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Tools of the Table Crackers: Using Quantitative Methods to Analyze Historical Numerical Tables

Glen VAN BRUMMELEN, Matthieu HUSSON
and Clemency MONTELLE

1. Introduction: Defining the Issues

The content of numerical tables presents technical and historical challenges for those who seek to analyse them. However, such endeavors can be rewarding to historians. Table values provide a unique insight into the practices of historical table makers, beyond the usual textual content. As well as illuminating the practices of historical table-makers, tabular entries can also reveal something of the users of the tables and their priorities.

Tabular manuscripts can reveal new historical information that may not be explicit in other historical sources. One might be able to retrieve information about the original numerical methods, algorithms, and hand computations used to construct the entries. Other historical information might also be restored, including the author's sources, the abilities of those who compiled them and their priorities, the intended audience (who used it), the table's application (how it was used), the underlying theory (astronomical, physical or mathematical), transmission both to and from other sources, and dependence on other tables and traditions. In some cases, analyses of tables can even prompt scholars to propose new chronologies or dating.

Typically, table analysis efforts are directed primarily toward ancient and early sources, rather than modern. Generally the older the table, the less information we have surrounding its numerical data. With the passing of time, historical documents may be destroyed, damaged, separated, or corrupted. Documentation that surrounds the tables might not survive; tables might be incomplete. Furthermore, ancient cultures of inquiry may have had different priorities and incentives from their modern counterparts in presenting and promulgating scientific results. Often historical table makers were little motivated to reveal their techniques and methods of construction. All of this makes recourse to the results of table cracking even more valuable.

To reflect the complexity of the process of seeking the underlying structure of a table, as well as the epiphany once the content has been unlocked, table analysis has often been dubbed 'table cracking'. Table cracking involves

the manipulation of a table's entries to retrieve information about the table. Typically this includes the use of modern quantitative methods, arithmetical or statistical, to glean underlying features of the numerical data. This line of inquiry is often far removed from the original historical context in which the table was constructed. Nonetheless these features can be relevant to a proper historical understanding of the table and its context.

Despite the new information table cracking offers, many historians remain skeptical. Some feel uneasy or unqualified to assess the appropriateness of the sophisticated statistical procedures that are often employed. Others object to the number or scope of the assumptions that must be made, the apparent *ad hoc* nature of the process, or of the occasional imprecision or excessive precision of the results. Some are wary of the historical interpretations that have been proposed as a result of table cracking analysis.

We seek here to consider and classify the various processes that have fallen under the rubric of 'table cracking'. We will analyze through selected examples the procedures involved and the results they have generated. We will outline some of the advantages and pitfalls of this approach, advance some preliminary general standards for those who want to appreciate table crackers' results, present guidelines to evaluate them critically, and provide references to help readers begin to develop their own skills.

1.1. Accounting for Tabular Errors

The manual copying of literary texts invariably introduces errors or discrepancies; textual critics use this principle when they produce a critical edition of a text and determine the stemmatic relation between manuscripts. Errors also can be illuminating to table crackers.¹ Numerical tables are subject to the same discrepancies when they are copied. These types of error can help determine stemmatic relations between various copies of the same table and other related tables. Moreover, these discrepancies can take on additional significance. Indeed, errors in tabular values can be caused by significant factors other than unintentional copying mistakes.

The most obvious errors in tabulated data are scribal errors. These are accidental mistakes and alterations that are introduced when a table is copied. Usually the result of a moment of inattention or carelessness by the scribe, they can reveal something of the process of copying and other aspects of the original sources, such as common confusions between various symbolic notations or poor layouts. Another class of error is computational errors. These discrepancies can indicate idiosyncratic decisions made at intermediary steps

¹ See van Dalen, *Ancient and Mediaeval Astronomical Tables*, pp. 12-18; Neugebauer, *Astronomical Cuneiform Texts*, p. 27.

of computation, simple arithmetic mistakes, rounding conventions, the precision desired, or even differences in the nature of the function supposed to have been tabulated. These errors can be evidence of the difference between the algorithm expressed in the text and its tabular implementation.

Common ways of detecting errors include: comparing a recomputed tabular value with the original, examining first (or sometimes second) differences, and plotting the function values on a graph and looking for aberrations or irregularities. Sometimes one can trace the cause of an error back to some intermediary computational step, although given the variety of decisions that are made when implementing an algorithm, this is often very difficult in practice. Correcting errors can be more contentious; in some situations (especially where there may be several alternative computational models), it can be unclear precisely what the correct value corresponding to a given entry should be. In fact, computational discrepancies can challenge the notion that there is an original correct table underlying the existing one. One can then imagine the challenges that might arise in producing a critical edition of such a table.

2. Table Cracking I: Restoring the author's sources and methods

The term 'table cracking' has been interpreted in a variety of ways; almost the only constant between them is the use of the table's entries to gain historical information, usually in a quantitative manner. These meanings divide roughly into two categories. Firstly, one may learn about the means by which the table was constructed. This includes ascertaining the theoretical models underlying the table, determining the use of certain historically attested numerical parameters or other underlying tables, and reconstructing the process of computation. These goals reflect the table's origins, the scientific activity and authorial process that resulted in the table's production. Secondly, table crackers attempt to restore information about how the table must have been used or evaluated *after* it was constructed. Activities related to authorial process, in particular, tend to share certain methodological features that deserve close attention. Thus we survey efforts to uncover a table's mathematical origins in Section 2.2 and make evaluative methodological observations in Section 2.3, before proceeding to restoring the table's applications in Section 2.4.

2.1. Understanding the theoretical models underlying a table

Reconstructing the theoretical model according to which a table was computed is often challenging. The text that accompanies the table, known as paratext, can offer some evidence about the underlying structure. But the task can be more difficult in the typical case when the historical sources

Tabular data				First differences	
<i>Date</i>	<i>sign</i>	<i>deg</i>	<i>min</i>	<i>deg</i>	<i>min</i>
1 Jan	9	20	22		
2 Jan	9	21	24	1	2
3 Jan	9	22	25	1	1
4 Jan	9	24	27	1	2
⋮	⋮	⋮	⋮	⋮	⋮
1 March	11	20	55		
2 March	11	21	55	1	0
3 March	11	22	54	0	59
4 March	11	23	54	1	1
⋮	⋮	⋮	⋮	⋮	⋮
1 July	3	18	26		
2 July	3	19	23	0	57
3 July	3	20	20	0	57
4 July	3	21	17	0	57

Table 1: An excerpt from Cortés’ *Table of the true place*, with first differences

describing the tables give contradictory, vague, or misleading reports. The numerical entries themselves are usually the starting point; they often provide table crackers with sufficient information to recover the table’s mathematical framework. We consider in detail an example of a reconstruction of the model underlying a pair of tables in which the layout of the tables proves to be misleading.

The Spanish cosmographer Martin Cortés de Albarca’s (1510–1582) *Arte de Navegar* (1551) became one of the most popular navigation manuals by the beginning of the seventeenth century. Many aspects of astronomical navigation are detailed in this treatise. We concentrate here on two tables designed for the computation of the sun’s true longitude at noon, a necessary step in the determination of one’s terrestrial latitude.

In this example, as with most others in this chapter, the author was working in a geocentric astronomical system inspired by Ptolemy’s *Almagest*. In this tradition, the standard procedure to determine the true solar longitude involves two steps. The first employs a set of tables that give the sun’s mean longitude. Since it is assumed that the sun travels along the ecliptic at a constant velocity, the table computes a linear function with respect to time. The second step uses a table that computes a correction factor, called the solar equation, that adjusts the longitude from mean to true based on the sun’s position in its orbit (so that both the argument and the function are measured in degrees). A geometric model (see Figure 2 on p. 31) allows the solar equation to be calculated trigonometrically for any argument. Did Cortés follow this model of computation in the *Arte de Navegar*?

Tabular data			First differences		Tabular data			First differences	
<i>years</i>	<i>deg</i>	<i>min</i>	<i>deg</i>	<i>min</i>	<i>years</i>	<i>deg</i>	<i>min</i>	<i>deg</i>	<i>min</i>
1545	1	0			1558	0	50	0	-15
1546	0	45	0	-15	1559	0	35	0	-15
1547	0	30	0	-15	1560	0	21	0	-14
1548	0	15	0	-15	1561	1	7	0	+46
1549	1	2	0	+47	1562	0	52	0	-15
1550	0	47	0	-15	1563	0	37	0	-15
1551	0	32	0	-15	1564	0	23	0	-14
1552	0	18	0	-14	1565	1	9	0	+46
1553	1	4	0	+46	1566	0	54	0	-15
1554	0	49	0	-15	1567	0	39	0	-15
1555	0	34	0	-15	1568	0	25	0	-14
1556	0	19	0	-15	1569	1	11	0	+46
1557	1	5	0	+46					

Table 2: An excerpt from Cortés’ ‘solar equation’ table, with first differences

The caption of the first table in the Spanish edition is *Tabla del verdadero lugar del sol*, which was incompletely translated in the first English edition (1561) as *The table of the true place*. Just as in mean motion tables, this table’s argument is in time units and its entries are in degrees of arc, giving values for each day of an unspecified year. However, a quick examination of the first differences of the entries verifies that the tabulated function is not linear (see Table 1).

The caption of the second table is *Tabla de las equaciones del sol*, correctly translated in the first English edition as *Table of the equations of the sunne*. But this table is nothing like a solar equation. Its argument is in calendar years (running from 1545 to 1688), rather than degrees; and the table’s entries form a roughly linear trend rather than trigonometric (see Table 2). Thus, despite the titles, Cortés does not follow Ptolemy.

Cortés instructs his readers how to use the tables using a worked example. To determine the sun’s true longitude on 22 February 1568, select the entry corresponding to 22 February in the first table; then select the entry corresponding to 1568 in the second table. Add these two entries together, and the result is the sun’s true longitude. This method of computation confirms what we just saw: Cortés is not using Ptolemy’s tabular approach. However, this does not imply that the underlying geometrical model is fundamentally different. Further exploration is needed. When Cortés’ tables were compiled, the Parisian Alfonsine tables were dominant in Europe and would have been readily available to Cortés or his sources. The models underlying Alfonsine astronomy are well known.² So we begin by hypothesizing that the tables can

² See for example Chabás and Goldstein, *The Alfonsine Tables of Toledo*.

Tabular data				Alfonsine position (1483)			Difference
<i>Date</i>	<i>sign</i>	<i>deg</i>	<i>min</i>	<i>sign</i>	<i>deg</i>	<i>min</i>	<i>min</i>
1 Jan	9	20	22	9	20	23	1
2 Jan	9	21	24	9	21	24	0
3 Jan	9	22	25	9	22	25	0
4 Jan	9	23	26	9	23	27	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1 March	11	20	55	11	20	57	2
2 March	11	21	55	11	21	57	2
3 March	11	22	54	11	22	56	2
4 March	11	23	54	11	23	56	2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1 July	3	18	26	3	18	27	1
2 July	3	19	23	3	19	24	1
3 July	3	20	20	3	20	21	1
4 July	3	21	17	3	21	18	1

Table 3: Sample of Cortés’ *Table of the true place* compared with the true position of the sun computed according to the *editio princeps* of the Parisian Alfonsine Tables

be reconstructed from Alfonsine material. If this hypothesis proves correct, then both the model and parameters originate in Alfonsine astronomy.

Let us begin with the *Table of the true place*, which gives the position of the sun for each day of a full year. The first differences of the entries in this table vary, indicating that the table does not provide mean positions. But it may provide true solar positions. In fact, it appears that it does. We recomputed the true solar position for each day of the year 1545, the first argument of the equation table, according to the Parisian Alfonsine Tables as they are presented in the 1483 Ratdolt edition (*princeps*). The difference between these recomputations and the values given by Cortés are within 2 arcminutes (see Table 3).³ More advanced table-cracking techniques may help us analyse whether the discrepancies between Cortés’ values and the recomputed ones show any significant pattern,⁴ but this fit is good enough to demonstrate that the table computes Alfonsine true solar positions for 1545.

Turning to the entries of the equation table, we find a clear pattern: each value is about 15 minutes smaller than the preceding one for a sequence of three entries; then, at the fourth entry (a leap year) the value increases by 46 or 47 minutes; and the cycle repeats. So this table does not represent

³ In this table we have expressed the Alfonsine positions using 30° increments in order to follow Cortés’ practice and to avoid confusion. The Parisian Alfonsine Tables, however, use 60° increments.

⁴ For instance, one may notice that all differences are positive. This may be caused by a different epoch value.

a solar equation, but rather a quantity that makes an adjustment every leap year. It seems likely that the table gives the total displacement of the mean (or true) sun on the ecliptic for the year: that is, the difference between the mean (or true) sun's longitude at the beginnings of years n and $n + 1$. The numerical effect of the distinction between mean and true displacement of the sun is only around 2 arcminutes in longitude. So, to confirm the link between Cortés' tables and those of the Parisian Alfonsine tradition, we may choose either hypothesis. Tentatively, we guess mean displacement.⁵

A variety of methods have been developed to derive mean motion parameters from mean motion tables.⁶ Here we adopt a simple calculation, dubbed 'squeezing' by Neugebauer and Kennedy,⁷ and refer the reader to the next section for descriptions of more sophisticated parameter derivation techniques. Computing the difference between mean solar positions over the largest possible range consisting of a multiple of four years in Cortés' equation table (1545 to 1685), we can derive a daily mean motion parameter by simply adding this difference to 50,400 (360×104 , i.e., the 140 complete revolutions of the sun in 140 years) and dividing the result by 51,135 ($365;25 \times 140$, i.e., the number of days in 140 Julian years). This gives us a parameter value of $0;59,8,19,38^\circ/\text{day}$. If we assume that the table values were rounded to the last place, we obtain lower and upper bounds for the solar mean motion parameter of $0;59,8,19,36^\circ/\text{day}$ and $0;59,8,19,40^\circ/\text{day}$.⁸ The parameter used in the Alfonsine tradition of $0;59,8,19,37,19,13,56^\circ/\text{day}$ lies within this interval.⁹ On the other hand, the second plausible historical parameter, the sidereal motion of $0;59,8,11,28,27^\circ/\text{day}$ used in the Toledan tradition, does not lie within the interval. The same holds for Ptolemy's $0;59,8,17,13,12,31^\circ/\text{day}$ and al-Battānī's $0;59,8,20,47^\circ/\text{day}$ (corresponding to a solar year of 365;14,26 days). Therefore this rough estimate is enough to confirm our hypothesis that the equation table depends on Alfonsine mean motions.

Historians often encounter considerably more complex cases. When studying a set of fourteenth-century Latin tables by John Vimond, José Chabás and Bernard R. Goldstein were faced with peculiar layouts, headings, and

⁵ The hypothesis that the equation table gives the true position (rather than the mean) is less likely because it would make little sense to use a position specific for the first day of the year as an adjustment for every position of that same year.

⁶ See, for instance, Mielgo, 'A Method of Analysis', and van Dalen, 'Origin of the Mean Motion Tables'.

⁷ See, for example, Kennedy, 'A Survey', p. 20.

⁸ These bounds do not constitute a confidence interval as they are not derived from statistical analysis. However they can be heuristically used in a similar fashion.

⁹ This number may seem unreasonably precise to us but it actually reflects the actors' practices.

entries, and no accompanying instructions.¹⁰ These tables includes elements to compute latitudes and longitudes of the planets and luminaries, syzygies, and material on the fixed stars. Using mathematical tools similar to those described below (reconstruction of computations, parameter estimation, comparison of results with historically attested sources), Chabás and Goldstein were able to uncover the underlying theoretical model and parameters. This analysis revealed that Vimond's set is the earliest known Parisian tabular material related to the Alfonsine tradition.

In other cases, issues can arise from the number and complexity of the possible underlying models. For instance, in sixteenth-century Vienna, John Angelus computed ephemerides on the basis of Peurbach's planetary tables. Richard Kremer and Jerzy Dobrzycki analyzed the discrepancies between the ephemerides and the planetary positions derived from the standard Alfonsine tradition.¹¹ The differences revealed that Peurbach must have known of some of the geometrical models developed in Maragha, at least in the form of a diagram.

Restoring missing entries in tables

Especially in ancient studies, the historian may be confronted with a numerical table in a fragmentary state. Cuneiform tables, for instance, are frequently broken, manuscripts can be torn or crumbling, or the surface of a document may be damaged by wear, water, or mould, rendering part of the text illegible. Despite this, table crackers often are able to reconstruct the missing entries. This is because the table's numerical content, unlike (say) the text of prose or poetical works, generally has an underlying theoretical model and computational algorithm that determines the entries. Thus, in principle, it can take as little as one entry and its argument (and sometimes partial at that) to establish the underlying pattern, and consequently restore the missing entries.

The mathematical corpus of Mesopotamia has benefited especially from such reconstruction efforts. Cuneiform texts from this region were inscribed on clay tablets, by now several millennia old. When excavated, more often than not they are broken (sometimes into many pieces), the clay surface is worn, and the imprinted cuneiform signs are illegible. In the case of numerical tables, reconstruction of the missing content is sometimes a trivial task (e.g., fragmentary multiplication tables or reciprocal tables); however, in cases where the underlying mathematical relation is unexpressed or unclear, it can be difficult or impossible. If the table is the only one of its kind, scholars usually are able to restore it only after prolonged and painstaking analysis, if at all.

¹⁰ See Chabás and Goldstein, 'Early Alfonsine Astronomy in Paris.'

¹¹ Dobrzycki and Kremer, 'Peurbach and Maragha Astronomy?'

BM 55557	$p, q, r(N)$	N	N^4	Text
obv 1	- 4 10	1 1 2 6 33 45	1 4 14 56 27 48 28 7 58 24 34 22 30 13 23 22 30 52 44 3 45	
2	13 3 -	1 1 26 24	1 5 58 14 30 41 45 31 1 24 5 45 36	
3	- 12 2	1 1 30 33 45	1 6 16 8 59 52 16 43 11 54 24 18 45 52 44 3 45	
4	11 - 8	1 1 43 42 13 20	1 7 13 5 8 12 12 11 32 4 28 26 29 16 11 44 31 36 17 46 40	
5	11 8 -	1 2 12 28 48	1 9 19 53 44 58 13 47 33 33 25 41 0 59 40 24 57 36	
6	6 18 - 5	1 3 12 35 33 20	1 13 54 25 28 17 40 25 18 37 59 52 58 6 47 7 59 0 44 26 40	
7	1 7 5	1 3 16 52 30	1 14 14 29 13 12 43 52 58 10 47 45 14 3 45	
8	4 5 -	1 4 48	1 21 37 45 36 55 17 45 36	
9	- 1 7	1 5 6 15	1 23 10 22 7 26 32 44 29 37 44 3 45	
10	2 10 -	1 5 36 36	1 25 47 16 41 53 18 39 58 35 36 57 36	
11	17 - 8	1 5 50 37 2 13 20	1 27 0 59 1 14 19 36 11 10 13 14 59 53 42 41 10 28 37 25 16 2 57 46 40	
12	- 7 8	1 5 55 4 41 15	1 27 24 36 17 6 6 18 54 2 42 14 14 34 29 52 44 56 29 3 45	
13	17 8 -	1 6 21 18 43 12	1 29 45 8 39 39 55 50 37 43 46 35 5 45 55 7 56 41 51 21 36	
14a	5 - 3	1 6 40	1 31 26 58 6 25 1 6 40 ?	
14b	1 4 2	1 7 30	1 36 6 30 14 3 45	
15a	12 -	1 8 16	1 40 32 59 8 12 10 44 16 ?	
15b	1 10 3	1 8 20 37 30	1 41 0 16 49 24 21 47 21 33 31 48 59 3 45	
16	10 5 -	1 9 7 12	1 45 40 20 17 8 51 7 38 15 56 9 36	
17	8 10 -	1 9 59 2 24	1 51 3 20 55 31 30 54 41 17 4 22 3 25 3 21 36	
18	- 4 5	1 10 18 45	1 53 9 20 29 38 53 33 3 41 29 3 45	
19	- 10 6	1 11 11 29 3 45	1 58 55 13 36 42 19 18 54 52 21 40 41 3 20 26 44 0 14 3 45	
20a	3 2 -	1 12	2 4 24 57 36 ?	
20b	18 -	1 12 49 4	2 10 9 55 50 45 36 24 12 52 17 8 16	
21				
rev 1'	21 6 -	1 57 57 53 16 48	14 56 30 42 20 35 17 25 27 7 19 14 33 8 4 18 36 3 27 21 36	
2'	13 - 5	1 58 31 6 40	15 13 27 47 53 42 32 30 13 52 11 24 46 25 11 6 40	

Figure 1: John Britton reconstructs a table of fourth powers based on a very small fragment. Reproduced from the *Journal of Cuneiform Studies* 43–45 (1991–93), p. 75.

practitioners; for instance, there exist in the cuneiform corpus tables of reciprocals of regular numbers, as well as tables of the squares of regular numbers.

Armed with this insight, Britton consulted a massive tabulation of 11-digit regular sexagesimal numbers $N = 2^p 3^q 5^r$ and their reciprocals compiled by Gingerich, over 36 pages in length, listed according to various values of the powers p, q, r .¹³ For those N that ended in the string ...44,3,45, Britton noticed similarities in the corresponding values of p, q, r . In particular, the value $N(0, 16, 40) = \dots 23,22,30,52,44,3,45$ matched line 1 of his text. In the same way, Britton identified the numbers in lines 3 and 9, and from this followed yet more lines. Once a handful of numbers had been identified, Britton noticed that the values of p, q, r were all divisible by 4. This leads naturally to the conclusion that the numbers are the tail ends of fourth powers of regular sexagesimal numbers. At this point it was easy to reconstruct the leading sexagesimal places (around eight) of the broken entries, and complete entries that had been broken off of the table altogether. Furthermore, based on the tablet's physical dimensions he also proposed that there were three additional columns that preceded the fourth powers.¹⁴ These columns, he argued, included a row count, the regular numbers themselves, and their factorisation details.

Because of the damage to the tablet (only 2 partial edges remained: top right and far right edge), Britton could only hypothesize where the fourth powers began and where they ended. Now that the nature of the table was confirmed, he could compare with tables of similar functions (such as reciprocal tables and the so-called 'double six place' tables). This led to the surmise that the original tablet contained fourth powers of regular numbers running from 1,1,2,6,33,45 to 1,58,31,6,40. Britton noted that while contextual considerations such as these can be helpful, they can also raise more questions than they solve. BM 55557 is the only fourth power tablet of its kind in the extant record, and there are no mathematical problems in the entire corpus that call for the computation of a fourth power or its root. Thus, despite the reliable reconstruction of the tablet's numerical content, issues concerning its purpose still remain outstanding.

Mathieu Ossendrijver recently faced a similar challenge.¹⁵ The fragments he was studying appeared to be without parallel in the primary literature and exhibited remarkable mathematical virtuosity, seemingly unconnected to any practical application. Some tablets contained numbers with up to 30 sexagesimal places, making them the longest numbers appearing in the cuneiform

¹³ Gingerich, *Eleven-Digit Regular Sexagesimals*.

¹⁴ This was supported by the curvature of the existing fragment, which suggested the overall size of the original table. See Britton, 'A Table of Fourth Powers', pp. 71–72.

¹⁵ Ossendrijver, 'Powers of 9'.

corpus (and possibly antiquity!). Ossendrijver noticed that the final sexagesimal places in one of these lists of numbers, so-called text A, alternated between 9 and 21, suggesting a relation to powers of 9 expressed in sexagesimal form. From this clue, he identified text A as a special sort of factorization table for 9^{46} , namely sexagesimal representations of 9^n , $0 \leq n \leq 46$.¹⁶ In a like manner, the final places of a similar text, the so-called text B, alternated between 12, 36, 48, and 24, which suggested factors of 5. (The factors of 5 had emerged because 12 is the reciprocal of 5 in base 60.) Text B turned out to be a factorization table for $9^{11} \cdot 12^n$ ($n \geq 39$). These tablets reveal a new sort of mathematical activity in the ancient Near East, hitherto unknown to modern scholars.

A contrasting technique for reconstructing the contents and circumstances of cuneiform tabular texts was developed by Neugebauer when he was working on what would eventually be called the Astronomical Cuneiform Texts (ACT) in the 1930s. Neugebauer was faced with many fragments of tabular astronomical ephemerides, some dated, some not. In cases where he had both a dated fragment and a fragment of the same type but undated, he developed a method of dating the undated fragment called the ‘Linear Diophant.’¹⁷ In a nutshell, Neugebauer computed the function tabulated on the dated text forward and backward in time until he reached a value (or modulo thereof, for periodic functions) that corresponded to a value in the undated text. From this he could calculate the time interval from the dated tablet to the undated one, and thus establish the date of the undated fragment. Of course, many of the fragments concerned functions that were periodic. These were typically computed via linear zigzag functions or step functions, so that the Linear Diophant method would furnish infinitely many possible dates for the undated fragment. However, almost always only one solution was historically plausible. Concerning his procedure and its connection to the tasks of the historian of astronomy, he was quite emphatic:

The method ... has nothing to do with astronomy, nor with history. It only fulfills a task for a certain group of astronomical cuneiform texts which would otherwise fall on the custodian of a museum [namely, joining tablet fragments] ... It is essential to emphasize that the solution of this task becomes possible here without any hypothesis about the content of the texts, since nothing else is used but the generative laws of the series of numbers that are empirically derived from the fragments.¹⁸

¹⁶ Here, Ossendrijver postulates that a table such as this one could have been a way to double check 9^{46} is correctly computed by repeatedly multiplying it by $6;40$, i.e. the reciprocal of 9, until 1 is reached; this is the sense in which ‘factorisation’ is invoked.

¹⁷ For details, see Neugebauer, *Astronomical Cuneiform Texts*, pp. 35–37.

¹⁸ Ossendrijver, ‘Translating Babylonian Mathematical Astronomy’, p. 335.

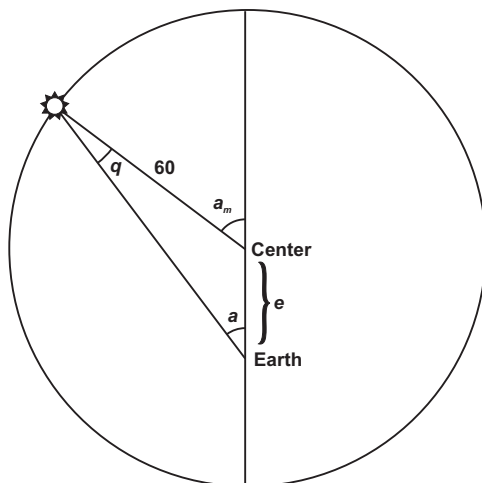


Figure 2: The solar equation $q(a_m)$

2.2. Reconstructing a table’s numerical parameters

Most mathematically computed astronomical tables rely on functions with built-in numerical parameters. For instance, the solar equation, the difference between mean solar motion and true solar motion, is found in Ptolemaic astronomy according to

$$q(a_m) = \arctan \frac{e \sin a_m}{60 + e \cos a_m}, \tag{1}$$

where the radius of the deferent (the circle on which the sun orbits the earth) is set equal to 60, the mean anomaly a_m increases at a constant rate, and e is the distance from the earth to the center of the deferent (see Figure 2). Different astronomers used different values for e ; Ptolemy used 2;29,30, while al-Battānī used 2;4,45. A common parameter in Islamic tables, related to the *Zīj al-‘Alā’i*, was to choose e so that the maximum solar equation (which occurs at $a_m \approx 92^\circ$) has exactly the value 1;59; this occurs when $e \approx 2;4,35,29,51$. However, medieval authors did not often report their parameter values; they simply presented the completed table. Determining the parameter used by a certain astronomer from his table of the solar equation would help to place that astronomer into a tradition of astronomical inquiry.

Van Dalen has developed a pair of statistical tools that allow the scholar to input a historical table and receive back an estimate of a numerical parameter embedded within it.¹⁹ Briefly, the first method works as follows: take the 180 entries in the solar equation table, substitute them one at a time with

¹⁹ van Dalen, ‘A Statistical Method.’

the appropriate value of a_m into (1), and solve for e . The resulting values of e will differ slightly from each other, due to errors caused by rounding and approximation at various stages of the calculation. Some values will be more reliable than others; for instance, $q(1^\circ) = 0;2,0$ is a small number and rounding will have a larger relative effect on it than on other entries. Van Dalen thus computes a weighted average of the estimates, with the weights chosen corresponding to a measure of the reliability of the estimate. He then uses a technique related to least squares²⁰ to generate an interval around the estimator that is 95 % likely to contain the true value of e (provided certain statistical assumptions are satisfied²¹).

Van Dalen's second approach is a maximum likelihood estimator. Suppose that the table has $q(1^\circ) = 0;2,0$. Presumably this is the rounded result of a calculation that produced a number somewhere between $0;1,59,30$ and $0;2,0,30$. Back calculating from these two values produces an interval of possible values for e . In a perfect world, intersecting the intervals produced in this way from every entry in the table provides a very small interval of values of e that could have led to this table. Unfortunately, due to errors in computation and rounding, this seldom occurs. Instead one takes an estimate derived from the values of e that correspond to the largest number of these intervals. Van Dalen points out that this criterion is most effective for tables with few errors, and especially for mean motion tables (for which the underlying function is linear).

Van Dalen has applied these methods to a number of situations with success. We report here one case in his original paper, the solar equation table in the popular 13th-century *Shāmīl Zij*. For the solar eccentricity his weighted estimator yields $e = 2;4,35,29,29$, with the 95 % confidence interval $(2;4,35,26, 2;4,35,35)$. The maximum likelihood estimator yields similar results, with an interval of $(2;4,35,29,29, 2;4,35,32,56)$. This clearly rejects both e values given by Ptolemy and al-Battānī, but confirms strongly the parameter generated by a maximum solar equation of $1;59$. Van Dalen has published

²⁰ Other table cracking researchers have used least squares techniques; see for instance Kremer, 'Marcus Schinnagel's Winged Polyptych' (cf. p. 45). The basic idea is to fit a mathematical model to a set of data, and choose the parameters of the model in order to minimize the sum of the squares of the deviations of the data from the model's prediction.

²¹ The most important of these assumptions is statistical independence of the individual estimates for e , which can lead to problems. For instance, if a stretch of ten entries was computed by interpolating linearly between the two 'node' entries preceding and following them, then the estimates of e generated by these ten entries will be related to each other for two reasons: all ten entries are affected by the errors in the entries at the two nodes, and the quantity being tabulated is a linear function between the nodes rather than the function itself. In this situation these estimates would fail the assumption of statistical independence. Van Dalen is careful to check for the use of interpolation before applying his methods.

results based on these and similar methods elsewhere,²² and has also reconstructed several parameters at once from al-Khwārizmī's table for the equation of time.²³

2.3. Determining dependences between tables

The nature and process of the transmission of knowledge is fundamental to the study of the history of science. However, transmission of numerical tables often occurred through means that left no documentary evidence behind: for instance, an author finds a table of interest to his current project in his personal library, and uses it to compute a new table for some other purpose. Seldom is the debt to the original table acknowledged in the manuscripts. Such a transmission might happen within or across cultures, between two different stages of a single author's career, or even within a collection of tables. The latter is the case for the sine and tangent tables in the 13th-century *Baghdādī Zij*, which van Dalen ascribes to Abū l-Wafā'.²⁴ As the argument of the tangent function approaches 90°, its values grow without bound. If the tangent is calculated the conventional way, according to $\tan \vartheta = \sin \vartheta / \sin(90^\circ - \vartheta)$, the values for the sine in the denominator become very small, and rounding produces a large relative error. Through recomputation, van Dalen shows that the large errors in the tangent table (up to more than eleven units in the second-last sexagesimal place) are almost completely accounted for by computation from the values in the sine table. This verifies that the tangent table derived from the sine table.

Usually, however, the errors in table entries are much smaller, and the dependence is not so obvious. One such situation is found in the works of 14th-century Syrian astronomical timekeeper Shams al-Dīn al-Khalīlī, whose occupation was to use mathematical astronomy to guide the Muslim faithful to pray at the appointed times of day, in the direction of Mecca (the *qibla*). Al-Khalīlī composed one of the most accurate and thorough tables for the *qibla* of the medieval period.²⁵ Separately, he constructed a set of *auxiliary tables*, which in various combinations allow the reader to solve a variety of problems in spherical astronomy, including the *qibla*.²⁶ Al-Khalīlī's auxiliary functions were

$$f(\varphi, \vartheta) = \frac{\sin \vartheta}{\cos \varphi}, \quad g(\varphi, \vartheta) = \sin \vartheta \tan \varphi, \quad \text{and} \quad G(x, y) = \arccos \frac{x}{\cos y}. \quad (2)$$

²² For examples, see van Dalen, *Islamic Astronomical Tables*.

²³ van Dalen, 'Al-Khwārizmī's Astronomical Tables'.

²⁴ van Dalen, 'Islamic and Chinese Astronomy', pp. 349–51.

²⁵ King, 'Al-Khalīlī's Qibla Table'.

²⁶ King, 'Al-Khalīlī's Auxiliary Tables'.

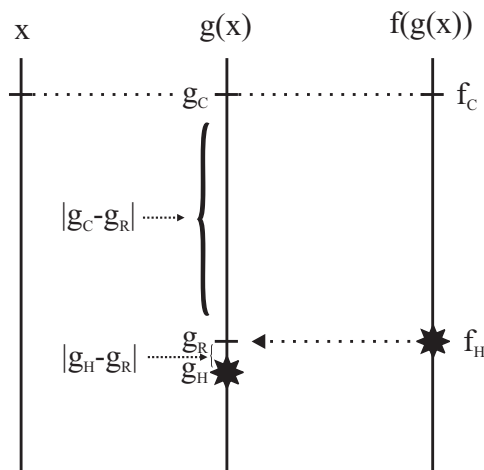


Figure 3: The dependence of the values in one table on the values of another. Consider the three vertical number lines aligned so that moving horizontally between them corresponds to precise computation. The \star symbols represent the historical table values; g_R is back computed from f_H . In this instance g_R is much closer to g_H than to g_C , providing evidence for dependence.

Van Brummelen²⁷ hypothesized that al-Khalilī's table for f was computed from the entries in the table for g , according to the relation

$$f(\varphi, \bar{\varphi} \pm n) = \cos n \pm g(\varphi, n).$$

If this were to be verified, it would effectively demonstrate that Muslim astronomers were sufficiently aware of issues in numerical computation to devote intellectual resources to seeking out mathematical techniques for improving computational speed: the same sort of thinking that in Europe produced prosthaphairesis and, eventually, logarithms. Since the relation is mathematically correct, recomputation of the entries proves nothing — any mathematically valid method should produce correct function values. Rather, to demonstrate the hypothesis one must locate traces of the errors in the entries of the table for g in the entries of the table for f .

A general procedure to test hypotheses such as this was developed by Van Brummelen.²⁸ Applied to this example, it works as follows. Let $g(x)$ be the function of the purported underlying table, and let $f(g(x))$ be the function of the hypothesized dependent table with argument $g(x)$. For each entry of g and the conjectured dependent entry in f , define the following quantities:

- f_C and g_C , the correct values of the functions;

²⁷ Van Brummelen, 'The Numerical Structure'.

²⁸ Van Brummelen and Butler, 'Determining the Interdependence'.

- f_H and g_H , the values of the entries in the historical tables;
- $g_R = f^{-1}(f_H)$, a reconstructed value of the underlying g derived from f_H .

The question hinges on whether g_R lies closer to g_H or to g_C (see Figure 3).

- If g_R is closer to g_H than to g_C , a trace of the error in g is present in f , and we have evidence for dependence.
- If g_R is closer to g_C than to g_H , the dependent entry is more accurate than one would expect from the use of the underlying entry, and we have evidence that the dependent table is too accurate to derive from the underlying table.

For each pair of underlying and dependent entries the quantity $|g_H - g_R| - |g_C - g_R|$ is computed, and the resulting data are tested for a mean different from zero. For this purpose Van Brummelen chooses the non-parametric Wilcoxon signed-rank test; while it is less powerful than the traditional statistical procedures described elsewhere in this article, it is robust against the possibility of scribal errors and other disturbances in the data.

Each page of the auxiliary tables gives entries for f and g for a fixed value of φ and $\vartheta = 1^\circ, 2^\circ, \dots, 90^\circ$, so a separate test was performed for the columns with $\varphi = 5^\circ, 10^\circ, \dots, 55^\circ$ (the highest value of φ in the table). The test for $\varphi = 5^\circ$ gave a p -value of 0.2% in favor of dependence,²⁹ and for $\varphi = 45^\circ$ the p -value was 1.4%; all other p -values were less than 0.05. Thus Van Brummelen concluded strongly in favor of dependence.

This episode has a surprising epilogue. In 2000, David King discovered another set of tables authored by al-Khalilī. In a rare instance of statistical methods verified by later historical sources coming to light, the new manuscript confirmed that al-Khalilī had computed his tables according to the method asserted by the statistical procedure.

2.4. Reconstructing the process of computation

For some tables, it is possible to reconstruct the computational process that generated the table in more detail than identifying parameters or underlying tables. The possibilities here are endless and depend on the context. They might include deciding between several possible mathematical paths to the solution, detecting the use of interpolation, or identifying the use of an approximate method at a certain moment in the process of computation. Since most tables represent functions defined mathematically, the errors in the entries are usually the only basis on which to decide these questions, along with

²⁹ A p -value measures the probability of obtaining a result at least as far from the expected value as the observed result, under the null hypothesis that no effect exists.

the collection of historically plausible techniques that bear on the problem. We seldom read of table construction in the primary literature or see a table partly completed. One nice exception to this (see Figure 4), part of a double-argument table of latitudes for Venus in Jamshīd al-Kāshī's early 15th-century *Khāqānī Zīj*, illustrates that we may not assume that a computer started with the first entry and simply worked his way through to the end.³⁰

Since the possibilities for how a table may have been constructed are so varied, general techniques are not usually available. Usually researchers work by comparing the results of the various plausible methods of calculation with the pattern of errors in the table. It is both easy and difficult to measure the success of a certain method: easy, since one may declare victory by choosing the method that best fits the entries among the historically plausible alternatives; difficult, because one's confidence in that assertion cannot be measured easily in a quantitative way. Fortunately, in many cases the result of the re-computation is so clear that an unequivocal decision is easy to make.

One simple example of this is the sine table in a manuscript of a Latin translation of al-Khwārizmī's *zīj*. This table uses a base circle radius of $R = 150$, of Indian origin. But Benno van Dalen has pointed out that almost all the entries in the table end in 0, 2, 5, or 7.³¹ This suggests that the table was generated by multiplying a sine table with the Ptolemaic parameter $R = 60$ by $2\frac{1}{2}$.³² In this case the generated table is in fact precisely equal to the original table scaled by a constant. Other instances have been discovered where a new table was generated from an existing table by means of a scaling factor, but the results are only approximately correct.³³ In these cases, presumably the table's author either wanted to save time and effort, or was not capable of computing the new table directly.

Another example of a clear reconstruction is the sine table in al-Samaw'al's 12th-century *Exposure of the Errors of the Astronomers*. Al-Samaw'al criticizes traditional sine tables for using approximative methods, forced on table makers by the use of a circle divided into 360 parts, i.e., degrees. Al-Samaw'al's table uses 480° , seemingly bypassing the problem. However, a distinctive error pattern (every fourth entry correct, and errors bulging in the negative direction between them) is almost precisely matched by computation of the table by interpolating between entries in a traditional sine table.³⁴

³⁰ See Van Brummelen, 'The Tables of Planetary Latitudes'.

³¹ van Dalen, 'Al-Khwārizmī's Astronomical Tables', p. 206.

³² Hogendijk, 'Al-Khwārizmī's Table of the "Sine of the Hours"', p. 11, reconstructs the sine table used by al-Khwārizmī to generate his table of the 'sine of the hours'; this underlying sine table actually does use $R = 150$.

³³ See for instance Van Brummelen, *Mathematical Tables*, pp. 176-79, and Dorce, 'The *Tāj al-azyāj*'.

³⁴ Van Brummelen, Mimura and Kerai, 'Al-Samaw'al's Curious Approach'.

c	$g(c)$	Error
12	0,24	[-7]
24	2,16	[+12]
36	4,32	[-6]
48	8,18	[+7]
60	12,26	[-11]
72	17,44	[-7]
84	23,24	[-19]
96	29,49	[-10]
108	36,14	[-9]
120	42,38	[+1]
132	48,18	[-1]
144	53,12	[+2]
156	56,36	[-17]
168	59, 4	[-8]
180	60, 0	0

Table 5: Excerpt from the *Almagest* lunar interpolation table. (The full table is given for arguments in multiples of 6° up to $c = 90^\circ$, and in multiples of 3° thereafter.) Errors, given in square brackets, are in terms of the last place. Thus, for instance, the correct value of $g(c)$ is 0,31.

A typical example of a slightly more sophisticated reconstruction comes from Ptolemy's *Almagest*. Most of the *Almagest*'s tables are difficult to study for their computational secrets, because the tables' entries tend to be accurate to within one or two units in the last place. This is not true of one group of tables. To determine the longitudes of the moon or one of the planets, the *equation of anomaly* p must be tabulated. However, it is a function of two arguments. To avoid tabulating a gigantic rectangular grid of entries, Ptolemy devises an approximative process that requires the tabulation of four single-argument functions. One set of these tables, the *interpolation coefficient* $g(c)$, contains unusually large errors in the last place (see Figure 5).

Now, the function $g(c)$ is computed fairly directly from $p_{\max}(c)$, the *maximum equation of anomaly*. For the lunar table, Van Brummelen³⁵ back computed a set of values of $p_{\max}(c)$ from the values of $g(c)$, and discovered a clear pattern: the third sexagesimal places of these values cluster strongly around 0, 15, 30, and 45 (see Figure 5). Thus Ptolemy used values of $p_{\max}(c)$ rounded to units of 0;0,15. (Further examination revealed that this step size was caused by an application of linear interpolation; Ptolemy had directly computed values of $p_{\max}(c)$ only for multiples of 12° .) Thus Van Brummelen was able to reconstruct a small table of $p_{\max}(c)$ for the moon. Similar results were obtained when the same method was applied to the planetary tables.

³⁵ Van Brummelen, 'Lunar and Planetary Interpolation Tables.'

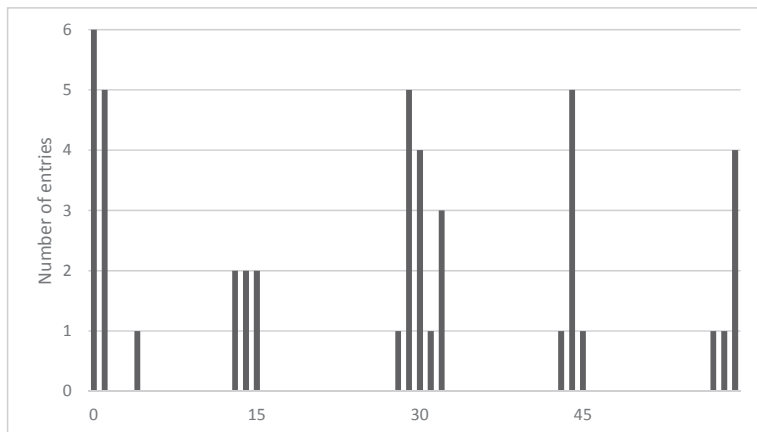


Figure 5: Clustering of the third sexagesimal place of back-computed values of $p_{\max}(c)$.

3. Methodological Considerations

3.1. The moment of decision

Mathematical methods applied to answer questions in the history of science have been at a minimum controversial, and at worst have been ignored by the historical community. This is due partly to the technical ‘smoke screen’ that often makes it very difficult to understand the procedures without months of training. Skepticism has only increased when some of the conclusions reached through quantitative methods have been wildly at odds with results achieved by conventional historical inquiry. It is irrational to reject these methods outright simply because one does not understand them; nevertheless, it is incumbent on quantitative researchers to make their results, and the strengths and weaknesses of their conclusions, available to the community. This has seldom been done, so we provide a road map here.

The general practice of a table cracker usually follows the same path. The researcher hopes to draw a historical conclusion from a set of historical quantitative data (say, the reconstruction of an underlying table of p_{\max} in the *Almagest* planetary interpolation tables in the preceding section). This data is manipulated in some way, often to undo mathematical processing of the data that had been performed by the historical author (e.g., back computing from the interpolation table’s values to a set of values for p_{\max}). The transformed data reveals, or does not reveal, a pattern (e.g., clustering of the last sexagesimal place of the reconstructed p_{\max} values around multiples of 15). A moment of decision is reached. The researcher concludes that the pattern is not a coincidence, and must therefore be explained (if it is decided the pattern isn’t there, the result usually is not published). Finally, the researcher

asserts a cause for the pattern (e.g., the p_{max} values originally had multiples of 15 in the last place), and makes a historical interpretation (e.g., Ptolemy used the underlying table of p_{max}).

The moment of decision must be parsed carefully to evaluate the reliability of the researcher's conclusion. Two distinct stages must be considered. The first is an evaluation of the assumptions made when concluding that the pattern is a genuine artifact; the second is a consideration of the integration of the pattern with the available historical context. The quality of a result seldom relies on the details of the researcher's computations: the devil is either in the assumptions made prior to the calculations, or in the historical interpretations when they are done.

Every identification of a pattern involves certain assumptions, implied or not. Although many of these assumptions are benign, they are not always obvious, and it is difficult to bring them to the surface. In the case of Ptolemy's interpolation tables, we assumed that the back calculation of p_{max} should produce a uniform distribution of values in the third sexagesimal place. More subtly, we assumed that each back calculation is an independent witness to the underlying event. If for some reason the fact that the last sexagesimal place of the first p_{max} value is close to a multiple of 15 would imply that the next one will also be, then we might look at Figure 5 with more skeptical eyes.

To agree upon a conclusion, the pattern must be sufficiently clear that the researcher and the reader agree that it must be there. In the case of general statistical procedures such as van Dalen's parameter estimation and Van Brummelen's table dependence test, the assumptions embedded in the moment of decision are more explicit — part of the discipline of statistics — and thus are associated with standard practices of evaluation. Usually they are the following:

- The data points are statistically independent; that is, the value of any one data point does not influence the value of another.
- The data points are identically distributed.
- The data set is normally distributed.

The first of these assumptions is the most difficult in practice. For instance, if a table was computed with the aid of linear interpolation, then most entries might be in error in the same direction, and van Dalen's parameter estimation may produce a confidence interval for the parameter that is systematically too low or too high. Thus, before employing his method, van Dalen checks for the use of interpolation procedures. The second assumption is often safer in table cracking than it is in, say, matches of historical data to physical phenomena, but it must still be considered. For instance, in Van

Brummelen's table dependence test, the back calculation from f_H to the underlying g_R , rather than the more intuitive forward calculation of g_H to an f_R value, is done to avoid the potential skewing of distributions that might result from the computation from g to f . Finally, the normal distribution assumption is not difficult to satisfy due to the Central Limit Theorem; if a data set contains 40 or more data points, this assumption is almost always benign. However, if the data set is small, or if scribal errors possibly affect a number of tabular values, the assumption can be bypassed using non-parametric tests (such as that used by Van Brummelen with respect to table dependence).

Statistical conclusions come in two varieties, both often misinterpreted. A 95 % confidence interval (such as that produced by van Dalen's parameter estimation) gives an interval of values that contains the correct parameter 95 % of the time the procedure is performed, *as long as* the assumptions are met. A test (such as Van Brummelen's table dependence procedure) concludes with a p -value. This number reflects the probability that a result as unusual as the observed one would arise by random chance. It is *not* the probability that the result is false. In typical scientific practice, a p -value of $< 5\%$ is considered strong enough evidence to reject the random chance hypothesis.

No historical investigation, quantitative or not, is free of assumptions or doubt. The advantage of quantitative methods is that the reliability of the assumptions may often be evaluated directly. It is the responsibility of the researcher to make these assumptions as explicit and verifiable as possible. Studies of tabular data tend to be the most straightforward in this respect, since the data have a strict mathematical structure and are not usually subject to the vagaries of physical phenomena.

The second stage, the historical interpretation of the pattern, depends on the specific situation being studied, and usually cannot be framed in a quantitative analysis. Often the interpretation is obvious. In the case of the p_{max} calculations, one may question whether or not Ptolemy actually compiled the reconstructed p_{max} values into tabular form, or whether it was Ptolemy himself who performed these computations, but there is no further controversy. Generally, one hopes that the quantitative result is coherent with historical evidence from other sources. In some cases the result may lean against the weight of previously-established historical analysis. In these situations, *both* the quantitative and the historical assumptions must be examined for a resolution. In some cases the result might be explained by more than one historical interpretation.³⁶ The more dramatically the quantitative result varies from

³⁶ See, for example, a debate most recently between Chabás/Goldstein, Samsó/Castelló, and Poulle on the origin of a value for precession of $17;8^\circ$ in the star catalogue in the *Libro de las estrellas de la ochuaua espera*, a 13th-century text. Poulle, 'The Alfonsine Tables', assigns the parameter to a pre-Alfonsine theory of precession; Samsó and Castelló, 'An Hypothesis

established wisdom, the clearer that result must be; as the aphorism says, extraordinary claims require extraordinary evidence.

3.2. General versus specific techniques

By now the reader may have recognized an important methodological distinction. Some procedures (Sections 2.2 and 2.3) apply generally to a class of tables, while others (especially from Sections 2.1 and 2.4) apply to specific situations. General techniques can be applied uniformly to multiple contexts, can sometimes be automated, and allow easy comparison between different sources. Specific techniques have complementary advantages: they are adaptable to particular sources and particular historical scenarios.

Each type of procedure has its cautions. While general methods tend to rely on clearly stated assumptions, a single unstated and flawed assumption might undermine many conclusions at once. Also, while interpreting results applied to a spectrum of tables, one runs the risk of assuming (dangerously) that all the tables were constructed using similar computational norms. Finally, general procedures have tended to be more cautious and conservative, less likely to make historically dramatic assertions.

On the other hand, the reliability of *ad hoc* methods designed to apply to specific situations can be inherently difficult to evaluate. Assumptions are not often stated clearly, and the moment of decision must occur through the instinct of the table cracker rather than the result of a statistical test. Nevertheless, it should be noted that disagreements have seldom arisen in practice due to this shortcoming; when a specific method has identified a pattern, respondents usually have agreed that the pattern is in fact there. When controversies arise they are almost always at the stage of historical analysis, in cases where more than one explanation may be proposed to reconcile the pattern with the historical narrative.

The choice between the two types of method will always depend on local conditions (the nature of the table, the question that is being asked). Arguments will be more convincing if they rely on well tested general methods whenever possible, and use specific ones only when no other option is available. Regardless of the procedure used, the researcher must make explicit the underlying assumptions and demonstrate their validity (or at least their insignificance).

on the Epoch', p. 118, shows that it might arise from an erroneous dating of Ptolemy's star catalogue by the Alfonsine astronomers; while Chabás and Goldstein, *The Alfonsine Tables of Toledo*, pp. 234-35, derive it from a Castilian source.

4. Table Cracking II: Restoring Tabular Context

Coming to terms with a historical numerical table requires the determination of the processes that led to its creation, but it is also crucial to understand the role that the completed table played for its author and its users. This information can be elusive; sometimes all that remains in the manuscript record is the table itself (or just part of it), with little or no trace of the paratext. Even when the paratext exists it must be treated with caution. (One example is al-Kāshī's double-argument planetary latitude table, part of which is shown in Figure 4; the table's instructions have the arguments reversed.) Finally, even an intact and correct paratext may not be sufficient to answer contextual questions: the author may not have felt the need to describe the users and purpose of the table, or the users may themselves have found their own new applications.

Secondly, many tables were designed to model or represent some sort of physical phenomenon, often (but not always) astronomical. The extent of a table's predictive success often can provide useful information concerning the table's context. What specific phenomenon was the table meant to reproduce? How concerned would the authors and users have been about successes and failures in the table's predictive power? What concerns, other than accuracy, might have motivated judgments about a table's quality? This section considers the activities of table crackers in answering these contextual questions.

4.1. Reconstructing a table's purpose

Many numerical tables from early sources have come down to us as they are, with no accompanying information. In these cases, establishing a plausible interpretation of the tabular data is not always straightforward: sometimes, numerical patterns may correspond to multiple interpretations. As the following example illustrates, table crackers have had difficulties for centuries. In the eighteenth century, the Sanskrit astronomer Kevalarāma was commissioned by Jayasiṃha, regent of Jaipur (1699-1743), to translate Philippe de La Hire's *Tabulae Astronomicae*, which had come to the court of Jayasiṃha in the possession of Portuguese astronomer Pedro da Silva, into Sanskrit.³⁷ The part of Kevalarāma's translation (*Dr̥kpaḥśasārīṇī*, 1725) on how to compute true planetary positions with de La Hire's tables was a disaster. Kevalarāma failed to understand the proper use of de La Hire's tables, especially their use of logarithms. Kevalarāma, unfamiliar with logarithms (and probably poorly advised by the visiting European scholars), simply ignored the steps that invoked them. As a result, his reasoning and procedures became completely

³⁷ See Pingree, 'Philippe de La Hire's Planetary Theories,' for an account.

senseless. Since de La Hire's tables of logarithms came without any explanation of the rules of how to manipulate them, Kevalarāma's confusion wasn't entirely his own fault.³⁸

More recently, perhaps the most famous mathematical document subjected to this sort of table cracking is the Old Babylonian tablet Plimpton 322. The surviving fragment contains four columns of numbers. Neugebauer³⁹ instigated interest in the tablet by noticing that three of the columns (once the first has been heavily reconstructed) follow the pattern

$$\frac{d^2}{l^2}, b, d,$$

where b and d are Pythagorean numbers (integer solutions to $d^2 = b^2 + l^2$).

This was pursued further by Aaboe, Neugebauer, and Sachs, arguing for an underlying theory of generating functions where the columns were sums and differences of certain squares (p and q), their selection subject to mathematical criteria.⁴⁰ Other scholars, perhaps prompted by a short remark by Neugebauer and Sachs, argued that Plimpton 322 was in fact a trigonometric table, since it appears to measure the lengths of sides of right triangles.⁴¹

Eleanor Robson has taken these accounts to task for various technical and historical shortcomings.⁴² However, before delving into the arguments, she enumerates six conditions that must be satisfied in any hypothesis that purports to explain any tabular text: historical sensitivity, cultural consistency, calculational plausibility, physical reality (respect for the physical dimensions of the original archaeological artifact), textual completeness (the explanation should account for the paratext as well as the entries), and tabular order (sensitivity to the logical order of the columns).⁴³ None of these criteria should be emphasized at the expense of another. These categories reflect a more general movement in the history of mathematics in recent decades toward a greater sensitivity to contexts outside of the purely scientific content of the text.

Robson's interpretation of Plimpton 322 (taking a cue from work by Bruins and Høyrup) considers the context of administrative tabular documents written in the early second millennium BCE. Readings of the column titles

³⁸ Not long afterward, de La Hire's work was translated again twice. One of these translations, *Phiraṅgicandracchedyopayogika* (1734 or 1735), correctly reported the procedures, possibly due to the influence of Father Boudier, who was visiting Jayasimha's court at the time.

³⁹ Neugebauer, *The Exact Sciences in Antiquity*, pp. 36 ff.

⁴⁰ Aaboe, *Episodes*, pp. 30–31; Neugebauer and Sachs, *Mathematical Cuneiform Texts*, pp. 38–41.

⁴¹ See Joyce, 'Plimpton 322,' and Calinger, *A Contextual History*.

⁴² Robson, 'Neither Sherlock Holmes nor Babylon'; Robson, 'Words and Pictures.'

⁴³ Robson, 'Neither Sherlock Holmes nor Babylon,' p. 176.

turned out to provide significant inspiration in her analysis. She argues that the numbers in the table relate to reciprocal pairs (numbers whose product is 60), shows that they apply in procedures of the so-called concrete geometry,⁴⁴ and concludes that they provide numerical examples to help teachers generate problems for their students. Robson's work has been followed by analyses also inspired by contextual considerations, such as Friberg and Britton, Proust, and Shnider.⁴⁵ In the latter, the authors argue that Plimpton 322 is a problem text that includes a complete worked solution. The problem concerns 'normalized sexagesimal rectangles' — that is, this tablet was produced from a desire to generate a series of finite sexagesimal rectangles from a basic algorithm. Their interpretation is that Plimpton 322 was generated via the following problem:⁴⁶ 'Make a list of all the rectangles with length equal to 1 and width and diagonal equal to finite sexagesimal numbers, and represent the dimensions in reduced form, without a common sexagesimal factor.' The mathematical theory underlying the numbers uses the diagonal rule for rectangles (how to find the diagonal of a rectangle in terms of the sides) and the process of completing the square. Their account is supported by a comparison with related non-tabular mathematical documents such as MS 3971, which describes related procedures based on reciprocal pairs and produces a sexagesimal rectangle.

In the case of Plimpton 322, since so little information accompanies the table, questions regarding its purpose arise mostly by examining its numerical content. In other cases, an abundance of accompanying information can (paradoxically) lead to even more issues for the table cracker. Such is the case with the 1489 polyptych⁴⁷ by Marcus Schinnagel, recently studied by Richard Kremer.⁴⁸ Fully opened, the polyptych is more than 3 meters wide; its central panel measures 140 by 130 cm (see Plate 1). This polyptych is filled not with the usual Christian iconography, but rather with calendrical, astronomical and astrological tables and related material. Such a unique document gives rise to several questions. What sources participated in its realization? What could have been the author's and supporters' purpose? Was it intended as a practical astronomical tool?

Kremer begins his study of the polyptych not with table cracking, but with an analysis of the prose found within it. Other than Schinnagel's signature in

⁴⁴ Cut-and-paste geometrical procedures that manipulate geometrical shapes in a concrete manner.

⁴⁵ Friberg, *A Remarkable Collection*, and Britton et al., 'Plimpton 322: A Review'.

⁴⁶ Britton et al., 'Plimpton 322: A Review', pp. 558–559.

⁴⁷ A polyptych is a large artistic display, usually a collection of paintings, divided into panels. Common in early modern central Europe, polyptychs are often found as altarpieces in cathedrals.

⁴⁸ Kremer, 'Marcus Schinnagel's Winged Polyptych'.

Latin at the top of the central panel, the entire text is in German; it constitutes a primer for the horoscope-maker or the phlebotomist (a medical specialist on blood-letting). Next, turning to the tables, Kremer identifies their layout and a sequence of oddities and inconsistencies, mostly consequences of graphical constraints in a panel's layout or of mistakes by the artist. Once the tables have been identified, Kremer sets about determining the sorts of computations that can be done with them (along with a geometric instrument also found on the polyptych). He concludes that the polyptych may be used to determine planetary longitudes (to degrees) from 1489 to 1526, times of true syzygy (to minutes) from 1475 to 1512, and times and magnitudes of eclipses from 1489 to 1551. However, certain parts of the polyptych are inconsistent with practical use. For instance, without a solar mean motion table, the solar equation table that appears on the polyptych is useless. Next, using techniques similar to those outlined in Section 2.2, Kremer identifies the sources and numerical parameters underlying the tables. He concludes that the various sources were compiled with little care for mathematical and astronomical coherence. When predictions can be achieved, the polyptych produces results comparable to the Parisian Alfonsine tables with an accuracy of degrees rather than minutes. In this respect it matches the accuracies of almanachs and ephemerides made in Europe at this time for horoscope making and medicine.

Next, Kremer turns to the history of art to gauge the cultural meaning of the polyptych, concluding that the work was commissioned by the von Reischach family to convey a message about the harmony of the cosmos. As for Schinnagel, his mission was likely one of self-promotion, seeking patronage as he attempted to establish himself in Swabia.

The two examples of this subsection illustrate that in very different settings, table cracking may be used to understand the role that the table played among its practitioners, the table's link to other tables of the period, and the mathematical practices that would have been needed in its implementation. Along with insights gained by more familiar historical techniques, this information helps the researcher to frame a proper historical interpretation.

4.2. The table's fit with its associated physical phenomenon

A number of modern studies of early astronomical tables compare the results produced by the table with actual, physical celestial positions. This is done for several reasons. The most obvious is to gauge a measure of the table's predictive success. For instance, one might compare a table of computed eclipse possibilities with actual eclipses, or compare planetary visibility tables with the actual planets' visibilities. Some celestial phenomena are more amenable to such comparisons (for instance, did an eclipse occur or did it not?), while

some are more delicate (such as acronychal risings, planetary positions, and stationary points; these are impossible to observe directly with any precision). Researchers who seek such comparisons usually recompute estimates of these quantities using modern scientific theory (taking into account effects such as gravitational influences, refraction, and so on) and match these, sometimes value for value, with those given in a historical table. The results allow investigators to make conclusions regarding the effectiveness of the historical astronomical models and procedures.

However, caution must be applied when pursuing this sort of inquiry. Conformity with a physical phenomenon was not always the primary goal. In some cases the historical scientist might have been attempting to conform instead with a dominant scientific theory, or even with observational data generated by themselves or illustrious predecessors. In these cases it is all too easy to leap to a conclusion of scientific fraud. As in modern times, historical authors lived and worked in cultures where the interactions between theory, observation, and authority were more nuanced than the textbook account of the scientific method would have one believe.

One such case is the planetary theory and related tables of Ibn al-Shāṭir, a 14th-century Syrian astronomer. Ibn al-Shāṭir's work is part of a lengthy scientific tradition within medieval Islam questioning Ptolemy's planetary models. Their critique was not the fit of these models to the planetary data; rather, astronomers aimed their criticisms at Ptolemy's violations of Aristotelian physics. In particular, to reproduce certain planetary phenomena Ptolemy had been forced to introduce a new point in his models, called the *equant*. The center of the epicycle was asserted to move around its orbit circle (the *deferent*) uniformly, not around its own center (as required by Aristotle), but around the equant point. Ibn al-Shāṭir's tables instead built upon a geometric model of his own invention that avoids any such violations. Ibn al-Shāṭir's tables were considered an improvement on Ptolemy not because they fit the observations any better, but because they conformed more closely to the general principles of natural philosophy.

In another case, the priority of computability outranked that of fit with the phenomenon. In the first half of the second century BCE, Hypsicles of Alexandria composed the *Anaphorikos*, within which he set out a scheme to compute a table of all oblique ascensions for a given local latitude (see Figure 6).⁴⁹ Hypsicles' scheme was based on the assumption that rising times can be computed arithmetically — that is, by a linear sequence of values. The results produced by such a scheme could not hope to do more than model the true oblique ascensions qualitatively, but the values would be very easy to

⁴⁹ For a translation and technical commentary of this work, see Montelle, 'The Anaphoricus of Hypsicles.'

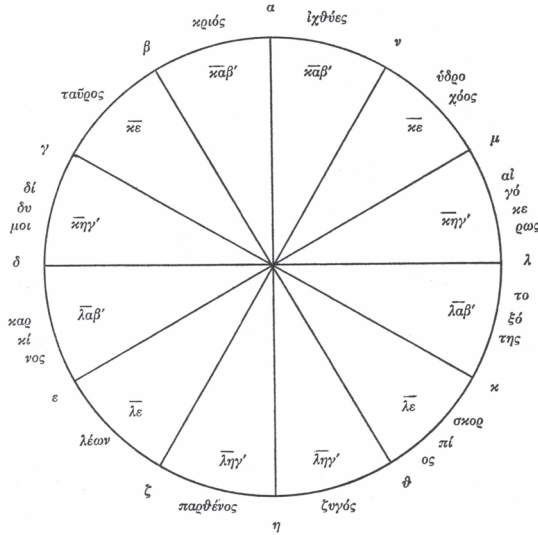


Figure 6: A transcription of the circular table displaying oblique ascensions for Alexandria found in MS Vat. gr. 204, f. 135v, from Hypsicles’ *Anaphorikos*. Reproduced from V. de Falco and Max Krause, ‘Hypsikles. Die Aufgangszeiten der Gestirne’, *Abhandlungen der Akademie der Wissenschaften in Göttingen, philologisch-historische Klasse*, Dritte Folge, Nr. 62 (1966), p. 37.

compute. Using the local latitude of Alexandria, Hypsicles presents a worked example; for successive zodiacal signs (and applying symmetry for signs 7 to 12), they are:

sign	rising time	sign
1	21;40	12
2	25	11
3	28;20	10
4	31;40	9
5	35	8
6	38;20	7

Note that the difference between oblique ascensions for successive signs is the constant value 3;20.

Clearly, this scheme was very rough indeed.⁵⁰ However, to dismiss it for this reason is to neglect the elegance of the approach and its primary purpose

⁵⁰ Several of the factors that contribute to this ‘roughness’ include: i) it is a linear arithmetical scheme; ii) the value for the obliquity of the ecliptic ε is not made explicit by Hypsicles; and iii) the ratio of 5:7 for the shortest to longest day (equivalent to a latitude of $\varphi \approx 35;32^\circ$) is too high for Alexandria, which is closer to $\varphi = 30^\circ$. For the purpose of comparison, we have recomputed the oblique ascensions using spherical trigonometry, assuming ε to be Ptolemy’s $23;51^\circ$. For $\varphi = 30^\circ$, there results: 20;59, 24;17, 29;57, 34;35, 35;31, 34;41, 34;41. When $\varphi = 35;32^\circ$, there results: 19;21, 22;54, 29;21, 35;11, 36;54, 36;19, 36;19.

of computability. Hypsicles and many after him modelled rising times linearly, and continued to do so even when more accurate spherical approaches became available. Astrologers used it to compute oblique ascensions for their local circumstances using only a single empirically derivable fact (the ratio of longest to shortest day). Therefore, rather than rejecting a table or computational scheme that poorly reproduces a physical phenomenon as scientifically inferior, the table cracker should evaluate the historical context to consider the table maker's goals in the light of the computing tools and empirical data available, as well as the quantitative abilities of the table's audience.

As delicate as the matter is, comparing tabular data with physical phenomena has produced important results, especially when comparing two different historical techniques against each other. For instance, scholars such as Steele have investigated Babylonian tables used for the prediction of eclipses, to determine whether the later methods improved in their ability to predict the timing and circumstances of eclipses over the last millennium that they were astronomically active. Using modern retrodictions, comparisons, and statistical trends, Steele concluded, surprisingly, that there is no evidence for an improvement in accuracy as time progressed.⁵¹ Building on Steele's work, Montelle considers the reckoning of eclipse possibilities in the so-called 'ACT' tabular cuneiform sources. Her comparison with actual eclipse possibilities also suggests that these later ACT methods (which could contain up to 18 columns of intermediary tabulated data) were not able to predict correctly the timing and circumstances of eclipses.⁵² In fact these tabulated predictions were no more accurate than the earlier non-tabular sources, despite being more deliberate, taking into account more factors, working to greater precision, being technically more elaborate, and showing more mathematical reasoning. This reveals the prodigious difficulty early investigators were facing in producing schemes that fit actual observed eclipses.

5. Concluding Remarks: Approaching a Table with Due Caution

When used appropriately, table cracking can reliably enhance existing historical methods and studies. It can help inform historians in their key lines of inquiry: how tables were computed, how they were read and understood, and how they were used. In some instances, table cracking is the only recourse we have when generating historical information about the table's creation, purpose, and effectiveness. However, table cracking techniques are most powerful when they are used in tandem with other historical information. Forwarding a claim based on analytical means alone can lead to unwarranted conclusions. Table crackers may initially produce results in apparent conflict with

⁵¹ Steele, *Observations and Predictions*.

⁵² Montelle, *Chasing Shadows*, pp. 94–97 and Appendix B.

other more traditional inquiries. Through a process of balancing, evaluating, and re-interpreting these sources of information within a wider context, the table cracker can form a clearer impression of the contents and use of the table.

Clearly, reliability of quantitative results will aid the researcher in convincing her colleagues of the validity of her conclusions. General procedures can accomplish this by using tried and tested statistical techniques. However, every table is unique. When general methods fail to adapt to the local conditions of the table or to the question asked by the researcher, techniques can be adjusted to fit the situation.

Beyond the arrays of numbers lies the diligence of a compiler, the assiduousness of a scribe, the expectations of a patron, the industry of a user. The table embodies a combination of these influences in both direct and nuanced ways. Traces of these are detectable to table crackers; however, no amount of quantitative analysis will be able on its own to reconstruct fully the historical circumstances of a table. Thus table cracking efforts form but a part of the tools historians can draw upon when investigating a numerical table.

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