Stochastic maintenance for large numbers of bridges

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Abstract

Maintenance of a large number of public bridges needs a statistical approach. Following [OCR], we use Markov chain degradation models with randomized maintenance strategies. Statistically, a randomized strategy represents the frequency of bridges on which a maintenance plan is implemented. We develop a Dynamic Programming approach for stationary optimal randomized strategies. In this setting, the Bellman equation becomes a functional equation, which we solve completely using the value iteration and policy iteration classical methodologies. They lead to algorithms to find the optimal solution which are used to the maintenance of a portfolio of bridges.

Key words: infinite state dynamic programming; stochastic models; maintenance; replacement

Area of review: Stochastic Models

1 INTRODUCTION

1.1 GENERAL DESCRIPTION

The issue of maintenance has become a very important topic for a lot of industrial and public sectors. In industry, it has followed naturally the development of reliability for components, then for equipments and systems. As systems became more and more complex, more costly and lived longer, maintenance decisions

could not remain a matter of practitioners, with empirical decisions in terms of when to perform maintenance and what level of maintenance to implement.

For instance, in aeronautics, the maintenance costs can represent 20% of operating costs ([ASR]) which is huge and compels using optimisation techniques to reduce costs. The objective of reducing costs without jeopardizing safety is overwhelming, but the problems to solve may be specific to the industrial sector. In aeronautics, one needs to solve an aircraft maintenance routing problem. It consists in assigning aircraft to a sequence of flights that allow them to be routed to a maintenance station within a given number of days to perform a daily check. One builds flight schedules that satisfy maintenance constraints, such as the maintenance base locations for the fleet. We refer to [ASR] and the program ReMap [REM] for a good overview of Maintenance Optimization in aeronautics (see also [QDE]). Another example is maintenance of railways. Railway transport is becoming an even more relevant mean of transportation as an alternative to road and air vehicles. To fulfill these growing needs it is important to improve reliability and reduce the life-cycle cost of railway infrastructures in terms of building and maintenance. In this context, the issue is to implement an adequate maintenance plan based on track geometry degradation predictive models. We also refer to the program SMARTE [WIS] and to [LPS].

In this work we focus on bridges for which the problematics are different. In the presentation of the system PONTIS ([GOS]), the situation in the US in 1997 is described. At this time, the US bridge system represents 565 thousand bridges. In fact, 70% of the nation's existing bridges, were built prior to 1935. Some of them, still carrying large volumes of traffic, date back to the 1890s. The fact that most were designed for less traffic, slower speeds, and lighter roads, combined with the gradual effects of weather and de-icing chemicals, have caused substantial problems. Conversely to aeronautics, the responsibility of maintenance lies in the public sector, States and the Federal Government. At the federal level, there is the Federal Highway Bridge Replacement and Rehabilitation Program. Because no federal funds are available for the routine maintenance of bridges, generally the states have little incentive to maintain bridges. Instead, there is an inherent incentive to allow bridges to deteriorate to the point at which rehabilitation or replacement is necessary. For years, this problem had seemingly been dealt with by concentrating on the larger and more costly projects for rehabilitation or replacement as those activities were eligible for federal funds. This led to ignoring maintenance programs in favor of major rehabilitation. Given the magnitude of the problem and the enormous funds needed, it was obvious that different approaches were necessary. It has led to the development of BMS, Bridge Management System, and SHM, Structural Health Monitoring, see [SCG]. A BMS is defined as a rational and systematic approach to organizing and carrying out all the activities related to managing a network of bridges. It includes optimizing the selection of maintenance and improvement actions to maximize the benefits while minimizing the costs.

As described in [ROM], there are two types of approaches. The first one develops deterioration models in which the history of deterioration is taken into account. This is particularly important when the deterioration originates from chemical processes. In view of the complexity of the models, only few decision variables are generally introduced. Degradation models based on physical aspects and history can be found in [CSBS], [YXI], [ROM], [FDS],[KOF]. The other approach is that of Markov Decision processes. In this approach, the deterioration is described by a Markov chain, where the state represents the condition of the facility. The use of MDP can be found in [SCG], [GOS], [GKO], [MKN].

Deterioration models methods based on reliability are the most powerful ones. The fundamental concept is that of reliability index of a structural system. It is evaluated in terms of demand, such as loading effect, and capacity, such as resistance. Both demand and capacity vary during the system lifetime. Therefore, the reliability index varies with time. The variation of reliability index with time is called the reliability index profile. Efforts to maintain the performance of a system above a minimum prescribed reliability level need inspection, maintenance, and rehabilitation.

The Markov Decision processes are widely used in Operations Research. Besides their intrinsic importance, a new momentum has arised from their use in Reinforcement Learning. For instance, in [WBL], deep reinforcement learning is used to solve the MDP optimization problem. Although they do not take into account the physical aspects of deterioration, they have become popular for optimizing maintenance decisions for bridges. One issue is to define the "reward" function associated to the evolution of the condition of the facility. It contains the cost of maintenance actions, not difficult to establish, but also the risk and impact of each degradation level. One way to handle this issue is to consider only the maintenance costs, while imposing constraints on probabilities of degradation levels. Following [OCR], we use MDP in this work. Unlike [OCR], we consider a stationary dynamic problem with infinite horizon. The assumption of stationarity is reasonable, in view of the fact that the transition probability matrices are not dependent on time. At the same time, it is not a myopic policy, since it takes account of all future periods. It gives the optimal decision as a condition based rule, which is of importance. The main drawback is that Bellman equation is a functional equation and not a sequence of functions as in the finite horizon case. We provide the full theory for randomized maintenance strategies, based on value iteration and policy iteration algorithms.

Due to the presence of constraints in the modeling, the optimal strategies are indeed randomized. Randomized strategies are interpreted in terms of frequencies of bridges for which a specific maintenance plan is implemented. Although randomized strategies have been introduced in the literature, in general they are not so needed. The paper [OCR] shows their importance and interest when dealing with constraints and large numbers of structures. We are close to this paper, which we extend in several directions. We first solve a dynamic stationary optimization problem, with infinite horizon, using the value iteration and policy

iteration methodologies and prove the convergence of algorithms. Then, we apply it to the maintenance of a portfolio of public bridges.

2 DESCRIPTION OF THE MODEL

2.1 MARKOV CHAIN OF DEGRADATION

The park of bridges is composed of similar units. The condition of the structure is described by a sequence of states of degradation, numbered as $(1, \cdots, N)$ where 1 is the best one and N is the worst one. The increasing order reflects the increase of degradation. In one period of time (a month, a year) the state of degradation can go from i to $j \geq i$, $1 \leq i, j \leq N$, according to a probability law $\pi(i, j)$. So

$$\pi(i,j) = 0, \text{ if } j < i, \ 2 \leq i \leq N \tag{2.1}$$

$$\pi(i,j) \geq 0, \text{if } j \geq i, \sum_{j \geq i}^{N} \pi(i,j) = 1$$

This matrix is valid when there is no maintenance and reflects the natural degradation process. It defines the probability transition matrix of a stationary Markov chain. The Markov chain is described by a sequence of random variables X_n constructed on a probability space $(\Omega, \mathcal{A}, P)$, where n stands for the time. For any (i, j) and any n, it means that $P(X_{n+1} = j | X_n = i) = \pi(i, j)$. The first condition in (2.1) captures the obvious fact that the state of degradation cannot improve naturally, without maintenance.

2.2 MAINTENANCE PLAN AND MAINTENANCE STRATEGY

Following [OCR], we introduce maintenance plans $a = 1, \cdots, m$. A maintenance plan is characterized by a subset $A_a \subset \{2, \cdots, N\}$ of states of degradation, for which an action of maintenance is performed. The complement $\bar{A}_a$ in $\{1, \cdots, N\}$ contains the states for which no action is performed. Clearly the state 1 cannot be improved, so it is always in $\bar{A}_a$, for any a. If $i \in A_a$, then the plan defines a state $a(i) \in \{1, \cdots, i-1\}$ to which the state i is rehabilitated. This description of maintenance actions seems to be common in practice. For convenience we denote by $a = 0$, the no maintenance plan. So $A_0 = \emptyset$ and $\bar{A}_0 = \{1, \ldots, N\}$. When a maintenance plan $a \neq 0$ is operated, then the probability transition matrix π is clearly changed into π^a as follows

$$\pi^a(i,j) = \left| \begin{array}{ll} \delta_{a(i),j} & \text{if } i \in A_a \\ \pi(i,j) & \text{if } i \notin A_a \end{array} \right. \tag{2.2}$$

Of course, $\pi^0(i,j)$=$\pi(i,j)$. A maintenance strategy is a function $i \in \{1, \cdots, N\} \to x_a(i)$, which is a probability on $\{0, 1, \cdots, m\}$ the set of maintenance plans (including no maintenance action). So $0 \leq x_a(i) \leq 1$ and $\sum_{a=0}^{m} x_a(i) = 1$. A maintenance strategy is a randomized strategy in the terminology of MDP. To implement a maintenance strategy, we interpret $x_a(i)$ as the proportion of structures in the state i for which the maintenance plan a is operated. We denote a maintenance strategy by x or $x(.)$ to emphasize that $i \to x(i)$ is a function with values in the set of probabilities on the maintenance plans. When a maintenance strategy $x(.)$ is used then the evolution of the condition of the set of structures is characterized by the transition probability matrix

$$\pi^{x(i)}(i,j) = \sum_{a=0}^{m} x_a(i)\pi^a(i,j) \tag{2.3}$$

Note that

$$\sum_{j=1}^{N} \pi^{x(i)}(i,j) = \sum_{a=0}^{m} x_a(i) = 1$$

2.3 CONTROLLED MARKOV CHAIN

The next step is to define a controlled Markov chain. We first define a control policy as a sequence of maintenance strategies indexed by time. So we have a sequence of maintenance strategies $x^n(.)$ with values $x_a^n(i)$. The transition probability matrix at time n is

$$\pi^{x^n(i)}(i,j) = \sum_{a=0}^{m} x_a^n(i)\pi^a(i,j)$$

Representing the Markov chain by the sequence of random variables X_n, we have the evolution

$$P(X_{n+1} = j | X_n = i) = \sum_{a=0}^{m} x_a^n(i)\pi^a(i,j) \tag{2.4}$$

We have defined in this way a controlled Markov chain. We denote by p^n the probability distribution of the variable X_n, i.e. $p_i^n = P(X_n = i)$. This is also the proportion of structures in the state i at time n. We can then write the relation

$$p_i^{n+1} = \sum_{k=1}^{N} p_k^n \sum_{a=0}^{m} x_a^n(k)\pi^a(k,i) \tag{2.5}$$

with the initial condition

$$p_i^0 = p_i, \tag{2.6}$$

which represents the proportion of structures in the state i at time 0. We will also use the notation

$$P_i(x(.),p) = \sum_{k=1}^{N} p_k \sum_{a=0}^{m} x_a(k)\pi^a(k,i) \tag{2.7}$$

$P_i(x(.),p)$ represents the proportion of structures in the state i after implementing the maintenance strategy $x(.)$, to a state distribution characterized by the proportions p. It is natural to impose constraints on $x(.)$ to limit the proportions of bridges with high degradation state. So we may write

$$P_i(x(.),p) \leq \overline{P}_i,\ 1 \leq i \leq N \tag{2.8}$$

In the sequel, $P(x(.),p)$ will denote the vector of components $P_i(x(.),p)$, $i = 1, \cdots, N$.

2.4 COST STRUCTURE

In order to define an optimal control problem, we first have to describe the various costs. A natural way is to introduce an increasing function $i \to f(i)$ representing the cost incurred during a unit period of time of a structure being in the state of degradation i. The more the structure is degraded, the more costly it is. This cost captures in particular the risk of operating a structure in the state i. We could have more inspections, more accidents or more unavailability when the structure deterioration is high. Let us remark that if there are no constraints like (2.8), we need such a function, otherwise there is no need for maintenance.

The second category of costs concerns the cost of maintenance. For any (i,j) with $i > j = 1, \cdots, N-1$, $i = 2, \cdots, N$, we pay $C(i,j)$ which represents the cost of improving the state of the structure from i to j. We have $C(i,j) = 0$ if $i \leq j$. In addition, $j \to C(i,j)$ is decreasing, since it costs more to bring a given state i to j_1 than into j_2 if $j_1 < j_2$.

From the function $C(i,j)$, we can derive the cost of a maintenance plan a activated on a structure i, which we denote by $C^a(i)$. We have

$$C^a(i) = \left| \begin{array}{cc} 0 & \text{if } i \notin A_a \\ C(i,a(i)) & \text{if } i \in A_a \end{array} \right. \tag{2.9}$$

The cost of a maintenance strategy $x(.)$ operated on the category of structures i becomes

$$C^{x(.)}(i) = \sum_{a=0}^{m} x_a(i) C^a(i) \tag{2.10}$$

So the total cost of a maintenance strategy $x(.)$ is

$$C(x(.)) = \sum_{i=1}^{N} C^{x(.)}(i) = \sum_{i=1}^{N} \sum_{a=0}^{m} x_a(i) C^a(i) \tag{2.11}$$

We now implement a control policy $\mathcal{X} = (x^0(.), x^1(.), \cdots, x^n(.), \cdots)$ on an infinite horizon. We use a discount factor $\rho < 1$ to compute the discounted flow of expenses on the park of structures. Recalling the Markov chain X_n describing the health of the park as a function of time, the pay-off functional to be minimized is given by

$$J(p, \mathcal{X}) = \sum_{n=0}^{\infty} \rho^n E[f(X_n) + C^{x^n(.)}(X_n)] \tag{2.12}$$

in which p represents the initial distribution of probabilities of the states of health at time 0. The analytic version of (2.12) is written as follows

$$J(p, \mathcal{X}) \quad = \quad \sum_{n=0}^{\infty} \rho^n \sum_{i=1}^{N} p_i^n [f(i) + \sum_{a=0}^{m} x_a^n(i) C^a(i)] \tag{2.13}$$

in which p^n evolves with time, according to the formula (2.5). The cost of maintenance at time n is the cost the maintenance strategy $x^n(.)$, namely $C(x^n(.))$. We add the constraints

$$P_i(x^n(.), p^n) \leq \overline{P}_i, \ 1 \leq i \leq N \tag{2.14}$$

$$C(x^n(.)) \leq R \tag{2.15}$$

The last constraint captures a global budget constraint. We now summarize the constraints by introducing the set of maintenance strategies depending on the parameter p (p being a fixed probability distribution on $\{1, \cdots, N\}$)

$$\mathcal{U}(p) = \left\{ x(.) \left| \begin{array}{ll} P_i(x(.), p) \leq & \overline{P}_i, \ 1 \leq i \leq N \\ C((x(.)) \leq R & \end{array} \right. \right\} \tag{2.16}$$

Thus, (2.14) can be rewritten as

$$x^n(.) \in \mathcal{U}(p^n), \quad \forall n \tag{2.17}$$

3 DYNAMIC PROGRAMMING

3.1 BELLMAN EQUATION

We define the value function

$$V(p) = \inf_{\{\mathcal{X}|x^n(.)\in\mathcal{U}(p^n),\forall n\}} J(p,\mathcal{X}) \tag{3.1}$$

From Dynamic Programming theory, $V(p)$ satisfies the Bellman equation

$$V(p) = \inf_{\{x(.)\,|x(.)\in\mathcal{U}(p)\}} \left[\sum_{i=1}^{N} p_i(f(i) + C^{x(.)}(i)) + \rho V(P(x(.),p))\right] \tag{3.2}$$

3.2 EXISTENCE AND UNIQUENESS

For the study of the functional equation (3.2) it is useful to consider that the argument p lies in a bounded subspace of R^N, the space $\mathcal{P}$ of probabilities distribution on $\{1,\cdots,N\}$. The concepts of continuity or measurability are those related to R^N. We look for bounded measurable functions $p \to V(p)$. The space of bounded measurable functions on probabilities on maintenance plans, *i.e.* on the set $\{0,1,\cdots,m\}$ is equipped with the norm

$$||V|| = \inf_{p\in\mathcal{P}} |V(p)| \tag{3.3}$$

We state the result

Theorem 1. *There exists one and only one lower semi-continuous bounded positive function which solves Bellman equation (3.2). It coincides with the value function (3.1). There exists an optimal control policy, which is stationary, i.e at each time, it is the same maintenance strategy which achieves the infimum in the bracket on the right hand side of (3.2).*

3.3 PROOF OF THEOREM 1: VALUE ITERATION

Proof. We begin with an a priori estimate for $||V||$. If $V(p)$ sastisfies (3.2) , then necessarily

$$||V|| \leq \frac{\max_i(f(i) + \max_a C^a(i))}{1-\rho} \tag{3.4}$$

It also holds that $V(p) \geq 0$. Since $V(p)$ is l.s.c. bounded below and $x(,)$ lies in a compact, the infimum in the right hand side of (3.2) is attained. We denote it by $\hat{x}(.;p)$. Next, we call $\hat{p}^n$ the sequence defined by induction as follows

$$\widehat{p}^{n+1} = P(\widehat{x}(.;\widehat{p}^n), \widehat{p}^n) \tag{3.5}$$

$$\hat{p}^0 = p$$

and we also define the control $\widehat{x}_n(.) = \widehat{x}(.;\widehat{p}^n)$ and the control policy $\widehat{\chi} = (\widehat{x}_1(.), \cdots, \widehat{x}_n(.), \cdots)$. Note that (3.2) writes

$$V(p) = \sum_{i=1}^{N} p_i(f(i) + C^{\hat{x}(.;p)}(i)) + \rho V(P(\widehat{x}(.;p), p)) \tag{3.6}$$

We next insert $p = \hat{p}^n$ in equation (3.5). Multiplying both sides by ρ^n and using (3.5), we get

$$\rho^n V(\hat{p}^n) = \rho^n \sum_{i=1}^{N} \hat{p}_i^n (f(i) + C^{\hat{x}_n(.)}(i)) + \rho^{n+1} V(\hat{p}^{n+1})$$

Adding up, for $n = 0$ to $n = M$, we get

$$V(p) = \sum_{n=0}^{M} \rho^n \sum_{i=1}^{N} \hat{p}_i^n (f(i) + C^{\hat{x}_n(.)}(i)) + \rho^{M+1} V(\hat{p}^{M+1})$$

We let $M \to +\infty$. Since $V(p)$ is bounded, we have $\rho^{M+1} V(\hat{p}^{M+1}) \to 0$. Hence we obtain

$$V(p) = \sum_{n=0}^{+\infty} \rho^n \sum_{i=1}^{N} \hat{p}_i^n (f(i) + C^{\hat{x}_n(.)}(i)) =$$

$$\sum_{n=0}^{+\infty} \rho^n E(f(X_n) + C^{\hat{x}_n(.)}(X_n)) = J(p, \hat{\mathcal{X}})$$

If we take now any control policy $\mathcal{X} = (x^1(.), \cdots\ x^n(.), \cdots)$, which is admissible, i.e. satisfying (2.14), then by a similar reasoning, we obtain $V(p) \leq J(p, \mathcal{X})$. Therefore we claim that $V(p)$ coincides with the value function (3.1). Therefore the solution of (3.2) which is positive, l.s.c, bounded is unique.

It remains to prove the existence. The proof will be constructive. We first consider the problem (3.2) without the budget constraint. So we look for a function $\bar{V}(p)$ solving

$$\bar{V}(p) = \inf_{x(.)} \left[\sum_{i=1}^{N} p_i (f(i) + \sum_{a=0}^{m} x_a(i) C^a(i)) + \rho \bar{V}(\sum_{k=1}^{N} p_k \sum_{a=0}^{m} x_a(k) \pi^a(k, .)) \right] \tag{3.7}$$

This equation has an explicit solution

$$\bar{V}(p) = \sum_{i=1}^{N} p_i v_i \tag{3.8}$$

in which the numbers v_i are the solutions of the system of nonlinear algebraic equations

$$v_i = f(i) + \min_a [C^a(i) + \rho \sum_{k=1}^{N} v_k \pi^a(i,k)], i = 1, \cdots N \tag{3.9}$$

which has a unique solution. Note that, when $f(i) = 0$, the solution is simply $v_i = 0$. Defining $\hat{a}_i$ as the minimum in the right hand side, we can also write

$$v_i = f(i) + C^{\hat{a}_i}(i) + \rho \sum_{k=1}^{N} v_k \pi^{\hat{a}_i}(i,k) \tag{3.10}$$

We consider then the iteration, called **value iteration**

$$V^{l+1}(p) = \inf_{\{\{x(.)\,|x(.)\in\mathcal{U}(p)\}\}} \left[\sum_{i=1}^{N} p_i (f(i) + C^{x(.)}(i)) + \rho V^l(P(x(.),p)) \right] \tag{3.11}$$

$$V^0(p) = \bar{V}(p)$$

We first check that

$$V^1(p) \geq V^0(p) \tag{3.12}$$

and it is clear from (3.11) that if $V^l(p) \geq V^{l-1}(p)$, then $V^{l+1}(p) \geq V^l(p)$. So the sequence $V^l(p)$ is monotone increasing. On the other hand, we can also check that

$$V^l(p) \leq \frac{\max_i (f(i) + \max_a C^a(i))}{1-\rho} \tag{3.13}$$

Therefore, we can claim that $V^l(p) \uparrow V(p)$. We can also check that the functions $V^l(p)$ are l.s.c. We prove it by induction. Suppose $V^l(p)$ is l.s.c. and let consider a sequence p^h such that $p^h \to p$. The function

$$x(.) \to \sum_{i=1}^{N} p_i^h (f(i) + \sum_{a=0}^{m} x_a(i) C^a(i)) + \rho V^l(P(x(.), p^h))$$

is also l.s.c. and the infimum on the set $\{x(.)\,|x(.) \in \mathcal{U}(p^h)\}$ is attained at some $x^h(.)$. Therefore

$$V^{l+1}(p^h) = \sum_{i=1}^{N} p_i^h (f(i) + \sum_{a=0}^{m} x_a^h(i) C^a(i)) + \rho V^l(P(x^h(.), p^h))$$

Since the sequence $x^h(.)$ remains in a compact set, there exists a subsequence of the pair $p^h, x^h(.)$ which converges. To simplify the notation we assume that $p^h, x^h(.)$ converges to $p, x(.)$. By the l.s.c property of V^l, we have

$$\sum_{i=1}^{N} p_i(f(i) + \sum_{a=0}^{m} x_a(i)C^a(i)) + \rho V^l(P(x(.), p)) \leq \liminf_h V^{l+1}(p^h)$$

hence also

$$V^{l+1}(p) \leq \sum_{i=1}^{N} p_i(f(i) + \sum_{a=0}^{m} x_a(i)C^a(i)) + \rho V^l(P(x(.), p)) \leq \liminf_h V^{l+1}(p^h)$$

which proves that $V^{l+1}(p)$ is l.s.c. It follows that $V(p) := \lim \uparrow V^l(p)$ is also l.s.c and clearly satisfies

$$V(p) \leq \inf_{\{x(.)\,|x(.)\in\mathcal{U}(p)\}} \left[\sum_{i=1}^{N} p_i(f(i) + \sum_{a=0}^{m} x_a(i)C^a(i)) + \rho V(P(x(.), p))\right] \tag{3.14}$$

Let us prove that the opposite inequality is also true. First and from the l.s.c property of $V^l(p)$, there exists $x^l(.)$ depending on p, which attains the infimum in the right hand side of (3.11). Therefore, we may write

$$V^{l+1}(p) = \sum_{i=1}^{N} p_i(f(i) + \sum_{a=0}^{m} x_a^l(i)C^a(i)) + \rho V^l(P(x^l(.), p)) \tag{3.15}$$

Let $L > M$, we have

$$V^{L+1}(p) \geq \sum_{i=1}^{N} p_i(f(i) + \sum_{a=0}^{m} x_a^L(i)C^a(i)) + \rho V^M(P(x^L(.), p)) \tag{3.16}$$

hence also

$$V(p) \geq \sum_{i=1}^{N} p_i(f(i) + \sum_{a=0}^{m} x_a^L(i)C^a(i)) + \rho V^M(P(x^L(.), p)) \tag{3.17}$$

Since $x^L(.)$ is bounded, we can extract a subsequence which we still denote by $x^L(.)$ and which converges to $x^*(.)$. Since V^M is l.s.c. we obtain

$$V(p) \geq \sum_{i=1}^{N} p_i(f(i) + \sum_{a=0}^{m} x_a^*(i)C^a(i)) + \rho V^M(P(x^*(.), p))$$

Letting $M \to +\infty$, we get

$$V(p) \geq \sum_{i=1}^{N} p_i(f(i) + \sum_{a=0}^{m} x_a^*(i)C^a(i)) + \rho V(P(x^*(.), p)$$

and thus the inequality opposite to (3.14) is also true. Therefore $V(p)$ is solution of (3.2), which completes the proof of Theorem 1. ■ □

3.4 POLICY ITERATION

There exists another iteration technique, which can also prove Theorem 1 and provides a different algorithm. It is called **Policy Iteration.** We fix a maintenance strategy $x^l(.;p)$ and solve the linear problem

$$V^{l+1}(p) = \sum_{i=1}^{N} p_i(f(i) + C^{x^l(.;p)}(i)) + \rho V^{l+1}(P(x^l(.;p),p)) \tag{3.18}$$

The function $V^{l+1}(p)$ is uniquely defined. Let us define recursively the sequence $p^l = (p_n^l)_n$ as follows

$$p_{n+1}^l = P(x^l(.;p_n^l), p_n^l) \tag{3.19}$$

$$p_0^l = p$$

and $x_n^l(.) = x^l(.;p_n^l)$. Due to (2.12) and (2.13), we obtain

$$V^{l+1}(p) = \sum_{n=0}^{+\infty} \rho^n \sum_{i=1}^{N} p_{ni}^l (f(i) + C^{x_n^l(.)}(i)) \tag{3.20}$$

We then find $x^{l+1}(.;p)$ by solving the minimization problem

$$\min_{\{x(.)|x(.)\in\mathcal{U}(p)\}} \left\{ \sum_{i=1}^{N} p_i(f(i) + C^{x(.)}(i)) + \rho V^{l+1}(P(x(.),p)) \right\} \tag{3.21}$$

and we start the iteration with $x^0(.;p)$ which solves the minimization problem

$$\min_{\{x(.)|x(.)\in\mathcal{U}(p)\}} \sum_{i=1}^{N} p_i(f(i) + C^{x(.)}(i)) \tag{3.22}$$

We can then state

Theorem 2. *The sequence $V^l(p)$ defined by (3.18) is monotone decreasing and converges towards the solution of Bellman equation (3.2).*

Proof. First and by induction we prove that

$$0 \le V^l(p) \le \frac{\max_i(f(i) + \max_a C^a(i))}{1-\rho} = b \tag{3.23}$$

We next check that

$$V^2(p) \le V^1(p) \tag{3.24}$$

We have, by definition of the iteration,

$$x^1(.;p) \text{ solves } \min_{\{x(.)|x(.)\in\mathcal{U}(p)\}} \left\{ \sum_{i=1}^{N} p_i(f(i) + C^{x(.)}(i)) + \rho V^1(P(x(.),p)) \right\}$$

and

$$V^2(p) = \sum_{i=1}^{N} p_i(f(i) + C^{x^1(.;p)}(i)) + \rho V^2(P(x^1(.;p),p)) =$$

$$\sum_{i=1}^{N} p_i(f(i) + C^{x^1(.;p)}(i)) + \rho V^1(P(x^1(.;p),p)) + \rho(V^2(P(x^1(.;p),p)) - V^1(P(x^1(.;p),p))) \leq$$

$$\sum_{i=1}^{N} p_i(f(i) + C^{x^0(.;p)}(i)) + \rho V^1(P(x^0(.;p),p)) + \rho(V^2(P(x^1(.;p),p)) - V^1(P(x^1(.;p),p))) =$$

$$V^1(p) + \rho(V^2(P(x^1(.;p),p)) - V^1(P(x^1(.;p),p)))$$

Therefore, we have shown that

$$V^2(p) - V^1(p) \leq \rho(V^2(P(x^1(.;p),p)) - V^1(P(x^1(.;p),p))) \tag{3.25}$$

Let us set

$$p^2 = P(x^1(.;p),p),\ p^3 = P(x^1(.;p^2),p^2)$$

and more generally, define the sequence $p^{k+1} = P(x^1(.;p^k),p^k)$. Taking $p = p^k$ in (3.25), we have

$$V^2(p^k) - V^1(p^k) \leq \rho(V^2(p^{k+1}) - V^1(p^{k+1}))$$

which implies

$$V^2(p) - V^1(p) \leq \rho^k(V^2(p^{k+1}) - V^1(p^{k+1}))$$

$$\leq \rho^k b$$

from (3.23). Letting $k \to +\infty$, we obtain (3.24). By similar arguments, we obtain $V^{l+1}(p) \leq V^l(p)$. Therefore $V^l(p) \downarrow V^*(p)$ and we directly get the following inequality

$$V^*(p) \geq \inf_{\{x(.)\,|x(.)\in\mathcal{U}(p)\}} \left[\sum_{i=1}^{N} p_i(f(i) + C^{x(.)}(i)) + \rho V^*(P(x(.),p))\right] \tag{3.26}$$

On the other hand, using both (3.18) and the decreasing property of $(V_l)_l$, we can write

$$V^{l+1}(p) \leq \sum_{i=1}^{N} p_i(f(i) + C^{x^l(.;p)}(i)) + \rho V^l(P(x^l(.;p),p))$$

From the definition of $x^l(.;p)$ given in (3.21) it gives

$$V^{l+1}(p) \leq \sum_{i=1}^{N} p_i(f(i) + C^{x(.)}(i)) + \rho V^l(P(x(.),p)),\ \forall x(.)$$

which implies

$$V^*(p) \leq \sum_{i=1}^{N} p_i(f(i) + C^{x(.)}(i)) + \rho V^*(P(x(.),p)),\ \forall x(.)$$

and also

$$V^*(p) \leq \inf_{\{x(.)\,|x(.)\in\mathcal{U}(p)\}} \left[\sum_{i=1}^{N} p_i(f(i) + C^{x(.)}(i)) + \rho V^*(P(x(.),p))\right]$$

which, combined with (3.26) proves that $V^*(p)$ solves Bellman equation, and thus it is unique since it coincides with the value function in (3.1). This completes the proof. ■ □

4 Algorithms

In both iterations (3.11) or (3.21), we have to solve a minimization problem with constraints

$$\inf_{\{\{x(.)\,|x(.)\in\mathcal{U}(p)\}\}} \left[\sum_{i=1}^{N} p_i(f(i) + C^{x(.)}(i)) + \rho\Psi(P(x(.),p))\right] \tag{4.1}$$

where $\Psi(p)$ is is a given function (namely $V^l(p)$ or $V^{l+1}(p)$). The set of constraints is defined by

$$P_i(x(.),p) \leq \overline{P}_i,\ 1 \leq i \leq N \tag{4.2}$$

$$\sum_{i=1}^{N} C^{x(.)}(i) \leq R$$

$$x_a(i) \geq 0, \sum_{a=0}^{m} x_a(i) = 1,\ \forall i$$

We treat the constraints of the two first lines by Kuhn Tucker multipliers. There exist $\lambda_i \geq 0$, $1 \leq i \leq N$ and $\mu \geq 0$, such that solving the minimization problem

$$\min_{\{x(.)\geq 0 | \sum_{a=0}^{m} x_a(i)=1, \forall i\}} \sum_{i=1}^{N} (p_i + \mu) C^{x(.)}(i) + \sum_{i=1}^{N} \lambda_i P_i(x(.), p) + \rho \Psi(P(x(.), p)) \tag{4.3}$$

with optimal solution $\widehat{x}(.)$, it is also the solution of the constrained problem, provided it is admissible (*i.e.* it satisfies the two first lines of constraints in (4.2)). Therefore it satisfies the complementarity conditions

$$\lambda_i (P_i(x(.), p) - \overline{P}_i) = 0, \ 1 \leq i \leq N \tag{4.4}$$

$$\mu (\sum_{i=1}^{N} C^{x(.)}(i) - R) = 0$$

When there is no budget constraint and $f(i)$=0, the myopic solution, which corresponds to the first iteration, solves the following problem

$$\min_{\{x(.)\geq 0 | \sum_{a=0}^{m} x_a(i)=1, \forall i\}} \sum_{i=1}^{N} p_i C^{x(.)}(i) + \sum_{i=I}^{N} \lambda_i P_i(x(.), p) \tag{4.5}$$

$$\lambda_i (P_i(x(.), p) - \overline{P}_i) = 0, \ 1 \leq i \leq N$$

$$P_i(x(.), p) \leq \overline{P}_i, \ 1 \leq i \leq N$$

We mention that the same optimization problem has also been considered in [OCR].

5 Application on IQOA

The IQOA notation (Image Qualité Ouvrages d'Art) is numbered $\{1, \ldots 5\}$ ($N = 5$). The degradation (see [OCR]) follows

$$\pi = \pi^0 = \begin{pmatrix} 0.72 & 0.26 & 0 & 0.02 & 0 \\ 0 & 0.8 & 0.18 & 0.02 & 0 \\ 0 & 0 & 0.98 & 0.02 & 0 \\ 0 & 0 & 0 & 0.89 & 0.11 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We successively consider strategies with a single maintenance plan ($m = 1$) in Section 5.1 and with two

maintenance plans ($m = 2$) in Section 5.2. We also fix the discounting parameter to $\rho = \frac{1}{1+0.06}$.

5.1 Single maintenance plan

For a single maintenance plan $m = 1$, we consider $A_1 = \{3, 5\}$. For the plan $a = 1$, $a(3) = 2$ and $a(5) = 2$ (maintenance up to the notation 2 for bridges in condition 3 or 5) and

$$\pi^1 = \begin{pmatrix} 0.72 & 0.26 & 0 & 0.02 & 0 \\ 0 & 0.8 & 0.18 & 0.02 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.89 & 0.11 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We are looking for the function $i \longmapsto x(i)$ valued in probability distribution on $\{0, 1\}$. There is only a real action on states $i \in \{3, 5\}$, for the other states plan 0 or plan 1 are equivalent. Moreover, the cost of maintenance is given by $C^1(3) = 2$ and $C^1(5) = 12$ and $C^1(i) = 0$, $i \in \overline{A}_1$ as in [OCR].

5.1.1 Comparison to the Orcesi-Cremona myopic maintenance

As in [OCR], we fix two constraints on the acceptable proportions of bridges in state 3 and 5, namely $\overline{P}_3 = 0.20$ and $\overline{P}_5 = 0.05$. We also consider the following initial distribution of bridges in the portfolio

$$p = \begin{pmatrix} 0.1 \\ 0.5 \\ 0.3 \\ 0.07 \\ 0.03 \end{pmatrix}$$

For the simulation and the comparison to the result of [OCR], the cost fonction of being in a degraded state is set to $f(i) = 0$, $i = 1, \ldots, 5$. In this setting, the myopic maintenance of Orcesi-Cremona (OC policy) and our stationary maintenance (with respect to V_1) are equal and the cost is 6.32. The maintenance control policy is illustrated in the Figure 1.

It is worth emphasizing that the Orcesi-Cremona methodology cannot handle a non-zero cost function f for which their solution is naturally suboptimal. For instance, with $f(i) = 0.1 \cdot i^2$, $i = 1, \ldots, 5$ our maintenance policy and OC policy differ with respective costs 18.60 and 19.06.

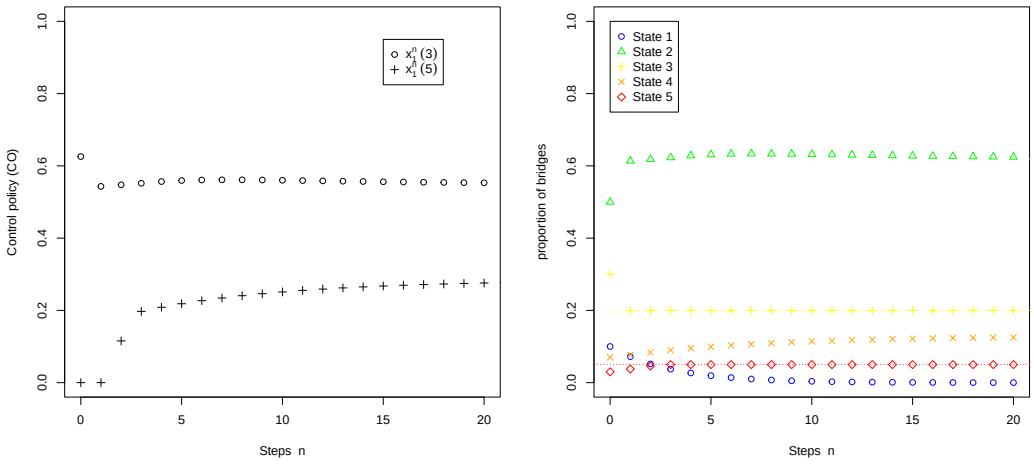

Figure 1: Maintenance control policy (on the left) and proportions of bridges in time (on the right) for the Cremona-Orcesi program

5.1.2 Sensibility analysis

For the single maintenance plan, we propose a sensibility analysis on the initial condition and the budget in order to visualize the stationary solution. The initial proportion of bridges is fixed as follows

$$p^0 = \begin{pmatrix} 0.1 \\ 0.53 - \theta \\ 0.3 \\ 0.07 \\ \theta \end{pmatrix}$$

and it is parametrized by θ which is the proportion of the worst conditioned bridges (notation 5). In our simulation, the parameter θ varies from 0.01 to 0.5. Secondly, the budget constraint R can vary from 4 to 14. The results are illustrated in Figure 2

Our numerical methodology allows to compute the region (see Figure 2) where no strategy can be found that satisfies a maximal budget R and constraints on the proportions of bridges in states 3 and 5.

The optimal proportions of bridges in state 3 and state 5 needed to be repaired are computed for initial distribution of bridges given by p^0. On the Figure 2, we can see that when the proportion of initial degraded bridges (in state 5) is sufficiently big, the budget is mainly used to repair it and less maintenance of bridges in state 3 is done.

Moreover, we can remark on Figure 2, for a budget up to 4 and with $\theta = 0.03$, that no maintenance plan for bridges in state 5 is done as in the first maintenance policy step in Figure 1.

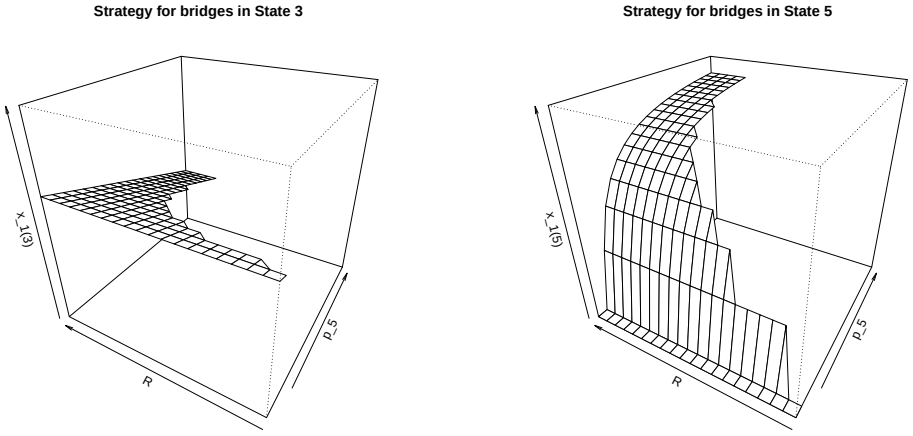

Figure 2: Optimal strategies for V_1, namely $x_1(3)$ (on the left) and $x_1(5)$ (on the right) for different values of budget constraint R and initial proportions of bridges in state 5. The graphics windows in height is set to $[0, 1]$.

5.2 Two maintenance plans

Let us now consider two maintenance plans ($m = 2$) for which the optimal policy has not been computed in [OCR]. We consider $A_1 = \{3, 5\}$ and the plan $a = 1$ such that $a(3) = 2$ and $a(5) = 2$ (maintenance up to the notation 2 for bridges in condition 3 or 5) which is associated with

$$\pi^1 = \begin{pmatrix} 0.72 & 0.26 & 0 & 0.02 & 0 \\ 0 & 0.8 & 0.18 & 0.02 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.89 & 0.11 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We also consider $A_2 = \{3, 5\}$ and the plan $a = 2$ such that $a(3) = 1$ and $a(5) = 1$ (maintenance up to the notation 1 for bridges in condition 3 or 5) which is associated with

$$\pi^2 = \begin{pmatrix} 0.72 & 0.26 & 0 & 0.02 & 0 \\ 0 & 0.8 & 0.18 & 0.02 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.89 & 0.11 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The cost of maintenance is given by $C^1(3) = 2$, $C^1(5) = 12$, $C^1(i) = 0$, $i \in \overline{A}_1$, $C^2(3) = 3$, $C^2(5) = 36$

and $C^2(i) = 0$, $i \in \overline{A}_2$ as in [OCR].

We are looking for the fonction $i \longmapsto x(i)$ valued in probability distribution on $\{0, 1, 2\}$. There is only a real action on states $i \in \{3, 5\}$, for the other states plan 0, plan 1 and plan 2 are equivalent.

We again fix two constraints on the acceptable proportions of bridges in state 3 and 5, namely $\overline{P}_3 = 0.20$ and $\overline{P}_5 = 0.05$. We also impose $\underline{P}_1 = 0.20$ to force the maintainer to keep a lower proportion of bridge in state 1, namely

$$\underline{P}_1 \leq P_1(x(.), p).$$

Once again and for the two maintenance plans, we propose a sensibility analysis on the initial condition and the budget and we fix the initial proportion of bridges as follows

$$p^0 = \begin{pmatrix} 0.15 \\ 0.33 - \theta \\ 0.45 \\ 0.07 \\ \theta \end{pmatrix}$$

It is always parametrized by θ which is the proportion of the worst conditioned bridges (notation 5). In our simulation, the parameter θ varies from 0.01 to 0.3. Secondly, the budget constraint R can vary from 4 to 36. The results are illustrated in Figure 3

Our numerical methodology allows to compute the region (see Figure 3) where no strategy with plans 0, 1, 2 can be found that satisfies a maximal budget R and constraints on the marginals $P_1(.,.)$, $P_3(.,.)$ and $P_5(.,.)$.

The optimal proportions of bridges in state 3 and in state 5 needed to be repaired for both maintenance plans 1 and 2 are computed. On the Figure 3, we can see that when the initial proportion of degraded bridges (in State 5) is sufficiently big in the initial distribution, a big part of the budget is used to repair it with the maintenance plan 1.

The plan 2 is used for bridges in state 3 to fulfill the lower proportion of bridge in state 1 required. The strategy 2 for bridges in state 5 is never used with our condition since it is too expensive.

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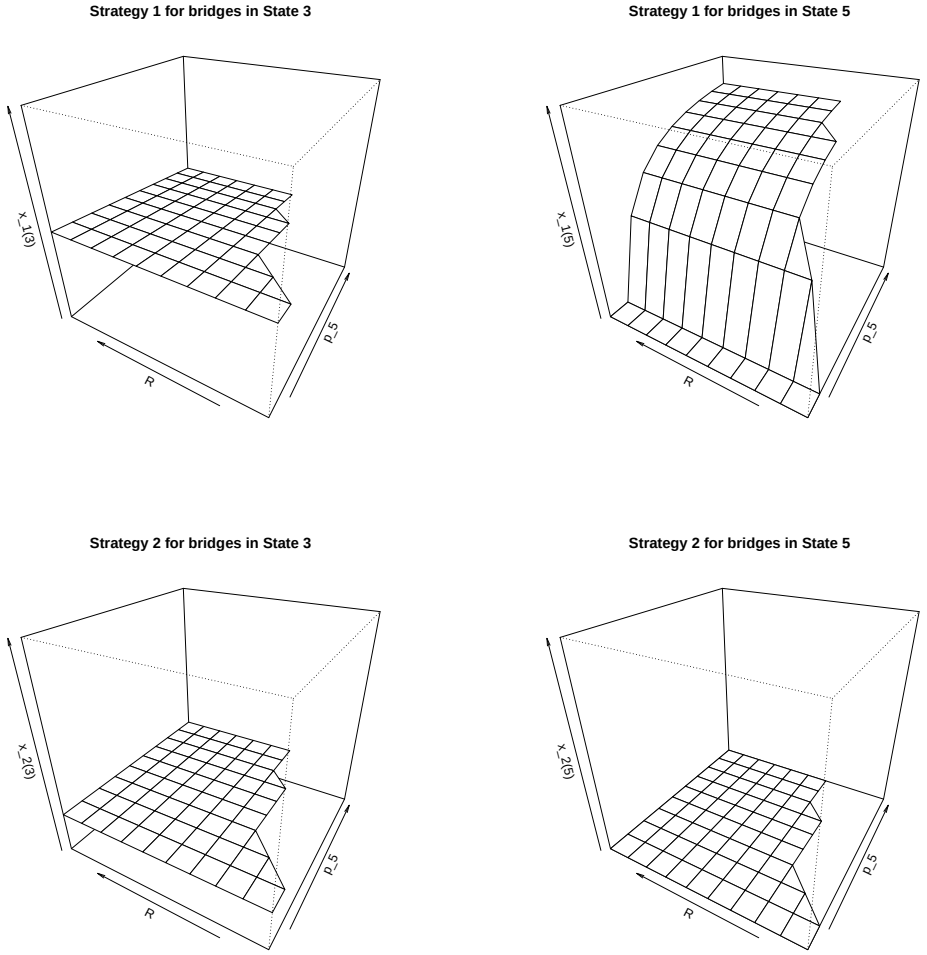

Figure 3: Optimal strategies for V_1, namely $x_1(3)$ and $x_2(3)$ (on the left) and $x_1(5)$ and $x_2(5)$ (on the right) for different values of market constraint R and initial proportions of bridges in state 5. The graphics windows in height is set to $[0, 1]$.

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