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Relations between values of arithmetic Gevrey series, and applications to values of the Gamma function

S. Fischler and T. Rivoal

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Abstract

We investigate the relations between the rings \mathbf{E} , \mathbf{G} and \mathbf{D} of values taken at algebraic points by arithmetic Gevrey series of order either -1 (E -functions), 0 (analytic continuations of G -functions) or 1 (renormalization of divergent series solutions at ∞ of E -operators) respectively. We prove in particular that any element of \mathbf{G} can be written as multivariate polynomial with algebraic coefficients in elements of \mathbf{E} and \mathbf{D} , and is the limit at infinity of some E -function along some direction. This prompts to defining and studying the notion of mixed functions, which generalizes simultaneously E -functions and arithmetic Gevrey series of order 1 . Using natural conjectures for arithmetic Gevrey series of order 1 and mixed functions (which are analogues of a theorem of André and Beukers for E -functions) and the conjecture $\mathbf{D} \cap \mathbf{E} = \overline{\mathbb{Q}}$ (but not necessarily all these conjectures at the same time), we deduce a number of interesting Diophantine results such as an analogue for mixed functions of Beukers' linear independence theorem for values of E -functions, the transcendence of the values of the Gamma function and its derivatives at all non-integral algebraic numbers, the transcendence of Gompertz constant as well as the fact that Euler's constant is not in \mathbf{E} .

1 Introduction

A power series $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \in \overline{\mathbb{Q}}[[x]]$ is said to be an E -function when it is solution of a linear differential equation over $\overline{\mathbb{Q}}(x)$ (holonomic), and $|\sigma(a_n)|$ (for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) and the least common denominator of a_0, a_1, \dots, a_n grow at most exponentially in n . They were defined and studied by Siegel in 1929, who also defined the class of G -functions: a power series $\sum_{n=0}^{\infty} a_n x^n \in \overline{\mathbb{Q}}[[x]]$ is said to be a G -function when $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ is an E -function. In this case, $\sum_{n=0}^{\infty} n! a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is called an Θ -function, following the terminology introduced by André in [1]. E -functions are entire, while G -functions have a positive radius of convergence, which is finite except for polynomials. Here and below, we see $\overline{\mathbb{Q}}$ as embedded into \mathbb{C} . Following André again, E -functions, G -functions and Θ -fonctions are exactly arithmetic Gevrey series of order $s = -1, 0, 1$ respectively. Actually André defines arithmetic

Gevrey series of any order $s \in \mathbb{Q}$, but the set of values at algebraic points is the same for a given $s \neq 0$ as for $s/|s|$ using [1, Corollaire 1.3.2].

\mathfrak{E} -functions are divergent series, unless they are polynomials. Given an \mathfrak{E} -function f and any $\theta \in \mathbb{R}$, except finitely many values mod 2π (namely anti-Stokes directions of f), one can perform Ramis' 1-summation of $f(1/z)$ in the direction θ , which coincides in this setting with Borel-Laplace summation (see [14] or [9]). This provides a function denoted by $f_\theta(1/z)$, holomorphic on the open subset of \mathbb{C} consisting in all $z \neq 0$ such that $\theta - \frac{\pi}{2} - \varepsilon < \arg z < \theta + \frac{\pi}{2} + \varepsilon$ for some $\varepsilon > 0$, of which $f(1/z)$ is the asymptotic expansion in this sector (called a large sector bisected by θ). Of course $f(1/z)$ can be extended further by analytic continuation, but this asymptotic expansion may no longer be valid. When an \mathfrak{E} -function is denoted by f_j , we shall denote by $f_{j,\theta}$ or $\mathfrak{f}_{j;\theta}$ its 1-summation and we always assume (implicitly or explicitly) that θ is not an anti-Stokes direction.

In [8], [9] and [10, §4.3], we have studied the sets \mathbf{G} , \mathbf{E} and \mathbf{D} defined respectively as the sets of all the values taken by all (analytic continuations of) G -functions at algebraic points, of all the values taken by all E -functions at algebraic points and of all values $f_\theta(1)$ where f is an \mathfrak{E} -function ($\theta = 0$ if it is not an anti-Stokes direction, and $\theta > 0$ is very small otherwise.) These three sets are countable sub-rings of \mathbb{C} that all contain $\overline{\mathbb{Q}}$; conjecturally, they are related to the set of periods and exponential periods, see §3. (The ring \mathbf{D} is denoted by \mathfrak{D} in [10].)

We shall prove the following result in §3.

Theorem 1. *Every element of \mathbf{G} can be written as a multivariate polynomial (with coefficients in $\overline{\mathbb{Q}}$) in elements of \mathbf{E} and \mathbf{D} .*

Moreover, \mathbf{G} coincides with the set of all convergent integrals $\int_0^\infty F(x)dx$ where F is an E -function, or equivalently with the set of all finite limits of E -functions at ∞ along some direction.

Above, a convergent integral $\int_0^\infty F(x)dx$ means a finite limit of the E -function $\int_0^z F(x)dx$ as $z \rightarrow \infty$ along some direction; this explains the equivalence of both statements.

We refer to Eq. (3.2) in §3 for an expression of $\log(2)$ as a polynomial in elements in \mathbf{E} and \mathbf{D} ; the number π could be similarly expressed by considering z and iz instead of z and $2z$ there. Examples of the last statement are the identities (see [12] for the second one):

$$\int_0^{+\infty} \frac{\sin(x)}{x} dx = \frac{\pi}{2} \quad \text{and} \quad \int_0^{+\infty} J_0(ix)e^{-3x} dx = \frac{\sqrt{6}}{96\pi^3} \Gamma\left(\frac{1}{24}\right) \Gamma\left(\frac{5}{24}\right) \Gamma\left(\frac{7}{24}\right) \Gamma\left(\frac{11}{24}\right).$$

It is notoriously difficult to prove/disprove that a given element of \mathbf{G} is transcendental; it is known that a Siegel-Shidlovskii type theorem for G -functions can not hold *mutatis mutandis*. Theorem 1 suggests that an alternative approach to the study of the Diophantine properties of elements of \mathbf{G} can be through a better understanding of joint study of the elements of \mathbf{E} and \mathbf{D} , modulo certain conjectures to begin with. Our applications will not be immediately directed to the elements of \mathbf{G} but rather to the understanding of the (absence of) relations between the elements of \mathbf{E} and \mathbf{D} .

It seems natural (see [9, p. 37]) to conjecture that $\mathbf{E} \cap \mathbf{G} = \overline{\mathbb{Q}}$, and also that $\mathbf{G} \cap \mathbf{D} = \overline{\mathbb{Q}}$, though both properties seem currently out of reach. In this paper, we suggest (see §2) a possible approach towards the following analogous conjecture.

Conjecture 1. *We have $\mathbf{E} \cap \mathbf{D} = \overline{\mathbb{Q}}$.*

In §2 we shall make a functional conjecture, namely Conjecture 3, that implies Conjecture 1. We also prove that Conjecture 1 has very important consequences, as the following result shows.

Theorem 2. *Assume that Conjecture 1 holds. Then $\Gamma^{(s)}(a)$ is a transcendental number for any rational number $a > 0$ and any integer $s \geq 0$, except of course if $s = 0$ and $a \in \mathbb{N}$.*

One of the aims of this paper is to show that combining \mathfrak{D} - and E -functions may lead to very important results in transcendental number theory. Let us recall now briefly the main known results on E -functions.

Point (i) in the following result is due to André [2] for E -functions with rational Taylor coefficients, and to Beukers [6] in the general case. André used this property to obtain a new proof of the Siegel-Shidlovskii Theorem, and Beukers to prove an optimal refinement of this theorem (namely, (ii) below).

Theorem A. (i) [André, Beukers] *If an E -function $F(z)$ is such that $F(1) = 0$, then $\frac{F(z)}{z-1}$ is an E -function.*

(ii) [Beukers] *Let $\underline{F}(z) := {}^t(f_1(z), \dots, f_n(z))$ be a vector of E -functions solution of a differential system $\underline{F}'(z) = A(z)\underline{F}(z)$ for some matrix $A(z) \in M_n(\overline{\mathbb{Q}}(z))$.*

Let $\xi \in \overline{\mathbb{Q}}^$ which is not a pole of a coefficient of A . Let $P \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ be a homogeneous polynomial such that*

$$P(f_1(\xi), \dots, f_n(\xi)) = 0.$$

Then there exists $Q \in \overline{\mathbb{Q}}[Z, X_1, \dots, X_n]$, homogeneous in the X_i , such that

$$Q(z, f_1(z), \dots, f_n(z)) = 0 \text{ identically and } P(X_1, \dots, X_n) = Q(\xi, X_1, \dots, X_n).$$

In particular, we have

$$\text{trdeg}_{\overline{\mathbb{Q}}}(f_1(\xi), \dots, f_n(\xi)) = \text{trdeg}_{\overline{\mathbb{Q}}(z)}(f_1(z), \dots, f_n(z)).$$

The Siegel-Shidlovskii Theorem itself is the final statement about equality of transcendence degrees.

In this paper we state conjectural analogues of these results, involving \mathfrak{D} -functions. The principal difficulty is that these functions are divergent power series, and the exact analogue of Theorem A is meaningless. André discussed the situation in [2] and even though he did not formulate exactly the following conjecture, it seems plausible to us. From it, we will show how to deduce an analogue of the Siegel-Shidlovskii theorem for \mathfrak{D} -functions. Ferguson [7, p. 171, Conjecture 1] essentially stated this conjecture when $\mathfrak{f}(z)$ has *rational* coefficients and when $\theta = 0$.

Conjecture 2. Let $f(z)$ be an \mathfrak{A} -function and $\theta \in (-\pi/2, \pi/2)$ be such that $f_\theta(1) = 0$. Then $\frac{f(z)}{z-1}$ is an \mathfrak{A} -function.

In other words, the conclusion of this conjecture asserts that $\frac{z}{1-z}f(1/z)$ is an \mathfrak{A} -function in $1/z$; this is equivalent to $\frac{f(1/z)}{z-1}$ being an \mathfrak{A} -function in $1/z$ (since we have $\frac{f(1/z)}{z-1} = O(1/z)$ unconditionally as $|z| \rightarrow \infty$).

Following Beukers' proof [6] yields the following result (see [3, §4.6] for a related conjecture).

Theorem 3. Assume that Conjecture 2 holds.

Let $\underline{f}(z) := {}^t(f_1(z), \dots, f_n(z))$ be a vector of \mathfrak{A} -functions solution of a differential system $\underline{f}'(z) = A(z)\underline{f}(z)$ for some matrix $A(z) \in M_n(\overline{\mathbb{Q}}(z))$. Let $\xi \in \overline{\mathbb{Q}}^*$ and $\theta \in (\arg(\xi) - \pi/2, \arg(\xi) + \pi/2)$; assume that ξ is not a pole of a coefficient of A , and that θ is anti-Stokes for none of the f_j .

Let $P \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ be a homogeneous polynomial such that

$$P(f_{1,\theta}(1/\xi), \dots, f_{n,\theta}(1/\xi)) = 0.$$

Then there exists $Q \in \overline{\mathbb{Q}}[Z, X_1, \dots, X_n]$, homogeneous in the X_i , such that

$$Q(z, f_1(z), \dots, f_n(z)) = 0 \text{ identically and } P(X_1, \dots, X_n) = Q(1/\xi, X_1, \dots, X_n).$$

In particular, we have

$$\text{trdeg}_{\overline{\mathbb{Q}}}(\underline{f}_{1,\theta}(1/\xi), \dots, \underline{f}_{n,\theta}(1/\xi)) = \text{trdeg}_{\overline{\mathbb{Q}}(z)}(\underline{f}_1(z), \dots, \underline{f}_n(z)).$$

As an application of Theorem 3, we shall prove the following corollary. Note that under his weaker version of Conjecture 2, Ferguson [7, p. 171, Theorem 2] proved that Gompertz's constant is an irrational number.

Corollary 1. Assume that Conjecture 2 holds. Then for any $\alpha \in \overline{\mathbb{Q}}$, $\alpha > 0$, and any $s \in \mathbb{Q} \setminus \mathbb{Z}_{\geq 0}$, the number $\int_0^\infty (t + \alpha)^s e^{-t} dt$ is a transcendental number.

In particular, Gompertz's constant $\delta := \int_0^\infty e^{-t}/(t + 1) dt$ is a transcendental number.

In this text we suggest an approach towards Conjecture 1, based on the new notion of *mixed functions* which enables one to consider E - and \mathfrak{A} -functions at the same time. In particular we shall state a conjecture about such functions, namely Conjecture 3 in §2, which implies both Conjecture 1 and Conjecture 2. The following result is a motivation for this approach.

Proposition 1. Assume that both Conjectures 1 and 2 hold. Then neither Euler's constant $\gamma := -\Gamma'(1)$ nor $\Gamma(a)$ (with $a \in \mathbb{Q}^+ \setminus \mathbb{N}$) are in \mathbf{E} .

It is likely that none of these numbers is in \mathbf{G} , but (as far as we know) there is no "functional" conjecture like Conjecture 3 that implies this. It is also likely that none is in \mathbf{D} as well, but we don't know if this can be deduced from Conjecture 3.

The structure of this paper is as follows. In §2 we define and study mixed functions, a combination of E - and \mathfrak{E} -functions. Then in §3 we express any value of a G -function as a polynomial in values of E - and \mathfrak{E} -functions, thereby proving Theorem 1. We study derivatives of the Γ function at rational points in §4, and prove Theorem 2 and Proposition 1. At last, §5 is devoted to adapting Beukers' method to our setting: this approach yields Theorem 3 and Corollary 1.

2 Mixed functions

2.1 Definition and properties

In view of Theorem 1, it is natural to study polynomials in E - and \mathfrak{E} -functions. We can prove a Diophantine result that combines both Theorems A(ii) and 3 but under a very complicated polynomial generalization of Conjecture 2. We opt here for a different approach to mixing E - and \mathfrak{E} -functions for which very interesting Diophantine consequences can be deduced from a very easy to state conjecture which is more in the spirit of Conjecture 2. We refer to §2.3 for proofs of all properties stated in this section (including Lemma 1 and Proposition 2), except Theorem 4.

Definition 1. We call mixed (arithmetic Gevrey) function any formal power series

$$\sum_{n \in \mathbb{Z}} a_n z^n$$

such that $\sum_{n \geq 0} a_n z^n$ is an E -function in z , and $\sum_{n \geq 1} a_{-n} z^{-n}$ is an \mathfrak{E} -function in $1/z$.

In other words, a mixed function is defined as a formal sum $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ where F is an E -function and \mathfrak{f} is an \mathfrak{E} -function. In particular, such a function is zero if, and only if, both F and \mathfrak{f} are constants such that $F + \mathfrak{f} = 0$; obviously, F and \mathfrak{f} are uniquely determined by Ψ upon assuming (for instance) that $\mathfrak{f}(0) = 0$. The set of mixed functions is a $\overline{\mathbb{Q}}$ -vector space stable under multiplication by z^n for any $n \in \mathbb{Z}$. Unless $\mathfrak{f}(z)$ is a polynomial, such a function $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ is purely formal: there is no $z \in \mathbb{C}$ such that $\mathfrak{f}(1/z)$ is a convergent series. However, choosing a direction θ which is not anti-Stokes for \mathfrak{f} allows one to evaluate $\Psi_\theta(z) = F(z) + \mathfrak{f}_\theta(1/z)$ at any z in a large sector bisected by θ . Here and below, such a direction will be said *not anti-Stokes for Ψ* and whenever we write \mathfrak{f}_θ or Ψ_θ we shall assume implicitly that θ is not anti-Stokes.

Definition 1 is a formal definition, but one may identify a mixed function with the holomorphic function it defines on a given large sector by means of the following lemma.

Lemma 1. Let Ψ be a mixed function, and $\theta \in \mathbb{R}$ be a non-anti-Stokes direction for Ψ . Then Ψ_θ is identically zero (as a holomorphic function on a large sector bisected by θ) if, and only if, Ψ is equal to zero (as a formal power series in z and $1/z$).

Any mixed function $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ is solution of an E -operator. Indeed, this follows from applying [1, Theorem 6.1] twice: there exist an E -operator L such that $L(\mathfrak{f}(1/z)) = 0$, and an E -operator M such that $M(L(F(z))) = 0$ (because $L(F(z))$ is an E -function). Hence $ML(F(z) + \mathfrak{f}(1/z)) = 0$ and by [1, p. 720, §4.1], ML is an E -operator.

We formulate the following conjecture, which implies both Conjecture 1 and Conjecture 2.

Conjecture 3. *Let $\Psi(z)$ be an mixed function, and $\theta \in (-\pi/2, \pi/2)$ be such that $\Psi_\theta(1) = 0$. Then $\frac{\Psi(z)}{z-1}$ is an mixed function.*

The conclusion of this conjecture is that $\Psi(z) = (z-1)\Psi_1(z)$ for some mixed function Ψ_1 . This conclusion can be made more precise as follows; see §2.3 for the proof.

Proposition 2. *Let $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ be an mixed function, and $\theta \in (-\pi/2, \pi/2)$ be such that $\Psi_\theta(1) = 0$. Assume that Conjecture 3 holds for Ψ and θ .*

Then both $F(1)$ and $\mathfrak{f}_\theta(1)$ are algebraic, and $\frac{\mathfrak{f}(1/z) - \mathfrak{f}_\theta(1)}{z-1}$ is an \mathcal{O} -function.

Of course, in the conclusion of this proposition, one may assert also that $\frac{F(z) - F(1)}{z-1}$ is an E -function using Theorem A(i).

Conjecture 3 already has far reaching Diophantine consequences: Conjecture 2 and Theorem 2 stated in the introduction, and also the following result that encompasses Theorem 3 in the linear case.

Theorem 4. *Assume that Conjecture 3 holds.*

Let $\Psi(z) := {}^t(\Psi_1(z), \dots, \Psi_n(z))$ be a vector of mixed functions solution of a differential system $\Psi'(z) = A(z)\Psi(z)$ for some matrix $A(z) \in M_n(\overline{\mathbb{Q}}(z))$. Let $\xi \in \overline{\mathbb{Q}}^$ and $\theta \in (\arg(\xi) - \pi/2, \arg(\xi) + \pi/2)$; assume that ξ is not a pole of a coefficient of A , and that θ is anti-Stokes for none of the Ψ_j .*

Let $\lambda_1, \dots, \lambda_n \in \overline{\mathbb{Q}}$ be such that

$$\sum_{i=1}^n \lambda_i \Psi_{i,\theta}(\xi) = 0.$$

Then there exist $L_1, \dots, L_n \in \overline{\mathbb{Q}}[z]$ such that

$$\sum_{i=1}^n L_i(z) \Psi_i(z) = 0 \text{ identically and } L_i(\xi) = \lambda_i \text{ for any } i.$$

In particular, we have

$$\text{rk}_{\overline{\mathbb{Q}}}(\Psi_{1,\theta}(\xi), \dots, \Psi_{n,\theta}(\xi)) = \text{rk}_{\overline{\mathbb{Q}}(z)}(\Psi_1(z), \dots, \Psi_n(z)).$$

The proof of Theorem 4 follows exactly the linear part of the proof of Theorem 3 (see §5.1), which is based on [6, §3]. The only difference is that \mathcal{O} -functions have to be replaced

with mixed functions, and Conjecture 2 with Conjecture 3. In particular Proposition 4 stated in §5.1 remains valid with these modifications.

However a product of mixed functions is not, in general, a mixed function. Therefore the end of [6, §3] does not adapt to mixed functions, and there is no hope to obtain in this way a result on the transcendence degree of a field generated by values of mixed functions.

As an application of Theorem 4, we can consider the mixed functions $1, e^{\beta z}$ and $\mathfrak{f}(1/z) := \sum_{n=0}^{\infty} (-1)^n n! z^{-n}$, where β is a fixed non-zero algebraic number. These three functions are linearly independent over $\mathbb{C}(z)$ and form a solution of a differential system with only 0 for singularity (because $(\mathfrak{f}(1/z))' = (1 + 1/z)\mathfrak{f}(1/z) - 1$), hence for any $\alpha \in \overline{\mathbb{Q}}, \alpha > 0$ and any $\varrho \in \overline{\mathbb{Q}}^*$, the numbers $1, e^{\varrho}, \mathfrak{f}_0(1/\alpha) := \int_0^{\infty} e^{-t}/(1 + \alpha t) dt$ are $\overline{\mathbb{Q}}$ -linearly independent (for a fixed α , take $\beta = \varrho/\alpha$).

2.2 Values of mixed functions

We denote by \mathbf{M}_G the set of values $\Psi_{\theta}(1)$, where Ψ is a mixed function and $\theta = 0$ if it is not anti-Stokes, $\theta > 0$ is sufficiently small otherwise. This set is obviously equal to $\mathbf{E} + \mathbf{D}$.

Proposition 3. *For every integer $s \geq 0$ and every $a \in \mathbb{Q}^+, a \neq 0$, we have $\Gamma^{(s)}(a) \in e^{-1}\mathbf{M}_G$.*

This result follows immediately from Eq. (4.4) below (see §4.2), written in the form

$$\Gamma^{(s)}(a) = e^{-1}((-1)^s es! E_{a,s+1}(-1) + \mathfrak{f}_{a,s+1;0}(1)),$$

because $e^z E_{a,s+1}(-z)$ is an E -function and $\mathfrak{f}_{a,s+1;0}(1)$ is the 1-summation in the direction 0 of an \mathfrak{A} -function.

It would be interesting to know if $\Gamma^{(s)}(a)$ belongs to \mathbf{M}_G . We did not succeed in proving it does, and we believe it does not. Indeed, for instance if we want to prove that $\gamma \in \mathbf{M}_G$, a natural strategy would be to construct an E -function $F(z)$ with asymptotic expansion of the form $\gamma + \log(z) + \mathfrak{f}(1/z)$ in a large sector, and then to evaluate at $z = 1$. However this strategy cannot work since there is no such E -function (see the footnote in the proof of Lemma 1 in §2.3).

2.3 Proofs concerning mixed functions

To begin with, let us take Proposition 2 for granted and prove that Conjecture 3 implies both Conjecture 1 and Conjecture 2. Concerning Conjecture 2 it is clear. To prove that it implies Conjecture 1, let $\xi \in \mathbf{D}$, i.e. $\xi = \mathfrak{f}_{\theta}(1)$ is the 1-summation of an \mathfrak{A} -function $\mathfrak{f}(z)$ in the direction $\theta = 0$ if it is not anti-Stokes, and $\theta > 0$ close to 0 otherwise. Assume that ξ is also in \mathbf{E} : we have $\xi = F(1)$ for some E -function $F(z)$. Therefore, $\Psi(z) = F(z) - \mathfrak{f}(1/z)$ is a mixed function such that $\Psi_{\theta}(1) = 0$. By Conjecture 3 and Proposition 2, we have $\xi = \mathfrak{f}_{\theta}(1) \in \overline{\mathbb{Q}}$. This concludes the proof that Conjecture 3 implies Conjecture 1.

Let us prove Proposition 2 now. Assuming that Conjecture 3 holds for Ψ and θ , there exists a mixed function $\Psi_1(z) = F_1(z) + \mathfrak{f}_1(1/z)$ such that $\Psi(z) = (z-1)\Psi_1(z)$. We have

$$F(z) - (z-1)F_1(z) + \mathfrak{f}(1/z) - (z-1)\mathfrak{f}_1(1/z) = 0 \quad (2.1)$$

as a formal power series in z and $1/z$. Now notice that $z-1 = z(1 - \frac{1}{z})$, and that we may assume \mathfrak{f} and \mathfrak{f}_1 to have zero constant terms. Denote by α the constant term of $\mathfrak{f}(1/z) - z(1 - \frac{1}{z})\mathfrak{f}_1(1/z)$. Then we have

$$F(z) - (z-1)F_1(z) + \alpha + \mathfrak{f}_2(1/z) = 0$$

for some \mathfrak{E} -function \mathfrak{f}_2 without constant term, so that $\mathfrak{f}_2 = 0$, $F(z) = (z-1)F_1(z) - \alpha$ and $F(1) = -\alpha \in \overline{\mathbb{Q}}$. This implies $\mathfrak{f}_\theta(1) = \alpha$, and $\frac{\mathfrak{f}(1/z) - \mathfrak{f}_\theta(1)}{z-1} = \mathfrak{f}_1(1/z)$ is an \mathfrak{E} -function since $\mathfrak{f}_2 = 0$. This concludes the proof of Proposition 2.

At last, let us prove Lemma 1. We write $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ and assume that Ψ_θ is identically zero. Modifying θ slightly if necessary, we may assume that the asymptotic expansion $-\mathfrak{f}(1/z)$ of $F(z)$ in a large sector bisected by θ is given explicitly by [9, Theorem 5] applied to $F(z) - F(0)$; recall that such an asymptotic expansion is unique (see [9]). As in [9] we let $g(z) = \sum_{n=1}^{\infty} a_n z^{-n-1}$ where the coefficients a_n are given by $F(z) - F(0) = \sum_{n=1}^{\infty} \frac{a_n}{n!} z^n$. For any $\sigma \in \mathbb{C} \setminus \{0\}$ there is no contribution in $e^{\sigma z}$ in the asymptotic expansion of $F(z)$, so that $g(z)$ is holomorphic at σ . At $\sigma = 0$, the local expansion of g is of the form $g(z) = h_1(z) + h_2(z) \log(z)$ with G -functions h_1 and h_2 , and the coefficients of h_2 are related to those of \mathfrak{f} ; however we shall not use this special form ⁽¹⁾. Now recall that $g(z) = G(1/z)/z$ where G is a G -function; then G is entire and has moderate growth at infinity (because ∞ is a regular singularity of G), so it is a polynomial due to Liouville's theorem. This means that $F(z)$ is a polynomial in z . Recall that asymptotic expansions in large sectors are unique. Therefore both F and \mathfrak{f} are constant functions, and $F + \mathfrak{f} = 0$. This concludes the proof of Lemma 1.

3 Proof of Theorem 1: values of G -functions

In this section we prove Theorem 1. Let us begin with an example, starting with the relation proved in [15, Proposition 1] for $z \in \mathbb{C} \setminus (-\infty, 0]$:

$$\gamma + \log(z) = zE_{1,2}(-z) - e^{-z}\mathfrak{f}_{1,2;0}(1/z) \quad (3.1)$$

where $E_{1,2}$ is an E -function, and $\mathfrak{f}_{1,2}$ is an \mathfrak{E} -function, both defined below in §4.2.

Apply Eq. (3.1) at both z and $2z$, and then subtract one equation from the other. This provides a relation of the form

$$\log(2) = F(z) + e^{-z}\mathfrak{f}_{1,0}(1/z) + e^{-2z}\mathfrak{f}_{2,0}(1/z) \quad (3.2)$$

¹Actually we are proving that the asymptotic expansion of a non-polynomial E -function is never a \mathbb{C} -linear combination of functions $z^\alpha \log^k(z)\mathfrak{f}(1/z)$ with $\alpha \in \mathbb{Q}$, $k \in \mathbb{N}$ and \mathfrak{E} -functions \mathfrak{f} : some exponentials have to appear.

valid in a large sector bisected by 0, with an E -function F and \mathfrak{E} -functions \mathfrak{f}_1 and \mathfrak{f}_2 . Choosing arbitrarily a positive real algebraic value of z yields an explicit expression of $\log(2) \in \mathbf{G}$ as a multivariate polynomial in elements of \mathbf{E} and \mathbf{D} . But this example shows also that a polynomial in E - and \mathfrak{E} -functions may be constant even though there does not seem to be any obvious reason. In particular, the functions 1, $F(z)$, $e^{-z}\mathfrak{f}_{1;0}(1/z)$, and $e^{-2z}\mathfrak{f}_{2;0}(1/z)$ are linearly dependent over \mathbb{C} . However we see no reason why they would be linearly dependent over $\overline{\mathbb{Q}}$. This could be a major drawback to combine in E - and \mathfrak{E} -functions, since functions that are linearly dependent over \mathbb{C} but not over $\overline{\mathbb{Q}}$ can not belong to any Picard-Vessiot extension over $\overline{\mathbb{Q}}$.

Let us come now to the proof of Theorem 1. We first prove the second part, which runs as follows (it is reproduced from the unpublished note [16]).

From the stability of the class of E -functions by $\frac{d}{dz}$ and \int_0^z , we deduce that the set of convergent integrals $\int_0^\infty F(x)dx$ of E -functions and the set of finite limits of E -functions along some direction as $z \rightarrow \infty$ are the same. Theorem 2(iii) in [9] implies that if an E -function has a finite limit as $z \rightarrow \infty$ along some direction, then this limit must be in \mathbf{G} . Conversely, let $\beta \in \mathbf{G}$. By Theorem 1 in [8], there exists a G -function $G(z) = \sum_{n=0}^\infty a_n z^n$ of radius of convergence ≥ 2 (say) such that $G(1) = \beta$. Let $F(z) = \sum_{n=0}^\infty \frac{a_n}{n!} z^n$ be the associated E -function. Then for any z such that $\operatorname{Re}(z) > \frac{1}{2}$, we have

$$\frac{1}{z}G\left(\frac{1}{z}\right) = \int_0^{+\infty} e^{-xz}F(x)dx.$$

Hence, $\beta = \int_0^{+\infty} e^{-x}F(x)dx$ where $e^{-z}F(z)$ is an E -function.

We shall now prove the first part of Theorem 1. In fact, we shall prove a slightly more general result, namely Theorem 5 below. We first recall a few notations. Denote by \mathbf{S} the \mathbf{G} -module generated by all derivatives $\Gamma^{(s)}(a)$ (with $s \in \mathbb{N}$ and $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$), and by \mathbf{V} the \mathbf{S} -module generated by \mathbf{E} . Recall that \mathbf{G} , \mathbf{S} and \mathbf{V} are rings. Conjecturally, $\mathbf{G} = \mathcal{P}[1/\pi]$ and $\mathbf{V} = \mathcal{P}_e[1/\pi]$ where \mathcal{P} and \mathcal{P}_e are the ring of periods and the ring of exponential periods over $\overline{\mathbb{Q}}$ respectively (see [8, §2.2] and [10, §4.3]). We have proved in [10, Theorem 3] that \mathbf{V} is the \mathbf{S} -module generated by the numbers $e^\rho \chi$, with $\rho \in \overline{\mathbb{Q}}$ and $\chi \in \mathbf{D}$.

Theorem 5. *The ring \mathbf{V} is the ring generated by \mathbf{E} and \mathbf{D} . In particular, all values of G -functions belong to the ring generated by \mathbf{E} and \mathbf{D} .*

In other words, the elements of \mathbf{V} are exactly the sums of products ab with $a \in \mathbf{E}$ and $b \in \mathbf{D}$.

Proof of Theorem 5. We already know that \mathbf{V} is a ring, and that it contains \mathbf{E} and \mathbf{D} . To prove the other inclusion, denote by U the ring generated by \mathbf{E} and \mathbf{D} . Using Proposition 3 proved in §2.2 and the functional equation of Γ , we have $\Gamma^{(s)}(a) \in U$ for any $s \in \mathbb{N}$ and any $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Therefore for proving that $\mathbf{V} \subset U$, it is enough to prove that $\mathbf{G} \subset U$.

Let $\xi \in \mathbf{G}$. Using [11, Theorem 3] there exists an E -function $F(z)$ such that for any $\theta \in [-\pi, \pi)$ outside a finite set, ξ is a coefficient of the asymptotic expansion of $F(z)$

in a large sector bisected by θ . As the proof of [11, Theorem 3] shows, we can assume that ξ is the coefficient of e^z in this expansion.

Denote by L an E -operator of which F is a solution, and by μ its order. André has proved [1] that there exists a basis $(H_1(z), \dots, H_\mu(z))$ of formal solutions of L at infinity such that for any j , $e^{-\rho_j z} H_j(z) \in \text{NGA}\{1/z\}_1^{\overline{\mathbb{Q}}}$ for some algebraic number ρ_j . We recall that elements of $\text{NGA}\{1/z\}_1^{\overline{\mathbb{Q}}}$ are arithmetic Nilsson-Gevrey series of order 1 with algebraic coefficients, i.e. $\overline{\mathbb{Q}}$ -linear combinations of functions $z^k(\log z)^\ell \mathfrak{f}(1/z)$ with $k \in \mathbb{Q}$, $\ell \in \mathbb{N}$ and \mathfrak{D} -functions \mathfrak{f} . Expanding in this basis the asymptotic expansion of $F(z)$ in a large sector bisected by θ (denoted by \widetilde{F}), there exist complex numbers $\kappa_1, \dots, \kappa_\mu$ such that $\widetilde{F}(z) = \kappa_1 H_1(z) + \dots + \kappa_\mu H_\mu(z)$. Then we have $\xi = \kappa_1 c_1 + \dots + \kappa_\mu c_\mu$, where c_j is the coefficient of e^z in $H_j(z) \in e^{\rho_j z} \text{NGA}\{1/z\}_1^{\overline{\mathbb{Q}}}$. We have $c_j = 0$ if $\rho_j \neq 1$, and otherwise c_j is the constant coefficient of $e^{-z} H_j(z)$: in both cases c_j is an algebraic number. Therefore to conclude the proof that $\xi \in U$, it is enough to prove that $\kappa_1, \dots, \kappa_\mu \in U$.

For simplicity let us prove that $\kappa_1 \in U$. Given solutions F_1, \dots, F_μ of L , we denote by $W(F_1, \dots, F_\mu)$ the corresponding wronskian matrix. Then for any z in a large sector bisected by θ we have

$$\kappa_1 = \frac{\det W(F(z), H_{2,\theta}(z), \dots, H_{\mu,\theta}(z))}{\det W(H_{1,\theta}(z), \dots, H_{\mu,\theta}(z))}$$

where $H_{j,\theta}(z)$ is the 1-sommation of $H_j(z)$ in this sector. The determinant in the denominator belongs to $e^{az} \text{NGA}\{1/z\}_1^{\overline{\mathbb{Q}}}$ with $a = \rho_1 + \dots + \rho_\mu \in \overline{\mathbb{Q}}$. As the proof of [10, Theorem 6] shows, there exist $b, c \in \overline{\mathbb{Q}}$, with $c \neq 0$, such that

$$\det W(H_{1,\theta}(z), \dots, H_{\mu,\theta}(z)) = cz^b e^{az}.$$

We take $z = 1$, and choose $\theta = 0$ if it is not anti-Stokes for L (and $\theta > 0$ sufficiently small otherwise). Then we have

$$\kappa_1 = c^{-1} e^{-a} \left(\det W(F(z), H_{2,\theta}(z), \dots, H_{\mu,\theta}(z)) \right)_{|z=1} \in U.$$

This concludes the proof. □

Remark 1. The second part of Theorem 1 suggests the following comments. It would be interesting to have a better understanding (in terms of \mathbf{E} , \mathbf{G} and \mathbf{D}) of the set of convergent integrals $\int_0^\infty R(x)F(x)dx$ where R is a rational function in $\overline{\mathbb{Q}}(x)$ and F is an E -function, which are thus in \mathbf{G} when $R = 1$ (see [16] for related considerations). Indeed, classical examples of such integrals are $\int_0^{+\infty} \frac{\cos(x)}{1+x^2} dx = \pi/(2e) \in \pi\mathbf{E}$, Euler's constant $\int_0^{+\infty} \frac{1-(1+x)e^{-x}}{x(1+x)} dx = \gamma \in \mathbf{E} + e^{-1}\mathbf{D}$ (using Eq. (3.1) and [20, p. 248, Example 2]) and Gompertz constant $\delta := \int_0^{+\infty} \frac{e^{-x}}{1+x} dx \in \mathbf{D}$. A large variety of behaviors can thus be expected here.

For instance, using various explicit formulas in [13, Chapters 6.5–6.7], it can be proved that

$$\int_0^{+\infty} R(x)J_0(x)dx \in \mathbf{G} + \mathbf{E} + \gamma\mathbf{E} + \log(\overline{\mathbb{Q}}^*)\mathbf{E}$$

for any $R(x) \in \overline{\mathbb{Q}}(x)$ without poles on $[0, +\infty)$, where $J_0(x) = \sum_{n=0}^{\infty} (ix/2)^{2n}/n!^2$ is a Bessel function.

A second class of examples is when $R(x)F(x)$ is an even function without poles on $[0, +\infty)$ and such that $\lim_{|x| \rightarrow \infty, \text{Im}(x) \geq 0} x^2 R(x)F(x) = 0$. Then by the residue theorem,

$$\int_0^{+\infty} R(x)F(x)dx = i\pi \sum_{\rho, \text{Im}(\rho) > 0} \text{Res}_{x=\rho}(R(x)F(x)) \in \pi\mathbf{E}$$

where the summation is over the poles of R in the upper half plane.

4 Derivatives of the Γ function at rational points

In this section we prove Theorem 2 and Proposition 1 stated in the introduction, dealing with $\Gamma^{(s)}(a)$. To begin with, we define E -functions $E_{a,s}(z)$ in §4.1 and prove a linear independence result concerning these functions. Then we prove in §4.2 a formula for $\Gamma^{(s)}(a)$, namely Eq. (4.4), involving $E_{a,s+1}(-1)$ and the 1-summation of an \mathfrak{E} -function. This enables us to prove Theorem 2 in §4.3 and Proposition 1 in §4.4.

4.1 Linear independence of a family of E -functions

To study derivatives of the Γ function at rational points, we need the following lemma. For $s \geq 1$ and $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, we consider the E -function $E_{a,s}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!(n+a)^s}$.

Lemma 2. (i) For any $a \in \mathbb{Q} \setminus \mathbb{Z}$ and any $s \geq 1$, the functions

$$1, e^z, e^z E_{a,1}(-z), e^z E_{a,2}(-z), \dots, e^z E_{a,s}(-z)$$

are linearly independent over $\mathbb{C}(z)$.

(ii) For any $a \in \mathbb{N}^*$ and any $s \geq 2$, the functions

$$1, e^z, e^z E_{a,2}(-z), \dots, e^z E_{a,s}(-z)$$

are linearly independent over $\mathbb{C}(z)$.

Remark 2. Part (i) of the lemma is false if $a \in \mathbb{N}^*$ because $1, e^z, e^z E_{a,1}(-z)$ are $\mathbb{Q}(z)$ -linearly dependent in this case (see the proof of Part (ii) below).

Proof. (i) For simplicity, we set $\psi_s(z) := e^z E_{a,s}(-z)$. We proceed by induction on $s \geq 1$. Let us first prove the case $s = 1$. The derivative of $\psi_1(z)$ is $(1 + (z - a)\psi_1(z))/z$. Let us assume the existence of a relation $\psi_1(z) = u(z)e^z + v(z)$ with $u, v \in \mathbb{C}(z)$ (a putative relation $U(z) + V(z)e^z + W(z)\psi_1(z) = 0$ forces $W \neq 0$ because $e^z \notin \mathbb{C}(z)$). Then after differentiation of both sides, we end up with

$$\frac{1 + (z - a)\psi_1(z)}{z} = (u(z) + u'(z))e^z + v'(z).$$

Hence,

$$\frac{1 + (z - a)(u(z)e^z + v(z))}{z} = (u(z) + u'(z))e^z + v'(z).$$

Since $e^z \notin \mathbb{C}(z)$, the function $v(z)$ is a rational solution of the differential equation $zv'(z) = (z - a)v(z) + 1$: $v(z)$ cannot be identically 0, and it cannot be a polynomial (the degrees do not match on both sides). It must then have a pole at some point ω , of order $d \geq 1$ say. We must have $\omega = 0$ because otherwise the order of the pole at ω of $zv'(z)$ is $d + 1$ while the order of the pole of $(z - a)v(z) + 1$ is at most d . Writing $v(z) = \sum_{n \geq -d} v_n z^n$ with $v_{-d} \neq 0$ and comparing the term in z^{-d} of $zv'(z)$ and $(z - a)v(z) + 1$, we obtain that $d = a$. This forces a to be an integer ≥ 1 , which is excluded. Hence, $1, e^z, e^z E_{a,1}(-z)$ are $\mathbb{C}(z)$ -linearly independent.

Let us now assume that the case $s - 1 \geq 1$ holds. Let us assume the existence of a relation over $\mathbb{C}(z)$

$$\psi_s(z) = v(z) + u_0(z)e^z + \sum_{j=1}^{s-1} u_j(z)\psi_j(z). \quad (4.1)$$

(A putative relation $V(z) + U_0(z)e^z + \sum_{j=1}^s U_j(z)\psi_j(z) = 0$ forces $U_s \neq 0$ by the induction hypothesis). Differentiating (4.1) and because $\psi'_j(z) = (1 - \frac{a}{z})\psi_j(z) + \frac{1}{z}\psi_{j-1}(z)$ for all $j \geq 1$ (where we have let $\psi_0(z) = 1$), we have

$$\begin{aligned} A(z)\psi_s(z) + \frac{1}{z}\psi_{s-1}(z) &= v'(z) + (u_0(z) + u'_0(z))e^z + \sum_{j=1}^{s-1} u'_j(z)\psi_j(z) \\ &\quad + \sum_{j=1}^{s-1} u_j(z)\left(A(z)\psi_j(z) + \frac{1}{z}\psi_{j-1}(z)\right), \end{aligned} \quad (4.2)$$

where $A(z) := 1 - a/z$. Substituting the right-hand side of (4.1) for $\psi_s(z)$ on the left-hand side of (4.2), we then deduce that

$$\begin{aligned} v'(z) - A(z)v(z) + (u'_0(z) + (1 - A(z))u_0(z))e^z \\ + \frac{1}{z}(z - a)u_1(z)\psi_1(z) + \sum_{j=1}^{s-1} u'_j(z)\psi_j(z) + \frac{1}{z}\sum_{j=1}^{s-1} u_j(z)\psi_{j-1}(z) - \frac{1}{z}\psi_{s-1}(z) = 0. \end{aligned}$$

This is a non-trivial $\mathbb{C}(z)$ -linear relation between $1, e^z, \psi_1(z), \psi_2(z), \dots, \psi_{s-1}(z)$ because the coefficient of $\psi_{s-1}(z)$ is $u'_{s-1}(z) - 1/z$ and it is not identically 0 because $u'_{s-1}(z)$ cannot have a pole of order 1. But by the induction hypothesis, we know that such a relation is impossible.

(ii) The proof can be done by induction on $s \geq 2$ similarly. In the case $s = 2$, assume the existence of a relation $\psi_2(z) = u(z)e^z + v(z)$ with $u(z), v(z) \in \mathbb{C}(z)$. By differentiation, we obtain

$$\left(1 - \frac{a}{z}\right)\psi_2(z) = -\frac{1}{z}\psi_1(z) + (u(z) + u'(z))e^z + v'(z).$$

By induction on $a \geq 1$, we have $\psi_1(z) = (a-1)!e^z/z^a + w(z)$ for some $w(z) \in \mathbb{Q}(z)$. Hence, we have

$$\left(1 - \frac{a}{z}\right)u(z) = -\left(\frac{(a-1)!}{z^{a+1}} + 1\right)u(z) + u'(z)$$

which is not possible. Let us now assume that the case $s-1 \geq 2$ holds, as well as the existence of a relation over $\mathbb{C}(z)$

$$\psi_s(z) = v(z) + u_0(z)e^z + \sum_{j=2}^{s-1} u_j(z)\psi_j(z). \quad (4.3)$$

We proceed exactly as above by differentiation of both sides of (4.3). Using the relation $\psi'_j(z) = (1 - \frac{a}{z})\psi_j(z) + \frac{1}{z}\psi_{j-1}(z)$ for all $j \geq 2$ and the fact that $\psi_1(z) = (a-1)!e^z/z^a + w(z)$, we obtain a relation $\tilde{v}(z) + \tilde{u}_0(z)e^z + \sum_{j=2}^{s-1} \tilde{u}_j(z)\psi_j(z) = 0$ where $\tilde{u}_{s-1}(z) = u'_{s-1}(z) - 1/z$ cannot be identically 0. The induction hypothesis rules out the existence of such a relation. \square

4.2 A formula for $\Gamma^{(s)}(a)$

Let $z > 0$ and $a \in \mathbb{Q}^+$, $a \neq 0$. We have

$$\Gamma^{(s)}(a) = \int_0^\infty t^{a-1} \log(t)^s e^{-t} dt = \int_0^z t^{a-1} \log(t)^s e^{-t} dt + \int_z^\infty t^{a-1} \log(t)^s e^{-t} dt.$$

On the one hand,

$$\begin{aligned} \int_0^z t^{a-1} \log(t)^s e^{-t} dt &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^z t^{a+n-1} \log(t)^s dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{k=0}^s (-1)^k \frac{s!}{(s-k)!} \frac{z^{n+a} \log(z)^{s-k}}{(n+a)^{k+1}} \\ &= \sum_{k=0}^s \frac{(-1)^k s!}{(s-k)!} z^a \log(z)^{s-k} E_{a,k+1}(-z); \end{aligned}$$

recall that $E_{a,j}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!(n+a)^j}$. On the other hand,

$$\begin{aligned} \int_z^{\infty} t^{a-1} \log(t)^s e^{-t} dt &= e^{-z} \int_0^{\infty} (t+z)^{a-1} \log(t+z)^s e^{-t} dt \\ &= z^{a-1} e^{-z} \sum_{k=0}^s \binom{s}{k} \log(z)^{s-k} \int_0^{\infty} (1+t/z)^{a-1} \log(1+t/z)^k e^{-t} dt. \end{aligned}$$

Now $z > 0$ so that

$$\mathfrak{f}_{a,k+1;0}(z) := \int_0^{\infty} (1+tz)^{a-1} \log(1+tz)^k e^{-t} dt = \frac{1}{z} \int_0^{\infty} (1+x)^{a-1} \log(1+x)^k e^{-x/z} dx$$

is the 1-summation at the origin in the direction 0 of the \mathfrak{A} -function

$$\sum_{n=0}^{\infty} n! u_{a,k,n} z^n,$$

where the sequence $(u_{a,k,n})_{n \geq 0} \in \mathbb{Q}^{\mathbb{N}}$ is defined by the expansion of the G -function:

$$(1+x)^{a-1} \log(1+x)^k = \sum_{n=0}^{\infty} u_{a,k,n} x^n.$$

Note that if $k = 0$ and $a \in \mathbb{N}^*$, then $u_{a,k,n} = 0$ for any $n \geq a$, and $\mathfrak{f}_{a,k+1;0}(1/z)$ is a polynomial in $1/z$. Therefore, we have for any $z > 0$:

$$\Gamma^{(s)}(a) = \sum_{k=0}^s \frac{(-1)^k s!}{(s-k)!} z^a \log(z)^{s-k} E_{a,k+1}(-z) + z^{a-1} e^{-z} \sum_{k=0}^s \binom{s}{k} \log(z)^{s-k} \mathfrak{f}_{a,k+1;0}(1/z).$$

In particular, for $z = 1$, this relation reads

$$\Gamma^{(s)}(a) = (-1)^s s! E_{a,s+1}(-1) + e^{-1} \mathfrak{f}_{a,s+1;0}(1). \quad (4.4)$$

Since $\gamma = -\Gamma'(1)$ we obtain as a special case the formula

$$\gamma = E_{1,2}(-1) - e^{-1} \mathfrak{f}_{1,2;0}(1), \quad (4.5)$$

which is also a special case of Eq. (3.1) proved in [15].

4.3 Proof of Theorem 2

Let us assume that $\Gamma^{(s)}(a) \in \overline{\mathbb{Q}}$ for some $a \in \mathbb{Q}^+ \setminus \mathbb{N}$ and $s \geq 0$. Then $e^z \Gamma^{(s)}(a) + (-1)^{s+1} s! e^z E_{a,s+1}(-z)$ is an E -function. The relation $e \Gamma^{(s)}(a) + (-1)^{s+1} s! e E_{a,s+1}(-1) = \mathfrak{f}_{a,s+1;0}(1)$ proved at the end of §4.2 shows that $\alpha := e \Gamma^{(s)}(a) + (-1)^{s+1} s! e E_{a,s+1}(-1) \in \mathbf{E} \cap \mathbf{D}$. Hence α is in $\overline{\mathbb{Q}}$ by Conjecture 1 and we have a non-trivial $\overline{\mathbb{Q}}$ -linear relation

between $1, e$ and $eE_{a,s+1}(-1)$: we claim that this is not possible. Indeed, consider the vector

$$Y(z) := {}^t(1, e^z, e^z E_{a,1}(-z), \dots, e^z E_{a,s+1}(-z)).$$

It is solution of a differential system $Y'(z) = M(z)Y(z)$ where 0 is the only pole of $M(z) \in M_{s+3}(\overline{\mathbb{Q}}(z))$ (see the computations in the proof of Lemma 2 above). Since the components of $Y(z)$ are $\overline{\mathbb{Q}}(z)$ -linearly independent by Lemma 2(i), we deduce from Beukers' [6, Corollary 1.4] that

$$1, e, eE_{a,1}(-1), \dots, eE_{a,s+1}(-1)$$

are $\overline{\mathbb{Q}}$ -linearly independent, and in particular that $1, e$ and $eE_{a,s+1}(-1)$ are $\overline{\mathbb{Q}}$ -linearly independent. This concludes the proof if $a \in \mathbb{Q}^+ \setminus \mathbb{N}$.

Let us assume now that $\Gamma^{(s)}(a) \in \overline{\mathbb{Q}}$ for some $a \in \mathbb{N}^*$ and $s \geq 1$. Then $e^z \Gamma^{(s)}(a) + (-1)^{s+1} s! e^z E_{a,s+1}(-z)$ is an E -function. The relation $\Gamma^{(s)}(a) + (-1)^{s+1} s! E_{a,s+1}(-1) = e^{-1} \mathfrak{f}_{a,s+1;0}(1)$ shows that $\alpha := e\Gamma^{(s)}(a) + (-1)^{s+1} s! eE_{a,s+1}(-1) \in \mathbf{E} \cap \mathbf{D}$. Hence α is in $\overline{\mathbb{Q}}$ by Conjecture 1 and we have a non-trivial $\overline{\mathbb{Q}}$ -linear relation between $1, e$ and $eE_{a,s+1}(-1)$: we claim that this is not possible. Indeed, consider the vector $Y(z) := {}^t(1, e^z, e^z E_{a,2}(-z), \dots, e^z E_{a,s+1}(-z))$: it is solution of a differential system $Y'(z) = M(z)Y(z)$ where 0 is the only pole of $M(z) \in M_{s+2}(\overline{\mathbb{Q}}(z))$. Since the components of $Y(z)$ are $\overline{\mathbb{Q}}(z)$ -linearly independent by Lemma 2(ii), we deduce again from Beukers' theorem that

$$1, e, eE_{a,2}(-1), \dots, eE_{a,s+1}(-1)$$

are $\overline{\mathbb{Q}}$ -linearly independent, and in particular that $1, e$ and $eE_{a,s+1}(-1)$ are $\overline{\mathbb{Q}}$ -linearly independent. This concludes the proof of Theorem 2.

4.4 Proof of Proposition 1

Recall that Eq. (4.5) proved in §4.2 reads $eE_{1,2}(-1) - e\gamma = \mathfrak{f}_{1,2;0}(1)$. Assuming that $\gamma \in \mathbf{E}$, the left-hand side is in \mathbf{E} while the right-hand side is in \mathbf{D} . Hence both sides are in $\overline{\mathbb{Q}}$ by Conjecture 1. Note that, by integration by parts,

$$\mathfrak{f}_{1,2;0}(1) = \int_0^\infty \log(1+t)e^{-t} dt = \int_0^\infty \frac{e^{-t}}{1+t} dt$$

is Gompertz's constant. Hence, by Corollary 1 (which holds under Conjecture 2), the number $\mathfrak{f}_{1,2;0}(1)$ is not in $\overline{\mathbb{Q}}$. Consequently, $\gamma \notin \mathbf{E}$.

Similarly, Eq. (4.4) with $a \in \mathbb{Q} \setminus \mathbb{Z}$ and $s = 0$ reads $e\Gamma(a) - eE_{a,1}(-1) = \mathfrak{f}_{a,1;0}(1)$. Assuming that $\Gamma(a) \in \mathbf{E}$, the left-hand side is in \mathbf{E} while the right-hand side is in \mathbf{D} . Hence both sides are in $\overline{\mathbb{Q}}$ by Conjecture 1. But by Corollary 1 (which holds under Conjecture 2), the number $\mathfrak{f}_{a,1;0}(1) = \int_0^\infty (1+t)^{a-1} e^{-t} dt$ is not in $\overline{\mathbb{Q}}$. Hence, $\Gamma(a) \notin \mathbf{E}$.

5 Application of Beukers' method and consequence

In this section we prove Theorem 3 and Corollary 1 stated in the introduction.

5.1 Proof of Theorem 3

The proof of Theorem 3 is based on the arguments given in [6], except that E -functions have to be replaced with \mathfrak{D} -functions, and 1-summation in non-anti-Stokes directions is used for evaluations. Conjecture 2 is used as a substitute for Theorem A(i).

The main step is the following result, the proof of which is analogous to the end of the proof of [6, Corollary 2.2].

Proposition 4. *Assume that Conjecture 2 holds.*

Let \mathfrak{f} be an \mathfrak{D} -function, $\xi \in \overline{\mathbb{Q}}^$ and $\theta \in (\arg(\xi) - \pi/2, \arg(\xi) + \pi/2)$. Assume that θ is not anti-Stokes for \mathfrak{f} , and that $\mathfrak{f}_\theta(1/\xi) = 0$. Denote by $Ly = 0$ a differential equation, of minimal order, satisfied by $\mathfrak{f}(1/z)$.*

Then all solutions of $Ly = 0$ are holomorphic and vanish at ξ ; the differential operator L has an apparent singularity at ξ .

To deduce Theorem 3 from Proposition 4, it is enough to follow [6, §3].

5.2 Proof of Corollary 1

Let $s \in \mathbb{Q} \setminus \mathbb{Z}_{\geq 0}$. The \mathfrak{D} -function $\mathfrak{f}(z) := \sum_{n=0}^{\infty} s(s-1)\dots(s-n+1)z^n$ is solution of the inhomogeneous differential equation $z^2\mathfrak{f}'(z) + (1-sz)\mathfrak{f}(z) - 1 = 0$, which can be immediately transformed into a differential system satisfied by the vector of \mathfrak{D} -functions ${}^t(1, \mathfrak{f}(z))$. The coefficients of the matrix have only 0 as pole. Moreover, $\mathfrak{f}(z)$ is a transcendental function because $s \notin \mathbb{Z}_{\geq 0}$. Hence, by Theorem 3, $\mathfrak{f}_0(1/\alpha) \notin \overline{\mathbb{Q}}$ when $\alpha \in \overline{\mathbb{Q}}$, $\alpha > 0$, because 0 is not an anti-Stokes direction of $\mathfrak{f}(z)$. It remains to observe that this 1-summation is

$$\int_0^\infty (1+tz)^s e^{-t} dt.$$

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