n))_{n \in \mathbb{N}}$  still hold with  $s^2$  given by (5.5) (or equivalently by (5.6)).

**Remark.** When it is assumed that  $\tilde{\mu}$  admits a moment of order 2, we do not know whether  $s^2$  is also equal to any the above limits other than (5.5) or (5.6) (we even do not know whether the limits themselves exist) but we will see that the CLTs still hold.

**Proof.** We start with the case where  $\mu$  is strictly contracting and admits a moment of order 2. The different expressions of  $s^2$  follow from (5.2), Proposition 3.2 and the fact that for every real random variables  $U, V$ ,  $|\mathbb{E}(U^2) - \mathbb{E}(V^2)| \leq \|U - V\|_2(\|U\|_2 + \|V\|_2)$ . Next, note that we can deduce the CLT for  $(\sigma(A_n, x))_{n \in \mathbb{N}}$  by using (3.3). To get the CLT for  $(\log \|A_n\|)_{n \in \mathbb{N}}$ , it suffices to notice that, for any  $x \in S^+$ ,

$$\log \|A_n\| - \log v(A_n) \geq \log \|A_n\| - \sigma(A_n, x) \geq 0, \quad (5.7)$$

and to use the fact that by (3.4),  $\sup_{n \in \mathbb{N}} (\log \|A_n\| - \log v(A_n))$  is in  $L^2$ . The CLT for  $(\log v(A_n))_{n \in \mathbb{N}}$  follows from the CLT for  $(\log \|A_n\|)_{n \in \mathbb{N}}$  and (3.4). Finally, the CLT for  $(\log \kappa(A_n))_{n \in \mathbb{N}}$  follows from the fact that  $v(A_n) \leq \kappa(A_n) \leq \|A_n\|$  and (3.4). To get the last convergence, we use previous arguments and Inequality (5.8) below which is stated in [20, equation (1)]: Let  $U, V$  and  $R$  be random variables with  $|U - V| \leq R$   $\mathbb{P}$ -a.s. For any  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  and any  $\varepsilon > 0$ , we have

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(U \leq t) - \psi(t)| \leq \sup_{t \in \mathbb{R}} |\mathbb{P}(V \leq t) - \psi(t)| + \mathbb{P}(R > \varepsilon) + \sup_{t \in \mathbb{R}} |\psi(t - \varepsilon) - \psi(t + \varepsilon)|. \quad (5.8)$$

Assume now that  $\tilde{\mu}$  is strictly contracting and admits a moment of order 2. Applying the first part of Proposition 5.3 to  $\tilde{\mu}$ , we obtain a CLT for  $(\|Y_n^t \cdots Y_1^t\|)_{n \in \mathbb{N}}$  which, by item (iii) of



Lemma 2.2, implies a CLT for  $(\|Y_1 \cdots Y_n\|)_{n \in \mathbb{N}}$ . Similarly, since for any matrix  $\kappa(g^t) = \kappa(g)$ , we infer the convergence in law for  $(\log \kappa(A_n))_{n \in \mathbb{N}}$ .

The fact that  $s^2$  is still given by (5.5) or (5.6) also follows from the above arguments.  $\square$

We also have a (functional) CLT for the coefficients. As noticed in the previous section, one cannot expect in general to identify  $s^2$  thanks to the matrix coefficients as in Proposition 5.3.

**Proposition 5.4.** *Assume that  $\mu$  or  $\tilde{\mu}$  is strictly contracting and admits a moment of order 2. Then, with  $s^2$  be given either by (5.5) or (5.6),*

$$\begin{aligned} \sup_{x, y \in S^+} \sup_{t \in \mathbb{R}} \left| \mathbb{P}(\log \langle x, A_n y \rangle - n\lambda_\mu \leq t\sqrt{n}) - \phi_s(t) \right| &\xrightarrow{n \rightarrow +\infty} 0, \\ \sup_{t \in \mathbb{R}} \left| \mathbb{P}(\inf_{x, y \in S^+} \log \langle x, A_n y \rangle - n\lambda_\mu \leq t\sqrt{n}) - \phi_s(t) \right| &\xrightarrow{n \rightarrow +\infty} 0. \end{aligned} \quad (5.9)$$

In particular, we also have a CLT for  $(\sigma(A_n, W_0))_{n \in \mathbb{N}}$ ,  $(\log v(A_n))_{n \in \mathbb{N}}$  or  $(\sigma(A_n, x))_{n \in \mathbb{N}}$ .

**Proof.** We prove (5.9), the other convergences follow from the fact that for every  $u, v \in S^+$  and any  $n \in \mathbb{N}$ ,

$$\inf_{x, y \in S^+} \log \langle x, A_n y \rangle \leq \log \langle u, A_n v \rangle \leq \sigma(A_n, v) \leq \log \|A_n\|, \quad (5.10)$$

$$\inf_{x, y \in S^+} \log \langle x, A_n y \rangle \leq \log v(A_n) \leq \log \|A_n\|. \quad (5.11)$$

We start with the case where  $\tilde{\mu}$  is strictly contracting and admits a moment of order 2. We proceed as for the proof of Theorem 4.4. By Proposition 3.2 applied to  $\tilde{\mu}$ ,

$$\begin{aligned} &\sum_{n \in \mathbb{N}} \mathbb{P}(\sup_{y \in S^+} |\log \|Y_1^t \cdots Y_n^t\| - \log \|Y_1^t \cdots Y_n^t y\|| \geq \varepsilon \sqrt{n}) \\ &\leq \sum_{n \in \mathbb{N}} \mathbb{P}(\sup_{y \in S^+} \sup_{m \in \mathbb{N}} |\log \|Y_m^t \cdots Y_1^t\| - \log \|Y_m^t \cdots Y_1^t y\|| \geq \varepsilon \sqrt{n}) < \infty. \end{aligned}$$

In particular, since for any  $g \in G$ ,  $\|g\| \leq d\|g^t\|$ ,

$$\mathbb{P}(\sup_{y \in S^+} |\log \|A_n\| - \log \|Y_1^t \cdots Y_n^t y\|| \geq \varepsilon \sqrt{n}) \xrightarrow{n \rightarrow +\infty} 0. \quad (5.12)$$

To conclude it remains to use Inequality (5.8) below with  $U := (\inf_{x, y \in S^+} \log \langle x, A_n y \rangle) - n\lambda_\mu / \sqrt{n}$ ,  $V := (\log \|A_n\| - n\lambda_\mu) / \sqrt{n}$  and

$$R := (|U| + |V|)\mathbf{1}_{\{T > n\}} + |\log W| + \log d + \sup_{y \in S^+} |\log \|A_n\| - \log \|Y_1^t \cdots Y_n^t y\||,$$

where  $T$  is defined by (4.2) and  $W$  is the positive random variable defined in (4.5). By (4.5) again,  $|U - V| \leq R$  and (5.9) follows from Inequality (5.8), using Proposition 5.2 and the fact that  $\mathbb{P}(T < \infty) = 1$ .

Assume now that  $\mu$  is strictly contracting and admits a moment of order 2. Notice that, for every  $n \in \mathbb{N}$ ,  $\inf_{x,y \in S^+} \log \langle x, Y_n \cdots Y_1 y \rangle = \inf_{x,y \in S^+} \log \langle Y_1^t \cdots Y_n^t x, y \rangle$  and that the latter has the same law as  $\inf_{x,y \in S^+} \log \langle x, Y_n^t \cdots Y_1^t y \rangle$ . Hence, it suffices to apply the already proven part of the proposition to  $\tilde{\mu}$ , using (5.10) and (5.11).  $\square$

## 6 The almost sure invariance principle

**Theorem 6.1.** *Let  $p \geq 2$ . Assume that  $\mu$  is strictly contracting and admits a moment of order  $p$ . Let  $s^2$  be as in Proposition 5.2. Then, one can redefine the process  $(\sigma(A_n, W_0))_{n \in \mathbb{N}}$  on another probability space on which there exist iid variables  $(N_n)_{n \in \mathbb{N}}$  with law  $\mathcal{N}(0, s^2)$ , such that*

$$\begin{aligned} |\sigma(A_n, W_0) - n\lambda_\mu - (N_1 + \dots + N_n)| &= o(\sqrt{n \log \log n}) \quad \mathbb{P}\text{-a.s. if } p = 2 \\ \text{and } |\sigma(A_n, W_0) - n\lambda_\mu - (N_1 + \dots + N_n)| &= o(n^{1/p}) \quad \mathbb{P}\text{-a.s. if } p > 2 \end{aligned}$$

**Remark.** It is not necessary here that  $s^2 > 0$ .

**Proof.** When  $p > 2$ , the result follows from Theorem 1 of [10] by taking into account (3.2). The case  $p = 2$  follows from (5.4) and the ASIP for martingales with stationary and ergodic increments in  $L^2$ , see [23].  $\square$

Proceeding as in the proof of Theorem 4.2 (using in particular the argument yielding (6.1) below) and using Lemma 4.1 of [2], Proposition 3.2 and (5.7), one can prove that the above theorem holds if we replace  $(\sigma(A_n, W_0))_{n \in \mathbb{N}}$  with any of the following sequences:  $(\sigma(A_n, x))_{n \in \mathbb{N}}$  (for a given  $x \in S^+$ ),  $(\log \|A_n\|)_{n \in \mathbb{N}}$ ,  $(\log \kappa(A_n))_{n \in \mathbb{N}}$  or  $(\log v(A_n))_{n \in \mathbb{N}}$ .

Let us give the ASIP for the matrix coefficients.

**Theorem 6.2.** *Let  $p \geq 2$ . Assume that  $\mu$  is strictly contracting and that  $\mu$  and  $\tilde{\mu}$  admit a moment of order  $p$ . Then, for every  $x, y \in S^+$ , one can redefine the process  $(\log \langle y, A_n x \rangle)_{n \in \mathbb{N}}$  on another probability space on which there exist iid variables  $(N_n)_{n \in \mathbb{N}}$  with law  $\mathcal{N}(0, s^2)$ , such that*

$$\begin{aligned} |\log \langle y, A_n x \rangle - n\lambda_\mu - (N_1 + \dots + N_n)| &= o(\sqrt{n \log \log n}) \quad \mathbb{P}\text{-a.s. if } p = 2 \\ \text{and } |\log \langle y, A_n x \rangle - n\lambda_\mu - (N_1 + \dots + N_n)| &= o(n^{1/p}) \quad \mathbb{P}\text{-a.s. if } p > 2. \end{aligned}$$

**Proof.** We proceed as for the proof of Theorem 4.4. Since  $\tilde{\mu}$  almost admits a moment of order  $p \geq 1$ , using (3.3), for every  $\varepsilon > 0$ , we have

$$\sum_{n \geq 1} \mathbb{P} \left( \sup_{y \in S^+} \left| \log \|Y_1^t \cdots Y_n^t\| - \log \|Y_1^t \cdots Y_n^t y\| \right| \geq \varepsilon n^{1/p} \right) < \infty.$$

By the Borel-Cantelli lemma, we then infer that

$$\frac{\sup_{y \in S^+} \left| \log \|Y_1^t \cdots Y_n^t\| - \log \|Y_1^t \cdots Y_n^t y\| \right|}{n^{1/p}} \xrightarrow[n \rightarrow +\infty]{} 0 \quad \mathbb{P}\text{-a.s.} \quad (6.1)$$

We finish the proof by using similar arguments as those developed in the proof of Theorem 4.4 replacing (4.6) by (6.1).  $\square$

**Remark.** In the proof we used that  $\tilde{\mu}$  almost admits a moment of order  $p$ , hence it may seem that one can weaken the conditions of Theorem 6.2. It turns out that if  $\mu$  admits a moment of order  $p$  and if  $\tilde{\mu}$  almost admit a moment of order  $p$ , then  $\tilde{\mu}$  admits a moment of order  $p$ . This follows from the fact that for every  $g \in G$ ,  $v(g^t) \leq \|g^t\| \leq d\|g\|$  and  $\frac{1}{v(g^t)} \leq \frac{\|g\|}{v(g^t)} \frac{1}{v(g)}$ .

In the case of exponential moments, combining ideas from [10] and [8], it is possible to obtain logarithmic rates in the ASIP. This is done in the preprint [11] where it is proved that if  $\mu$  is strictly contracting and has a subexponential moment of order  $\gamma \in (0, 1]$  then the conclusion of Theorem 6.1 holds with rate  $O((\log n)^{2+1/\gamma})$ .

## 7 The Berry-Esseen theorem

In this section, we obtain the Berry-Esseen theorem for the norm cocycle and the matrix norm, when  $\mu$  admits a moment of order  $p \in [2, 3]$ . We get the rate of convergence  $n^{1-p/2}$  which corresponds to the rate in the setting of sums of iid random variables.

As far as we know the only rate of that type under polynomial moment condition has been obtained by Hennion and Hervé [16]. More precisely, they required a moment of order  $p > 4$  for  $\mu$  to get the rate  $n^{-1/2}$ .

We also obtain Berry-Esseen type results (with possibly suboptimal rates) for the spectral radius and the quantity  $\log v(A_n)$  under stronger moment assumptions. In addition, we get Berry-Esseen type results for the matrix coefficients under exponential moment conditions. Finally, assuming that  $\mu$  has a moment moment of order 3 and satisfies an extra (quite restrictive) condition, we prove that the spectral radius and the matrix coefficients satisfy Berry-Esseen type estimates with rate of order  $n^{-1/2}$ . The latter result has been obtained independently by Xiao et al. [27] by a completely different manner. Note that the method of Xiao et al. [27] allows them to obtain more precise results such as Edgeworth expansions.

In this section, we use the notation  $\phi_s(t) = \mathbb{P}(sZ \leq t)$  with  $Z$  a standard normal variable.

## 7.1 Berry-Esseen for the norm cocycle and the matrix norm

**Theorem 7.1.** *Let  $p \in (2, 3]$ . Assume that  $\mu$  is strictly contracting and admits a moment of order  $p$ . Assume that  $s^2 > 0$  with  $s^2$  as in Proposition 5.2. Then, setting  $v_n = \left(\frac{1}{n}\right)^{p/2-1}$ , we have*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(\sigma(A_n, W_0) - n\lambda_\mu \leq t\sqrt{n}) - \phi_s(t) \right| = O(v_n), \quad (7.1)$$

$$\sup_{x \in S^+} \sup_{t \in \mathbb{R}} \left| \mathbb{P}(\sigma(A_n, x) - n\lambda_\mu \leq t\sqrt{n}) - \phi_s(t) \right| = O(v_n), \quad (7.2)$$

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(\log \|A_n\| - n\lambda_\mu \leq t\sqrt{n}) - \phi_s(t) \right| = O(v_n). \quad (7.3)$$

Hennion and Hervé [16] obtained the rate (7.1) with  $v_n = 1/\sqrt{n}$  when  $p > 4$ , see their Theorem 3.3.

**Proof.** The proof of (7.1) and of (7.2) follow the one of Theorem 2.1 of [12] with  $T = n^{p/2-1}$ , using the estimate (3.2) instead of their estimate [12, (3.12)]. Indeed, using (3.2), one can prove that for  $R_1$  and  $U_2 - U_2^*$  defined in [12, (3.4), (3.5) and (4.15)] we have, for any  $p \geq 2$ ,  $\|R_1\|_p = O(1)$  and  $\|U_2 - U_2^*\|_p = O(1)$  provided that  $\mu$  has a moment of order  $p$ , whereas in the case of  $GL_d(\mathbb{R})$ , under the same moment condition on  $\mu$ , the above quantities were of order  $m^{1/p}$  in [12] (see their Lemmas 4.3 and 4.6). Consequently for positive matrices, analyzing the proofs of Lemmas 4.10 and 4.11 of [12], we infer that when  $\mu$  has a moment of order  $q = r$ , the inequalities stated in [12, Lemmas 4.10 and 4.11] hold by replacing their right hand sides by  $|t|^r/m^{(p-2)/2} + |t|/m^{1/2+\eta}$  (with  $\eta > 0$ ). Following the proof of [12, Theorem 2.1] by taking into account the previous upper bounds and selecting  $T = n^{p/2-1}$ , the result follows.

The proof of (7.3) requires some extra arguments. Notice that for every  $x \in S^+$  and every  $n \in \mathbb{N}$ ,  $\|A_n x\| = \langle e, A_n x \rangle = \langle A_n^t e, x \rangle$  and that, by items (ii) and (iii) of Lemma 2.2,  $\|A_n^t e\|/d \leq \|A_n\| \leq d^2 \|A_n^t e\|$ . Hence,

$$\int_{S^+} \left| \log \|A_n\| - \log \|A_n x\| \right| d\nu(x) \leq 2 \log d + \sup_{y \in S^+} |\log \langle y, x \rangle| d\nu(x) < \infty. \quad (7.4)$$

Hence, we are in position to redo the proof of the bound [12, (2.4)] (see their Section 3.1.2) since (7.4) is the precise analogue of [12, (3.30)].  $\square$

**Remarks.** By some arguments already mentionned, (7.3) also holds if  $\tilde{\mu}$  is strictly contracting and admits a moment of order  $p \in (2, 3]$ . Let us notice that (7.1) follows also from Theorem 2.3 of [18], since the Assumptions 2.1 there are satisfied due to the exponential convergence of the coefficients  $\delta_{\infty,p}$  in Proposition 3.2.

Finally, let us mention that Xiao et al. [27] obtained (7.2) and (7.3) for  $p = 3$  under their condition A2, see their Theorem 1.2 (see also Theorem 2.1 of [26] by the same authors, when

$\mu$  has a subexponential moment). It easily follows from Lemma 2.1 of [27] that their condition A2 is equivalent to the condition used in Theorem 7.8 below (with  $r = 1$ ). However, it should be emphasized that [27] and [26] provide Berry-Esseen type results including “target functions” or Edgeworth expansions, see their papers for further details. Our approach does not seem to allow to obtain such results.

## 7.2 Berry-Esseen for the spectral radius and the matrix coefficients

**Proposition 7.2.** *Let  $p \in (2, 3]$ . Assume that  $\mu$  is strictly contracting, admits a moment of order  $p$  and almost admits a moment of order  $q \in [p, \max(p, (p-2)/(3-p))]$ . Assume that  $s^2 > 0$  with  $s^2$  as in Proposition 5.2. Set  $v_n = \left(\frac{1}{n}\right)^{p/2-1}$  if  $p \in (2, 1 + \sqrt{3}]$  and  $v_n = \left(\frac{1}{n}\right)^{q/2(q+1)}$  if  $p \in (1 + \sqrt{3}, 3]$ . Then,*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(\log v(A_n) - n\lambda_\mu \leq t\sqrt{n}) - \phi_s(t) \right| = O(v_n) \quad (7.5)$$

and

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(\log \kappa(A_n) - n\lambda_\mu \leq t\sqrt{n}) - \phi_s(t) \right| = O(v_n). \quad (7.6)$$

**Remark.** When  $p \leq 1 + \sqrt{3}$  the condition on  $q$  reads  $q = p$  hence is satisfied. When  $p = 3$  the condition on  $q$  reads  $q \geq p$ . (7.6) also hold if  $\tilde{\mu}$  satisfies the assumptions of the proposition, by the arguments developed in the proof of Proposition 5.3.

**Proof.** Since  $\mu$  admits a moment of order  $p$ , by Proposition 3.2 and Markov’s inequality, there exists  $C > 0$  such that for every  $x > 0$  and every  $n \in \mathbb{N}$ ,  $\mathbb{P}(|\log \|A_n\| - \log v(A_n)| \geq x) \leq C/x^q$ . Hence, (7.5) follows from Theorem 7.1 and Lemma 7.3 below (which is [13, Lemma 2]) with  $U_n = \log v(A_n) - n\lambda_\mu$ ,  $V_n = \log \|A_n\| - n\lambda_\mu$ ,  $R_n = \log v(A_n) - \log \|A_n\|$ , and (up to some multiplicative constants)  $a_n = n^{(p-2)/2}$ ,  $b_n = n^{q/2(q+1)}$  and  $c_n = (\sqrt{n}/b_n)^q$ . Finally, (7.6) follows from the fact that  $v(A_n) \leq \kappa(A_n) \leq \|A_n\|$  and the same arguments as above.

**Lemma 7.3.** *Let  $(U_n)_{n \in \mathbb{N}}$ ,  $(V_n)_{n \in \mathbb{N}}$  and  $(R_n)_{n \in \mathbb{N}}$  be three sequences of random variables. Assume that  $|U_n - V_n| \leq |R_n|$   $\mathbb{P}$ -a.s. and that there exist three sequences of positive numbers  $(a_n)_{n \in \mathbb{N}}$ ,  $(b_n)_{n \in \mathbb{N}}$  and  $(c_n)_{n \in \mathbb{N}}$  going to infinity as  $n \rightarrow \infty$ , and a positive constant  $s$  such that, for any integer  $n$ ,*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(V_n \leq t\sqrt{n}) - \phi(t/s) \right| \leq \frac{1}{a_n}, \quad \text{and} \quad \mathbb{P}(|R_n| \geq \sqrt{2\pi n s}/b_n) \leq \frac{1}{c_n}.$$

Then, for any integer  $n$ ,

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(U_n \leq t\sqrt{n}) - \phi(t/s) \right| \leq \frac{1}{a_n} + \frac{1}{b_n} + \frac{1}{c_n}.$$

□

We shall now improve the rates under a strengthening of our integrability condition. The proof will rely on the following large deviation result.

**Lemma 7.4.** *Assume that  $\mu$  is strictly contracting and almost admits some exponential moment of order  $\gamma \in (0, 1]$ . Then, there exist  $\eta, \delta > 0$  such that*

$$\mathbb{P}(\max_{1 \leq k \leq n} |\log v(A_k) - \log \|A_k\|| \geq \eta n) \leq e^{-\delta n^\gamma}.$$

**Proof.** For every  $n \in \mathbb{N}$ , using that  $\|\cdot\|$  is submultiplicative and that  $v$  is supermultiplicative, we see that, setting  $\tau := \mathbb{E}(\log \|Y_1\|/v(Y_1))$ ,

$$\max_{1 \leq k \leq n} |(\log(\|A_k\|) - \log(v(A_k)))| \leq \max_{1 \leq k \leq n} \left| \sum_{i=1}^k [\log(\|Y_i\|/v(Y_i)) - \tau] \right| + n\tau.$$

Then the desired result follows from Theorem 2.1 of [14], see their estimate (2.7) applied in the independent case (in particular the quantities in (2.3) and (2.4) of [14] are identical). □

**Proposition 7.5.** *Assume that  $\mu$  is strictly contracting, admits a moment of order  $p \in (2, 3]$  and almost admits an exponential moment of order  $\gamma \in (0, 1]$ . Assume that  $s^2 > 0$  with  $s^2$  as in Proposition 5.2. Set  $v_n = \frac{(\log n)^{1/\gamma}}{n^{(p-2)/2}}$ . Then,*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(\log v(A_n) - n\lambda_\mu \leq t\sqrt{n}) - \phi_s(t) \right| = O(v_n)$$

and

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(\log \kappa(A_n) - n\lambda_\mu \leq t\sqrt{n}) - \phi_s(t) \right| = O(v_n). \quad (7.7)$$

**Remark.** (7.7) also holds if  $\tilde{\mu}$  satisfies the assumptions of the proposition.

**Proof.** Let  $\varepsilon \in (0, 1)$  be such that (3.5) holds. Let  $x, y \in S^+$ . Let  $n \in \mathbb{N}$ . Let  $\omega \in \Omega$ . Let  $1 \leq m < [n/r]$  be such that  $c(Y_{mr} \cdots Y_{(m-1)r+1})(\omega) \leq 1 - \varepsilon$ . Using the cocycle property and several items of Proposition 2.1 (in particular item (iv)), we see that

$$\begin{aligned} & |\sigma(A_n, x) - \sigma(A_n, y)| \\ & \leq |\sigma(Y_n \cdots Y_{mr+1}, A_{mr} \cdot x) - \sigma(Y_n \cdots Y_{mr+1}, A_{mr} \cdot y)| + |\sigma(A_{mr}, x) - \sigma(A_{mr}, y)| \\ & \leq 2 \ln(1/(1 - d(A_{mr} \cdot x, A_{mr} \cdot y))) + \log \|A_{mr}\| - \log v(A_{mr}) \\ & \leq 2 \ln(1/\varepsilon) + \log \|A_{mr}\| - \log v(A_{mr}). \end{aligned}$$

Define

$$\Gamma_m := \{\exists k \in 1, \dots, m : c(Y_{kr} \cdots Y_{(k-1)r+1}) \leq 1 - \varepsilon\}. \quad (7.8)$$

Taking the supremum over  $x$  and the infimum over  $y$ , we infer that on  $\Gamma_m$ ,

$$\log \|A_n\| - \log v(A_n) \leq 2 \ln(1/\varepsilon) + \max_{1 \leq k \leq m} (\log \|A_{kr}\| - \log v(A_{kr})).$$

Hence, for  $\eta m \geq 4 \ln(1/\varepsilon)$ , using Lemma 7.4, we have

$$\begin{aligned} \mathbb{P}(\log \|A_n\| - \log v(A_n) \geq \eta m) &\leq \mathbb{P}(\Gamma_m^c) + \mathbb{P}\left(\max_{1 \leq k \leq m} (\log \|A_{kr}\| - \log v(A_{kr})) \geq \eta m/2\right) \\ &\leq \alpha^m + C_\eta e^{-\delta_\eta m^\gamma}, \end{aligned}$$

where  $\alpha := \mathbb{P}(\Gamma_1^c)$ .

Taking  $m = \lceil C(\log n)^{1/\gamma} \rceil + 1$ , with  $C$  large enough, we infer that the right-hand side is bounded by  $D/\sqrt{n}$ , and we conclude using Theorem 7.1 and Lemma 7.3 applied with  $U_n = \log \|A_n\|$ ,  $V_n = \log v(A_n)$ ,  $R_n = \log \|A_n\| - \log v(A_n)$  and (up to some multiplicative constants)  $a_n = n^{(p-2)/2}$ ,  $b_n = \sqrt{n}/(\log n)^{1/\gamma}$  and  $c_n = \sqrt{n}$ .  $\square$

**Proposition 7.6.** *Let  $p \in (2, 3]$ . Assume that  $\mu$  is strictly contracting and admits a moment of order  $p$ . Assume that  $s^2 > 0$  with  $s^2$  as in Proposition 5.2. Assume moreover that  $\tilde{\mu}$  almost admits a moment of order  $q \in [p, \max(p, (p-2)/(3-p))]$  (resp. an exponential moment of order  $\gamma \in (0, 1]$ ). Then, for every  $x \in S^+$ , the conclusion of Proposition 7.2 (resp. Proposition 7.5) holds with  $\inf_{y \in S^+} \langle y, A_n x \rangle$  in place of  $\kappa(A_n)$ .*

**Proof.** For every  $0 < \delta \leq 1$ , define

$$G_\delta := \{g \in G : \langle y, g \cdot x \rangle \geq \delta \quad \forall x, y \in S^+\}. \quad (7.9)$$

Notice that  $\cup_{\delta \in (0, 1]} G_\delta = G^+$ , so that when  $\mu$  is strictly contracting, there exist  $r \geq 1$  and  $\delta \in (0, 1]$  for which  $\mu^{*r}(G_\delta) > 0$ .

Let  $p_0 = \mathbb{P}(\langle y, A_r \cdot x \rangle < 1/n_0 : x, y \in S^+)$ . Note that  $p_0 \in [0, 1)$  for  $n_0$  large enough.

For  $n > r$ , let  $1 \leq m \leq \lfloor n/r \rfloor$  be a positive integer.

Next note that, for any  $g \in G_\delta$  and any  $g' \in G$  and any  $x, y \in S^+$ , setting  $x' = g'x/\|g'x\|$ ,

$$\langle y, gg' \cdot x \rangle = \left\langle y, \frac{gg'x}{\|gg'x\|} \right\rangle = \langle y, g \cdot x' \rangle \geq \delta. \quad (7.10)$$

This implies that if, for some integer  $k \in [m, \lfloor n/r \rfloor]$   $Y_{kr} \dots Y_{(k-1)r+1} \in G_{1/n_0}$ , for  $x, y \in S^+$ , we have

$$\langle y, A_n x \rangle \geq \langle Y_{kr+1}^t \dots Y_n^t y, \frac{A_{kr}x}{\|A_{kr}x\|} \rangle \|A_{kr}x\| \geq (1/n_0) \|Y_{kr+1}^t \dots Y_n^t y\| \frac{\|A_n x\|}{\|Y_n \dots Y_{kr+1}\|}. \quad (7.11)$$

Therefore, if we define

$$\Delta_{n,m} := \{\omega \in \Omega \mid \exists k \in [m, [n/r] - 1] : (Y_{kr} \cdots Y_{(k-1)r+1})(\omega) \in G_{1/n_0}\}, \quad (7.12)$$

we get that, on the set  $\Delta_{n,m}$  and using  $\|g\| \leq d\|g^t\|$ ,

$$\begin{aligned} & \inf_{x, y \in S^+} (\log \langle y, A_n x \rangle - \log \|A_n x\|) \\ & \geq -\log(n_0) - \log d + \min_{mr \leq \ell \leq n-1} (\log v(Y_{\ell+1}^t \cdots Y_n^t) - \log \|Y_{\ell+1}^t \cdots Y_n^t\|). \end{aligned} \quad (7.13)$$

Notice that all the above quantities are non positive and that  $\min_{mr \leq \ell \leq n} (\log v(Y_{\ell+1}^t \cdots Y_n^t) - \log \|Y_{\ell+1}^t \cdots Y_n^t\|)$  has the same law as  $\min_{1 \leq \ell \leq n-mr} (\log v(Y_\ell^t \cdots Y_1^t) - \log \|Y_\ell^t \cdots Y_1^t\|)$ .

Note that

$$\mathbb{P}(\Delta_{n,m}^c) = p_0^{[n/r-m]}. \quad (7.14)$$

Next, assume that  $\tilde{\mu}$  almost admits a moment of order  $q$ , with  $q$  as in the proposition and take  $m = 1$ . Combining the above computations, for every  $a > \log n_0 + \log d$  and every  $x \in S^+$ , we have

$$\begin{aligned} & \mathbb{P}\left(\left| \inf_{y \in S^+} \log \langle y, A_n x \rangle - \log \|A_n x\| \right| \geq 2a\right) \\ & \leq \mathbb{P}(\Delta_{n,m}^c) + \frac{\mathbb{E}\left(\sup_{n \in \mathbb{N}} \left| \log(v(Y_n^t \cdots Y_1^t) - \log \|Y_n^t \cdots Y_1^t\|) \right|^q\right)}{a^q}. \end{aligned}$$

Hence, using Proposition 3.2, one may finish the proof as the proof of Proposition 7.2.

Assume now that  $\tilde{\mu}$  almost admits some exponential moment of order  $\gamma \in (0, 1]$  and let  $x \in S^+$  be fixed. We wish to apply Theorem 7.1 combined with Lemma 7.3 applied to  $U_n = \log \|A_n x\|$ ,  $V_n = \inf_{y \in S^+} \log \langle y, A_n x \rangle$ ,  $R_n = U_n - V_n$  and, up to some multiplicative constants, the sequences  $a_n$ ,  $b_n$  and  $c_n$  given at the end of the proof of Proposition 7.5.

To do so, it is enough to find  $K > 0$  large enough (independent from  $n$ ) and  $m$ , such that

$$\mathbb{P}(\Delta_{n,m}^c) + (\mathbb{P}(\max_{1 \leq \ell \leq n-mr} |\log \|Y_\ell^t \cdots Y_1^t\| - \log v(Y_\ell^t \cdots Y_1^t)| \geq \eta[K(\log n)^{1/\gamma}]) = O(1/\sqrt{n}), \quad (7.15)$$

where  $\eta$  is given in Lemma 7.4.

Taking  $m = [(n - K(\log n)^{1/\gamma})/r] - 1$  and using Lemma 7.4 we have

$$\mathbb{P}(\max_{1 \leq \ell \leq n-mr} |\log \|Y_\ell^t \cdots Y_1^t\| - \log v(Y_\ell^t \cdots Y_1^t)| \geq \eta[K(\log n)^{1/\gamma}]) \leq e^{-\delta[(K(\log n)^{1/\gamma})]^\gamma}.$$

To conclude one may take  $K = \max((2\delta)^{-1/\gamma}, 2^{-1}r(\log(1/p_0))^{-1})$  that implies also that  $\mathbb{P}(\Delta_{n,m}^c) = O(1/\sqrt{n})$ .  $\square$

To get the results of Proposition 7.6 for the quantity  $\inf_{x, y \in S^+} \langle y, A_n x \rangle$  instead of  $\inf_{y \in S^+} \langle y, A_n x \rangle$ , we make an additional assumption on  $\mu$ .



**Proposition 7.7.** *Let  $p \in (2, 3]$ . Assume that  $\mu$  is strictly contracting and admits a moment of order  $p$ . Assume that  $s^2 > 0$  with  $s^2$  as in Proposition 5.2. Assume moreover that  $\mu$  and  $\tilde{\mu}$  almost admit a moment of order  $q \in [p, \max(p, (p-2)/(3-p))]$  (resp. an exponential moment of order  $\gamma \in (0, 1]$ ). Then the conclusion of Proposition 7.2 (resp. Proposition 7.5) holds with  $\inf_{x, y \in S^+} \langle y, A_n x \rangle$  in place of  $\kappa(A_n)$ .*

**Proof.** The proof is very close the proof of Proposition 7.6, hence we only give the main step. We keep the same notations. Starting from (7.11), we get that, on the set where  $Y_{mr} \cdots Y_{(m-1)r+1} \in G_{1/n_0}$  (recall that  $G_{1/n_0}$  is defined in (7.9)),

$$\begin{aligned} \inf_{x, y \in S^+} (\log \langle y, A_n x \rangle) - \log \|A_n\| &\geq \\ &- \log n_0 - \log d + (\log v(A_n) - \log \|A_n\|) + (\log v(Y_{mr+1}^t \cdots Y_n^t) - \log \|Y_{mr+1}^t \cdots Y_n^t\|). \end{aligned}$$

Hence, the only difference with the proof of Proposition 7.6 is that we need to handle the term  $\log v(A_n) - \log \|A_n\|$  but this may be done, as in the proof of Proposition 7.6, using Lemma 3.2 when  $\mu$  almost admits a moment of order  $q$  and Lemma 7.4 when  $\mu$  almost admits some exponential moment of order  $\gamma \in (0, 1]$ .  $\square$

We shall now obtain the rate  $O(1/\sqrt{n})$  for the spectral radius and the coefficients under a much stronger condition on  $\mu$ , also considered in [27]. Actually, in Theorem 1.2 of [27] the authors obtain the results of Theorem 7.8 for  $p = 3$ . Recall that the set  $G_\delta$  has been defined in (7.9).

**Theorem 7.8.** *Let  $p \in (2, 3]$ . Assume that  $\mu$  is strictly contracting and admits a moment of order  $p$ . Assume that  $s^2 > 0$  with  $s^2$  as in Proposition 5.2. Assume that there exist  $0 < \delta \leq 1$  and  $r \in \mathbb{N}$  such that  $\mu^{*r}(G_\delta) = 1$ . Then the conclusion of Theorem 7.1 holds with  $\log(\inf_{y \in S^+} \langle y, A_n x \rangle)$  (for every  $x \in S^+$ ),  $\log v(A_n^t)$ , or  $\log \kappa(A_n)$  instead of  $\log \|A_n\|$ .*

**Proof.** By assumption, for every  $n \geq r$  and  $x \in S^+$ , using that  $\frac{A_n x}{\|A_n x\|} = (Y_n \cdots Y_{n+1-r}) \cdot (A_{n-r} x)$ , we have, for every  $x, y \in S^+$

$$1 \geq \frac{\langle y, A_n x \rangle}{\|A_n x\|} \geq \delta \quad \mathbb{P}\text{-a.s.} \quad (7.16)$$

Then, the result for  $(\inf_{y \in S^+} \langle y, A_n x \rangle)_{n \in \mathbb{N}}$  follows from Theorem 7.1 and the fact that  $\|A_n\| \geq \inf_{y \in S^+} \langle y, A_n x \rangle \geq \delta \|A_n x\|$ .

To prove the remaining cases, first observe that  $G_\delta$  is a right ideal, i.e.  $G_\delta G \subset G_\delta$ , which was already observed in [25] (see also (7.10) for a short proof). Hence, by Lemma 7.10 below, there exist  $C > 0$  and  $0 \leq \gamma < 1$ , such that for every  $n \geq r$ ,  $\mu^{*n}(G_{C, \gamma}) = 1$ , with  $G_{C, \gamma}$  defined by (7.17). In particular, for every  $n \geq r$ ,  $v(A_n^t) \geq \|A_n\|/C$ .

This proves the result for  $(v(A_n^t))_{n \in \mathbb{N}}$ , by Theorem 7.1, since  $d\|A_n\| \geq v(A_n^t)$ . Then, the last case follows since  $\|A_n\| \geq \kappa(A_n) = \kappa(A_n^t) \geq v(A_n^t)$ , for every  $n \in \mathbb{N}$ .  $\square$

To get the conclusion of Theorem 7.8 for the quantities  $\inf_{x, y \in S^+} \langle y, A_n x \rangle$  and  $\log(v(A_n))$  instead of  $\inf_{y \in S^+} \langle y, A_n x \rangle$  and  $\log(v(A_n^t))$ , we impose the same strong condition on  $\mu$  and  $\tilde{\mu}$ .

**Theorem 7.9.** *Let  $p \in (2, 3]$ . Assume that  $\mu$  is strictly contracting and admits a moment of order  $p$ . Assume that  $s^2 > 0$  with  $s^2$  as in Proposition 5.2. Assume that there exist  $0 < \delta \leq 1$  and  $r \in \mathbb{N}$  such that  $\mu^{*r}(G_\delta) = 1$  and  $\tilde{\mu}^{*r}(G_\delta) = 1$ . Then the conclusion of Theorem 7.1 holds with  $\log(\inf_{x, y \in S^+} \langle y, A_n x \rangle)$ , or  $\log v(A_n)$  instead of  $\log \|A_n\|$ .*

**Remark.** It follows from Lemma 2.1 of [27] that if  $\mu^{*r}(G_\delta) = 1$  and  $\tilde{\mu}^{*r}(G_\delta) = 1$ , all matrices of the support of  $\mu^{*r}$  satisfy Item (1.1) of their condition A1.

**Proof.** The proof for  $(v(A_n))_{n \in \mathbb{N}}$  may be done as the proof for  $(v(A_n^t))_{n \in \mathbb{N}}$  in the previous theorem using that  $\tilde{\mu}^r(G_\delta) = 1$ . The case of  $(\inf_{x, y \in S^+} \langle y, A_n x \rangle)_{n \in \mathbb{N}}$  then follows since, by (7.16),  $\|A_n\| \geq \inf_{x, y \in S^+} \langle y, A_n x \rangle \geq \delta v(A_n)$  for every  $n \in \mathbb{N}$ .  $\square$

We now give a condition that is equivalent to the condition  $\mu^{*r}(G_\delta) = 1$  (or  $\mu^{*r}(G_\delta) > 0$ ). An equivalent condition, specific to the case of positive matrices (hence not valid in the general situation considered in Section 10), has been obtained in [27], see their Lemma 2.1.

For every  $C > 0$  and  $0 \leq \gamma < 1$ , set

$$G_{C, \gamma} := \{g \in G : c(g) \leq \gamma \text{ and } \|g\| \leq Cv(g^t)\}. \quad (7.17)$$

**Lemma 7.10.** *For every  $0 < \delta \leq 1$ , there exist  $0 \leq \gamma < 1$  and  $C > 0$  such that  $G_\delta \subset G_{C, \gamma}$ . Conversely, for every  $0 \leq \gamma' < 1$  and every  $C' > 0$  there exists  $0 < \delta' \leq 1$  such that  $G_{C', \gamma'} \subset G_{\delta'}$ . Hence, there exists  $0 < \delta \leq 1$  such that  $\mu(G_\delta) > 0$  (resp.  $\mu(G_\delta) = 1$ ) if and only if there exists  $0 \leq \gamma < 1$  and  $C > 0$  such that  $\mu(G_{C, \gamma}) > 0$  (resp.  $\mu(G_{C, \gamma}) = 1$ ).*

**Proof.** The proof relies on the following observations: for every  $x \in S^+$ ,  $\langle x, ge \rangle = \|g^t x\|$  and  $\|g^t x\|/\|g\| \geq \langle x, g \cdot e \rangle/d \geq \|g^t x\|/(d\|g\|)$ .

Let  $g \in G_\delta$ , with  $\delta > 0$ . By the previous computations,  $\|g\| \leq v(g^t)/\delta$ .

Let  $x, y \in S^+$ . Let us bound  $d(g \cdot x, g \cdot y)$ . For every  $u \in S^+$ , we have

$$\delta \langle u, g \cdot y \rangle \leq \delta \leq \langle u, g \cdot x \rangle.$$

This implies that  $m(g \cdot x, g \cdot y) \geq \delta$  (notice that then we must have  $\delta \leq 1$ ). Similarly,  $m(g \cdot y, g \cdot x) \geq \delta$  and  $d(g \cdot x, g \cdot y) \leq \frac{1-\delta^2}{1+\delta^2} =: \gamma < 1$ . So,  $G_\delta \subset G_{1/\delta, \gamma}$ .

Let  $0 \leq \gamma < 1$  and  $C > 0$ . Let  $g \in G_{C,\gamma}$ . Let  $x, y \in S^+$ . Notice that  $m(g \cdot x, g \cdot y) \leq 1$ . Hence,  $\gamma \geq d(g \cdot x, g \cdot y) \geq \frac{1-m(y,x)}{1+m(y,x)}$  and  $m(y, x) \geq \frac{1-\gamma}{1+\gamma}$ . We infer that  $g \cdot y - \frac{1-\gamma}{1+\gamma}g \cdot x$  has non negative coordinates. Taking,  $x = e$ , we see that for every  $u \in S^+$ ,

$$\langle u, g \cdot y \rangle \geq \frac{1-\gamma}{1+\gamma} \langle u, g \cdot e \rangle \geq \frac{1-\gamma}{1+\gamma} \|g^t x\| / (d\|g\|) \geq \frac{1-\gamma}{Cd(1+\gamma)}.$$

□

## 8 Regularity of the invariant measure

We prove here regularity properties of the invariant measure under various moment conditions.

**Theorem 8.1.** *Assume that  $\tilde{\mu}$  is strictly contracting and almost admits a moment of order  $p \geq 1$ . Then*

$$\int_{S^+} \sup_{y \in S^+} |\log \langle y, x \rangle|^p d\nu(x) < \infty. \quad (8.1)$$

**Remark.** In the case of invertible matrices, Benoist and Quint [1] proved that if  $\mu$  has a moment of order  $p > 1$ , then  $\sup_{y \in X} \int_X |\log \langle y, x \rangle|^{p-1} d\nu(x) < \infty$ .

**Proof.** By Fubini's theorem, it is enough to prove that

$$\sum_{n \geq 1} n^{p-1} \nu(\{x \in S^+ : \sup_{y \in S^+} |\log \langle y, x \rangle| \geq cn\}) < \infty,$$

for some  $c > 0$ . Using that  $\nu$  is  $\mu$ -invariant, it suffices to prove that

$$\sum_{n \geq 1} n^{p-1} \mathbb{P}\left(\sup_{x, y \in S^+} |\log \langle y, A_n \cdot x \rangle| \geq cn\right) < \infty. \quad (8.2)$$

Now, on  $\Delta_{n,1}$  (recall its definition (7.12)), by (7.13), we have

$$|\log \langle y, A_n \cdot x \rangle| \leq \log n_0 + \log d + \max_{1 \leq k \leq n} |\log v(Y_k^t \cdots Y_n^t) - \log \|Y_k^t \cdots Y_n^t\||. \quad (8.3)$$

Since  $\mathbb{P}(\Delta_{n,1}^c) = \eta^{[n/r-1]}$  with  $\eta \in [0, 1)$ , it is clear that (8.2) will hold if we can find some  $c > 0$  such that

$$\sum_{n \geq 1} n^{p-1} \mathbb{P}\left(\max_{1 \leq k \leq n} |\log v(Y_k^t \cdots Y_n^t) - \log \|Y_k^t \cdots Y_n^t\|| \geq cn\right) < \infty. \quad (8.4)$$

By Proposition 3.2, since  $\tilde{\mu}$  almost admits a moment of order  $p$ ,  $\sup_{n \geq 1} |\log v(Y_n^t \cdots Y_1^t) - \log \|Y_n^t \cdots Y_1^t\|| \in L^p$  which yields (8.4). □

**Theorem 8.2.** *Assume that  $\tilde{\mu}$  is strictly contracting and almost admits an exponential moment of order  $\gamma \in (0, 1]$ . Then, there exists  $\delta > 0$  such that*

$$\int_{S^+} \sup_{y \in S^+} e^{\delta |\log \langle y, x \rangle|^\gamma} d\nu(x) < \infty. \quad (8.5)$$

**Remark.** Inequality (8.5) has been proved in Proposition 3.3 of [26] with the supremum outside the integral, under stronger conditions. On another hand, they also obtained (8.5) with respect to their measures  $\nu_s$ , see [26] for the definition.

**Proof.** Proceeding as above, the theorem will be proved if we can show that there exist  $\delta, \eta > 0$  such that

$$\sum_{n \geq 1} e^{\delta n^\gamma} \mathbb{P} \left( \sup_{x, y \in S^+} |\log \langle y, A_n \cdot x \rangle| \geq \eta n \right) < \infty. \quad (8.6)$$

We conclude thanks to (8.3) and Lemma 7.4.  $\square$

## 9 Deviation inequalities

We now provide deviation estimates, in the style of Baum-Katz.

**Proposition 9.1.** *Assume that  $\mu$  is strictly contracting and admits a moment of order  $p \geq 1$ . Let  $\alpha \in (1/2, 1]$  such that  $\alpha \geq 1/p$ . For any  $\varepsilon > 0$ , we have*

$$\sum_{n \geq 1} n^{\alpha p - 2} \sup_{x \in S^+} \mathbb{P} \left( \max_{1 \leq k \leq n} |\sigma(A_k, x) - k\lambda_\mu| \geq n^\alpha \varepsilon \right) < \infty. \quad (9.1)$$

**Remark.** Using Proposition 3.2, Inequality (5.7) and the fact that for  $Z \in L^p$ ,  $p \geq 1$ ,  $\sum_{n \geq 1} n^{p\alpha-1} \mathbb{P}(Z \geq n^\alpha \varepsilon) < \infty$ , for any  $\varepsilon > 0$  and any  $\alpha > 0$ , one can prove similar results for  $\log \|A_n\| - n\lambda_\mu$ ,  $\log \kappa(A_n) - n\lambda_\mu$ ,  $\log v(A_n) - n\lambda_\mu$  or  $\sup_{x \in S^+} |\log \|A_n x\| - n\lambda_\mu|$ . In addition to its own interest, let us recall that Proposition 9.1 applied with  $\alpha = 1/p$  (hence  $1 \leq p < 2$ ) implies the Marcinkiewicz-Zygmund strong law of large numbers: for every  $x \in S^+$ ,

$$\frac{\sigma(A_n, x) - n\lambda_\mu}{n^{1/p}} \xrightarrow[n \rightarrow +\infty]{} 0 \quad \mathbb{P}\text{-a.s.} \quad (9.2)$$

Indeed, by (9.1) with  $\alpha = 1/p$ ,  $\sum_{n \geq 0} \mathbb{P}(\max_{1 \leq k \leq 2^n} |\sigma(A_k, x) - k\lambda_\mu| \geq 2^{n/p}) < \infty$  and (9.2) follows by the Borel-Cantelli lemma.

Proposition 9.1 is the version for positive matrices of Theorem 4.1 of [9], stated for invertible matrices. The proof is exactly the same. Let us mention the key ingredients: The result concerns a cocycle for which, when  $p \geq 2$ , the function  $\psi$  in (5.1) is well defined and bounded

and  $\sup_{k \geq 1} \sup_{x \in S^+} \|\mathbb{E}((\sigma(Y_k, A_{k-1} \cdot x))^2 | \mathcal{F}_{k-1})\|_\infty < \infty$ ; and, when  $1 \leq p < 2$ , one can control the coefficients  $\delta_{1,\infty}(n)$ .

Concerning the matrix coefficients, the following result holds.

**Proposition 9.2.** *Assume that  $\mu$  is strictly contracting and that  $\mu$  and  $\tilde{\mu}$  admit a moment of order  $p \geq 1$ . Let  $\alpha \in (1/2, 1]$  such that  $\alpha \geq 1/p$ . For any  $\varepsilon > 0$ , we have*

$$\sum_{n \geq 1} n^{\alpha p - 2} \mathbb{P}\left(\sup_{x, y \in S^+} |\log \langle y, A_n x \rangle - n \lambda_\mu| \geq n^\alpha \varepsilon\right) < \infty.$$

**Remark.** One cannot expect to have a maximum over  $1 \leq k \leq n$  inside the probability, since one may have  $\mathbb{P}(\log \langle y, A_1 x \rangle = -\infty) > 0$ , for some  $x, y \in S^+$ .

**Proof.** On the set  $\Delta_{n,1}$  defined by (7.12), we get by using (7.13) with  $m = 1$ ,

$$\begin{aligned} \sup_{x, y \in S^+} |\log \langle y, A_n x \rangle - n \lambda_\mu| \\ \leq \sup_{x \in S^+} |\log \|A_n x\| - n \lambda_\mu| + \max_{1 \leq k \leq n} |\log v(Y_k^t \cdots Y_n^t) - \log \|Y_k^t \cdots Y_n^t\||. \end{aligned}$$

To conclude, we apply the remark after Proposition 9.1 and the fact that the random variables  $\max_{1 \leq k \leq n} |\log v(Y_k^t \cdots Y_n^t) - \log \|Y_k^t \cdots Y_n^t\||$  and  $\max_{1 \leq k \leq n} |\log v(Y_k^t \cdots Y_1^t) - \log \|Y_k^t \cdots Y_1^t\||$  have the same law, combined with Proposition 3.2 applied to  $\tilde{\mu}$ .  $\square$

## 10 Generalization to cones

In this section we show how to extend the previous results to general cones. In the previous sections we studied products of positive matrices, that is products of matrices leaving invariant the cone  $(\mathbb{R}^+)^d$ . In this section we consider more general cones. This type of generalization was also investigated in [4].

There are many examples of closed solid cones as the ones considered below. For instance, the Lorentz (or ice-cream) cone:  $\{(x_1, \dots, x_n, z) \in \mathbb{R}^{n+1} : z \geq 0, x_1^2 + \dots + x_n^2 \leq z^2\}$ . The linear operators (of matrices) leaving invariant the Lorentz cone have been studied in details by Loewy and Schneider [22].

Another example is the cone  $K_S$  of positive semi-definite matrices of order  $n$  viewed as a cone of the vector space of symmetric matrices of order  $n$ . Examples of operators leaving invariant  $K_S$  are given by  $M \mapsto A^t M A$  where  $A$  is a matrix of size  $n$  or  $M \mapsto \text{tr}(M R_0) S_0$ , with  $R_0, S_0 \in K_S$  and convex combinations of those.

Let  $d \geq 2$ . We endow  $V = \mathbb{R}^d$  with its usual inner product  $\langle \cdot, \cdot \rangle$  and the associated norm  $\|\cdot\|_2$ .

Let  $K$  be a closed proper convex cone with non empty interior of  $\mathbb{R}^d$ . We recall that a cone of  $\mathbb{R}^d$  is a set of  $\mathbb{R}^d$  stable by multiplication by non-negative real numbers and that it is proper if  $K \cap (-K) = \{0\}$ .

We shall call such cones *closed solid cones*, as in [21], page 3.

We associate with  $K$  its dual cone  $K^* := \{x^* \in V^* : \langle x^*, x \rangle \geq 0 \quad \forall x \in K\}$ .

By Lemma 1.2.4 of [21],  $K^*$  is also a closed solid cone. Moreover, for every  $x^* \in \text{int}(K^*)$ , (the interior of  $K^*$ )  $\langle x^*, x \rangle > 0$  for every  $x \in K \setminus \{0\}$  and  $\Sigma_{x^*} := \{x \in K : \langle x^*, x \rangle = 1\}$  is a compact convex set.

We define a partial order on  $V$  by setting for every  $x, y \in V$ ,  $x \preceq_K y$  if  $y - x \in K$ .

In the sequel we will need to work with a monotone norm for  $K$ , that is a norm compatible with  $\preceq_K$  in the sense of (10.2) below.

Let us fix once and for all  $x_0^* \in \text{int}(K^*)$ . Then, for every  $x \in V$ , set

$$\|x\|_{x_0^*} = \sup_{x^* \in K^* : x^* \preceq_{K^*} x_0^*} |\langle x^*, x \rangle|. \quad (10.1)$$

By Lemma 11.4,  $\|\cdot\|_{x_0^*}$  is a norm on  $V$  and, using the definition of  $K^*$ ,

$$\|x\|_{x_0^*} \leq \|y\|_{x_0^*} \quad \text{for } x, y \text{ such that } 0 \preceq_K x \preceq_K y. \quad (10.2)$$

Notice also that

$$\|x\|_{x_0^*} = \langle x_0^*, x \rangle \quad \forall x \in K. \quad (10.3)$$

Recall that  $(K^*)^* = K$ . Hence fixing once and for all some  $x_0 \in \text{int}(K)$ , with  $\langle x_0^*, x_0 \rangle = 1$ , one defines also a monotone norm on  $V^*$  by setting

$$\|x^*\|_{x_0} := \sup_{x \preceq_K x_0} |\langle x^*, x \rangle| \quad \forall x^* \in V^*.$$

Then, for every  $x^* \in K^*$ ,  $\|x^*\|_{x_0} = \langle x^*, x_0 \rangle$ .

Set

$$S^+ := K \cap \{x \in V : \|x\|_{x_0^*} = 1\} = \{x \in K : \langle x_0^*, x \rangle = 1\}$$

and

$$S^{++} := \text{int}(K) \cap \{x \in V : \|x\|_{x_0^*} = 1\} = \{x \in \text{int}(K) : \langle x_0^*, x \rangle = 1\}.$$

Notice that those definitions are consistent with (1.1) and (2.6), taking  $x_0^* = (1, \dots, 1)$ .

We shall now define an application  $d$  on  $(K \setminus \{0\})^2$  that will make  $(S^+, d)$  a metric space.

We first define an equivalence relation  $\sim_K$  on  $K$ , by setting for every  $x, y$ ,  $x \sim_K y$  if there exists  $0 < \alpha \leq \beta$  such that  $\alpha x \preceq_K y \preceq_K \beta x$ . The equivalence classes for  $\sim_K$  are called *parts* of  $K$ . By Lemma 11.2,  $\text{int}(K)$  is a part of  $K$ .

Given  $x, y \in K \setminus \{0\}$ , set

$$m(x, y) = \sup\{\lambda \geq 0 : \lambda y \preceq_K x\}.$$

This definition is consistent with the definition of the function  $m$  defined in Section 1 when  $K = (\mathbb{R}^+)^d$ .

Notice that if some  $\lambda > 0$  is such that  $\lambda y \preceq_K x$  then  $x - \lambda y \in K$ , hence  $x/\lambda - y \in K$ . So  $m(x, y) < +\infty$  since  $K$  is closed and  $K \cap (-K) = \{0\}$ .

In particular, using again that  $K$  is closed,  $m(y, x)m(x, y)y \preceq_K m(y, x)x \preceq_K y$  so that  $m(y, x)m(x, y) \leq 1$ .

Then, we define for every  $x, y \in K \setminus \{0\}$ ,

$$d(x, y) = \varphi(m(x, y)m(y, x)),$$

where  $\varphi$  is given by (2.3).

It follows from the definition of  $\sim_K$  that  $x \sim_K y$  if and only if  $m(x, y)m(y, x) > 0$  if and only if  $d(x, y) < 1$ .

Note that  $d(x, y) = \tanh((1/2)d_H(x, y))$  where  $d_H$  is introduced page 26 of [21]. Actually,  $d_H$  is defined when  $x \sim_K y$  and when one does not have  $x \sim_K y$  then one sets  $d_H(x, y) = +\infty$ .

**Proposition 10.1.**  *$(S^+, d)$  is a complete metric space and  $S^{++}$  is closed. Moreover, there exists  $C_{x_0} > 0$  such that*

$$\|x - y\|_{x_0^*} \leq C_{x_0^*} \frac{d(x, y)}{1 - d(x, y)} \quad \forall (x, y) \in S^+. \quad (10.4)$$

**Remark.** When  $x \sim_K y$  the right-hand side of (10.4) is finite. Otherwise,  $d(x, y) = 1$  and the right-hand side of (10.4) has to be interpreted as  $+\infty$ .

**Proof.** We first prove that  $(S^+, d)$  is a metric space. Let  $x, y, z \in S^+$  be such that  $x \sim_K y$  and  $y \sim_K z$ . By Proposition 2.1.1 of [21],  $d_H(x, z) \leq d_H(x, y) + d_H(y, z)$ . Using that  $u \mapsto \tanh(u/2)$  is subadditive, the inequality remains true with  $d$  in place of  $d_H$ . If we do not have  $x \sim_K y$  and  $y \sim_K z$ , then  $m(x, y)m(y, x) = 0$  or  $m(y, z)m(z, y) = 0$ , hence  $d(x, y) = 1$  or  $d(y, z) = 1$  so that the triangle inequality is still satisfied.

The fact that  $d$  is a distance on  $S^+$  then follows from (other statements of) Proposition 2.1.1 of [21]. The fact that  $(S^+, d)$  is complete follows from Lemma 2.5.4 of [21]. Indeed, if

$(x_n)_{n \in \mathbb{N}} \subset S^+$  is a Cauchy sequence for  $d$ , then  $d(x_p, x_q) < 1$ , say for  $q, p \geq N$ , so that  $(x_n)_{n \geq N}$  is included in a part  $P$  of  $K$ . But, by Lemma 2.5.4 of [21],  $S^+ \cap P$  is complete for  $d$ .

Let us explain why  $S^{++}$  is closed. Using similar arguments as above we see that it is enough to prove that  $\text{int}(K)$  is a part of  $K$ , but this follows from Lemma 11.2.

Inequality (10.4) follows from (2.21) page 47 of [21], using the relation between  $d_H$  and  $d$ .  $\square$

We shall now define the analogue of the positive matrices.

Let

$$G := \{g \in M_d(\mathbb{R}) : g(K \setminus \{0\}) \subset K \setminus \{0\}, g(\text{int}(K)) \subset \text{int}(K)\}.$$

It follows from Lemma 11.3 below that

$$G := \{g \in M_d(\mathbb{R}) : g^t(K^* \setminus \{0\}) \subset K^* \setminus \{0\}, g^t(\text{int}(K^*)) \subset \text{int}(K^*)\}.$$

In particular,  $g \in G$  is allowable in the sense of [4] (see *a*) page 1527). Hence, the allowability condition in [4] is redundant.

We endow  $M_d(\mathbb{R})$  with the norm:  $\|g\|_{x_0^*} := \sup_{x \in K, \|x\|_{x_0^*}=1} \|gx\|_{x_0^*}$ . The fact that this is indeed a norm follows from the fact that  $K$  has non empty interior (i.e.  $K - K = V$ ). Notice that for  $g \in G$ ,

$$\|g\|_{x_0^*} = \sup_{x \in K, \langle x_0^*, x \rangle = 1} \langle x_0^*, gx \rangle.$$

Define also

$$G^+ := \{g \in G : g(K \setminus \{0\}) \subset \text{int}(K)\}.$$

By Lemma 10.1,

$$G^+ := \{g \in G : g^t(K^* \setminus \{0\}) \subset \text{int}(K^*)\}.$$

Define for every  $g \in G$

$$v_{x_0^*}(g) = \inf_{x \in K, \|x\|_{x_0^*}=1} \|gx\|_{x_0^*},$$

Notice that for  $g \in G$ ,  $v(g) = \inf_{x \in K, \langle x_0^*, x \rangle = 1} \langle x_0^*, gx \rangle$ .

We then define  $N_{x_0^*}(g) := \max(\|g\|_{x_0^*}, 1/v_{x_0^*}(g))$  and  $L_{x_0^*}(g) := \frac{\|g\|_{x_0^*}}{v_{x_0^*}(g)}$ .

The semi-group  $G$  is acting on  $S^+$  as follows.

$$g \cdot x = \frac{gx}{\|gx\|_{x_0^*}} = \frac{gx}{\langle x_0^*, gx \rangle} \quad \forall (g, x) \in G \times S^+.$$

We then define a cocycle by setting  $\sigma(g, x) = \log(\|gx\|_{x_0^*})$  for every  $(g, x) \in G \times S^+$ .



For every  $g \in G$  set

$$c(g) := \sup_{x, y \in K \setminus \{0\}} d(gx, gy).$$

**Proposition 10.2.** *For every  $(g, g', x, y) \in G^2 \times (S^+)^2$  we have*

- (i)  $|\sigma(g, x)| \leq \log N(g)$ ;
- (ii)  $|\sigma(g, x) - \sigma(g, y)| \leq 2C_{x_0^*} L(g) d(x, y)$  if  $d(x, y) \leq 1/2$ ;
- (iii)  $|\sigma(g, x) - \sigma(g, y)| \leq 2 \ln(1/(1 - d(x, y)))$  ;
- (iv)  $c(gg') \leq c(g)c(g')$ ;
- (v)  $c(g) \leq 1$  and  $c(g) < 1$  iff  $g \in G^+$ ;
- (vi)  $d(g \cdot x, g \cdot y) \leq c(g)d(x, y)$ .

**Remark.** The constant  $C > 0$  appearing in item (ii) is the same as in (10.4).

**Proof.** Item (i) is obvious. Item (ii) may be proved exactly as item (i) of Lemma 5.3 of [15], using (10.4).

Let us prove Item (iii). Let  $x, y \in S^+$ . Assume that  $x \sim_K y$ , since otherwise the right-hand side in item (iii) equals  $+\infty$  and the inequality is clear. We have  $m(x, y)y \preceq_K x$  and  $m(y, x)x \preceq_K y$ . Since  $g \in G$ ,  $m(x, y)gy \preceq_K gx$  and  $m(y, x)gx \preceq_K gy$ . Using that  $\|\cdot\|_{x_0^*}$  is monotone we infer that  $m(x, y)\|gy\|_{x_0^*} \leq \|gx\|_{x_0^*}$  and  $m(y, x)\|gx\|_{x_0^*} \leq \|y\|_{gx_0^*}$ . Hence

$$m(x, y) \leq \frac{\|gx\|_{x_0^*}}{\|y\|_{x_0^*}} \leq 1/m(y, x).$$

Then, the proof may be finished as the proof of item (ii) of Lemma 5.3 of [15].

The proof of Item (iv) may be done exactly as in [15].

Item (v) follows from Proposition 10.3 below and Item (vi) may be proved as in [15].  $\square$

We may define as above a distance  $d^*$  on  $K^*$  to which we associate a function  $c^*$  on the set

$$G^* := \{g \in M_d(\mathbb{R}) : g(K^* \setminus \{0\}) \subset K^* \setminus \{0\}, g(\text{int}(K)) \subset \text{int}(K)\}.$$

Notice that by Lemma 11.3,  $G^* = \{g^t : g \in G\}$ .

Set  $S^{*+} := \{x^* \in K^* : \langle x^*, x_0 \rangle = 1\}$  and denote by  $\mathcal{E}(S^{*+})$  the extreme points of  $S^{*+}$ . Denote also  $\mathcal{E}(S^+)$  the extreme points of  $S^+$ .

**Proposition 10.3.** *For every  $g \in G$ , we have*

$$c(g) = \sup_{x,y \in S^+, x^*, y^* \in S^{*+}} \frac{\langle x^*, gx \rangle \langle y^*, gy \rangle - \langle x^*, gy \rangle \langle y^*, gx \rangle}{\langle x^*, gx \rangle \langle y^*, gy \rangle + \langle x^*, gy \rangle \langle y^*, gx \rangle} \quad (10.5)$$

$$= \sup_{x,y \in \mathcal{E}(S^+), x^*, y^* \in \mathcal{E}(S^{*+})} \frac{\langle x^*, gx \rangle \langle y^*, gy \rangle - \langle x^*, gy \rangle \langle y^*, gx \rangle}{\langle x^*, gx \rangle \langle y^*, gy \rangle + \langle x^*, gy \rangle \langle y^*, gx \rangle}. \quad (10.6)$$

The suprema in (10.5) and (10.6) are taken over the  $(x, y, x^*, y^*)$  such that  $\langle x^*, gx \rangle \langle y^*, gy \rangle > 0$ . In particular  $c(g) \leq 1$  and  $c(g) < 1$  if and only if  $g \in G^+$ .

**Remarks.** When  $K = (\mathbb{R}^+)^d$  (10.6) is just (2.5). For  $g \in G$ , (10.8) implies that  $c^*(g^t) = c(g)$ .

**Proof.** As in (2.7) page 35 of [21], noticing that they denote by  $m(x/y)$  the quantity  $m(x, y)$ , we have

$$m(x, y) = \inf_{x^* \in S^{*+}} \frac{\langle x^*, x \rangle}{\langle x^*, y \rangle} = \inf_{x^* \in \mathcal{E}(S^{*+})} \frac{\langle x^*, x \rangle}{\langle x^*, y \rangle}.$$

Here and in the sequel, it is implicit that we take the infimum over the  $x^*$  such that  $\langle x^*, y \rangle > 0$ .

Hence, we have

$$m(x, y)m(y, x) = \inf_{x^*, y^* \in S^{*+}} \frac{\langle x^*, x \rangle \langle y^*, y \rangle}{\langle x^*, y \rangle \langle y^*, x \rangle} = \inf_{x^*, y^* \in \mathcal{E}(S^{*+})} \frac{\langle x^*, x \rangle \langle y^*, y \rangle}{\langle x^*, y \rangle \langle y^*, x \rangle}.$$

Extending naturally  $\varphi$  to a non decreasing function on  $[0, +\infty[$ , we infer that

$$d(x, y) = \sup_{x^*, y^* \in S^{*+}} \varphi\left(\frac{\langle x^*, x \rangle \langle y^*, y \rangle}{\langle x^*, y \rangle \langle y^*, x \rangle}\right) = \sup_{x^*, y^* \in \mathcal{E}(S^{*+})} \varphi\left(\frac{\langle x^*, x \rangle \langle y^*, y \rangle}{\langle x^*, y \rangle \langle y^*, x \rangle}\right). \quad (10.7)$$

For every  $g \in G$ , we have

$$\sup_{x,y \in S^+} d(gx, gy) = \sup_{x,y \in S^+, x^*, y^* \in S^{*+}} \varphi\left(\frac{\langle x^*, gx \rangle \langle y^*, gy \rangle}{\langle x^*, gy \rangle \langle y^*, gx \rangle}\right) \quad (10.8)$$

$$\begin{aligned} &= \sup_{x,y \in S^+, x^*, y^* \in \mathcal{E}(S^{*+})} \varphi\left(\frac{\langle x^*, gx \rangle \langle y^*, gy \rangle}{\langle x^*, gy \rangle \langle y^*, gx \rangle}\right) \\ &= \sup_{x^*, y^* \in \mathcal{E}(S^{*+})} d^*(g^t x^*, g^t y^*) \\ &= \sup_{x^*, y^* \in \mathcal{E}(S^{*+}), x, y \in \mathcal{E}(S^+)} \varphi\left(\frac{\langle x^*, gx \rangle \langle y^*, gy \rangle}{\langle x^*, gy \rangle \langle y^*, gx \rangle}\right), \end{aligned} \quad (10.9)$$

where we used (10.7) for  $d^*$  to obtain the last equality.

Then, (10.5) and (10.6) follow by noticing that for every  $s, t, u, v \geq 0$ , with  $uv > 0$

$$\varphi(st/uv) = \frac{uv - st}{st + uv}.$$

The fact that  $c(g) \leq 1$  is obvious.

Let  $g \in G^+$ . Then,  $\langle x^*, gx \rangle > 0$  for every  $x \in K \setminus \{0\}$  and  $x^* \in K^* \setminus \{0\}$ . Hence, the continuous function (for either  $d$  or  $\|\cdot\|$ )  $(x, y, x^*, y^*) \mapsto \frac{\langle x^*, gx \rangle \langle y^*, gy \rangle - \langle x^*, gy \rangle \langle y^*, gx \rangle}{\langle x^*, gx \rangle \langle y^*, gy \rangle + \langle x^*, gy \rangle \langle y^*, gx \rangle}$  defined on the compact  $(S^+)^2 \times (S^{*+})^2$  takes values in  $[-1, 1[$ . So,  $c(g) < 1$ .

Assume now that  $g \in G \setminus G^+$ . By assumption, there exists  $x \in S^+$  such that  $gx \in K \setminus \text{int}(K)$ . By Lemma 11.1, there exists  $y^* \in S^{*+}$  such that  $\langle y^*, gx \rangle = 0$ . Since  $gx \neq 0$  and  $g^t y^* \neq 0$ , there exist  $x^* \in S^{*+}$  and  $y \in S^+$  such that  $\langle y^*, gy \rangle > 0$  and  $\langle x^*, gx \rangle > 0$ . Hence,  $c(g) = 1$ .  $\square$

We shall now consider the analogous statements as those given in Lemma 2.2. Only item (ii) requires a proof.

**Lemma 10.4.** *There exists  $C > 0$  such that for every  $g \in G$ ,*

$$\|gx_0\|_{x_0^*} \leq \|g\|_{x_0^*} \leq C\|gx_0\|_{x_0^*}.$$

**Proof.** Since  $\langle x_0^*, x_0 \rangle = 1$ ,  $\|gx_0\|_{x_0^*} \leq \|g\|_{x_0^*}$ . Let  $x \in K$  be such that  $\langle x_0^*, x \rangle = 1$ . Let  $g \in G$ . Using Lemma 11.2 with the cone  $K^*$  there exists  $\varepsilon > 0$  such that  $g^t x_0^* \preceq_{K^*} \frac{\|g^t x_0^*\|_{x_0}}{\varepsilon} x_0^*$ . Hence, using that  $gx \in K$  and Lemma 11.1,

$$\begin{aligned} \|gx\|_{x_0^*} &= \langle x_0^*, gx \rangle = \langle g^t x_0^*, x \rangle \leq \frac{\|g^t x_0^*\|_{x_0}}{\varepsilon} \langle x_0^*, x \rangle \\ &= \frac{\langle g^t x_0^*, x_0 \rangle}{\varepsilon} = \frac{\langle x_0^*, gx_0 \rangle}{\varepsilon} = \frac{\|gx_0\|_{x_0^*}}{\varepsilon}. \end{aligned}$$

$\square$

All the results of the previous sections hold true for a cocycle satisfying all the properties listed in Proposition 2.1 and Lemma 2.2, replacing the quantities  $N(g)$  and  $L(g)$  in the moment conditions by the quantities  $N_{x_0^*}(g)$  and  $L_{x_0^*}(g)$ .

## 11 Technical results

The next lemma is just Lemma 1.2.4 of [21].

**Lemma 11.1.** *Let  $K$  be a closed solid cone. Then*

$$\text{int}(K^*) = \{x^* \in K^* : \langle x^*, x \rangle > 0, \forall x \in K \setminus \{0\}\}.$$

The next lemma follows from the proof Lemma 1.2.4 of [21]. We recall the arguments.

**Lemma 11.2.** *Let  $\|\cdot\|$  be a norm on  $V = \mathbb{R}^d$ . Let  $K$  be a closed solid cone. Then, for every  $x \in \text{int}(K)$ , there exists  $\varepsilon > 0$ , such that for every  $y \in K \cap \bar{B}_{\|\cdot\|}(0, 1)$ , where  $\bar{B}_{\|\cdot\|}(0, 1)$  is the closure of the unit ball  $B_{\|\cdot\|}(0, 1)$ , we have  $y \preceq \frac{1}{\varepsilon}x$ . Then  $\|y\| \leq \frac{1}{\varepsilon}$ . In particular,  $\text{int}(K)$  is a part of  $K$ .*

**Proof.** Let  $x \in \text{int}(K)$ . There exists  $\varepsilon > 0$  such that  $\bar{B}_{\|\cdot\|}(x, \varepsilon) \subset \text{int}(K)$ . Let  $y \in \bar{B}_{\|\cdot\|}(0, 1)$ . Then,  $x - \varepsilon y \in K$ , which means precisely that  $y \preceq \frac{1}{\varepsilon}x$ . In particular, if  $x, y \in \text{int}(K)$ ,  $x \sim_K y$ .

It remains to prove that for every  $(x, y) \in \text{int}(K) \times K$ ,  $x \sim_K y \Rightarrow y \in \text{int}(K)$ .

Hence, let  $x \in \text{int}(K)$ . There exists  $\varepsilon > 0$  such that  $B_{\|\cdot\|}(x, \varepsilon) \subset K$ .

Let  $y \in K$  be such that  $y \sim_K x$ . There exists  $\alpha > 0$  such that  $x \preceq_K \alpha y$ . So  $\alpha y - x \in K$  and

$$\alpha y = x + \alpha y - x \in \cup_{z \in K}(z + B_{\|\cdot\|}(x, \varepsilon)),$$

which is an open subset of  $K$ . □

**Lemma 11.3.** *Let  $g \in M_d(\mathbb{R})$  and let  $K$  be a closed solid cone of  $E$ .*

(i)  $g(K \setminus \{0\}) \subset K \setminus \{0\}$  if and only if  $g^t(\text{int}(K^*)) \subset \text{int}(K^*)$ ;

(ii)  $g(\text{int}(K)) \subset \text{int}(K)$  if and only if  $g^t(K^* \setminus \{0\}) \subset K^* \setminus \{0\}$ .

**Proof.** Assume that  $g(K \setminus \{0\}) \subset K \setminus \{0\}$ . Let  $x^* \in \text{int}(K^*)$  and  $x \in K \setminus \{0\}$ . We have

$$\langle g^t x^*, x \rangle = \langle x^*, gx \rangle > 0,$$

by Lemma 11.1. Using Lemma 11.1 again, we see that  $g^t x^* \in \text{int}(K^*)$ .

Assume that  $g^t(\text{int}(K^*)) \subset \text{int}(K^*)$ . Let  $x \in K \setminus \{0\}$  and  $x^* \in \text{int}(K^*)$ . We have

$$\langle x^*, gx \rangle = \langle g^t x^*, x \rangle > 0.$$

Hence  $gx \in K^{**} = K$  (see Exercise 2.31 of [3]) and  $gx \neq 0$ , which proves item (i).

Item (ii) is just item (i) for  $K^*$  using that  $K^{**} = K$ . □

**Lemma 11.4.**  $\|\cdot\|_{x_0^*}$  defined by (10.1) is a norm for every  $x_0^* \in \text{int}(K^*)$ .

**Proof.** By Lemma 1.2.5 of [21], the set  $\{x^* \in K : x^* \preceq_{K^*} x_0^*\}$  is bounded, hence  $\|\cdot\|_{x_0^*}$  is finite on  $V$ . The fact that  $\|\cdot\|_{x_0^*}$  satisfies the triangular inequality and is positively homogeneous are obvious.

Assume that  $x \in E$ , is such that  $\|x\|_{x_0^*} = 0$ . By Lemma 11.2 applied to  $K^*$  (with  $x = x_0^*$ ), for every  $x^* \in K^*$ ,  $\langle x^*, x \rangle = 0$ . Since  $K^*$  has non empty interior,  $K^* - K^* = V^*$  and  $x = 0$ . □

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