Non-cutoff Boltzmann equation with soft potentials in the whole space

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Abstract. We prove the existence and uniqueness of global solutions to the Boltzmann equation with non-cutoff soft potentials in the whole space when the initial data is a small perturbation of a Maxwellian with polynomial decay in velocity. Our method is based in the decomposition of the desired solution into two parts: one with polynomial decay in velocity satisfying the Boltzmann equation with only a dissipative part of the linearized operator ; the other with Gaussian decay in velocity verifying the Boltzmann equation with a coupling term.

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1. Introduction

Consider the Boltzmann equation for the unknown $F = F(t,x,v)$, with $t \geqslant 0$, $x \in \mathbf{R}^3$, and $v \in \mathbf{R}^3$:

$$\partial_t F + v \cdot \nabla_x F = Q(F,F) \tag{1.1}$$

complemented with an initial data $F_0 = F_0(x,v)$. The collision operator Q is bilinear and acts only on the velocity variable $v \in \mathbf{R}^3$, which represents the fact that collisions are supposed to be localized in space, and it reads

$$Q(f,g)(v) = \int_{\mathbf{R}^3}\int_{\mathbf{S}^2} B(v-v_*,\sigma)\left[f(v'_*)g(v') - f(v_*)g(v)\right]\,\mathrm{d}\sigma\,\mathrm{d}v_*. \tag{1.2}$$

The pre- and post-collision velocities (v', v'_*) and (v, v_*) are given by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \quad \text{and} \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma \tag{1.3}$$

which is one possible parametrization of the conservation of momentum and energy in an elastic collision

$$v' + v'_* = v + v_* \quad \text{and} \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$

Date: December 8, 2022.

The collision kernel $B(v-v_*,\sigma)$ encodes the physics of the interaction between particles. It is assumed to be nonnegative and to depend only on the relative velocity $|v-v_*|$ and the angle $\cos\theta = \sigma\cdot\frac{(v-v_*)}{|v-v_*|}$ as

$$B(v-v_*,\sigma) = |v-v_*|^\gamma b(\cos\theta),$$

where $-3<\gamma\leqslant 1$ and the angular part b is a smooth function (except maybe at $\theta=0$). As it is standard now, we may suppose, without loss of generality, that $\theta\in[0,\pi/2]$ by replacing B by its symmetrized version if necessary.

In this paper we shall consider the case of *non-cutoff soft potentials*, more precisely we assume that b is an implicit function that is locally smooth and has a non-integrable singularity at $\theta=0$ as

$$\sin\theta\, b(\cos\theta) \underset{\theta\sim 0}{\approx} C_b\,\theta^{-1-2s} \quad \text{with} \quad s\in(0,1), \tag{1.4}$$

for some constant $C_b>0$, and

$$-1<\gamma+2s<0 \quad \text{and} \quad -3/2-s<\gamma<0. \tag{1.5}$$

Let $\mu(v)=(2\pi)^{-3/2}e^{-|v|^2/2}$ be the standard Maxwellian and define the perturbation

$$F=\mu+f$$

which satisfies

$$\begin{cases}\partial_t f + v\cdot\nabla_x f = \mathscr{L}f + Q(f,f)\\ f_{|t=0}=f_0=F_0-\mu\end{cases} \tag{1.6}$$

where $\mathscr{L}$ is the linearized collision operator given by

$$\mathscr{L}f = Q(\mu,f)+Q(f,\mu).$$

We also denote by Λ the full linearized operator

$$\Lambda := \mathscr{L} - v\cdot\nabla_x.$$

It is well known (see for instance [40]) that $\mathscr{L}$ is a nonnegative self-adjoint operator on the space $L^2_v(\mu^{-1}\,\mathrm{d}v)$ with kernel given by

$$\ker(\mathscr{L}) = \operatorname{span}\{\mu, v\mu, |v|^2\mu\}.$$

We define π to be the orthogonal projection onto $\ker(\mathscr{L})$ so that we can decompose

$$f=\pi f + f^\perp, \quad f^\perp := f-\pi f$$

with

$$\pi f = \left\{\rho[f]+u[f]\cdot v+\theta[f]\frac{(|v|^2-3)}{2}\right\}\mu \tag{1.7}$$

where

$$\rho[f]=\int_{\mathbf{R}^3} f\,\mathrm{d}v, \quad u[f]=\int_{\mathbf{R}^3} vf\,\mathrm{d}v, \quad \theta[f]=\int_{\mathbf{R}^3}\frac{(|v|^2-3)}{3}f\,\mathrm{d}v.$$

1.1. **Main result.** Before stating our main result we shall introduce the functional spaces we work with. If X is a function space and w a non-negative function, we define the weighted space $X(w)$ as the space associated to the norm

$$\|f\|_{X(w)} := \|wf\|_X.$$

In particular, for a weight function $m=m(v)$, we consider the weighted Lebesgue space $L^2_xL^2_v(m)$ as the space associated to the inner product

$$\langle f,g\rangle_{L^2_xL^2_v(m)} := \langle mf, mg\rangle_{L^2_{x,v}}$$

and the corresponding norm

$$\|f\|_{L^2_xL^2_v(m)} := \|mf\|_{L^2_{x,v}},$$

where $\langle\cdot,\cdot\rangle_{L^2_{x,v}}$ and $\|\cdot\|_{L^2_{x,v}}$ denote the usual inner product and norm of $L^2(\mathbf{R}^3_x\times\mathbf{R}^3_v)$.

We consider polynomial weight functions $m(v) = \langle v\rangle^k := (1+|v|^2)^{k/2}$ with $k > 0$, and we introduce the anisotropic dissipation space in velocity $H^{s,*}_v(m)$, inspired from the one presented in [28], as the space associated to the norm

$$\|f\|^2_{H^{s,*}_v(m)} := \|\langle v\rangle^{\gamma/2} f\|^2_{L^2_v(m)} + \|f\|^2_{\dot H^{s,*}_v(m)}, \tag{1.8}$$

$$\|f\|^2_{\dot H^{s,*}_v(m)} := \int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} b(\cos\theta)\,\mu(v_*)\langle v_*\rangle^{\gamma}\left(\mathcal{F}-\mathcal{F}'\right)^2 \mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v, \tag{1.9}$$

where we use the shorthand $\mathcal{F} = \mathcal{F}(v) := m(v)\langle v\rangle^{\gamma/2} f(v)$ and $\mathcal{F}' = \mathcal{F}(v')$ recalling that v' is defined in (1.3), and which satisfies the following bound (see Lemma 2.7):

$$\|\langle v\rangle^{\gamma/2} f\|_{H^s_v(m)} \lesssim \|f\|_{H^{s,*}_v(m)} \lesssim \|\langle v\rangle^{\gamma/2+s} f\|_{H^s_v(m)}.$$

For functions $f = f(x,v)$ depending on the position x and velocity v variables, we also define the polynomially weighted spaces $\mathbf{X}(m)$, $\mathbf{X}^*(m)$ and $\mathbf{Y}(m)$ as the spaces associated to the norms

$$\|f\|^2_{\mathbf{X}(m)} := \|f\|^2_{L^2_{x,v}(m)} + \|\langle v\rangle^{-6s}\nabla^3_x f\|^2_{L^2_{x,v}(m)}, \tag{1.10}$$

$$\|f\|^2_{\mathbf{X}^*(m)} := \|f\|^2_{L^2_x H^{s,*}_v(m)} + \|\langle v\rangle^{-6s}\nabla^3_x f\|^2_{L^2_x H^{s,*}_v(m)}, \tag{1.11}$$

and

$$\|f\|^2_{\mathbf{Y}(m)} := \|f^\perp\|^2_{\mathbf{X}^*(m)} + \|\nabla_x \pi f\|^2_{H^2_x L^2_v}. \tag{1.12}$$

respectively.

We can now state our main result:

Theorem 1.1. *Assume (1.4)–(1.5) hold. Consider $k > 13/2 + 7|\gamma|/2 + 8s$ and define the weight function $m = \langle v\rangle^k$. There exists $\varepsilon_0 > 0$ small enough such that any initial data $f_0 \in \mathbf{X}(m)$ satisfying $\|f_0\|_{\mathbf{X}(m)} \leqslant \varepsilon_0$ gives rise to a unique global weak solution $f \in L^\infty(\mathbf{R}_+;\mathbf{X}(m)) \cap L^2(\mathbf{R}_+;\mathbf{Y}(m))$ to (1.6), which satisfies the energy estimate*

$$\sup_{t\geqslant 0}\|f(t)\|^2_{\mathbf{X}(m)} + \int_0^\infty \|f(t)\|^2_{\mathbf{Y}(m)}\,\mathrm{d}t \lesssim \|f_0\|^2_{\mathbf{X}(m)}.$$

We now briefly review the known results for the Boltzmann equation near Maxwellian in the torus $\mathbf{T}^3$ and the whole space $\mathbf{R}^3$.

We star by considering the case of cutoff potentials, which corresponds to angular kernels b for which the singularity in (1.4) is removed by assuming b integrable. By working near equilibrium, Grad [22] constructed in 1965 the first spatially inhomogeneous solutions for short times. Ukai [37, 38] gave in the 1970's a new impulse to the Cauchy problem and established, in the case of hard potentials $\gamma \in [0,1]$, the existence of global solutions in $L^\infty_v H^s_x\left(\langle v\rangle^k \mu^{-1}\,\mathrm{d}v\right)$, first in the periodic box $\mathbf{T}^3$ in 1974 [37], then in the whole space $\mathbf{R}^3$ in 1976 [38], by relying on spectral studies of the linearized equation [37, 20, 40] (let us mention also [32]). The case of soft potentials was then treated in 1980 by Caflisch [11], then in 1982 by Asano and Ukai [40] only for $\gamma \in (-1,0)$, but this approach was recently extended to the full range $\gamma \in (-3,0)$ by Sun and Wu [42] in 2021 and then Deng [16] in 2022. These results were then proven using energy methods in spaces of the form $L^2_v H^s_x\left(\mu^{-1}\,\mathrm{d}v\,\mathrm{d}x\right)$ by Kawashima [29], Liu, Yang and Yu [30], Guo [26] and Guo and Strain [35, 36], as well as Duan [17].

Concerning the non-cutoff case, the first existence result near equilibrium attributed to Ukai [39] ; he constructed local solutions for analytic initial data in (x,v) having Gaussian decay using the Cauchy-Kowalewski theorem. Between 2011 and 2012, Gressman and Strain [24, 23] (in the torus), and Alexandre, Morimoto, Ukai, Xu, Yang [6, 4, 7] (in the whole space) constructed the first global solutions in spaces of the form $H^s_{x,v}\left(\langle v\rangle^k\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v\right)$ by working with anisotropic norms. In the whole space framework, Strain [34] obtained the optimal time-decay for solutions in the whole space. Later Sohinger and Strain [33] extended these results to some Besov spaces in 2014, and Fang and Wang [41] relaxed some technical regularity and integrability assumptions in 2022. Recently, in the case of the torus, Duan, Liu, Sakamoto and Strain [18] obtained the existence of small-amplitude

solutions, that is, in the space $L^1_k L^\infty_t L^2_v(\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v)$ where k denotes the Fourier variable in space. Let us also mention two very recent works in the case of the whole space: Deng [15], in the case of hard potentials, who worked with an anisotropic norm defined from the pseudo-differential study of Alexandre, Hérau and Li [2]; and also Duan, Sakamoto and Ueda [19] who constructed with small-amplitude solutions (as in [18]) in the case of hard and moderately soft potentials.

All the above results concern solutions with Gaussian decay in velocity, that is, they hold in functional spaces with a weight in velocity of the form $\mu^{-1}\,\mathrm{d}v$. In 2017, Gualdani, Mischler and Mouhot [25], in the line of [31], constructed solutions with polynomial decay in velocity. More precisely they relaxed the integrability conditions of previous results and constructed solutions in $W^{\ell,p}_v W^{s,q}_x\left(\langle v\rangle^k\,\mathrm{d}v\,\mathrm{d}x\right)$, in the case of hard spheres in the torus $\mathbf{T}^3$. In the same framework, the case of non-cutoff hard potentials was treated in [28, 9], and that of non-cutoff soft potentials in [13]. Very recently, still in the torus and also in spaces with polynomial weights, the case of cutoff soft potentials was studied by Cao [12].

Our result in Theorem 1.1 gives then, up to our knowledge, the first result of existence of global solutions with polynomial decay in velocity to the non-cutoff Boltzmann equation in the whole space. Inspired by the strategy of [10], we shall construct a solution f to (1.6) by considering a decomposition of the form $f = h + g$, where $h(t) \in \mathbf{X}(m)$ has polynomial decay in velocity and satisfies a "nice" semilinear equation in which only a dissipative part of the linearized operator Λ is present, and $g(t)$ has Gaussian decay in velocity and evolves according to the Boltzmann equation plus some coupling term coming from h, with convenient decay properties in time and velocity. This system will then be solved using an iterative scheme and an energy method. In Section 2 we prove the necessary estimates on Q, and in Section 3 we prove the necessary coercive-type estimate on the linear part of the equation for h and recall those related to the equation for g. We then proceed to prove the existence of a global unique solution to (1.6) in Section 4.

1.2. **Notations.** The relation denoted $A \lesssim B$ is to be understood as $A \leqslant CB$ for some uniform constant $C > 0$, and $A \approx B$ as both $A \lesssim B$ and $B \lesssim A$.

When considering a function $f(v)$ depending on the velocity variable, we shall use the standard shorthand notations

$$f = f(v), \quad f' = f(v'), \quad f_* = f(v_*), \quad f'_* = f(v'_*), \tag{1.13}$$

where we recall that the pre- and post-collision velocities (v', v'_*) and (v, v_*) are defined in (1.3).

2. Estimates on the collision operator

This section is devoted to estimates on the collision operator Q. We shall prove homogeneous estimates in Section 2.2, and deduce from them inhomogeneous estimates in Section 2.3

2.1. **Auxiliary results.** We state a few results that will be useful in the sequel. This first lemma will be used to estimates integrals against the kinetic part $|v - v_*|^\gamma$ of the collision kernel $B(v - v_*, \sigma)$.

Lemma 2.1. *Let $\alpha \in (0, 3)$ and $s \in (0, 1]$. For any smooth enough function $f = f(v)$ one has:*

(1) If $0 < \alpha < \frac{3}{2}$ then for any $\ell > \frac{3}{2}$ there holds, for any $v \in \mathbf{R}^3$,

$$\int_{\mathbf{R}^3} |v - v_*|^{-\alpha} |f(v_*)|\,\mathrm{d}v_* \lesssim \langle v\rangle^{-\alpha} \|\langle v\rangle^\ell f\|_{L^2_v}. \tag{2.1}$$

(2) If $0 < \alpha < \frac{3}{2} + s$ then for any $\ell > \frac{3}{2} + s$ there holds, for any $v \in \mathbf{R}^3$,

$$\int_{\mathbf{R}^3} |v - v_*|^{-\alpha} |f(v_*)|\,\mathrm{d}v_* \lesssim \langle v\rangle^{-\alpha} \|\langle v\rangle^\ell f\|_{H^s_v}. \tag{2.2}$$

Proof. From [14, Lemma 3.3] for instance one has, for any $0 < \beta < 3$ and $\vartheta > 3$,

$$\int_{\mathbf{R}^3} |v - v_*|^{-\beta} \langle v_* \rangle^{-\vartheta} \, \mathrm{d}v_* \lesssim \langle v \rangle^{-\beta}, \quad \forall v \in \mathbf{R}^3. \tag{2.3}$$

We now write for $p \in [1, \infty]$, thanks to Hölder's inequality,

$$\int_{\mathbf{R}^3} |v - v_*|^{-\alpha} |f(v_*)| \, \mathrm{d}v_* \leqslant \left(\int_{\mathbf{R}^3} |v - v_*|^{-\alpha \frac{p}{p-1}} \langle v_* \rangle^{-\ell \frac{p}{p-1}} \, \mathrm{d}v_* \right)^{\frac{p-1}{p}} \| \langle v \rangle^{\ell} f \|_{L^p_v}.$$

We then conclude by using (2.3) with: $p = 2$ if $0 < \alpha < \frac{3}{2}$; and $p = \frac{6}{3-2s}$ if $0 < \alpha < \frac{3}{2} + s$ by using the Sobolev embedding $H^s(\mathbf{R}^3) \hookrightarrow L^{\frac{6}{3-2s}}(\mathbf{R}^3)$. □

Lemma 2.2. *Let $\varphi = \varphi(v)$ be a Schwartz function and $\alpha \in (0, 3)$. For any $\eta \in (0, 1)$ and $\ell > 0$ there is $C > 0$ such that there holds, for any $v \in \mathbf{R}^3$,*

$$\int_{\mathbf{R}^3} |v - v_*|^{-\alpha} \varphi(v_*) \, \mathrm{d}v_* \leqslant \eta \langle v \rangle^{-\alpha} + C \langle v \rangle^{-\ell}.$$

Proof. Let $v \in \mathbf{R}^3$ be fixed. We split the integral for some $M \geqslant 1$:

$$\begin{aligned} &\int_{\mathbf{R}^3} |v - v_*|^{-\alpha} |\varphi(v_*)| \, \mathrm{d}v_* \\ &\quad \leqslant \int_{|v - v_*| \geqslant M} |v - v_*|^{-\alpha} |\varphi(v_*)| \, \mathrm{d}v_* + \int_{|v - v_*| \leqslant M} |v - v_*|^{-\alpha} |\varphi(v_*)| \, \mathrm{d}v_* \\ &\quad =: I_1 + I_2. \end{aligned}$$

For I_2 we write $\langle v_* \rangle^{-\ell} \leqslant \langle v - v_* \rangle^{\ell} \langle v \rangle^{-\ell}$ for some arbitrary $\ell \geqslant 0$, thus

$$I_2 \lesssim \langle v \rangle^{-\ell} \| \langle v \rangle^{\ell} \varphi \|_{L^\infty_v} \int_{|v - v_*| \leqslant M} |v - v_*|^{-\alpha} \langle v - v_* \rangle^{\ell} \, \mathrm{d}v_* \lesssim \langle v \rangle^{-\ell}.$$

For I_1 we take $p > 3/(3 - \alpha)$, so that $p/(p-1) < 3/\alpha$, and apply Hölder's inequality to get

$$\begin{aligned} I_1 &\leqslant \left(\int_{\mathbf{R}^3} |v - v_*|^{-\alpha \frac{p}{p-1}} \langle v_* \rangle^{-q \frac{p}{p-1}} \, \mathrm{d}v_* \right)^{\frac{p-1}{p}} \left(\int_{\mathbf{R}^3} \mathbf{1}_{|v - v*| \geqslant M} \langle v_* \rangle^{qp} |\varphi(v_*)|^p \, \mathrm{d}v_* \right)^{\frac{1}{p}} \\ &\lesssim \langle v \rangle^{-\alpha} \left(\int_{\mathbf{R}^3} \mathbf{1}_{|v - v*| \geqslant M} \, \langle v_* \rangle^{qp} |\varphi(v_*)|^p \, \mathrm{d}v_* \right)^{\frac{1}{p}}, \end{aligned}$$

where $q > 0$ is such that $q \frac{p}{p-1} > 3$. We now observe that for $M \geqslant 2|v|$, if $|v - v_*| \geqslant M$ then $|v_*| \geqslant |v - v_*| - |v| \geqslant M/2$, and thus we get

$$\begin{aligned} \int_{\mathbf{R}^3} \mathbf{1}_{|v - v_*| \geqslant M} \, \langle v_* \rangle^{qp} |\varphi(v_*)|^p \, \mathrm{d}v_* &\lesssim \int_{\mathbf{R}^3} \mathbf{1}_{|v_*| \geqslant M/2} \, \langle v_* \rangle^{qp} |\varphi(v_*)|^p \, \mathrm{d}v_* \\ &\lesssim M^{-p} \int_{\mathbf{R}^3} \langle v_* \rangle^{(q+1)p} |\varphi(v_*)|^p \, \mathrm{d}v_* \\ &\lesssim M^{-p}. \end{aligned}$$

We then conclude taking $M \geqslant \max(1, 2|v|)$ large enough. □

Proposition 2.3. *The following change of variables formulas hold:*

$$\begin{aligned} &\int b(\cos\theta) |v - v_*|^{\gamma} f(v, v', \theta) \, \mathrm{d}\sigma \, \mathrm{d}v_* \\ &\qquad \approx \int b\left(\cos(\pi - 2\theta)\right) |v - v_*|^{\gamma} \sin\left(\frac{\pi}{2} - \theta\right)^{-2-\gamma} f\left(v, v_*, \pi - 2\theta\right) \, \mathrm{d}\sigma \, \mathrm{d}v_*, \end{aligned} \tag{2.4}$$

$$\int b(\cos\theta) |v - v_*|^{\gamma} f(v', v_*, \theta) \, \mathrm{d}\sigma \, \mathrm{d}v \approx \int b(\cos 2\theta) |v - v_*|^{\gamma} f\left(v, v_*, 2\theta\right) \, \mathrm{d}\sigma \, \mathrm{d}v, \tag{2.5}$$

$$\int B(v - v_*, \sigma) f(v, v_*, v', v'_*, \theta) \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v = \int B(v - v_*, \sigma) f(v', v'_*, v, v_*, \theta) \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v. \tag{2.6}$$

Proof. The pre-post collisional change of variables (2.6) is known to be involutive with Jacobian 1 and it is easy to check that $|v - v_*| = |v' - v'_*|$. We only deal with the first two change of variables. Recall the definition of v':

$$v' = \frac{v + v_*}{2} + \sigma \frac{|v - v_*|}{2}.$$

Denote $k := \dfrac{v - v_*}{|v - v_*|}$ and recall that θ is the angle (k, σ). The differentials of v' with respect to v and v_* writes in the basis (k, σ, w) where $w \perp k, \sigma$ (at least when k and σ are not colinear)

$$\frac{\mathrm{d}v'}{\mathrm{d}v} = \frac{1}{2}\left(\mathrm{Id} + \langle \cdot\,, k\rangle \sigma\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1+\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.7}$$

$$\frac{\mathrm{d}v'}{\mathrm{d}v_*} = \frac{1}{2}\left(\mathrm{Id} - \langle \cdot\,, k\rangle \sigma\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1-\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.8}$$

Thus, the following identities hold:

$$\left|\frac{\mathrm{d}v'}{\mathrm{d}v}\right| = \frac{1}{8}(1 + \cos\theta) = \frac{1}{4}\cos^2\left(\frac{\theta}{2}\right),$$
$$\left|\frac{\mathrm{d}v'}{\mathrm{d}v_*}\right| = \frac{1}{8}(1 - \cos\theta) = \frac{1}{4}\sin^2\left(\frac{\theta}{2}\right).$$

Furthermore, the definition of v' also implies

$$|v' - v_*|^2 = \frac{1}{2}\cos\left(\frac{\theta}{2}\right)^2 |v - v_*|^2,$$
$$|v' - v|^2 = \frac{1}{2}\sin\left(\frac{\theta}{2}\right)^2 |v - v_*|^2.$$

The angle φ formed by $v' - v_*$ and σ and the angle ψ formed by $v' - v$ and σ are related to θ by

$$\varphi = \frac{\theta}{2}, \quad \psi = \frac{\pi - \theta}{2},$$

thus the integrals are estimated as follows:

$$\int b(\cos\theta)|v - v_*|^\gamma f(v, v', \theta)\,\mathrm{d}\sigma\,\mathrm{d}v_* \approx \int b\left(\cos(\pi - 2\psi)\right)|v - v|^\gamma \sin\left(\frac{\pi}{2} - \psi\right)^{-2-\gamma} f\left(v, v_*, \pi - 2\psi\right)\,\mathrm{d}\sigma\,\mathrm{d}v',$$

$$\int b(\cos\theta)|v - v_*|^\gamma f(v', v_*, \theta)\,\mathrm{d}\sigma\,\mathrm{d}v \approx \int b(\cos 2\varphi)|v - v_*|^\gamma f\left(v', v_*, 2\varphi\right)\,\mathrm{d}\sigma\,\mathrm{d}v'.$$

We conclude to (2.5) and (2.4) by renaming the integration variables. □

We state the so-called *Cancellation lemma* from [1, Lemma 1].

Proposition 2.4 (Cancellation lemma)**.** *The following cancellation formula holds:*

$$\int B(v - v_*, \sigma)(f' - f)\,\mathrm{d}v_*\,\mathrm{d}\sigma = (f * S)(v),$$

where the function $S = S(z)$ is defined as

$$S(z) := 2\pi \int_0^{\pi/2} \sin\theta \left(\frac{1}{\cos^3\theta} B\left(\frac{|z|}{\cos(\theta/2)}, \cos\theta\right) - B(|z|, \cos\theta)\right)\mathrm{d}\theta.$$

In the particular case of the collision kernel $B_\delta(v - v_*, \sigma) = B(v - v_*, \sigma)\mathbf{1}_{|\theta| \leqslant \delta}$, the corresponding function S_δ satisfies

$$S_\delta(z) \approx \delta^{2-2s}|z|^\gamma.$$

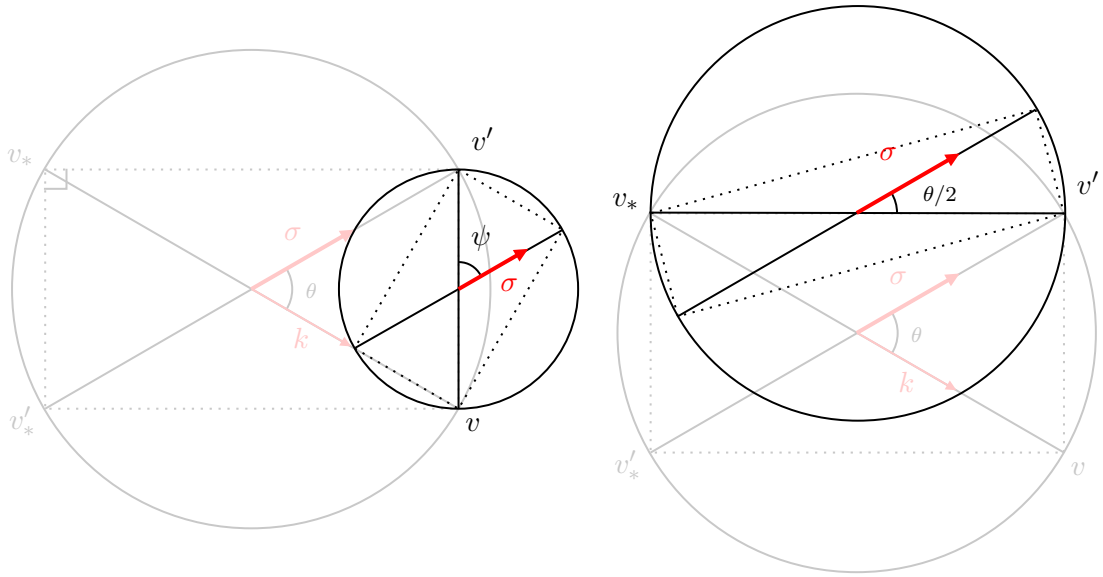

FIGURE 1. The changes of variables $(v, v') \to (v, v_*)$ and $(v', v_*) \to (v, v_*)$.

The following lemma is from [3, Lemma 2.3].

Lemma 2.5. *Let $m = \langle v\rangle^k$ with $k \geqslant 0$, then there holds*

$$m(v) \lesssim m(v') + m(v'_*), \tag{2.9}$$

$$|m(v) - m(v')| \lesssim \theta m(v')\langle v'_*\rangle + \theta^k m(v'_*), \tag{2.10}$$

where the pre- and post-collisional velocities (v', v'_*) and (v, v_*) are defined in (1.3).

This lemma will serve to remove the kinetic singularity $|v - v_*|^\gamma$ in some integrals involving the collision kernel $B(v - v_*, \sigma)$.

Lemma 2.6. *For any $a > -3$, $b \in \mathbf{R}$, $q > 3 + 2s + a + b$ and any smooth enough function f there holds*

$$\begin{aligned}&\int b(\cos\theta)|v - v_*|^a\langle v - v_*\rangle^b\langle v_*\rangle^{-q} f(v, v')\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &\quad\approx \int b(\cos\theta)\langle v - v_*\rangle^{a+b}\langle v_*\rangle^{-q} f(v, v')\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\end{aligned}$$

Proof. As we cannot simply control $|v - v_*|^a\langle v - v_*\rangle^b$ by $\langle v - v_*\rangle^{a+b}$, we resort to using Carleman's representation:

$$\begin{aligned}&\int b(\cos\theta)|v - v_*|^a\langle v - v_*\rangle^b\langle v_*\rangle^{-q} f(v, v')\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v_*\\ &\quad\approx \int_{\substack{v,h\in\mathbf{R}^3,\\ y\perp h,\ |y|\geqslant|h|}} \frac{|y|^{1+2s+a}}{|h|^{1+2s}}\langle y\rangle^b\langle v + y\rangle^{-q} f(v, v + h)\,\mathrm{d}y\,\mathrm{d}h\,\mathrm{d}v,\\ &\quad= \int K_{\mathrm{sing}}(v, h)\frac{f(v, v + h)}{|h|^{1+2s}}\,\mathrm{d}h\,\mathrm{d}v,\end{aligned}$$

where we denoted the singular y-integral, which is well-defined because $1+2s+a+b-q < -2$ and $1 + 2s + a > -2$

$$K_{\mathrm{sing}}(v, h) := \int_{\substack{y\perp h,\\ |y|\geqslant|h|}} |y|^{1+2s+a}\langle y\rangle^b\langle v + y\rangle^{-q}\,\mathrm{d}y$$

and aim to prove

$$K_{\mathrm{sing}}(v, h) \approx K_{\mathrm{reg}}(v, h) := \int_{\substack{y\perp h,\\ |y|\geqslant|h|}} |y|^{1+2s}\langle y\rangle^{a+b}\langle v + y\rangle^{-q}\,\mathrm{d}y.$$

To do so, we split K_{sing} for $|y| \leqslant \varepsilon$ and $|y| \geqslant \varepsilon$ where $\varepsilon > 0$ will be chosen later:

$$K_{\text{sing}}(v,h) = \int_{\substack{y\perp h,\\ \varepsilon \geqslant |y| \geqslant |h|}} |y|^{1+2s+a}\langle y\rangle^b \langle v+y\rangle^{-q}\,\mathrm{d}y + \int_{\substack{y\perp h,\\ |y|\geqslant \max\{|h|,\varepsilon\}}} |y|^{1+2s+a}\langle y\rangle^b \langle v+y\rangle^{-q}\,\mathrm{d}y.$$

Concerning the first part, if ε is small enough, the assumption $|y| \leqslant \varepsilon$ implies $\langle v+y\rangle^{-q} \approx \langle v\rangle^{-q}$ uniformly in y. Concerning the second part, the assumption $|y| \geqslant \varepsilon$ implies $\langle y\rangle \approx |y|$ uniformly in y. Thus we have

$$\begin{aligned} K_{\text{sing}}(v,h) &= \int_{\substack{y\perp h,\\ \varepsilon \geqslant |y| \geqslant |h|}} |y|^{1+2s+a}\langle y\rangle^b \langle v+y\rangle^{-q}\,\mathrm{d}y + \int_{\substack{y\perp h,\\ |y|\geqslant \max\{|h|,\varepsilon\}}} |y|^{1+2s+a}\langle y\rangle^b \langle v+y\rangle^{-q}\,\mathrm{d}y \\ &\approx \langle v\rangle^{-q} \int_{\substack{y\perp h,\\ \varepsilon \geqslant |y| \geqslant |h|}} |y|^{1+2s+a}\langle y\rangle^b \,\mathrm{d}y + \int_{\substack{y\perp h,\\ |y|\geqslant \max\{|h|,\varepsilon\}}} |y|^{1+2s}\langle y\rangle^{a+b} \langle v+y\rangle^{-q}\,\mathrm{d}y \\ &\approx \langle v\rangle^{-q} + \int_{\substack{y\perp h,\\ |y|\geqslant \max\{|h|,\varepsilon\}}} |y|^{1+2s}\langle y\rangle^{a+b} \langle v+y\rangle^{-q}\,\mathrm{d}y \end{aligned}$$

With the same reasoning, we have

$$\begin{aligned} K_{\text{reg}}(v,h) &= \int_{\substack{y\perp h,\\ \varepsilon \geqslant |y| \geqslant |h|}} |y|^{1+2s}\langle y\rangle^{a+b} \langle v+y\rangle^{-q}\,\mathrm{d}y + \int_{\substack{y\perp h,\\ |y|\geqslant \max\{|h|,\varepsilon\}}} |y|^{1+2s}\langle y\rangle^{a+b} \langle v+y\rangle^{-q}\,\mathrm{d}y \\ &\approx \langle v\rangle^{-q} \int_{\substack{y\perp h,\\ \varepsilon \geqslant |y| \geqslant |h|}} |y|^{1+2s}\langle y\rangle^{a+b} \,\mathrm{d}y + \int_{\substack{y\perp h,\\ |y|\geqslant \max\{|h|,\varepsilon\}}} |y|^{1+2s}\langle y\rangle^{a+b} \langle v+y\rangle^{-q}\,\mathrm{d}y \\ &\approx \langle v\rangle^{-q} + \int_{\substack{y\perp h,\\ |y|\geqslant \max\{|h|,\varepsilon\}}} |y|^{1+2s}\langle y\rangle^{a+b} \langle v+y\rangle^{-q}\,\mathrm{d}y. \end{aligned}$$

We conclude that $K_{\text{sing}} \approx K_{\text{reg}}$, which concludes this step. □

Lemma 2.7. *For any $\gamma \in (-\infty, 2]$ and $s \in (0,1)$, the anisotropic norm defined as*

$$\|f\|^2_{H^{s,*}_v(m)} = \|\langle v\rangle^{\gamma/2} f\|^2_{L^2_v(m)} + \int b(\cos\theta)(\mu\langle v\rangle^{-\gamma})_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v,$$

where we denoted $\mathcal{F} = m\langle v\rangle^{\gamma/2} f$, is equivalent to the following norms involving a weight function $\varphi \in L^1_v(\langle v\rangle) \cap L\log L$:

(1) $\|\langle v\rangle^{\gamma/2} f\|^2_{L^2_v(m)} + \int b(\cos\theta)\varphi_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v$, *if also* $\varphi \in L^1_v\left(\langle v\rangle^{2s}\right)$,

(2) $\|\langle v\rangle^{\gamma/2} f\|^2_{L^2_v(m)} + \int b(\cos\theta)\varphi_*\langle v\rangle^{\gamma}(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v$, *if also* $\varphi \in L^1_v\left(\langle v\rangle^{4-\gamma}\right)$,

(3) $\|\langle v\rangle^{\gamma/2} f\|^2_{L^2_v(m)} + \int b(\cos\theta)\langle v-v_*\rangle^{\gamma}\varphi_*(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v$, *if also* $\varphi \in L^1_v\left(\langle v\rangle^{4-\gamma+|\gamma|}\right)$,

where we denoted $F = mf$. Furthermore, the anisotropic norm can be compared to isotropic Sobolev norms as follows:

$$\|\langle v\rangle^{\gamma/2} f\|^2_{H^s_v(m)} \lesssim \|f\|^2_{H^{s,*}_v(m)} \lesssim \|\langle v\rangle^{\gamma/2+2s} f\|^2_{H^s_v(m)}, \tag{2.11}$$

and we have the general bound

$$\int b(\cos\theta)\langle v\rangle^{\gamma}\varphi_*(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \lesssim \|\langle v\rangle^{4-\gamma}\varphi\|_{L^1_v}\|f\|^2_{H^{s,*}_v(m)}. \tag{2.12}$$

Proof. We first establish the comparison (2.11) followed by the equivalence $\|f\|^2_{H^{s,*}_v(m)} \approx (1)$, then proceed to show (1) ≈ (2) and (2) ≈ (3).

Step 1: Proof of (2.11) and $\|f\|^2_{H^{s,*}_v(m)} \approx (1)$. From the proof of [6, Lemma 2.7, estimate of A], for some constant $c_\psi > 0$ depending on $\|\psi\|_{L\log L}$ and $\|\psi\|_{L^1_v(\langle v\rangle)}$, we have

$$\|\mathcal{F}\|^2_{L^2_v} + \int b(\cos\theta)\psi_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \geqslant c_\psi\|\mathcal{F}\|^2_{H^s_v}, \tag{2.13}$$

and from the proof of [6, Lemma 2.8], we also have

$$\int b(\cos\theta)\left(\mu\langle v\rangle^{-\gamma}\right)_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\lesssim\|\langle v\rangle^{2s}\mathcal{F}\|^2_{H^s_v}.$$

The comparison (2.11) follows from these two bounds with $\psi=\mu\langle v\rangle^{-\gamma}$. Furthermore, it was established in the proof of [28, Lemma 2.3-*(ii)*, estimate of I^δ_{11}] that

$$\begin{aligned}\int b(\cos\theta)\varphi_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v=&\frac{\|\varphi\|_{L^1_v}}{\|\mu\langle v\rangle^{-\gamma}\|_{L^1_v}}\|f\|^2_{\dot H^{s,*}_v(m)}\\&+\mathcal{O}\Big(\|\varphi\|_{L^1_v(\langle v\rangle^{2s})}\|\mathcal{F}\|^2_{H^s_v}\Big).\end{aligned}\tag{2.14}$$

Thus, interpolating this estimate with (2.13) with $\psi=\varphi$ as to absorb the $\mathcal{O}$ term, we deduce $\|f\|^2_{H^{s,*}_v(m)}\approx(1)$.

Step 2: Proof of (1) $\approx$ (2) *and* (2.12). On the one hand, we have

$$\begin{aligned}&\int b(\cos\theta)\varphi_*\langle v\rangle^\gamma(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\&\qquad=\int b(\cos\theta)\varphi_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\&\qquad\quad+\int b(\cos\theta)\varphi_*(F')^2\Big(\langle v'\rangle^{\gamma/2}-\langle v\rangle^{\gamma/2}\Big)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\&\qquad\quad+2\int b(\cos\theta)\varphi_*F'(\mathcal{F}'-\mathcal{F})\Big(\langle v'\rangle^{\gamma/2}-\langle v\rangle^{\gamma/2}\Big)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v,\end{aligned}$$

thus, using Young's inequality, we have

$$\begin{aligned}&\frac12\int b(\cos\theta)\varphi_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v-\int b(\cos\theta)\varphi_*(F')^2\Big(\langle v'\rangle^{\gamma/2}-\langle v\rangle^{\gamma/2}\Big)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\&\qquad\leqslant\int b(\cos\theta)\varphi_*\langle v\rangle^\gamma(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\&\leqslant2\int b(\cos\theta)\varphi_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v+2\int b(\cos\theta)\varphi_*(F')^2\Big(\langle v'\rangle^{\gamma/2}-\langle v\rangle^{\gamma/2}\Big)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v.\end{aligned}$$

Next, the inequality [28, (2.7)] valid for any $\alpha\leqslant1$:

$$|\langle v\rangle^\alpha-\langle v'\rangle^\alpha|\lesssim\sin\left(\tfrac{\theta}{2}\right)\langle v'\rangle^\alpha\langle v_*\rangle^{2-\alpha},$$

used with $\alpha=\gamma/2\leqslant1$ allows to tame the angular singularity, making it integrable:

$$\begin{aligned}&\int b(\cos\theta)\varphi_*(F')^2\Big(\langle v'\rangle^{\gamma/2}-\langle v\rangle^{\gamma/2}\Big)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\&\qquad\lesssim\int b(\cos\theta)\sin\left(\tfrac{\theta}{2}\right)^2\left(\varphi\langle v\rangle^{4-\gamma}\right)_*(\mathcal{F}')^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\&\qquad\lesssim\int\left(\varphi\langle v\rangle^{4-\gamma}\right)_*\mathcal{F}^2\,\mathrm{d}v_*\,\mathrm{d}v=\|\langle v\rangle^{4-\gamma}\varphi\|_{L^1_v}\|\mathcal{F}\|^2_{L^2_v},\end{aligned}$$

where we used (2.5) and then integrated in σ in the second inequality. In conclusion, we have shown

$$\begin{aligned}\int b(\cos\theta)\varphi_*\langle v\rangle^\gamma(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\approx&\int b(\cos\theta)\varphi_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\&+\mathcal{O}\left(\|\varphi^{4-\gamma}\|_{L^1_v}\|\mathcal{F}\|^2_{H^s_v}\right),\end{aligned}\tag{2.15}$$

from which (1) $\approx$ (2) follows. Combining (2.15) with (2.14), and observing that $2s\leqslant4-\gamma$, we obtain

$$\int b(\cos\theta)\varphi_*\langle v\rangle^\gamma(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\approx\|\varphi\|_{L^1_v}\|f\|^2_{\dot H^{s,*}_v(m)}+\mathcal{O}\left(\|\langle v\rangle^{4-\gamma}\varphi\|_{L^1_v}\|\mathcal{F}\|^2_{H^s_v}\right),$$

from which we deduce (2.12) thanks to (2.11).

Step 3: Proof of (2) $\approx$ (3). The equivalence is immediate thanks to the previous steps and

$$\langle v_*\rangle^{-|\gamma|}\langle v\rangle^\gamma\leqslant\langle v-v_*\rangle^\gamma\leqslant\langle v_*\rangle^{|\gamma|}\langle v\rangle^\gamma,$$

as it leads to the comparison

$$\begin{aligned}\int b(\cos\theta)\left(\varphi\langle v\rangle^{-|\gamma|}\right)_*\langle v\rangle^\gamma(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ \leqslant\int b(\cos\theta)\varphi_*\langle v-v_*\rangle^\gamma(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ \leqslant\int b(\cos\theta)\left(\varphi\langle v\rangle^{|\gamma|}\right)_*\langle v\rangle^\gamma(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v.\end{aligned}$$

This concludes the proof. □

We have seen in Lemma 2.7 that only the strength of the angular singularity and the growth of the weight in v (i.e. $\langle v\rangle^\gamma$ or $\langle v-v_*\rangle^\gamma$) up to the change of weight $\mathcal{F}\leftrightarrow F$ are the defining features of the norm $\|\cdot\|_{H^{s,*}_v}$. One could combine this result with Lemma 2.6 to show yet another equivalence:

$$\|f\|^2_{H^{s,*}_v(m)}\approx\|\langle v\rangle^{\gamma/2}f\|^2_{L^2_v(m)}+\int B(v-v_*,\sigma)\mu_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}v_*\,\mathrm{d}v\,\mathrm{d}\sigma,$$

which is the definition chosen in the series [6, 4, 5, 7, 8, 3].

2.2. **Homogeneous estimates.** We cite below the main nonlinear estimate of [7] established by Alexandre-Morimoto-Ukai-Xu-Yang, in a slightly simplified version using the fact that $\gamma+2s<0$. In order to do this we shall first introduce the exponentially weighted spaces used in [7], namely we define the spaces E and E^* as the spaces associated to the norms

$$\|f\|^2_E:=\|f\|^2_{L^2_v(\mu^{-1/2})},\tag{2.16}$$

$$\|f\|^2_{E^*}:=\|\langle v\rangle^{\gamma/2+s}f\|^2_{L^2_v(\mu^{-1/2})}+\int|v-v_*|^\gamma b(\cos\theta)\mu_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v,\tag{2.17}$$

where we denoted $\mathcal{F}:=\mathcal{F}(v)=\mu^{-1/2}(v)f(v)$ and $\mathcal{F}'=\mathcal{F}(v')$. Let us recall that this norm can be compared to isotropic Sobolev norms [7, Proposition 2.2]:

$$\|\langle v\rangle^{\gamma/2}f\|_{H^s_v(\mu^{-1/2})}\lesssim\|f\|_{E^*}\lesssim\|\langle v\rangle^{\gamma/2+s}f\|_{H^s_v(\mu^{-1/2})}.$$

Lemma 2.8 ([7, Theorem 1.2])**.** *For any smooth enough functions $f=f(v)$, $g=g(v)$ and $h=h(v)$ the following bound holds*

$$\langle Q(f,g),h\rangle_E\lesssim\|h\|_{E^*}\left(\|f\|_E\|g\|_{E^*}+\|f\|_{E^*}\|g\|_E\right).$$

In particular we have

$$\langle Q(g,f),f\rangle_E\lesssim\|g\|_E\|f\|^2_{E^*}+\|g\|_{E^*}\|f\|_{E^*}\|f\|_E.$$

The goal of this section is to establish similar estimates in spaces with polynomial weights, which we state below.

Proposition 2.9. *Assume $k>9/2-|\gamma|/2+2s$ and consider $m=\langle v\rangle^k$. For any $\ell>13/2+2|\gamma|$ and smooth enough functions f,g,h there holds*

$$\begin{aligned}&\langle Q(f,g),h\rangle_{L^2_v(m)}\\ &\quad\lesssim\left(\|\langle v\rangle^{\gamma/2}f\|_{L^2_v(m)}\|g\|_{H^{s,*}_v(\langle v\rangle^\ell)}+\|f\|_{H^{s,*}_v(\langle v\rangle^\ell)}\|g\|_{H^{s,*}_v(m)}\right)\|\langle v\rangle^{\gamma/2}h\|_{L^2_v(m)}\\ &\qquad+\|f\|_{L^2_v(\langle v\rangle^\ell)}\|\langle v\rangle^{2s}g\|_{H^{s,*}_v(m)}\|h\|_{H^{s,*}_v(m)}\end{aligned}\tag{2.18}$$

Moreover there holds

$$\begin{aligned}&\langle Q(g,f),f\rangle_{L^2_v(m)}\\ &\quad\lesssim\|g\|_{L^2_v(\langle v\rangle^\ell)}\|f\|^2_{H^{s,*}_v(m)}+\|g\|_{H^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}f\|_{L^2_v(m)}\|f\|_{H^{s,*}_v(m)}\\ &\qquad+\|\langle v\rangle^{\gamma/2}g\|_{L^2_v(m)}\|f\|_{H^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}f\|_{L^2_v(m)}.\end{aligned}\tag{2.19}$$

These estimates will be proved by combining commutator estimates (Lemma 2.11) with He's estimates in L^2_v (Lemma 2.10, for (2.18)) or new anisotropic estimates in $L^2(m)$ (for (2.19)). Let us start by recalling the estimate established in He [27].

Lemma 2.10 ([27, Theorem 1.1]). *Assume $-1 < \gamma + 2s < 0$. For any $w_1, w_1 \geqslant 0$ such that $w_1 + w_2 = \gamma + 2s$ and $a, b \in [0, 2s]$ such that $a + b = 2s$, any $\ell_0 > 3/2 + |\gamma + 2s|$, there holds*

$$\langle Q(f,g), h\rangle_{L^2_v} \lesssim \|f\|_{L^2_v(\langle v\rangle^{\ell_0})}\|g\|_{H^a_v(\langle v\rangle^{w_1})}\|h\|_{H^b_v(\langle v\rangle^{w_2})}.$$

Let us now state and prove the commutator estimates required to prove Proposition 2.9.

Lemma 2.11. *Suppose $k > 9/2 - |\gamma|/2 + 2s$ and consider $m = \langle v\rangle^k$. For any smooth enough functions f, g, h and any $\ell > 13/2 + 2|\gamma|$ there holds*

$$\begin{aligned}|\langle Q(f,g), h\rangle_{L^2_v(m)} - \langle Q(f, mg), mh\rangle_{L^2_v}| \\ \lesssim \|\langle v\rangle^{\gamma/2}h\|_{L^2_v(m)}\Big(\|g\|_{H^{s,*}_v(m)}\|f\|_{H^{s,*}_v(\langle v\rangle^\ell)} + \|g\|_{H^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}f\|_{L^2_v(m)} \\ + \|g\|_{H^{s,*}_v(m)}\|f\|_{L^2_v(\langle v\rangle^\ell)}\Big).\end{aligned}$$

Proof. We shall adapt the proof of [9, Proposition 3.1] where the hard potentials case $\gamma \in [0, 1]$ was considered. We start back from their decomposition:

$$\langle Q(f,g), h\rangle_{L^2_v(m)} - \langle Q(f, mg), mh\rangle_{L^2_v} = \int B(v - v_*, \sigma) f'_* g' H(m - m') \,\mathrm{d}\sigma \,\mathrm{d}v_* \,\mathrm{d}v = \sum_{j=1}^6 \Gamma_j,$$

where the terms Γ_j are defined in the proof of [9, Proposition 3.1] and, under the assumption $k > 9/2 - |\gamma|/2 + 2s$, were shown to satisfy

$$\Gamma_1 = k \int b(\cos\theta)|v - v_*|^\gamma \langle v\rangle^{k-2}|v - v_*|(v_* \cdot \omega)\cos^{k-1}\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\theta}{2}\right) f_* g \,(mh)' \,\mathrm{d}\sigma \,\mathrm{d}v_* \,\mathrm{d}v,$$

where

$$\omega = \frac{\sigma - (\sigma\cdot\xi)\xi}{|\sigma - (\sigma\cdot\xi)\xi|} \quad \text{and} \quad \xi = \frac{v - v_*}{|v - v_*|},$$

as well as the bounds

$$\begin{aligned}\Gamma_2 &\lesssim I\left(g; (mf)^2\right)^{1/2} \times I\left(g; (mh)^2\right), \\ \Gamma_3 &\lesssim I\left(\langle v\rangle g; \left(\langle v\rangle^{k-1} f\right)^2\right)^{1/2} \times I\left(\langle v\rangle g; (mh)^2\right)^{1/2}, \\ \Gamma_4 &\lesssim I\left(\langle v\rangle^2 f; (\langle v\rangle^{k-2} g)^2\right) \times I\left(\langle v\rangle^2 f; (mh)^2\right), \\ \Gamma_5 &\lesssim I\left(\langle v\rangle^4 f; \left(\langle v\rangle^{k-4} g\right)^2\right)^{1/2} \times I\left(\langle v\rangle^4 f; (mh)^2\right)^{1/2}, \\ \Gamma_6 &\lesssim I\left(f; (mg)^2\right)^{1/2} \times I\left(f; (mh)^2\right)^{1/2},\end{aligned}$$

where we have denoted for compactness

$$I(\varphi; \Phi) := \int |v - v_*|^\gamma \varphi_* \Phi \,\mathrm{d}v \,\mathrm{d}v_*.$$

First, in virtue of (2.2), we have for $\ell_0 > 4 + 3/2 + s$ the following estimate:

$$\sum_{j=2}^6 \Gamma_j \lesssim \|\langle v\rangle^{\gamma/2}h\|_{L^2_v(m)}\left(\|f\|_{H^s_v(\langle v\rangle^{\ell_0})}\|\langle v\rangle^{\gamma/2}g\|_{L^2_v(m)} + \|g\|_{H^s_v(\langle v\rangle^{\ell_0})}\|\langle v\rangle^{\gamma/2}f\|_{L^2_v(m)}\right).$$

It remains to estimate the term Γ_1. Still following the proof of [9, Proposition 3.1]), we denote $\tilde\omega = \frac{v' - v}{|v' - v|}$, so that $\tilde\omega$ is orthogonal to $v' - v_*$, and thus we split $\Gamma_1 = \Gamma_{1,1} + \Gamma_{1,2}$ with

$$\Gamma_{1,1} = k \int b(\cos\theta)\cos^k\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\theta}{2}\right)|v - v_*|^{\gamma+1}\langle v\rangle^{k-2}(v_* \cdot \omega) f_* g \,(mh)' \,\mathrm{d}\sigma \,\mathrm{d}v_* \,\mathrm{d}v,$$

and

$$\Gamma_{1,2} = k \int b(\cos\theta)\cos^{k-1}\left(\tfrac{\theta}{2}\right)\sin^2\left(\tfrac{\theta}{2}\right)|v - v_*|^{\gamma+1}\langle v\rangle^{k-2}(v_* \cdot \tilde\omega) f_* g \,(mh)' \,\mathrm{d}\sigma \,\mathrm{d}v_* \,\mathrm{d}v.$$

For the term $\Gamma_{1,2}$ we can then argue as for the terms $(\Gamma_j)_{2\leqslant j\leqslant 6}$ and we obtain

$$\begin{aligned}\Gamma_{1,2} &\lesssim I\left(\langle v\rangle^2 f;(\langle v\rangle^{k-1}g)^2\right)\times I\left(\langle v\rangle^2 f;(mh)^2\right)\\ &\lesssim \|\langle v\rangle^{\gamma/2}h\|_{L^2_v(m)}\|f\|_{H^s_v(\langle v\rangle^{\ell_0})}\|\langle v\rangle^{\gamma/2}g\|_{L^2_v(m)}.\end{aligned}$$

Moreover the term $\Gamma_{1,1}$ is shown to satisfy, denoting $\mathcal{G}:=\langle v\rangle^{k-2}g$,

$$\Gamma_{1,1}\lesssim\int b(\cos\theta)\sin\left(\tfrac{\theta}{2}\right)|v-v_*|^\gamma\langle v_*\rangle^2|f_*||\mathcal{G}-\mathcal{G}'||m'h'|\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v.$$

Using Hölder's inequality for some $n>0$ to choose later,

$$\begin{aligned}\Gamma_{1,1}\lesssim&\int b^{1/2}(\cos\theta)|v-v_*|^{\gamma/2}\sin\left(\tfrac{\theta}{2}\right)(mh)'\left(\langle v\rangle^2 f\right)_*^{1/2}\\ &\times b^{1/4}(\cos\theta)\langle v-v_*\rangle^{\gamma/4}\langle v\rangle^{1/2}\left|\mathcal{G}-\mathcal{G}'\right|^{1/2}\left(\langle v\rangle^{2+n}f\right)_*^{1/2}\\ &\times b^{1/4}(\cos\theta)|v-v_*|^{\gamma/2}\langle v-v_*\rangle^{-\gamma/4}\langle v\rangle^{1/2}\left|\mathcal{G}-\mathcal{G}'\right|^{1/2}\langle v_*\rangle^{-n/2}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ \lesssim&\left(\int b(\cos\theta)|v-v_*|^\gamma\sin^2\left(\tfrac{\theta}{2}\right)\left[(mh)'\right]^2\left(\langle v\rangle^2 f\right)_*\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/2}\\ &\times\left(\int b(\cos\theta)\langle v-v_*\rangle^\gamma\langle v\rangle^2\left(\mathcal{G}-\mathcal{G}'\right)^2\left(\langle v\rangle^{2+n}f\right)_*^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/4}\\ &\times\left(\int b(\cos\theta)|v-v_*|^{2\gamma}\langle v-v_*\rangle^{-\gamma}\langle v\rangle^2\left(\mathcal{G}-\mathcal{G}'\right)^2\langle v_*\rangle^{-2n}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/4}.\end{aligned}$$

The change of variable (2.5) followed by (2.2) in the first integral with $\ell_1>2+3/2+s$ gives

$$\begin{aligned}\Gamma_{1,1}\lesssim&\|f\|^{1/2}_{H^s_v(\langle v\rangle^{\ell_1})}\|\langle v\rangle^{\gamma/2}h\|_{L^2_v(m)}\\ &\times\left(\int b(\cos\theta)\langle v-v_*\rangle^\gamma\langle v\rangle^2\left(\mathcal{G}-\mathcal{G}'\right)^2\left(\langle v\rangle^{2+n}f\right)_*^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/4}\\ &\times\left(\int b(\cos\theta)|v-v_*|^{2\gamma}\langle v-v_*\rangle^{-\gamma}\langle v\rangle^2\left(\mathcal{G}-\mathcal{G}'\right)^2\langle v_*\rangle^{-2n}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/4}.\end{aligned}$$

Next, using Lemma 2.6 in the third integral, taking $2n>1+2s+\gamma$ (note that $2\gamma>-3$), we have

$$\begin{aligned}\Gamma_{1,1}\lesssim&\|f\|^{1/2}_{H^s_v(\langle v\rangle^{\ell_1})}\|\langle v\rangle^{\gamma/2}h\|_{L^2_v(m)}\\ &\times\left(\int b(\cos\theta)\langle v-v_*\rangle^\gamma\langle v\rangle^2\left(\mathcal{G}-\mathcal{G}'\right)^2\left(\langle v\rangle^{2+n}f\right)_*^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/4}\\ &\times\left(\int b(\cos\theta)\langle v-v_*\rangle^\gamma\langle v\rangle^2\left(\mathcal{G}-\mathcal{G}'\right)^2\langle v_*\rangle^{-2n}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/4}.\end{aligned}$$

The inequality $\langle v\rangle\leqslant\langle v-v_*\rangle\langle v_*\rangle$ and the fact that $\gamma\leqslant 0$ then imply

$$\begin{aligned}\Gamma_{1,1}\lesssim&\|f\|^{1/2}_{H^s_v(\langle v\rangle^{\ell_1})}\|\langle v\rangle^{\gamma/2}h\|_{L^2_v(m)}\\ &\times\left(\int b(\cos\theta)\langle v\rangle^{2+\gamma}\left(\mathcal{G}-\mathcal{G}'\right)^2\left(\langle v\rangle^{2+n-\gamma/2}f\right)_*^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/4}\\ &\times\left(\int b(\cos\theta)\langle v\rangle^{2+\gamma}\left(\mathcal{G}-\mathcal{G}'\right)^2\langle v_*\rangle^{-2n-\gamma}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/4}.\end{aligned}$$

As $2+\gamma\leqslant 2$, we may use (2.12) to bound these two integrals:

$$\begin{aligned}\Gamma_{1,1}&\lesssim\|f\|^{1/2}_{H^s_v(\langle v\rangle^{\ell_1})}\|\langle v\rangle^{\gamma/2}h\|_{L^2_v(m)}\|g\|_{H^{s,*}_v(m)}\|\langle v\rangle^{2-\gamma}\left(\langle v\rangle^{2+n-\gamma/2}f\right)^2\|^{1/4}_{L^1_v}\|\langle v\rangle^{-2(n-2+\gamma)}\|^{1/4}_{L^1_v}\\ &\lesssim\|f\|^{1/2}_{H^s_v(\langle v\rangle^{\ell_1})}\|\langle v\rangle^{\gamma/2}h\|_{L^2_v(m)}\|g\|_{H^{s,*}_v(m)}\|\langle v\rangle^{\ell_2}f\|^{1/2}_{L^2_v},\end{aligned}$$

where we considered $n>7/2+|\gamma|$ and $\ell_2=3+n+|\gamma|>13/2+2|\gamma|$. Note that since $13/2+2|\gamma|=11/2-\gamma/2+s+(-\gamma-\frac{\gamma+2s}{2}+1)$ and $\gamma+2s<0$, we have $\ell_2>\ell_0+|\gamma|/2$.

We conclude the proof by gathering previous estimates, using (2.11), and taking $\ell = \max(\ell_0 + |\gamma|/2, \ell_1, \ell_2) = \ell_2$. □

We can now prove the main estimates of this subsection, that is to say those of Proposition 2.9.

Proof of Proposition 2.9. In this proof, we denote $F := mf$, $G := mg$ and $H := mh$.

Step 1: Proof of (2.18). The first estimate (2.18) is a combination of Lemmas 2.10 and 2.11. We first observe that Lemma 2.10 with $w_1 = k + \gamma/2 + 2s$, $w_2 = k + \gamma/2$ and $a = b = s$ yields, for $\ell_0 > 3/2 + |\gamma + 2s|$,

$$\begin{aligned}\langle Q(f,G),H\rangle_{L^2_v} &\lesssim \|f\|_{L^2_v(\langle v\rangle^{\ell_0})}\|\langle v\rangle^{\gamma/2+2s}g\|_{H^s_v(m)}\|\langle v\rangle^{\gamma/2}h\|_{H^s_v(m)}\\ &\lesssim \|f\|_{L^2(\langle v\rangle^{\ell_0})}\|\langle v\rangle^{2s}g\|_{H^{s,*}_v(m)}\|h\|_{H^{s,*}_v(m)},\end{aligned}$$

where we used (2.11) in the last line. We then deduce (2.18) by putting this estimate together with Lemma 2.11 for $\ell > 13/2 + 2|\gamma| > \ell_0$.

Step 2: Reductions for the proof of (2.19). First, we decompose the trilinear form using a commutator:

$$\langle Q(g,f),f\rangle_{L^2_v(m)} = \langle Q(g,F),F\rangle_{L^2_v} + \mathbf{I}_3,$$

where we denoted

$$\mathbf{I}_3 := \langle Q(g,f),f\rangle_{L^2_v(m)} - \langle Q(g,F),F\rangle_{L^2_v}.$$

Second, we decompose the remaining term as

$$\begin{aligned}\langle Q(g,F),F\rangle_{L^2_v(m)} &= \int B(v-v_*,\sigma)(g'_*F' - g_*F)F\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v_*\\ &= \frac12\int B(v-v_*,\sigma)(2g'_*F'F - g_*F^2 - g_*(F')^2)\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v_*\\ &\quad + \frac12\int B(v-v_*,\sigma)g_*((F')^2 - F^2)\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v_*.\end{aligned}$$

Using the change of variables (2.6) in the first term of the first integral and the cancellation lemma 2.4 in the second integral, we obtain for some $c > 0$

$$\begin{aligned}\langle Q(g,F),F\rangle_{L^2_v(m)} &= -\int B(v-v_*,\sigma)g_*(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v_* - c\int |v-v_*|^\gamma g_*F^2\,\mathrm{d}v\,\mathrm{d}v_*\\ &=: \mathbf{I}_1 + \mathbf{I}_2,\end{aligned}$$

To sum up, we have the decomposition

$$\langle Q(g,f),f\rangle_{L^2_v(m)} = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3.$$

The term $\mathbf{I}_2$ satisfies by Lemma 2.1, for any $\ell_0 > 3/2 + s$,

$$\mathbf{I}_2 \lesssim \|g\|_{H^s_v(\langle v\rangle^{\ell_0})}\|\langle v\rangle^{\gamma/2}f\|^2_{L^2(m)},$$

and the term $\mathbf{I}_3$ satisfies by Lemma 2.11, for any $\ell > 13/2 + 2|\gamma|$,

$$\begin{aligned}\mathbf{I}_3 \lesssim \|\langle v\rangle^{\gamma/2}f\|_{L^2_v(m)}\Big(&\|f\|_{H^{s,*}_v(m)}\|g\|_{H^{s,*}_v(\langle v\rangle^\ell)} + \|f\|_{H^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}g\|_{L^2_v(m)}\\ &+ \|f\|_{H^{s,*}_v(m)}\|g\|_{L^2_v(\langle v\rangle^\ell)}\Big),\end{aligned}$$

so that, since $\ell > \ell_0 + |\gamma|/2$, the sum $\mathbf{I}_2 + \mathbf{I}_3$ satisfies the same estimate as $\mathbf{I}_3$.

Let us turn to $\mathbf{I}_1$. Using the Cauchy-Schwarz inequality with some positive $q > 0$ to be chosen later, we have

$$\begin{aligned}\mathbf{I}_1 &= \int b(\cos\theta)|v-v_*|^\gamma g_*(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v_*\\ &\leqslant \left(\int b(\cos\theta)|v-v_*|^{2\gamma}\langle v-v_*\rangle^{-\gamma}\langle v_*\rangle^{-q}(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v_*\right)^{1/2}\\ &\quad\times\left(\int b(\cos\theta)\langle v-v_*\rangle^\gamma\left(\langle v\rangle^{q/2}g\right)_*^2(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v_*\right)^{1/2}.\end{aligned}$$

Assuming $q > 3+\gamma+2s$, we remove the singularity in the integral prefactor using Lemma 2.6, and then use the inequality $\langle v - v_*\rangle^\gamma \leqslant \langle v\rangle^\gamma \langle v_*\rangle^{-\gamma}$ in both integrals:

$$\begin{aligned}
\mathbf{I}_1 &\lesssim \left(\int b(\cos\theta) \langle v - v_*\rangle^\gamma \langle v_*\rangle^{-q} (F' - F)^2 \,\mathrm{d}\sigma \,\mathrm{d}v \,\mathrm{d}v_* \right)^{1/2} \\
&\quad \times \left(\int b(\cos\theta) \langle v - v_*\rangle^\gamma \left(\langle v\rangle^{q/2} g \right)_*^2 (F' - F)^2 \,\mathrm{d}\sigma \,\mathrm{d}v \,\mathrm{d}v_* \right)^{1/2} \\
&\lesssim \left(\int b(\cos\theta) \langle v_*\rangle^{-q-\gamma} \langle v\rangle^\gamma (F' - F)^2 \,\mathrm{d}\sigma \,\mathrm{d}v \,\mathrm{d}v_* \right)^{1/2} \\
&\quad \times \left(\int b(\cos\theta) \left(\langle v\rangle^{q/2-\gamma/2} g \right)_*^2 \langle v\rangle^\gamma (F' - F)^2 \,\mathrm{d}\sigma \,\mathrm{d}v \,\mathrm{d}v_* \right)^{1/2}.
\end{aligned}$$

Using (2.12) and imposing $q > 7 - 2\gamma$ so that $\langle v\rangle^{4-q-2\gamma}$ is integrable, we then obtain

$$\begin{aligned}
\mathbf{I}_1 &\lesssim \|f\|^2_{H^{s,*}_v(m)} \| \langle v\rangle^{4-\gamma} \left(\langle v\rangle^{q/2} g \right)^2 \|^{1/2}_{L^1_v} \\
&\lesssim \|f\|^2_{H^{s,*}_v(m)} \| \langle v\rangle^{\ell_1} g \|_{L^2_v},
\end{aligned}$$

where $\ell_1 = 2 - \gamma/2 + q/2$ satisfies $\ell_1 > 11/2 + 2|\gamma|$ and $\ell_1 > 7/2 + 3|\gamma|/2 + s$. The estimate (2.19) is then proved by putting together previous estimates and observing that $\max(\ell, \ell_1) = \ell$. □

2.3. **Inhomogeneous estimates in the non-cutoff case.** We shall prove inhomogeneous nonlinear estimates for the collision operator Q by using the homogeneous estimates proven in previous Subsection 2.2. More precisely we shall prove bilinear estimates (Proposition 2.12) and trilinear estimates (Proposition 2.13 and Proposition 2.14), for polynomially weighted spaces $\mathbf{X}(m)$ as well as for exponentially weighted spaces $\mathbf{E}$ (defined below). It is worth mentioning that, because of our strategy employed in Section 4, some of these estimates are of *mixed type*, that is, they involve one function in a polynomially weighted space $\mathbf{X}(m)$ and another function in a exponentially weighted space $\mathbf{E}$.

We introduce the exponentially weighted spaces $\mathbf{E}$ and $\mathbf{E}^*$ as the spaces associated to the norms

$$\|f\|^2_{\mathbf{E}} := \|f\|^2_{L^2_x L^2_v(\mu^{-1/2})} + \|\nabla^3_x f\|^2_{L^2_x L^2_v(\mu^{-1/2})} \tag{2.20}$$

and, respectively,

$$\|f\|^2_{\mathbf{E}^*} := \|\nabla_x \pi f\|^2_{H^2_x L^2_v(\mu^{-1/2})} + \|f^\perp\|^2_{L^2_x E^*} + \|\nabla^3_x f^\perp\|^2_{L^2_x E^*}, \tag{2.21}$$

where we recall that E^* is defined in (2.17), which are similar to those studied in [34].

We start by proving the estimates we will use to prove the stability of the iterative scheme from Section 4.

Proposition 2.12. *Assume $k > 13/2 + 2|\gamma| + 6s$ and consider $m = \langle v\rangle^k$. For any $f, g \in \mathbf{X}(m) \cap \mathbf{X}^*(m)$ there holds*

$$\langle Q(g,f), f\rangle_{\mathbf{X}(m)} \lesssim \|f\|^2_{\mathbf{X}^*(m)} \|g\|_{\mathbf{X}(m)} + \|f\|_{\mathbf{X}(m)} \|f\|_{\mathbf{X}^*(m)} \|g\|_{\mathbf{X}^*(m)}. \tag{2.22}$$

Moreover for any $f \in \mathbf{X}(m) \cap \mathbf{X}^*(m)$ and $g \in \mathbf{E} \cap \mathbf{E}^*$ there holds

$$\langle Q(g,f), f\rangle_{\mathbf{X}(m)} \lesssim \|f\|^2_{\mathbf{X}^*(m)} \|g\|_{\mathbf{E}} + \|f\|_{\mathbf{X}(m)} \|f\|_{\mathbf{X}^*(m)} \|g\|_{\mathbf{E}^*}. \tag{2.23}$$

Proof. Let us start by expanding the inner product defining the norm of $\mathbf{X}(m)$ in (1.10)

$$\langle Q(g,f), f\rangle_{\mathbf{X}(m)} = \langle Q(g,f), f\rangle_{L^2_x L^2_v(m)} + \langle \nabla^3_x Q(g,f), \nabla^3_x f\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})},$$

thus we get

$$\begin{aligned}
&\langle Q(g,f), f\rangle_{\mathbf{X}(m)} \\
&\lesssim \left| \langle Q(g,f), f\rangle_{L^2_x L^2_v(m)} \right| + \sum_{|\alpha|=3} \sum_{0 \leqslant \beta \leqslant \alpha} \left| \langle Q(\partial_x^{\alpha-\beta} g, \partial_x^\beta f), \partial_x^\alpha f\rangle_{L^2_x L^2_v(m\langle v\rangle^{-2|\alpha|s})} \right|,
\end{aligned} \tag{2.24}$$

and we shall estimate each term separately.

We fix some $\ell > 13/2 + 2|\gamma|$ such that $k \geqslant \ell + 6s$, and observe that in particular we can apply both estimates of Proposition 2.9 with the weight $\langle v\rangle^{k-6s}$ in the sequel.

Step 1: General estimates of (2.24) in L^p_x-norms. The first term in (2.24) is bounded by integrating estimate (2.19) in space and using Hölder's inequality $L^\infty_x - L^2_x - L^2_x$, which yields:

$$\begin{aligned}
&\langle Q(g,f),f\rangle_{L^2_xL^2_v(m)}\\
&\quad\lesssim \|g\|_{L^\infty_x L^2_v(\langle v\rangle^\ell)}\|f\|^2_{L^2_xH^{s,*}_v(m)}\\
&\qquad+\|g\|_{L^\infty_x H^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}f\|_{L^2_xL^2_v(m)}\|f\|_{L^2_xH^{s,*}_v(m)}\\
&\qquad+\|\langle v\rangle^{\gamma/2}g\|_{L^2_xL^2_v(m)}\|f\|_{L^\infty_x H^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}f\|_{L^2_xL^2_v(m)}.
\end{aligned}\tag{2.25}$$

We bound the second term in (2.24) depending on the value of β. When $\beta = \alpha$, we obtain using (2.19) a similar estimate as before:

$$\begin{aligned}
&\langle Q(g,\partial^\alpha_x f),\partial^\alpha_x f\rangle_{L^2_xL^2_v(m\langle v\rangle^{-6s})}\\
&\quad\lesssim \|g\|_{L^\infty_x L^2_v(\langle v\rangle^\ell)}\|\nabla^3_x f\|^2_{L^2_xH^{s,*}_v(m\langle v\rangle^{-6s})}\\
&\qquad+\|g\|_{L^\infty_x H^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}\nabla^3_x f\|_{L^2_xL^2_v(m\langle v\rangle^{-6s})}\|\nabla^3_x f\|_{L^2_xH^{s,*}_v(m\langle v\rangle^{-6s})}\\
&\qquad+\|\langle v\rangle^{\gamma/2}g\|_{L^\infty_x L^2_v(m\langle v\rangle^{-6s})}\|\nabla^3_x f\|_{L^2_xH^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}\nabla^3_x f\|_{L^2_xL^2_v(m\langle v\rangle^{-6s})}.
\end{aligned}\tag{2.26}$$

When $\beta = 0$, we use (2.18) that we integrate in space and using again Hölder's inequality $L^\infty_x - L^2_x - L^2_x$, which gives

$$\begin{aligned}
&\langle Q(\partial^\alpha_x g,f),\partial^\alpha_x f\rangle_{L^2_xL^2_v(m\langle v\rangle^{-6s})}\\
&\quad\lesssim \|\langle v\rangle^{\gamma/2}\nabla^3_x g\|_{L^2_xL^2_v(m\langle v\rangle^{-6s})}\|f\|_{L^\infty_x H^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}\nabla^3_x f\|_{L^2_xL^2_v(m\langle v\rangle^{-6s})}\\
&\qquad+\|\nabla^3_x g\|_{L^2_xH^{s,*}_v(\langle v\rangle^\ell)}\|f\|_{L^\infty_x H^{s,*}_v(m\langle v\rangle^{-6s})}\|\langle v\rangle^{\gamma/2}\nabla^3_x f\|_{L^2_xL^2_v(m\langle v\rangle^{-6s})}\\
&\qquad+\|\nabla^3_x g\|_{L^2_xL^2_v(\langle v\rangle^\ell)}\|f\|_{L^\infty_x H^{s,*}_v(m\langle v\rangle^{-4s})}\|\nabla^3_x f\|_{L^2_xH^{s,*}_v(m\langle v\rangle^{-6s})}.
\end{aligned}\tag{2.27}$$

When $|\beta| = 1$, we we integrate estimate (2.18) using Hölder's inequality $L^2_x - L^4_x - L^4_x$:

$$\begin{aligned}
&\langle Q(\partial^{\alpha-\beta}_x g,\partial^\beta_x f),\partial^\alpha_x f\rangle_{L^2_xL^2_v(m\langle v\rangle^{-6s})}\\
&\quad\lesssim \|\langle v\rangle^{\gamma/2}\nabla^2_x g\|_{L^4_xL^2_v(m\langle v\rangle^{-6s})}\|\nabla_x f\|_{L^4_xH^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}\nabla^3_x f\|_{L^2_xL^2_v(m\langle v\rangle^{-6s})}\\
&\qquad+\|\nabla^2_x g\|_{L^4_xH^{s,*}_v(\langle v\rangle^\ell)}\|\nabla_x f\|_{L^4_xH^{s,*}_v(m\langle v\rangle^{-6s})}\|\langle v\rangle^{\gamma/2}\nabla^3_x f\|_{L^2_xL^2_v(m\langle v\rangle^{-6s})}\\
&\qquad+\|\nabla^2_x g\|_{L^4_xL^2_v(\langle v\rangle^\ell)}\|\nabla_x f\|_{L^4_xH^{s,*}_v(m\langle v\rangle^{-4s})}\|\nabla^3_x f\|_{L^2_xH^{s,*}_v(m\langle v\rangle^{-6s})}.
\end{aligned}\tag{2.28}$$

For $|\beta| = 2$ we integrate again (2.18) in space using Hölder's inequality $L^\infty_x - L^2_x - L^2_x$, which yields

$$\begin{aligned}
&\langle Q(\partial^{\alpha-\beta}_x g,\partial^\beta_x f),\partial^\alpha_x f\rangle_{L^2_xL^2_v(m\langle v\rangle^{-6s})}\\
&\quad\lesssim \|\langle v\rangle^{\gamma/2}\nabla_x g\|_{L^\infty_x L^2_v(m\langle v\rangle^{-6s})}\|\nabla^2_x f\|_{L^2_xH^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}\nabla^3_x f\|_{L^2_xL^2_v(m\langle v\rangle^{-6s})}\\
&\qquad+\|\nabla_x g\|_{L^\infty_x H^{s,*}_v(\langle v\rangle^\ell)}\|\nabla^2_x f\|_{L^2_xH^{s,*}_v(m\langle v\rangle^{-6s})}\|\langle v\rangle^{\gamma/2}\nabla^3_x f\|_{L^2_xL^2_v(m\langle v\rangle^{-6s})}\\
&\qquad+\|\nabla_x g\|_{L^\infty_x L^2_v(\langle v\rangle^\ell)}\|\nabla^2_x f\|_{L^2_xH^{s,*}_v(m\langle v\rangle^{-4s})}\|\nabla^3_x f\|_{L^2_xH^{s,*}_v(m\langle v\rangle^{-6s})}.
\end{aligned}\tag{2.29}$$

Step 2: Sobolev embeddings for (2.22). We first observe that

$$\begin{aligned}
\|f\|_{\mathbf{X}(m)} \approx{}& \|f\|_{L^2_xL^2_v(m)} + \|\langle v\rangle^{-2s}\nabla_x f\|_{L^2_xL^2_v(m)}\\
&+\|\langle v\rangle^{-4s}\nabla^2_x f\|_{L^2_xL^2_v(m)} + \|\langle v\rangle^{-6s}\nabla^3_x f\|_{L^2_xL^2_v(m)}
\end{aligned}$$

and

$$\begin{aligned}
\|f\|_{\mathbf{X}^*(m)} \approx{}& \|f\|_{L^2_xH^{s,*}_v(m)} + \|\langle v\rangle^{-2s}\nabla_x f\|_{L^2_xH^{s,*}_v(m)}\\
&+\|\langle v\rangle^{-4s}\nabla^2_x f\|_{L^2_xH^{s,*}_v(m)} + \|\langle v\rangle^{-6s}\nabla^3_x f\|_{L^2_xH^{s,*}_v(m)}
\end{aligned}$$

Moreover, since $k \geqslant \ell + 6s$, we have

$$\|g\|_{H^2_xL^2_v(\langle v\rangle^\ell)} + \|\nabla_x g\|_{H^1_xL^2_v(\langle v\rangle^\ell)} + \|\nabla^2_x g\|_{H^1_xL^2_v(\langle v\rangle^\ell)} + \|\nabla^3_x g\|_{L^2_xL^2_v(\langle v\rangle^\ell)} \lesssim \|g\|_{\mathbf{X}(m)}$$

and

$$\|g\|_{H^2_x H^{s,*}_v(\langle v\rangle^\ell)} + \|\nabla_x g\|_{H^1_x H^{s,*}_v(\langle v\rangle^\ell)} + \|\nabla^2_x g\|_{H^1_x H^{s,*}_v(\langle v\rangle^\ell)} + \|\nabla^3_x g\|_{L^2_x H^{s,*}_v(\langle v\rangle^\ell)} \lesssim \|g\|_{\mathbf{X}^*(m)}.$$

From (2.25), using the Sobolev embedding $H^2_x(\mathbf{R}^3) \hookrightarrow L^\infty_x(\mathbf{R}^3)$ we thus get

$$\langle Q(g,f), f\rangle_{L^2_x L^2_v(m)} \lesssim \|f\|^2_{\mathbf{X}^*(m)}\|g\|_{\mathbf{X}(m)} + \|f\|_{\mathbf{X}(m)}\|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{X}^*(m)}.$$

Arguing similarly from (2.26) we obtain

$$\langle Q(g,\partial^\alpha_x f), \partial^\alpha_x f\rangle_{L^2_x L^2_v(m)} \lesssim \|f\|^2_{\mathbf{X}^*(m)}\|g\|_{\mathbf{X}(m)} + \|f\|_{\mathbf{X}(m)}\|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{X}^*(m)}.$$

For (2.27), we use again $H^2_x(\mathbf{R}^3) \hookrightarrow L^\infty_x(\mathbf{R}^3)$ and

$$\|f\|_{H^2_x H^{s,*}_v(m\langle v\rangle^{-4s})} \lesssim \|f\|_{\mathbf{X}^*(m)}$$

to deduce

$$\langle Q(\partial^\alpha_x g, f), \partial^\alpha_x f\rangle_{L^2_x L^2_v(m)} \lesssim \|f\|^2_{\mathbf{X}^*(m)}\|g\|_{\mathbf{X}(m)} + \|f\|_{\mathbf{X}(m)}\|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{X}^*(m)}.$$

For $|\beta| = 1$, from (2.28) and the Sobolev embedding $H^1_x(\mathbf{R}^3) \hookrightarrow L^4_x(\mathbf{R}^3)$, we remark that

$$\|\nabla_x f\|_{H^1_x H^{s,*}_v(m\langle v\rangle^{-4s})} \lesssim \|f\|_{\mathbf{X}^*(m)},$$

hence we get

$$\langle Q(\partial^{\alpha-\beta}_x g, \partial^\beta_x f), \partial^\alpha_x f\rangle_{L^2_x L^2_v(m)} \lesssim \|f\|^2_{\mathbf{X}^*(m)}\|g\|_{\mathbf{X}(m)} + \|f\|_{\mathbf{X}(m)}\|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{X}^*(m)}.$$

Finally, for the case $|\beta| = 2$, estimate (2.29) together with $H^2_x(\mathbf{R}^3) \hookrightarrow L^\infty_x(\mathbf{R}^3)$ yields

$$\langle Q(\partial^{\alpha-\beta}_x g, \partial^\beta_x f), \partial^\alpha_x f\rangle_{L^2_x L^2_v(m)} \lesssim \|f\|^2_{\mathbf{X}^*(m)}\|g\|_{\mathbf{X}(m)} + \|f\|_{\mathbf{X}(m)}\|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{X}^*(m)}.$$

This concludes the proof of (2.22).

Step 2: Proof of estimate (2.23). We first remark that

$$\|g\|_{H^2_x H^{s,*}_v(\langle v\rangle^\ell)} \lesssim \|\pi g\|_{H^2_x L^2_v} + \|g^\perp\|_{H^2_x H^{s,*}_v(\langle v\rangle^\ell)} \lesssim \|g\|_{\mathbf{E}} + \|g\|_{\mathbf{E}^*},$$

and that

$$\|\langle v\rangle^{\gamma/2} g\|_{L^2_x L^2_v(m)} + \|\langle v\rangle^{\gamma/2} g\|_{H^2_x L^2_v(m\langle v\rangle^{-6s})} \lesssim \|g\|_{\mathbf{E}}.$$

Moreover

$$\|\langle v\rangle^{\gamma/2} f\|_{L^2_x L^2_v(m)} + \|\langle v\rangle^{\gamma/2}\nabla^3_x f\|_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \lesssim \min\{\|f\|_{\mathbf{X}(m)}, \|f\|_{\mathbf{X}^*(m)}\}.$$

Therefore from (2.25) we get

$$\langle Q(g,f), f\rangle_{L^2_x L^2_v(m)} \lesssim \|f\|^2_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}} + \|f\|_{\mathbf{X}(m)}\|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}^*},$$

and furthermore, for the case $\beta = 0$, we deduce from (2.26)

$$\langle Q(g,\partial^\alpha_x f), \partial^\alpha_x f\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \lesssim \|f\|^2_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}} + \|f\|_{\mathbf{X}(m)}\|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}^*}.$$

For all the other cases $|\beta| = 1$, $|\beta| = 2$ and $\beta = \alpha$, we can argue as in Step 1 by observing that

$$\|\nabla_x g\|_{H^2_x H^{s,*}_v(\langle v\rangle^\ell)} + \|\nabla^2_x g\|_{H^1_x H^{s,*}_v(\langle v\rangle^\ell)} + \|\nabla^3_x g\|_{L^2_x H^{s,*}_v(\langle v\rangle^\ell)} \lesssim \|g\|_{\mathbf{E}^*},$$

which thus implies from (2.27)–(2.28)–(2.29) that

$$\langle Q(\partial^{\alpha-\beta}_x g, \partial^\alpha_x f), \partial^\alpha_x f\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \lesssim \|f\|^2_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}} + \|f\|_{\mathbf{X}(m)}\|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}^*}.$$

This concludes the proof of (2.23). □

We now prove the estimates which we will use to prove the convergence of the iterative scheme in Section 4.

Proposition 2.13. *Assume $k > 13/2 + 5|\gamma|/2 + 6s$ and consider $m = \langle v\rangle^k$. For any $f, g, h \in \mathbf{X}(m) \cap \mathbf{X}^*(m)$ there holds*

$$\langle Q(f,g), h\rangle_{\mathbf{X}(m)} \lesssim \|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{X}^*(m)}\|h\|_{\mathbf{X}(m)} + \|f\|_{\mathbf{X}(m)}\|\langle v\rangle^{2s} g\|_{\mathbf{X}^*(m)}\|h\|_{\mathbf{X}^*(m)}. \tag{2.30}$$

Moreover for any $f, h \in \mathbf{X}(m) \cap \mathbf{X}^*(m)$ and $g \in \mathbf{E} \cap \mathbf{E}^*$ there holds

$$\begin{aligned}\langle Q(f,g), h\rangle_{\mathbf{X}(m)} &\\ \lesssim \|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}}\|h\|_{\mathbf{X}^*(m)} &+ \|f\|_{\mathbf{X}(m)}\|g\|_{\mathbf{E}^*}\|h\|_{\mathbf{X}^*(m)} + \|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}^*}\|h\|_{\mathbf{X}(m)},\end{aligned} \tag{2.31}$$

and

$$\langle Q(g,f), h\rangle_{\mathbf{X}(m)} \lesssim \|\langle v\rangle^{2s} f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}}\|h\|_{\mathbf{X}^*(m)} + \|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}^*}\|h\|_{\mathbf{X}(m)}, \tag{2.32}$$

Proof. By expanding the inner product of $\mathbf{X}(m)$, we are led to estimate

$$\begin{aligned}\langle Q(f,g), h\rangle_{\mathbf{X}(m)} \lesssim &\left|\langle Q(f,g), h\rangle_{L^2_x L^2_v(m)}\right| \\ &+ \sum_{|\alpha|=3}\sum_{0\leqslant\beta\leqslant\alpha}\left|\langle Q(\partial_x^{\alpha-\beta} f, \partial_x^\beta g), \partial_x^\alpha h\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})}\right|.\end{aligned} \tag{2.33}$$

The proof of each one of the estimates (2.30), (2.31) and (2.32) then follows the same approach: For each term appearing in (2.33) we integrate in space the corresponding homogeneous estimate and then use Hölder's inequality and Sobolev embeddings arguing similarly as in Step 1 of the proof of Proposition 2.12.

We fix some $\ell > 13/2 + 2|\gamma|$ such that $k \geqslant \ell + 6s + |\gamma|/2$, and remark that we can apply estimate (2.18) of Proposition 2.9 with the weight $\langle v\rangle^{k-6s}$ in the sequel.

Step 1: General estimates of (2.33) in L^p_x-norms. The first term in (2.33) is estimated using (2.18) and Hölder's inequality $L^\infty_x - L^2_x - L^2_x$, which yields

$$\begin{aligned}&\langle Q(f,g), h\rangle_{L^2_x L^2_v(m)} \\ &\quad\lesssim \|\langle v\rangle^{\gamma/2} f\|_{L^2_x L^2_v(m)}\|g\|_{L^\infty_x H^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2} h\|_{L^2_x L^2_v(m)} \\ &\qquad + \|f\|_{L^\infty_x H^{s,*}_v(\langle v\rangle^\ell)}\|g\|_{L^2_x H^{s,*}_v(m)}\|\langle v\rangle^{\gamma/2} h\|_{L^2_x L^2_v(m)} \\ &\qquad + \|f\|_{L^\infty_x L^2_v(\langle v\rangle^\ell)}\|\langle v\rangle^{2s} g\|_{L^2_x H^{s,*}_v(m)}\|h\|_{L^2_x H^{s,*}_v(m)}.\end{aligned} \tag{2.34}$$

The second term in (2.33) is then estimated depending on the value of β. For the case $\beta = 0$, we also have by using (2.18) and Hölder's inequality $L^\infty_x - L^2_x - L^2_x$ that

$$\begin{aligned}&\langle Q(\partial_x^\alpha f, g), \partial_x^\alpha h\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\ &\quad\lesssim \|\langle v\rangle^{\gamma/2}\nabla_x^3 f\|_{L^2_x L^2_v(m\langle v\rangle^{-6s})}\|g\|_{L^\infty_x H^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}\nabla_x^3 h\|_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\ &\qquad + \|\nabla_x^3 f\|_{L^2_x H^{s,*}_v(\langle v\rangle^\ell)}\|g\|_{L^\infty_x H^{s,*}_v(m\langle v\rangle^{-6s})}\|\langle v\rangle^{\gamma/2}\nabla_x^3 h\|_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\ &\qquad + \|\nabla_x^3 f\|_{L^2_x L^2_v(\langle v\rangle^\ell)}\|\langle v\rangle^{2s} g\|_{L^\infty_x H^{s,*}_v(m\langle v\rangle^{-6s})}\|\nabla_x^3 h\|_{L^2_x H^{s,*}_v(m\langle v\rangle^{-6s})}.\end{aligned} \tag{2.35}$$

When $|\beta| = 1$ we use Hölder's inequality $L^4_x - L^4_x - L^2_x$ to get

$$\begin{aligned}&\langle Q(\partial_x^{\alpha-\beta} f, \partial_x^\beta g), \partial_x^\alpha h\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\ &\quad\lesssim \|\langle v\rangle^{\gamma/2}\nabla_x^2 f\|_{L^4_x L^2_v(m\langle v\rangle^{-6s})}\|\nabla_x g\|_{L^4_x H^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}\nabla_x^3 h\|_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\ &\qquad + \|\nabla_x^2 f\|_{L^4_x H^{s,*}_v(\langle v\rangle^\ell)}\|\nabla_x g\|_{L^4_x H^{s,*}_v(m\langle v\rangle^{-6s})}\|\langle v\rangle^{\gamma/2}\nabla_x^3 h\|_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\ &\qquad + \|\nabla_x^2 f\|_{L^4_x L^2_v(\langle v\rangle^\ell)}\|\langle v\rangle^{2s}\nabla_x g\|_{L^4_x H^{s,*}_v(m\langle v\rangle^{-6s})}\|\nabla_x^3 h\|_{L^2_x H^{s,*}_v(m\langle v\rangle^{-6s})}.\end{aligned} \tag{2.36}$$

and for $|\beta| = 2$:

$$\begin{aligned}&\langle Q(\partial_x^{\alpha-\beta} f, \partial_x^\beta g), \partial_x^\alpha h\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\ &\quad\lesssim \|\langle v\rangle^{\gamma/2}\nabla_x f\|_{L^4_x L^2_v(m\langle v\rangle^{-6s})}\|\nabla_x^2 g\|_{L^4_x H^{s,*}_v(\langle v\rangle^\ell)}\|\langle v\rangle^{\gamma/2}\nabla_x^3 h\|_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\ &\qquad + \|\nabla_x f\|_{L^4_x H^{s,*}_v(\langle v\rangle^\ell)}\|\nabla_x^2 g\|_{L^4_x H^{s,*}_v(m\langle v\rangle^{-6s})}\|\langle v\rangle^{\gamma/2}\nabla_x^3 h\|_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\ &\qquad + \|\nabla_x f\|_{L^4_x L^2_v(\langle v\rangle^\ell)}\|\langle v\rangle^{2s}\nabla_x^2 g\|_{L^4_x H^{s,*}_v(m\langle v\rangle^{-6s})}\|\nabla_x^3 h\|_{L^2_x H^{s,*}_v(m\langle v\rangle^{-6s})}.\end{aligned} \tag{2.37}$$

Finally, for $\beta = \alpha$ we get, using again Hölder's inequality $L^\infty_x - L^2_x - L^2_x$,

$$\begin{aligned}
&\langle Q(f, \partial_x^\alpha g), \partial_x^\alpha h\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\
&\quad\lesssim \|\langle v\rangle^{\gamma/2} f\|_{L^\infty_x L^2_v(m\langle v\rangle^{-6s})} \|\nabla_x^3 g\|_{L^2_x H^{s,*}_v(\langle v\rangle^\ell)} \|\langle v\rangle^{\gamma/2}\nabla_x^3 h\|_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\
&\qquad + \|f\|_{L^\infty_x H^{s,*}_v(\langle v\rangle^\ell)} \|\nabla_x^3 g\|_{L^2_x H^{s,*}_v(m\langle v\rangle^{-6s})} \|\langle v\rangle^{\gamma/2}\nabla_x^3 h\|_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\
&\qquad + \|f\|_{L^\infty_x L^2_v(\langle v\rangle^\ell)} \|\langle v\rangle^{2s}\nabla_x^3 g\|_{L^2_x H^{s,*}_v(m\langle v\rangle^{-6s})} \|\nabla_x^3 h\|_{L^2_x H^{s,*}_v(m\langle v\rangle^{-6s})}.
\end{aligned} \tag{2.38}$$

We now split the proof into three steps.

Step 2: Sobolev embeddings to prove (2.30). It follows from estimates (2.34)–(2.35)–(2.36)–(2.37)–(2.38) by using the Sobolev embeddings $H^2_x(\mathbf{R}^3) \hookrightarrow L^\infty_x(\mathbf{R}^3)$ and $H^1_x(\mathbf{R}^3) \hookrightarrow L^4_x(\mathbf{R}^3)$ as in Step 1 of Proposition 2.12.

Step 3: Proof of (2.31). We first observe that

$$\|g\|_{H^2_x H^{s,*}_v(\langle v\rangle^\ell)} \lesssim \|\pi g\|_{H^2_x L^2_v} + \|g^\perp\|_{L^2_x H^{s,*}_v(\langle v\rangle^\ell)} \lesssim \|g\|_{\mathbf{E}} + \|g\|_{\mathbf{E}^*},$$

and

$$\|\langle v\rangle^{2s} g\|_{L^2_x H^{s,*}_v(m)} \lesssim \|\pi g\|_{L^2_x L^2_v} + \|\langle v\rangle^{2s} g^\perp\|_{L^2_x H^{s,*}_v(m)} \lesssim \|g\|_{\mathbf{E}} + \|g\|_{\mathbf{E}^*}.$$

Moreover

$$\|\langle v\rangle^{\gamma/2} f\|_{L^2_x L^2_v(m)} \lesssim \min\left\{\|f\|_{\mathbf{X}(m)}, \|f\|_{\mathbf{X}^*(m)}\right\}$$

and

$$\|\langle v\rangle^{\gamma/2} f\|_{H^2_x L^2_v(\langle v\rangle^\ell)} \lesssim \min\left\{\|f\|_{\mathbf{X}(m)}, \|f\|_{\mathbf{X}^*(m)}\right\}$$

since $k \geqslant \ell + 6s$. Therefore from (2.34) we get

$$\begin{aligned}
&\langle Q(f, g), h\rangle_{L^2_x L^2_v(m)} \\
&\quad\lesssim \|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}}\|h\|_{\mathbf{X}^*(m)} + \|f\|_{\mathbf{X}(m)}\|g\|_{\mathbf{E}^*}\|h\|_{\mathbf{X}^*(m)} + \|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}^*}\|h\|_{\mathbf{X}(m)}.
\end{aligned}$$

For the case $\beta = 0$ we remark that, similarly as above, we have

$$\|\langle v\rangle^{2s} g\|_{H^2_x H^{s,*}_v(m\langle v\rangle^{-6s})} \lesssim \|\pi g\|_{H^2_x L^2_v} + \|\langle v\rangle^{2s} g^\perp\|_{H^2_x H^{s,*}_v(m\langle v\rangle^{-6s})} \lesssim \|g\|_{\mathbf{E}} + \|g\|_{\mathbf{E}^*}.$$

moreover

$$\|\langle v\rangle^{\gamma/2}\nabla_x^3 f\|_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \lesssim \min\left\{\|f\|_{\mathbf{X}(m)}, \|f\|_{\mathbf{X}^*(m)}\right\},$$

as well as

$$\|\nabla_x^3 f\|_{L^2_x L^2_v(\langle v\rangle^\ell)} \lesssim \min\left\{\|f\|_{\mathbf{X}(m)}, \|f\|_{\mathbf{X}^*(m)}\right\},$$

since $k + \gamma/2 \geqslant \ell + 6s$. Hence we deduce from (2.35)

$$\begin{aligned}
&\langle Q(\partial_x^\alpha f, g), \partial_x^\alpha h\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\
&\quad\lesssim \|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}}\|h\|_{\mathbf{X}^*(m)} + \|f\|_{\mathbf{X}(m)}\|g\|_{\mathbf{E}^*}\|h\|_{\mathbf{X}^*(m)} + \|f\|_{\mathbf{X}^*(m)}\|g\|_{\mathbf{E}^*}\|h\|_{\mathbf{X}(m)}.
\end{aligned}$$

For all the other cases $|\beta| = 1$, $|\beta| = 2$ and $\beta = \alpha$, we can argue as in Step 1 by observing that

$$\|\nabla_x g\|_{L^4_x H^{s,*}_v(\langle v\rangle^\ell)} + \|\nabla_x^2 g\|_{L^4_x H^{s,*}_v(\langle v\rangle^\ell)} + \|\nabla_x^3 g\|_{L^2_x H^{s,*}_v(\langle v\rangle^\ell)} \lesssim \|g\|_{\mathbf{E}^*},$$

as well as

$$\|\nabla_x g\|_{L^4_x H^{s,*}_v(m\langle v\rangle^{-4s})} + \|\nabla_x^2 g\|_{L^4_x H^{s,*}_v(m\langle v\rangle^{-4s})} + \|\nabla_x^3 g\|_{L^2_x H^{s,*}_v(m\langle v\rangle^{-4s})} \lesssim \|g\|_{\mathbf{E}^*},$$

which thus implies from (2.36)–(2.37)–(2.38) that

$$\langle Q(\partial_x^{\alpha-\beta} f, \partial_x^\alpha g), \partial_x^\alpha h\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \lesssim \|g\|_{\mathbf{E}^*}\left(\|f\|_{\mathbf{X}^*(m)}\|h\|_{\mathbf{X}(m)} + \|f\|_{\mathbf{X}(m)}\|h\|_{\mathbf{X}^*(m)}\right).$$

Step 4: Proof of (2.32). It follows similarly as in Step 2 above, so we omit the proof. □

We now introduce an equivalent norm $|||\cdot|||_{\mathbf{E}} \approx \|\cdot\|_{\mathbf{E}}$ induced by the scalar product

$$\langle\!\langle f, g\rangle\!\rangle_{\mathbf{E}} := \langle f, g\rangle_{\mathbf{E}} + \Psi[f, g], \tag{2.39}$$

for some bilinear form Ψ which is defined in [34, Lemma 2.1] and is of the form, recalling the definition of the projection $\pi[f]$ as well as $\rho[f]$, $u[f]$ and $\theta[f]$ in (1.7),

$$\begin{aligned}\Psi[f,g] := \eta_1 \left\langle \nabla_x \theta[f], \Lambda[g^\perp] \right\rangle_{H^2_x} + \eta_1 \left\langle \Lambda[f^\perp], \nabla_x \theta[g] \right\rangle_{H^2_x} \\ + \eta_2 \left\langle \nabla_x u[f] + \nabla_x u[f]^\top, \Theta[g^\perp] + I_3 \theta[g] \right\rangle_{H^2_x} \\ + \eta_2 \left\langle \Theta[f^\perp] + I_3 \theta[f], \nabla_x u[g] + \nabla_x u[g]^\top \right\rangle_{H^2_x} \\ + \eta_3 \left\langle \nabla_x \rho[f], u[g] \right\rangle_{H^2_x} + \eta_3 \left\langle u[f], \nabla_x \rho[g] \right\rangle_{H^2_x}\end{aligned} \tag{2.40}$$

with $0 < \eta_3 \ll \eta_2 \ll \eta_1 \ll 1$, I_3 the identity matrix and

$$\Lambda[f] = \int_{\mathbf{R}^3} v(|v|^2 - 5) f \, \mathrm{d}v, \quad \Theta[f] = \int_{\mathbf{R}^3} (v \otimes v - I_3) f \, \mathrm{d}v.$$

Proposition 2.14. *The following estimates holds for the new scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathbf{E}}$ on E: For any $f, g, h \in \mathbf{E} \cap \mathbf{E}^*$ one has*

$$\langle\!\langle Q(f,g), h \rangle\!\rangle_{\mathbf{E}} \lesssim \|h\|_{\mathbf{E}^*} \left(\|f\|_{\mathbf{E}} \|g\|_{\mathbf{E}^*} + \|f\|_{\mathbf{E}^*} \|g\|_{\mathbf{E}} \right). \tag{2.41}$$

Furthermore, for any bounded compactly supported function $\chi = \chi(v)$, we also have for any $\ell > 0$

$$\langle\!\langle g, \chi f \rangle\!\rangle_{\mathbf{E}} \lesssim \|g\|_{\mathbf{E}} \|f\|_{\mathbf{X}(\langle v \rangle^\ell)}, \tag{2.42}$$

$$\langle\!\langle g, \chi f \rangle\!\rangle_{\mathbf{E}} \lesssim \|g\|_{\mathbf{E}^*} \|f\|_{\mathbf{X}(\langle v \rangle^\ell)} + \|\pi g\|_{L^2_x} \|f\|_{\mathbf{X}^*(\langle v \rangle^\ell)}. \tag{2.43}$$

Proof. We start by proving (2.41) then comment on how to prove (2.42) and (2.43).

Step 1: Bound of $\langle \cdot, \cdot \rangle_E$ for (2.41). First, since Q is $\mathbf{E}$-orthogonal to πh, we have

$$\langle Q(f,g), h \rangle_{\mathbf{E}} = \langle Q(f,g), h^\perp \rangle_{L^2_x L^2_v(\mu^{-1/2})} + \langle \nabla_x^3 Q(f,g), \nabla_x^3 h^\perp \rangle_{L^2_x L^2_v(\mu^{-1/2})}.$$

The proof of Proposition 2.13 can then be adapted (and simplified) to get from the homogeneous estimate of Lemma 2.8 in the current setting

$$\langle Q(f,g), h \rangle_{\mathbf{E}} \lesssim \|h\|_{\mathbf{E}^*} \left(\|f\|_{\mathbf{E}} \|g\|_{\mathbf{E}^*} + \|f\|_{\mathbf{E}^*} \|g\|_{\mathbf{E}} \right).$$

A detailed proof is given in [34, Section 3.1] (note that the authors require more derivatives because of their homogeneous nonlinear estimates).

Step 2: Bound of $\Psi[\cdot, \cdot]$ for (2.41). In this step, we use the notation from the definition (2.40) of Ψ. Before considering any estimate, note that because Q is $L^2_v\left(\mu^{-1/2}\right)$-orthogonal to $\ker(\mathscr{L})$, we have

$$\rho[Q(f,g)] = u[Q(f,g)] = \theta[Q(f,g)] = 0,$$

which leaves us with fewer terms that we estimate using the Cauchy-Schwarz inequality:

$$\begin{aligned}\Psi[Q(f,g), h] &= \eta_1 \left\langle \Lambda[Q(f,g)], \nabla_x \theta[h] \right\rangle_{H^2_x} + \eta_2 \left\langle \Theta[Q(f,g)], \nabla_x u[h] + \nabla_x u[h]^\top \right\rangle_{H^2_x} \\ &\lesssim \|\Lambda[Q(f,g)]\|_{H^2_x} \|\nabla_x \theta[h]\|_{H^2_x} + \|\Theta[Q(f,g)]\|_{H^2_x} \|\nabla_x u[h]\|_{H^2_x}\end{aligned}$$

Using Lemma 2.8 and the fact that $v_i(|v|^2 - 5)\mu, (v_i v_j - \delta_{ij})\mu \in E^*$ for any $1 \leqslant i, j \leqslant 3$, we have

$$\|\Lambda[Q(f,g)]\|^2_{H^2_x} + \|\Theta[Q(f,g)]\|^2_{H^2_x} \lesssim \sum_{0 \leqslant p+q \leqslant 2} \int \left\{ \|\nabla_x^p f\|_E \|\nabla_x^q g\|_{E^*} + \|\nabla_x^p f\|_{E^*} \|\nabla_x^q f\|_E \right\}^2 \mathrm{d}x.$$

Combined with the embedding $H^2_x(\mathbf{R}^3) \hookrightarrow L^\infty_x(\mathbf{R}^3)$ or $H^1_x(\mathbf{R}^3) \hookrightarrow L^4_x(\mathbf{R}^3)$ and arguing as in Step 2 of Proposition 2.13, this yields

$$\|\Lambda[Q(f,g)]\|^2_{H^2_x} + \|\Theta[Q(f,g)]\|^2_{H^2_x} \lesssim \|f\|_{\mathbf{E}} \|g\|_{\mathbf{E}^*} + \|f\|_{\mathbf{E}^*} \|g\|_{\mathbf{E}}.$$

Finally we get

$$\|\nabla_x \theta[h]\|_{H^2_x} + \|\nabla_x u[h]\|_{H^2_x} \lesssim \|\nabla_x \pi h\|_{H^2_x E} \lesssim \|h\|_{\mathbf{E}^*}.$$

To sum up, we have shown

$$\Psi\left[Q(f,g), h\right] \lesssim \|h\|_{\mathbf{E}^*} \left(\|f\|_{\mathbf{E}} \|g\|_{\mathbf{E}^*} + \|f\|_{\mathbf{E}^*} \|g\|_{\mathbf{E}} \right).$$

Step 3: Proof of and (2.42) *and* (2.43). Brutal bounds lead to (2.42). The proof of (2.43) is immediate from (2.11) which implies

$$\langle g, \chi f\rangle_{\mathbf{E}} \lesssim \|\langle v\rangle^{\gamma/2+s} g\|_{\mathbf{E}} \|\chi\langle v\rangle^{\ell} f\|_{\mathbf{E}} \lesssim \left(\|g\|_{\mathbf{E}^*} + \|\pi g\|_{L^2_x L^2_v}\right) \|f\|_{\mathbf{X}(\langle v\rangle^{\ell})}.$$

Similar arguments from the previous step also lead to

$$\Psi\left[g, \chi f\right] \lesssim \|g\|_{\mathbf{E}^*} \|f\|_{\mathbf{X}(\langle v\rangle^{\ell})},$$

which completes the proof. □

3. Linear theory

3.1. Estimates on $\mathscr{L}$.

The goal of this subsection is to prove the following proposition.

Proposition 3.1. *Let $k > 7/2 - |\gamma|/2 + 2s$ and denote the weight function $m = \langle v\rangle^k$. For any smooth enough function f there holds*

$$\int (\mathscr{L} f) f m^2 \,\mathrm{d}x\,\mathrm{d}v \leqslant -c\|f\|^2_{L^2_x H^{s,*}_v(m)} + C\|f\|^2_{L^2_{x,v}}, \tag{3.1}$$

for some positive constants $c, C > 0$.

We introduce a splitting of the angular cross section $b(\cos\theta)$ so as to decompose the linearized operator $\mathscr{L}$ as a singular regularizing part and a weakly coercive non-singular part, namely we define for any $\delta \in (0,1]$

$$b(\cos\theta) = b(\cos\theta)\mathbf{1}_{|\theta|\leqslant \delta\pi/2} + b(\cos\theta)\mathbf{1}_{|\theta|>\delta\pi/2} =: b_\delta(\cos\theta) + b^c_\delta(\cos\theta),$$

which induces the following splitting of the linearized operator:

$$\mathscr{L} = \mathscr{L}_\delta + \mathscr{L}^c_\delta.$$

Denote ν_δ the approximate collision frequency defined as

$$\nu_\delta(v) = \int_{\mathbf{R}^3\times\mathbf{S}^2} |v - v_*|^\gamma b^c_\delta(\cos\theta)\mu(v_*)\,\mathrm{d}\sigma\,\mathrm{d}v_*$$

which satisfies, according to the cutoff case (see for instance [21]), for some positive constants $\nu_0, \nu_1 > 0$

$$\nu_0 \delta^{-2s}\langle v\rangle^\gamma \leqslant \nu_\delta(v) \leqslant \nu_1\delta^{-2s}\langle v\rangle^\gamma, \quad \forall v \in \mathbf{R}^3. \tag{3.2}$$

The cutoff part of the linearized collision operator then splits

$$\mathscr{L}^c_\delta f = -\nu_\delta f + \int_{\mathbf{R}^3\times\mathbf{S}^2} |v - v_*|^\gamma b^c_\delta(\cos\theta)\left[f(v'_*)\mu(v') - f(v_*)\mu(v) + \mu(v'_*)f(v')\right]\,\mathrm{d}\sigma\,\mathrm{d}v_*. \tag{3.3}$$

Lemma 3.2 (Non-grazing collisions)**.** *Suppose $k > 3/2 + |\gamma|/2 + s$ and let $m = \langle v\rangle^k$. For any $\delta, \varepsilon \in (0,1]$ there holds*

$$\int (\mathscr{L}^c_\delta g)\, g m^2 \,\mathrm{d}v \leqslant -c\delta^{-2s}\|\langle v\rangle^{\gamma/2} g\|^2_{L^2_v(m)} + \varepsilon\|\langle v\rangle^{\gamma/2} g\|^2_{H^s_v(m)} + C_{\delta,\varepsilon}\|g\|^2_{L^2_v},$$

for some positive constants $c, C_{\delta,\varepsilon} > 0$.

Proof. Firstly we consider

$$\begin{aligned}\int (\mathscr{L}^c_\delta g)\, g m^2 \,\mathrm{d}v = &- \|\nu^{1/2}_\delta g\|^2_{L^2_v(m)} + \int |v - v_*|^\gamma b^c_\delta(\cos\theta) g'_* \mu' g m^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ &- \int |v - v_*|^\gamma b^c_\delta(\cos\theta) g_* \mu g m^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ &+ \int |v - v_*|^\gamma b^c_\delta(\cos\theta) \mu'_* g' g m^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v,\end{aligned}$$

so that, using the bounds (3.2) on ν_δ, we have

$$\int (\mathscr{L}^c_\delta g)\, g m^2 \,\mathrm{d}v + \nu_0\delta^{-2s}\|\langle v\rangle^{\gamma/2} g\|^2_{L^2_v(m)} \leqslant \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3,$$

with, denoting $G = mg$,

$$\mathbf{I}_1 := \int |v-v_*|^\gamma b_\delta^c(\cos\theta) g'_* \mu' G m \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v,$$
$$\mathbf{I}_2 := \int |v-v_*|^\gamma b_\delta^c(\cos\theta) g_* \mu G m \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v,$$
$$\mathbf{I}_3 := \int |v-v_*|^\gamma b_\delta^c(\cos\theta) \mu'_* g' G m \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v.$$

In Step 1, we prove that the terms $\mathbf{I}_1$ and $\mathbf{I}_3$ satisfy the bound

$$\mathbf{I}_1 + \mathbf{I}_3 \lesssim \delta^{-1-\gamma/2-2s} \int |v-v_*|^\gamma \, G^2 \, \varphi_* \,\mathrm{d}v\,\mathrm{d}v_*, \tag{3.4}$$

where φ denotes a Schwartz function (typically of the form $\mu^a\langle v\rangle^b$) from which we will deduce using Lemma 2.2 with $\eta = \varepsilon\delta^{1+\gamma/2}$ that there holds

$$\mathbf{I}_1 + \mathbf{I}_3 \leqslant \varepsilon\delta^{-2s}\|\langle v\rangle^{\gamma/2} g\|^2_{L^2_v(m)} + C_{\varepsilon,\delta}\|g\|^2_{L^2_v}. \tag{3.5}$$

In Step 2, we will prove that $\mathbf{I}_2$ satisfies

$$\mathbf{I}_2 \leqslant \varepsilon\delta^{-2s}\|\langle v\rangle^{\gamma/2} g\|_{H^s_v(m)} + C_{\varepsilon,\delta}\|g\|^2_{L^2_v},$$

so that taking ε small enough, we obtain

$$\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 \leqslant \frac{\nu_0}{2}\delta^{-2s}\|\langle v\rangle^{\gamma/2} g\|^2_{L^2_v(m)} + \varepsilon'\|\langle v\rangle^{\gamma/2} g\|^2_{H^s_v(m)} + C_{\delta,\varepsilon}\|g\|^2_{L^2_v},$$

where ε' is arbitrarily small. This will indeed prove the lemma by taking ε' small enough.

Step 2: Proof of (3.4) *for* $\mathbf{I}_1$ *and* $\mathbf{I}_3$. We start by splitting $\mathbf{I}_1$ using (2.9):

$$\begin{aligned}\mathbf{I}_1 &= \int |v-v_*|^\gamma b_\delta^c(\cos\theta) g'_* \mu' m G \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ &\lesssim \int |v-v_*|^\gamma b_\delta^c(\cos\theta) \left(\langle v\rangle g\right)'_* (\mu m)' G \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ &\quad + \int |v-v_*|^\gamma b_\delta^c(\cos\theta) G'_* \mu' G \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v =: \mathbf{I}_{11} + \mathbf{I}_{12},\end{aligned}$$

Rewriting $\mathbf{I}_{11}$ thanks to (2.6), using the Cauchy-Schwarz inequality, then integrating in σ one obtains

$$\begin{aligned}\mathbf{I}_{11} &= \int |v-v_*|^\gamma b_\delta^c(\cos\theta) \left(\langle v\rangle g\right)_* (\mu m) G' \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ &\leqslant \left(\int |v-v_*|^\gamma b_\delta^c(\cos\theta) \left(\langle v\rangle g\right)^2_* (\mu m) \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/2} \\ &\quad \times \left(\int |v-v_*|^\gamma b_\delta^c(\cos\theta) (\mu m) (G')^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/2}.\end{aligned}$$

Using the change of variables (2.4) in the post-factor, then integrating in σ we get

$$\begin{aligned}\mathbf{I}_{11} &\lesssim \left(\int b_\delta^c(\cos\theta)|v-v_*|^\gamma \left(\langle v\rangle g\right)^2_* (\mu m) \,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/2} \\ &\quad \times \left(\int b_\delta^c(\cos(\pi-2\theta))|v-v_*|^\gamma (\pi-2\theta)^{-2-\gamma} (\mu m) G^2_* \,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/2} \\ &\lesssim \delta^{-1-\gamma/2-2s} \int |v-v_*|^\gamma (\mu m) G^2_* \,\mathrm{d}v_*\,\mathrm{d}v.\end{aligned}$$

To bound the part $\mathbf{I}_{12}$, we start again with the Cauchy-Schwarz inequality:

$$\begin{aligned}\mathbf{I}_{12} &= \int |v-v_*|^\gamma b_\delta^c(\cos\theta) G'_* \mu' G \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ &\leqslant \left(\int |v-v_*|^\gamma b_\delta^c(\cos\theta) (G'_*)^2 \mu' \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/2} \\ &\quad \times \left(\int |v-v_*|^\gamma b_\delta^c(\cos\theta) \mu' G^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{1/2}.\end{aligned}$$

Up to the pre-post change of variables (2.6) in the prefactor, this term is dealt with in the same way as $\mathbf{I}_{11}$. Similar computations (using (2.5) this time) lead to

$$\mathbf{I}_3 \lesssim \delta^{-2s} \int |v-v_*|^\gamma (\mu m) G_*^2 \, \mathrm{d}v_* \, \mathrm{d}v.$$

This concludes this step.

Step 2: Proof of (3.5) for $\mathbf{I}_2$. For $\mathbf{I}_2$, we integrate in σ to get the factor δ^{-2s}, then in v_* using the estimate (2.1) with the power $\theta = k + \gamma/2 > 3/2 + s$, which yields

$$\begin{aligned}
\mathbf{I}_2 &= \int b_\delta^c(\cos\theta) |v-v_*|^\gamma g_* \mu G \, \mathrm{d}v_* \, \mathrm{d}v \\
&\lesssim \delta^{-2s} \| \langle v \rangle^\theta g \|_{H^s_v} \int \langle v \rangle^\gamma \mu G \, \mathrm{d}v \\
&\lesssim \varepsilon \delta^{-2s} \| \langle v \rangle^{\gamma/2} g \|_{H^s_v(m)} + C_{\varepsilon,\delta} \, \|g\|_{L^2_v} \, ,
\end{aligned}$$

where the last line comes from Young's inequality. □

Lemma 3.3 (Grazing collisions)**.** *Let $k > 13/2 + 2|\gamma|$ and define $m = \langle v \rangle^k$. There exists some $c > 0$ such that for any $\varepsilon > 0$ and $\delta > 0$*

$$\langle \mathscr{L}_\delta f, f \rangle_{L^2_v(m)} \leqslant \varepsilon \| f \|^2_{H^{s,*}} + C_\varepsilon \| \langle v \rangle^{\gamma/2} f \|^2_{L^2_v(m)}, \tag{3.6}$$

$$\langle \mathscr{L} f, f \rangle_{L^2_v(m)} \leqslant -c \| f \|^2_{H^{s,*}_v(m)} + C \| \langle v \rangle^{\gamma/2} f \|^2_{L^2_v(m)}, \tag{3.7}$$

for some $C_\varepsilon > 0$.

Proof. We start by splitting the Dirichlet form using commutators:

$$\begin{aligned}
\langle \mathscr{L}_\delta f, f \rangle_{L^2_v(m)} &= \langle Q_\delta(\mu, f), f \rangle_{L^2_v(m)} + \langle Q_\delta(f, \mu), f \rangle_{L^2_v(m)} \\
&= \langle Q_\delta(\mu, F), F \rangle_{L^2_v} + R_1 + R_2 + R_3,
\end{aligned}$$

where we denoted

$$\begin{aligned}
R_1 &:= \langle Q_\delta(f, m\mu), F \rangle_{L^2_v}, \\
R_2 &:= \langle Q_\delta(\mu, f), f \rangle_{L^2_v(m)} - \langle Q_\delta(\mu, F), F \rangle_{L^2_v}, \\
R_3 &:= \langle Q_\delta(f, \mu), f \rangle_{L^2_v(m)} - \langle Q_\delta(f, m\mu), F \rangle_{L^2_v}.
\end{aligned}$$

The first term is estimated using Lemma 2.10 (where we choose $w_1 = \gamma/2 + 2s$, $w_2 = \gamma/2$ for the weights, $a = 2s$ and $b = 0$ for the derivatives):

$$R_1 \lesssim \| \langle v \rangle^{\gamma/2} f \|^2_{L^2_v(m)},$$

and the two other ones using Lemma 2.11 and Young's inequality:

$$\begin{aligned}
R_2 + R_3 &\leqslant C \left(\| \langle v \rangle^{\gamma/2} f \|^2_{L^2_v(m)} + \| \langle v \rangle^{\gamma/2} f \|_{L^2_v(m)} \| f \|_{H^{s,*}_v(m)} \right) \\
&\leqslant C_\varepsilon \| \langle v \rangle^{\gamma/2} f \|^2_{L^2_v(m)} + \varepsilon \| f \|^2_{H^{s,*}_v(m)}.
\end{aligned}$$

We then focus on the first term which provides the anisotropic dissipation $\dot{H}^{s,*}_v(m)$:

$$\begin{aligned}
\langle Q_\delta(\mu, F), F \rangle_{L^2_v(m)} &= \int B_\delta(v - v_*, \sigma)(\mu'_* F' - \mu_* F) F \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \\
&= \frac{1}{2} \int B_\delta(v - v_*, \sigma)(2\mu'_* F' F - \mu_* F^2 - \mu'_* F^2) \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \\
&\quad + \frac{1}{2} \int B_\delta(v - v_*, \sigma)(\mu'_* - \mu_*) F^2 \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v.
\end{aligned}$$

We use (2.6) to change the term $\mu'_* F^2$ of the first integral into $\mu_* (F')^2$, and the cancellation lemma (Proposition 2.4) in the second integral:

$$\begin{aligned}
\langle Q_\delta(\mu, F), F \rangle_{L^2_v(m)} &= -\frac{1}{2} \int B_\delta(v - v_*, \sigma) \mu'_* (F' - F)^2 \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \\
&\quad + C_\delta \int |v - v_*|^\gamma \mu_* F^2 \, \mathrm{d}v_* \, \mathrm{d}v,
\end{aligned}$$

where $C_\delta \lesssim 1$. We thus have in virtue of (2.1)

$$\langle Q_\delta(\mu, F), F\rangle_{L^2_v(m)} + \frac{1}{2}\int B_\delta(v-v_*,\sigma)\mu'_*(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \lesssim \|\langle v\rangle^{\gamma/2} f\|^2_{L^2_v(m)}.$$

Next, using (2.6), and then $|v-v_*| \leqslant \langle v\rangle\langle v_*\rangle$ combined with the fact that $\gamma \leqslant 0$, we have

$$\begin{aligned}
-\int B_\delta(v-v_*,\sigma)\mu'_*(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v& \\
&= -\int b_\delta(\cos\theta)|v-v_*|^\gamma \mu_*(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&\leqslant -\int b_\delta(\cos\theta)\langle v\rangle^\gamma \left(\mu\langle v\rangle^{-\gamma}\right)_* (F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v.
\end{aligned}$$

One shows as in the proof of (1) $\approx$ (2) from Lemma 2.7 that

$$\begin{aligned}
-\int B_\delta(v-v_*,\sigma)\mu'_*(F'-F)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v + \frac{1}{2}\int b_\delta(\cos\theta)\left(\mu\langle v\rangle^\gamma\right)_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v& \\
\lesssim \|\langle v\rangle^{\gamma/2} f\|^2_{L^2_v(m)}.&
\end{aligned}$$

Together with the previous estimates, we conclude that

$$\begin{aligned}
\langle \mathscr{L}_\delta f, f\rangle_{L^2_v(m)} \leqslant& -\frac{1}{2}\int b_\delta(\cos\theta)\left(\mu\langle v\rangle^\gamma\right)_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&+ \varepsilon\|f\|^2_{H^{s,*}_v(m)} + C_\varepsilon\|\langle v\rangle^{\gamma/2} f\|^2_{L^2_v(m)}.
\end{aligned}$$

The second term being non-positive, we conclude that (3.6) holds.

Furthermore, this proof works when replacing b_δ by b (which corresponds, in a way, to taking δ large), thus for $\mathscr{L}$ we get

$$\begin{aligned}
\langle \mathscr{L} f, f\rangle_{L^2_v(m)} \leqslant& -\frac{1}{2}\int b(\cos\theta)\left(\mu\langle v\rangle^\gamma\right)_*(\mathcal{F}'-\mathcal{F})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&+ \varepsilon\|f\|^2_{H^{s,*}_v(m)} + C_\varepsilon\|\langle v\rangle^{\gamma/2} f\|^2_{L^2_v(m)}.
\end{aligned}$$

Recalling the definition of the norm $\dot{H}^{s,*}_v(m)$ in (1.9), we therefore deduce that (3.7) also holds by taking ε small enough. □

We are now able to complete the proof of Proposition 3.1.

Proof of Proposition 3.1. We get from Lemmas 3.2 and (3.6) that for δ small enough

$$\begin{aligned}
\langle \mathscr{L} f, f\rangle_{L^2_v(m)} &= \langle \mathscr{L}^c_\delta f, f\rangle_{L^2_v(m)} + \langle \mathscr{L}_\delta f, f\rangle_{L^2_v(m)} \\
&\leqslant \varepsilon\|f\|^2_{\dot{H}^{s,*}_v(m)} + C_\varepsilon\|\langle v\rangle^{\gamma/2} f\|_{L^2_v(m)} - c\delta^{-2s}\|\langle v\rangle^{\gamma/2} f\|^2_{L^2_v(m)} + C\|f\|^2_{L^2_v} \\
&\leqslant \varepsilon\|f\|^2_{\dot{H}^{s,*}_v(m)} - \|\langle v\rangle^{\gamma/2} f\|^2_{L^2_v(m)} + C\|f\|^2_{L^2_v(m)}.
\end{aligned}$$

We interpolate this estimate with (3.7): for any $\theta \in [0,1]$

$$\begin{aligned}
\langle \mathscr{L} f, f\rangle_{L^2_v(m)} \leqslant& \left[\theta\varepsilon - (1-\theta)c\right]\|f\|^2_{\dot{H}^{s,*}_v(m)} \\
&+ \left[-\theta + (1-\theta)C\right]\|\langle v\rangle^{\gamma/2} f\|_{L^2_v(m)} + \theta C\|f\|^2_{L^2_v}.
\end{aligned}$$

We deduce (3.1) by taking θ close enough to 1, ε small enough, and integrating in space. □

3.2. **Estimates on Λ.** We already know from [34, (2.20)] that in the gaussian space $\mathbf{E}$, the full linearized operator $\Lambda = \mathscr{L} - v\cdot\nabla_x$ dissipates the $\mathbf{E}^*$-norm. We recall here this result.

Proposition 3.4 ([34])**.** *The equivalent scalar product defined in* (2.39) *satisfy the coercive-type estimate:*

$$\langle\!\langle \Lambda f, f\rangle\!\rangle_{\mathbf{E}} \lesssim -\|f\|^2_{\mathbf{E}^*}.$$

Concerning the polynomial space $\mathbf{X}(m)$, we will rely on the following splitting of the linearized operator Λ:

$$\Lambda = \mathcal{A} + \mathcal{B},$$
$$\mathcal{A} := M\chi_R, \quad \mathcal{B} := \Lambda - \mathcal{A},$$

with constants $M, R > 0$ and $\chi_R(v) = \chi(v/R)$, where $\chi \in C_c^\infty(\mathbf{R}^3)$ is a smooth function satisfying $\mathbf{1}_{|v|\leqslant 1} \leqslant \chi \leqslant \mathbf{1}_{|v|\leqslant 2}$. The parameters $M, R > 0$ will be tuned later (to be chosen large enough) in order to make $\mathcal{B}$ dissipative.

Proposition 3.5. *Assume $k > 13/2 + 2|\gamma|$ and define $m = \langle v\rangle^k$. There are $M_0, R_0 > 0$ large enough such that for any $M \geqslant M_0$ and $R \geqslant R_0$ there holds, for any smooth enough function f,*

$$\langle \mathcal{B}f, f\rangle_{L^2_{x,v}(m)} \lesssim -\|f\|^2_{L^2_x H^{s,*}_v(m)}.$$

Proof. We write

$$\langle \mathcal{B}f, f\rangle_{L^2_{x,v}(m)} = \langle \mathscr{L} - \mathcal{A}f, f\rangle_{L^2_{x,v}(m)} - \langle v\cdot\nabla_x f, f\rangle_{L^2_{x,v}(m)}.$$

The second term vanishes in virtue of its gradient structure $(\nabla_x f)\, f = \nabla_x(|f|^2)$, thus by Proposition 3.1, we have for some constants $c, C > 0$

$$\begin{aligned}\langle \mathcal{B}f, f\rangle_{L^2_{x,v}(m)} &\leqslant -c\|f\|^2_{L^2_x H^{s,*}_v(m)} + C\|f\|^2_{L^2_x L^2_v} - M\|\chi_R(v) f\|^2_{L^2_x L^2_v(m)} \\ &\leqslant -\frac{c}{2}\|f\|^2_{L^2_x H^{s,*}_v(m)} - \int \Big(\frac{c}{2}\langle v\rangle^\gamma - Cm^{-2} + M\chi_R(v)\Big)\,|f|^2 m^2\,\mathrm{d}v\,\mathrm{d}x,\end{aligned}$$

by using that $\|f\|_{L^2_x H^{s,*}_v(m)} \geqslant \|\langle v\rangle^{\gamma/2} f\|_{L^2_x L^2_v(m)}$. For large values of $|v|$, we have that $\frac{c}{2}\langle v\rangle^\gamma - Cm^{-2} > 0$ by the assumption $k > |\gamma|/2$, thus there are $M_0, R_0 > 0$ large enough such that for all $M \geqslant M_0$ and $R \geqslant R_0$ we have

$$\frac{c}{2}\langle v\rangle^\gamma - Cm^{-2} + M\chi_R(v) > 0,$$

from which we deduce the desired estimate. □

As an immediate consequence of Proposition 3.5 and the fact that $\mathcal{B}$ commutes with ∇_x, we obtain the following dissipative estimate for $\mathcal{B}$ in spaces of the type $\mathbf{X}(m)$ and $\mathbf{X}^*(m)$, recalling the definition in (1.10) and (1.11), respectively:

Corollary 3.6. *Assume $k > 13/2 + 2|\gamma| + 6s$ and define $m = \langle v\rangle^k$. There are $M_0, R_0 > 0$ large enough such that for any $M \geqslant M_0$ and $R \geqslant R_0$ there holds, for any smooth enough function f,*

$$\langle \mathcal{B}f, f\rangle_{\mathbf{X}(m)} \lesssim -\|f\|^2_{\mathbf{X}^*(m)}.$$

4. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Recall that we assume non-cutoff soft potentials (1.4)–(1.5). We fix

$$k > 13/2 + 7|\gamma|/2 + 8s$$

and define the weight function $m = \langle v\rangle^k$. We also consider $\theta, k_0 > 0$ such that

$$2 < \theta < \frac{2(k-k_0)}{|\gamma|} \quad \text{and} \quad k_0 > 13/2 + 5|\gamma|/2 + 8s$$

and define the weight function $m_0 = \langle v\rangle^{k_0}$. Finally we define the weight functions $\underline{m} = m\langle v\rangle^{-2s} = \langle v\rangle^{k-2s}$ as well as $\underline{m_0} = m_0\langle v\rangle^{-2s} = \langle v\rangle^{k_0-2s}$. Observe that we have

$$k_0 - 2s > 13/2 + 5|\gamma|/2 + 6s$$

so that we may apply in the sequel Propositions 2.12 and 2.13 as well as Corollary 3.6 with the weight function $\underline{m_0}$, and consequently also with the weights $\underline{m}$, m_0 and m.

Drawing inspiration from [10], we seek a solution to (1.6) of the form

$$f(t) = h(t) + g(t) \in \mathbf{X}(m) + \mathbf{E}$$

where the two parts evolve according to the differential system

$$\begin{cases} \partial_t h = \mathcal{B}h + Q(h,h) + Q(g,h) + Q(h,g), \\ \partial_t g = \Lambda g + Q(g,g) + \mathcal{A}h, \\ h(0,x,v) = f_0(x,v), \quad g(0,x,v) = 0. \end{cases} \tag{4.1}$$

We will construct a solution to this system by building a sequence of approximate solutions $(h_N, g_N)_{N=0}^{\infty}$ initialized as $(h_0, g_0) = 0$ and defined inductively by

$$\begin{cases} \partial_t h_{N+1} = \mathcal{B}h_{N+1} + Q(h_N, h_{N+1}) + Q(g_N, h_{N+1}) + Q(h_{N+1}, g_N), \\ \partial_t g_{N+1} = \Lambda g_{N+1} + Q(g_N, g_{N+1}) + \mathcal{A}h_N, \\ h_{N+1}(0,x,v) = f_0(x,v), \quad g_{N+1}(0,x,v) = 0. \end{cases} \tag{4.2}$$

To do so, we introduce the functional spaces $\mathscr{X}(m)$, $\mathscr{Y}(m_0)$ and $\mathscr{E}$ as the spaces associated to the norms, respectively:

$$\|h\|^2_{\mathscr{X}(m)} := \sup_{t\geqslant 0} \|h(t)\|^2_{\mathbf{X}(m)} + \int_0^\infty \|h(t)\|^2_{\mathbf{X}^*(m)}\,\mathrm{d}t, \tag{4.3}$$

$$\|h\|^2_{\mathscr{Y}(m_0)} := \sup_{t\geqslant 0} (1+t)^\theta \|h(t)\|^2_{\mathbf{X}(m_0)} + \int_0^\infty (1+t)^\theta \|h(t)\|^2_{\mathbf{X}^*(m_0)}\,\mathrm{d}t, \tag{4.4}$$

$$\|g\|^2_{\mathscr{E}} := \sup_{t\geqslant 0} |||g(t)|||^2_{\mathbf{E}} + \int_0^\infty \|g(t)\|^2_{\mathbf{E}^*}\,\mathrm{d}t, \tag{4.5}$$

where we recall that the norm $|||\cdot|||_{\mathbf{E}}$ is defined in (2.39) and it is equivalent to $\|\cdot\|_{\mathbf{E}}$, and that the spaces $\mathbf{X}(m)$, $\mathbf{X}^*(m)$, $\mathbf{Y}(m)$, $\mathbf{E}$ and $\mathbf{E}^*$ are defined respectively in (1.10), (1.11), (1.12), (2.20) and (2.21). Similarly, we also consider the spaces $\mathscr{X}(\underline{m})$ and $\mathscr{Y}(\underline{m_0})$ defined respectively by (4.3) and (4.4) but with the weights $\underline{m}$ and $\underline{m_0}$.

4.1. **Stability of the scheme.** In this subsection, we will show by induction that if the initial data satisfies

$$\|f_0\|_{\mathbf{X}(m)} \leqslant \varepsilon_0$$

with $\varepsilon_0 > 0$ small enough, then the following bound holds for all $N \geqslant 0$:

$$\|h_N\|_{\mathscr{X}(m)} + \|h_N\|_{\mathscr{Y}(m_0)} + \|g_N\|_{\mathscr{E}} \lesssim \|f_0\|_{\mathbf{X}(m)} \leqslant \varepsilon_0. \tag{4.6}$$

This is of course true for $N = 0$. Assume this bound for some $N \geqslant 0$ and let us deduce it for $N+1$.

4.1.1. *Stability of h_{N+1} in norm $\mathscr{X}(m)$.* We start with the first equation of (4.2). The weak coercivity estimate on $\mathcal{B}$ from Corollary 3.6 gives for some $\lambda > 0$

$$\langle \mathcal{B}h_{N+1}, h_{N+1}\rangle_{\mathbf{X}(m)} \leqslant -\lambda \|h_{N+1}\|^2_{\mathbf{X}^*(m)},$$

and the nonlinear terms are estimated using Proposition 2.12:

$$\begin{aligned} \langle Q(h_N, h_{N+1}), h_{N+1}\rangle_{\mathbf{X}(m)} \lesssim & \|h_{N+1}\|^2_{\mathbf{X}^*(m)} \|h_N\|_{\mathbf{X}(m)} \\ & + \|h_{N+1}\|_{\mathbf{X}(m)} \|h_{N+1}\|_{\mathbf{X}^*(m)} \|h_N\|_{\mathbf{X}^*(m)}, \end{aligned}$$

$$\begin{aligned} \langle Q(g_N, h_{N+1}), h_{N+1}\rangle_{\mathbf{X}(m)} \lesssim & \|h_{N+1}\|^2_{\mathbf{X}^*(m)} \|g_N\|_{\mathbf{E}} \\ & + \|h_{N+1}\|_{\mathbf{X}(m)} \|h_{N+1}\|_{\mathbf{X}^*(m)} \|g_N\|_{\mathbf{E}^*}, \end{aligned}$$

as well as the bound (2.31):

$$\begin{aligned} \langle Q(h_{N+1}, g_N), h_{N+1}\rangle_{\mathbf{X}(m)} \lesssim & \|h_{N+1}\|^2_{\mathbf{X}^*(m)} \|g_N\|_{\mathbf{E}} \\ & + \|h_{N+1}\|_{\mathbf{X}(m)} \|h_{N+1}\|_{\mathbf{X}^*(m)} \|g_N\|_{\mathbf{E}^*}. \end{aligned}$$

To sum up, we have the following energy estimate for h_{N+1}:

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h_{N+1}\|^2_{\mathbf{X}(m)} + \lambda \|h_{N+1}\|^2_{\mathbf{X}^*(m)} \lesssim & \|h_{N+1}\|^2_{\mathbf{X}^*(m)} \left(\|g_N\|_{\mathbf{E}} + \|h_N\|_{\mathbf{X}(m)}\right) \\ & + \|h_{N+1}\|_{\mathbf{X}(m)} \|h_{N+1}\|_{\mathbf{X}^*(m)} \left(\|h_N\|_{\mathbf{X}^*(m)} + \|g_N\|_{\mathbf{E}^*}\right). \end{aligned}$$

Integrating in time and simplifying by $\|h_{N+1}\|_{\mathscr{X}(m)}$, we obtain using the bounds (4.6)

$$\|h_{N+1}\|_{\mathscr{X}(m)} \lesssim \varepsilon_0 \|h_{N+1}\|_{\mathscr{X}(m)} + \|f_0\|_{\mathbf{X}(m)},$$

which, assuming $\varepsilon_0 > 0$ small enough, simplifies as

$$\|h_{N+1}\|_{\mathscr{X}(m)} \lesssim \|f_0\|_{\mathbf{X}(m)} \leqslant \varepsilon_0. \tag{4.7}$$

This concludes this step.

4.1.2. *Stability of h_{N+1} in norm $\mathscr{Y}(m_0)$.* As for the estimates with the weight m, we have for some $\lambda > 0$

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h_{N+1}\|^2_{\mathbf{X}(m_0)} + \lambda \|h_{N+1}\|^2_{\mathbf{X}^*(m_0)} \\
&\quad \lesssim \|h_{N+1}\|^2_{\mathbf{X}^*(m_0)} \left(\|g_N\|_{\mathbf{E}} + \|h_N\|_{\mathbf{X}(m_0)}\right) \\
&\qquad + \|h_{N+1}\|_{\mathbf{X}(m_0)} \|h_{N+1}\|_{\mathbf{X}^*(m_0)} \left(\|h_N\|_{\mathbf{X}^*(m_0)} + \|g_N\|_{\mathbf{E}^*}\right),
\end{aligned}$$

which, using the bounds (4.6) and assuming $\varepsilon_0 > 0$ small enough in order to absorb the first term in the right-hand side by the left-hand side, simplifies as

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h_{N+1}\|^2_{\mathbf{X}(m_0)} + \frac{\lambda}{2} \|h_{N+1}\|^2_{\mathbf{X}^*(m_0)} \\
&\quad \lesssim \|h_{N+1}\|_{\mathbf{X}(m_0)} \|h_{N+1}\|_{\mathbf{X}^*(m_0)} \left(\|h_N\|_{\mathbf{X}^*(m_0)} + \|g_N\|_{\mathbf{E}^*}\right),
\end{aligned}$$

Moreover, since we have $\|\langle v\rangle^{\gamma/2} h\|_{\mathbf{X}(m_0)} \leqslant \|h\|_{\mathbf{X}^*(m_0)}$, the following interpolation inequality holds for any $R > 0$:

$$\langle R\rangle^{-|\gamma|} \|h\|^2_{\mathbf{X}(m_0)} \leqslant \|h\|^2_{\mathbf{X}^*(m_0)} + \langle R\rangle^{-|\gamma|-2(k-k_0)} \|h\|^2_{\mathbf{X}(m)},$$

thus, taking $\langle R\rangle = \left(\frac{\lambda}{2\theta}\right)^{1/|\gamma|} (1+t)^{1/\gamma|}$, we have

$$\frac{2\theta}{\lambda}(1+t)^{-1} \|h\|^2_{\mathbf{X}(m_0)} \leqslant \|h\|^2_{\mathbf{X}^*(m_0)} + \frac{2\theta}{\lambda}(1+t)^{-1-\frac{2(k-k_0)}{|\gamma|}} \|h\|^2_{\mathbf{X}(m)}.$$

We now plug this control in the energy estimate:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h_{N+1}\|^2_{\mathbf{X}(m_0)} + \frac{\lambda}{4} \|h_{N+1}\|^2_{\mathbf{X}^*(m_0)} + \frac{\theta}{2}(1+t)^{-1} \|h_{N+1}\|^2_{\mathbf{X}^*(m_0)} \\
&\quad \lesssim (1+t)^{-1-\frac{2(k-k_0)}{|\gamma|}} \|h_{N+1}\|^2_{\mathbf{X}(m)} \\
&\qquad + \|h_{N+1}\|_{\mathbf{X}(m_0)} \|h_{N+1}\|_{\mathbf{X}^*(m_0)} \left(\|h_N\|_{\mathbf{X}^*(m_0)} + \|g_N\|_{\mathbf{E}^*}\right),
\end{aligned}$$

and then multiply both sides by $(1+t)^\theta$:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+t)^\theta \|h_{N+1}\|^2_{\mathbf{X}(m_0)}\right\} + \frac{\lambda}{2}(1+t)^\theta \|h_{N+1}\|^2_{\mathbf{X}^*(m_0)} \\
&\quad \lesssim (1+t)^{\theta-1-\frac{2(k-k_0)}{|\gamma|}} \|h_{N+1}\|^2_{\mathbf{X}(m)} \\
&\qquad + (1+t)^\theta \|h_{N+1}\|_{\mathbf{X}(m_0)} \|h_{N+1}\|_{\mathbf{X}^*(m_0)} \left(\|h_N\|_{\mathbf{X}^*(m_0)} + \|g_N\|_{\mathbf{E}^*}\right).
\end{aligned}$$

Integrating in time and using (4.6), we get

$$\|h_{N+1}\|^2_{\mathscr{Y}(m_0)} \lesssim \|h_{N+1}\|^2_{\mathscr{X}(m)} + \varepsilon_0 \|h_{N+1}\|^2_{\mathscr{Y}(m_0)} + \|f_0\|^2_{\mathbf{X}(m_0)},$$

where we used the fact that $\theta - 1 - \frac{2(k-k_0)}{|\gamma|} < -1$ so that $(1+t)^{\theta-1-\frac{2(k-k_0)}{|\gamma|}}$ is integrable. Assuming $\varepsilon_0 > 0$ small enough and plugging in (4.7), we finally get

$$\|h_{N+1}\|_{\mathscr{Y}(m_0)} \lesssim \|f_0\|_{\mathbf{X}(m_0)} \leqslant \varepsilon_0. \tag{4.8}$$

This concludes this step.

4.1.3. *Stability of g_{N+1} in norm $\mathscr{E}$.* We now turn to the second equation of (4.2). The weak coercivity estimate from Proposition 3.4 gives us for some $\lambda > 0$

$$\langle\!\langle \Lambda g_{N+1}, g_{N+1}\rangle\!\rangle_{\mathbf{E}} \leqslant -\lambda \|g_{N+1}\|^2_{\mathbf{E}^*}.$$

The nonlinear term is estimated using (2.41):

$$\langle\!\langle Q(g_N, g_{N+1}), g_{N+1}\rangle\!\rangle_{\mathbf{E}} \lesssim \|g_N\|_{\mathbf{E}}\|g_{N+1}\|^2_{\mathbf{E}^*} + \|g_{N+1}\|_{\mathbf{E}^*}\|g_{N+1}\|_{\mathbf{E}}\|g_N\|_{\mathbf{E}^*}.$$

The coupling term is estimated using (2.42):

$$\langle\!\langle \mathcal{A}h_N, g_{N+1}\rangle\!\rangle_{\mathbf{E}} \lesssim \|h_N\|_{\mathbf{X}(m_0)}\|g_{N+1}\|_{\mathbf{E}}.$$

Putting these bounds together et recalling that $|||\cdot|||_{\mathbf{E}}$ and $\|\cdot\|_{\mathbf{E}}$ are equivalent, we obtain the following energy estimate for g_{N+1}:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||g_{N+1}|||^2_{\mathbf{E}} + \lambda\|g_{N+1}\|^2_{\mathbf{E}^*} \lesssim& |||g_N|||_{\mathbf{E}}\|g_{N+1}\|^2_{\mathbf{E}^*} + \|g_{N+1}\|_{\mathbf{E}^*}|||g_{N+1}|||_{\mathbf{E}}\|g_N\|_{\mathbf{E}^*}\\ &+ \|h_N\|_{\mathbf{X}(m_0)}|||g_{N+1}|||_{\mathbf{E}},\end{aligned}$$

which, using (4.6) and assuming $\varepsilon_0 > 0$ small enough in order to absorb the first term in the right-hand side by the left-hand sid, simplifies into

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||g_{N+1}|||^2_{\mathbf{E}} + \frac{\lambda}{2}\|g_{N+1}\|^2_{\mathbf{E}^*} \lesssim \|g_{N+1}\|_{\mathbf{E}^*}|||g_{N+1}|||_{\mathbf{E}}\|g_N\|_{\mathbf{E}^*} + \|h_N\|_{\mathbf{X}(m_0)}|||g_{N+1}|||_{\mathbf{E}}.$$

Integrating in time and using (4.6) and simplifying by $\|g_{N+1}\|_{\mathscr{E}}$, we finally get

$$\begin{aligned}\|g_{N+1}\|^2_{\mathscr{E}} &\lesssim \|g_N\|_{\mathscr{E}}\|g_{N+1}\|^2_{\mathscr{E}} + \|h_N\|_{\mathscr{Y}(m_0)}\|g_{N+1}\|_{\mathscr{E}}\\ &\lesssim \varepsilon_0\|g_{N+1}\|^2_{\mathscr{E}} + \|f_0\|_{\mathbf{X}(m)}\|g_{N+1}\|_{\mathscr{E}},\end{aligned}$$

which implies, assuming $\varepsilon_0 > 0$ small enough, that

$$\|g_{N+1}\|_{\mathscr{E}} \lesssim \|f_0\|_{\mathbf{X}(m)} \leqslant \varepsilon_0. \tag{4.9}$$

This concludes this step.

We therefore deduce (4.6) for $N+1$ by gathering estimates (4.7), (4.8) and (4.9), which completes the stability part of the proof.

4.2. **Convergence of the scheme.** Consider the successive differences of $(h_N)_{N=0}^{\infty}$ denoted by $d_{N+1} := h_{N+1} - h_N$, that of $(g_N)_{N=0}^{\infty}$ by $e_{N+1} := g_{N+1} - g_N$, and consider the equation satisfied by d_{N+1}

$$\begin{cases}\partial_t d_{N+1} = \mathcal{B}d_{N+1} + Q(h_N, d_{N+1}) + Q(d_N, h_N)\\ \qquad\qquad\qquad + Q(g_N, d_{N+1}) + Q(e_N, h_N)\\ \qquad\qquad\qquad + Q(d_{N+1}, g_N) + Q(h_N, e_N),\\ d_{N+1}(0,x,v) = 0,\end{cases} \tag{4.10}$$

as well as the one satisfied by e_{N+1}:

$$\begin{cases}\partial_t e_{N+1} = \Lambda e_{N+1} + Q(g_N, e_{N+1}) + Q(e_N, g_N) + \mathcal{A}d_N,\\ e_{N+1}(0,x,v) = 0.\end{cases} \tag{4.11}$$

In this subsection, we shall establish that for $\varepsilon_0 > 0$ small enough the following bound holds, for some $C_0 > 0$ and all $N \geqslant 0$:

$$\|e_N\|_{\mathscr{E}} + \|d_N\|_{\mathscr{X}(\underline{m})} + \|d_N\|_{\mathscr{Y}(\underline{m_0})} \lesssim (C_0\varepsilon_0)^{N/2}. \tag{4.12}$$

4.2.1. *Recursive estimates for d_{N+1} in norm $\mathscr{X}(\underline{m})$.* Let $N \in \mathbf{N}$. We start by considering the first equation (4.10). The estimates of Proposition 2.12 give the following control:

$$\begin{aligned}\langle Q(h_N, d_{N+1}), d_{N+1}\rangle_{\mathbf{X}(\underline{m})} \lesssim & \|d_{N+1}\|^2_{\mathbf{X}^*(\underline{m})}\|h_N\|_{\mathbf{X}(\underline{m})} \\ & + \|h_N\|_{\mathbf{X}^*(\underline{m})}\|d_{N+1}\|_{\mathbf{X}(\underline{m})}\|d_{N+1}\|_{\mathbf{X}^*(\underline{m})},\end{aligned}$$

and also

$$\begin{aligned}\langle Q(g_N, d_{N+1}), d_{N+1}\rangle_{\mathbf{X}(\underline{m})} \lesssim & \|d_{N+1}\|^2_{\mathbf{X}^*(\underline{m})}\|g_N\|_{\mathbf{E}} \\ & + \|g_N\|_{\mathbf{E}^*}\|d_{N+1}\|_{\mathbf{X}(\underline{m})}\|d_{N+1}\|_{\mathbf{X}^*(\underline{m})}.\end{aligned}$$

Moreover, estimate (2.31) gives

$$\begin{aligned}\langle Q(d_{N+1}, g_N), d_{N+1}\rangle_{\mathbf{X}(\underline{m})} \lesssim & \|d_{N+1}\|^2_{\mathbf{X}^*(\underline{m})}\|g_N\|_{\mathbf{E}} \\ & + \|d_{N+1}\|_{\mathbf{X}^*(\underline{m})}\|g_N\|_{\mathbf{E}^*}\|d_{N+1}\|_{\mathbf{X}(\underline{m})},\end{aligned}$$

as well as

$$\begin{aligned}\langle Q(h_N, e_N), d_{N+1}\rangle_{\mathbf{X}(\underline{m})} \lesssim & \|d_{N+1}\|_{\mathbf{X}^*(\underline{m})}\|h_N\|_{\mathbf{X}^*(\underline{m})}\|e_N\|_{\mathbf{E}} \\ & + \|d_{N+1}\|_{\mathbf{X}^*(\underline{m})}\|e_N\|_{\mathbf{E}^*}\|h_N\|_{\mathbf{X}(\underline{m})} \\ & + \|d_{N+1}\|_{\mathbf{X}(\underline{m})}\|e_N\|_{\mathbf{E}^*}\|h_N\|_{\mathbf{X}^*(\underline{m})}.\end{aligned}$$

Finally, estimates (2.32) and (2.30) give respectively the following bounds, which force to work in the larger space $\mathbf{X}(\underline{m})$ instead of $\mathbf{X}(m)$:

$$\begin{aligned}\langle Q(e_N, h_N), d_{N+1}\rangle_{\mathbf{X}(\underline{m})} \lesssim & \|d_{N+1}\|_{\mathbf{X}^*(\underline{m})}\|e_N\|_{\mathbf{E}^*}\|h_N\|_{\mathbf{X}(\underline{m})} \\ & + \|d_{N+1}\|_{\mathbf{X}^*(\underline{m})}\|h_N\|_{\mathbf{X}^*(m)}\|e_N\|_{\mathbf{E}},\end{aligned}$$

and

$$\begin{aligned}\langle Q(d_N, h_N), d_{N+1}\rangle_{\mathbf{X}(\underline{m})} \lesssim & \|d_{N+1}\|_{\mathbf{X}(\underline{m})}\|d_N\|_{\mathbf{X}^*(\underline{m})}\|h_N\|_{\mathbf{X}^*(\underline{m})} \\ & + \|d_{N+1}\|_{\mathbf{X}^*(\underline{m})}\|h_N\|_{\mathbf{X}^*(m)}\|d_N\|_{\mathbf{X}(\underline{m})}.\end{aligned}$$

As in the step of stability in section 4.1.1, we put these bounds together and integrate the resulting energy estimate to obtain the following control:

$$\|d_{N+1}\|^2_{\mathscr{X}(\underline{m})} \lesssim \varepsilon_0\|d_{N+1}\|^2_{\mathscr{X}(\underline{m})} + \varepsilon_0\|d_{N+1}\|_{\mathscr{X}(\underline{m})}\|e_N\|_{\mathscr{E}} + \varepsilon_0\|d_{N+1}\|_{\mathscr{X}(\underline{m})}\|d_N\|_{\mathscr{X}(\underline{m})}$$

where we used the stability estimates (4.6). Assuming $\varepsilon_0 > 0$ small enough, this simplifies as

$$\|d_{N+1}\|_{\mathscr{X}(\underline{m})} \lesssim \varepsilon_0\|e_N\|_{\mathscr{E}} + \varepsilon_0\|d_N\|_{\mathscr{X}(\underline{m})}. \tag{4.13}$$

4.2.2. *Recursive estimate for d_{N+1} in norm $\mathscr{Y}(\underline{m_0})$.* Let $N \in \mathbf{N}$. Arguing as in the step of stability in section 4.1.2, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{(1+t)^\theta\|d_{N+1}\|^2_{\mathbf{X}(\underline{m_0})}\Big\} + \frac{\lambda}{2}(1+t)^\theta\|d_{N+1}\|^2_{\mathbf{X}^*(\underline{m_0})} \\ &\quad\lesssim (1+t)^\theta\|d_{N+1}\|_{\mathbf{X}(\underline{m_0})}\|d_{N+1}\|_{\mathbf{X}^*(\underline{m_0})}\left(\|h_N\|_{\mathbf{X}^*(m_0)} + \|g_N\|_{\mathbf{E}^*}\right) \\ &\qquad + (1+t)^\theta\|d_{N+1}\|_{\mathbf{X}^*(\underline{m_0})}\|h_N\|_{\mathbf{X}^*(m_0)}\left(\|e_N\|_{\mathbf{E}} + \|d_N\|_{\mathbf{X}(\underline{m_0})}\right) \\ &\qquad + (1+t)^\theta\|d_{N+1}\|_{\mathbf{X}^*(\underline{m_0})}\|h_N\|_{\mathbf{X}(\underline{m_0})}\left(\|e_N\|_{\mathbf{E}^*} + \|d_N\|_{\mathbf{X}^*(\underline{m_0})}\right) \\ &\qquad + (1+t)^\theta\|d_{N+1}\|_{\mathbf{X}(\underline{m_0})}\|h_N\|_{\mathbf{X}^*(\underline{m_0})}\|e_N\|_{\mathbf{E}^*} \\ &\qquad + (1+t)^{\theta-1-\frac{2(k-k_0)}{|\gamma|}}\|d_{N+1}\|^2_{\mathbf{X}(\underline{m})}.\end{aligned}$$

After integrating and using the bounds (4.6) from the stability estimate, we are left with

$$\begin{aligned}\|d_{N+1}\|^2_{\mathscr{Y}(\underline{m_0})} &\lesssim \varepsilon_0\|d_{N+1}\|^2_{\mathscr{Y}(\underline{m_0})} + \varepsilon_0\|d_{N+1}\|_{\mathscr{Y}(\underline{m_0})}\left(\|e_N\|_{\mathscr{E}} + \|d_N\|_{\mathscr{Y}(\underline{m_0})}\right) + \|d_{N+1}\|^2_{\mathscr{X}(\underline{m})} \\ &\lesssim \varepsilon_0\|d_{N+1}\|^2_{\mathscr{Y}(\underline{m_0})} + \varepsilon_0\left(\|e_N\|^2_{\mathscr{E}} + \|d_N\|^2_{\mathscr{Y}(\underline{m_0})}\right) + \|d_{N+1}\|^2_{\mathscr{X}(\underline{m})},\end{aligned}$$

where we used Young's inequality in the last line. Assuming $\varepsilon_0 > 0$ small enough and plugging (4.13) in, this bound simplifies as

$$\|d_{N+1}\|_{\mathscr{Y}(\underline{m_0})} \lesssim \varepsilon_0 \|e_N\|_{\mathscr{E}} + \varepsilon_0 \|d_N\|_{\mathscr{X}(\underline{m_0})} + \varepsilon_0 \|d_N\|_{\mathscr{Y}(\underline{m_0})}. \tag{4.14}$$

4.2.3. *Recursive estimates for e_N in norm $\mathscr{E}$.* Let $N \in \mathbf{N}$. We now considering the second equation (4.11). Using (2.41) we have

$$\langle\!\langle Q(g_N, e_{N+1}), e_{N+1} \rangle\!\rangle_{\mathbf{E}} \lesssim \|g_N\|_{\mathbf{E}} \|e_{N+1}\|_{\mathbf{E}^*}^2 + \|e_{N+1}\|_{\mathbf{E}^*} \|e_{N+1}\|_{\mathbf{E}} \|g_N\|_{\mathbf{E}^*},$$

as well as

$$\langle\!\langle Q(e_N, g_N), e_{N+1} \rangle\!\rangle_{\mathbf{E}} \lesssim \|e_{N+1}\|_{\mathbf{E}^*} \left(\|e_N\|_{\mathbf{E}^*} \|g_N\|_{\mathbf{E}} + \|e_N\|_{\mathbf{E}} \|g_N\|_{\mathbf{E}^*} \right),$$

and using (2.42) we get

$$\langle\!\langle \mathcal{A} d_N, e_{N+1} \rangle\!\rangle_{\mathbf{E}} \lesssim \|d_N\|_{\mathbf{X}(\underline{m_0})} \|e_{N+1}\|_{\mathbf{E}}.$$

Arguing as in section 4.1.3, we gather these estimates and integrate the resulting energy estimate in time to obtain the following bound:

$$\|e_{N+1}\|_{\mathscr{E}} \lesssim \|g_N\|_{\mathscr{E}} \|e_{N+1}\|_{\mathscr{E}} + \|g_N\|_{\mathscr{E}} \|e_N\|_{\mathscr{E}} + \|d_N\|_{\mathscr{Y}(\underline{m_0})},$$

which, using the stability estimates (4.6) and assuming $\varepsilon_0 > 0$ small enough, simplifies as

$$\|e_{N+1}\|_{\mathscr{E}} \lesssim \varepsilon_0 \|e_N\|_{\mathscr{E}} + \|d_N\|_{\mathscr{Y}(\underline{m_0})}. \tag{4.15}$$

4.2.4. *Proof of convergence.* We first prove (4.12) by using previous estimates. It is clearly true for $N = 0$, so we assume that (4.12) holds for all integers up to some $N \geqslant 0$, and we shall deduce it for $N + 1$. Thanks to estimates (4.13), (4.14) and (4.15) we have obtained, for all $N \geqslant 0$,

$$\begin{aligned}
\|d_{N+1}\|_{\mathscr{X}(\underline{m})} &\lesssim \varepsilon_0 \left(\|d_N\|_{\mathscr{X}(\underline{m})} + \|e_N\|_{\mathscr{E}} \right) \\
\|d_{N+1}\|_{\mathscr{Y}(\underline{m_0})} &\lesssim \varepsilon_0 \left(\|d_N\|_{\mathscr{X}(\underline{m})} + \|d_N\|_{\mathscr{Y}(\underline{m_0})} + \|e_N\|_{\mathscr{E}} \right) \\
\|e_{N+1}\|_{\mathscr{E}} &\lesssim \varepsilon_0 \|e_N\|_{\mathscr{E}} + \|d_N\|_{\mathscr{Y}(\underline{m_0})}.
\end{aligned}$$

This implies that

$$\|d_{N+1}\|_{\mathscr{X}(\underline{m})} + \|d_{N+1}\|_{\mathscr{Y}(\underline{m_0})} + \|e_{N+1}\|_{\mathscr{E}} \lesssim \varepsilon_0 (\|d_{N-1}\|_{\mathscr{X}(\underline{m})} + \|d_{N-1}\|_{\mathscr{Y}(\underline{m_0})} + \|e_{N-1}\|_{\mathscr{E}})$$

and thus, using (4.12) for $N - 1$, we deduce

$$\|d_{N+1}\|_{\mathscr{X}(\underline{m})} + \|d_{N+1}\|_{\mathscr{Y}(\underline{m_0})} + \|e_{N+1}\|_{\mathscr{E}} \lesssim \varepsilon_0 (C_0 \varepsilon_0)^{\frac{N-1}{2}} \lesssim (C_0 \varepsilon_0)^{\frac{N+1}{2}},$$

which proves (4.12).

Therefore, assuming $\varepsilon_0 > 0$ small enough, the sequence $(h_N, g_N)_{N \geqslant 0}$ is a Cauchy sequence in $\mathscr{X}(\underline{m}) \times \mathscr{E}$ and thus converges to some limit (h, g) in $\mathscr{X}(\underline{m}) \times \mathscr{E}$. In virtue of the stability estimates, the limit thus satisfies the bounds

$$\begin{aligned}
\sup_{t \geqslant 0} \|h(t)\|_{\mathbf{X}(m)}^2 + \int_0^\infty \|h(t)\|_{\mathbf{X}^*(m)}^2 \, \mathrm{d}t &\lesssim \|f_0\|_{\mathbf{X}(m)}^2, \\
\sup_{t \geqslant 0} (1+t)^\theta \|h(t)\|_{\mathbf{X}(m_0)}^2 + \int_0^\infty (1+t)^\theta \|h(t)\|_{\mathbf{X}^*(m_0)}^2 \, \mathrm{d}t &\lesssim \|f_0\|_{\mathbf{X}(m)}^2, \\
\sup_{t \geqslant 0} \|g(t)\|_{\mathbf{E}}^2 + \int_0^\infty \|g(t)\|_{\mathbf{E}^*}^2 \, \mathrm{d}t &\lesssim \|f_0\|_{\mathbf{X}(m)}^2.
\end{aligned}$$

The solution thus constructed to the original perturbation equation (1.6) is given by letting $f := h + g \in L^\infty(\mathbf{R}_+; \mathbf{X}(m)) \cap L^2(\mathbf{R}_+; \mathbf{Y}(m))$, which thus satisfies

$$\sup_{t \geqslant 0} \|f(t)\|_{\mathbf{X}(m)}^2 + \int_0^\infty \left(\|f^\perp(t)\|_{\mathbf{X}^*(m)}^2 + \|\nabla_x \pi f(t)\|_{H_x^2 L_v^2}^2 \right) \mathrm{d}t \lesssim \|f_0\|_{\mathbf{X}(m)}^2.$$

4.3. **Uniqueness of the solution.** Consider two solutions $f, \widetilde{f} \in L^\infty(\mathbf{R}_+;\mathbf{X}(m)) \cap L^2(\mathbf{R}_+;\mathbf{Y}(m))$ to (1.6) with the same initial condition f_0 and verifying

$$\sup_{t\geqslant 0}\|f(t)\|^2_{\mathbf{X}(m)} + \int_0^\infty \|f(t)\|^2_{\mathbf{Y}(m)}\,\mathrm{d}t \lesssim \|f_0\|^2_{\mathbf{X}(m)} \leqslant \varepsilon_0^2,$$
$$\sup_{t\geqslant 0}\|\widetilde{f}(t)\|^2_{\mathbf{X}(m)} + \int_0^\infty \|\widetilde{f}(t)\|^2_{\mathbf{Y}(m)}\,\mathrm{d}t \lesssim \|f_0\|^2_{\mathbf{X}(m)} \leqslant \varepsilon_0^2,$$

with $\varepsilon_0 > 0$ small enough. Denote $d := f - \widetilde{f}$ the difference of these solutions, which satisfies

$$\partial_t d = \mathcal{B}d + \mathcal{A}d + Q(d,f) + Q(\widetilde{f},d).$$

Arguing as in the previous steps, one gets for some $\lambda > 0$ the control

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|d\|^2_{\mathbf{X}(\underline{m})} + \lambda\|d\|^2_{\mathbf{X}^*(\underline{m})} \lesssim & \|d\|^2_{\mathbf{X}(\underline{m})} + \|d\|^2_{\mathbf{X}^*(\underline{m})}\|f\|_{\mathbf{X}(\underline{m})} + \|d\|_{\mathbf{X}^*(\underline{m})}\|d\|_{\mathbf{X}(\underline{m})}\|f\|_{\mathbf{Y}(m)} \\ & + \|d\|^2_{\mathbf{X}^*(\underline{m})}\|\widetilde{f}\|_{\mathbf{X}(\underline{m})} + \|d\|_{\mathbf{X}^*(\underline{m})}\|d\|_{\mathbf{X}(\underline{m})}\|\widetilde{f}\|_{\mathbf{X}^*(\underline{m})},\end{aligned}$$

which, once integrated from $t = 0$ to $t = T < \infty$, gives (with obvious notation)

$$\begin{aligned}\|d\|^2_{\mathscr{X}(\underline{m};T)} \lesssim & \Big(T + \|f\|_{L^\infty([0,T];\mathbf{X}(m))} + \|f\|_{L^2([0,T];\mathbf{X}^*(m))} \\ & + \|\widetilde{f}\|_{L^\infty([0,T];\mathbf{X}(m))} + \|\widetilde{f}\|_{L^2([0,T];\mathbf{X}^*(m))}\Big)\|d\|^2_{\mathscr{X}(\underline{m};T)}.\end{aligned}$$

Observing that

$$\|\phi\|_{\mathbf{X}^*(m)} \lesssim \|\pi\phi\|_{H^3_x L^2_v} + \|\phi^\perp\|_{\mathbf{X}^*(m)} \lesssim \|\pi\phi\|_{L^2_x L^2_v} + \|\phi\|_{\mathbf{Y}(m)} \lesssim \|\phi\|_{\mathbf{X}(m)} + \|\phi\|_{\mathbf{Y}(m)}$$

we have

$$\|f\|_{L^2([0,T];\mathbf{X}^*(m))} \lesssim \sqrt{T}\|f\|_{L^\infty([0,T];\mathbf{X}(m))} + \|f\|_{L^2([0,T];\mathbf{Y}(m))} \lesssim \sqrt{T}\varepsilon_0 + \varepsilon_0$$

and similarly for $\widetilde{f}$. Using the uniform bounds on f and $\widetilde{f}$, this becomes

$$\|d\|^2_{\mathscr{X}(\underline{m};T)} \lesssim \Big(T + \varepsilon_0 + \sqrt{T}\varepsilon_0\Big)\|d\|^2_{\mathscr{X}(\underline{m};T)}.$$

Assuming $T > 0$ small enough and $\varepsilon_0 > 0$ small enough, we have (for instance)

$$\|d\|^2_{\mathscr{X}(\underline{m};T)} \leqslant \frac{1}{2}\|d\|^2_{\mathscr{X}(\underline{m};T)},$$

which means that $d = 0$, or equivalently $f = \widetilde{f}$, on interval $[0,T]$. By continuing this argument, we deduce that f_0 gives rise to a unique (global) solution, namely, f. This concludes the proof of Theorem 1.1.

References

[1] Alexandre, R., Desvillettes, L., Villani, C., and Wennberg, B. Entropy dissipation and long-range interactions. *Arch. Ration. Mech. Anal. 152*, 4 (2000), 327–355.

[2] Alexandre, R., Hérau, F., and Li, W.-X. Global hypoelliptic and symbolic estimates for the linearized Boltzmann operator without angular cutoff. *J. Math. Pures Appl. (9) 126* (2019), 1–71.

[3] Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., and Yang, T. Regularizing effect and local existence for the non-cutoff Boltzmann equation. *Arch. Ration. Mech. Anal. 198* (2009), 39–123.

[4] Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., and Yang, T. The Boltzmann equation without angular cutoff in the whole space. II: Global existence for hard potential. *Anal. Appl., Singap. 9*, 2 (2011), 113–134.

[5] Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., and Yang, T. The Boltzmann equation without angular cutoff in the whole space: qualitative properties of solutions. *Arch. Ration. Mech. Anal. 202*, 2 (2011), 599–661.

[6] Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., and Yang, T. Global existence and full regularity of the Boltzmann equation without angular cutoff. *Commun. Math. Phys. 304*, 2 (2011), 513–581.

[7] Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., and Yang, T. The Boltzmann equation without angular cutoff in the whole space. I: global existence for soft potential. *J. Funct. Anal. 262*, 3 (2012), 915–1010.

[8] Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., and Yang, T. Local existence with mild regularity for the Boltzmann equation. *Kinet. Relat. Models 6*, 4 (2013), 1011–1041.
[9] Alonso, R., Morimoto, Y., Sun, W., and Yang, T. Non-cutoff Boltzmann equation with polynomial decay perturbations. *Rev. Mat. Iberoam. 37*, 1 (2021), 189–292.
[10] Briant, M., Merino-Aceituno, S., and Mouhot, C. From Boltzmann to incompressible Navier-Stokes in Sobolev spaces with polynomial weight. *Anal. Appl., Singap. 17*, 1 (2019), 85–116.
[11] Caflisch, R. E. The Boltzmann equation with a soft potential. II: Nonlinear, spatially- periodic. *Commun. Math. Phys. 74* (1980), 97–109.
[12] Cao, C. Cutoff Boltzmann equation with polynomial perturbation near Maxwellian, 2022. arXiv:2207.10272.
[13] Cao, C., He, L.-B., and Ji, J. Propagation of moments and sharp convergence rate for inhomogeneous non-cutoff Boltzmann equation with soft potentials, 2022. arXiv:2204.01394.
[14] Carrapatoso, K., Tristani, I., and Wu, K.-C. Cauchy problem and exponential stability for the inhomogeneous Landau equation. *Arch. Ration. Mech. Anal. 221*, 1 (2016), 363–418.
[15] Deng, D. Global existence of non-cutoff Boltzmann equation in weighted Sobolev space. *J. Stat. Phys. 188*, 3 (2022), 51. Id/No 25.
[16] Deng, D. On the Cauchy problem of Boltzmann equation with a very soft potential. *J. Evol. Equ. 22*, 1 (2022), 58. Id/No 17.
[17] Duan, R. On the Cauchy problem for the Boltzmann equation in the whole space: Global existence and uniform stability in $L^2_\xi(H^N_x)$. *J. Differ. Equations 244*, 12 (2008), 3204–3234.
[18] Duan, R., Liu, S., Sakamoto, S., and Strain, R. M. Global mild solutions of the Landau and non-cutoff Boltzmann equations. *Comm. Pure Appl. Math. 74*, 5 (2021), 932–1020.
[19] Duan, R., Sakamoto, S., and Ueda, Y. An $L^1_k \cap L^p_k$ approach for the non-cutoff Boltzmann equation in $\mathbb{R}^3$. arXiv:2209.10815, 2022.
[20] Ellis, R. S., and Pinsky, M. A. The first and second fluid approximation to the linearized Boltzmann equation. *J. Math. Pures Appl. (9) 54* (1975), 125–156.
[21] Glassey, R. T. *The Cauchy problem in kinetic theory*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.
[22] Grad, H. Asymptotic equivalence of the Navier-Stokes and nonlinear Boltzmann equations. Proc. Sympos. Appl. Math. 17, 154-183 (1965)., 1965.
[23] Gressman, P. T., and Strain, R. M. Global classical solutions of the Boltzmann equation without angular cut-off. *J. Am. Math. Soc. 24*, 3 (2011), 771–847.
[24] Gressman, P. T., and Strain, R. M. Sharp anisotropic estimates for the Boltzmann collision operator and its entropy production. *Adv. Math. 227*, 6 (2011), 2349–2384.
[25] Gualdani, M. P., Mischler, S., and Mouhot, C. Factorization of non-symmetric operators and exponential H-theorem. *Mém. Soc. Math. Fr., Nouv. Sér. 153* (2017), 3–137.
[26] Guo, Y. Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Ration. Mech. Anal. 169*, 4 (2003), 305–353.
[27] He, L.-B. Sharp bounds for Boltzmann and Landau collision operators. *Ann. Sci. Éc. Norm. Supér. (4) 51*, 5 (2018), 1253–1341.
[28] Hérau, F., Tonon, D., and Tristani, I. Regularization estimates and Cauchy theory for inhomogeneous Boltzmann equation for hard potentials without cut-off. *Commun. Math. Phys. 377*, 1 (2020), 697–771.
[29] Kawashima, S. The Boltzmann equation and thirteen moments. *Japan J. Appl. Math. 7*, 2 (1990), 301–320.
[30] Liu, T.-P., Yang, T., and Yu, S.-H. Energy method for Boltzmann equation. *Physica D 188*, 3-4 (2004), 178–192.
[31] Mouhot, C. Rate of convergence to equilibrium for the spatially homogeneous Boltzmann equation with hard potentials. *Commun. Math. Phys. 261*, 3 (2006), 629–672.
[32] Shizuta, Y. On the classical solutions of the Boltzmann equation. *Commun. Pure Appl. Math. 36* (1983), 705–754.
[33] Sohinger, V., and Strain, R. M. The Boltzmann equation, Besov spaces, and optimal time decay rates in $\mathbb{R}^n_x$. *Adv. Math. 261* (2014), 274–332.
[34] Strain, R. M. Optimal time decay of the non cut-off Boltzmann equation in the whole space. *Kinet. Relat. Models 5*, 3 (2012), 583–613.
[35] Strain, R. M., and Guo, Y. Almost exponential decay near Maxwellian. *Commun. Partial Differ. Equations 31*, 1-3 (2006), 417–429.
[36] Strain, R. M., and Guo, Y. Exponential decay for soft potentials near Maxwellian. *Arch. Ration. Mech. Anal. 187*, 2 (2008), 287–339.
[37] Ukai, S. On the existence of global solutions of mixed problem for non-linear Boltzmann equation. *Proc. Japan Acad. 50* (1974), 179–184.
[38] Ukai, S. Les solutions globales de l'équation de Boltzmann dans l'espace tout entier et dans le demi-espace. *C. R. Acad. Sci., Paris, Sér. A 282* (1976), 317–320.
[39] Ukai, S. Local solutions in Gevrey classes to the nonlinear Boltzmann equation without cutoff. *Japan J. Appl. Math. 1*, 1 (1984), 141–156.

[40] Ukai, S., and Asano, K. On the Cauchy problem of the Boltzmann equation with a soft potential. *Publ. Res. Inst. Math. Sci. 18* (1982), 477–519.
[41] Wang, H., and Fang, Z. Global existence and time decay of the non-cutoff Boltzmann equation with hard potential. *Nonlinear Anal., Real World Appl. 63* (2022), 32. Id/No 103416.
[42] Wu, Y., and Sun, J. Asymptotic behavior of linearized Boltzmann equations for soft potentials with cut-off. *Electron. J. Differ. Equ. 2021* (2021), 21. Id/No 46.

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