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ON SOME GENERIC CLASSES OF ERGODIC MEASURE PRESERVING TRANSFORMATIONS

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Abstract. We answer positively a question of Ryzhikov, namely we show that being a relatively weakly mixing extension is a comeager property in the Polish group of measure preserving transformations. We study some related classes of ergodic transformations and their interrelations. In the second part of the paper we show that for a fixed ergodic T with property **A**, a generic extension $\widehat{T}$ of T also has property **A**. Here **A** stands for each of the following properties: (i) having the same entropy as T, (ii) Bernoulli, (iii) K, and (iv) loosely Bernoulli.

Introduction

Motivated by a question of Valery Ryzhikov [37], regarding the nature of the class of ergodic transformations $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ which admit a proper factor $\mathbf{X} \to \mathbf{Y}$ with $\mathbf{Y} = (Y, \mathcal{Y}, \nu, T)$ nontrivial and where the extension is relatively weakly mixing (we call this class RWM), we consider in this paper the RWM property and some related classes as follows:

We view the collection MPT (measure preserving transformations) of invertible measure preserving transformations on a nonatomic standard probability space $(X, \mathcal{X}, \mu)$, as the group $\mathrm{Aut}\,(\mu)$ of automorphisms of $(X, \mathcal{X}, \mu)$. The topology on $\mathrm{Aut}\,(X, \mu)$ is induced by a complete metric

$$D(S,T) = \sum_{n\in\mathbb{N}} 2^{-n}(\mu(SA_n \triangle TA_n) + \mu(S^{-1}A_n \triangle T^{-1}A_n)),$$

with $\{A_n\}_{n\in\mathbb{N}}$ a dense sequence in the measure algebra $(\mathcal{X}, d_\mu)$, where $d_\mu(A,B) = \mu(A \triangle B)$. Equipped with this topology $\mathrm{Aut}\,(X, \mu)$ is a Polish topological group. The set $\mathrm{MPT}_e \subset \mathrm{Aut}\,(\mu)$, comprising the ergodic transformations, is a dense G_δ subset of $\mathrm{Aut}\,(\mu)$. Thus $\mathrm{MPT} = \mathrm{Aut}\,(\mu)$ and we use the latter when we want to emphasise the group structure of this space.

Definition 0.1. Given an ergodic system $\mathbf{X}$ we say that a factor map $\pi\colon \mathbf{X} \to \mathbf{Y}$ is *nontrivial* when $\mathbf{Y}$ is not the trivial one-point system and π is not an isomorphism. (We often refer to a factor map $\pi\colon \mathbf{X} \to \mathbf{Y}$ from $\mathbf{X}$ to $\mathbf{Y}$ also as an extension of $\mathbf{Y}$.) An ergodic dynamical system $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ is:

(1) RD (*relatively distal*) if there exists a nontrivial factor map $\pi\colon \mathbf{X} \to \mathbf{Y}$ such that the extension is relatively distal. Every ergodic distal system which is not isomorphic to a cyclic permutation on $\mathbb{Z}_p$ with p a prime number is RD.
(2) RWM (*relatively weakly mixing*) if there exists a nontrivial factor map $\pi\colon \mathbf{X} \to \mathbf{Y}$ such that the extension is relatively weakly mixing; i.e. such that the relative

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product $\mathbf{X} \times_{\mathbf{Y}} \mathbf{X}$ is ergodic. By the Furstenberg–Zimmer theorem, every ergodic system which is neither distal nor weakly mixing is RWM.

(3) TRD (*totally relatively distal*) if every nontrivial factor map $\pi\colon \mathbf{X} \to \mathbf{Y}$ is relatively distal. Every distal system is TRD. Every prime system is (vacuously) TRD.

(4) TRD$^\sim$ if it is TRD, but not prime. Every distal system which is not isomorphic to $\mathbb{Z}_p$ for some prime number p is TRD$^\sim$. By Veech's theorem every nonprime weakly mixing simple system is in TRD$^\sim$ (see [15, Theorem 12.3]). The example in [7] is a weakly mixing system in TRD$^\sim$ with no prime factor.

(5) TRWM (*totally relatively weakly mixing*) if every factor map $\pi\colon \mathbf{X} \to \mathbf{Y}$ which is nontrivial is relatively weakly mixing. Every prime system is (vacuously) TRWM, and probably also products of disjoint prime systems are TRWM, but it is not clear to us what else there is in this class. An ergodic system $\mathbf{X}$ in TRWM is either weakly mixing, or it admits the finite system $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ as a factor, for some prime number $p \geq 2$, such that the extension $\mathbf{X} \to \mathbb{Z}_p$ is relatively weakly mixing and $\mathbf{X}$ is not of the form $\mathbf{X} = \mathbb{Z}_p \times \mathbf{W}$ for a weakly mixing system $\mathbf{W}$. (An example of a system which has the latter form—but is not TRWM—is provided in [28].) In fact, by the Furstenberg–Zimmer structure theorem an ergodic system $\mathbf{X} \in$ TRWM has the structure $\mathbf{X} \to \mathbf{Y}$, where $\mathbf{Y}$ is the maximal distal factor of $\mathbf{X}$ and the extension is relatively weakly mixing. If $\mathbf{Y}$ is the trivial system then $\mathbf{X}$ is weakly mixing. Otherwise, as $\mathbf{X}$ is TRWM, we must have $\mathbf{Y} = \mathbb{Z}_p$ for some prime $p \geq 2$. Finally $\mathbf{X}$ cannot have the form $\mathbf{X} = \mathbb{Z}_p \times \mathbf{W}$ for a weakly mixing system $\mathbf{W}$, since if it were the projection map $\mathbf{X} \to \mathbf{W}$ would not be a relatively weakly mixing extension.

(6) An ergodic system $\mathbf{X}$ is called 2-*fold quasi-simple* (2-*fold distally simple*) if every ergodic non-product 2-fold self-joining of it is compact (relatively distal, respectively) over the marginals. Every simple system (hence every weakly mixing MSJ system) is 2-fold quasi-simple.

For more details and results concerning these notions see (among other publications) the following list:

(i) In [32] Rudolph introduced the notion of MSJ (minimal self-joinings), and in [46] Veech introduced 2-simple systems. Simple systems of higher order and their joinings were studied in [8]. In [23] King has shown that simplicity of order 4 implies simplicity of all orders, and in [16] this was improved to showing that order 3 suffices.

(ii) Various generalizations of simplicity were introduced and previous results were sharpened in [10] and [36].

(iii) The first genericity type theorem was King's paper [24] where he has shown that having roots of all orders is generic. This was followed by Ageev who first proved that, for every finite abelian group G, having G in the centraliser is generic [2], thereby showing that the class of prime transformations is meager, and then in [4], that the generic system is neither simple nor semisimple.

(iv) Stronger and stronger results of this nature are to be found in [12, 42, 38, 11, 40], and [6].

For information concerning the Furstenberg–Zimmer theorem and structure theory in ergodic theory we refer to [15].

In Section 1 we present a preliminary study of the nature of these classes and their interrelations. In Section 2 we answer positively Ryzhikov's question [37], namely we show that the property RWM is comeager.

In the second part of the paper, we change slightly the framework of our discussion. We mainly fix an ergodic transformation T with a certain property, and then consider either its family of factors or its family of ergodic extensions.

In Section 3 we show that the relatively weakly mixing factors of any fixed positive entropy T form a dense G_δ set. In Section 4 we show that a generic extension of any ergodic T does not add entropy. In Section 5 we prove that for any fixed Bernoulli transformation of finite entropy the generic extension is Bernoulli. In Section 6 we show that for any fixed K" automorphism T, the generic extension is K, and it is relatively mixing over T. Finally, in Section 7 we show that for any fixed loosely Bernoulli transformation T, the generic extension is loosely Bernoulli.

We remind the readers that the classes of weakly mixing, rank-1, rigid, χ-mixing and zero-entropy systems, are all comeager sets in MPT (see e.g. [20, 22] and [41]).[1]

Note that, whereas by Austin [5], every positive entropy ergodic T decomposes non-trivially as a direct product, Friedman's result [13] that, for $0 < \chi < 1$, a χ-mixing system admits no non-trivial product as a factor, combined with the fact that χ-mixing is comeager [41], show that the generic (zero-entropy ergodic) T does not split (and moreover cannot have a non-trivial product as a factor).

We would like to thank Valery Ryzhikov for bringing his question in [37] to our attention and for several helpful e-conversations. We also thank Oleg Ageev for a careful and thorough reading of a previous draft of the paper. His useful comments and corrections considerably improved our work. We also thank Tim Austin for pointing out an inaccuracy in the proof of Theorem 6.1 in an earlier version.

1. Some general results concerning the classes mentioned in the Introduction

Claim 1. *Every ergodic system with positive entropy is* RD.

Proof. It is well known that every Bernoulli system is RD. The case of a positive entropy ergodic system follows from the weak Pisnsker theorem [5]. □

Claim 2. *The property* TRD *is inherited by factors.*

Proof. Let $\mathbf{X}$ be a TRD system and $\mathbf{X} \to \mathbf{Y}$ a nontrivial factor. Suppose $\mathbf{Y} \to \mathbf{Z}$ is a nontrivial factor of $\mathbf{Y}$. Let $\mathbf{Y} \to \mathbf{Z}_{\mathrm{rd}} \to \mathbf{Z}$ be the diagram obtained by the Furstenberg–Zimmer structure theorem (the relative version); i.e. $\mathbf{Z}_{\mathrm{rd}}$ is the largest distal extension of $\mathbf{Z}$ in $\mathbf{Y}$ and $\mathbf{Y} \to \mathbf{Z}_{\mathrm{rd}}$ is a relatively weakly mixing extension. Now from the combined diagram

$$\mathbf{X} \to \mathbf{Y} \to \mathbf{Z}_{\mathrm{rd}} \to \mathbf{Z}$$

and the fact that $\mathbf{X}$ is TRD, we deduce that the map $\mathbf{X} \to \mathbf{Z}_{\mathrm{rd}}$ is a relatively distal extension, and it follows that the intermediate extension $\mathbf{Y} \to \mathbf{Z}_{\mathrm{rd}}$ is also distal (this is a nontrivial result; see [15, Theorem 10.18]). Thus the extension $\mathbf{Y} \to \mathbf{Z}_{\mathrm{rd}}$ is both relatively weakly mixing and relatively distal, hence an isomorphism. This means that indeed $\mathbf{Y} \to \mathbf{Z}$ is a relatively distal extension, as claimed. □

Claim 3. *An ergodic* $\mathbf{X}$ *is not* RWM *iff it is* TRD. *Thus*

$$\mathrm{MPT}_e = \mathit{RWM} \sqcup \mathit{TRD}.$$

[1] For $\chi \in [0,1]$, T is said to be *χ-mixing* if there is a sequence $n_k \nearrow \infty$ in $\mathbb{N}$ such that for any two measurable sets $A, B \in \mathcal{X}$ $\lim_{k\to\infty} \mu(T^{n_k} A \cap B) = \chi\mu(A)\mu(B) + (1-\chi)\mu(A \cap B)$.

Proof. Suppose $\mathbf{X}$ is not RWM and $\mathbf{X} \to \mathbf{Y}$ is a factor map. Then by the Furstenberg–Zimmer structure theorem (the relative version) there is a canonical diagram $\mathbf{X} \to \mathbf{Y}_{\mathrm{rd}} \to \mathbf{Y}$, where $\mathbf{Y}_{\mathrm{rd}} \to \mathbf{Y}$ is relatively distal and $\mathbf{X} \to \mathbf{Y}_{\mathrm{rd}}$ is relatively weakly mixing. However, by assumption the map $\mathbf{X} \to \mathbf{Y}_{\mathrm{rd}}$ is an isomorphism, so that $\mathbf{X} = \mathbf{Y}_{\mathrm{rd}} \to \mathbf{Y}$ is relatively distal. The other inclusion is trivial: if $\mathbf{X}$ is TRD it cannot be RWM. □

Claim 4. *Every* 2*-fold distally simple system is* TRD*. A fortiori, every* 2*-fold quasi-simple system is* TRD*.*

Proof. Suppose $\mathbf{X} \to \mathbf{Y}$ is a proper factor of a 2-fold distally simple system $\mathbf{X}$. Let $\mathbf{X} \to \mathbf{Y}_{\mathrm{rd}} \to \mathbf{Y}$ be the corresponding Furstenberg–Zimmer diagram. Consider the factor map $\mathbf{X} \times_{\mathbf{Y}_{\mathrm{rd}}} \mathbf{X} \to \mathbf{X}$ defined by the projection map (on either the first or the second coordinate). By definition the system $\mathbf{X} \times_{\mathbf{Y}_{\mathrm{rd}}} \mathbf{X}$ is ergodic and it is a non-product self-joining of $\mathbf{X}$. By the definition of 2-fold distal simplicity the extensions $\mathbf{X} \times_{\mathbf{Y}_{\mathrm{rd}}} \mathbf{X} \to \mathbf{X}$ and hence also the intermediate extension

$$\mathbf{X} \times_{\mathbf{Y}_{\mathrm{rd}}} \mathbf{X} \to \mathbf{Y}_{\mathrm{rd}} \times_{\mathbf{Y}_{\mathrm{rd}}} \mathbf{X} \cong \mathbf{X}$$

are distal extensions.

On the other hand, by [15, Theorem 9.23], the extension $\mathbf{X} \times_{\mathbf{Y}_{\mathrm{rd}}} \mathbf{X} \to \mathbf{X}$ is weakly mixing, and therefore it is an isomorphism. In turn, this means that $\mathbf{X} \to \mathbf{Y}_{\mathrm{rd}}$ is an isomorphism, so that the extension $\mathbf{X} \to \mathbf{Y}$ is relatively distal, as required. □

Problem 1.1. Is the example in [18] TRD?

Problem 1.2. What is the extent of the class TRWM?

Claim 5. *Every* TRWM *system has zero entropy.*

Proof. By Sinai's theorem every ergodic positive entropy system, say $\mathbf{X}$, admits a Bernoulli factor $\mathbf{X} \to \mathbf{B}$. The Bernoulli system $\mathbf{B}$ has a nontrivial factor $\sigma\colon \mathbf{B} \to \mathbf{Z}$, such that the extension σ is a compact group extension. Now the resulting factor map $\mathbf{X} \to \mathbf{Z}$ is not relatively weakly mixing. It follows that every TRWM system has zero entropy. □

By Krieger's theorem every factor of an ergodic system $\mathbf{X} \to \mathbf{Y}$, with $\mathbf{Y}$ having entropy $< \log 2$, is determined by a partition $\alpha = \{A, X \setminus A\}$ for some $A \in \mathcal{X}$ of positive measure, so that $\mathcal{Y} = \vee_{N\in\mathbb{N}} \vee_{j=-N}^{N} T^j \alpha$.

Proposition 1.3. *If a dynamical system* $\mathbf{X}$ *is* RWM*, then, for any* $\delta > 0$*, it has a factor* $\pi\colon \mathbf{X} \to \mathbf{Y}$ *such that* (i) $\mathbf{Y}$ *is infinite with entropy* $< \delta$*, and* (ii) *the extension* π *is nontrivial and relatively weakly mixing. In particular, taking* $\delta = \log 2$*, we conclude, by Krieger's theorem, that* $\mathbf{Y}$ *can be taken as a subshift on two symbols; i.e. that* $\mathbf{Y}$ *admits a generator of the form* $\{A, A^c\}$*.*

Proof. We give two proofs as follows:

(a) There is by definition a factor map $\sigma\colon \mathbf{X} \to \mathbf{Z}$ with $\mathbf{Z}$ infinite and such that the extension σ is relatively weakly mixing. If $\mathbf{Z}$ has zero entropy there is nothing to prove. Otherwise there is a factor map $\mathbf{Z} \to \mathbf{Y}'$ with $\mathbf{Y}'$ infinite and having entropy $< \delta$. Let $\mathbf{Z} \to \mathbf{Y} \to \mathbf{Y}'$ be the relative Fustenberg–Zimmer tower, so that $\mathbf{Y}$ is the maximal distal extension of $\mathbf{Y}'$ within $\mathbf{Z}$. Now as the extensions $\mathbf{Z} \to \mathbf{Y}$ and $\mathbf{X} \to \mathbf{Z}$ are both relatively weakly mixing extensions, so is the iterated extension $\mathbf{X} \to \mathbf{Y}$. Finally, as distal extensions do not raise entropy, the entropy of $\mathbf{Y}$ is $< \delta$.

(b) By definition there is a nontrivial factor map $\sigma\colon \mathbf{X} \to \mathbf{Z}$ such that the extension σ is relatively weakly mixing. By the weak Pinsker property [5], we have a decomposition $\mathbf{Z} = \mathbf{B} \times \mathbf{Y}$ with $\mathbf{B}$ a Bernoulli system and $\mathbf{Y}$ having entropy $< \delta$. Now clearly the composed map $\mathbf{X} \to \mathbf{Y}$ satisfies the assertion of the proposition. □

We denote by $\mathbf{M}$ the measure algebra associated to $(\mathcal{X}, \mu)$ consisting of equivalence classes of the relation $A \sim B \iff \mu(A \triangle B) = 0$, equipped with the complete metric $d_{\mathbf{M}}(A, B) = \mu(A \triangle B)$.

In [2] Ageev shows that the generic automorphism of a Lebesgue space is conjugate to a G-extension for any finite abelian group G. Thus, in particular, it follows that the generic automorphism is not prime. In [4] he shows that the collection of 2-fold quasi-simple systems (and a fortiori that of simple systems) forms a meager subset of MPT or MPT_e, the spaces of measure preserving and ergodic measure preserving transformations, respectively.

Proposition 1.4. RWM *is an analytic subset of* MPT.

Proof. We consider the space MPT consisting of the invertible measure preserving transformations of a nonatomic standard probability space $(X, \mathcal{X}, \mu)$. Given an ergodic T in MPT, each positive set $A \in \mathcal{X},\ 0 < \mu(A) < 1$, determines a partition $\alpha = \{A, X \setminus A\}$ which, in turn, defines a T-invariant σ-algebra $\mathcal{A} = \vee_{N\in\mathbb{N}} \vee_{j=-N}^{N} T^j\alpha$. We let $\pi\colon \mathbf{X} \to \mathbf{Y} = (Y, \mathcal{Y}, \nu, T)$ be the corresponding factor map, so that $\mathcal{A} = \pi^{-1}(\mathcal{Y})$. By Proposition 1.3 every RWM system $\mathbf{X}$ admits a nontrivial factor $\pi\colon \mathbf{X} \to \mathbf{Y}$ with π relatively weakly mixing and $\mathbf{Y}$ a two-set generator as above.

Let $\mu = \int_Y \mu_y \, d\nu(y)$ be the disintegration of μ over ν, and let

$$\lambda = \int_Y (\mu_y \times \mu_y) \, d\nu(y)$$

be the relative product measure of μ with itself over ν. Using a Rohklin skew product representation for the ergodic system $\mathbf{X}$ as $(X, \mu) = (Y \times Z, \nu \times \eta)$, we have $\mu_y = \eta$ for ν-a.e. y, and for functions f and g in $L^\infty(\mu)$, we have

$$\int_{X\times X} f(x_1)g(x_2)\, d\lambda(x_1, x_2) = \int_Y \left(\int_Z f(y, z_1)\, d\mu_y(z_1) \cdot \int_Z g(y, z_2)\, d\mu_y(z_2) \right) d\nu(y).$$

Another way of writing $\int_Z f(y, z)\, d\mu_y(z)$ is $\mathbb{E}(f|\mathcal{A})$, a function on X measurable with respect to $\mathcal{A}$. Therefore

$$\int_{X\times X} f(x_1)g(x_2)\, d\lambda(x_1, x_2) = \int_X \mathbb{E}(f|\mathcal{A})(x) \cdot \mathbb{E}(g|\mathcal{A})(x)\, d\mu(x). \tag{1}$$

If

$$\mathcal{P}_1 \prec \ldots \prec \mathcal{P}_n \prec \mathcal{P}_{n+1} \prec \ldots$$

is a sequence of finite partitions such that the corresponding algebras $\widehat{\mathcal{P}}_n$ satisfy $\vee_{n\in\mathbb{N}}\widehat{\mathcal{P}}_n = \mathcal{A}$, then, by the martingale convergence theorem,

$$\mathbb{E}(f|\mathcal{P}_n) \to \mathbb{E}(f|\mathcal{A}). \tag{2}$$

Now by definition the extension $\pi\colon \mathbf{X} \to \mathbf{Y}$ is a relatively weakly mixing extension when the measure λ is ergodic. This is the case if, for a dense sequence of pairs of sets $\{(C_n, D_n)\}_{n\in\mathbb{N}}$ in $\mathbf{M} \times \mathbf{M}$, we have for all $n \in \mathbb{N}$

$$\frac{1}{L}\sum_{i=0}^{L-1} (T \times T)^i(\mathbf{1}_{C_n} \times \mathbf{1}_{D_n}) \xrightarrow{L_2} \int (\mathbf{1}_{C_n} \times \mathbf{1}_{D_n})\, d\lambda,$$

i.e. with $a_n = \int (\mathbf{1}_{C_n} \times \mathbf{1}_{D_n})\, d\lambda$,

$$\int \Big| \frac{1}{L}\sum_{i=0}^{L-1} (T \times T)^i(\mathbf{1}_{C_n} \times \mathbf{1}_{D_n}) - a_n \Big|^2 d\lambda \to 0.$$

Now expanding the expression

$$\int \Big|\frac{1}{L}\sum_{i=0}^{L-1}(T\times T)^i(\mathbf{1}_{C_n}\times\mathbf{1}_{D_n})-a_n\Big|^2 d\lambda,$$

writing $\mathbf{1}_{C_n}=f_n$ and $\mathbf{1}_{D_n}=g_n$, and using (1), we get

$$\begin{aligned}
&\int \Big|\frac{1}{L}\sum_{i=0}^{L-1}(T\times T)^i(\mathbf{1}_{C_n}\times\mathbf{1}_{D_n})-a_n\Big|^2 d\lambda\\
&\qquad=\int \Big|\frac{1}{L}\sum_{i=0}^{L-1}(T\times T)^i(f_n\times g_n)-a_n\Big|^2 d\lambda\\
&\qquad=\frac{1}{L_2}\sum_{i,j}\int\big((T\times T)^i(f_n\times g_n)(T\times T)^j(f_n\times g_n)\\
&\qquad\qquad -a_n(T\times T)^i(f_n\times g_n)-a_n(T\times T)^j(f_n\times g_n)+a_n^2\big)\,d\lambda\\
&\qquad=\frac{1}{L_2}\sum_{i,j}\int(T^if_n\cdot T^jf_n)\times(T^ig_n\cdot T^jg_n)\,d\lambda-a_n^2\\
&\qquad=\frac{1}{L_2}\sum_{i,j}\int\mathbb{E}(T^if_n\cdot T^jf_n|\mathcal{A})\cdot\mathbb{E}(T^ig_n\cdot T^jg_n|\mathcal{A})\,d\mu-a_n^2.
\end{aligned}$$

Taking $\mathcal{P}_M=\vee_{j=-M}^{M}T^j\alpha$ and applying (2) we can approximate this by the corresponding sum

$$\frac{1}{L_2}\sum_{i,j}^{L-1}\int\left[\mathbb{E}\Big(T^if_n\cdot T^jf_n\,\Big|\bigvee_{j=-M}^{M}T^j\alpha\Big)\cdot\mathbb{E}\Big(T^ig_n\cdot T^jg_n\,\Big|\bigvee_{j=-M}^{M}T^j\alpha\Big)\right]d\mu-a_n^2,$$

which we denote by

$$EA(C_n,D_n,L,M).$$

Let $\{E_m\}_{m=1}^{\infty}$ be a sequence of sets in $\mathcal{X}$ which is dense in the subspace of $\mathbf{M}$ comprising sets E with $\mu(E)>1/10$. Let $\{(C_n,D_n)\}_{n=1}^{\infty}$ be a dense sequence in $\mathbf{M}\times\mathbf{M}$. For positive integers, m,n,k,L,M,N, we consider the set

$$U(m,n,k,L,N,M)\subset\mathbf{M}\times\mathrm{MPT}_e,$$

comprising those pairs (A,T) of $\mathbf{M}\times\mathrm{Aut}_e(\mu)$ that, with $\alpha=\{A,X\setminus A\}$ $0<\mu(A)<1$, satisfy the following inequalities:

(1) $d_{\mathbf{M}}\Big(E_m,\vee_{j=-N}^{N}T^j\alpha\Big)>1/100$,
(2) $EA(C_n,D_n,L,M)<1/k$,

where the distance $d_{\mathbf{M}}\Big(E_m,\vee_{j=-N}^{N}T^j\alpha\Big)$ is defined as the minimum of the distances $d(E_m,B)$, when B ranges over the elements of the finite algebra generated by the partition $\vee_{j=-N}^{N}T^j\alpha$.

The set $U(m,n,k,L,N,M)$ is open and we let

$$\mathrm{rwm}=\bigcup_{m=1}^{\infty}\bigcap_{N=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{L=1}^{\infty}\bigcup_{M_0=1}^{\infty}\bigcap_{M=M_0}^{\infty}U(m,n,k,L,N,M).$$

Now, in view of Proposition 1.3, RWM is the projection of the set rwm in $\mathrm{Aut}_e(\mu)$; i.e.

$$\mathrm{RWM}=\{T\in\mathrm{Aut}(\mu)\colon\exists A\in\mathbf{M},\ (A,T)\in\mathrm{rwm}\}.$$

As we have shown that the set rwm is Borel, it follows that the set RWM is analytic. □

The next proposition is proved similarly and we use the notation of the previous proof.

Proposition 1.5. TRWM *is a co-analytic subset of* MPT.

Proof. By Claim 5 it suffices to show that TRWM is a co-analytic subset of the G_δ set of 0-entropy ergodic transformations. We also recall that by Krieger's theorem every such transformation admits a two-set generator.

Let $\{(C_n, D_n)\}_{n=1}^\infty$ be a dense sequence in $\mathbf{M} \times \mathbf{M}$. For positive integers, n, k, L, M we consider the set

$$U(n,k,L,M) \subset \mathbf{M} \times \mathrm{Aut}_e(\mu),$$

comprising those pairs $(A,T) \in \mathbf{M} \times \mathrm{Aut}_e(\mu)$ that, with $\alpha = \{A, X \setminus A\},\ 0 < \mu(A) < 1$, satisfy the following inequality:

$$EA(C_n, D_n, L, M) < 1/k.$$

The set $U(n,k,L,M)$ is open and we let

$$\mathrm{trwm} = \bigcap_{n=1}^\infty \bigcap_{k=1}^\infty \bigcup_{L=1}^\infty \bigcup_{M_0=1}^\infty \bigcap_{M=M_0}^\infty U(n,k,L,M).$$

Now TRWM is the collection of $T \in \mathrm{Aut}_e(\mu)$ such that T has zero entropy and $(A,T) \in$ trwm for all $A \in \mathbf{M}$, i.e.

$$\mathrm{Aut}_e(\mu) \setminus \mathrm{TRWM} = \{T \in \mathrm{Aut}(\mu) \colon \exists A \in \mathbf{M},\ (A,T) \notin \mathrm{trwm}\}. \qquad \square$$

Problem 1.6. Ryzhikov's question in [37] is whether the set RWM is co-meager.

Remark 1.7. In view of Definition 0.1(4) and Claim 3 we have that

$$\mathrm{MPT}_e = \mathrm{RWM} \sqcup \mathrm{TRD} = \mathrm{RWM} \sqcup \mathrm{TRD}^\sim \sqcup \mathrm{PRIME}$$

and the last union is disjoint (note that the prime systems of the form $\mathbb{Z}_p$ do not belong to $\mathrm{Aut}(\mu)$). Since by [2] the collection PRIME (which is shown to be co-analytic in [17]) is meager, we conclude, by the zero-one law (see [17]), that one of the, clearly invariant, collections RWM and TRD (or RWM and TRD$^\sim$) is meager and the other comeager. In the next section we will show that the set RWM is the comeager one.

In the next proposition we determine the, relatively low, complexity of the class PROP consisting of elements of MPT_e which admit a proper factor.

Proposition 1.8. (1) *The set*

$$\mathrm{prop} = \left\{(A,T) \colon \alpha = \{A, X \setminus A\},\ \mathcal{Y} = \bigvee_{N \in \mathbb{N}} \bigvee_{j=-N}^{N} T^j \alpha \subsetneq \mathcal{X}\right\}$$

is a $G_{\delta,\sigma}$ *subset of* $\mathbf{M} \times \mathrm{MPT}_e$ *which is invariant under the diagonal action of* $G = \mathrm{Aut}(\mu)$ *on* $\mathbf{M} \times \mathrm{MPT}_e$ *defined by*

$$g \cdot (A,T) = (gA, gTg^{-1}), \quad A \in \mathbf{M},\ T \in \mathrm{MPT}_e\,.$$

(2) *Its projection*

$$\mathrm{PROP} = \left\{T \in \mathrm{MPT}_e \colon \exists \alpha,\ \mathcal{Y} = \bigvee_{N \in \mathbb{N}} \bigvee_{j=-N}^{N} T^j \alpha \subsetneq \mathcal{X}\right\}$$

is a nonempty $G_{\delta,\sigma}$ *subset of* MPT_e.

Proof. (1) Let $\{E_m\}_{m=1}^\infty$ be a sequence of sets in $\mathcal{X}$ which is dense in the subspace of $\mathbf{M}$ comprising sets E with $\mu(E) \geq 1/10$.

For positive integers, m, N set

$$U(m,N) = \left\{(A,T) \in \mathbf{M} \times \mathrm{MPT}_e \colon d_{\mathbf{M}}\left(E_m, \bigvee_{j=-N}^{N} T^j\alpha\right) > 1/100\right\},$$

with $\alpha = \alpha(A) = \{A, X \setminus A\}$ (for $A \in \mathcal{X}$ of positive measure), The $U(m,N)$ is an open set and the set

$$U(m) = \bigcap_{N=1}^{\infty} U(m,N) = \left\{(A,T) \in \mathbf{M} \times \mathrm{MPT}_e \colon d_{\mathbf{M}}\left(E_m, \bigvee_{N\in\mathbb{N}} \bigvee_{j=-N}^{N} T^j\alpha\right) > 1/100\right\}$$

is a G_δ set. Finally we have that

$$\mathrm{prop} = \bigcup_{m=1}^{\infty} U(m) = \left\{(A,T) \in \mathbf{M} \times \mathrm{MPT}_e) \colon \bigvee_{N\in\mathbb{N}} \bigvee_{j=-N}^{N} T^j\alpha \neq \mathcal{X}\right\}$$

is a $G_{\delta,\sigma}$ set. The invariance is clear.

(2) The projection $P \colon \mathbf{M} \times \mathrm{MPT}_e \to \mathrm{MPT}_e$ is an open homomorphism of Polish dynamical systems. Therefore each $P(U(m,N))$ is an open set, so that the image $\mathrm{PROP} = P(\mathrm{prop})$ is a non-empty invariant $G_{\delta,\sigma}$ subset of MPT_e. □

2. RWM is generic

Let $(X, \mathcal{X}, \mu)$ be a standard Lebesgue space, were the probability measure μ has no atoms. Let $\mathbf{B}$ denote the W^*-algebra of bounded linear operators on the Hilbert space $L_2(\mu)$, equipped with the strong operator topology. It is easy to see that the space MPT, comprising the Koopman operators corresponding to the invertible measure preserving transformations of $(X, \mathcal{X}, \mu)$, is a closed subset of $\mathbf{B}$. Let $\mathbf{P}$ be the set of positive projections, i.e. those elements P of $\mathbf{B}$ such that $\|P\| = 1, P^2 = P$, $P(\mathbf{1}) = 1$, and $Pf \geq 0$ for every $0 \leq f \in L_2(\mu)$. This is a closed subset of $\mathbf{B}$.

A sub-σ-algebra $\mathcal{Y} \subset \mathcal{X}$ determines a standard probability space $(Y, \mathcal{Y}, \nu)$, a closed subspace $H = H(\mathcal{Y}) \subset L_2(\mu)$, and a positive projection $P \colon L_2(\mu) \to H$. More precisely, a positive projection P can be identified with a *conditional expectation operator* over the subspace $H = PL_2(\mu)$ that can be described as the set of functions in $L_2(\mu)$ which are measurable with respect to a sub-σ-algebra $\mathcal{Y} \subset \mathcal{X}$. As was shown by Rokhlin, we can realize the latter inclusion as a map $(X, \mathcal{X}, \mu) \to (Y, \mathcal{Y}, \nu)$, where $(Y, \mathcal{Y}, \nu)$ is a standard Lebesgue space, so that the projection P has the form

$$(Pf)(y) = \int_Y f(x)\, d\mu_y(x), \quad \text{for } \nu\text{-a.e. } y \in Y.$$

Clearly the collection of projections P whose range has dimension $\leq k$ for $k \in \mathbb{N}$ is a closed set and it follows that the collection $\mathbf{P}_i$ of the projection whose range is infinite dimensional forms a G_δ subset of $\mathbf{P}$. We denote by $\mathbf{P}_{ci}$ the set of $P \in \mathbf{P}$ such that $P \neq \mathrm{Id}$, and such that in the corresponding Rokhlin representation, in the disintegration $\mu = \int_Y \mu_y\, d\nu(y)$, a.e. μ_y has no atoms. Finally let $\mathbf{Q} = \mathbf{P}_i \cap \mathbf{P}_{ci}$.

We denote by $\mathbf{T}$ the collection of weakly mixing MPT's, which we often identify with their Koopman operators. By Halmos' theorem $\mathbf{T}$ is a dense G_δ subset of MPT. Thus we view the elements of both $\mathbf{P}$ and $\mathbf{T}$ as operators in $\mathbf{B}$. Note that if T is weakly mixing and $(X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, T \restriction \mathcal{Y})$ is a nontrivial factor, then the corresponding P is necessarily in $\mathbf{P}_i$. The group $G = \mathrm{Aut}\,(\mu)$ acts continuously on $\mathbf{B}$ by conjugations, and both $\mathbf{T}$ and $\mathbf{Q}$ are invariant subsets under this action. Let

$$\mathbf{L} = \{(T,P) \colon PT = TP,\ P \in \mathbf{Q},\ T \in \mathbf{T}\}$$

and we will consider the diagonal action of $\mathrm{Aut}\,(\mu)$ on $\mathbf{L}$. Denote by π_1 and π_2 the projections of $\mathbf{L}$ on its first and second coordinates, respectively. Note that if $T = \pi_1(T,P)$ for $(T,P) \in \mathbf{L}$, then T is weakly mixing and is not prime, and if $P = \pi_2(T,P)$ for $(T,P) \in \mathbf{L}$, then the operator TP, defined on the infinite-dimensional space $PL_2(\mu)$, is a factor of T and is therefore weakly mixing on $L_2(Y,\mathcal{Y},\nu)$ with ν atomless.

Lemma 2.1. *A pair (T,P) is in $\mathbf{L}$ iff T, as a transformation in* MPT*, is weakly mixing and it admits a factor map $\mathbf{X} = (X,\mathcal{X},\mu,T) \to \mathbf{Y} = (Y,\mathcal{Y},\nu,S)$ with $PL_2(\mu) \cong \cong L_2(\nu) = L_2(\mathcal{Y})$, where ν has no atoms, and $\mathbf{X}$ has the form of a Rokhlin skew product*

$$(X,\mathcal{X},\mu,T) = (Y \times Z, \mathcal{Y} \otimes \mathcal{Z}, \nu \times \eta, \widehat{S}),$$

where $(Z,\mathcal{Z},\eta,T)$ is a standard Lebesgue space with η atomless, and $S\colon Y \to \mathrm{Aut}\,(Z,\eta)$ is a measurable cocycle with $\widehat{S}(y,z) = (Ty, S_y z)$ μ-a.e.

Proof. If $(T,P) \in \mathbf{L}$ then T is weakly mixing, hence ergodic, and it then follows that the Rokhlin presentation has this form (see e.g. [15, Theorem 3.18]). Conversely, clearly if T is a weakly mixing skew product as described, then the corresponding pair (T,P) is in $\mathbf{L}$. □

Proposition 2.2.

(1) $\mathbf{Q}$ *is a G_δ subset of* $\mathbf{B}$.
(2) G *acts transitively on* $\mathbf{Q}$.
(3) $\mathbf{T}$ *is a G_δ subset of* $\mathbf{B}$.
(4) *The action of G on the Polish space* $\mathbf{T}$ *is minimal.*
(5) $\mathbf{L}$ *is a closed subset of the Polish space* $\mathbf{T} \times \mathbf{Q}$.
(6) G *acts minimally on* $\mathbf{L}$.

Proof. (1) We already observed that the condition (i) that $PL_2(\mu)$ be infinite dimensional defines a G_δ subset, $\mathbf{P}_i$ of $\mathbf{B}$. Thus a positive projection $P \in \mathbf{P}_i$ is in $\mathbf{Q}$ iff it is in $\mathbf{P}_{ci}$. We claim that this is equivalent to the following condition:

For all positive measure sets $A \in \mathcal{X}$ there exist sets B and C such that

(i) $\mathbf{1}_A = \mathbf{1}_B + \mathbf{1}_C$,
(ii) $P(\mathbf{1}_B) = P(\mathbf{1}_C)$.

Now we can express this property as the intersection of a countable collection of open sets as follows. Let $\{A_n\}$ be a dense sequence in the measure algebra and define

$$U(N,n,i,j) = \big\{P \in \mathbf{P}\colon \mu(A_n \triangle (A_i \cup A_j)) < 1/N, \|P(\mathbf{1}_{A_i}) - P(\mathbf{1}_{A_j})\| < 1/N, \mu(A_i \cap A_j) < 1/N\big\}.$$

Now set

$$\mathbf{P}_{ci} = \bigcap_{n\geq 1} \bigcap_{N\geq 1} \bigcup_{(i,j)} U(N,n,i,j).$$

To see that this intersection captures the property that the conditional measures are continuous we argue by contradiction. If $\{\mu_y\}_{y\in Y}$ are the conditional measures associated to the projection P, and for a set $B \subset Y$ with $\nu(B) > 0$, there are atoms of μ_y for all $y \in B$ with measure $\geq c > 0$, then there is a measurable function $f\colon B \to X$ such that $\mu(\{f(y)\}) \geq c,\ y \in B$. The range $A = f(B)$ is a measurable subset of X and

$$(P\mathbf{1}_A)(y) = \mu_y(\{f(y)\}).$$

Now we take $1/N \ll \mu(A)$ and A_{n_0} such that $\mu(A \triangle A_{n_0}) < 1/N$. We see that if

$$\mu(A \triangle (A_{i_0} \cup A_{j_0})) < 1/N,$$

then the approximate equality $P\mathbf{1}_{A_{i_0}} \approx P\mathbf{1}A_{j_0}$ cannot hold. Thus P is not in the intersection $\mathbf{P}_{ci}$, as required.

To prove claim (2) observe that P determines a Rokhlin decomposition of $\mathcal{X}$ over $\mathcal{Y}$ as explained above, and that clearly any two such decompositions are measurably isomorphic. Claim (3) is well known. Claim (4) follows from Halmos' Conjugacy Lemma [20, page 77], which asserts that the conjugacy class of each aperiodic element of $\mathrm{Aut}\,(\mu)$ is dense. Claim (5) is easy to check. Finally claim (6) can be easily deduced from claim (2) and [19, Proposition 2.3], which is a relative analogue of Halmos' conjugacy lemma. □

Remark 2.3. In the definition of $\mathbf{Q}$ we excluded the two extreme cases, when $H = L_2(\mu)$ (i.e. when $P = \mathrm{Id}$) and when H is finite dimensional. We note however that due to this exclusion, in the projection of $\mathbf{L}$ on the $\mathbf{T}$ coordinate, the weakly mixing transformations that are missing are: (i) The prime transformations, which according to Ageev [2] form a meager subset of $\mathbf{T}$. (ii) Those weakly mixing transformations for which every factor map $\mathbf{X} \to \mathbf{Y}$ is finite to one. Such transformations are in TRD, and e.g. by Ageev [3], or by Stepin and Eremenko—who show in [42] that the generic ergodic transformation admits the infinite torus in its centralizer, form a meager set.

Recall the following (see [27, Apendix A] and [26, Definition 2.7, Definition 2.4]).

Definition 2.4. Let Y, Z be Polish spaces and let $f\colon Y \to Z$ be a continuous map.

(1) We say that f is *category preserving* if it satisfies the following condition: For any comeager $A \subseteq Z$, $f^{-1}(A)$ is comeager in Y.
(2) A point $y \in Y$ is a point of *local density for* f if for any neighborhood U of y the set $\overline{f(U)}$ is a neighborhood of $f(y)$.

We then have the following ([26, Proposition 2.8]; see also Tikhonov's work [44]):

Proposition 2.5. *Let Y, Z be Polish spaces and $f\colon Y \to Z$ a continuous map. Then f is category preserving if and only if the set of points which are locally dense for f is dense in Y.*

We also recall the following well-known observation called Dougherty's lemma (see e.g. [24] and [26, Proposition 2.5]).

Proposition 2.6. *Let Y, Z be Polish spaces and let $f\colon Y \to Z$ be a continuous map such that the set of points which are locally dense for f is dense in Y. Let $B \subset Y$ be a comeager subset of Y. Then $f(B)$ is not meager in Z.*

The following statement is shown in [27].

Theorem 2.7. *Let Y, Z be Polish spaces, and $f\colon Y \to Z$ be a category preserving map. Also, let A be a subset of Y with the property of Baire. Then the following assertions are equivalent*:

(i) *A is comeager in Y,*
(ii) *$\{z \in Z\colon A \cap f^{-1}(z)$ is comeager in $f^{-1}(z)\}$ is comeager in Z.*

Next we have a dynamical version of the Kuratowski–Ulam theorem [26, Proposition 2.10]:

Proposition 2.8. *Let H be a Polish group, Y, Z be two Polish H-spaces and $f\colon Y \to Z$ an H-map. Assume that Y is minimal (i.e. every orbit is dense) and $f(Y)$ is not meager. Then f is category preserving.*

Remark 2.9. An older result of Veech [45, Proposition 3.1] shows this result for compact topological systems. See also [14, Lemma 5.2] where a topological analogue of simplicity was studied.

The next theorem answers positively Ryzhikov's question.

Theorem 2.10. *The property RWM is generic.*

Proof. We first show that RWM is comeager. Consider the closed set $\mathbf{L} \subset \mathbf{T} \times \mathbf{Q}$. By [39] (see also [19, Theorem 1.3]) we know that for every fixed $P \in \mathbf{Q}$ the set of elements $T \in \mathbf{T}$ such that T is a relatively weakly mixing extension of $T \restriction \mathcal{Y}$, where $\mathcal{Y}$ is the σ-algebra determined by P, is a dense G_δ subset of $\mathbf{L}^P$, where

$$\mathbf{L}^P = \{T \in \mathbf{T} \colon (T, P) \in \mathbf{L}\}.$$

Consider the set

$$\mathbf{L}_{\mathrm{RWM}} = \{(T, P) \in \mathbf{L} \colon T \text{ is relatively weakly mixing over the factor determined by } P\}.$$

By Proposition 1.4 the set $\mathrm{RWM} \cap \mathbf{T}$ is an analytic subset of $\mathbf{T}$, hence has the BP. Denoting by π_1 the projection from $\mathbf{T} \times \mathbf{Q}$ onto $\mathbf{T}$ we observe that $\mathbf{L}_{\mathrm{RWM}} = \mathbf{L} \cap \pi_1^{-1}(\mathrm{RWM})$, and as $\mathbf{L}$ is a closed subset of $\mathbf{T} \times \mathbf{Q}$, we conclude that $\mathbf{L}_{\mathrm{RWM}}$ has the BP.

Applying Proposition 2.8 to the minimal Polish system $(\mathbf{L}, G)$ and the projection map $\pi_2 \colon (\mathbf{L}, G) \to (\mathbf{Q}, G)$, we conclude that the map π_2 is category preserving. (Note that $\pi_2(L) = \mathbf{Q}$.) Applying Theorem 2.7 to $\mathbf{L}_{\mathrm{RWM}} \subset \mathbf{L}$, we conclude that $\mathbf{L}_{\mathrm{RWM}}$ is comeager in $\mathbf{L}$.

Now by the same token also the other projection map $\pi_1 \colon (\mathbf{L}, G) \to (\mathbf{T}, G)$ is category preserving and so, by Dougherty's lemma, Proposition 2.6, the image of the set $\mathbf{L}_{\mathrm{RWM}}$ under π_1, namely the subset $\pi_1(\mathbf{L}_{\mathrm{RWM}}) \subset \mathbf{T}$, is not meager in $\pi_1(\mathbf{L}) \subset \mathbf{T}$. Since this subset is also invariant, it follows from the zero-one law that it is comeager in $\mathbf{T}$. We now note that this subset is exactly the set of nonprime, weakly mixing systems which are RWM and, in view of Remark 2.3, it follows that the set $\mathrm{RWM} \cap \mathbf{T}$ is indeed comeager in $\pi_1(\mathbf{L}) \subset \mathbf{T}$. □

Remark 2.11. Our proof of Theorem 2.10 uses Ageev's result, which asserts that the class PRIME is meager, so of course it does not provide a new proof of this statement. However note that whereas Ageev's proof demonstrates the prevalence of group extensions, ours demonstrates the prevalence of relatively weakly mixing extensions.

3. The relatively weakly mixing factors of a positive entropy T form a dense G_δ set

In the second part of the paper, Sections 3 to 7, we slightly change the framework of our discussion. We mainly fix an ergodic transformation T with a certain property, and then consider either its family of factors, as in the present section, or its family of ergodic extensions, as in Sections 4 to 7.

Recall that $\mathbf{P}$ denotes the G_δ set of positive projections in $\mathbf{B}$, the W^*-algebra of bounded linear operators on $L_2(\mu)$.

Proposition 3.1. *For a fixed $T \in \mathrm{MPT}_e$, the set $\mathbf{W}_T$ of those $P \in \mathbf{P}$ such that $TP = PT$, and such that T is relatively weakly mixing over the factor determined by P, is a G_δ set.*

Proof. Let $\mathbf{P}_T$ denote the closed subset of $\mathbf{P}$ comprising the projections P satisfying $TP = PT$. First note that for two bounded functions f, g in $L_2(\mu)$ the integral of the product $f(x_1)g(x_2)$ with respect to the relative product measure $\lambda = \mu \underset{\nu}{\times} \mu$ over the factor $\mathbf{Y}$ defined by $P \in \mathbf{P}_T$ is given by

$$\int f(x_1)g(x_2)\, d\lambda(x_1, x_2) = \int Pf(x)Pg(x)\, d\mu(x).$$

Now we fix a sequence $\{f_i\}$ of bounded functions that are dense in $L_2(\mu)$ and then define the sets

$$U(k,N,i,j)=\left\{P\in\mathbf{P}_T\colon\ \int\left|1/N\sum_{n=0}^{N-1}T^nPf_i(x)T^nPf_j(x)-a_{ij}\right|^2d\mu(x)<1/k\right\}$$

where $a_{ij}=\int Pf_i(x)Pf_j(x)d\mu(x)$. These are open sets and we let

$$\mathbf{W}=\bigcap_{(i,j)}\bigcap_{k\geq 1}\bigcup_{N\geq 1}U(N,n,i,j).$$

Now $\mathbf{W}$ is a G_δ set and if $P\in\mathbf{W}$, then T is RWM. Clearly also when $P\in\mathbf{P}$ is such that $PT=TP$ and the extension over the factor determined by P is weakly mixing, then $P\in\mathbf{W}$. Thus $\mathbf{W}=\mathbf{W}_T$. □

Theorem 3.2. *If $T\in\mathrm{MPT}_e$ has positive and finite entropy, then among the positive projections P which commute with T (i.e. among the factors of T), for a dense G_δ set the corresponding extension over the factor determined by P is relatively weakly mixing.*

Proof. This is just the set $\mathbf{W}_T$ above and hence, by Proposition 3.1, it is a G_δ set.

To see that it is dense we use the fact that is proved by Thouvenot for a T with the weak-Pinsker property [43, Lemma 7]. Namely, given $\varepsilon>0$ one can perturb by no more than ε any partition $\mathcal{Q}$ to a partition $\overline{\mathcal{Q}}$ so that the new partition splits off with a Bernoulli complement $\overline{\mathcal{B}}$:

(1) $|\overline{\mathcal{Q}}-\mathcal{Q}|<\varepsilon$,
(2) $(\overline{\mathcal{Q}})_T\perp(\overline{\mathcal{B}})_T$,
(3) $(\overline{\mathcal{Q}})_T\vee(\overline{\mathcal{B}})_T=\mathcal{X}$,
(4) the partitions $T^i\overline{\mathcal{B}},\ i\in\mathbb{Z}$, are independent.

Now given a projection $P\in\mathbf{P}_T$ that one wants to approximate, we find a generator $\mathcal{Q}$ for the factor determined by P and apply Thouvenot's lemma.

However, to make sure that the Bernoulli part of Thouvenot's splitting is not trivial we first modify the generator $\mathcal{Q}$ by a little bit so that the entropy of the new $\mathcal{Q}$ is strictly less than the entropy of T. Then we choose the ε in the conclusion of [43, Lemma 7] so small that the factor defined by $\overline{\mathcal{Q}}$ still has less than full entropy. Now we are sure that the Bernoulli complement is non-trivial. Finally, by Austin's recent result [5], the weak Pinsker property always holds and our proof is complete. □

Problem 3.3. Can one prove an analogous claim for relative Bernoulli extensions?

4. A generic extension does not add entropy

In the last few sections of the paper we show that for a fixed ergodic T with property **A**, a generic extension $\widehat{T}$ of T also has the property **A**. Here **A** stands for each of the following properties: (i) having the same entropy as T, (ii) Bernoulli, (iii) K, and (iv) loosely Bernoulli.

Theorem 4.1. *For any fixed ergodic transformation T, the generic extension does not increase entropy.*

Proof. Let $\mathbf{X}=(X,\mathcal{X},\mu,T)$ be an ergodic system with finite entropy, which for convenience we assume equals 1. Let $\mathcal{R}\subset\mathcal{X}$ be a finite generating partition with entropy 1. Let $\mathcal{S}$ be the collection of Rokhlin cocycles with values in $\mathrm{MPT}(I,\lambda)$, where λ is the normalized Lebesgue measure on the unit interval $I=[0,1]$. Thus an element $S\in\mathcal{S}$

is a measurable map $x \mapsto S_x \in \mathrm{MPT}(I,\lambda)$, and we associate to it the *skew product transformation*

$$\widehat{S}(x,u) = (Tx, S_x u), \quad x \in X, u \in I.$$

Let $Y = X \times I$ and set $\mathbf{Y} = (Y, \mathcal{Y}, \mu \times \lambda)$, with $\mathcal{Y} = \mathcal{X} \otimes \mathcal{B}$.

We recall that, by Rokhlin's theorem, every ergodic extension $\mathbf{Y} \to \mathbf{X}$ either has this form or it is n to 1 a.e. for some $n \in \mathbb{N}$ (see e.g. [15, Theorem 3.18]). Thus the collection $\mathcal{S}$ parametrises the ergodic extensions of $\mathbf{X}$ with infinite fibers. This defines a Polish topology on $\mathcal{S}$ which is inherited from $\mathrm{MPT}(X \times I, \mu \times \lambda)$. Of course a finite-to-one extension does not add entropy and thus it suffices to show that for a dense G_δ subset $\mathcal{S}_0 \subset \mathcal{S}$ we have $h(\widehat{S}) = 1$ for every $S \in \mathcal{S}_0$.

For each $n \in \mathbb{N}$ let $\mathcal{Q}_n$ denote the dyadic partition of $[0,1]$ into intervals of size $1/2^n$, and let

$$\mathcal{P}_n = \mathcal{R} \times \mathcal{Q}_n.$$

Clearly, for any $S \in \mathcal{S}$ we have $h(\mathcal{P}_n, \widehat{S}) \geq 1$.

Lemma 4.2. *For any $\varepsilon > 0$, the set*

$$U(n,\varepsilon) = \{S \in \mathcal{S} \colon h(\mathcal{P}_n, \widehat{S}) < 1 + \varepsilon\}$$

is open.

Proof. Let $S_0 \in U(n,\varepsilon)$. Then, there exists N such that

$$\frac{H\left(\vee_0^{N-1} S_0^{-i}\mathcal{P}_n\right)}{N} = 1 + a,$$

with $0 \leq a < \varepsilon$. If $\delta = \frac{\varepsilon - a}{2}$ and S is sufficiently close to S_0, then

$$\left| \frac{H\left(\vee_0^{N-1} S^{-i}\mathcal{P}_n\right)}{N} - \frac{H\left(\vee_0^{N-1} S_0^{-i}\mathcal{P}_n\right)}{N} \right| < \delta,$$

so that S will be in $U(n,\varepsilon)$. □

It follows from the lemma that

$$\mathcal{S}_0 = \bigcap_{n\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} U(n, 1/k)$$

is a G_δ subset of $\mathcal{S}$ consisting of those $S \in \mathcal{S}$ such that $h(\widehat{S}) = 1$. Clearly $\mathcal{S}_0$ is nonempty; e.g. we can take S to be the constant cocycles with a fixed value R_α, an irrational rotation on the circle. Thus by the relative Halmos theorem [19, Proposition 2.3], it follows that $\mathcal{S}_0$ is a dense G_δ subset of $\mathcal{S}$, as claimed. □

Using the classical Kuratowski–Ulam theorem we can "lift" this theorem to the collection of ergodic transformations which commute with a fixed $P \in \mathbf{Q}$ (see Section 2).

As usual let $(X, \mathcal{X}, \mu)$ be a standard atomless Lebesgue space and let $\mathcal{Y} \subset \mathcal{X}$ be a σ-algebra such that $\nu = \mu \restriction \mathcal{Y}$ is atomless and such that

$$(X, \mathcal{X}, \mu) = (Y \times Z, \mathcal{Y} \otimes \mathcal{Z}, \nu \times \eta),$$

where $(Z, \mathcal{Z}, \eta)$ is a standard Lebesgue space with η atomless. Let $P \in \mathbf{P}$ be the projection of $L_2(\mu)$ on $L_2(\mathcal{Y})$.

Set

$$\begin{aligned}\mathcal{N} &= \{T \in \mathrm{Aut}\,(\mu)\colon T \text{ leaves the } \sigma\text{-algebra } \mathcal{Y} \text{ invariant}\}\\ &= \{T \in \mathrm{Aut}\,(\mu)\colon TP = PT\},\end{aligned}$$

and let

$$\mathcal{N}_0 = \{T \in \mathcal{N}\colon T \text{ is ergodic and does not add entropy to } T \restriction \mathcal{Y}\}.$$

As the collection of ergodic transformations is a dense G_δ subset of $\mathrm{Aut}\,(\nu)$, Theorem 4.1, combined with the Kuratowski–Ulam theorem [30, Theorem 15.4], immediately yield the following result.

Corollary 4.3. *The collection $\mathcal{N}_0$ forms a residual subset of $\mathcal{N}$.*

5. A generic extension of a Bernoulli system is Bernoulli

Theorem 5.1. *For any fixed Bernoulli transformation T of finite entropy the generic extension is Bernoulli.*

Proof. We first recall the following

Definition 5.2. A finite measurable partition $\mathcal{P}$ of X is called *very weak Bernoulli* (VWB for short) if for every $\varepsilon > 0$, there is a positive integer N such that for all $k \geq 1$, there is a collection $\mathcal{G}_k$ of atoms of the partition $\vee_{i=-k}^{-1} T^{-i}\mathcal{P}$ such that

(1) $\mu\big(\cup\{A \in \mathcal{G}_k\}\big) > 1 - \varepsilon$,
(2) $\bar{d}_N\left(\vee_{i=0}^{N-1} T^{-i}\mathcal{P}, \vee_{i=0}^{N-1} T^{-i}\mathcal{P} \restriction A\right) < \varepsilon$, for every atom A of $\mathcal{G}_k$.

Here $\bar{d}_N$ is the normalized Hamming distance of distributions, and $\vee_{i=0}^{N-1} T^{-i}\mathcal{P} \restriction A$ is the conditional distribution of $\vee_{i=0}^{N-1} T^{-i}\mathcal{P}$ restricted to the atom A of the partition $\vee_{i=-k}^{-1} T^{-i}\mathcal{P}$.

As the inverse limit of Bernoulli systems is Bernoulli, to show that a transformation T on $(X, \mathcal{X}, \mu)$ is Bernoulli it suffices to show that for a refining sequence of partitions

$$\mathcal{P}_1 \prec \ldots \prec \mathcal{P}_n \prec \mathcal{P}_{n+1} \prec \ldots$$

such that the corresponding algebras $\widehat{\mathcal{P}}_n$ satisfy $\vee_{n\in\mathbb{N}} \widehat{\mathcal{P}}_n = \mathcal{X}$, for each n, the process $(T, \mathcal{P}_n)$ is VWB.

With no loss of generality we assume that the entropy of the Bernoulli transformation T is 1. We keep the notations of the previous section (Section 4), and we will show that the generic element of the space $\mathcal{S}_0$ is Bernoulli. By Theorem 4.1 for every $S \in \mathcal{S}$ the corresponding extension $\widehat{S}$ of T has entropy 1. We want to show that for a generic set of $S \in \mathcal{S}_0$, the corresponding $\widehat{S}$ is Bernoulli. To do this we need to express the fact that the partition $\mathcal{P}_n$ is VWB in a "finite way".

The problem is that in the definition of the VWB property the inequality

$$\bar{d}_N\left(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{P}_n, \bigvee_{i=0}^{N-1} T^{-i}\mathcal{P}_n \restriction A\right) < \varepsilon,$$

for $A \in \mathcal{G}_k$, has to hold for all large k. The key to being able to express this with one value of k_0 lies in the fact that we know that the entropy of $\mathcal{P}_n$ with respect to $\widehat{S}$ for $S \in \mathcal{S}_0$ equals 1. This is done as follows.

Definition 5.3. For two labeled partitions $\mathcal{P} = \{P_1, \ldots, P_a\}$, $\mathcal{Q} = \{Q_1, \ldots, Q_b\}$ of (X, μ) we say that $\mathcal{P}$ is ε-independent of $\mathcal{Q}$ if there is a set of indices $J \subset \{1, \ldots, b\}$ such that

(1) $\sum_{j\in J} \mu(Q_j) > 1 - \varepsilon$,

(2) $\sum_{i=1}^{a} |\mu(P_i|Q_j) - \mu(P_i)| < \varepsilon$ for each $j \in J$.

One can use entropy to express independence since $H(\mathcal{P}|\mathcal{Q}) = H(\mathcal{P})$ if and only if $\mathcal{P}$ is independent of $\mathcal{Q}$. If the number of elements of $\mathcal{P}$ is fixed, then the following is also well known. Given $\varepsilon > 0$ there is a $\delta > 0$ such that if $H(\mathcal{P}) < H(\mathcal{P}|\mathcal{Q}) + \delta$, then $\mathcal{P}$ is ε-independent of $\mathcal{Q}$. The conditional version of this also holds. Namely, if $\mathcal{R}$ is a third partition, given $\varepsilon > 0$ there is a $\delta > 0$ such that if

$$H(\mathcal{P}|\mathcal{R}) < H(\mathcal{P}|\mathcal{R} \vee \mathcal{Q}) + \delta, \tag{3}$$

then $\mathcal{P}$ conditioned on $\mathcal{R}$ is ε-independent of $\mathcal{Q}$. That is, there is a set of atoms A of $\mathcal{R}$ whose union has total measure $> 1 - \varepsilon$, such that conditioned on A, $\mathcal{P}$ is ε-inepended of $\mathcal{Q}$.

Here is another general fact. If the entropy of a partition $\mathcal{P}$ in $(X, \mathcal{X}, \mu, T)$ equals h, then for all N,

$$H\Big(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{P} \,\Big|\, \bigvee_{i=-\infty}^{-1} T^{-i}\mathcal{P}\Big) = Nh, \tag{4}$$

hence

$$\begin{aligned} Nh &\le H\Big(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{P} \,\Big|\, \bigvee_{i=-k}^{-1} T^{-i}\mathcal{P}\Big) \\ &= H\Big(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{P} \,\Big|\, \Big(\bigvee_{i=-k_0}^{-1} T^{-i}\mathcal{P}\Big) \bigvee \Big(\bigvee_{i=-k}^{-k_0-1} T^{-i}\mathcal{P}\Big)\Big), \end{aligned}$$

for all $k > k_0 \ge 1$.

Now in the inequality (3) we let $\mathcal{P} = \vee_{i=0}^{N-1} T^{-i}\mathcal{P}$, $\mathcal{R} = \vee_{i=-k_0}^{-1} T^{-i}\mathcal{P}$ and $\mathcal{Q} = \vee_{i=-k}^{-k_0-1} T^{-i}\mathcal{P}$. It follows that, given $\varepsilon > 0$, there is a $\delta > 0$ such that, if k_0 is sufficiently large so that

$$H\Big(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{P} \,\Big|\, \bigvee_{i=-k_0}^{-1} T^{-i}\mathcal{P}\Big) < Nh + \delta,$$

then, by (4),

$$\begin{aligned} &H\Big(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{P} \,\Big|\, \bigvee_{i=-k_0}^{-1} T^{-i}\mathcal{P}\Big) \\ &\quad < H\Big(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{P} \,\Big|\, \Big(\bigvee_{i=-k_0}^{-1} T^{-i}\mathcal{P}\Big) \bigvee \Big(\bigvee_{i=-k}^{-k_0-1} T^{-i}\mathcal{P}\Big)\Big) + \delta; \end{aligned}$$

hence, by (3), $\vee_{i=0}^{N-1} T^{-i}\mathcal{P}$ conditioned on $\vee_{i=-k_0}^{-1} T^{-i}\mathcal{P}$, is ε-independent of $\vee_{i=-k}^{-k_0-1} T^{-i}\mathcal{P}$, for all $k > k_0$.

With this background we come to the main step of the proof. Define the set $U(n, N_1, N_2, \varepsilon, \delta)$ to consist of those $S \in \mathcal{S}_0$ that satisfy:

(1) $H\left(\vee_{i=0}^{N_1-1} \widehat{S}^{-i}\mathcal{P}_n \,\middle|\, \vee_{i=-N_2}^{-1} \widehat{S}^{-i}\mathcal{P}_n\right) < N_1 + \delta$,

(2) $\bar{d}_{N_1}\left(\vee_{i=0}^{N_1-1} \widehat{S}^{-i}\mathcal{P}_n, \vee_{i=0}^{N_1-1} \widehat{S}^{-i}\mathcal{P}_n \restriction A\right) < \varepsilon$, for a set $\mathcal{G}_{N_2}$ of $A \in \vee_{i=-N_2}^{-1} \widehat{S}^{-i}\mathcal{P}_n$, such that $(\mu \times \lambda)(\cup\{A\colon A \in \mathcal{G}_{N_2}\}) > 1 - \varepsilon$.

We claim that the sets $U(n, N_1, N_2, \varepsilon, \delta)$ are open (easy to check) and that the G_δ set

$$\mathcal{S}_1 = \bigcap_{n,k,l} \bigcup_{N_1,N_2} U(n, N_1, N_2, 1/k, 1/l)$$

comprises exactly the elements $S \in \mathcal{S}_0$ for which the corresponding $\widehat{S}$ is Bernoulli. Thus, if $S \in \mathcal{S}_0$ is such that $\widehat{S}$ is Bernoulli, then for every n, ε, δ, there are N_1, N_2 such that $S \in U(n, N_1, N_2, \varepsilon, \delta)$, and conversely, every for Bernoulli $\widehat{S}$ the corresponding S is in $\mathcal{S}_1$.

Note that (1) holds for any N_1 and δ if only N_2 is sufficiently large, because for $S \in \mathcal{S}_0$

$$H\Big(\bigvee_{i=0}^{N_1-1} \widehat{S}^{-i}\mathcal{P}_n \,\Big|\, \bigvee_{i=-\infty}^{-1} \widehat{S}^{-i}\mathcal{P}_n\Big) = N_1.$$

Finally the collection $\mathcal{S}_1$ is nonempty; e.g., by a deep result of Rudolph [33, 34], every weakly mixing group extension of T is in $\mathcal{S}_1$. In fact an explicit example of such an extension of the 2-shift is given by Adler and Shields, [1].

Again we now apply the relative Halmos theorem [19, Proposition 2.3] to deduce that $\mathcal{S}_1$ is a dense G_δ subset of $\mathcal{S}$, as claimed. □

6. A generic extension of a K system is K

Theorem 6.1. *For any fixed* K *transformation T, the generic extension is* K.

Proof. The process $(T, \mathcal{P})$ is a K-process if it has a trivial tail, i.e.

$$\bigcap_{k=1}^{\infty} \bigvee_{i=-\infty}^{-k} T^{-i}\mathcal{P}$$

is the trivial σ-field. Thus this can be expressed by the property that for all N and $\varepsilon > 0$ there is a k_0 such that $\vee_{i=0}^{N-1} T^{-i}\mathcal{P}$ is ε-independent of $\vee_{i=-\infty}^{-k} T^{-i}\mathcal{P}$ for all $k \geq k_0$. As we have seen one can express the ε-independence in terms of entropy so that the K-property can also be expressed by saying that for all N and $\varepsilon > 0$, there is a k_0 such that for all $k \geq k_0$

$$H(\vee_{i=0}^{N-1} T^{-i}\mathcal{P} \,|\, \vee_{i=-k}^{-k_0} T^{-i}\mathcal{P}) > H(\vee_{i=0}^{N-1} T^{-i}\mathcal{P}) - \varepsilon. \tag{5}$$

To rid ourselves of the need for (5) for all $k \geq k_0$, notice that for all n if k_1 is sufficiently large then

$$H(\vee_{i=0}^{n-1} T^{-i}\mathcal{P} \,|\, \vee_{i=-k_1}^{-1} T^{-i}\mathcal{P}) < nh + \delta = n + \delta, \tag{6}$$

where $h = 1$ is the entropy of the process $(T, \mathcal{P})$. Now given ε if we take k_0 as above and for $n = N + k_0$ and δ sufficiently small find k_1 so that (6) holds then (5) for $k = k_0 + k_1$ will imply (5) with 2ε , instead of ε for all $k \geq k_0 + k_1$. Remark too that if a system is K for each member of a refining sequence of partitions, then it is K, as the inverse limit of K automorphisms is K.

So, keeping the notations of Section 5, we define $U(n, N, k_0, k_1, \varepsilon, \delta)$ to consist of those $S \in \mathcal{S}_0$ that satisfy

(1) $H(\vee_{-k_0}^{N-1} \hat{S}^{-i}\mathcal{P}_n \,|\, \vee_{i=-k_1}^{-1} \hat{S}^{-i}\mathcal{P}_n) < N + k_0 + \delta,$

(2) $H\left(\vee_{i=0}^{N-1} \widehat{S}^{-i}\mathcal{P}_n, \,|\, \vee_{i=-k_1}^{-k_0} \widehat{S}^{-i}\mathcal{P}_n\right) > H\left(\vee_{i=0}^{N-1} \widehat{S}^{-i}\mathcal{P}_n\right) - \varepsilon.$

It is now easy to check that the set $U(n, N, k_0, k_1, \varepsilon, \delta)$ is open and that the G_δ set

$$\mathcal{S}_1 = \bigcap_{n,l,m,N} \bigcup_{k_0,k_1} U(n, N, k_0, k_1, 1/l, 1/m)$$

consists of those $S \in \mathcal{S}_0$ for which $\widehat{S}$ is K.

To see that $\mathcal{S}_1$ is not empty when the base T is an arbitrary K-automorphism, we use a result of Parry [31, Theorem 6] who showed that if a circle extension of a K-automorphism is weakly mixing, then it is K. By a theorem of Jones and Parry [21,

Theorem 8], a generic circle extension of a weakly mixing system is weakly mixing and we conclude that the generic circle extension of the K-automorphism T is K.

Finally, again applying the relative Halmos theorem [19, Proposition 2.3], we deduce that $\mathcal{S}_1$ is a dense G_δ subset of $\mathcal{S}$, as claimed. □

In the context of factors of Bernoulli shifts the following question is open. Can there be a full entropy factor (Y, S) of a Bernoulli shift (X, T) such that the relative product $X \times_Y X$ is ergodic but not a K-automorphism—i.e. can this relative product have a nontrivial zero entropy factor? We will next show that for any finite entropy K-automorphism (X, T) the generic extension $(\widehat{X}, \widehat{T})$ is such that the relative product $\widehat{X} \times_X \widehat{X}$ is also K. We use (a bit modified) notation as in Section 4.

Theorem 6.2. *Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be a finite entropy K-automorphism, and S a Rokhlin cocycle with values in* $\mathrm{MPT}(I, \lambda)$*, where $I = [0, 1]$ and λ is a Lebesgue measure on I. We denote by $\widehat{S}$ the transformation on the relative independent product $X \times I \times I$ defined by*

$$\widehat{S}(x, u, v) = (Tx, S_x u, S_x v), \quad (x, u, v) \in X \times I \times I.$$

Then for a generic $S \in \mathcal{S}$ the transformation $\widehat{S}$ is a K*-automorphism.*

Proof. As usual the proof divides into two parts, showing that our property is a G_δ subset of $\mathcal{S}$ and then showing that it is nonempty.

Now $\mathcal{Q}_n$ will denote the product dyadic partition of $I \times I$ into squares of size $\frac{1}{2^n} \times \frac{1}{2^n}$, and if $\mathcal{R}$ is a generating partition of (X, T), we denote it by $\mathcal{P}_n = \mathcal{R} \times \mathcal{Q}_n$. It follows from Section 4 that for a dense G_δ subset $\mathcal{S}_0 \subset \mathcal{S}$, the corresponding $\widehat{S}$ for $S \in \mathcal{S}_0$ has the same entropy as T. With this change of notation the proof of Theorem 6.1 shows that the set of $S \in \mathcal{S}_0$ such that $\widehat{S}$ is a K-automorphism is a G_δ set. It remains to show that it is not empty.

Fix a mixing zero entropy system $\mathbf{Z} = (Z, \mathcal{Z}, \nu, R)$ and an independent partition $\{C_0, C_1, C_{-1}\}$ of X such that $\mu(C_1) = \mu(C_{-1}) < 1/4$, define $f(x)$ by setting $f(x) = i$ for $x \in C_i,\ i = 0, 1, -1$, so that $\int f \, d\mu = 0$, and define $T_f \colon X \times Z \times Z \to X \times Z \times Z$ by

$$T_f(x, z_1, z_2) = (Tx, R^{f(x)} z_1, R^{f(x)} z_2).$$

An easy version of the proof by Meilijson [25] shows that T_f is ergodic. Now, according to Rudolph [35, Corollary 8], when T is a K-automorphism and R (hence also $R \times R$) is mixing, then if T_f is ergodic, it must be K.

Another way to see this is as follows. Use Austin's result [5] to write the K-system $\mathbf{X}$ as a product $X = \mathbf{Y} \times \mathbf{B}$ with $\mathbf{Y}$ a K-system and $\mathbf{B}$ Bernoulli. Let $\mathbf{Z}$ be a mixing system and apply Meilijson's construction [25] to obtain a skew product extension on $B \times Z$ which is K. Now the system $\widehat{\mathbf{X}}$ on $\widehat{X} = X \times Z = Y \times B \times Z$ is K and the independent relative product

$$\widehat{X} \underset{X}{\times} \widehat{X} = (Y \times B \times Z) \underset{Y \times B}{\times} (Y \times B \times Z) \cong Y \times ((B \times Z) \underset{B}{\times} (B \times Z))$$

is also a K system.

Again an application of the relative Halmos theorem [19, Proposition 2.3] finishes the proof. □

Let us recall the definition of relative mixing.

Definition 6.3. A factor map $\pi \colon \mathbf{X} \to \mathbf{Y}$ is called *relatively mixing* if for $f, g \in L_\infty(X)$

$$\lim_{n \to \infty} \|\mathbb{E}(T^n f \cdot g \mid Y) - \mathbb{E}(T^n f \mid Y)\mathbb{E}(g \mid Y)\|_2 = 0. \tag{7}$$

When g is $\mathcal{Y}$-measurable then

$$\mathbb{E}(T^n f \cdot g \mid Y) - \mathbb{E}(T^n f \mid Y)\mathbb{E}(g \mid Y) = g\mathbb{E}(T^n f \mid Y) - g\mathbb{E}(T^n f \mid Y) = 0,$$

so we can replace in equation (7), g by $g - \mathbb{E}(g|Y)$, whose conditional expectation is zero. Thus an equivalent condition is: for $f, g \in L_\infty(X)$ with $\mathbb{E}(g \mid Y) = 0$,

$$\lim_{n\to\infty} \|\mathbb{E}(T^n f \cdot g \mid Y)\|_2 = 0. \tag{8}$$

It is well known that the set of mixing transformations is meager (see [20]). In the paper [39] by Mike Schnurr, the following relative version is proved (theorem 6): the set of transformations T acting on the product $X_1 \times X_2$ and leaving the σ-algebra X_2 invariant, in such a way that T is relatively mixing with respect to X_2, is meager.

However, as a consequence of Theorem 6.2, we will show that, when T acting on X is K, then the generic extension of T will be relatively mixing over X.

Let us mention that this result sheds some light on the following old question, originally due to D. Ornstein: given a Bernoulli shift T, does there exist a factor of T relative to which T is weakly mixing but not strongly mixing? Now, in view of the following theorem, there is no hope of using Baire category arguments to resolve this question, as was done historically in the "absolute" case.

Theorem 6.4. *Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be a* K*-automorphism, then the generic extension of $\mathbf{X}$ is relatively mixing over $\mathbf{X}$.*

Proof. This result is a consequence of Theorem 6.2 and the following lemma

Lemma 6.5. *Let $\mathbf{X}$ be ergodic and $\mathbf{Y}$ be a factor of $\mathbf{X}$ with factor map $\pi\colon \mathbf{X} \to \mathbf{Y}$. Then the following are equivalent*:

(1) *$\mathbf{X}$ is a relatively mixing extension of $\mathbf{Y}$,*
(2) *in the relatively independent product $X \times_Y X$, the Koopman operator restricted to $L_2(Y)^\perp$ is mixing.*

Proof. Let $X = Y \times Z$ be the Rohlin representation of $\mathbf{X}$ over $\mathbf{Y}$ and let $\mu = \int_Y \mu_y \, d\nu(y)$ be the disintegration of μ over ν. Let

$$W := X \underset{Y}{\times} X = \{(x_1, x_2) \in X \times X \colon \pi(x_1) = \pi(x_2)\},$$

and let the relative product measure λ, supported on W, be given by

$$\lambda = \int_Y \mu_y \times \mu_y \, d\nu(y).$$

Thus for $F \in L_2(\lambda)$

$$\int F(x_1, x_2)\, d\lambda(x_1, x_2) = \int_Y \left(\int_{X\times X} F(x_1, x_2)\, d\mu_y(x_1)\, d\mu_y(x_2) \right) d\nu(y).$$

Note that a function $F(x_1, x_2) \in L_2(\lambda)$ is in $L_2(Y)^\perp$ iff

$$\mathbb{E}(F \mid Y)(y) = \int_X F(x_1, x_2)\, d\mu_y(x_1) d\mu_y(x_2)$$

is ν-a.e. 0.

(1) $\Rightarrow$ (2). Given $f_1, f_2, g_1, g_2 \in L_\infty(X)$, let

$$f_1 \otimes f_2(x_1, x_2) = f_1(x_1) f_2(x_2), \quad g_1 \otimes g_2(x_1, x_2) = g_1(x_1) g_2(x_2),$$

and note that, e.g., $\mathbb{E}(f_1 \otimes f_2 \mid Y) = \mathbb{E}(f_1 \mid Y)\mathbb{E}(f_2 \mid Y)$. Let

$$F = f_1 \otimes f_2 - \mathbb{E}(f_1 \mid Y)\mathbb{E}(f_2 \mid Y),$$
$$G = g_1 \otimes g_2 - \mathbb{E}(g_1 \mid Y)\mathbb{E}(g_2 \mid Y).$$

Since linear combinations of functions of the form $f_1 \otimes f_2$ are dense in $L_2(\lambda)$, mixing in $L_2(Y)^\perp$ will follow from

$$\Big| \int_W T^n F \cdot G \, d\lambda \Big| \to 0 \tag{9}$$

(where we write $T^n F$ for the diagonal action of T on $X \times X$). Expanding the left-hand side of the last formula we get

$$\int_Y \mathbb{E}\left([T^n(f_1 \otimes f_2) - T^n(\mathbb{E}(f_1 \mid Y)\mathbb{E}(f_2 \mid Y))] \, [g_1 \otimes g_2 - \mathbb{E}(g_1 \mid Y)\mathbb{E}(g_2 \mid Y)] \mid Y \right) d\nu.$$

There are four terms in the product $T^n F \cdot G$. The first one is

$$\int_W T^n(f_1 \otimes f_2) \cdot g_1 \otimes g_2 \, d\lambda = \int_Y \Big(\int_X T^n f_1(x_1) g_1(x_1) \, d\mu_y(x_1) \Big) \Big(\int_X T^n f_2(x_2) g_2(x_2) \, d\mu_y(x_2) \Big) d\nu(y).$$

By (1), i. e. by relative mixing (7) (and boundedness of the functions), for large n, we can replace, with only a small error, the expression $\int_X T^n f_1(x_1) g_1(x_1) \, d\mu_y(x_1)$ by $\mathbb{E}(T^n f_1 \mid Y)\mathbb{E}(g_1 \mid Y)(y)$.

Similarly the other three terms can be replaced by the same expression, one with a plus sign and two with a minus sign, and thus the total is indeed close to zero, as claimed.

(2) $\Rightarrow$ (1) Assuming $\mathbb{E}(g \mid Y) = 0$, we need to show that equation (8) holds. Now under this assumption, with $f = f_1 = f_2$ and $g = g_1 = g_2$ real-valued functions, equation (9) reads

$$\int_W T^n f(x_1) T^n f(x_2) \cdot g(x_1) g(x_2) \, d\lambda = \int_Y \mathbb{E}(T^n f \cdot g \mid Y) \cdot \mathbb{E}(T^n f \cdot g \mid Y) \, d\nu(y) = \|\mathbb{E}(T^n f \cdot g \mid Y)\|_2^2 \to 0. \quad \square$$

Applying the lemma to the relative independent product in Theorem 6.2, and recalling the fact that every K-automorphism is mixing, these arguments complete the proof of Theorem 6.4. $\square$

7. A generic extension of a LB system is LB

Theorem 7.1. *For any fixed loosely Bernoulli transformation T, the generic extension is loosely Bernoulli.*

Proof. We recall first one of the definitions of Loosely Bernoulli (LB) transformations. There are two kinds, one with zero entropy and the other having positive entropy.

For the latter, $(X, \mathcal{X}, \mu, T)$ is called *loosely Bernoulli* if there is some set $A \in \mathcal{X}$ such that the induced transformation T_A is isomorphic to a Bernoulli shift. The zero entropy loosely Bernoulli are defined in a similar fashion except that now T_A is required to be isomorphic to an irrational rotation. For the basic theory and facts that we will use see [29].

There is a characterization of process that define LB transformations similar to the VWB condition, but with the $\bar{d}_n$ metric replaced by the $\bar{f}_n$ metric which we proceed to define.

For two words $u, v \in \{1, 2, \ldots, a\}^n$ we define

$$\bar{f}_n(u, v) = 1 - \frac{k}{n},$$

where k is the maximal integer for which we can find subsequences $0 \le i_1 < i_2 < \dots$ $\dots < i_k \le n-1$ and $0 \le j_1 < j_2 < \dots < j_k \le n-1$, with

$$u(i_r) = v(j_r), \quad 1 \le r \le k.$$

This defines a metric on words, and using it instead of the normalized Hamming metric we also define the $\bar{f}_n$ metric on probability distributions on $\{1, 2, \dots, a\}^n$.

Definition 7.2. A finite measurable partition $\mathcal{P}$ of X is called *very loosely Bernoulli* (VLB for short) if for every $\varepsilon > 0$, there is a positive integer N such that for all $k \ge 1$, there is a collection $\mathcal{G}_k$ of atoms of the partition $\vee_{i=-k}^{-1} T^{-i}\mathcal{P}$ such that

(1) $\mu\big(\cup\{A \in \mathcal{G}_k\}\big) > 1 - \varepsilon$,

(2) $\bar{f}_N\Big(\vee_{i=0}^{N-1} T^{-i}\mathcal{P}, \vee_{i=0}^{N-1} T^{-i}\mathcal{P} \restriction A\Big) < \varepsilon$, for every atom A of $\mathcal{G}_k$.

Here $\bar{f}_N$ is the normalized $\bar{f}_N$ distance of distributions, and $\vee_{i=0}^{N-1} T^{-i}\mathcal{P} \restriction A$ is the conditional distribution of $\vee_{i=0}^{N-1} T^{-i}\mathcal{P}$ restricted to the atom A of the partition $\vee_{i=-k}^{-1} T^{-i}\mathcal{P}$.

For zero entropy LB there is a simpler formulation as follows.

Definition 7.3. Let $\mathcal{P} = \{P_1, \dots, P_a\}$ be a measurable partition of X. Given $\alpha \in \{1, 2, \dots, a\}^n$, we set $[\alpha] = \cap_{i=0}^{N-1} T^{-i} P_{\alpha_i}$. A zero entropy process, defined by $\mathcal{P} = \{P_1, \dots, P_a\}$, is *very loosely Bernoulli* if given $\varepsilon > 0$ there is an N and a subset $G_N \subset \{1, 2, \dots, a\}^N$ such that

(1) $\sum\{\mu([\alpha])\colon \alpha \in G_N\} > 1 - \varepsilon$,

(2) for all $\alpha, \alpha' \in G_N$, $\bar{f}_N(\alpha, \alpha') < \varepsilon$.

Once again it suffices to show that there is a refining sequence of partitions

$$\mathcal{P}_1 \prec \dots \prec \mathcal{P}_n \prec \mathcal{P}_{n+1} \prec \dots$$

such that the corresponding algebras $\widehat{\mathcal{P}}_n$ satisfy $\vee_{n\in\mathbb{N}} \widehat{\mathcal{P}}_n = \mathcal{X}$, and such that for each n, the process $(T, \mathcal{P}_n)$ is VLB.

In zero entropy, to see that a generic extension of an LB T is LB, we keep notations from previous sections and define, for each n, N, ε,

$$U(n, N, \varepsilon) = \{S \in \mathcal{S}_0\colon \text{the process } (\widehat{S}, \mathcal{P}_n) \text{ satisfies (1) and (2)}\}.$$

These are open sets and the set

$$\bigcap_{n,k} \bigcup_{N=1}^{\infty} U(n, N, 1/k)$$

is a G_δ set and consists of LB transformations. To see that it is not empty we can use the fact that for any LB transformation T with zero entropy, a compact abelian group extension is also LB [29, Theorem 7.3].

For the positive entropy case we repeat exactly the same argument as in Section 5 with $\bar{f}_N$ replacing $\bar{d}_N$. The fact that there are relative zero entropy extensions of any LB transformation of positive entropy again follows from the fact that, in positive entropy, any ergodic isometric extension of an LB transformation is LB [33, Corollary 8]. □

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