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Relations between values of arithmetic Gevrey series, and applications to values of the Gamma function

S. Fischler and T. Rivoal

January 3, 2023

Abstract

We investigate the relations between the rings \mathbf{E} , \mathbf{G} and \mathbf{D} of values taken at algebraic points by arithmetic Gevrey series of order either -1 (E -functions), 0 (analytic continuations of G -functions) or 1 (renormalization of divergent series solutions at ∞ of E -operators) respectively. We prove in particular that any element of \mathbf{G} can be written as multivariate polynomial with algebraic coefficients in elements of \mathbf{E} and \mathbf{D} , and is the limit at infinity of some E -function along some direction. This prompts to defining and studying the notion of mixed functions, which generalizes simultaneously E -functions and arithmetic Gevrey series of order 1 . Using natural conjectures for arithmetic Gevrey series of order 1 and mixed functions (which are analogues of a theorem of André and Beukers for E -functions) and the conjecture $\mathbf{D} \cap \mathbf{E} = \overline{\mathbb{Q}}$ (but not necessarily all these conjectures at the same time), we deduce a number of interesting Diophantine results such as an analogue for mixed functions of Beukers' linear independence theorem for values of E -functions, the transcendence of the values of the Gamma function and its derivatives at all non-integral algebraic numbers, the transcendence of Gompertz constant as well as the fact that Euler's constant is not in \mathbf{E} .

1 Introduction

A power series $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \in \overline{\mathbb{Q}}[[x]]$ is said to be an E -function when it is solution of a linear differential equation over $\overline{\mathbb{Q}}(x)$ (holonomic), and $|\sigma(a_n)|$ (for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) and the least common denominator of a_0, a_1, \dots, a_n grow at most exponentially in n . They were defined and studied by Siegel in 1929, who also defined the class of G -functions: a power series $\sum_{n=0}^{\infty} a_n x^n \in \overline{\mathbb{Q}}[[x]]$ is said to be a G -function when $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ is an E -function. In this case, $\sum_{n=0}^{\infty} n! a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is called an \mathfrak{A} -function, following the terminology introduced by André in [1]. E -functions are entire, while G -functions have a positive radius of convergence, which is finite except for polynomials. Here and below, we see $\overline{\mathbb{Q}}$ as embedded into \mathbb{C} . Following André again, E -functions, G -functions and \mathfrak{A} -functions are exactly arithmetic Gevrey series of order $s = -1, 0, 1$ respectively. Actually André defines arithmetic

Gevrey series of any order $s \in \mathbb{Q}$, but the set of values at algebraic points is the same for a given $s \neq 0$ as for $s/|s|$ using [1, Corollaire 1.3.2].

\mathfrak{E} -functions are divergent series, unless they are polynomials. Given an \mathfrak{E} -function f and any $\theta \in \mathbb{R}$, except finitely many values mod 2π (namely anti-Stokes directions of f), one can perform Ramis' 1-summation of $f(1/z)$ in the direction θ , which coincides in this setting with Borel-Laplace summation (see [14] or [9]). This provides a function denoted by $f_\theta(1/z)$, holomorphic on the open subset of \mathbb{C} consisting in all $z \neq 0$ such that $\theta - \frac{\pi}{2} - \varepsilon < \arg z < \theta + \frac{\pi}{2} + \varepsilon$ for some $\varepsilon > 0$, of which $f(1/z)$ is the asymptotic expansion in this sector (called a large sector bisected by θ). Of course $f(1/z)$ can be extended further by analytic continuation, but this asymptotic expansion may no longer be valid. When an \mathfrak{E} -function is denoted by f_j , we shall denote by $f_{j,\theta}$ or $\mathfrak{f}_{j;\theta}$ its 1-summation and we always assume (implicitly or explicitly) that θ is not an anti-Stokes direction.

In [8], [9] and [10, §4.3], we have studied the sets \mathbf{G} , \mathbf{E} and \mathbf{D} defined respectively as the sets of all the values taken by all (analytic continuations of) G -functions at algebraic points, of all the values taken by all E -functions at algebraic points and of all values $f_\theta(1)$ where f is an \mathfrak{E} -function ($\theta = 0$ if it is not an anti-Stokes direction, and $\theta > 0$ is very small otherwise.) These three sets are countable sub-rings of \mathbb{C} that all contain $\overline{\mathbb{Q}}$; conjecturally, they are related to the set of periods and exponential periods, see §3. (The ring \mathbf{D} is denoted by \mathfrak{D} in [10].)

We shall prove the following result in §3.

Theorem 1. *Every element of \mathbf{G} can be written as a multivariate polynomial (with coefficients in $\overline{\mathbb{Q}}$) in elements of \mathbf{E} and \mathbf{D} .*

Moreover, \mathbf{G} coincides with the set of all convergent integrals $\int_0^\infty F(x)dx$ where F is an E -function, or equivalently with the set of all finite limits of E -functions at ∞ along some direction.

Above, a convergent integral $\int_0^\infty F(x)dx$ means a finite limit of the E -function $\int_0^z F(x)dx$ as $z \rightarrow \infty$ along some direction; this explains the equivalence of both statements.

We refer to Eq. (3.2) in §3 for an expression of $\log(2)$ as a polynomial in elements in \mathbf{E} and \mathbf{D} ; the number π could be similarly expressed by considering z and iz instead of z and $2z$ there. Examples of the last statement are the identities (see [12] for the second one):

$$\int_0^{+\infty} \frac{\sin(x)}{x} dx = \frac{\pi}{2} \quad \text{and} \quad \int_0^{+\infty} J_0(ix)e^{-3x} dx = \frac{\sqrt{6}}{96\pi^3} \Gamma\left(\frac{1}{24}\right) \Gamma\left(\frac{5}{24}\right) \Gamma\left(\frac{7}{24}\right) \Gamma\left(\frac{11}{24}\right).$$

It is notoriously difficult to prove/disprove that a given element of \mathbf{G} is transcendental; it is known that a Siegel-Shidlovskii type theorem for G -functions can not hold *mutatis mutandis*. Theorem 1 suggests that an alternative approach to the study of the Diophantine properties of elements of \mathbf{G} can be through a better understanding of joint study of the elements of \mathbf{E} and \mathbf{D} , modulo certain conjectures to begin with. Our applications will not be immediately directed to the elements of \mathbf{G} but rather to the understanding of the (absence of) relations between the elements of \mathbf{E} and \mathbf{D} .

It seems natural (see [9, p. 37]) to conjecture that $\mathbf{E} \cap \mathbf{G} = \overline{\mathbb{Q}}$, and also that $\mathbf{G} \cap \mathbf{D} = \overline{\mathbb{Q}}$, though both properties seem currently out of reach. In this paper, we suggest (see §2) a possible approach towards the following analogous conjecture.

Conjecture 1. *We have $\mathbf{E} \cap \mathbf{D} = \overline{\mathbb{Q}}$.*

In §2 we shall make a functional conjecture, namely Conjecture 3, that implies Conjecture 1. We also prove that Conjecture 1 has very important consequences, as the following result shows.

Theorem 2. *Assume that Conjecture 1 holds. Then $\Gamma^{(s)}(a)$ is a transcendental number for any rational number $a > 0$ and any integer $s \geq 0$, except of course if $s = 0$ and $a \in \mathbb{N}$.*

One of the aims of this paper is to show that combining \mathfrak{D} - and E -functions may lead to very important results in transcendental number theory. Let us recall now briefly the main known results on E -functions.

Point (i) in the following result is due to André [2] for E -functions with rational Taylor coefficients, and to Beukers [6] in the general case. André used this property to obtain a new proof of the Siegel-Shidlovskii Theorem, and Beukers to prove an optimal refinement of this theorem (namely, (ii) below).

Theorem A. (i) [André, Beukers] *If an E -function $F(z)$ is such that $F(1) = 0$, then $\frac{F(z)}{z-1}$ is an E -function.*

(ii) [Beukers] *Let $\underline{F}(z) := {}^t(f_1(z), \dots, f_n(z))$ be a vector of E -functions solution of a differential system $\underline{F}'(z) = A(z)\underline{F}(z)$ for some matrix $A(z) \in M_n(\overline{\mathbb{Q}}(z))$.*

Let $\xi \in \overline{\mathbb{Q}}^$ which is not a pole of a coefficient of A . Let $P \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ be a homogeneous polynomial such that*

$$P(f_1(\xi), \dots, f_n(\xi)) = 0.$$

Then there exists $Q \in \overline{\mathbb{Q}}[Z, X_1, \dots, X_n]$, homogeneous in the X_i , such that

$$Q(z, f_1(z), \dots, f_n(z)) = 0 \text{ identically and } P(X_1, \dots, X_n) = Q(\xi, X_1, \dots, X_n).$$

In particular, we have

$$\text{trdeg}_{\overline{\mathbb{Q}}}(f_1(\xi), \dots, f_n(\xi)) = \text{trdeg}_{\overline{\mathbb{Q}}(z)}(f_1(z), \dots, f_n(z)).$$

The Siegel-Shidlovskii Theorem itself is the final statement about equality of transcendence degrees.

In this paper we state conjectural analogues of these results, involving \mathfrak{D} -functions. The principal difficulty is that these functions are divergent power series, and the exact analogue of Theorem A is meaningless. André discussed the situation in [2] and even though he did not formulate exactly the following conjecture, it seems plausible to us. From it, we will show how to deduce an analogue of the Siegel-Shidlovskii theorem for \mathfrak{D} -functions. Ferguson [7, p. 171, Conjecture 1] essentially stated this conjecture when $\mathfrak{f}(z)$ has *rational* coefficients and when $\theta = 0$.

Conjecture 2. Let $f(z)$ be an \mathfrak{A} -function and $\theta \in (-\pi/2, \pi/2)$ be such that $f_\theta(1) = 0$. Then $\frac{f(z)}{z-1}$ is an \mathfrak{A} -function.

In other words, the conclusion of this conjecture asserts that $\frac{z}{1-z}f(1/z)$ is an \mathfrak{A} -function in $1/z$; this is equivalent to $\frac{f(1/z)}{z-1}$ being an \mathfrak{A} -function in $1/z$ (since we have $\frac{f(1/z)}{z-1} = O(1/z)$ unconditionally as $|z| \rightarrow \infty$).

Following Beukers' proof [6] yields the following result (see [3, §4.6] for a related conjecture).

Theorem 3. Assume that Conjecture 2 holds.

Let $\underline{f}(z) := {}^t(f_1(z), \dots, f_n(z))$ be a vector of \mathfrak{A} -functions solution of a differential system $\underline{f}'(z) = A(z)\underline{f}(z)$ for some matrix $A(z) \in M_n(\overline{\mathbb{Q}}(z))$. Let $\xi \in \overline{\mathbb{Q}}^*$ and $\theta \in (\arg(\xi) - \pi/2, \arg(\xi) + \pi/2)$; assume that ξ is not a pole of a coefficient of A , and that θ is anti-Stokes for none of the f_j .

Let $P \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ be a homogeneous polynomial such that

$$P(f_{1,\theta}(1/\xi), \dots, f_{n,\theta}(1/\xi)) = 0.$$

Then there exists $Q \in \overline{\mathbb{Q}}[Z, X_1, \dots, X_n]$, homogeneous in the X_i , such that

$$Q(z, f_1(z), \dots, f_n(z)) = 0 \text{ identically and } P(X_1, \dots, X_n) = Q(1/\xi, X_1, \dots, X_n).$$

In particular, we have

$$\text{trdeg}_{\overline{\mathbb{Q}}}(\underline{f}_{1,\theta}(1/\xi), \dots, \underline{f}_{n,\theta}(1/\xi)) = \text{trdeg}_{\overline{\mathbb{Q}}(z)}(\underline{f}_1(z), \dots, \underline{f}_n(z)).$$

As an application of Theorem 3, we shall prove the following corollary. Note that under his weaker version of Conjecture 2, Ferguson [7, p. 171, Theorem 2] proved that Gompertz's constant is an irrational number.

Corollary 1. Assume that Conjecture 2 holds. Then for any $\alpha \in \overline{\mathbb{Q}}$, $\alpha > 0$, and any $s \in \mathbb{Q} \setminus \mathbb{Z}_{\geq 0}$, the number $\int_0^\infty (t + \alpha)^s e^{-t} dt$ is a transcendental number.

In particular, Gompertz's constant $\delta := \int_0^\infty e^{-t}/(t+1)dt$ is a transcendental number.

In this text we suggest an approach towards Conjecture 1, based on the new notion of *mixed functions* which enables one to consider E - and \mathfrak{A} -functions at the same time. In particular we shall state a conjecture about such functions, namely Conjecture 3 in §2, which implies both Conjecture 1 and Conjecture 2. The following result is a motivation for this approach.

Proposition 1. Assume that both Conjectures 1 and 2 hold. Then neither Euler's constant $\gamma := -\Gamma'(1)$ nor $\Gamma(a)$ (with $a \in \mathbb{Q}^+ \setminus \mathbb{N}$) are in \mathbf{E} .

It is likely that none of these numbers is in \mathbf{G} , but (as far as we know) there is no "functional" conjecture like Conjecture 3 that implies this. It is also likely that none is in \mathbf{D} as well, but we don't know if this can be deduced from Conjecture 3.

The structure of this paper is as follows. In §2 we define and study mixed functions, a combination of E - and \mathfrak{E} -functions. Then in §3 we express any value of a G -function as a polynomial in values of E - and \mathfrak{E} -functions, thereby proving Theorem 1. We study derivatives of the Γ function at rational points in §4, and prove Theorem 2 and Proposition 1. At last, §5 is devoted to adapting Beukers' method to our setting: this approach yields Theorem 3 and Corollary 1.

2 Mixed functions

2.1 Definition and properties

In view of Theorem 1, it is natural to study polynomials in E - and \mathfrak{E} -functions. We can prove a Diophantine result that combines both Theorems A(ii) and 3 but under a very complicated polynomial generalization of Conjecture 2. We opt here for a different approach to mixing E - and \mathfrak{E} -functions for which very interesting Diophantine consequences can be deduced from a very easy to state conjecture which is more in the spirit of Conjecture 2. We refer to §2.3 for proofs of all properties stated in this section (including Lemma 1 and Proposition 2), except Theorem 4.

Definition 1. We call mixed (arithmetic Gevrey) function any formal power series

$$\sum_{n \in \mathbb{Z}} a_n z^n$$

such that $\sum_{n \geq 0} a_n z^n$ is an E -function in z , and $\sum_{n \geq 1} a_{-n} z^{-n}$ is an \mathfrak{E} -function in $1/z$.

In other words, a mixed function is defined as a formal sum $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ where F is an E -function and \mathfrak{f} is an \mathfrak{E} -function. In particular, such a function is zero if, and only if, both F and \mathfrak{f} are constants such that $F + \mathfrak{f} = 0$; obviously, F and \mathfrak{f} are uniquely determined by Ψ upon assuming (for instance) that $\mathfrak{f}(0) = 0$. The set of mixed functions is a $\overline{\mathbb{Q}}$ -vector space stable under multiplication by z^n for any $n \in \mathbb{Z}$. Unless $\mathfrak{f}(z)$ is a polynomial, such a function $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ is purely formal: there is no $z \in \mathbb{C}$ such that $\mathfrak{f}(1/z)$ is a convergent series. However, choosing a direction θ which is not anti-Stokes for \mathfrak{f} allows one to evaluate $\Psi_\theta(z) = F(z) + \mathfrak{f}_\theta(1/z)$ at any z in a large sector bisected by θ . Here and below, such a direction will be said *not anti-Stokes for Ψ* and whenever we write \mathfrak{f}_θ or Ψ_θ we shall assume implicitly that θ is not anti-Stokes.

Definition 1 is a formal definition, but one may identify a mixed function with the holomorphic function it defines on a given large sector by means of the following lemma.

Lemma 1. *Let Ψ be a mixed function, and $\theta \in \mathbb{R}$ be a non-anti-Stokes direction for Ψ . Then Ψ_θ is identically zero (as a holomorphic function on a large sector bisected by θ) if, and only if, Ψ is equal to zero (as a formal power series in z and $1/z$).*

Any mixed function $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ is solution of an E -operator. Indeed, this follows from applying [1, Theorem 6.1] twice: there exist an E -operator L such that $L(\mathfrak{f}(1/z)) = 0$, and an E -operator M such that $M(L(F(z))) = 0$ (because $L(F(z))$ is an E -function). Hence $ML(F(z) + \mathfrak{f}(1/z)) = 0$ and by [1, p. 720, §4.1], ML is an E -operator.

We formulate the following conjecture, which implies both Conjecture 1 and Conjecture 2.

Conjecture 3. *Let $\Psi(z)$ be an mixed function, and $\theta \in (-\pi/2, \pi/2)$ be such that $\Psi_\theta(1) = 0$. Then $\frac{\Psi(z)}{z-1}$ is an mixed function.*

The conclusion of this conjecture is that $\Psi(z) = (z-1)\Psi_1(z)$ for some mixed function Ψ_1 . This conclusion can be made more precise as follows; see §2.3 for the proof.

Proposition 2. *Let $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ be an mixed function, and $\theta \in (-\pi/2, \pi/2)$ be such that $\Psi_\theta(1) = 0$. Assume that Conjecture 3 holds for Ψ and θ .*

Then both $F(1)$ and $\mathfrak{f}_\theta(1)$ are algebraic, and $\frac{\mathfrak{f}(1/z) - \mathfrak{f}_\theta(1)}{z-1}$ is an \mathcal{O} -function.

Of course, in the conclusion of this proposition, one may assert also that $\frac{F(z) - F(1)}{z-1}$ is an E -function using Theorem A(i).

Conjecture 3 already has far reaching Diophantine consequences: Conjecture 2 and Theorem 2 stated in the introduction, and also the following result that encompasses Theorem 3 in the linear case.

Theorem 4. *Assume that Conjecture 3 holds.*

Let $\Psi(z) := {}^t(\Psi_1(z), \dots, \Psi_n(z))$ be a vector of mixed functions solution of a differential system $\Psi'(z) = A(z)\Psi(z)$ for some matrix $A(z) \in M_n(\overline{\mathbb{Q}}(z))$. Let $\xi \in \overline{\mathbb{Q}}^$ and $\theta \in (\arg(\xi) - \pi/2, \arg(\xi) + \pi/2)$; assume that ξ is not a pole of a coefficient of A , and that θ is anti-Stokes for none of the Ψ_j .*

Let $\lambda_1, \dots, \lambda_n \in \overline{\mathbb{Q}}$ be such that

$$\sum_{i=1}^n \lambda_i \Psi_{i,\theta}(\xi) = 0.$$

Then there exist $L_1, \dots, L_n \in \overline{\mathbb{Q}}[z]$ such that

$$\sum_{i=1}^n L_i(z) \Psi_i(z) = 0 \text{ identically and } L_i(\xi) = \lambda_i \text{ for any } i.$$

In particular, we have

$$\text{rk}_{\overline{\mathbb{Q}}}(\Psi_{1,\theta}(\xi), \dots, \Psi_{n,\theta}(\xi)) = \text{rk}_{\overline{\mathbb{Q}}(z)}(\Psi_1(z), \dots, \Psi_n(z)).$$

The proof of Theorem 4 follows exactly the linear part of the proof of Theorem 3 (see §5.1), which is based on [6, §3]. The only difference is that \mathcal{O} -functions have to be replaced

with mixed functions, and Conjecture 2 with Conjecture 3. In particular Proposition 4 stated in §5.1 remains valid with these modifications.

However a product of mixed functions is not, in general, a mixed function. Therefore the end of [6, §3] does not adapt to mixed functions, and there is no hope to obtain in this way a result on the transcendence degree of a field generated by values of mixed functions.

As an application of Theorem 4, we can consider the mixed functions $1, e^{\beta z}$ and $\mathfrak{f}(1/z) := \sum_{n=0}^{\infty} (-1)^n n! z^{-n}$, where β is a fixed non-zero algebraic number. These three functions are linearly independent over $\mathbb{C}(z)$ and form a solution of a differential system with only 0 for singularity (because $(\mathfrak{f}(1/z))' = (1 + 1/z)\mathfrak{f}(1/z) - 1$), hence for any $\alpha \in \overline{\mathbb{Q}}, \alpha > 0$ and any $\varrho \in \overline{\mathbb{Q}}^*$, the numbers $1, e^{\varrho}, \mathfrak{f}_0(1/\alpha) := \int_0^{\infty} e^{-t}/(1 + \alpha t) dt$ are $\overline{\mathbb{Q}}$ -linearly independent (for a fixed α , take $\beta = \varrho/\alpha$).

2.2 Values of mixed functions

We denote by \mathbf{M}_G the set of values $\Psi_{\theta}(1)$, where Ψ is a mixed function and $\theta = 0$ if it is not anti-Stokes, $\theta > 0$ is sufficiently small otherwise. This set is obviously equal to $\mathbf{E} + \mathbf{D}$.

Proposition 3. *For every integer $s \geq 0$ and every $a \in \mathbb{Q}^+, a \neq 0$, we have $\Gamma^{(s)}(a) \in e^{-1}\mathbf{M}_G$.*

This result follows immediately from Eq. (4.4) below (see §4.2), written in the form

$$\Gamma^{(s)}(a) = e^{-1}((-1)^s es! E_{a,s+1}(-1) + \mathfrak{f}_{a,s+1;0}(1)),$$

because $e^z E_{a,s+1}(-z)$ is an E -function and $\mathfrak{f}_{a,s+1;0}(1)$ is the 1-summation in the direction 0 of an \mathfrak{A} -function.

It would be interesting to know if $\Gamma^{(s)}(a)$ belongs to \mathbf{M}_G . We did not succeed in proving it does, and we believe it does not. Indeed, for instance if we want to prove that $\gamma \in \mathbf{M}_G$, a natural strategy would be to construct an E -function $F(z)$ with asymptotic expansion of the form $\gamma + \log(z) + \mathfrak{f}(1/z)$ in a large sector, and then to evaluate at $z = 1$. However this strategy cannot work since there is no such E -function (see the footnote in the proof of Lemma 1 in §2.3).

2.3 Proofs concerning mixed functions

To begin with, let us take Proposition 2 for granted and prove that Conjecture 3 implies both Conjecture 1 and Conjecture 2. Concerning Conjecture 2 it is clear. To prove that it implies Conjecture 1, let $\xi \in \mathbf{D}$, i.e. $\xi = \mathfrak{f}_{\theta}(1)$ is the 1-summation of an \mathfrak{A} -function $\mathfrak{f}(z)$ in the direction $\theta = 0$ if it is not anti-Stokes, and $\theta > 0$ close to 0 otherwise. Assume that ξ is also in \mathbf{E} : we have $\xi = F(1)$ for some E -function $F(z)$. Therefore, $\Psi(z) = F(z) - \mathfrak{f}(1/z)$ is a mixed function such that $\Psi_{\theta}(1) = 0$. By Conjecture 3 and Proposition 2, we have $\xi = \mathfrak{f}_{\theta}(1) \in \overline{\mathbb{Q}}$. This concludes the proof that Conjecture 3 implies Conjecture 1.

Let us prove Proposition 2 now. Assuming that Conjecture 3 holds for Ψ and θ , there exists a mixed function $\Psi_1(z) = F_1(z) + \mathfrak{f}_1(1/z)$ such that $\Psi(z) = (z-1)\Psi_1(z)$. We have

$$F(z) - (z-1)F_1(z) + \mathfrak{f}(1/z) - (z-1)\mathfrak{f}_1(1/z) = 0 \quad (2.1)$$

as a formal power series in z and $1/z$. Now notice that $z-1 = z(1 - \frac{1}{z})$, and that we may assume \mathfrak{f} and \mathfrak{f}_1 to have zero constant terms. Denote by α the constant term of $\mathfrak{f}(1/z) - z(1 - \frac{1}{z})\mathfrak{f}_1(1/z)$. Then we have

$$F(z) - (z-1)F_1(z) + \alpha + \mathfrak{f}_2(1/z) = 0$$

for some \mathfrak{E} -function \mathfrak{f}_2 without constant term, so that $\mathfrak{f}_2 = 0$, $F(z) = (z-1)F_1(z) - \alpha$ and $F(1) = -\alpha \in \overline{\mathbb{Q}}$. This implies $\mathfrak{f}_\theta(1) = \alpha$, and $\frac{\mathfrak{f}(1/z) - \mathfrak{f}_\theta(1)}{z-1} = \mathfrak{f}_1(1/z)$ is an \mathfrak{E} -function since $\mathfrak{f}_2 = 0$. This concludes the proof of Proposition 2.

At last, let us prove Lemma 1. We write $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ and assume that Ψ_θ is identically zero. Modifying θ slightly if necessary, we may assume that the asymptotic expansion $-\mathfrak{f}(1/z)$ of $F(z)$ in a large sector bisected by θ is given explicitly by [9, Theorem 5] applied to $F(z) - F(0)$; recall that such an asymptotic expansion is unique (see [9]). As in [9] we let $g(z) = \sum_{n=1}^{\infty} a_n z^{-n-1}$ where the coefficients a_n are given by $F(z) - F(0) = \sum_{n=1}^{\infty} \frac{a_n}{n!} z^n$. For any $\sigma \in \mathbb{C} \setminus \{0\}$ there is no contribution in $e^{\sigma z}$ in the asymptotic expansion of $F(z)$, so that $g(z)$ is holomorphic at σ . At $\sigma = 0$, the local expansion of g is of the form $g(z) = h_1(z) + h_2(z) \log(z)$ with G -functions h_1 and h_2 , and the coefficients of h_2 are related to those of \mathfrak{f} ; however we shall not use this special form ⁽¹⁾. Now recall that $g(z) = G(1/z)/z$ where G is a G -function; then G is entire and has moderate growth at infinity (because ∞ is a regular singularity of G), so it is a polynomial due to Liouville's theorem. This means that $F(z)$ is a polynomial in z . Recall that asymptotic expansions in large sectors are unique. Therefore both F and \mathfrak{f} are constant functions, and $F + \mathfrak{f} = 0$. This concludes the proof of Lemma 1.

3 Proof of Theorem 1: values of G -functions

In this section we prove Theorem 1. Let us begin with an example, starting with the relation proved in [15, Proposition 1] for $z \in \mathbb{C} \setminus (-\infty, 0]$:

$$\gamma + \log(z) = zE_{1,2}(-z) - e^{-z}\mathfrak{f}_{1,2;0}(1/z) \quad (3.1)$$

where $E_{1,2}$ is an E -function, and $\mathfrak{f}_{1,2}$ is an \mathfrak{E} -function, both defined below in §4.2.

Apply Eq. (3.1) at both z and $2z$, and then subtract one equation from the other. This provides a relation of the form

$$\log(2) = F(z) + e^{-z}\mathfrak{f}_{1,0}(1/z) + e^{-2z}\mathfrak{f}_{2,0}(1/z) \quad (3.2)$$

¹Actually we are proving that the asymptotic expansion of a non-polynomial E -function is never a \mathbb{C} -linear combination of functions $z^\alpha \log^k(z)\mathfrak{f}(1/z)$ with $\alpha \in \mathbb{Q}$, $k \in \mathbb{N}$ and \mathfrak{E} -functions \mathfrak{f} : some exponentials have to appear.

valid in a large sector bisected by 0, with an E -function F and \mathfrak{E} -functions \mathfrak{f}_1 and \mathfrak{f}_2 . Choosing arbitrarily a positive real algebraic value of z yields an explicit expression of $\log(2) \in \mathbf{G}$ as a multivariate polynomial in elements of \mathbf{E} and \mathbf{D} . But this example shows also that a polynomial in E - and \mathfrak{E} -functions may be constant even though there does not seem to be any obvious reason. In particular, the functions 1, $F(z)$, $e^{-z}\mathfrak{f}_{1;0}(1/z)$, and $e^{-2z}\mathfrak{f}_{2;0}(1/z)$ are linearly dependent over \mathbb{C} . However we see no reason why they would be linearly dependent over $\overline{\mathbb{Q}}$. This could be a major drawback to combine in E - and \mathfrak{E} -functions, since functions that are linearly dependent over \mathbb{C} but not over $\overline{\mathbb{Q}}$ can not belong to any Picard-Vessiot extension over $\overline{\mathbb{Q}}$.

Let us come now to the proof of Theorem 1. We first prove the second part, which runs as follows (it is reproduced from the unpublished note [16]).

From the stability of the class of E -functions by $\frac{d}{dz}$ and \int_0^z , we deduce that the set of convergent integrals $\int_0^\infty F(x)dx$ of E -functions and the set of finite limits of E -functions along some direction as $z \rightarrow \infty$ are the same. Theorem 2(iii) in [9] implies that if an E -function has a finite limit as $z \rightarrow \infty$ along some direction, then this limit must be in \mathbf{G} . Conversely, let $\beta \in \mathbf{G}$. By Theorem 1 in [8], there exists a G -function $G(z) = \sum_{n=0}^\infty a_n z^n$ of radius of convergence ≥ 2 (say) such that $G(1) = \beta$. Let $F(z) = \sum_{n=0}^\infty \frac{a_n}{n!} z^n$ be the associated E -function. Then for any z such that $\operatorname{Re}(z) > \frac{1}{2}$, we have

$$\frac{1}{z}G\left(\frac{1}{z}\right) = \int_0^{+\infty} e^{-xz}F(x)dx.$$

Hence, $\beta = \int_0^{+\infty} e^{-x}F(x)dx$ where $e^{-z}F(z)$ is an E -function.

We shall now prove the first part of Theorem 1. In fact, we shall prove a slightly more general result, namely Theorem 5 below. We first recall a few notations. Denote by \mathbf{S} the \mathbf{G} -module generated by all derivatives $\Gamma^{(s)}(a)$ (with $s \in \mathbb{N}$ and $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$), and by \mathbf{V} the \mathbf{S} -module generated by \mathbf{E} . Recall that \mathbf{G} , \mathbf{S} and \mathbf{V} are rings. Conjecturally, $\mathbf{G} = \mathcal{P}[1/\pi]$ and $\mathbf{V} = \mathcal{P}_e[1/\pi]$ where \mathcal{P} and \mathcal{P}_e are the ring of periods and the ring of exponential periods over $\overline{\mathbb{Q}}$ respectively (see [8, §2.2] and [10, §4.3]). We have proved in [10, Theorem 3] that \mathbf{V} is the \mathbf{S} -module generated by the numbers $e^\rho \chi$, with $\rho \in \overline{\mathbb{Q}}$ and $\chi \in \mathbf{D}$.

Theorem 5. *The ring \mathbf{V} is the ring generated by \mathbf{E} and \mathbf{D} . In particular, all values of G -functions belong to the ring generated by \mathbf{E} and \mathbf{D} .*

In other words, the elements of \mathbf{V} are exactly the sums of products ab with $a \in \mathbf{E}$ and $b \in \mathbf{D}$.

Proof of Theorem 5. We already know that \mathbf{V} is a ring, and that it contains \mathbf{E} and \mathbf{D} . To prove the other inclusion, denote by U the ring generated by \mathbf{E} and \mathbf{D} . Using Proposition 3 proved in §2.2 and the functional equation of Γ , we have $\Gamma^{(s)}(a) \in U$ for any $s \in \mathbb{N}$ and any $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Therefore for proving that $\mathbf{V} \subset U$, it is enough to prove that $\mathbf{G} \subset U$.

Let $\xi \in \mathbf{G}$. Using [11, Theorem 3] there exists an E -function $F(z)$ such that for any $\theta \in [-\pi, \pi)$ outside a finite set, ξ is a coefficient of the asymptotic expansion of $F(z)$

in a large sector bisected by θ . As the proof of [11, Theorem 3] shows, we can assume that ξ is the coefficient of e^z in this expansion.

Denote by L an E -operator of which F is a solution, and by μ its order. André has proved [1] that there exists a basis $(H_1(z), \dots, H_\mu(z))$ of formal solutions of L at infinity such that for any j , $e^{-\rho_j z} H_j(z) \in \text{NGA}\{1/z\}_1^{\overline{\mathbb{Q}}}$ for some algebraic number ρ_j . We recall that elements of $\text{NGA}\{1/z\}_1^{\overline{\mathbb{Q}}}$ are arithmetic Nilsson-Gevrey series of order 1 with algebraic coefficients, i.e. $\overline{\mathbb{Q}}$ -linear combinations of functions $z^k(\log z)^\ell \mathfrak{f}(1/z)$ with $k \in \mathbb{Q}$, $\ell \in \mathbb{N}$ and \mathfrak{D} -functions \mathfrak{f} . Expanding in this basis the asymptotic expansion of $F(z)$ in a large sector bisected by θ (denoted by \widetilde{F}), there exist complex numbers $\kappa_1, \dots, \kappa_d$ such that $\widetilde{F}(z) = \kappa_1 H_1(z) + \dots + \kappa_\mu H_\mu(z)$. Then we have $\xi = \kappa_1 c_1 + \dots + \kappa_\mu c_\mu$, where c_j is the coefficient of e^z in $H_j(z) \in e^{\rho_j z} \text{NGA}\{1/z\}_1^{\overline{\mathbb{Q}}}$. We have $c_j = 0$ if $\rho_j \neq 1$, and otherwise c_j is the constant coefficient of $e^{-z} H_j(z)$: in both cases c_j is an algebraic number. Therefore to conclude the proof that $\xi \in U$, it is enough to prove that $\kappa_1, \dots, \kappa_\mu \in U$.

For simplicity let us prove that $\kappa_1 \in U$. Given solutions F_1, \dots, F_μ of L , we denote by $W(F_1, \dots, F_\mu)$ the corresponding wronskian matrix. Then for any z in a large sector bisected by θ we have

$$\kappa_1 = \frac{\det W(F(z), H_{2,\theta}(z), \dots, H_{\mu,\theta}(z))}{\det W(H_{1,\theta}(z), \dots, H_{\mu,\theta}(z))}$$

where $H_{j,\theta}(z)$ is the 1-sommation of $H_j(z)$ in this sector. The determinant in the denominator belongs to $e^{az} \text{NGA}\{1/z\}_1^{\overline{\mathbb{Q}}}$ with $a = \rho_1 + \dots + \rho_\mu \in \overline{\mathbb{Q}}$. As the proof of [10, Theorem 6] shows, there exist $b, c \in \overline{\mathbb{Q}}$, with $c \neq 0$, such that

$$\det W(H_{1,\theta}(z), \dots, H_{\mu,\theta}(z)) = cz^b e^{az}.$$

We take $z = 1$, and choose $\theta = 0$ if it is not anti-Stokes for L (and $\theta > 0$ sufficiently small otherwise). Then we have

$$\kappa_1 = c^{-1} e^{-a} \left(\det W(F(z), H_{2,\theta}(z), \dots, H_{\mu,\theta}(z)) \right)_{|z=1} \in U.$$

This concludes the proof. □

Remark 1. The second part of Theorem 1 suggests the following comments. It would be interesting to have a better understanding (in terms of \mathbf{E} , \mathbf{G} and \mathbf{D}) of the set of convergent integrals $\int_0^\infty R(x)F(x)dx$ where R is a rational function in $\overline{\mathbb{Q}}(x)$ and F is an E -function, which are thus in \mathbf{G} when $R = 1$ (see [16] for related considerations). Indeed, classical examples of such integrals are $\int_0^{+\infty} \frac{\cos(x)}{1+x^2} dx = \pi/(2e) \in \pi\mathbf{E}$, Euler's constant $\int_0^{+\infty} \frac{1-(1+x)e^{-x}}{x(1+x)} dx = \gamma \in \mathbf{E} + e^{-1}\mathbf{D}$ (using Eq. (3.1) and [20, p. 248, Example 2]) and Gompertz constant $\delta := \int_0^{+\infty} \frac{e^{-x}}{1+x} dx \in \mathbf{D}$. A large variety of behaviors can thus be expected here.

For instance, using various explicit formulas in [13, Chapters 6.5–6.7], it can be proved that

$$\int_0^{+\infty} R(x)J_0(x)dx \in \mathbf{G} + \mathbf{E} + \gamma\mathbf{E} + \log(\overline{\mathbb{Q}}^*)\mathbf{E}$$

for any $R(x) \in \overline{\mathbb{Q}}(x)$ without poles on $[0, +\infty)$, where $J_0(x) = \sum_{n=0}^{\infty} (ix/2)^{2n}/n!^2$ is a Bessel function.

A second class of examples is when $R(x)F(x)$ is an even function without poles on $[0, +\infty)$ and such that $\lim_{|x| \rightarrow \infty, \text{Im}(x) \geq 0} x^2 R(x)F(x) = 0$. Then by the residue theorem,

$$\int_0^{+\infty} R(x)F(x)dx = i\pi \sum_{\rho, \text{Im}(\rho) > 0} \text{Res}_{x=\rho}(R(x)F(x)) \in \pi\mathbf{E}$$

where the summation is over the poles of R in the upper half plane.

4 Derivatives of the Γ function at rational points

In this section we prove Theorem 2 and Proposition 1 stated in the introduction, dealing with $\Gamma^{(s)}(a)$. To begin with, we define E -functions $E_{a,s}(z)$ in §4.1 and prove a linear independence result concerning these functions. Then we prove in §4.2 a formula for $\Gamma^{(s)}(a)$, namely Eq. (4.4), involving $E_{a,s+1}(-1)$ and the 1-summation of an \mathfrak{E} -function. This enables us to prove Theorem 2 in §4.3 and Proposition 1 in §4.4.

4.1 Linear independence of a family of E -functions

To study derivatives of the Γ function at rational points, we need the following lemma. For $s \geq 1$ and $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, we consider the E -function $E_{a,s}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!(n+a)^s}$.

Lemma 2. (i) For any $a \in \mathbb{Q} \setminus \mathbb{Z}$ and any $s \geq 1$, the functions

$$1, e^z, e^z E_{a,1}(-z), e^z E_{a,2}(-z), \dots, e^z E_{a,s}(-z)$$

are linearly independent over $\mathbb{C}(z)$.

(ii) For any $a \in \mathbb{N}^*$ and any $s \geq 2$, the functions

$$1, e^z, e^z E_{a,2}(-z), \dots, e^z E_{a,s}(-z)$$

are linearly independent over $\mathbb{C}(z)$.

Remark 2. Part (i) of the lemma is false if $a \in \mathbb{N}^*$ because $1, e^z, e^z E_{a,1}(-z)$ are $\mathbb{Q}(z)$ -linearly dependent in this case (see the proof of Part (ii) below).

Proof. (i) For simplicity, we set $\psi_s(z) := e^z E_{a,s}(-z)$. We proceed by induction on $s \geq 1$. Let us first prove the case $s = 1$. The derivative of $\psi_1(z)$ is $(1 + (z - a)\psi_1(z))/z$. Let us assume the existence of a relation $\psi_1(z) = u(z)e^z + v(z)$ with $u, v \in \mathbb{C}(z)$ (a putative relation $U(z) + V(z)e^z + W(z)\psi_1(z) = 0$ forces $W \neq 0$ because $e^z \notin \mathbb{C}(z)$). Then after differentiation of both sides, we end up with

$$\frac{1 + (z - a)\psi_1(z)}{z} = (u(z) + u'(z))e^z + v'(z).$$

Hence,

$$\frac{1 + (z - a)(u(z)e^z + v(z))}{z} = (u(z) + u'(z))e^z + v'(z).$$

Since $e^z \notin \mathbb{C}(z)$, the function $v(z)$ is a rational solution of the differential equation $zv'(z) = (z - a)v(z) + 1$: $v(z)$ cannot be identically 0, and it cannot be a polynomial (the degrees do not match on both sides). It must then have a pole at some point ω , of order $d \geq 1$ say. We must have $\omega = 0$ because otherwise the order of the pole at ω of $zv'(z)$ is $d + 1$ while the order of the pole of $(z - a)v(z) + 1$ is at most d . Writing $v(z) = \sum_{n \geq -d} v_n z^n$ with $v_{-d} \neq 0$ and comparing the term in z^{-d} of $zv'(z)$ and $(z - a)v(z) + 1$, we obtain that $d = a$. This forces a to be an integer ≥ 1 , which is excluded. Hence, $1, e^z, e^z E_{a,1}(-z)$ are $\mathbb{C}(z)$ -linearly independent.

Let us now assume that the case $s - 1 \geq 1$ holds. Let us assume the existence of a relation over $\mathbb{C}(z)$

$$\psi_s(z) = v(z) + u_0(z)e^z + \sum_{j=1}^{s-1} u_j(z)\psi_j(z). \quad (4.1)$$

(A putative relation $V(z) + U_0(z)e^z + \sum_{j=1}^s U_j(z)\psi_j(z) = 0$ forces $U_s \neq 0$ by the induction hypothesis). Differentiating (4.1) and because $\psi'_j(z) = (1 - \frac{a}{z})\psi_j(z) + \frac{1}{z}\psi_{j-1}(z)$ for all $j \geq 1$ (where we have let $\psi_0(z) = 1$), we have

$$\begin{aligned} A(z)\psi_s(z) + \frac{1}{z}\psi_{s-1}(z) &= v'(z) + (u_0(z) + u'_0(z))e^z + \sum_{j=1}^{s-1} u'_j(z)\psi_j(z) \\ &\quad + \sum_{j=1}^{s-1} u_j(z)\left(A(z)\psi_j(z) + \frac{1}{z}\psi_{j-1}(z)\right), \end{aligned} \quad (4.2)$$

where $A(z) := 1 - a/z$. Substituting the right-hand side of (4.1) for $\psi_s(z)$ on the left-hand side of (4.2), we then deduce that

$$\begin{aligned} v'(z) - A(z)v(z) + (u'_0(z) + (1 - A(z))u_0(z))e^z \\ + \frac{1}{z}(z - a)u_1(z)\psi_1(z) + \sum_{j=1}^{s-1} u'_j(z)\psi_j(z) + \frac{1}{z}\sum_{j=1}^{s-1} u_j(z)\psi_{j-1}(z) - \frac{1}{z}\psi_{s-1}(z) = 0. \end{aligned}$$

This is a non-trivial $\mathbb{C}(z)$ -linear relation between $1, e^z, \psi_1(z), \psi_2(z), \dots, \psi_{s-1}(z)$ because the coefficient of $\psi_{s-1}(z)$ is $u'_{s-1}(z) - 1/z$ and it is not identically 0 because $u'_{s-1}(z)$ cannot have a pole of order 1. But by the induction hypothesis, we know that such a relation is impossible.

(ii) The proof can be done by induction on $s \geq 2$ similarly. In the case $s = 2$, assume the existence of a relation $\psi_2(z) = u(z)e^z + v(z)$ with $u(z), v(z) \in \mathbb{C}(z)$. By differentiation, we obtain

$$\left(1 - \frac{a}{z}\right)\psi_2(z) = -\frac{1}{z}\psi_1(z) + (u(z) + u'(z))e^z + v'(z).$$

By induction on $a \geq 1$, we have $\psi_1(z) = (a-1)!e^z/z^a + w(z)$ for some $w(z) \in \mathbb{Q}(z)$. Hence, we have

$$\left(1 - \frac{a}{z}\right)u(z) = -\left(\frac{(a-1)!}{z^{a+1}} + 1\right)u(z) + u'(z)$$

which is not possible. Let us now assume that the case $s-1 \geq 2$ holds, as well as the existence of a relation over $\mathbb{C}(z)$

$$\psi_s(z) = v(z) + u_0(z)e^z + \sum_{j=2}^{s-1} u_j(z)\psi_j(z). \quad (4.3)$$

We proceed exactly as above by differentiation of both sides of (4.3). Using the relation $\psi'_j(z) = (1 - \frac{a}{z})\psi_j(z) + \frac{1}{z}\psi_{j-1}(z)$ for all $j \geq 2$ and the fact that $\psi_1(z) = (a-1)!e^z/z^a + w(z)$, we obtain a relation $\tilde{v}(z) + \tilde{u}_0(z)e^z + \sum_{j=2}^{s-1} \tilde{u}_j(z)\psi_j(z) = 0$ where $\tilde{u}_{s-1}(z) = u'_{s-1}(z) - 1/z$ cannot be identically 0. The induction hypothesis rules out the existence of such a relation. \square

4.2 A formula for $\Gamma^{(s)}(a)$

Let $z > 0$ and $a \in \mathbb{Q}^+$, $a \neq 0$. We have

$$\Gamma^{(s)}(a) = \int_0^\infty t^{a-1} \log(t)^s e^{-t} dt = \int_0^z t^{a-1} \log(t)^s e^{-t} dt + \int_z^\infty t^{a-1} \log(t)^s e^{-t} dt.$$

On the one hand,

$$\begin{aligned} \int_0^z t^{a-1} \log(t)^s e^{-t} dt &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^z t^{a+n-1} \log(t)^s dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{k=0}^s (-1)^k \frac{s!}{(s-k)!} \frac{z^{n+a} \log(z)^{s-k}}{(n+a)^{k+1}} \\ &= \sum_{k=0}^s \frac{(-1)^k s!}{(s-k)!} z^a \log(z)^{s-k} E_{a,k+1}(-z); \end{aligned}$$

recall that $E_{a,j}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!(n+a)^j}$. On the other hand,

$$\begin{aligned} \int_z^{\infty} t^{a-1} \log(t)^s e^{-t} dt &= e^{-z} \int_0^{\infty} (t+z)^{a-1} \log(t+z)^s e^{-t} dt \\ &= z^{a-1} e^{-z} \sum_{k=0}^s \binom{s}{k} \log(z)^{s-k} \int_0^{\infty} (1+t/z)^{a-1} \log(1+t/z)^k e^{-t} dt. \end{aligned}$$

Now $z > 0$ so that

$$\mathfrak{f}_{a,k+1;0}(z) := \int_0^{\infty} (1+tz)^{a-1} \log(1+tz)^k e^{-t} dt = \frac{1}{z} \int_0^{\infty} (1+x)^{a-1} \log(1+x)^k e^{-x/z} dx$$

is the 1-summation at the origin in the direction 0 of the \mathfrak{A} -function

$$\sum_{n=0}^{\infty} n! u_{a,k,n} z^n,$$

where the sequence $(u_{a,k,n})_{n \geq 0} \in \mathbb{Q}^{\mathbb{N}}$ is defined by the expansion of the G -function:

$$(1+x)^{a-1} \log(1+x)^k = \sum_{n=0}^{\infty} u_{a,k,n} x^n.$$

Note that if $k = 0$ and $a \in \mathbb{N}^*$, then $u_{a,k,n} = 0$ for any $n \geq a$, and $\mathfrak{f}_{a,k+1;0}(1/z)$ is a polynomial in $1/z$. Therefore, we have for any $z > 0$:

$$\Gamma^{(s)}(a) = \sum_{k=0}^s \frac{(-1)^k s!}{(s-k)!} z^a \log(z)^{s-k} E_{a,k+1}(-z) + z^{a-1} e^{-z} \sum_{k=0}^s \binom{s}{k} \log(z)^{s-k} \mathfrak{f}_{a,k+1;0}(1/z).$$

In particular, for $z = 1$, this relation reads

$$\Gamma^{(s)}(a) = (-1)^s s! E_{a,s+1}(-1) + e^{-1} \mathfrak{f}_{a,s+1;0}(1). \quad (4.4)$$

Since $\gamma = -\Gamma'(1)$ we obtain as a special case the formula

$$\gamma = E_{1,2}(-1) - e^{-1} \mathfrak{f}_{1,2;0}(1), \quad (4.5)$$

which is also a special case of Eq. (3.1) proved in [15].

4.3 Proof of Theorem 2

Let us assume that $\Gamma^{(s)}(a) \in \overline{\mathbb{Q}}$ for some $a \in \mathbb{Q}^+ \setminus \mathbb{N}$ and $s \geq 0$. Then $e^z \Gamma^{(s)}(a) + (-1)^{s+1} s! e^z E_{a,s+1}(-z)$ is an E -function. The relation $e \Gamma^{(s)}(a) + (-1)^{s+1} s! e E_{a,s+1}(-1) = \mathfrak{f}_{a,s+1;0}(1)$ proved at the end of §4.2 shows that $\alpha := e \Gamma^{(s)}(a) + (-1)^{s+1} s! e E_{a,s+1}(-1) \in \mathbf{E} \cap \mathbf{D}$. Hence α is in $\overline{\mathbb{Q}}$ by Conjecture 1 and we have a non-trivial $\overline{\mathbb{Q}}$ -linear relation

between $1, e$ and $eE_{a,s+1}(-1)$: we claim that this is not possible. Indeed, consider the vector

$$Y(z) := {}^t(1, e^z, e^z E_{a,1}(-z), \dots, e^z E_{a,s+1}(-z)).$$

It is solution of a differential system $Y'(z) = M(z)Y(z)$ where 0 is the only pole of $M(z) \in M_{s+3}(\overline{\mathbb{Q}}(z))$ (see the computations in the proof of Lemma 2 above). Since the components of $Y(z)$ are $\overline{\mathbb{Q}}(z)$ -linearly independent by Lemma 2(i), we deduce from Beukers' [6, Corollary 1.4] that

$$1, e, eE_{a,1}(-1), \dots, eE_{a,s+1}(-1)$$

are $\overline{\mathbb{Q}}$ -linearly independent, and in particular that $1, e$ and $eE_{a,s+1}(-1)$ are $\overline{\mathbb{Q}}$ -linearly independent. This concludes the proof if $a \in \mathbb{Q}^+ \setminus \mathbb{N}$.

Let us assume now that $\Gamma^{(s)}(a) \in \overline{\mathbb{Q}}$ for some $a \in \mathbb{N}^*$ and $s \geq 1$. Then $e^z \Gamma^{(s)}(a) + (-1)^{s+1} s! e^z E_{a,s+1}(-z)$ is an E -function. The relation $\Gamma^{(s)}(a) + (-1)^{s+1} s! E_{a,s+1}(-1) = e^{-1} \mathfrak{f}_{a,s+1;0}(1)$ shows that $\alpha := e \Gamma^{(s)}(a) + (-1)^{s+1} s! e E_{a,s+1}(-1) \in \mathbf{E} \cap \mathbf{D}$. Hence α is in $\overline{\mathbb{Q}}$ by Conjecture 1 and we have a non-trivial $\overline{\mathbb{Q}}$ -linear relation between $1, e$ and $eE_{a,s+1}(-1)$: we claim that this is not possible. Indeed, consider the vector $Y(z) := {}^t(1, e^z, e^z E_{a,2}(-z), \dots, e^z E_{a,s+1}(-z))$: it is solution of a differential system $Y'(z) = M(z)Y(z)$ where 0 is the only pole of $M(z) \in M_{s+2}(\overline{\mathbb{Q}}(z))$. Since the components of $Y(z)$ are $\overline{\mathbb{Q}}(z)$ -linearly independent by Lemma 2(ii), we deduce again from Beukers' theorem that

$$1, e, eE_{a,2}(-1), \dots, eE_{a,s+1}(-1)$$

are $\overline{\mathbb{Q}}$ -linearly independent, and in particular that $1, e$ and $eE_{a,s+1}(-1)$ are $\overline{\mathbb{Q}}$ -linearly independent. This concludes the proof of Theorem 2.

4.4 Proof of Proposition 1

Recall that Eq. (4.5) proved in §4.2 reads $eE_{1,2}(-1) - e\gamma = \mathfrak{f}_{1,2;0}(1)$. Assuming that $\gamma \in \mathbf{E}$, the left-hand side is in \mathbf{E} while the right-hand side is in \mathbf{D} . Hence both sides are in $\overline{\mathbb{Q}}$ by Conjecture 1. Note that, by integration by parts,

$$\mathfrak{f}_{1,2;0}(1) = \int_0^\infty \log(1+t)e^{-t} dt = \int_0^\infty \frac{e^{-t}}{1+t} dt$$

is Gompertz's constant. Hence, by Corollary 1 (which holds under Conjecture 2), the number $\mathfrak{f}_{1,2;0}(1)$ is not in $\overline{\mathbb{Q}}$. Consequently, $\gamma \notin \mathbf{E}$.

Similarly, Eq. (4.4) with $a \in \mathbb{Q} \setminus \mathbb{Z}$ and $s = 0$ reads $e\Gamma(a) - eE_{a,1}(-1) = \mathfrak{f}_{a,1;0}(1)$. Assuming that $\Gamma(a) \in \mathbf{E}$, the left-hand side is in \mathbf{E} while the right-hand side is in \mathbf{D} . Hence both sides are in $\overline{\mathbb{Q}}$ by Conjecture 1. But by Corollary 1 (which holds under Conjecture 2), the number $\mathfrak{f}_{a,1;0}(1) = \int_0^\infty (1+t)^{a-1} e^{-t} dt$ is not in $\overline{\mathbb{Q}}$. Hence, $\Gamma(a) \notin \mathbf{E}$.

5 Application of Beukers' method and consequence

In this section we prove Theorem 3 and Corollary 1 stated in the introduction.

5.1 Proof of Theorem 3

The proof of Theorem 3 is based on the arguments given in [6], except that E -functions have to be replaced with \mathfrak{D} -functions, and 1-summation in non-anti-Stokes directions is used for evaluations. Conjecture 2 is used as a substitute for Theorem A(i).

The main step is the following result, the proof of which is analogous to the end of the proof of [6, Corollary 2.2].

Proposition 4. *Assume that Conjecture 2 holds.*

Let \mathfrak{f} be an \mathfrak{D} -function, $\xi \in \overline{\mathbb{Q}}^$ and $\theta \in (\arg(\xi) - \pi/2, \arg(\xi) + \pi/2)$. Assume that θ is not anti-Stokes for \mathfrak{f} , and that $\mathfrak{f}_\theta(1/\xi) = 0$. Denote by $Ly = 0$ a differential equation, of minimal order, satisfied by $\mathfrak{f}(1/z)$.*

Then all solutions of $Ly = 0$ are holomorphic and vanish at ξ ; the differential operator L has an apparent singularity at ξ .

To deduce Theorem 3 from Proposition 4, it is enough to follow [6, §3].

5.2 Proof of Corollary 1

Let $s \in \mathbb{Q} \setminus \mathbb{Z}_{\geq 0}$. The \mathfrak{D} -function $\mathfrak{f}(z) := \sum_{n=0}^{\infty} s(s-1)\dots(s-n+1)z^n$ is solution of the inhomogeneous differential equation $z^2\mathfrak{f}'(z) + (1-sz)\mathfrak{f}(z) - 1 = 0$, which can be immediately transformed into a differential system satisfied by the vector of \mathfrak{D} -functions ${}^t(1, \mathfrak{f}(z))$. The coefficients of the matrix have only 0 as pole. Moreover, $\mathfrak{f}(z)$ is a transcendental function because $s \notin \mathbb{Z}_{\geq 0}$. Hence, by Theorem 3, $\mathfrak{f}_0(1/\alpha) \notin \overline{\mathbb{Q}}$ when $\alpha \in \overline{\mathbb{Q}}$, $\alpha > 0$, because 0 is not an anti-Stokes direction of $\mathfrak{f}(z)$. It remains to observe that this 1-summation is

$$\int_0^\infty (1+tz)^s e^{-t} dt.$$

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