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1 **DISTRIBUTED NULL CONTROLLABILITY OF SOME 1D CASCADE**
2 **PARABOLIC SYSTEMS**

3 FRANCK BOYER* AND MORGAN MORANCEY†

4 **Abstract.** We consider several coupled systems of one-dimensional linear parabolic equations
5 where only one equation is controlled with a distributed control. For these systems we study the
6 minimal null-control time that is the minimal time needed to drive any initial condition to zero.

7 In a previous work [*Comptes Rendus. Mathématique*, Tome 361 (2023)] we extended the block
8 moment method to obtain a complete characterization of the minimal null-control time in an abstract
9 setting encompassing such non-scalar controls. In this paper, we push forward the application of this
10 general approach to some classes of 1D parabolic systems with distributed controls whose analysis
11 is out of reach by the usual approaches in the literature like Carleman-based methods, fictitious
12 control and algebraic resolubility, or standard moment method. To achieve this goal, we need to
13 prove refined spectral estimates for Sturm-Liouville operators that have their own interest.

14 **Key words.** Control theory, parabolic partial differential equations, minimal null control time,
15 block moment method

16 **AMS subject classifications.** 93B05, 93C20, 35K40

17 **1. Introduction.**

18 **1.1. Problems under study.**

19 In the last 15 years different works exhibited that for some coupled systems of
20 parabolic partial differential equations (see for instance [3, 5, 16, 22, 23]) or degenerate
21 parabolic equations (see for instance [6, 8, 9, 7, 17, 2]) it might be needed to wait
22 for some positive minimal time for null controllability to hold even if, in a parabolic
23 context, the information propagates at infinite velocity.

24 This phenomenon, quite surprising at first sight since it is not related to any
25 constraint imposed on the state or on the control, is now better understood. It is
26 more related to the geometry of the high frequency eigenelements of the underlying
27 evolution operator relatively to the observation operator. For instance, it may occur
28 in the following (non exclusive) situations: if there is condensation of eigenvalues, if
29 the observation of eigenvectors is too small with respect to the parabolic dissipation
30 rate, or if the norm of suitably chosen generalized eigenvectors is asymptotically too
31 large.

32 In the previous works [10, 13], we developed the block moment method which is
33 well adapted to study the minimal null-control time for autonomous coupled linear
34 one-dimensional parabolic partial differential equations. Our goal in this paper is to
35 provide several applications of this approach to some classes of such systems whose
36 analysis is out of reach by using other techniques available in the literature. Based on
37 the general results obtained in [13] we first characterize the minimal null-control time
38 of such systems in terms of the asymptotic behavior of some explicit quantities based
39 on the eigenelements of the evolution operator. In a second step, an extra spectral
40 analysis is developed, extending the one given in [1], in order to obtain a tractable
41 expression of the involved quantities that can be computed for actual examples. With
42 this approach, we manage to compute the minimal null-control time for many systems,
43 extending the results in the literature.

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44 To be more precise, the first class of control problems that will be studied in this
 45 paper is the following one

$$46 \quad (1.1) \quad \begin{cases} \partial_t y + \begin{pmatrix} A & 0 \\ q(x) & A \end{pmatrix} y = \begin{pmatrix} \mathbf{1}_\omega u(t, x) \\ 0 \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), \end{cases}$$

47 where

- 48 • A is the unbounded Sturm-Liouville operator defined in $L^2(0, 1)$ by

$$49 \quad (1.2) \quad D(A) = H^2(0, 1) \cap H_0^1(0, 1), \quad A = -\partial_x(\gamma \partial_x \bullet) + c \bullet,$$

50 with $c \in L^\infty(0, 1)$, $\gamma \in C^1([0, 1])$ satisfying $c \geq 0$ and $\inf_{[0, 1]} \gamma > 0$.

- 51 • the coupling function q belongs to $L^\infty(0, 1)$
- 52 • $\omega \subset (0, 1)$ is a non empty open set.

53 This system is well-posed in the sense that for every $y_0 \in X = (L^2(0, 1))^2$, for
 54 every $u \in L^2((0, T) \times (0, 1))$ there exists a unique solution in $C^0([0, T]; X)$. The null
 55 controllability property we shall study for this system is defined as follows.

56 **DEFINITION 1.1.** *Let $T > 0$. The system (1.1) is said to be null controllable at*
 57 *time T if for any $y_0 \in X$, there exists a control $u \in L^2((0, T) \times (0, 1))$ such that the*
 58 *associated solution of (1.1) satisfies $y(T) = 0$.*

59 This problem is not straightforward since the control u is localized in space and
 60 only acts in the first equation of the system; therefore controlling both components is
 61 only possible through the action of the coupling term corresponding to the function
 62 q in the second equation. It is now well known that such system may not be short-
 63 time null-controllable and our goal is to go deeper into the understanding of this
 64 phenomenon.

65 **DEFINITION 1.2.** *The minimal null control time for system (1.1) in X is defined*
 66 *as the unique value $T_0(X) \in [0, +\infty]$ such that*

- 67 • for any $T > T_0(X)$, system (1.1) is null controllable at time T ;
- 68 • for any $0 < T < T_0(X)$, system (1.1) is not null controllable at time T .

69 *When no confusion is possible, we shall simply denote this minimal time as T_0 .*

70 It will be also useful to introduce, for any $y_0 \in X$, the number $T_0(y_0) \in [0, +\infty]$
 71 which is the minimal time that is necessary to drive the system to 0 starting from the
 72 particular initial data y_0 . Notice that $T_0(X) = \sup_{y_0 \in X} T_0(y_0)$.

73 The question we address is thus the computation of the minimal null control time
 74 (being possibly 0 or infinity) of system (1.1).

75 This question has already been answered in some particular geometric configura-
 76 tions: when ω intersects the support of the coupling function $\text{Supp}(q)$, by means
 77 of Carleman estimates or in the opposite setting when ω is an interval disjoint from
 78 the support of q , by solving the associated moment problem. We will discuss those
 79 results more in details in Section 1.2.

80 Our goal in this article concerning (1.1) is twofold. First we prove that applying
 81 directly the abstract results on block moment problems from [13] encompasses all
 82 the previously known results for this problem even though they were proved with
 83 completely different techniques. Then, improving the strategy developed in [1] to
 84 study spectral quantities of interest in this problem, we are able to extend these

85 results to any choice of coupling term q and control domain ω . We will emphasize the
 86 role of the geometry (that is of the relative position of the connected components of ω
 87 with respect to the support of q) in the determination of the minimal null control time
 88 for system (1.1). Moreover, contrary to the related results in the literature, our proof
 89 does not rely on the explicit expression of the eigenfunctions of A so that it applies
 90 for a general Sturm-Liouville operator (instead of the Dirichlet-Laplace operator that
 91 was considered in [4, 5]). All these results are precisely stated in Section 1.3.

92 To point out even more the ability of our approach to determine the minimal null-
 93 control time for such problems we propose the study of some other related systems.
 94 To begin with, we obtain new results for a similar cascade problem in which coupling
 95 terms in the second equation now contain first-order operators, as studied in [16].
 96 More precisely, we consider the following control problem

$$97 \quad (1.3) \quad \begin{cases} \partial_t y + \begin{pmatrix} A & 0 \\ q(x) + p(x)\partial_x & A \end{pmatrix} y = \begin{pmatrix} \mathbf{1}_\omega u(t, x) \\ 0 \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), \end{cases}$$

98 with $q \in L^\infty(0, 1)$, $p \in W^{1, \infty}(0, 1)$. As a consequence of our analysis we will give an
 99 example for which the system is not approximately controllable, even if the coupling
 100 terms are active inside the control domain ω .

101 Finally, we analyze the null-controllability of the following simultaneous control
 102 problem which has not been studied in the literature so far

$$103 \quad (1.4) \quad \begin{cases} \partial_t y + \begin{pmatrix} A & 0 & 0 \\ q_2(x) & A & 0 \\ q_3(x) & 0 & A \end{pmatrix} y = \begin{pmatrix} \mathbf{1}_\omega u(t, x) \\ 0 \\ 0 \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), \end{cases}$$

104 with $q_2, q_3 \in L^\infty(0, 1)$. This problem can indeed be seen as a simultaneous control-
 105 lability problem since we look for a single control u that simultaneously controls two
 106 systems of the form (1.1): the one satisfied by (y_1, y_2) and the one satisfied by (y_1, y_3) .

107 The expression we obtain for the minimal simultaneous null control-time for (1.4)
 108 shows that this time can be strictly larger than the two minimal null-control times
 109 associated to the two subsystems. This kind of phenomenon was already observed,
 110 for instance, in [22].

111 In the sequel of this introduction, we will set some notation and present the
 112 results available in the literature concerning the analysis of the control problems we
 113 are interested in that is (1.1), (1.3) and (1.4), then we will state precisely our main
 114 results.

115 1.2. State of the art.

116 1.2.1. Notation.

- 117 • For any $\omega \subset (0, 1)$, we set for convenience $\|\bullet\|_\omega = \|\bullet\|_{L^2(\omega)}$ and the associated
 118 inner product $\langle \bullet, \bullet \rangle_\omega$
- 119 • We denote by $(\nu_k)_{k \geq 1}$ the increasing sequence of eigenvalues of the Sturm-
 120 Liouville operator A defined in (1.2). Notice that the sign assumption we

121 make on c ensures that for any $k \geq 1$, we have $\nu_k > 0$. The associated
 122 normalized eigenvectors are denoted by $(\varphi_k)_{k \geq 1}$; they form a Hilbert basis of
 123 $L^2(0, 1)$.

124 • For any $k \geq 1$, we define $\tilde{\varphi}_k$ as the unique solution of the Cauchy problem

$$125 \quad (1.5) \quad \begin{cases} (A - \nu_k)\tilde{\varphi}_k = 0, \\ \tilde{\varphi}_k(0) = 1, \\ \tilde{\varphi}'_k(0) = 0. \end{cases}$$

126 • We will also need to introduce $\psi_{k,q}$, the unique solution of the boundary value
 127 problem

$$128 \quad (1.6) \quad \begin{cases} (A - \nu_k)\psi_{k,q} = (I_k(q) - q)\varphi_k, \\ \psi_{k,q}(0) = \psi_{k,q}(1) = 0, \\ \langle \varphi_k, \psi_{k,q} \rangle_\omega = 0, \end{cases}$$

129 where $I_k(q)$ is the integral defined by

$$130 \quad (1.7) \quad I_k(q) = \int_0^1 q(x)\varphi_k^2(x)dx.$$

Such a solution exists since, precisely by (1.7), the right-hand side of the equation is orthogonal to φ_k and it is unique thanks to the choice of normalization $\langle \varphi_k, \psi_{k,q} \rangle_\omega = 0$. This particular choice is possible thanks to the fact that $\|\varphi_k\|_\omega > 0$ and it implies that $\psi_{k,q}$ is, among all the solutions of the underdetermined problem

$$(A - \nu_k)\psi = (I_k(q) - q)\varphi_k, \quad \psi(0) = \psi(1) = 0,$$

131 the one with minimal $L^2(\omega)$ norm. This will simplify some computations in
 132 the paper since it ensures orthogonality between observations of (generalized)
 133 eigenvectors.

134 • Following [14], for any $F \in L^2(0, 1)$ and any \mathfrak{C} connected component of
 135 $(0, 1) \setminus \omega$ we define an element of \mathbb{R}^2 as follows

$$136 \quad M_k(F, \mathfrak{C}) = \begin{cases} \begin{pmatrix} \int_{\mathfrak{C}} F \varphi_k \\ 0 \end{pmatrix}, & \text{if } \mathfrak{C} \text{ touches the boundary of } (0, 1) \\ \begin{pmatrix} \int_{\mathfrak{C}} F \varphi_k \\ \int_{\mathfrak{C}} F \tilde{\varphi}_k \end{pmatrix}, & \text{otherwise,} \end{cases}$$

137 that we gather into a single collection defined by

$$138 \quad (1.8) \quad \mathfrak{M}_k(F, \omega) = (M_k(F, \mathfrak{C}))_{\mathfrak{C} \in \mathcal{C}(\overline{(0,1)} \setminus \omega)},$$

139 where $\mathcal{C}(U)$ stands for the set of all connected components of any $U \subset [0, 1]$.
 140 We finally set

$$141 \quad (1.9) \quad \mathcal{M}_k(F, \omega) = \|\mathfrak{M}_k(F, \omega)\|_\infty = \sup \left\{ |M_k(F, \mathfrak{C})|_\infty ; \mathfrak{C} \in \mathcal{C}(\overline{(0,1)} \setminus \omega) \right\}.$$

142 **1.2.2. About the cascade system (1.1).**

- 143 • *Approximate controllability:*

144 By using the Fattorini-Hautus test (see [20]), it is proved in [14, Theorem
145 3.2] that, if $\text{Supp}(q) \cap \omega = \emptyset$, approximate controllability of (1.1) holds if and
146 only if

147 (1.10)
$$\mathcal{M}_k(q\varphi_k, \omega) \neq 0, \quad \forall k \geq 1.$$

148 Notice also that applying [14, Theorem 2.2] we obtain that

- 149 • If $\text{Supp}(q) \cap \omega \neq \emptyset$, approximate controllability of (1.1) holds without
150 any other condition.
151 • If $\text{Supp}(q) \cap \omega = \emptyset$, the necessary and sufficient condition (1.10) for
152 approximate controllability of (1.1) can be rewritten as

153 (1.11)
$$\mathcal{M}_k((I_k(q) - q)\varphi_k, \omega) \neq 0, \quad \forall k \geq 1,$$

154 where $I_k(q)$ is introduced in (1.7). Rewriting the approximate control-
155 lability condition as (1.11) is more coherent with the expression of the
156 minimal null control time that we obtain below (see Section 2.2). The
157 equivalence between conditions (1.10) and (1.11) is proved in Lemma 3.2
158 (choosing there $p = 0$).

- 159 • *Null controllability under a sign assumption:*

160 If there exists $\omega_0 \subset \omega$ such that q has a strict sign inside ω_0 then it follows
161 from [21] that null controllability holds in any arbitrary small time. The proof
162 is based on Carleman estimates.

- 163 • *Null controllability with disjoint control and coupling domains:*

164 System (1.1) was then studied in the case where $A = -\partial_{xx}$ and $\omega = (a, b)$ is
165 an interval such that $\text{Supp}(q) \cap \omega = \emptyset$.

166 First, it was proved in [4] that if $\text{Supp}(q) \subset (b, 1)$ then, approximate control-
167 lability holds if and only if

168
$$I_k(q) \neq 0, \quad \forall k \geq 1.$$

169 This condition is just a rephrasing of (1.10). In this case the authors proved
170 that the minimal null-control time $T_{0,q}$ for this system is given by

171
$$T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln |I_k(q)|}{\nu_k}.$$

172 Later on, it was proved in [5] that if $\text{Supp}(q)$ is included in $(0, a) \cup (b, 1)$, then
173 approximate controllability holds if and only if

174
$$|I_{1,k}(q)| + |I_{2,k}(q)| \neq 0, \quad \forall k \geq 1,$$

175 where

176 (1.12)
$$I_{1,k}(q) = \int_0^a q(x)\varphi_k^2(x)dx, \quad I_{2,k}(q) = \int_b^1 q(x)\varphi_k^2(x)dx.$$

177 In that situation, this condition is also a rephrasing of (1.10) and it was also
178 proved in [5] that the minimal null-control time is

179 (1.13)
$$T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln \max \{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\}}{\nu_k}.$$

180 Moreover, it is proved that for any $\tau_0 \in [0, +\infty]$ there exists a coupling
 181 function $q \in L^\infty(0, 1)$ such that the corresponding minimal time is actually
 182 $T_{0,q} = \tau_0$. Let us underline that these results are the first results exhibiting a
 183 positive minimal null control time for a system of coupled parabolic equations
 184 with a distributed control.
 185 The proofs of those results are based on the moment method since, due to
 186 the assumption $\text{Supp}(q) \cap \omega = \emptyset$, the strategy based on Carleman estimates
 187 is inefficient.

188 **1.2.3. About the system with a first-order coupling term.**

189 Null controllability of system (1.3) with a coupling term of order one has been
 190 studied in [11, 16, 18, 19]. Among other things, the author proves in [16] that, when
 191 approximate controllability holds, the minimal null-control time $T_{0,q,p}$ of system (1.3)
 192 when $\omega = (a, b)$ is an interval and A is the Dirichlet Laplace operator is given by

193
$$T_{0,q,p} = \limsup_{k \rightarrow +\infty} \frac{-\ln \max \{|I_k(r)|, |I_{1,k}(r)|, |I_{2,k}(r)|\}}{\nu_k},$$

194 where $r = q - \frac{1}{2}p'$. Note that the value of $T_{0,q,p}$ only depends on r and is equal to
 195 $T_{0,r}$ as defined in (1.13). As proved in Section 3.4.1, this feature is specific to the case
 196 where ω is an interval since in general $T_{0,q,p}$ really depend on both q and p , and not
 197 only on r .

198 The proof given in [16] is also based on the moments method and follows that
 199 of [5]. More precisely, the analysis in this reference is reduced, thanks to well-suited
 200 manipulations, to the one of a scalar moment problem despite the fact that the control
 201 space is, by nature, infinite dimensional. Those computations are thus specific to the
 202 problem under study and makes use of the explicit formulas for the eigenfunctions of
 203 the 1D Laplace operator, which is not the case of our proof.

204 In [18], the authors give a sufficient condition for null controllability for general
 205 parabolic systems in any dimension with first-order coupling terms. They deal with
 206 coefficients depending both on space and time but their analysis does not apply when
 207 $p = 0$ in ω . In [19], the same authors study the influence of the position of the
 208 control domain on controllability for one dimensional parabolic systems with first-
 209 order coupling terms. Their result can hardly be compared with our study since the
 210 two equations they consider are associated to different evolution operators.

211 **1.2.4. About the simultaneous control problem.**

212 To the best of our knowledge, the only available result in the literature concerning
 213 the controllability of (1.4) is the necessary and sufficient condition for approximate
 214 controllability given in [14, Theorem 3.2] that we recall now: approximate controlla-
 215 bility for system (1.4) holds if and only if, for any $k \geq 1$,

216 (1.14) $\mathfrak{M}_k(q_2\varphi_k, \omega)$ and $\mathfrak{M}_k(q_3\varphi_k, \omega)$ are linearly independent in $(\mathbb{R}^2)^{\mathcal{C}(\overline{(0,1)\setminus\omega})}$,

217 where the notation \mathfrak{M}_k is introduced in (1.8).

218 This gave rise to unexpected geometric control conditions for this problem. For
 219 instance, if ω is an interval that does not touch the boundary of $(0, 1)$, approximate
 220 controllability of system (1.4) never holds when $\text{Supp}(q_2)$ and $\text{Supp}(q_3)$ are located
 221 in the same connected component of $\overline{(0, 1)\setminus\omega}$. However, if there are located in two
 222 distinct connected components then approximate controllability holds if and only if
 223 the two subsystems are approximately controllable (see [14, Section 3.4]).

224 **1.3. Main results of this paper.**

225 First, we obtain the following characterization of the minimal null-control time
 226 for system (1.1).

227 **THEOREM 1.1.** *Let $\omega \subset (0, 1)$ be a non empty open set and let $q \in L^\infty(0, 1)$.
 228 Assume that either $\text{Supp}(q) \cap \omega \neq \emptyset$ or that (1.11) holds. Then, the minimal null-
 229 control time for system (1.1) is given by*

$$230 \quad T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln(I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2)}{2\nu_k}.$$

231 We recall that $\psi_{k,q}$ is defined in (1.6) and $I_k(q)$ is defined by (1.7).

232 This theorem is proved in Section 2.1. It is valid without any geometric assump-
 233 tions on the control domain ω nor on the support of the coupling term q . In this
 234 respect, it unifies the different results obtained in the literature for the study of null
 235 controllability of system (1.1) recalled in Section 1.2.2. For example, even if it is not
 236 clear at first sight, we manage to prove, in Section 2.3, that the formula above reduces
 237 to $T_{0,q} = 0$ when q has a strict sign on $\omega_0 \subset \omega$ as proved in [21].

238 Theorem 1.1 will be obtained as a consequence of [13, Theorems 11, 14 and 18]
 239 where the minimal null control time issue is analyzed in an abstract general setting.
 240 It relies on a careful estimate of the cost of resolution for block moment problems
 241 associated to a general admissible control operator. With some additional work, based
 242 on the method developed in [1] to obtain spectral estimates for the eigenlements of
 243 A , we also obtain the following characterization of the minimal null control time in
 244 the case where $\text{Supp}(q) \cap \omega = \emptyset$.

245 **THEOREM 1.2.** *Let $\omega \subset (0, 1)$ be a non empty open set with a finite number of
 246 connected components. Let $q \in L^\infty(0, 1)$ be such that $\text{Supp}(q) \cap \omega = \emptyset$. Assume
 247 that (1.11) holds. Then, the minimal null-control time for system (1.1) is given by*

$$248 \quad T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)}{\nu_k}$$

249 where \mathcal{M}_k is defined by (1.9).

250 The proof is given in Section 2.2. Compared to the one in Theorem 1.1, the expression
 251 for $T_{0,q}$ above is more convenient to deal with since it does not involve the function
 252 $\psi_{k,q}$. As we shall prove in Section 2.3, this formula is a natural extension of the ones
 253 obtained in the literature in some particular cases. However it holds true in more
 254 general situations, so that we are able to compute $T_{0,q}$ in cases that were not covered
 255 in the literature (see Proposition 2.5 as an example).

256 This theorem also extends the previous works in the field by considering for A a
 257 general Sturm-Liouville operator (and not only the Dirichlet-Laplace operator) since
 258 it does not make use of the explicit expressions of its eigenvalues and eigenfunctions.

259 **REMARK 1.1.** *Notice that the assumption $\text{Supp}(q) \cap \omega = \emptyset$ is necessary for this
 260 theorem to be true. For instance, if $q = 1$ and ω is an interval then, from [21], null
 261 controllability holds in any time $T > 0$ whereas we have $I_k(q) - q = 0$ for any $k \geq 1$.*

262 *However, this is not restrictive for our study since it is well-known that when
 263 $\text{Supp}(q) \cap \omega \neq \emptyset$, the system is indeed null-controllable at every time $T > 0$ (see for
 264 instance [21]).*

265 The tools used to prove Theorems 1.1 and 1.2 allow for a similar analysis for
 266 system (1.3). The corresponding results are stated in Section 3.

267 We now turn to the simultaneous controllability problem (1.4). For $q_2, q_3 \in$
 268 $L^\infty(0, 1)$, we set $\mathbf{q} = (q_2, q_3)$ and denote by $T_{0,\mathbf{q}}$ the minimal null control time for
 269 system (1.4) in $(L^2(0, 1))^3$. Since simultaneous null controllability at a given time
 270 implies null controllability at the same time for both subsystems (1.1) with $q = q_2$
 271 and $q = q_3$ it directly comes that

$$272 \quad (1.15) \quad T_{0,\mathbf{q}} \geq \max(T_{0,q_2}, T_{0,q_3}).$$

273 Actually, by linearity of the system, simultaneous null controllability at a given
 274 time implies null controllability at the same time for system (1.1) with any q in
 275 $\text{Span}(q_2, q_3) \setminus \{0\}$ that is

$$276 \quad (1.16) \quad T_{0,\mathbf{q}} \geq \sup_{\substack{q \in \text{Span}(q_2, q_3) \\ q \neq 0}} T_{0,q}.$$

277 We give below general characterizations of $T_{0,\mathbf{q}}$ similar to those obtained in Theo-
 278 rems 1.1 and 1.2 for system (1.1) and give, in Section 4.4, an explicit example of
 279 system (1.4) for which the inequality in (1.16) is strict.

280 In order to state the results, it will be convenient to use some extra notation. For
 281 any $q \in L^\infty(0, 1)$, we set

$$282 \quad (1.17) \quad \zeta_{k,q} = \psi_{k,q} + I_k(q)\varphi_k,$$

283 where $\psi_{k,q}$ was introduced in (1.6), and

$$284 \quad (1.18) \quad \vartheta_{k,q} = (I_k(q) - q)\varphi_k.$$

285 The formulas obtained in Theorems 1.1 and 1.2 for the minimal null-control time of
 286 system (1.1) can now be rephrased respectively as follows

$$287 \quad T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln \|\zeta_{k,q}\|_\omega}{\nu_k}$$

288 and

$$289 \quad T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln \mathcal{M}_k(\vartheta_{k,q}, \omega)}{\nu_k}.$$

290 We shall generalize those expressions for system (1.4) as follows.

291 **THEOREM 1.3.** *Let $\omega \subset (0, 1)$ be a non empty open set and let $q_2, q_3 \in L^\infty(0, 1)$.
 292 Assume that (1.14) holds. Then, the minimal null control time $T_{0,\mathbf{q}}$ for system (1.4)
 293 in $(L^2(0, 1))^3$ is given by*

$$294 \quad T_{0,\mathbf{q}} = \limsup_{k \rightarrow +\infty} \frac{1}{2\nu_k} \ln \frac{\max\left(\|\zeta_{k,q_2}\|_\omega^2, \|\zeta_{k,q_3}\|_\omega^2\right)}{\|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2}.$$

295 This theorem is proved in Section 4.2. Though it is not obvious at first sight,
 296 we will show in the proof of Proposition 4.3 that the approximate controllability
 297 assumption (1.14) actually implies that

$$298 \quad \|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2 > 0, \quad \forall k \geq 1$$

299 and thus the formula defining $T_{0,\mathbf{q}}$ is well-defined. Since for any $k \geq 1$ we clearly have

$$300 \quad \|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2 \leq \|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2,$$

301 we immediately see, by Theorem 1.1, that this formula is compatible with the expected
 302 property (1.15).

303 Notice that, when $(\text{Supp}(q_2) \cup \text{Supp}(q_3)) \cap \omega = \emptyset$, we have

$$304 \quad I_k(q_j) = \sum_{\mathbf{e} \in \mathcal{C}(\overline{(0,1)} \setminus \omega)} \int_{\mathbf{e}} q_j \varphi_k^2, \quad \forall j \in \{1, 2\}.$$

305 Hence, it comes that the approximate controllability condition (1.14) is equivalent in
 306 that case to the condition

$$307 \quad (1.19) \quad \mathfrak{M}_k(\vartheta_{k,q_2}, \omega) \text{ and } \mathfrak{M}_k(\vartheta_{k,q_3}, \omega) \text{ are linearly independent}$$

308 where ϑ_{k,q_2} and ϑ_{k,q_3} are defined by (1.18).

309 **THEOREM 1.4.** *Let $\omega \subset (0, 1)$ be a non empty open set with a finite number of*
 310 *connected components. Let $q_2, q_3 \in L^\infty(0, 1)$ be such that*

$$311 \quad (\text{Supp}(q_2) \cup \text{Supp}(q_3)) \cap \omega = \emptyset.$$

312 *Assume that (1.19) holds. Then, the minimal null control time for system (1.4) in*
 313 *$(L^2(0, 1))^3$ is given by*

$$314 \quad T_{0,\mathbf{q}} = \limsup_{k \rightarrow +\infty} \frac{-\ln \min_{q \in \mathbb{S}[\mathbf{q}]} \mathcal{M}_k(\vartheta_{k,q}, \omega)}{\nu_k}$$

where \mathcal{M}_k is defined by (1.9), $\vartheta_{k,q}$ is defined by (1.18) and

$$\mathbb{S}[\mathbf{q}] = \{q \in \text{Span}(q_2, q_3), \|q\|_\infty = 1\}.$$

315 This theorem is proved in Section 4.3. Notice that, by compactness of $\mathbb{S}[\mathbf{q}]$, the
 316 min appearing in this formula is actually achieved and moreover, since the approxi-
 317 mate controllability condition (1.19) implies that

$$318 \quad \mathcal{M}_k(\vartheta_{k,q}, \omega) \neq 0, \quad \forall q \in \mathbb{S}[\mathbf{q}],$$

319 we know that this min is positive. Thus, the formula for $T_{0,\mathbf{q}}$ in the above theorem is
 320 well defined.

This formulation is more convenient to deal with than the one of Theorem 1.3 on
 actual systems. For instance, with this formulation, we prove that the minimal null
 control time is not related to the minimal null control times of the subsystems. Indeed,
 in Section 4.4, for any $\tau_0 \in [0, +\infty]$, we design a couple of functions $\mathbf{q} = (q_2, q_3)$ such
 that $T_{0,\mathbf{q}} = \tau_0$ and

$$\sup_{q \in \mathbb{S}[\mathbf{q}]} T_{0,q} = \sup_{\substack{q \in \text{Span}(q_2, q_3) \\ q \neq 0}} T_{0,q} = 0$$

321 which proves that the inequality in (1.16) can be strict.

322 **1.4. Outline of the article.**

323 Section 2 is dedicated to the proof of the two formulations of the minimal null
 324 control time for system (1.1) stated in Theorems 1.1 and 1.2.

325 In Section 2.3, we give some applications of the obtained formulas: we prove that
 326 they encompass previously known results and let us get precise results in more general
 327 new configurations.

328 Section 3 is dedicated to the analysis of system (1.3). We show that taking into
 329 account first-order coupling terms in our methodology is relatively straightforward,
 330 compared to the original proofs in [16].

331 In Section 4, we determine the minimal null control time for the simultaneous
 332 controllability problem (1.4) as stated in Theorems 1.3 and 1.4.

333 Finally, we have gathered in Appendix A some spectral properties of Sturm-
 334 Liouville operators that are used all along this article.

335 2. A system with a space varying zero order coupling term.

336 In this section we prove the characterizations of the minimal null control time
 337 for system (1.1). We prove Theorem 1.1 in Section 2.1 as an application of the
 338 results of [13]. Then, analyzing the behaviour of the spectral quantities arising in
 339 Theorem 1.1, we prove Theorem 1.2 in Section 2.2.

340 2.1. A first formula for the minimal time.

341 First, let us check that our system (1.1) fits in the formalism of [13]. There, we
 342 considered abstract control problems of the form

$$343 \begin{cases} y'(t) + \mathcal{A}y(t) = \mathcal{B}u(t), \\ y(0) = y_0. \end{cases}$$

344 Thus, for system (1.1), the evolution operator \mathcal{A} is defined by

$$345 \mathcal{A} = \begin{pmatrix} A & 0 \\ q & A \end{pmatrix}, \quad D(\mathcal{A}) = D(A)^2$$

346 and the control operator \mathcal{B} is defined by

$$347 \mathcal{B} : u \in U = L^2(0, 1) \mapsto \begin{pmatrix} \mathbf{1}_\omega u \\ 0 \end{pmatrix}.$$

348 In [13] the results involve a Gelfand triple of Hilbert spaces $X_\diamond^* \subset X \subset X_{-\diamond}$ in
 349 order to deal with possibly unbounded control operators. In the present article we
 350 only consider distributed control operators which implies that there are no particular
 351 subtleties on the functional framework and we shall set here $X_{-\diamond} = X_\diamond^* = X =$
 352 $L^2(0, 1; \mathbb{R})^2$ (see [13, Section 2.1.1]). This implies the wellposedness of system (1.1)
 353 in the sense of [13, Proposition 2].

354 Thus, to use the characterizations of the minimal null control time obtained in [13]
 355 we shall prove that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H) defined in [13,
 356 Section 2.1.2]. Roughly speaking this assumption states that the operator \mathcal{A}^* admits
 357 a complete family of generalized eigenvectors which are observable (*i.e.* not in the
 358 kernel of \mathcal{B}^*). It also requires that the associated family of eigenvalues, each of
 359 them having finite geometric multiplicity and globally bounded algebraic multiplicity,
 360 satisfies a weak-gap assumption (*i.e.* they can be gathered in well separated blocks of
 361 bounded diameter and cardinality) and appropriate estimates on its counting function
 362 (see (A.4) and (A.5)).

363 • Let us detail the spectral analysis of the operator \mathcal{A}^* .

364 Its spectrum is given by $\Lambda = (\nu_k)_{k \geq 1}$. Recall that $I_k(q)$ is defined by (1.7) and
 365 $\psi_{k,q}$ is defined by (1.6). We distinguish the following cases.

366 \star If $I_k(q) \neq 0$ then ν_k is algebraically double and geometrically simple. A Jordan
 367 chain is given by

$$368 \quad (2.1) \quad \phi_k^0 = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}, \quad \phi_k^1 = \frac{1}{I_k(q)} \begin{pmatrix} \psi_{k,q} \\ \varphi_k \end{pmatrix}.$$

369 \star If $I_k(q) = 0$ then ν_k is geometrically double and a basis of eigenvectors is given
 370 by

$$371 \quad (2.2) \quad \phi_{k,1}^0 = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}, \quad \phi_{k,2}^0 = \begin{pmatrix} \psi_{k,q} \\ \varphi_k \end{pmatrix}.$$

372 \bullet Properties of eigenvalues.

373 The eigenvalues of \mathcal{A}^* are real and, due to the assumption $c \geq 0$ they satisfy
 374 $\nu_k > 0$.

375 From (A.1), these eigenvalues satisfy a gap condition with parameter ϱ and thus
 376 a grouping in the sense of [13, Proposition 6] is given by $G_k = \{\nu_k\}$.

377 The associated counting function N satisfies (A.4) and (A.5).

378 Gathering all these properties, we have that the sequence of eigenvalues of \mathcal{A}^*
 379 satisfies

$$380 \quad \Lambda \in \mathcal{L}_w \left(1, \varrho, 0, \frac{1}{2}, \kappa \right)$$

381 as defined in [13, Section 2.1.2].

382 \bullet Properties of eigenvectors.

383 The eigenvalue ν_k is either geometrically simple and algebraically double or semi-
 384 simple with geometric multiplicity 2. Due to the expressions (2.1) and (2.2) we obtain
 385 that the family of (generalized) eigenvectors of \mathcal{A}^* forms a complete family in X .

386 As stated in Section 1.2, the approximate controllability assumption

$$387 \quad \text{Ker}(\mathcal{A}^* - \lambda) \cap \text{Ker} \mathcal{B}^* = \{0\}, \quad \forall \lambda \in \Lambda$$

388 follows from (1.11) and [14, Theorem 2.2].

389 Thus, the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H) stated in [13, Section
 390 2.1.2].

391 *Proof (of Theorem 1.1).* From [13, Theorem 11], for any $y_0 \in X$ the minimal null
 392 control time from y_0 is given by

$$393 \quad T_{0,q}(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln^+ \mathcal{C}(G_k, y_0)}{2\nu_k}$$

394 where $\ln^+ s = \max(0, \ln s)$, for any $s \geq 0$ and the cost of the k -th block is given by

$$395 \quad \mathcal{C}(G_k, y_0) = \inf \left\{ \|\Omega^0\|_U^2 + \|\Omega^1\|_U^2 ; \Omega^0, \Omega^1 \in U \text{ with } \langle \Omega^0, \mathcal{B}^* \phi_k^0 \rangle_U = \langle y_0, \phi_k^0 \rangle_X \right. \\
 396 \quad \left. \text{and } \langle \Omega^0, \mathcal{B}^* \phi_k^1 \rangle_U + \langle \Omega^1, \mathcal{B}^* \phi_k^0 \rangle_U = \langle y_0, \phi_k^1 \rangle_X \right\}$$

398 if $I_k(q) \neq 0$ and

$$399 \quad \mathcal{C}(G_k, y_0) = \inf \left\{ \|\Omega\|_U^2 ; \Omega \in U \text{ with } \langle \Omega, \mathcal{B}^* \phi_{k,j}^0 \rangle_U = \langle y_0, \phi_{k,j}^0 \rangle_X \text{ for } j \in \{1, 2\} \right\}$$

400

401 if $I_k(q) = 0$.

402 To compute $\mathcal{C}(G_k, y_0)$ we distinguish the two cases.

403 \star Assume that $I_k(q) \neq 0$. Then, from [13, Theorem 14], it comes that

$$404 \quad \mathcal{C}(G_k, y_0) = \langle M^{-1}\xi, \xi \rangle$$

405 where

$$406 \quad M = \text{Gram}(\mathcal{B}^* \phi_k^0, \mathcal{B}^* \phi_k^1) + \text{Gram}(0, \mathcal{B}^* \phi_k^0) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\varphi_k\|_\omega^2 + \frac{1}{I_k(q)^2} \|\psi_{k,q}\|_\omega^2 \end{pmatrix}$$

407 and

$$408 \quad \xi = \begin{pmatrix} \langle y_0, \phi_k^0 \rangle_X \\ \langle y_0, \phi_k^1 \rangle_X \end{pmatrix}.$$

409 Thus,

$$410 \quad \mathcal{C}(G_k, y_0) = \frac{1}{\|\varphi_k\|_\omega^2} \left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\rangle_X^2 + \frac{1}{I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2} \left\langle y_0, \begin{pmatrix} \psi_{k,q} \\ \varphi_k \end{pmatrix} \right\rangle_X^2.$$

411 \star Assume that $I_k(q) = 0$. Then, from [13, Theorem 18], it comes that

$$412 \quad \mathcal{C}(G_k, y_0) = \langle M^{-1}\xi, \xi \rangle$$

413 where

$$414 \quad M = \text{Gram}(\mathcal{B}^* \phi_{k,1}^0, \mathcal{B}^* \phi_{k,2}^0) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\psi_{k,q}\|_\omega^2 \end{pmatrix}$$

415 and

$$416 \quad \xi = \begin{pmatrix} \langle y_0, \phi_{k,1} \rangle_X \\ \langle y_0, \phi_{k,2} \rangle_X \end{pmatrix}.$$

417 Thus,

$$418 \quad \mathcal{C}(G_k, y_0) = \frac{1}{\|\varphi_k\|_\omega^2} \left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\rangle_X^2 + \frac{1}{\|\psi_{k,q}\|_\omega^2} \left\langle y_0, \begin{pmatrix} \psi_{k,q} \\ \varphi_k \end{pmatrix} \right\rangle_X^2.$$

419 Finally, in both cases, the cost corresponding to the group G_k is given by

$$420 \quad (2.3) \quad \mathcal{C}(G_k, y_0) = \frac{1}{\|\varphi_k\|_\omega^2} \left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\rangle_X^2 + \frac{1}{I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2} \left\langle y_0, \begin{pmatrix} \psi_{k,q} \\ \varphi_k \end{pmatrix} \right\rangle_X^2.$$

421 We now evaluate the different contributions of the terms in the right-hand side of
422 (2.3).

423 Recall that $\|\varphi_k\|_{(0,1)} = 1$, that $(\psi_{k,q})_k$ is bounded thanks to Lemma A.2, and
424 that, from (A.3), we have

$$425 \quad \|\varphi_k\|_\omega \geq C > 0, \quad \forall k \geq 1.$$

426 Thus, getting back to (2.3), we obtain that

$$427 \quad \mathcal{C}(G_k, y_0) \leq C \|y_0\|_X^2 \left(1 + \frac{1}{I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2} \right), \quad \forall k \geq 1, \forall y_0 \in X,$$

428 which proves that

$$429 \quad T_{0,q}(y_0) \leq \limsup_{k \rightarrow +\infty} \frac{-\ln(I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2)}{2\nu_k}.$$

430 This estimate holds for every y_0 , which gives the upper bound for $T_{0,q}$.

431 To prove the converse inequality let us choose

$$432 \quad y_0 = \sum_{k \geq 1} \frac{1}{\nu_k} \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix},$$

433 which is indeed a converging series in X . From (2.3) we obtain that for this particular
434 choice of y_0 ,

$$435 \quad \mathcal{C}(G_k, y_0) = \frac{1}{\nu_k^2} \frac{1}{I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2}, \quad \forall k \geq 1.$$

436 Thus,

$$437 \quad T_{0,q} \geq T_{0,q}(y_0) = \limsup_{k \rightarrow +\infty} \frac{-\ln(I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2)}{2\nu_k}.$$

438 This ends the proof of Theorem 1.1. \square

439 **2.2. A second formula for the minimal time with disjoint control and** 440 **coupling domains.**

441 The minimal null control time has been characterized in Theorem 1.1. Thus, the
442 proof of Theorem 1.2 consists in comparing the asymptotic behaviors of

$$443 \quad \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)$$

444 and

$$445 \quad I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2.$$

446 To do so we will use the following result whose proof is postponed to the end of the
447 section, to improve the readability.

448 **PROPOSITION 2.1.** *Let $\omega \subset (0, 1)$ be a non empty open set with a finite number of*
449 *connected components.*

450 *i. There exists $K \in \mathbb{N}^*$ and $C > 0$ such that for any $k \geq K$, any $F \in L^2(0, 1)$*
451 *and any u satisfying the differential equation*

$$452 \quad (A - \nu_k)u = F,$$

453 *we have*

$$454 \quad \mathcal{M}_k(F, \omega) \leq C \left(\sqrt{\nu_k} \|u\|_\omega + \sqrt{\nu_k} (|u(0)| + |u(1)|) + \|F\|_\omega \right).$$

455 *ii. There exists $K \in \mathbb{N}^*$ and $C > 0$ such that for any $k \geq K$ and any $F \in L^2(0, 1)$*
456 *such that $\int_0^1 F(x)\varphi_k(x)dx = 0$, there exists u satisfying the boundary value*
457 *problem*

$$458 \quad \begin{cases} (A - \nu_k)u = F, \\ u(0) = u(1) = 0, \end{cases}$$

459 *as well as the estimate*

$$460 \quad \sqrt{\nu_k} \|u\|_\omega \leq C (\mathcal{M}_k(F, \omega) + \|F\|_\omega).$$

461 We now turn to the proof of Theorem 1.2.

462 *Proof (of Theorem 1.2).* Recall that, from Theorem 1.1,

$$463 \quad T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln(I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2)}{2\nu_k}.$$

464 By (1.6), we can apply point i. of Proposition 2.1 to $u = \psi_{k,q}$ and $F = (I_k(q) - q)\varphi_k$
465 to get, for $k \geq K$,

$$466 \quad \begin{aligned} \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)^2 &\leq C(\nu_k \|\psi_{k,q}\|_\omega^2 + \|(I_k(q) - q)\varphi_k\|_\omega^2) \\ 467 &\leq C\nu_k(\|\psi_{k,q}\|_\omega^2 + I_k(q)^2 \|\varphi_k\|_\omega^2), \end{aligned}$$

469 since $\text{Supp}(q) \cap \omega = \emptyset$. Thus,

$$470 \quad T_{0,q} \leq \limsup_{k \rightarrow +\infty} \frac{-\ln \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)}{\nu_k}.$$

471 We now prove the converse inequality. For k large enough, let u be the function given
472 by the point ii. of Proposition 2.1 with $F = (I_k(q) - q)\varphi_k$ (which, by definition of
473 $I_k(q)$, satisfies $\int_0^1 F(x)\varphi_k(x) dx = 0$). We observe that there exists $\alpha \in \mathbb{R}$ such that we
474 can write $u = \psi_{k,q} + \alpha\varphi_k$. Recall that we have imposed in (1.6), that $\langle \varphi_k, \psi_{k,q} \rangle_\omega = 0$,
475 so that we have $\|\psi_{k,q}\|_\omega \leq \|u\|_\omega$. Thus, using the estimate given by point ii. of
476 Proposition 2.1 and the assumption $\text{Supp}(q) \cap \omega = \emptyset$, we obtain that, for any $k \geq K$,

$$477 \quad (2.4) \quad \nu_k \|\psi_{k,q}\|_\omega^2 \leq C(\mathcal{M}_k(F, \omega)^2 + \|F\|_\omega^2) \leq C(\mathcal{M}_k(F, \omega)^2 + I_k(q)^2 \|\varphi_k\|_\omega^2).$$

478 We denote by $\mathfrak{C}_1, \dots, \mathfrak{C}_N$ the connected components of $\overline{(0,1)\omega}$. As $\text{Supp}(q) \cap \omega = \emptyset$,
479 notice that

$$480 \quad \begin{aligned} \sum_{j=1}^N \int_{\mathfrak{C}_j} F(x)\varphi_k(x) dx &= I_k(q) \sum_{j=1}^N \int_{\mathfrak{C}_j} \varphi_k^2(x) dx - \sum_{j=1}^N \int_{\mathfrak{C}_j} q(x)\varphi_k^2(x) dx \\ 481 &= I_k(q)(1 - \|\varphi_k\|_\omega^2) - I_k(q) \\ 482 &= -I_k(q)\|\varphi_k\|_\omega^2. \end{aligned}$$

484 Thus, from (A.3) we deduce that

$$485 \quad |I_k(q)| \|\varphi_k\|_\omega \leq C\mathcal{M}_k(F, \omega).$$

486 Plugging this inequality into (2.4) we obtain

$$487 \quad \|\psi_{k,q}\|_\omega^2 + I_k(q)^2 \|\varphi_k\|_\omega^2 \leq C\mathcal{M}_k(F, \omega)^2.$$

488 This implies that

$$489 \quad T_{0,q} \geq \limsup_{k \rightarrow +\infty} \frac{-\ln \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)}{\nu_k}$$

490 and ends the proof of Theorem 1.2. \square

491 To conclude this section, it remains to prove Proposition 2.1. To do so, we start
492 with the following result that comes from Lemma A.1.

493 LEMMA 2.2. Let A be the Sturm-Liouville operator defined by (1.2) and let $\lambda_0 > 0$.
 494 There exists $C > 0$ depending on γ , c and λ_0 such that, for any $\lambda \geq \lambda_0$, for any
 495 $F \in L^2(0, 1)$, for any $0 < a < b < 1$, for any u satisfying

$$(A - \lambda)u = F \quad \text{on } [a, b],$$

497 and for any $x \in [a, b]$, we have

$$498 \quad |u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \leq \frac{C}{b-a} \left(1 + \frac{1}{\lambda(b-a)^2} \right) \|u\|_{(a,b)}^2 + C \frac{(b-a)}{\lambda} \|F\|_{(a,b)}^2.$$

499 *Proof.* Let $\chi_0 \in C^\infty(\mathbb{R}; \mathbb{R})$ be a cut-off function such that $0 \leq \chi_0 \leq 1$ and

- 500 • $\chi_0(x) = 1$ for every $x \in [1/4, 3/4]$,
- 501 • $\chi_0(x) = 0$ for every $x \notin (0, 1)$.

We then set

$$\chi(x) = \chi_0 \left(\frac{x-a}{b-a} \right),$$

502 in such a way that, if we set $\alpha = a + \frac{b-a}{4}$ and $\beta = b - \frac{b-a}{4}$, we have

- 503 • $\chi(x) = 1$ for every $x \in [\alpha, \beta]$,
- 504 • $\chi(x) = 0$ for every $x \notin (a, b)$.

505 Let $C_1 > 0$ be the constant given by Lemma A.1 and assume that $\lambda \geq \lambda_0$.

506 Let $x \in [a, b]$. We apply Lemma A.1 to obtain for any $y \in (a, b)$

$$507 \quad \left(|u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \right) \leq C_1 \left(|u(y)|^2 + \frac{\gamma(y)}{\lambda} |u'(y)|^2 + \frac{b-a}{\lambda} \|F\|_{(a,b)}^2 \right).$$

508 Integrating in the variable $y \in (\alpha, \beta)$ gives

$$509 \quad (2.5) \quad \frac{b-a}{2} \left(|u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \right) \leq C_1 \left(\|u\|_{(a,b)}^2 + \frac{(b-a)^2}{2\lambda} \|F\|_{(a,b)}^2 \right) \\ + \frac{C_1}{\lambda} \int_\alpha^\beta \gamma(y) |u'(y)|^2 dy.$$

510 Then integrating by parts, using $(A - \lambda)u = 0$ and Cauchy-Schwarz inequality yield

$$511 \quad \frac{1}{\lambda} \int_\alpha^\beta \gamma(y) |u'(y)|^2 dy \leq \frac{1}{\lambda} \int_a^b \chi(y) \gamma(y) |u'(y)|^2 dy \\ 512 \quad = -\frac{1}{\lambda} \int_a^b \chi'(y) (\gamma u')(y) u(y) dy + \frac{1}{\lambda} \int_a^b \chi(y) (\lambda - c(y)) |u(y)|^2 dy \\ 513 \quad \leq \frac{\|\chi'\|_{L^\infty} \|\sqrt{\gamma}\|_{L^\infty}}{\lambda} \|\sqrt{\gamma} u'\|_{(a,b)} \|u\|_{(a,b)} \\ 514 \quad + \int_a^b \chi(y) \left| 1 - \frac{c(y)}{\lambda} \right| |u(y)|^2 dy \\ 515 \quad \leq \left(\frac{1}{\sqrt{\lambda}} \|\sqrt{\gamma} u'\|_{(a,b)} \right) \left(\frac{\|\chi'\|_{L^\infty} \|\sqrt{\gamma}\|_{L^\infty}}{\sqrt{\lambda}} \|u\|_{(a,b)} \right) \\ 516 \quad + \left(1 + \frac{\|c\|_{L^\infty}}{\lambda_0} \right) \|u\|_{(a,b)}^2. \\ 517$$

518 Thus, for any $\bar{C} > 0$,

$$519 \quad \frac{1}{\lambda} \int_{\alpha}^{\beta} \gamma(y) |u'(y)|^2 dy \leq \left(1 + \frac{\|c\|_{L^\infty}}{\lambda_0} + \frac{\|\sqrt{\gamma}\|_{L^\infty}^2 \|\chi'\|_{L^\infty}^2}{4\bar{C}} \frac{1}{\lambda} \right) \|u\|_{(a,b)}^2$$

$$520 \quad + \frac{\bar{C}}{\lambda} \|\sqrt{\gamma}u'\|_{(a,b)}^2.$$

522 Plugging it into estimate (2.5) and using that $\|\chi'\|_{L^\infty} = \|\chi'_0\|_{L^\infty} (b-a)^{-1}$, we obtain
(2.6)

$$523 \quad \frac{b-a}{2} \left(|u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \right) \leq C_1 \left(2 + \frac{\|c\|_{L^\infty}}{\lambda_0} + \frac{\|\sqrt{\gamma}\|_{L^\infty}^2 \|\chi'_0\|_{L^\infty}^2}{4\bar{C}} \frac{1}{\lambda(b-a)^2} \right) \|u\|_{(a,b)}^2$$

$$+ \frac{C_1(b-a)^2}{2\lambda} \|F\|_{(a,b)}^2 + \frac{C_1\bar{C}}{\lambda} \|\sqrt{\gamma}u'\|_{(a,b)}^2.$$

524 Applying again Lemma A.1 gives that, for any $y \in (a, b)$,

$$525 \quad \frac{\gamma(y)}{\lambda} |u'(y)|^2 \leq C_1 \left(|u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \right) + C_1 \frac{b-a}{\lambda} \|F\|_{(a,b)}^2.$$

526 Integrating in the variable $y \in (a, b)$ and setting $\bar{C} = \frac{1}{4C_1^2}$, we obtain

$$527 \quad \frac{C_1\bar{C}}{\lambda} \|\sqrt{\gamma}u'\|_{(a,b)}^2 \leq C_1^2\bar{C}(b-a) \left(|u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \right) + \frac{C_1^2\bar{C}}{\lambda} (b-a)^2 \|F\|_{(a,b)}^2$$

$$528 \quad \leq \frac{b-a}{4} \left(|u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \right) + \frac{1}{4} \frac{(b-a)^2}{\lambda} \|F\|_{(a,b)}^2.$$

530 Plugging it into (2.6) ends the proof of Lemma 2.2. \square

531 We now have all the ingredients to prove Proposition 2.1.

532 *Proof (of Proposition 2.1).* We denote by $\omega_1, \dots, \omega_N$ the connected components
533 of ω labeled such that

$$534 \quad \sup \omega_j \leq \inf \omega_{j+1}, \quad \forall j \in \llbracket 1, N-1 \rrbracket.$$

535 Let

$$536 \quad (2.7) \quad \lambda_0 = \max_{j \in \llbracket 1, N \rrbracket} \frac{1}{|\omega_j|^2}$$

537 and, let $K > 0$ be such that

$$538 \quad k \geq K \implies \nu_k \geq \lambda_0.$$

539 *We start with the proof of item i.*

540 Let $\mathfrak{C} = [a, b]$ be a connected component of $\overline{(0, 1)} \setminus \omega$. Integrating by parts we
541 obtain

$$542 \quad \int_{\mathfrak{C}} F(x) \varphi_k(x) dx = -(\gamma u' \varphi_k)(b) + (\gamma u' \varphi_k)(a) + (u \gamma \varphi'_k)(b) - (u \gamma \varphi'_k)(a).$$

543 Recall that from (A.6),

$$544 \quad |\varphi_k(x)| + \frac{1}{\sqrt{\nu_k}} |\varphi'_k(x)| \leq C, \quad \forall x \in (0, 1), \forall k \geq 1.$$

545 Similarly, applying Lemma A.1 with $y = 0$ we obtain

$$546 \quad (2.8) \quad |\tilde{\varphi}_k(x)| + \frac{1}{\sqrt{\nu_k}} |\tilde{\varphi}'_k(x)| \leq C, \quad \forall x \in (0, 1), \forall k \geq 1.$$

547 Thus,

$$548 \quad \frac{1}{\sqrt{\nu_k}} \left| \int_{\mathfrak{C}} F(x) \varphi_k(x) dx \right| \leq C \left(|u(a)| + \frac{\sqrt{\gamma(a)}}{\sqrt{\nu_k}} |u'(a)| \right) + C \left(|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}} |u'(b)| \right).$$

549 • If $\mathfrak{C} \cap \{0, 1\} = \emptyset$, then there exists $j \in \llbracket 2, N \rrbracket$ such that $a \in \overline{\omega_{j-1}}$ and $b \in \overline{\omega_j}$.
 550 Applying twice Lemma 2.2 (recall that λ_0 is defined by (2.7)) we obtain

$$551 \quad |u(a)| + \frac{\sqrt{\gamma(a)}}{\sqrt{\nu_k}} |u'(a)| \leq C \left(\|u\|_{\omega_{j-1}} + \frac{1}{\sqrt{\nu_k}} \|F\|_{\omega_{j-1}} \right),$$

552 and

$$553 \quad |u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}} |u'(b)| \leq C \left(\|u\|_{\omega_j} + \frac{1}{\sqrt{\nu_k}} \|F\|_{\omega_j} \right)$$

554 where C now also depends on ω . This implies

$$555 \quad \left| \int_{\mathfrak{C}} F(x) \varphi_k(x) dx \right| \leq C (\sqrt{\nu_k} \|u\|_{\omega} + \|F\|_{\omega}).$$

556 The same computations hold for $\left| \int_{\mathfrak{C}} F(x) \tilde{\varphi}_k(x) dx \right|$.

557 • Now, if $a = 0$, taking into account the boundary condition $\varphi_k(a) = 0$, the
 558 same computations yields

$$559 \quad \frac{1}{\sqrt{\nu_k}} \left| \int_{\mathfrak{C}} F(x) \varphi_k(x) dx \right| \leq C |u(0)| + C \left(|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}} |u'(b)| \right).$$

560 As $b \in \overline{\omega}$, applying Lemma 2.2 (recall that λ_0 is defined by (2.7)) we obtain

$$561 \quad |u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}} |u'(b)| \leq C \left(\|u\|_{\omega} + \frac{1}{\sqrt{\nu_k}} \|F\|_{\omega} \right)$$

562 where C now also depends on ω . This implies

$$563 \quad \left| \int_{\mathfrak{C}} F(x) \varphi_k(x) dx \right| \leq C (\sqrt{\nu_k} \|u\|_{\omega} + \sqrt{\nu_k} |u(0)| + \|F\|_{\omega}).$$

564 • Similarly, if $b = 1$, we prove that

$$565 \quad \left| \int_{\mathfrak{C}} F(x) \varphi_k(x) dx \right| \leq C (\sqrt{\nu_k} \|u\|_{\omega} + \sqrt{\nu_k} |u(1)| + \|F\|_{\omega}).$$

566 Gathering these results proves item i.

567 We now turn to the proof of item ii.

568 • We start designing u a solution of

$$569 \quad \begin{cases} (A - \nu_k)u = F, \\ u(0) = u(1) = 0, \end{cases}$$

570 such that

$$571 \quad (2.9) \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}} |u'(x)| \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_{\omega_1}), \quad \forall x \in \overline{\omega_1}.$$

572 To this end let us take any solution \bar{u} of

$$573 \quad \begin{cases} (A - \nu_k)\bar{u} = F, \\ \bar{u}(0) = \bar{u}(1) = 0. \end{cases}$$

574 Such a solution exists since $\int_0^1 F(x)\varphi_k(x)dx = 0$.

575 If $0 \notin \overline{\omega_1}$ we set $b = \inf \omega_1$ whereas if $0 \in \overline{\omega_1}$ we set $b \in \omega_1$. Notice that in
576 both cases

$$577 \quad \int_0^b F(x)\varphi_k(x)dx = \bar{u}(b)\gamma(b)\varphi_k'(b) - \gamma(b)\bar{u}'(b)\varphi_k(b).$$

578 Applying Lemma A.1 with $y = b$, integrating with respect to the variable
579 $x \in (0, 1)$ and using $\|\varphi_k\|_{(0,1)} = 1$ we obtain that there exists $C > 0$ such
580 that

$$581 \quad |\varphi_k(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}} |\varphi_k'(b)| \geq C.$$

582 – If $|\varphi_k(b)| \geq \frac{C}{2}$, we set $u = \bar{u} - \frac{\bar{u}(b)}{\varphi_k(b)}\varphi_k$.

583 Thus, we have $u(b) = 0$ which implies

$$584 \quad \sqrt{\gamma(b)}u'(b) = \frac{-1}{\sqrt{\gamma(b)\varphi_k(b)}} \int_0^b F(x)\varphi_k(x)dx.$$

585 Thus,

$$586 \quad (2.10) \quad |u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}} |u'(b)| \leq \frac{C}{\sqrt{\nu_k}} \left| \int_0^b F(x)\varphi_k(x)dx \right|.$$

587 – Otherwise, we have $\frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}} |\varphi_k'(b)| \geq \frac{C}{2}$. Setting $u = \bar{u} - \frac{\bar{u}'(b)}{\varphi_k'(b)}\varphi_k$, the
588 same computations also imply (2.10).

589 We now prove that (2.10) implies (2.9).

590 As $b \in \overline{\omega_1}$, applying Lemma A.1 and (2.10) we obtain for any $x \in \overline{\omega_1}$,

$$591 \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}} |u'(x)| \leq C \left(|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}} |u'(b)| + \frac{1}{\sqrt{\nu_k}} \|F\|_{\omega_1} \right)$$

$$592 \quad \leq \frac{C}{\sqrt{\nu_k}} \left(\left| \int_0^b F(x)\varphi_k(x)dx \right| + \|F\|_{\omega_1} \right).$$

593

594 – Assume first that $0 \notin \overline{\omega_1}$ and recall that $b = \inf \omega_1$. Then, by definition
 595 of $\mathcal{M}_k(F, \omega)$ (see (1.9)), we have

$$596 \quad \left| \int_0^b F(x) \varphi_k(x) dx \right| \leq \mathcal{M}_k(F, \omega).$$

597 Thus, for any $x \in \overline{\omega_1}$,

$$598 \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}} |u'(x)| \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_{\omega_1}).$$

599 – Otherwise, $0 \in \overline{\omega_1}$ and we have set $b \in \omega_1$. Then, since $(0, b) \subset \omega_1$ and
 600 $\|\varphi_k\|_{(0,1)} = 1$, we have

$$601 \quad \left| \int_0^b F(x) \varphi_k(x) dx \right| \leq \|F\|_{\omega_1}.$$

602 Thus, for any $x \in \overline{\omega_1}$,

$$603 \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}} |u'(x)| \leq \frac{C}{\sqrt{\nu_k}} \|F\|_{\omega_1}.$$

604 Gathering these two cases proves (2.9).

605 • We prove by induction that the function u designed at the previous step
 606 satisfies

$$607 \quad (2.11) \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}} |u'(x)| \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_{\omega}), \quad \forall x \in \overline{\omega_j}.$$

608 The case $j = 1$ is exactly (2.9) that was proved in the previous step. Let
 609 $j \in \llbracket 2, N \rrbracket$ be such that

$$610 \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}} |u'(x)| \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_{\omega}), \quad \forall x \in \overline{\omega_{j-1}}.$$

611 Let $a_j = \sup \omega_{j-1}$ and $b_j = \inf \omega_j$. Integrating by parts we obtain

$$612 \quad \frac{1}{\sqrt{\nu_k}} \int_{a_j}^{b_j} F(x) \varphi_k(x) dx = u(b_j) \frac{\gamma(b_j) \varphi'_k(b_j)}{\sqrt{\nu_k}} - \frac{\gamma(b_j) u'(b_j)}{\sqrt{\nu_k}} \varphi_k(b_j)$$

$$613 \quad - u(a_j) \frac{\gamma(a_j) \varphi'_k(a_j)}{\sqrt{\nu_k}} + \frac{\gamma(a_j) u'(a_j)}{\sqrt{\nu_k}} \varphi_k(a_j).$$

615 The same computations hold replacing φ_k by $\tilde{\varphi}_k$. Using the notation in
 616 Appendix A, this can be rewritten in matrix form as

$$617 \quad (2.12) \quad W_k(b_j) \begin{pmatrix} u(b_j) \\ \frac{\gamma(b_j) u'(b_j)}{\sqrt{\nu_k}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\nu_k}} \int_{a_j}^{b_j} F(x) \varphi_k(x) dx \\ \frac{1}{\sqrt{\nu_k}} \int_{a_j}^{b_j} F(x) \tilde{\varphi}_k(x) dx \end{pmatrix}$$

$$+ W_k(a_j) \begin{pmatrix} u(a_j) \\ \frac{\gamma(a_j) u'(a_j)}{\sqrt{\nu_k}} \end{pmatrix}.$$

618 Using Lemma A.3 and the definition of $\mathcal{M}_k(F, \omega)$ (see (1.9)), we deduce that

$$619 \quad |u(b_j)| + \frac{\sqrt{\gamma(b_j)}}{\sqrt{\nu_k}} |u'(b_j)| \leq \frac{C}{\sqrt{\nu_k}} \mathcal{M}_k(F, \omega) + C|u(a_j)| + C \frac{\sqrt{\gamma(a_j)}}{\sqrt{\nu_k}} |u'(a_j)|.$$

620 As $a_j \in \overline{\omega_{j-1}}$ the induction hypothesis imply

$$621 \quad |u(a_j)| + \frac{\sqrt{\gamma(a_j)}}{\sqrt{\nu_k}} |u'(a_j)| \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_\omega)$$

622 and thus we conclude that

$$623 \quad |u(b_j)| + \frac{\sqrt{\gamma(b_j)}}{\sqrt{\nu_k}} |u'(b_j)| \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_\omega).$$

624 As $b_j \in \overline{\omega_j}$, applying Lemma A.1 we obtain for any $x \in \overline{\omega_j}$

$$625 \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}} |u'(x)| \leq C \left(|u(b_j)| + \frac{\sqrt{\gamma(b_j)}}{\sqrt{\nu_k}} |u'(b_j)| + \frac{1}{\sqrt{\nu_k}} \|F\|_{\omega_j} \right)$$

$$626 \quad \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_\omega).$$

627

628 This proves (2.11).

629 • Conclusion : from (2.11) we obtain

$$630 \quad |u(x)| \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_\omega), \quad \forall x \in \omega_j, \forall j \in \llbracket 1, N \rrbracket.$$

631 This leads to

$$632 \quad \|u\|_{\omega_j} \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_\omega), \quad \forall j \in \llbracket 1, N \rrbracket$$

633 with a new value of C and ends the proof of item ii. □

634 2.3. Application of the minimal null control time formulas.

635 In this section we apply the characterizations of the minimal null control time
636 obtained in Theorems 1.1 and 1.2 to different specific configurations.

637 In Section 2.3.1, we recover previous characterizations of the minimal null control
638 time proved in [4, 5] when ω is an interval. Note however that in the above references,
639 explicit computations of eigenelements when A is the Laplace Dirichlet operator are
640 used. Our analysis does not make use of such computations and thus extend those
641 results to any Sturm-Liouville operator as defined in (1.2).

642 In Section 2.3.2, we recover null controllability in arbitrary time when q has a
643 strict sign on a part of ω as proved in [21].

644 Finally, in Section 2.3.3 we prove a new null controllability result for an explicit
645 q when ω is the union of two intervals.

646 **2.3.1. Unification of previous formulas for the minimal null control**
647 **time.**

648 Let us prove that the obtained results unifies previous characterizations given in
649 the literature and stated in Section 1.2.

650 • Let us consider the setting studied in [4] *i.e.*, $\omega = (a, b)$ and $\text{Supp}(q) \subset (b, 1)$.

651 In this case, $(0, 1) \setminus \omega$ has at most two connected components both touching the
652 boundary of $(0, 1)$. Thus, setting

653
$$F = (I_k(q) - q)\varphi_k$$

654 we obtain

655
$$\mathcal{M}_k(F, \omega) = \max \left\{ \left| \int_0^a F(x)\varphi_k(x)dx \right|, \left| \int_b^1 F(x)\varphi_k(x)dx \right| \right\}.$$

656 Using the assumption $\text{Supp}(q) \subset (b, 1)$ we get

657
$$\left| \int_0^a F(x)\varphi_k(x)dx \right| = |I_k(q)| \int_0^a \varphi_k^2(x)dx,$$

658 and

659
$$\left| \int_b^1 F(x)\varphi_k(x)dx \right| = \left| I_k(q) \int_b^1 \varphi_k^2(x)dx - I_k(q) \right| = |I_k(q)| \int_0^b \varphi_k^2(x)dx.$$

660 This gives

661
$$\mathcal{M}_k(F, \omega) = |I_k(q)| \int_0^b \varphi_k^2(x)dx.$$

662 Recall that from (A.3)

663
$$\inf_{k \geq 1} \int_0^b \varphi_k^2(x)dx > 0.$$

664 This implies that approximate controllability holds if and only if

665
$$I_k(q) \neq 0, \quad \forall k \geq 1,$$

666 and in this case that

667
$$T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln |I_k(q)|}{\nu_k}.$$

668 Thus we have extended the result proved in [4] for the Dirichlet-Laplace operator to
669 a general Sturm-Liouville operator.

670 • Let us now consider the setting studied in [5] *i.e.*, $\omega = (a, b)$ and $\text{Supp}(q) \cap \omega = \emptyset$.

671 Again, setting

672
$$F = (I_k(q) - q)\varphi_k$$

673 we obtain

674
$$\mathcal{M}_k(F, \omega) = \max \left\{ \left| \int_0^a F(x)\varphi_k(x)dx \right|, \left| \int_b^1 F(x)\varphi_k(x)dx \right| \right\}.$$

675 Using the notations introduced in (1.12) we have

676 (2.13)
$$\int_0^a F(x)\varphi_k(x)dx = I_k(q) \int_0^a \varphi_k^2(x)dx - I_{1,k}(q)$$

678 and

$$679 \quad (2.14) \quad \int_b^1 F(x)\varphi_k(x)dx = I_k(q) \int_b^1 \varphi_k^2(x)dx - I_{2,k}(q).$$

680 Thus,

$$681 \quad \mathcal{M}_k(F, \omega) \leq 2 \max \{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\}.$$

682 Conversely, using (2.13) and (2.14) we have

$$\begin{aligned} 683 \quad & \int_0^a F(x)\varphi_k(x)dx + \int_b^1 F(x)\varphi_k(x)dx \\ 684 \quad & = I_k(q) \left(\int_0^a \varphi_k^2(x)dx + \int_b^1 \varphi_k^2(x)dx \right) - (I_{1,k}(q) + I_{2,k}(q)) \\ 685 \quad & = -I_k(q) \int_a^b \varphi_k(x)^2 dx \\ 686 \end{aligned}$$

687 where we have used that $I_k(q) = I_{1,k}(q) + I_{2,k}(q)$. Thus, from (A.3) we get

$$688 \quad |I_k(q)| \leq C\mathcal{M}_k(F, \omega).$$

689 Using (2.13) or (2.14) and the previous inequality we obtain

$$690 \quad |I_{j,k}(q)| \leq C\mathcal{M}_k(F, \omega), \quad \forall j \in \{1, 2\}.$$

691 Thus,

$$692 \quad \max \{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\} \leq C\mathcal{M}_k(F, \omega).$$

693 This implies that approximate controllability holds if and only if

$$694 \quad \max \{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\} \neq 0, \quad \forall k \geq 1$$

695 and in this case

$$696 \quad T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln \max \{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\}}{\nu_k}.$$

697 Thus we have extended the result proved in [5] for the Dirichlet-Laplace operator to
698 a general Sturm-Liouville operator.

699 **2.3.2. Null controllability in arbitrary time with intersecting control** 700 **and coupling regions.**

701 Let us here consider the setting studied in [21].

PROPOSITION 2.3. *Assume that there exists an open set $\omega_0 \subset \omega$ and $q_0 > 0$ such that*

$$\inf_{\omega_0} q \geq q_0 \quad \text{or} \quad \sup_{\omega_0} q \leq -q_0,$$

702 *then, system (1.1) is null controllable in any time $T > 0$.*

703 Even though this result is already known from [21], we provide here a proof without
704 Carleman estimates.

705 *Proof.* We assume that $\inf_{\omega_0} q \geq q_0$ since the other case is similar. Here we
706 consider the minimal time characterization given by Theorem 1.1 and we shall prove
707 that $I_k(q)^2 \|\varphi_k\|_{\omega}^2 + \|\psi_{k,q}\|_{\omega}^2$ does not tend to zero exponentially fast, with respect to
708 ν_k , as k goes to infinity.

709 We split $\psi_{k,q}$ into two parts $\psi_{k,q} = \psi_{k,q,1} + \psi_{k,q,2}$ where $\psi_{k,q,1}$ is the unique
710 solution of the Cauchy problem

$$711 \quad (2.15) \quad \begin{cases} (A - \nu_k)\psi_{k,q,1} = I_k(q)\varphi_k, \\ \psi_{k,q,1}(0) = 0, \\ \psi'_{k,q,1}(0) = 0. \end{cases}$$

712 From Lemma A.1, there exists $C > 0$ depending only on γ and c such that

$$713 \quad \|\psi_{k,q,1}\|_{L^\infty(0,1)} \leq \frac{C}{\sqrt{\nu_k}} |I_k(q)|.$$

714 Then, from (A.3), we deduce that, when $\nu_k \geq 1$,

$$715 \quad \begin{aligned} \|\psi_{k,q,2}\|_{\omega}^2 &\leq 2(\|\psi_{k,q}\|_{\omega}^2 + \|\psi_{k,q,1}\|_{L^\infty(0,1)}^2) \\ 716 \quad &\leq 2\left(\|\psi_{k,q}\|_{\omega}^2 + \frac{C}{\nu_k} |I_k(q)|^2\right) \\ 717 \quad &\leq C(\|\psi_{k,q}\|_{\omega}^2 + |I_k(q)|^2 \|\varphi_k\|_{\omega}^2). \end{aligned}$$

719 Thus, in order to prove the result, it is enough to find some explicit lower bound
720 $r_k > 0$ such that

$$721 \quad (2.16) \quad \|\psi_{k,q,2}\|_{\omega} \geq r_k \quad \text{with} \quad \limsup_{k \rightarrow +\infty} \frac{-\ln r_k}{\nu_k} = 0.$$

722 As we seek for a lower bound for $\psi_{k,q,2}$ on ω , and thanks to our assumption on q ,
723 we can restrict ω to an interval (a, b) such that $q(x) \geq q_0 > 0$ for almost every $x \in \omega$.
724 Taking some $\ell > 0$ small enough to be determined later (see (2.19)), we introduce the
725 following subsets of ω :

- 726 • $\omega_1 = (a, a + \ell)$;
- 727 • $\omega_2 = (b - \ell, b)$;
- 728 • $\tilde{\omega} = \omega_1 \cup \omega_2$;
- 729 • $\mathfrak{C}_0 = \left[\frac{a+b}{2} - \frac{b-a}{6}, \frac{a+b}{2} + \frac{b-a}{6}\right]$;
- 730 • $\mathfrak{C} = [a + \ell, b - \ell]$.

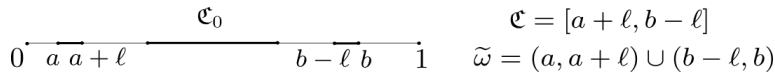


FIG. 1. *Splitting of $\omega = (a, b)$*

731 This configuration is pictured in Figure 1. Notice that $\tilde{\omega}$ is a subset of ω and thus
732 for any $k \geq 1$,

$$733 \quad \|\psi_{k,q,2}\|_{\omega}^2 \geq \|\psi_{k,q,2}\|_{\tilde{\omega}}^2.$$

734 From (A.3), there exists $\alpha_1 > 0$ depending on γ , c and \mathfrak{C}_0 such that

$$735 \quad (2.17) \quad \int_{\mathfrak{C}_0} \varphi_k^2(x) dx \geq \alpha_1, \quad \forall k \geq 1.$$

736 Following closely the proof of item i of Proposition 2.1 with a careful tracking of
 737 the dependency with respect to ℓ we can obtain the following lemma whose proof is
 738 postponed at the end of the section.

739 LEMMA 2.4. *There exists $\alpha_2 > 0$ depending on γ and c such that for any $\ell < \frac{b-a}{3}$,
 740 any $k \geq 1$ such that $\nu_k \geq 1$, any $F \in L^2(0, 1; \mathbb{R})$ and any u satisfying the differential
 741 equation*

$$(A - \nu_k)u = F,$$

742 we have

$$(2.18) \quad \sqrt{\ell} \left| \int_{\mathfrak{C}} F(x) \varphi_k(x) dx \right| \leq \alpha_2 \sqrt{\nu_k} \left(1 + \frac{1}{\ell \sqrt{\nu_k}} \right) \|u\|_{\tilde{\omega}} + \alpha_2 \ell \|F\|_{\tilde{\omega}}.$$

745 It is important to notice that the norms in the right-hand side of (2.18) are taken on
 746 the small set $\tilde{\omega}$ whereas the left-hand side is an integral on the large set $\mathfrak{C} = \omega \setminus \tilde{\omega}$.
 747 Hence, this inequality can be understood as an estimate of cancellations that occur
 748 in this integral.

749 Let $\alpha_2 > 0$ be the constant given in the above lemma and assume in all what
 750 follows that $\ell > 0$ is fixed such that

$$(2.19) \quad \sqrt{\ell} < \min \left\{ \sqrt{\frac{b-a}{3}}, \frac{q_0 \alpha_1}{2\alpha_2 \|q\|_{L^\infty(0,1)}} \right\}.$$

751 There exists $K \in \mathbb{N}^*$ such that

$$(2.20) \quad \nu_k \geq \frac{1}{\ell^2}, \quad \forall k \geq K.$$

752 In the rest of the proof, we assume that $k \geq K$.

753 Thanks to the equations (1.6) and (2.15) satisfied respectively by $\psi_{k,q}$ and $\psi_{k,q,1}$,
 754 we see that $\psi_{k,q,2}$ solves

$$(A - \nu_k) \psi_{k,q,2} = -q \varphi_k, \quad \text{in } (0, 1).$$

755 Applying Lemma 2.4, with $u = \psi_{k,q,2}$ and $F = -q \varphi_k$ we obtain

$$(2.21) \quad \sqrt{\ell} \left| \int_{\mathfrak{C}} q(x) \varphi_k^2(x) dx \right| \leq 2\alpha_2 \sqrt{\nu_k} \|\psi_{k,q,2}\|_{\tilde{\omega}} + \alpha_2 \ell \|q \varphi_k\|_{\tilde{\omega}}.$$

756 As $\ell < \frac{b-a}{3}$ we have $\mathfrak{C}_0 \subset \mathfrak{C}$ and thus

$$(2.22) \quad \left| \int_{\mathfrak{C}} q(x) \varphi_k^2(x) dx \right| \geq q_0 \int_{\mathfrak{C}_0} \varphi_k^2(x) dx \geq q_0 \alpha_1.$$

757 Notice also that, since $\|\varphi_k\|_{(0,1)} = 1$, we have

$$\|q \varphi_k\|_{\tilde{\omega}} \leq \|q\|_{L^\infty(0,1)}.$$

758 Gathering these estimates and using (2.19) we obtain the lower bound

$$(2.23) \quad \begin{aligned} 2\alpha_2 \sqrt{\nu_k} \|\psi_{k,q,2}\|_{\tilde{\omega}} &\geq \sqrt{\ell} \left| \int_{\mathfrak{C}} q(x) \varphi_k^2(x) dx \right| - \alpha_2 \ell \|q \varphi_k\|_{\tilde{\omega}} \\ &\geq \sqrt{\ell} \left(q_0 \alpha_1 - \alpha_2 \sqrt{\ell} \|q\|_{L^\infty(0,1)} \right) \geq \sqrt{\ell} \frac{q_0 \alpha_1}{2} \end{aligned}$$

759 which leads to (2.16) and ends the proof of Proposition 2.3. \square

769 It remains to prove the lemma.

770 *Proof (of Lemma 2.4).* From (A.6), there exists $C > 0$ depending on γ and c such
771 that

$$772 \quad (2.20) \quad \|\varphi_k\|_{L^\infty(0,1)} + \frac{1}{\sqrt{\nu_k}} \|\varphi'_k\|_{L^\infty(0,1)} \leq C, \quad \forall k \geq 1.$$

773 Integrating by parts, we obtain

$$774 \quad \int_{\mathfrak{e}} F(x) \varphi_k(x) dx = \int_{a+\ell}^{b-\ell} (A - \nu_k) u(x) \varphi_k(x) dx$$

$$775 \quad = -(\gamma u' \varphi_k)(b-\ell) + (\gamma u' \varphi_k)(a+\ell)$$

$$776 \quad + (u \gamma \varphi'_k)(b-\ell) - (u \gamma \varphi'_k)(a+\ell).$$

778 Using (2.20) we obtain

$$779 \quad \frac{1}{\sqrt{\nu_k}} \left| \int_{\mathfrak{e}} F(x) \varphi_k(x) dx \right| \leq C \|\sqrt{\gamma}\|_{L^\infty} \left(|u(a+\ell)| + \frac{\sqrt{\gamma(a+\ell)}}{\sqrt{\nu_k}} |u'(a+\ell)| \right)$$

$$780 \quad + C \|\sqrt{\gamma}\|_{L^\infty} \left(|u(b-\ell)| + \frac{\sqrt{\gamma(b-\ell)}}{\sqrt{\nu_k}} |u'(b-\ell)| \right).$$

782 Let $\lambda_0 = 1$ and let $K \in \mathbb{N}^*$ be such that

$$783 \quad k \geq K \implies \nu_k \geq \lambda_0.$$

784 Assume that $k \geq K$. As $a + \ell \in \overline{\omega}_1$ the application of Lemma 2.2 (recall that $\lambda_0 = 1$)
785 yields

$$786 \quad |u(a+\ell)| + \frac{\sqrt{\gamma(a+\ell)}}{\sqrt{\nu_k}} |u'(a+\ell)| \leq \frac{C}{\sqrt{\ell}} \left(1 + \frac{1}{\ell \sqrt{\nu_k}} \right) \|u\|_{\omega_1} + \frac{C\sqrt{\ell}}{\sqrt{\nu_k}} \|F\|_{\omega_1}.$$

787 As $b - \ell \in \overline{\omega}_2$ the application of Lemma 2.2 yields

$$788 \quad |u(b-\ell)| + \frac{\sqrt{\gamma(b-\ell)}}{\sqrt{\nu_k}} |u'(b-\ell)| \leq \frac{C}{\sqrt{\ell}} \left(1 + \frac{1}{\ell \sqrt{\nu_k}} \right) \|u\|_{\omega_2} + \frac{C\sqrt{\ell}}{\sqrt{\nu_k}} \|F\|_{\omega_2}$$

789 which concludes the proof. \square

790 2.3.3. Dealing with new geometric configurations.

791 We now illustrate that the minimal time formula obtained in Theorem 1.2 can be
792 successfully exploited to give an explicit value of this minimal time in more general
793 geometric configurations than the one available in the literature, for example when
794 ω is not an interval and $\text{Supp}(q) \cap \omega = \emptyset$. We provide below an example inspired
795 by [14].

796 **PROPOSITION 2.5.** *Let A be the Dirichlet Laplace operator (i.e., $\gamma = 1$ and $c = 0$)*
797 *and let*

$$798 \quad q : x \in (0, 1) \mapsto \left(x - \frac{1}{2} \right) \mathbf{1}_{\left(\frac{1}{4}, \frac{3}{4}\right)}(x).$$

799 *i. If $\omega \subset \left(\frac{3}{4}, 1\right)$, then approximate controllability for system (1.1) does not hold.*

800 *ii. If $\omega = (0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$, then system (1.1) is null controllable from X in any*
 801 *time $T > 0$.*

802 *Proof.* In this case, we have for any $k \geq 1$,

$$803 \quad \nu_k = k^2\pi^2, \quad \varphi_k = \sqrt{2}\sin(k\pi\bullet), \quad \tilde{\varphi}_k = \cos(k\pi\bullet).$$

804 The proof of item i can be found in [14, Section 3.3.1] and relies on explicit com-
 805 putations: due to symmetry it comes that $I_k(q) = 0$ for any $k \geq 1$. This implies
 806 that

$$807 \quad \int_0^{\inf(\omega)} q(x)\varphi_k(x)\varphi_k(x)dx = I_k(q) = 0.$$

808 Let \mathfrak{C} be any other connected component of $\overline{(0, 1) \setminus \omega}$ than $[0, \inf(\omega)]$. Then $\mathfrak{C} \subset (\frac{3}{4}, 1)$.
 809 This means that $q = 0$ on \mathfrak{C} which gives

$$810 \quad \int_{\mathfrak{C}} q(x)\varphi_k(x)\varphi_k(x)dx = \int_{\mathfrak{C}} q(x)\varphi_k(x)\tilde{\varphi}_k(x)dx = 0.$$

811 Thus,

$$812 \quad \mathcal{M}_k(q\varphi_k, \omega) = 0, \quad \forall k \geq 1.$$

813 We now turn to item ii. In this case $\overline{(0, 1) \setminus \omega}$ has only one connected component
 814 which is $[\frac{1}{4}, \frac{3}{4}]$ but the key point is that it does not touch the boundary of $(0, 1)$.
 815 Approximate controllability in this case was also studied in [14, Section 3.3.1]. Again
 816 for symmetry reasons we have

$$817 \quad \int_{\frac{1}{4}}^{\frac{3}{4}} q(x)\varphi_k(x)\varphi_k(x)dx = 0, \quad \forall k \geq 1,$$

818 but

$$819 \quad \int_{\frac{1}{4}}^{\frac{3}{4}} q(x)\varphi_k(x)\tilde{\varphi}_k(x)dx = \begin{cases} -\frac{(-1)^{\frac{k-1}{2}}}{2\sqrt{2}\pi^2k^2}, & \text{if } k \text{ is odd,} \\ -\frac{(-1)^{\frac{k}{2}}}{4\sqrt{2}\pi k}, & \text{if } k \text{ is even.} \end{cases}$$

820 This implies that for any $k \geq 1$,

$$821 \quad \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega) = \begin{cases} \frac{1}{2\sqrt{2}\pi^2k^2}, & \text{if } k \text{ is odd,} \\ \frac{1}{4\sqrt{2}\pi k}, & \text{if } k \text{ is even.} \end{cases}$$

822 Thus, from Theorem 1.2, we get

$$823 \quad T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)}{\nu_k} = 0,$$

824 which means that null-controllability holds at any time $T > 0$. □

825 **3. Cascade system with a first order coupling term.**

826 In this section we describe how the analysis conducted in Section 2 can be directly
 827 extended to system (1.3) that is when the coupling between the two equations operates
 828 through a zero order term and a first order term. This is for instance the setting
 829 studied in [16] and that we complete here.

830 **3.1. Setting and spectral analysis.**

831 To fit in the formalism of [13], we define

- 832 • the evolution operator \mathcal{A} by

$$833 \quad \mathcal{A} = \begin{pmatrix} A & 0 \\ q + p\partial_x & A \end{pmatrix}, \quad D(\mathcal{A}) = D(A)^2,$$

834 and the control operator \mathcal{B} by

$$835 \quad \mathcal{B} : u \in U = L^2(0, 1) \mapsto \begin{pmatrix} \mathbf{1}_\omega u \\ 0 \end{pmatrix}.$$

836 It will be convenient to separate the symmetric and skew-symmetric parts of
 837 the coupling terms in \mathcal{A} . In order to do so, we define a function r and an
 838 operator S_p as follows

$$839 \quad (3.1) \quad r = q - \frac{1}{2}p', \text{ and } S_p = \frac{1}{2}p' + p\partial_x.$$

840 We observe that S_p is skew-symmetric in $D(A)$ and that we can write

$$841 \quad \mathcal{A} = \begin{pmatrix} A & 0 \\ r + S_p & A \end{pmatrix}.$$

- 842 • The adjoint operator of \mathcal{A} is given by

$$843 \quad \mathcal{A}^* = \begin{pmatrix} A & q - \partial_x(p\bullet) \\ 0 & A \end{pmatrix} = \begin{pmatrix} A & r - S_p \\ 0 & A \end{pmatrix}, \quad D(\mathcal{A}^*) = D(\mathcal{A}).$$

844 Recall that $I_k(r)$ is defined by (1.7). In this section, $\psi_{k,r,p}$ denotes the unique
 845 solution of

$$846 \quad (3.2) \quad \begin{cases} (A - \nu_k)\psi_{k,r,p} = (I_k(r) - r)\varphi_k + S_p\varphi_k, \\ \psi_{k,r,p}(0) = \psi_{k,r,p}(1) = 0, \\ \langle \varphi_k, \psi_{k,r,p} \rangle_\omega = 0. \end{cases}$$

847 This system has indeed a unique solution since, due to the definition of $I_k(r)$ and the
 848 fact that S_p is skew-symmetric, the right-hand side of this equation is orthogonal to
 849 φ_k .

850 Let us detail the spectral analysis of the operator \mathcal{A}^* : its spectrum is given by $\Lambda =$
 851 $(\nu_k)_{k \geq 1}$ and we can distinguish the following cases.

- 852 • If $I_k(r) \neq 0$ then ν_k is algebraically double and geometrically simple. An
 853 associated Jordan chain is given by

$$854 \quad (3.3) \quad \phi_k^0 = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}, \quad \phi_k^1 = \frac{1}{I_k(r)} \begin{pmatrix} \psi_{k,r,p} \\ \varphi_k \end{pmatrix}.$$

855 • If $I_k(r) = 0$ then ν_k is geometrically double and an associated basis of eigen-
 856 vectors is given by

$$857 \quad (3.4) \quad \phi_{k,1}^0 = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}, \quad \phi_{k,2}^0 = \begin{pmatrix} \psi_{k,r,p} \\ \varphi_k \end{pmatrix}.$$

858 Except from the definition of $\psi_{k,r,p}$, the spectral analysis is the same as for system (1.1)
 859 (see Section 2.1). Thus, for the operators \mathcal{A} and \mathcal{B} to satisfy the assumption (H) stated
 860 in [13, Section 2.1.2] it only remains to study the approximate controllability condition

$$861 \quad \text{Ker}(\mathcal{A}^* - \lambda) \cap \text{Ker} \mathcal{B}^* = \{0\}, \quad \forall \lambda \in \Lambda.$$

862 This is the goal of the following section.

863 3.2. Approximate controllability.

864 From the Fattorini-Hautus test, we obtain the following characterization for ap-
 865 proximate controllability of system (1.3).

866 PROPOSITION 3.1. *Let $\omega \subset (0, 1)$ be a non empty set and let $q \in L^\infty(0, 1)$ and
 867 $p \in W^{1,\infty}(0, 1)$. Approximate controllability of system (1.3) holds if and only if*

$$868 \quad (3.5) \quad \mathcal{M}_k(r\varphi_k - S_p\varphi_k, \omega) \neq 0, \quad \forall k \geq 1 \text{ such that } r\varphi_k - S_p\varphi_k = 0 \text{ in } \omega.$$

869 The proof follows directly from [14, Theorems 2.1 and 2.2].

870 Notice that, for approximate controllability to hold, we have two very different
 871 situations.

- 872 • When $(\text{Supp}(q) \cup \text{Supp}(p)) \cap \omega = \emptyset$ condition (3.5) has to be checked for any
 873 $k \geq 1$.
- 874 • Whereas, when $(\text{Supp}(q) \cup \text{Supp}(p)) \cap \omega \neq \emptyset$ condition (3.5) has to be checked
 875 for at a most a single $k \geq 1$.

876 REMARK 3.1. *The question of approximate controllability for system (1.3) was
 877 already studied in [16, Theorems 1.1 and 1.2]. There it is stated that, if*

$$878 \quad (\text{Supp}(p) \cup \text{Supp}(q)) \neq \emptyset,$$

879 *approximate controllability holds in any time. In fact, this result is not correct since
 880 there can exist $k \geq 1$ such that*

$$881 \quad r\varphi_k - S_p\varphi_k = 0 \text{ in } \omega \quad \text{and} \quad \mathcal{M}_k(r\varphi_k - S_p\varphi_k, \omega) = 0.$$

882 *Such a counter-example was constructed by A. Dupouy in her Master Thesis [15],
 883 under the supervision of the first author.*

884 *We set $q = 0$, which implies that $r - S_p = -\partial_x(p\bullet)$. For a given $k \geq 1$, the
 885 idea is to select an interval $\omega = (a, b)$ such that $\varphi_k \neq 0$ on $\bar{\omega}$, which is possible since
 886 φ_k has only finitely many zeros in $(0, 1)$. Then, we choose $p = \frac{1}{\varphi_k}$ in ω so that, by
 887 construction $r\varphi_k - S_p\varphi_k = -\partial_x(p\varphi_k) = 0$ in ω . Finally, it is possible to extend p
 888 outside ω with appropriate regularity such that*

$$889 \quad \int_0^a \varphi_k(x) \partial_x(p\varphi_k)(x) dx = \int_b^1 \varphi_k(x) \partial_x(p\varphi_k)(x) dx = 0,$$

890 *i.e., $\mathcal{M}_k(r\varphi_k - S_p\varphi_k, \omega) = 0$.*

891 Due to the analysis conducted in Section 3.1, under the assumption (3.5), the
 892 operators \mathcal{A} and \mathcal{B} satisfy the assumption (H) stated in [13, Section 2.1.2].

893 For more coherence with the expression of the minimal null control time obtained
 894 in Theorem 3.5 below, instead of the approximate controllability condition (3.5), we
 895 use the following characterization.

896 LEMMA 3.2. *Let $\omega \subset (0, 1)$ be a non empty set and let $q \in L^\infty(0, 1)$ and $p \in$
 897 $W^{1, \infty}(0, 1)$. Assume that $(\text{Supp}(q) \cup \text{Supp}(p)) \cap \omega = \emptyset$. Then, for any $k \geq 1$,*

$$898 \quad \mathcal{M}_k(r\varphi_k - S_p\varphi_k, \omega) = 0 \iff \mathcal{M}_k((I_k(r) - r)\varphi_k + S_p\varphi_k, \omega) = 0.$$

899 Thus, approximate controllability of system (1.3) holds if and only if

$$900 \quad (3.6) \quad \mathcal{M}_k((I_k(r) - r)\varphi_k + S_p\varphi_k, \omega) \neq 0, \quad \forall k \geq 1 \text{ such that } r\varphi_k - S_p\varphi_k = 0 \text{ in } \omega.$$

901 *Proof.* Let $k \geq 1$. First of all notice that for any connected component \mathfrak{C} of
 902 $(0, 1) \setminus \omega$ we have

$$903 \quad (3.7) \quad \int_{\mathfrak{C}} (S_p\varphi_k)(x)\varphi_k(x)dx = 0.$$

904 Indeed, for any $a, b \in [0, 1]$ such that $p(a) = p(b) = 0$, integrating by parts we obtain

$$\begin{aligned} 905 \quad \int_a^b (S_p\varphi_k)(x)\varphi_k(x)dx &= \int_a^b \left(\frac{1}{2}\partial_x p(x)\varphi_k(x) + p(x)\partial_x \varphi_k(x) \right) \varphi_k(x)dx \\ 906 \quad &= - \int_a^b p(x)\partial_x \varphi_k(x)\varphi_k(x)dx + \int_a^b p(x)\partial_x \varphi_k(x)\varphi_k(x)dx \\ 907 \quad &= 0. \end{aligned}$$

909 Thus, the assumption $\text{Supp}(p) \cap \omega = \emptyset$ proves (3.7).

910 Now assume that $\mathcal{M}_k(r\varphi_k - S_p\varphi_k, \omega) = 0$. Then, using (3.7), for any connected
 911 component \mathfrak{C} of $(0, 1) \setminus \omega$ we have

$$912 \quad \int_{\mathfrak{C}} r(x)\varphi_k^2(x)dx = \int_{\mathfrak{C}} (S_p\varphi_k)(x)\varphi_k(x)dx = 0.$$

913 Since $\text{Supp}(r) \cap \omega = \emptyset$, this gives

$$914 \quad I_k(r) = \sum_{\mathfrak{C} \in \mathcal{C}((0, 1) \setminus \omega)} \int_{\mathfrak{C}} r(x)\varphi_k(x)^2 dx = 0$$

915 which proves that $\mathcal{M}_k((I_k(r) - r)\varphi_k + S_p\varphi_k, \omega) = 0$.

916 Finally assume that $\mathcal{M}_k((I_k(r) - r)\varphi_k + S_p\varphi_k, \omega) = 0$. Then, using (3.7), for
 917 any connected component \mathfrak{C} of $(0, 1) \setminus \omega$ we have

$$918 \quad I_k(r) \int_{\mathfrak{C}} \varphi_k(x)^2 dx = \int_{\mathfrak{C}} r(x)\varphi_k(x)^2 dx.$$

919 Since $\text{Supp}(r) \cap \omega = \emptyset$, this gives

$$\begin{aligned}
920 \quad I_k(r) \left(1 - \|\varphi_k\|_\omega^2\right) &= I_k(r) \sum_{\mathfrak{c} \in \mathcal{C}(\overline{(0,1)} \setminus \omega)} \int_{\mathfrak{c}} \varphi_k(x)^2 dx \\
921 \quad &= \sum_{\mathfrak{c} \in \mathcal{C}(\overline{(0,1)} \setminus \omega)} \int_{\mathfrak{c}} r(x) \varphi_k(x)^2 dx \\
922 \quad &= I_k(r).
\end{aligned}$$

924 Using (A.3) we obtain $I_k(r) = 0$ and thus $\mathcal{M}_k(r\varphi_k - S_p\varphi_k, \omega) = 0$. \square

925 3.3. Minimal null control time.

926 We now turn to the determination of the minimal null control time. For this
927 system, we have a result which is similar to Theorem 1.1 and that reads as follows.

928 **THEOREM 3.3.** *Let $\omega \subset (0, 1)$ be a non empty open set and let $q \in L^\infty(0, 1)$ and
929 $p \in W^{1,\infty}(0, 1)$. Assume that (3.5) holds. Then, the minimal null control time $T_{0,q,p}$
930 for system (1.3) is given by*

$$931 \quad T_{0,q,p} = \limsup_{k \rightarrow +\infty} \frac{-\ln(I_k(r)^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,r,p}\|_\omega^2)}{2\nu_k}$$

932 where $\psi_{k,r,p}$ is given by (3.2).

933 The proof follows exactly the proof of Theorem 1.1 and is left to the reader. The only
934 difference is that, due to the change of definition of $\psi_{k,r,p}$ one cannot use Lemma A.2
935 but shall instead use the following lemma.

936 **LEMMA 3.4.** *There exists $C > 0$ such that*

$$937 \quad \|\psi_{k,r,p}\|_{(0,1)} \leq C, \quad \forall k \geq 1$$

938 where $\psi_{k,r,p}$ is given by (3.2).

939 The proof follows the proof of Lemma A.2 with the use of the estimate

$$940 \quad \int_0^1 \partial_x(p\varphi_k)(x)^2 dx \leq 2\|p'\|_\infty^2 + 2 \int_0^1 p(x)^2 \varphi_k'(x)^2 dx \leq C\nu_k$$

941 due to (A.2).

942 Then, as in Theorem 1.2, we can simplify the formula in the case where the
943 coupling terms are not active in the control domain.

944 **THEOREM 3.5.** *Let $\omega \subset (0, 1)$ be a non empty open set with a finite number of
945 connected components. Let $q \in L^\infty(0, 1)$ and $p \in W^{1,\infty}(0, 1)$ be such that*

$$946 \quad (\text{Supp}(q) \cup \text{Supp}(p)) \cap \omega = \emptyset.$$

947 Assume that (3.6) holds. Then, the minimal null control time for system (1.1) is given
948 by

$$949 \quad T_{0,q,p} = \limsup_{k \rightarrow +\infty} \frac{-\ln \mathcal{M}_k((I_k(r) - r)\varphi_k + S_p\varphi_k, \omega)}{\nu_k}$$

950 where \mathcal{M}_k is defined by (1.9).

951 The proof follows exactly the proof of Theorem 1.2 and is left to the reader.

952 **3.4. Applications of the minimal null control time formulas.**

953 **3.4.1. When the coupling is not active in the control region.** In this
 954 section, we assume that

955 (3.8)
$$(\text{Supp}(p) \cup \text{Supp}(q)) \cap \omega = \emptyset.$$

956 • Assume first that $\omega = (a, b)$ is an interval.
 957 In that case, and when A is the Dirichlet-Laplace operator, it is proved in [16,
 958 Theorem 1.4] that, under the condition (3.8), when approximate controllability
 959 holds, the minimal null control time is given by

960 (3.9)
$$T_{0,q,p} = \limsup_{k \rightarrow +\infty} \frac{-\ln \max \{|I_k(r)|, |I_{1,k}(r)|, |I_{2,k}(r)|\}}{\nu_k},$$

961 where $I_{1,k}$ and $I_{2,k}$ are defined in (1.12).
 962 Let us show that the formulation given in Theorem 3.5 allows to recover this
 963 result, for a general diffusion operator A .
 964 Since ω is an interval, setting

965
$$F = (I_k(r)\varphi_k - r\varphi_k) + S_p\varphi_k,$$

966 we have

967
$$\mathcal{M}_k(F, \omega) = \max \left\{ \left| \int_0^a F(x)\varphi_k(x)dx \right|, \left| \int_b^1 F(x)\varphi_k(x)dx \right| \right\}.$$

Due to the assumption $\text{Supp}(p) \cap \omega = \emptyset$ we can use (3.7) to get

$$\int_0^a (S_p\varphi_k)\varphi_k dx = \int_b^1 (S_p\varphi_k)\varphi_k dx = 0.$$

968 Thus, it follows that

969
$$\int_0^a F(x)\varphi_k(x)dx = I_k(r) \int_0^a \varphi_k^2(x)dx - I_{1,k}(r)$$

970 and

971
$$\int_b^1 F(x)\varphi_k(x)dx = I_k(r) \int_b^1 \varphi_k^2(x)dx - I_{2,k}(r).$$

972 The rest of the proof follows that of Section 2.3.1, by using Theorem 3.5.
 973 • In the previous point it appears that the minimal control time given in (3.9)
 974 only depends on the quantity r . We will show now that when the control
 975 domain ω is not an interval, this may not be true any more. More precisely, we
 976 shall design an example such that $r = 0$, but nevertheless null controllability
 977 holds for any time $T > 0$.
 978 Assume that $\omega = (0, a) \cup (b, 1)$ with $0 < a < b < 1$. The main difference
 979 with the previous situation comes from the fact that $(0, 1) \setminus \omega$ has a (unique)
 980 connected component that does not touch the boundary of the domain, which
 981 makes an important difference in the definition of the quantities \mathfrak{M}_k , see
 982 Section 1.2.1.

983 We build our example as follows. We first choose a smooth function p sup-
 984 ported in (a, b) and such that

$$985 \quad (3.10) \quad \int_a^b \frac{p(x)}{\gamma(x)} dx \neq 0.$$

986 We now set $q = \frac{p'}{2}$ in such a way that $r = q - \frac{1}{2}p' = 0$. Moreover, by
 987 assumption on p , the condition (3.8) holds.

For any k , since r and p are supported outside ω , we immediately have that

$$r\varphi_k - S_p\varphi_k = 0, \text{ in } \omega,$$

and, by (1.8) and (3.7), we get

$$\mathcal{M}_k((I_k(r) - r)\varphi_k + S_p\varphi_k, \omega) = \left| \int_a^b (S_p\varphi_k)(x)\tilde{\varphi}_k(x)dx \right|.$$

By definition of S_p we can integrate by parts, using that $p(a) = p(b) = 0$, to find

$$\int_a^b (S_p\varphi_k)(x)\tilde{\varphi}_k(x)dx = \frac{1}{2} \int_a^b \frac{p(x)}{\gamma(x)} W_k(x)dx,$$

where $W_k = (\gamma\varphi_k')\tilde{\varphi}_k - \varphi_k(\gamma\tilde{\varphi}_k')$ is the Wronskian of φ_k and $\tilde{\varphi}_k$. Since φ_k and $\tilde{\varphi}_k$ solve the same second order linear ODE, this Wronskian is constant and we get

$$\int_a^b (S_p\varphi_k)(x)\tilde{\varphi}_k(x)dx = \frac{\gamma(0)\varphi_k'(0)}{2} \int_a^b \frac{p(x)}{\gamma(x)} dx.$$

Thanks to the assumption (3.10) we see that this quantity is not zero, which proves the approximate controllability condition (3.6). In addition, by using Theorem 3.5 and the asymptotics (A.2), it follows that the minimal null control time for our system is simply given by

$$T_{0,q,p} = \limsup_{k \rightarrow +\infty} \frac{-\ln |\varphi_k'(0)|}{\nu_k} = 0.$$

988 In this case, despite the fact that $r = 0$, we get that the system is null-
 989 controllable at any time $T > 0$.

990 Observe that if the control domain is restricted to $\omega = (0, a)$ (or $\omega = (b, 1)$)
 991 then this particular system is not even approximately controllable.

992 **3.4.2. When the coupling is active in the control region.** We now use the
 993 formulation given in Theorem 3.3 and the computations done in Section 2.3.2 to get
 994 the following sufficient condition for null controllability in arbitrary small time.

995 **PROPOSITION 3.6.** *Assume that the coefficients defining the Sturm-Liouville oper-
 996 ator A in (1.2) are sufficiently regular, i.e., $\gamma \in C^2([0, 1])$ and $c \in C^0([0, 1])$. Assume
 997 that there exists an open set $\omega_0 \subset \omega$ and $r_0 > 0$ such that*

$$998 \quad (3.11) \quad \inf_{\omega_0} r \geq r_0 \quad \text{or} \quad \sup_{\omega_0} r \leq -r_0$$

999 *and that the approximate controllability condition (3.5) holds. Then, system (1.3) is*
 1000 *null controllable at any time $T > 0$.*

1001 We observe that the approximate controllability condition is crucial in this result.
 1002 For instance, the example shown in Remark 3.1 is not approximately controllable even
 1003 if we have $r = -\frac{1}{2} \left(\frac{1}{\phi_k} \right)'$ which clearly satisfies (3.11).

1004 *Proof.* The proof follows closely the one in Section 2.3.2 but needs to be adapted to
 1005 handle some boundary terms coming from integration by parts in integrals involving
 1006 the first order coupling terms. We assume that $\inf_{\omega_0} r \geq r_0$, the other case being
 1007 similar.

1008 From Theorem 3.3, it is sufficient to prove that the quantity $I_k(r) \|\varphi_k\|_{\omega}^2 +$
 1009 $\|\psi_{k,r,p}\|_{\omega}^2$ does not tend exponentially fast to zero with respect to the eigenvalue
 1010 ν_k .

1011 The contribution of $I_k(r)$ is dealt with as in Section 2.3.2 by writing $\psi_{k,r,p} =$
 1012 $\psi_{k,r,p,1} + \psi_{k,r,p,2}$ with $\psi_{k,r,p,1}$ solving the Cauchy problem

$$1013 \quad \begin{cases} (A - \nu_k)\psi_{k,r,p,1} = I_k(r)\varphi_k, \\ \psi_{k,r,p,1}(0) = 0, \\ \psi'_{k,r,p,1}(0) = 0. \end{cases}$$

1014 It is thus sufficient to obtain a lower bound of the following form

$$1015 \quad (3.12) \quad \|\psi_{k,r,p,2}\|_{\omega}^2 \geq R_k \text{ with } \limsup_{k \rightarrow +\infty} \frac{-\ln R_k}{\nu_k} = 0,$$

1016 where $\psi_{k,r,p,2}$ satisfies the equation

$$1017 \quad (A - \nu_k)\psi_{k,r,p,2} = -r\varphi_k + S_p\varphi_k.$$

1018 As we seek for a lower bound it is sufficient to assume that $\omega = (a, b)$ is an interval
 1019 and that $r(x) \geq r_0$ for almost every $x \in \omega$.

1020 Due to Sturm oscillation theorem (see for instance [12, Corollary A.4.33]), there
 1021 exists $\ell_0 \in (0, 1)$ and $K \in \mathbb{N}^*$ depending on γ, c and $b - a$, such that for any $k \geq K$
 1022 there exists $c_k, d_k \in (a, b)$ satisfying

$$1023 \quad (3.13) \quad \begin{cases} \varphi_k(c_k) = \varphi_k(d_k) = 0, \\ |d_k - c_k| \geq \frac{3}{4}|b - a| \quad \text{and} \quad \min(|b - d_k|, |c_k - a|) \geq \ell_0, \\ \nu_k^2 \geq \frac{2}{\ell_0}. \end{cases}$$

1024 For every $k \geq K$, we now set

$$1025 \quad (3.14) \quad \ell_k = \frac{1}{\nu_k^2}, \quad \forall k \geq 1.$$

1026 To mimic the proof of Section 2.3.2, we introduce a_k and b_k such that $a_k + \ell_k = c_k$
 1027 and $b_k - \ell_k = d_k$. By the last point of (3.13) we see that $\ell_k \leq \frac{1}{2}\ell_0$ so that we have
 1028 $(a_k, b_k) \subset (a, b)$.

1029 We now operate a splitting of the interval (a_k, b_k) similar to that of Section 2.3.2
 1030 that is we set

- 1031 • $\tilde{\omega}_k = (a_k, a_k + \ell_k) \cup (b_k - \ell_k, b_k)$,
- 1032 • $\mathfrak{C}_k = [a_k + \ell_k, b_k - \ell_k]$
- 1033 • and $\mathfrak{C}_0 = \left[\frac{a+b}{2} - \frac{b-a}{6}, \frac{a+b}{2} + \frac{b-a}{6} \right]$.

1034 Notice that, by construction, we have $\mathfrak{C}_0 \subset \mathfrak{C}_k$ for every $k \geq 1$.

1035 From (A.3), there exists $\alpha_1 > 0$ depending on γ , c and \mathfrak{C}_0 such that

$$1036 \quad (3.15) \quad \int_{\mathfrak{C}_0} \varphi_k^2(x) dx \geq \alpha_1, \quad \forall k \geq 1.$$

1037 Applying Lemma 2.4, with $u = \psi_{k,r,p,2}$ and $F = -r\varphi_k + S_p\varphi_k$ we obtain

$$1038 \quad \sqrt{\ell_k} \left| \int_{\mathfrak{C}_k} F(x)\varphi_k(x) dx \right| \leq \alpha_2 \sqrt{\nu_k} \left(1 + \frac{1}{\ell_k \sqrt{\nu_k}} \right) \|\psi_{k,r,p,2}\|_{\bar{\omega}_k} + \alpha_2 \ell_k \|F\|_{\bar{\omega}_k}.$$

1039 Using (A.6) we obtain the existence of $\bar{C} > 0$ depending on γ , c , q and p such that

$$1040 \quad \|F\|_{\bar{\omega}_k} \leq 2 \left(\int_0^1 r(x)^2 \varphi_k(x)^2 + p'(x)^2 \varphi_k(x)^2 + p(x)^2 \varphi_k'(x)^2 dx \right)^{\frac{1}{2}} \leq \bar{C} \sqrt{\nu_k}.$$

1041 Thus,

$$1042 \quad \sqrt{\ell_k} \left| \int_{\mathfrak{C}_k} F(x)\varphi_k(x) dx \right| \leq \alpha_2 \sqrt{\nu_k} \left(1 + \frac{1}{\ell_k \sqrt{\nu_k}} \right) \|\psi_{k,r,p,2}\|_{\bar{\omega}_k} + \alpha_2 \bar{C} \ell_k \sqrt{\nu_k}.$$

1043 Since $\mathfrak{C}_0 \subset \mathfrak{C}_k$ we have

$$1044 \quad \int_{\mathfrak{C}_k} F(x)\varphi_k(x) dx = - \int_{\mathfrak{C}_k} r(x)\varphi_k(x)^2 dx,$$

1045 because the contribution of $S_p\varphi_k$ in this integral is zero, by integration by parts using
1046 the first point in (3.13). This integration by parts is the reason of the adjustments
1047 needed compared to Section 2.3.2. Thus,

$$1048 \quad \left| \int_{\mathfrak{C}_k} F(x)\varphi_k(x) dx \right| \geq r_0 \int_{\mathfrak{C}_0} \varphi_k^2(x) dx \geq r_0 \alpha_1.$$

1049 Gathering these estimates we obtain

$$1050 \quad \alpha_2 \sqrt{\nu_k} \left(1 + \frac{1}{\ell_k \sqrt{\nu_k}} \right) \|\psi_{k,r,p,2}\|_{\bar{\omega}_k} \geq \sqrt{\ell_k} \left(r_0 \alpha_1 - \alpha_2 \bar{C} \sqrt{\ell_k \nu_k} \right).$$

1051 Using the definition of ℓ_k in (3.14), it follows

$$1052 \quad \|\psi_{k,r,p,2}\|_{\bar{\omega}_k} \geq \frac{1}{\alpha_2 \nu_k^{3/2} (1 + \nu_k^{3/2})} \left(r_0 \alpha_1 - \frac{\alpha_2 \bar{C}}{\sqrt{\nu_k}} \right).$$

1053 This proves (3.12) and ends the proof of Proposition 3.6. \square

1054 **4. Simultaneous controllability of systems with a space varying zero** 1055 **order coupling term.**

1056 This section is dedicated to the analysis of the minimal null control time for the
1057 simultaneous null controllability problem stated in (1.4). In Section 4.1 we detail the
1058 spectral analysis of the underlying evolution operator. Section 4.2 is dedicated to the
1059 proof of the first formulation for the minimal null control time given in Theorem 1.3.
1060 Using the computations done in Section 2.2, we then deduce in Section 4.3 the second
1061 formulation given in Theorem 1.4. Finally an example is considered in Section 4.4.

1062 **4.1. Spectral analysis.**

1063 To fit again in the formalism of [13], we define the evolution operator \mathcal{A} in the
1064 state space $X = (L^2(0, 1))^3$ by

$$1065 \quad \mathcal{A} = \begin{pmatrix} A & 0 & 0 \\ q_2 & A & 0 \\ q_3 & 0 & A \end{pmatrix}, \quad D(\mathcal{A}) = D(A)^3$$

1066 and the control operator \mathcal{B} by

$$1067 \quad \mathcal{B} : u \in U = L^2(0, 1) \mapsto \begin{pmatrix} \mathbf{1}_\omega u \\ 0 \\ 0 \end{pmatrix}.$$

1068 The spectrum of \mathcal{A}^* is given by $\Lambda = \{\nu_k ; k \geq 1\}$ and thus, as proved in Sec-
1069 tion 2.1,

$$1070 \quad \Lambda \in \mathcal{L}_w \left(1, \varrho, 0, \frac{1}{2}, \kappa \right)$$

1071 as defined in [13, Section 2.1.2].

1072 In any case,

$$1073 \quad \phi_{k,1}^0 = \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix}$$

1074 is an eigenvector of \mathcal{A}^* associated to the eigenvalue ν_k . Recall that, for any $q \in$
1075 $L^\infty(0, 1)$, the function $\psi_{k,q}$ is defined by (1.6).

1076 \star Case *i*. If $I_k(q_2) = I_k(q_3) = 0$ then ν_k is geometrically triple. A basis of
1077 associated eigenvectors of \mathcal{A}^* is given by

$$1078 \quad (4.1) \quad \phi_{k,1}^0, \quad \phi_{k,2}^0 = \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}, \quad \phi_{k,3}^0 = \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}.$$

1079 \star Case *ii a*). If $I_k(q_2) = 0$ and $I_k(q_3) \neq 0$ then ν_k is geometrically double and
1080 algebraically double. A basis of the generalized eigenspace of \mathcal{A}^* is given by

$$1081 \quad (4.2) \quad \phi_{k,1}^0, \quad \phi_{k,2}^0 = \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}, \quad \phi_{k,1}^1 = \frac{1}{I_k(q_3)} \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}$$

1082 where $\phi_{k,1}^0$ and $\phi_{k,2}^0$ are eigenvectors and the generalized eigenvector $\phi_{k,1}^1$ satisfies

$$1083 \quad (\mathcal{A}^* - \nu_k)\phi_{k,1}^1 = \phi_{k,1}^0.$$

1084 \star Case *ii b*). If $I_k(q_2) \neq 0$ and $I_k(q_3) = 0$ then ν_k is geometrically double and
1085 algebraically double. A basis of the generalized eigenspace of \mathcal{A}^* is given by

$$1086 \quad (4.3) \quad \phi_{k,1}^0, \quad \phi_{k,2}^0 = \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}, \quad \phi_{k,1}^1 = \frac{1}{I_k(q_2)} \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}$$

1087 where $\phi_{k,1}^0$ and $\phi_{k,2}^0$ are eigenvectors and the generalized eigenvector $\phi_{k,1}^1$ satisfies

$$1088 \quad (\mathcal{A}^* - \nu_k)\phi_{k,1}^1 = \phi_{k,1}^0.$$

1089 \star Case *iii*. If $I_k(q_2) \neq 0$ and $I_k(q_3) \neq 0$ then ν_k is geometrically double and
 1090 algebraically double. A basis of the generalized eigenspace of \mathcal{A}^* is given by

$$1091 \quad (4.4) \quad \phi_{k,1}^0, \quad \phi_{k,2}^0 = I_k(q_3) \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} - I_k(q_2) \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}, \quad \phi_{k,1}^1 = \frac{1}{I_k(q_2)} \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}$$

1092 where $\phi_{k,1}^0$ and $\phi_{k,2}^0$ are eigenvectors and the generalized eigenvector $\phi_{k,1}^1$ satisfies

$$1093 \quad (\mathcal{A}^* - \nu_k)\phi_{k,1}^1 = \phi_{k,1}^0.$$

1094 Thus, using (4.1)-(4.4), we obtain that the family of (generalized) eigenvectors
 1095 forms a complete family in X .

1096 From [14, Theorem 3.2], the approximate controllability assumption

$$1097 \quad \text{Ker}(\mathcal{A}^* - \lambda) \cap \text{Ker} \mathcal{B}^* = \{0\}, \quad \forall \lambda \in \Lambda$$

1098 is equivalent to (1.14).

1099 Thus, the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H) stated in [13, Section
 1100 2.1.2].

1101 4.2. Characterization of the minimal null control time.

1102 This section is devoted to the proof of Theorem 1.3.

1103 4.2.1. An abstract characterization of the minimal null control time.

1104 Since the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H) stated in [13, Section
 1105 2.1.2] it comes from [13, Theorem 11] that, for any $y_0 \in X$, the minimal null control
 1106 time for system (1.4) from y_0 is given by

$$1107 \quad (4.5) \quad T_{0,q}(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln^+ \mathcal{C}(G_k, y_0)}{2\nu_k}$$

1108 where $\ln^+ s = \max(0, \ln s)$, for any $s \geq 0$ and the cost of the k -th block is given by

1109 \bullet in case *i*

$$1110 \quad (4.6) \quad \mathcal{C}(G_k, y_0) = \inf \left\{ \|\Omega\|_U^2 ; \Omega \in U \right. \\ \left. \text{with } \langle \Omega, \mathcal{B}^* \phi_{k,j}^0 \rangle_U = \langle y_0, \phi_{k,j}^0 \rangle_X \text{ for } j \in \{1, 2, 3\} \right\}$$

1111 \bullet and in cases *ii a)*, *ii b)* and *iii*

$$1112 \quad (4.7) \quad \mathcal{C}(G_k, y_0) = \inf \left\{ \|\Omega^0\|_U^2 + \|\Omega^1\|_U^2 ; \Omega^0, \Omega^1 \in U \right. \\ \left. \text{with } \langle \Omega^0, \mathcal{B}^* \phi_{k,j}^0 \rangle_U = \langle y_0, \phi_{k,j}^0 \rangle_X \text{ for } j \in \{1, 2\} \right. \\ \left. \text{and } \langle \Omega^0, \mathcal{B}^* \phi_{k,1}^1 \rangle_U + \langle \Omega^1, \mathcal{B}^* \phi_{k,1}^0 \rangle_U = \langle y_0, \phi_{k,1}^1 \rangle_X \right\}.$$

1113 The proof of Theorem 1.3 consists in computing the quantity $\mathcal{C}(G_k, y_0)$ and evaluating
 1114 its asymptotic behaviour.

1115 From [13, Theorem 18], in case *i*, an explicit expression of the cost $\mathcal{C}(G_k, y_0)$ of
 1116 the block is given by

$$1117 \quad (4.8) \quad \mathcal{C}(G_k, y_0) = \langle M^{-1}\xi, \xi \rangle$$

1118 where

$$\begin{aligned}
1119 \quad \xi &= \begin{pmatrix} \langle y_0, \phi_{k,1}^0 \rangle_X \\ \langle y_0, \phi_{k,2}^0 \rangle_X \\ \langle y_0, \phi_{k,3}^0 \rangle_X \end{pmatrix} \quad \text{and} \quad M = \text{Gram}_U (\mathcal{B}^* \phi_{k,1}^0, \mathcal{B}^* \phi_{k,2}^0, \mathcal{B}^* \phi_{k,3}^0) \\
1120 \quad &= \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 & 0 \\ 0 & \|\psi_{k,q_2}\|_\omega^2 & \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega \\ 0 & \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega & \|\psi_{k,q_3}\|_\omega^2 \end{pmatrix}. \\
1121
\end{aligned}$$

1122 Since cases *ii a)*, *ii b)* and *iii* involve algebraic and geometric multiplicities occurring
1123 simultaneously inside the same block, we cannot apply [13, Theorem 14] nor [13,
1124 Theorem 18] to get a similar expression. We compute such an explicit expression in
1125 the next subsection.

1126 4.2.2. An intermediate optimization argument.

1127 As detailed in [13, Section 5.4], when both algebraic and geometric multiplicities
1128 appear in the same group, one can repeat the arguments developed there to obtain
1129 an explicit expression of the cost of the block. This is what we do in the following
1130 proposition.

1131 **PROPOSITION 4.1.** *Let U be a real Hilbert space. Let $b_1^0, b_2^0, b_1^1 \in U$ be such that b_1^0
1132 and b_2^0 are linearly independent. Then, for any $\omega_1^0, \omega_2^0, \omega_1^1 \in \mathbb{R}$,*

$$\begin{aligned}
1133 \quad &\inf \left\{ \|\Omega^0\|_U^2 + \|\Omega^1\|_U^2 ; \Omega^0, \Omega^1 \in U \right. \\
1134 \quad &\quad \left. \text{with } \langle \Omega^0, b_j^0 \rangle_U = \omega_j^0 \text{ for } j \in \{1, 2\} \text{ and } \langle \Omega^0, b_1^1 \rangle_U + \langle \Omega^1, b_1^1 \rangle_U = \omega_1^1 \right\} \\
1135 \quad &= \langle M^{-1} \xi, \xi \rangle
\end{aligned}$$

1137 where

$$1138 \quad M = \text{Gram}_U (b_1^0, b_2^0, b_1^1) + \text{Gram}_U (0, 0, b_1^1) \quad \text{and} \quad \xi = \begin{pmatrix} \omega_1^0 \\ \omega_2^0 \\ \omega_1^1 \end{pmatrix}.$$

1139 *Proof.* First of all, notice that by projection the infimum can be computed for

$$1140 \quad \Omega^0, \Omega^1 \in \text{Span} (b_1^0, b_2^0, b_1^1).$$

1141 Thus, we are solving a finite dimensional optimization problem with a quadratic
1142 coercive functional and linear constraints. It admits a unique solution characterized
1143 by the existence of multipliers $m_1^0, m_2^0, m_1^1 \in \mathbb{R}$ such that

$$\begin{aligned}
1144 \quad (4.9) \quad \langle \Omega^0, H^0 \rangle_U + \langle \Omega^1, H^1 \rangle_U &= m_1^0 \langle H^0, b_1^0 \rangle_U + m_2^0 \langle H^0, b_2^0 \rangle_U \\
&\quad + m_1^1 \left(\langle H^0, b_1^1 \rangle_U + \langle H^1, b_1^1 \rangle_U \right)
\end{aligned}$$

1145 for any $H^0, H^1 \in U$.

1146 Using the constraints $\langle \Omega^0, b_j^0 \rangle_U = \omega_j^0$ for $j \in \{1, 2\}$ and $\langle \Omega^0, b_1^1 \rangle_U + \langle \Omega^1, b_1^1 \rangle_U = \omega_1^1$
1147 and choosing successively

- 1148 • $H^0 = b_1^0$ and $H^1 = 0$,
- 1149 • $H^0 = b_2^0$ and $H^1 = 0$,

1150 • $H^0 = b_1^1$ and $H^1 = b_1^0$
 1151 yields

$$1152 \quad (4.10) \quad \begin{pmatrix} \omega_1^0 \\ \omega_2^0 \\ \omega_1^1 \end{pmatrix} = M \begin{pmatrix} m_1^0 \\ m_2^0 \\ m_1^1 \end{pmatrix}$$

1153 with

$$1154 \quad M = \text{Gram}_U(b_1^0, b_2^0, b_1^1) + \text{Gram}_U(0, 0, b_1^0).$$

1155 We now prove that M is invertible. Let $x \in \mathbb{R}^3$ be such that $Mx = 0$. Then,

$$1156 \quad 0 = \langle Mx, x \rangle = \|x_1 b_1^0 + x_2 b_2^0 + x_3 b_1^1\|_U^2 + x_3^2 \|b_1^0\|_U^2.$$

1157 This implies $x_3 = 0$. Then, since b_1^0 and b_2^0 are assumed to be linearly independent,
 1158 we obtain $x_1 = x_2 = 0$. Getting back to (4.10), this gives

$$1159 \quad \begin{pmatrix} m_1^0 \\ m_2^0 \\ m_1^1 \end{pmatrix} = M^{-1}\xi.$$

1160 Finally, choosing $H^0 = \Omega^0$ and $H^1 = \Omega^1$ in (4.9) yields that the sought infimum is

$$1161 \quad \|\Omega^0\|_U^2 + \|\Omega^1\|_U^2 = \left\langle \begin{pmatrix} m_1^0 \\ m_2^0 \\ m_1^1 \end{pmatrix}, \xi \right\rangle = \langle M^{-1}\xi, \xi \rangle$$

1162 which ends the proof of Proposition 4.1. □

1163 4.2.3. Spectral characterization of the minimal null control time.

1164 To prove Theorem 1.3 we now give a more explicit expression for the quantity
 1165 $\mathcal{C}(G_k, y_0)$.

1166 LEMMA 4.2. *For any $k \geq 1$, let $\mathcal{C}(G_k, y_0)$ be defined by (4.6)-(4.7). Then,*

$$1167 \quad (4.11) \quad \mathcal{C}(G_k, y_0) = \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix} \right\rangle_X^2}{\|\varphi_k\|_\omega^2} + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \zeta_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X \zeta_{k,q_2} \right\|_\omega^2}{\|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2}$$

1168 where $\zeta_{k,\bullet}$ is defined in (1.17).

1169 *Proof.* The explicit expression of $\mathcal{C}(G_k, y_0)$ is given either by (4.8) or by Proposi-
 1170 tion 4.1.

1171 In all cases, we have

$$1172 \quad \mathcal{C}(G_k, y_0) = \langle M^{-1}\xi, \xi \rangle$$

1173 where, due to the choice of normalization $\langle \psi_{k,q_j}, \varphi_k \rangle_\omega = 0$, the matrix M has the
1174 form

$$1175 \quad M = \begin{pmatrix} m_{1,1} & 0 & 0 \\ 0 & m_{2,2} & m_{2,3} \\ 0 & m_{2,3} & m_{3,3} \end{pmatrix}.$$

1176 Thus, explicit computations yields

$$1177 \quad \langle M^{-1}\xi, \xi \rangle = \frac{1}{m_{1,1}} \xi_1^2 + \frac{1}{m_{2,2}m_{3,3} - m_{2,3}^2} (m_{3,3}\xi_2^2 - 2m_{2,3}\xi_2\xi_3 + m_{2,2}\xi_3^2).$$

1178 We now distinguish the different cases.

1179 • Case *i*.

1180 We have

$$1181 \quad \xi = \begin{pmatrix} \left\langle y_0, \phi_{k,1}^0 \right\rangle_X \\ \left\langle y_0, \phi_{k,2}^0 \right\rangle_X \\ \left\langle y_0, \phi_{k,3}^0 \right\rangle_X \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 & 0 \\ 0 & \|\psi_{k,q_2}\|_\omega^2 & \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega \\ 0 & \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega & \|\psi_{k,q_3}\|_\omega^2 \end{pmatrix}$$

1182 where $\phi_{k,1}^0$, $\phi_{k,2}^0$ and $\phi_{k,3}^0$ are defined in (4.1). Thus,

$$1183 \quad \langle M^{-1}\xi, \xi \rangle = \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix} \right\rangle_X^2}{\|\varphi_k\|_\omega^2} + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \psi_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X \psi_{k,q_2} \right\|_\omega^2}{\|\psi_{k,q_2}\|_\omega^2 \|\psi_{k,q_3}\|_\omega^2 - \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega^2}$$

1184 Notice that, due to the approximate controllability assumption

$$1185 \quad \text{Ker}(\mathcal{A}^* - \lambda) \cap \text{Ker} \mathcal{B}^* = \{0\}, \quad \forall \lambda \in \mathbb{R},$$

1186 we have $\|\psi_{k,q_2}\|_\omega^2 \|\psi_{k,q_3}\|_\omega^2 - \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega^2 > 0$.

1187 • Case *ii a*).

1188 We have

$$1189 \quad \xi = \begin{pmatrix} \left\langle y_0, \phi_{k,1}^0 \right\rangle_X \\ \left\langle y_0, \phi_{k,2}^0 \right\rangle_X \\ \left\langle y_0, \phi_{k,1}^1 \right\rangle_X \end{pmatrix}$$

1190 and

$$1191 \quad M = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 & 0 \\ 0 & \|\psi_{k,q_2}\|_\omega^2 & \frac{1}{I_k(q_3)} \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega \\ 0 & \frac{1}{I_k(q_3)} \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega & \frac{1}{I_k(q_3)^2} \|\psi_{k,q_3}\|_\omega^2 + \|\varphi_k\|_\omega^2 \end{pmatrix}$$

1192 where $\phi_{k,1}^0$, $\phi_{k,2}^0$ and $\phi_{k,1}^1$ are defined in (4.2). Thus,

$$\begin{aligned}
1193 \quad \langle M^{-1}\xi, \xi \rangle &= \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix} \right\rangle_X^2}{\|\varphi_k\|_\omega^2} + \frac{I_k(q_3)^2 \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X^2 \|\varphi_k\|_\omega^2}{\|\psi_{k,q_2}\|_\omega^2 \left(\|\psi_{k,q_3}\|_\omega^2 + I_k(q_3) \|\varphi_k\|_\omega^2 \right) - \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega^2} \\
1194 \quad &+ \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \psi_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X \psi_{k,q_2} \right\|_\omega^2}{\|\psi_{k,q_2}\|_\omega^2 \left(\|\psi_{k,q_3}\|_\omega^2 + I_k(q_3) \|\varphi_k\|_\omega^2 \right) - \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega^2}. \\
1195
\end{aligned}$$

1196 Using the normalization condition $\langle \psi_{k,q_j}, \varphi_k \rangle_\omega = 0$, this can be rewritten as

$$1197 \quad \langle M^{-1}\xi, \xi \rangle = \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix} \right\rangle_X^2}{\|\varphi_k\|_\omega^2} + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \zeta_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X \psi_{k,q_2} \right\|_\omega^2}{\|\psi_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \psi_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2}.$$

1198 Notice that, from Cauchy-Schwarz inequality,

$$\begin{aligned}
1199 \quad \|\psi_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \psi_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2 &= \|\psi_{k,q_2}\|_\omega^2 \left(\|\psi_{k,q_3}\|_\omega^2 + I_k(q_3)^2 \|\varphi_k\|_\omega^2 \right) \\
1200 \quad &\quad - \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega^2 \\
1201 \quad &\geq I_k(q_3)^2 \|\psi_{k,q_2}\|_\omega^2 \|\varphi_k\|_\omega^2.
\end{aligned}$$

1203 Then, due to the approximate controllability assumption

$$1204 \quad \text{Ker}(\mathcal{A}^* - \lambda) \cap \text{Ker} \mathcal{B}^* = \{0\}, \quad \forall \lambda \in \mathbb{R},$$

1205 we have $\|\psi_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \psi_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2 > 0$.

1206 • Case ii b).

1207 This case is exactly case ii a) when exchanging the roles of q_2 and q_3 . Thus,

$$1208 \quad \langle M^{-1}\xi, \xi \rangle = \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix} \right\rangle_X^2}{\|\varphi_k\|_\omega^2} + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \psi_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X \zeta_{k,q_2} \right\|_\omega^2}{\|\zeta_{k,q_2}\|_\omega^2 \|\psi_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \psi_{k,q_3} \rangle_\omega^2}$$

1209 and $\|\zeta_{k,q_2}\|_\omega^2 \|\psi_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \psi_{k,q_3} \rangle_\omega^2 > 0$.

1210 • Case iii.

1211 Recall that the eigenvectors are defined in (4.4). To preserve symmetry, we consider
1212 here the generalized eigenvector given by

$$1213 \quad \phi_{k,1}^1 = \frac{1}{2I_k(q_2)} \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} + \frac{1}{2I_k(q_3)} \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}.$$

1214 We have

$$1215 \quad \xi = \begin{pmatrix} \left\langle y_0, \begin{pmatrix} \phi_{k,1}^0 \\ 0 \\ 0 \end{pmatrix} \right\rangle_X \\ \left\langle y_0, \begin{pmatrix} \phi_{k,2}^0 \\ 0 \\ 0 \end{pmatrix} \right\rangle_X \\ \left\langle y_0, \begin{pmatrix} \phi_{k,1}^1 \\ 0 \\ 0 \end{pmatrix} \right\rangle_X \end{pmatrix} \quad \text{and} \quad M = M_1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \|\varphi_k\|_\omega^2 \end{pmatrix}$$

1216 with

$$1217 \quad M_1 = \text{Gram}_U \left(\varphi_k, I_k(q_3)\psi_{k,q_2} - I_k(q_2)\psi_{k,q_3}, \frac{1}{2I_k(q_2)}\psi_{k,q_2} + \frac{1}{2I_k(q_3)}\psi_{k,q_3} \right).$$

1218 As in the previous cases, straightforward computations (which are left to the reader)
1219 give

$$1221 \quad \langle M^{-1}\xi, \xi \rangle = \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix} \right\rangle_X^2}{\|\varphi_k\|_\omega^2}$$

$$1222 \quad + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \psi_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X \psi_{k,q_2} \right\|_\omega^2}{\|I_k(q_3)\psi_{k,q_2} - I_k(q_2)\psi_{k,q_3}\|_\omega^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,q_2}\|_\omega^2 \|\psi_{k,q_3}\|_\omega^2 - \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega^2}$$

$$1223 \quad + \frac{\left(I_k(q_3)^2 \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X^2 + I_k(q_2)^2 \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X^2 \right) \|\varphi_k\|_\omega^2}{\|I_k(q_3)\psi_{k,q_2} - I_k(q_2)\psi_{k,q_3}\|_\omega^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,q_2}\|_\omega^2 \|\psi_{k,q_3}\|_\omega^2 - \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega^2}.$$

1225 Using the normalization condition $\langle \psi_{k,q_j}, \varphi_k \rangle_\omega = 0$, this can be rewritten as

$$1227 \quad \langle M^{-1}\xi, \xi \rangle = \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix} \right\rangle_X^2}{\|\varphi_k\|_\omega^2}$$

$$1228 \quad + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \zeta_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X \zeta_{k,q_2} \right\|_\omega^2}{\|I_k(q_3)\psi_{k,q_2} - I_k(q_2)\psi_{k,q_3}\|_\omega^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,q_2}\|_\omega^2 \|\psi_{k,q_3}\|_\omega^2 - \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega^2}.$$

1230 Using again the normalization condition $\langle \psi_{k,q_j}, \varphi_k \rangle_\omega = 0$, we obtain

$$1231 \quad \|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2 = \|I_k(q_3)\psi_{k,q_2} - I_k(q_2)\psi_{k,q_3}\|_\omega^2 \|\varphi_k\|_\omega^2$$

$$1233 \quad + \|\psi_{k,q_2}\|_\omega^2 \|\psi_{k,q_3}\|_\omega^2 - \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega^2.$$

1234 Thus, from Cauchy-Schwarz inequality and the approximate controllability condition

$$1235 \quad \text{Ker}(\mathcal{A}^* - \lambda) \cap \text{Ker} \mathcal{B}^* = \{0\}, \quad \forall \lambda \in \mathbb{R},$$

1236 it comes that $\|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2 > 0$ and

$$1237 \quad \langle M^{-1}\xi, \xi \rangle = \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix} \right\rangle_X^2}{\|\varphi_k\|_\omega^2} + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \zeta_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X \zeta_{k,q_2} \right\|_\omega^2}{\|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2}.$$

1238 Notice that the last formula obtained in case *iii* degenerates as expected when
 1239 $I_k(q_2) = 0$ and / or $I_k(q_3) = 0$. Thus, gathering all cases proves (4.11) and ends the
 1240 proof of Lemma 4.2. \square

1241 We now have all the ingredients to prove Theorem 1.3.

1242 *Proof (of Theorem 1.3).* Recall that from (4.5) we have

$$1243 \quad T_{0,q}(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln^+ \mathcal{C}(G_k, y_0)}{2\nu_k}$$

1244 where, due to Lemma 4.2, we have for any $k \geq 1$,

$$1245 \quad \mathcal{C}(G_k, y_0) = \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix} \right\rangle_X^2}{\|\varphi_k\|_\omega^2} + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \zeta_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X \zeta_{k,q_2} \right\|_\omega^2}{\|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2}.$$

1246 We now estimate the previous right-hand side. As we will see in Section 2.1, we have

$$1247 \quad \|\psi_{k,q_2}\|_{(0,1)} + \|\psi_{k,q_3}\|_{(0,1)} \leq C, \quad \forall k \geq 1.$$

1248 Thus,

$$1249 \quad \left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \zeta_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X \zeta_{k,q_2} \right\|_\omega^2 \leq C \|y_0\|_X^2 \max\left(\|\zeta_{k,q_2}\|_\omega^2, \|\zeta_{k,q_3}\|_\omega^2\right).$$

1250 Recall that φ_k satisfies (A.3). This implies that

$$1251 \quad \mathcal{C}(G_k, y_0) \leq C \|y_0\|_X^2 \left(1 + \frac{\max\left(\|\zeta_{k,q_2}\|_\omega^2, \|\zeta_{k,q_3}\|_\omega^2\right)}{\|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2} \right)$$

1252 for any $k \geq 1$ and any $y_0 \in X$ which gives

$$1253 \quad T_{0,q} \leq \limsup_{k \rightarrow +\infty} \frac{1}{2\nu_k} \ln \frac{\max\left(\|\zeta_{k,q_2}\|_\omega^2, \|\zeta_{k,q_3}\|_\omega^2\right)}{\|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2}.$$

1254 We now prove the converse inequality. We define for all $k \geq 1$

$$1255 \quad \epsilon_k = \begin{cases} 1 & \text{if } \|\zeta_{k,q_2}\|_\omega > \|\zeta_{k,q_3}\|_\omega \\ 0 & \text{otherwise} \end{cases}$$

1256 and we choose the particular initial condition

$$1257 \quad y_0 = \sum_{k \geq 1} \frac{1}{\nu_k} \left(\epsilon_k \begin{pmatrix} 0 \\ 0 \\ \varphi_k \end{pmatrix} + (1 - \epsilon_k) \begin{pmatrix} 0 \\ \varphi_k \\ 0 \end{pmatrix} \right).$$

1258 From the expression (4.11) we obtain

$$1259 \quad \mathcal{C}(G_k, y_0) = \frac{1}{\nu_k^2} \frac{\max \left(\|\zeta_{k,q_2}\|_\omega^2, \|\zeta_{k,q_3}\|_\omega^2 \right)}{\|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2}.$$

1260 This gives that

$$1261 \quad T_{0,q} \geq T_{0,q}(y_0) = \limsup_{k \rightarrow +\infty} \frac{1}{2\nu_k} \ln \frac{\max \left(\|\zeta_{k,q_2}\|_\omega^2, \|\zeta_{k,q_3}\|_\omega^2 \right)}{\|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2}$$

1262 which ends the proof of Theorem 1.3. \square

1263 4.3. A second characterization of the minimal null control time.

1264 The goal of this section is to prove Theorem 1.4.

1265 We first notice that, by (1.19), we have that q_2 and q_3 are linearly independent
1266 and thus there exists $\underline{C}, \overline{C} > 0$ such that

$$1267 \quad (4.12) \quad \underline{C}(|\alpha_2| + |\alpha_3|) \leq \|\alpha_2 q_2 + \alpha_3 q_3\|_\infty \leq \overline{C}(|\alpha_2| + |\alpha_3|), \quad \forall \alpha_2, \alpha_3 \in \mathbb{R}.$$

1268 *Proof.* From Theorem 1.3 we now estimate, for any $k \geq 1$,

$$1269 \quad \max \left(\frac{\|\zeta_{k,q_2}\|_\omega^2}{\|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2}, \frac{\|\zeta_{k,q_3}\|_\omega^2}{\|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2} \right).$$

1270 Let $k \geq 1$ and assume that $\|\zeta_{k,q_3}\|_\omega > \|\zeta_{k,q_2}\|_\omega$. Notice that

$$1271 \quad \left\| \zeta_{k,q_3} \left\| \zeta_{k,q_2} - \frac{\langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega}{\|\zeta_{k,q_3}\|_\omega^2} \zeta_{k,q_3} \right\|_\omega \right\|_\omega^2 = \|\zeta_{k,q_3}\|_\omega^2 \|\zeta_{k,q_2}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2.$$

1272 Thus,

$$1273 \quad (4.13) \quad \frac{\max \left(\|\zeta_{k,q_2}\|_\omega^2, \|\zeta_{k,q_3}\|_\omega^2 \right)}{\|\zeta_{k,q_2}\|_\omega^2 \|\zeta_{k,q_3}\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2} = \frac{1}{\left\| \zeta_{k,q_2} - \frac{\langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega}{\|\zeta_{k,q_3}\|_\omega^2} \zeta_{k,q_3} \right\|_\omega^2} \\ = \frac{1}{\min_{\tau \in \mathbb{R}} \|\zeta_{k,q_2} - \tau \zeta_{k,q_3}\|_\omega^2}.$$

1274 By linearity we have, for any $\tau \in \mathbb{R}$,

$$1275 \quad \zeta_{k,q_2} - \tau \zeta_{k,q_3} = \zeta_{k,q_2 - \tau q_3}.$$

1276 We proved in Section 2.2 that there exists $K \in \mathbb{N}^*$, $C_1, C_2 > 0$ such that, for any
1277 $k \geq K$ and any $q \in L^\infty(0, 1)$ such that $\text{Supp}(q) \cap \omega = \emptyset$, we have

$$1278 \quad (4.14) \quad C_1 \|\zeta_{k,q}\|_\omega^2 \leq \mathcal{M}_k \left(\vartheta_{k,q}, \omega \right)^2 \leq C_2 \nu_k \|\zeta_{k,q}\|_\omega^2.$$

1279 where $\vartheta_{k,q}$ is defined by (1.18). The analysis is the same in the symmetric case
 1280 $\|\zeta_{k,q_2}\|_\omega > \|\zeta_{k,q_3}\|_\omega$.

1281 Thus, from Theorem 1.3, (4.13) and (4.14), it comes that

$$1282 \quad (4.15) \quad T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln \min \left\{ \min_{\tau \in \mathbb{R}} \mathcal{M}_k(\vartheta_{k,q_2 - \tau q_3}, \omega), \min_{\tau \in \mathbb{R}} \mathcal{M}_k(\vartheta_{k,q_3 - \tau q_2}, \omega) \right\}}{\nu_k}.$$

1283 To conclude the proof of Theorem 1.4, let us prove that the quantity

$$1284 \quad \min \left\{ \min_{\tau \in \mathbb{R}} \mathcal{M}_k(\vartheta_{k,q_2 - \tau q_3}, \omega), \min_{\tau \in \mathbb{R}} \mathcal{M}_k(\vartheta_{k,q_3 - \tau q_2}, \omega) \right\}$$

1285 appearing in the formula above has the same asymptotic behaviour as

$$1286 \quad \min_{q \in \mathbb{S}[q]} \mathcal{M}_k(\vartheta_{k,q}, \omega).$$

1287 Notice that, for any $\tau \in \mathbb{R}$, the function $q_\tau = \frac{q_2 - \tau q_3}{\|q_2 - \tau q_3\|_\infty}$ belongs to $\mathbb{S}[q]$ and thus

$$1288 \quad \mathcal{M}_k(\vartheta_{k,q_2 - \tau q_3}, \omega) = \|q_2 - \tau q_3\|_\infty \mathcal{M}_k(\vartheta_{k,q_\tau}, \omega)$$

$$1289 \quad \geq \underline{C} \min_{q \in \mathbb{S}[q]} \mathcal{M}_k(\vartheta_{k,q}, \omega),$$

$$1290$$

1291 where we have used (4.12). It follows that

$$1292 \quad \min_{\tau \in \mathbb{R}} \mathcal{M}_k(\vartheta_{k,q_2 - \tau q_3}, \omega) \geq \underline{C} \min_{q \in \mathbb{S}[q]} \mathcal{M}_k(\vartheta_{k,q}, \omega)$$

1293 and the exact same computation holds for $q_3 - \tau q_2$.

1294 Conversely, let $q = \alpha_2 q_2 + \alpha_3 q_3 \in \mathbb{S}[q]$. If $|\alpha_2| \geq |\alpha_3|$, then by (4.12), we have
 1295 $|\alpha_2| \geq \frac{1}{2\underline{C}}$ and thus

$$1296 \quad \mathcal{M}_k(\vartheta_{k,q}, \omega) = |\alpha_2| \mathcal{M}_k\left(\vartheta_{k,q_2 + \frac{\alpha_3}{\alpha_2} q_3}, \omega\right)$$

$$1297 \quad \geq \frac{1}{2\underline{C}} \min_{\tau \in \mathbb{R}} \mathcal{M}_k(\vartheta_{k,q_2 - \tau q_3}, \omega).$$

$$1298$$

1299 Otherwise, we have $|\alpha_3| > |\alpha_2|$ and a symmetric analysis gives

$$1300 \quad \mathcal{M}_k(\vartheta_{k,q}, \omega) \geq \frac{1}{2\underline{C}} \min_{\tau \in \mathbb{R}} \mathcal{M}_k(\vartheta_{k,q_3 - \tau q_2}, \omega).$$

1301 Finally, from the expression of the minimal null control time given in (4.15), the
 1302 claim of Theorem 1.4 is proved. \square

1303 4.4. An explicit example.

1304 In this section we consider A to be the Dirichlet Laplace operator (i.e., $\gamma = 1$ and
 1305 $c = 0$ in (1.2)) and $\omega = (0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$.

1306 PROPOSITION 4.3. *Let A and ω be defined as above. Let $\tau_0 \in [0, +\infty]$. There
 1307 exists $q_2, q_3 \in L^\infty(0, 1)$ such that*

- 1308 *i) approximate controllability of system (1.4) holds,*
- 1309 *ii) for any $(\alpha_2, \alpha_3) \in \mathbb{R}^2 \setminus \{0\}$, the minimal null control time for system (1.1)*
 1310 *with $q = \alpha_2 q_2 + \alpha_3 q_3$ is $T_{0,q} = 0$. In particular $T_{0,q_2} = T_{0,q_3} = 0$.*

1311 *iii) the minimal null control time for system (1.4) is $T_{0,q} = \tau_0$.*

1312 *Proof.* For $j \in \{2, 3\}$, we set $q_j = \mathbf{1}_{\mathcal{O}_j}$ with

$$1313 \quad \mathcal{O}_2 = \left(\frac{1}{2} - \delta_2, \frac{1}{2} + \delta_2 \right) \text{ and } \mathcal{O}_3 = (\eta_3 - \delta_3, \eta_3 + \delta_3),$$

1314 where η_3 , δ_2 and δ_3 are chosen such that

$$1315 \quad (4.16) \quad \text{Supp}(q_2) \cap \omega = \emptyset \quad \text{and} \quad \text{Supp}(q_3) \cap \omega = \emptyset.$$

1316 The approximate controllability of system (1.4) with these coupling functions has
 1317 been studied in [14, Section 3.4.2]. It is proved that approximate controllability holds
 1318 if and only if

$$1319 \quad (4.17) \quad \eta_3 \notin \mathbb{Q} \quad \text{and} \quad \delta_3 \notin \mathbb{Q}.$$

1320 Using for instance [5, Lemma 7.1], we can find $\eta_3 \notin \mathbb{Q}$ and $\delta_2, \delta_3 \notin \mathbb{Q}$ such that $2\eta_3$
 1321 and $2\delta_2$ are irrational algebraic numbers of degree 2 and

$$1322 \quad (4.18) \quad \limsup_{k \rightarrow +\infty} \frac{-\ln |\sin(2k\pi\delta_3)|}{k^2\pi^2} = \tau_0.$$

1323 These choices prove *i*).

1324 Let us now focus on *ii*) that is the determination of the minimal null control time
 1325 for system (1.1). Under the considered assumptions, we have explicit formulas for φ_k
 1326 and $\tilde{\varphi}_k$ as follows:

$$1327 \quad \varphi_k = \sqrt{2} \sin(k\pi\bullet) \quad \text{and} \quad \tilde{\varphi}_k = \cos(k\pi\bullet).$$

1328 From Theorem 1.2, for any $q \in L^\infty(0, 1)$, we have

$$1329 \quad T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)}{k^2\pi^2}.$$

1330 Since $\overline{(0, 1) \setminus \omega}$ has only one connected component $\mathfrak{C} = [\frac{1}{4}, \frac{3}{4}]$ it comes that

$$1331 \quad \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega) = \max \left\{ \left| \int_{\mathfrak{C}} (I_k(q) - q(x))\varphi_k(x)^2 dx \right|, \left| \int_{\mathfrak{C}} (I_k(q) - q(x))\varphi_k(x)\tilde{\varphi}_k(x) dx \right| \right\}.$$

1332 Then, for $j \in \{2, 3\}$, since $\text{Supp}(q_j) \subset \mathfrak{C}$, we have

$$1333 \quad \int_{\mathfrak{C}} (I_k(q_j) - q_j(x))\varphi_k(x)^2 dx = I_k(q_j) \left(1 - \|\varphi_k\|_\omega^2 \right) - I_k(q_j) = -\|\varphi_k\|_{\mathcal{O}_j}^2 \|\varphi_k\|_\omega^2.$$

1334 where we have used

$$1335 \quad I_k(q_j) = \int_{\mathcal{O}_j} \varphi_k(x)^2 dx = \|\varphi_k\|_{\mathcal{O}_j}^2.$$

1336 From (A.3) it comes that there exists $C > 0$ such that for any $k \geq 1$ and any $j \in \{2, 3\}$,

$$1337 \quad (4.19) \quad C \leq \left| \int_{\mathfrak{C}} (I_k(q_j) - q_j(x))\varphi_k(x)^2 dx \right| = \|\varphi_k\|_{\mathcal{O}_j}^2 \|\varphi_k\|_\omega^2 \leq 1.$$

1338 This already implies that $T_{0,q_2} = T_{0,q_3} = 0$. Let $(\alpha_2, \alpha_3) \in \mathbb{R}^2 \setminus \{0\}$ and $q = \alpha_2 q_2 +$
 1339 $\alpha_3 q_3$. We prove that

$$1340 \quad (4.20) \quad \limsup_{k \rightarrow +\infty} k^2 |I_k(q)| > 0$$

1341 which implies $T_{0,q} = 0$ since

$$1342 \quad \left| \int_{\mathfrak{C}} (I_k(q) - q(x)) \varphi_k(x)^2 dx \right| = |I_k(q)| \|\varphi_k\|_{\omega}^2.$$

1343 Explicit computations yield

$$1344 \quad I_k(q) = \alpha_2 \int_{\mathcal{O}_2} \sin^2(k\pi x) dx + \alpha_3 \int_{\mathcal{O}_3} \sin^2(k\pi x) dx$$

$$1345 \quad = \alpha_2 \delta_2 + \alpha_3 \delta_3 + \frac{(-1)^{k+1} \alpha_2}{2k\pi} \sin(2k\pi \delta_2) - \frac{\alpha_3}{2k\pi} \cos(2k\pi \eta_3) \sin(2k\pi \delta_3).$$

1347 If $\alpha_2 \delta_2 + \alpha_3 \delta_3 \neq 0$, the property (4.20) follows directly. Otherwise, we necessarily
 1348 have $\alpha_2 \neq 0$ and since $2\delta_2$ is an irrational algebraic number of degree 2 we have (see
 1349 for instance [5, Lemma 7.1])

$$1350 \quad \inf_{k \geq 1} k |\sin(2k\pi \delta_2)| > 0.$$

1351 Together with the choice of δ_3 in (4.18) this proves (4.20) and thus gives $T_{0,q} = 0$.

1352 We now turn to *iii*) that is the determination of the minimal null control time for
 1353 system (1.4). From Theorem 1.4 we have that the minimal null control time is given
 1354 by

$$1355 \quad (4.21) \quad T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln \min_{q \in \mathbb{S}[q]} \mathcal{M}_k(\vartheta_{k,q}, \omega)}{\nu_k}.$$

1356 Let $k \geq 1$. Since \mathfrak{C} is symmetric with respect to $\frac{1}{2}$, we have

$$1357 \quad \int_{\mathfrak{C}} \varphi_k(x) \tilde{\varphi}_k(x) dx = \sqrt{2} \int_{\frac{1}{4}}^{\frac{3}{4}} \sin(k\pi x) \cos(k\pi x) dx = 0.$$

1358 Thus, for $j \in \{2, 3\}$, we have

$$1359 \quad M_k(\vartheta_{k,q_j}, \mathfrak{C}) = \left(\begin{array}{c} -I_k(q_j) \|\varphi_k\|_{\omega}^2 \\ -\int_{\mathfrak{C}} q_j(x) \varphi_k(x) \tilde{\varphi}_k(x) dx \end{array} \right).$$

1360 Again a symmetry argument shows that

$$1361 \quad \int_{\mathfrak{C}} q_2(x) \varphi_k(x) \tilde{\varphi}_k(x) dx = \sqrt{2} \int_{\mathcal{O}_2} \sin(k\pi x) \cos(k\pi x) dx = 0.$$

1362 It follows that for any $q = \alpha_2 q_2 + \alpha_3 q_3 \in \mathbb{S}[q]$, we have

$$1363 \quad M_k(\vartheta_{k,q}, \mathfrak{C}) = \left(\begin{array}{c} -I_k(q) \|\varphi_k\|_{\omega}^2 \\ -\alpha_3 \int_{\mathcal{O}_3} \varphi_k(x) \tilde{\varphi}_k(x) dx \end{array} \right)$$

1364 and thus

$$1365 \quad (4.22) \quad \mathcal{M}_k(\vartheta_{k,q}, \omega) = \max \left\{ |I_k(q)| \|\varphi_k\|_\omega^2, |\alpha_3| \left| \int_{\mathcal{O}_3} \varphi_k(x) \tilde{\varphi}_k(x) dx \right| \right\}.$$

1366 Let us now prove that (4.21) reduces to

$$1367 \quad (4.23) \quad T_{0,q} = \limsup_{k \rightarrow +\infty} \frac{-\ln \left| \int_{\mathcal{O}_3} \varphi_k(x) \tilde{\varphi}_k(x) dx \right|}{\nu_k}.$$

• We set

$$\tilde{q}_k = I_k(q_3)q_2 - I_k(q_2)q_3, \quad \bar{q}_k = \frac{\tilde{q}_k}{\|\tilde{q}_k\|_\infty}$$

in such a way that $I_k(\bar{q}_k) = 0$ and $\|\bar{q}_k\|_\infty = 1$. By (4.22) and (4.12), we get

$$\mathcal{M}_k(\vartheta_{k,\bar{q}_k}, \omega) \leq \frac{1}{\underline{C}} \left| \int_{\mathcal{O}_3} \varphi_k(x) \tilde{\varphi}_k(x) dx \right|,$$

1368 so that

$$1369 \quad (4.24) \quad \min_{q \in \mathbb{S}[\mathbf{q}]} \mathcal{M}_k(\vartheta_{k,q}, \omega) \leq \frac{1}{\underline{C}} \left| \int_{\mathcal{O}_3} \varphi_k(x) \tilde{\varphi}_k(x) dx \right|.$$

1370 Recall that \underline{C} is the constant appearing in (4.12).

1371 • We now prove that, for some $C > 0$ that does not depend on k , we have

$$1372 \quad (4.25) \quad \min_{q \in \mathbb{S}[\mathbf{q}]} \mathcal{M}_k(\vartheta_{k,q}, \omega) \geq C \left| \int_{\mathcal{O}_3} \varphi_k(x) \tilde{\varphi}_k(x) dx \right|.$$

If it were not the case, we would have, up to a subsequence, the inequality

$$\min_{q \in \mathbb{S}[\mathbf{q}]} \mathcal{M}_k(\vartheta_{k,q}, \omega) \leq \varepsilon_k \left| \int_{\mathcal{O}_3} \varphi_k(x) \tilde{\varphi}_k(x) dx \right|,$$

1373 for some $\varepsilon_k \rightarrow 0$.

1374 In particular, from (4.22), it would exist for each k , a function $\tilde{q}_k = \alpha_{2,k}q_2 +$
1375 $\alpha_{3,k}q_3 \in \mathbb{S}[\mathbf{q}]$, such that

$$1376 \quad (4.26) \quad |I_k(\tilde{q}_k)| \|\varphi_k\|_\omega^2 \leq \varepsilon_k \left| \int_{\mathcal{O}_3} \varphi_k(x) \tilde{\varphi}_k(x) dx \right|,$$

1377 and

$$1378 \quad (4.27) \quad |\alpha_{3,k}| \left| \int_{\mathcal{O}_3} \varphi_k(x) \tilde{\varphi}_k(x) dx \right| \leq \varepsilon_k \left| \int_{\mathcal{O}_3} \varphi_k(x) \tilde{\varphi}_k(x) dx \right|.$$

From (4.27), we deduce first that $|\alpha_{3,k}| \leq \varepsilon_k$, and in particular $\alpha_{3,k} \rightarrow 0$.
Since $\|\tilde{q}_k\|_\infty = 1$, it follows that $|\alpha_{2,k}| \rightarrow \frac{1}{\|q_2\|_\infty}$, from which we deduce that

$$\lim_{k \rightarrow \infty} |I_k(\tilde{q}_k)| = \frac{1}{\|q_2\|_\infty} \lim_{k \rightarrow \infty} |I_k(q_2)| = \frac{|\mathcal{O}_2|}{\|q_2\|_\infty} > 0.$$

1379 By using (A.3), we obtain a contradiction with (4.26).

1380 Using (4.24) and (4.25) in (4.21) exactly proves (4.23).
 1381 Finally, explicit computations yield

$$1382 \quad \int_{\mathcal{O}_3} \varphi_k(x) \tilde{\varphi}_k(x) dx = \int_{\mathcal{O}_3} \sin(k\pi x) \cos(k\pi x) dx = \frac{\sin(2k\pi\eta_3) \sin(2k\pi\delta_3)}{2k\pi}.$$

1383 Since $2\eta_3$ is an irrational algebraic number of degree 2 we have (see for instance [5,
 1384 Lemma 7.1])

$$1385 \quad \inf_{k \geq 1} k |\sin(2k\pi\eta_3)| > 0.$$

1386 Together with the choice of δ_3 in (4.18) this ends the proof of Proposition 4.3. \square

1387 **Appendix A. Spectral properties of the Sturm-Liouville operator.**

1388 Let A be the Sturm-Liouville operator defined by (1.2). We recall here some
 1389 spectral properties that will be used in our study.

1390 From [1, Theorem 1.1 and Remark 2.1], there exist $\varrho > 0$ and $C > 0$ such that

$$1391 \quad (\text{A.1}) \quad \varrho < \nu_{k+1} - \nu_k, \quad \forall k \geq 1,$$

1392

$$1393 \quad (\text{A.2}) \quad \frac{1}{C} \sqrt{\nu_k} \leq |\varphi'_k(x)| \leq C \sqrt{\nu_k}, \quad \forall x \in \{0, 1\}, \forall k \geq 1,$$

1394 and, for any non-empty open set $\omega \subset (0, 1)$,

$$1395 \quad (\text{A.3}) \quad \inf_{k \geq 1} \|\varphi_k\|_\omega > 0.$$

1396 Let N be the counting function associated with the sequence of eigenvalues $(\nu_k)_{k \geq 1}$
 1397 *i.e.*,

$$1398 \quad N : r \in (0, +\infty) \mapsto \#\{\nu_k ; \nu_k \leq r\}.$$

1399 Using [12, Theorem IV.1.3], this counting function satisfies for some $\kappa > 0$,

$$1400 \quad (\text{A.4}) \quad N(r) \leq \kappa \sqrt{r}, \quad \forall r > 0,$$

1401 and

$$1402 \quad (\text{A.5}) \quad |N(r) - N(s)| \leq \kappa \left(1 + \sqrt{|r - s|}\right), \quad \forall r, s > 0.$$

1403 To estimate various quantities, we will make an intensive use of the following lemma
 1404 proved in [1, Lemma 2.3].

1405 **LEMMA A.1.** *Let A be the Sturm-Liouville operator defined by (1.2) and let $\lambda_0 > 0$.
 1406 There exists $C > 0$ depending on γ , c and λ_0 such that, for any $\lambda \geq \lambda_0$, for any
 1407 $F \in L^2(0, 1)$, for any $x, y \in [0, 1]$, for any u satisfying*

$$1408 \quad (A - \lambda)u = F \quad \text{in } [0, 1],$$

1409 *we have*

$$1410 \quad |u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \leq C \left(|u(y)|^2 + \frac{\gamma(y)}{\lambda} |u'(y)|^2 + \frac{1}{\lambda} \left| \int_x^y |F(s)| ds \right|^2 \right).$$

1411 Applying Lemma A.1 with $u = \varphi_k$, $F = 0$, $\lambda = \nu_k$ and integrating with respect to
 1412 the variable $y \in (0, 1)$ we obtain

$$1413 \quad |\varphi_k(x)|^2 + \frac{1}{\nu_k} |\varphi'_k(x)|^2 \leq C \left(1 + \frac{1}{\nu_k} \int_0^1 \gamma(y) |\varphi'_k(y)|^2 dy \right), \quad \forall x \in (0, 1), \forall k \geq 1.$$

1414 Integrating by parts leads to

$$1415 \quad \int_0^1 \gamma(y) |\varphi'_k(y)|^2 dy = \int_0^1 (\nu_k - c(y)) \varphi_k(y)^2 dy \leq \nu_k + \|c\|_\infty$$

1416 which yields the existence of $C > 0$ such that

$$1417 \quad (\text{A.6}) \quad |\varphi_k(x)|^2 + \frac{1}{\nu_k} |\varphi'_k(x)|^2 \leq C, \quad \forall x \in (0, 1), \forall k \geq 1.$$

1418 We shall also use this lemma to estimate $\psi_{k,q}$ (defined in (1.6)) as follows:

LEMMA A.2. *There exists $C > 0$ such that*

$$\|\psi_{k,q}\|_{(0,1)} \leq C, \quad \forall k \geq 1.$$

1419 *Proof.* The function $\tilde{\psi}_{k,q}$ defined by

$$1420 \quad \tilde{\psi}_{k,q} := \psi_{k,q} - \frac{\psi'_{k,q}(0)}{\varphi'_k(0)} \varphi_k,$$

1421 satisfies

$$1422 \quad \begin{cases} (A - \nu_k) \tilde{\psi}_{k,q} = (I_k(q) - q) \varphi_k, \\ \tilde{\psi}_{k,q}(0) = \tilde{\psi}_{k,q}(1) = 0, \\ \tilde{\psi}'_{k,q}(0) = 0. \end{cases}$$

1423 From Lemma A.1 with $y = 0$ it comes that

$$1424 \quad \left| \tilde{\psi}_{k,q}(x) \right|^2 + \frac{\gamma(x)}{\nu_k} \left| \tilde{\psi}'_{k,q}(x) \right|^2 \leq \frac{C}{\nu_k}, \quad \forall x \in (0, 1), \forall k \geq 1.$$

1425 which yields

$$1426 \quad \left\| \tilde{\psi}_{k,q} \right\|_{(0,1)} \leq C, \quad \forall k \geq 1.$$

1427 Notice that, by definition of $\tilde{\psi}_{k,q}$, we have $(\psi_{k,q} - \tilde{\psi}_{k,q}) \in \mathbb{R}\varphi_k$. Then, multiplying by
 1428 φ_k , integrating over ω and recalling that $\langle \psi_{k,q}, \varphi_k \rangle_\omega = 0$, we obtain that

$$1429 \quad \psi_{k,q} = \tilde{\psi}_{k,q} - \frac{\langle \tilde{\psi}_{k,q}, \varphi_k \rangle_\omega}{\|\varphi_k\|_\omega^2} \varphi_k.$$

1430 This implies that

$$1431 \quad \|\psi_{k,q}\|_{(0,1)} \leq \left\| \tilde{\psi}_{k,q} \right\|_{(0,1)} \left(1 + \frac{1}{\|\varphi_k\|_\omega} \right), \quad \forall k \geq 1.$$

1432 Then, estimate (A.3) ends the proof of Lemma A.2. □

1433 By definition, φ_k and $\tilde{\varphi}_k$ are solutions of the same linear second order ODE
 1434 $(A - \nu_k)\varphi_k = (A - \nu_k)\tilde{\varphi}_k = 0$. It is therefore natural to introduce the associated
 1435 Wronskian matrix

$$1436 \quad W_k(x) = \begin{pmatrix} \frac{\gamma(x)\varphi_k'(x)}{\sqrt{\nu_k}} & -\varphi_k(x) \\ \frac{\gamma(x)\tilde{\varphi}_k'(x)}{\sqrt{\nu_k}} & -\tilde{\varphi}_k(x) \end{pmatrix},$$

1437 for which we can prove the following estimate.

LEMMA A.3. *There exists $C > 0$ such that*

$$\|W_k(x)\| + \|W_k(x)^{-1}\| \leq C, \quad \forall x \in [0, 1], \forall k \geq 1.$$

Proof. Let us fix a $k \geq 1$. Applying Lemma A.1 to $u = \tilde{\varphi}_k$ and $y = 0$, we obtain

$$|\tilde{\varphi}_k(x)|^2 + \frac{1}{\nu_k} |\tilde{\varphi}_k'(x)|^2 \leq C, \quad \forall x \in (0, 1), \forall k \geq 1.$$

1438 Together with (A.6), this shows the uniform estimate on $\|W_k(x)\|$.

1439 Moreover, the determinant of $W_k(x)$ does not depend on x and is thus equal to
 1440 the determinant of $W_k(0)$ that is to $-\gamma(0)\varphi_k'(0)/\sqrt{\nu_k}$. By (A.2), we know that this
 1441 quantity is uniformly bounded from below. The bound for $W_k(x)^{-1}$ follows. \square

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