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Samy Skander Bahoura. An inequality for the solutions to an elliptic equation with weight. 2023.
hal-03917620

HAL Id: hal-03917620

<https://hal.science/hal-03917620>

Preprint submitted on 2 Jan 2023

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An inequality for the solutions to an elliptic equation with weight.

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Abstract

We derive a local uniform boundedness result for an equation with weight having interior singularity and log-holder weight. Also we have, $\sup u = f(\inf u)$.

Keywords: C^0 log-holder weight, interior singularity, a priori estimate, sup, inf, maximum principle, $\sup u = f(\inf u)$.

MSC: **35J60, 35B44, 35B45, 35B50**

1 Introduction and Main Results

We set $\Delta = \partial_{11} + \partial_{22}$ on open bounded domain Ω of \mathbb{R}^2 .

We consider the following equation:

$$(P) \begin{cases} -\Delta u = \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u \geq 0 & \text{in } \Omega. \end{cases}$$

Here:

$$u \in L^1_{loc}(\Omega),$$

$$0 \leq V \leq b, \int_{\Omega} \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta e^u dx \leq C,$$

and,

$$d = \text{diam}(\Omega), \beta \in (0, 1), 0 \in \Omega$$

The solutions of this equations are defined in the sense of distributions. Many authors studied this type of equations, on open sets of the two-dimensional space

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also for surfaces, with or without the boundary condition, see [1–20], here we can find existence and compactness results.

Among other results, we can see in [12] the following important Theorem

Theorem A (Brezis-Merle [12]). *If (u_i) is a sequence of solutions of problem (P) with (V_i) satisfying $0 < a \leq V_i \leq b < +\infty$ and without the term $\left(\frac{1}{-\log \frac{|x|}{2d}}\right)^\beta$, then, for any compact subset K of Ω , it holds:*

$$\sup_K u_i \leq c,$$

with c depending on a, b, K, Ω

One can find in [12] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i} , namely, we have:

Theorem B (Brezis-Merle [12]). *For $(u_i)_i$ and $(V_i)_i$ two sequences of functions relative to the problem (P) without the term $\left(\frac{1}{-\log \frac{|x|}{2d}}\right)^\beta$ and with,*

$$0 \leq V_i \leq b < +\infty \text{ and } \int_{\Omega} e^{u_i} dy \leq C,$$

then for all compact set K of Ω it holds;

$$\sup_K u_i \leq c,$$

with c depending on b, C, K and Ω .

With more regularity on V one can have another type of estimates, a sup + inf type inequalities. Shafrir see [19], proved that, a sup + C inf inequality for solutions the previous equation without assumption on the boundary and without

the term $\left(\frac{1}{-\log \frac{|x|}{2d}}\right)^\beta$ with V_i satisfying $0 < a \leq V_i \leq b < +\infty$.

Here, we have:

Theorem 1.1 *For sequences $(u_i)_i$ and $(V_i)_i$ of the Problem (P), for all compact set K of Ω , there exists a positive constant $c = c(b, \beta, C, K, \Omega)$ such that:*

$$\|u_i\|_{L^\infty(K)} \leq c(b, \beta, C, K, \Omega),$$

Consider a positive number $M > 0$, if we assume $u_i \geq -M$, we can extend the previous result to any function u bounded from below by $-M$. If we consider the function $v_i = u_i + M$, then v_i satisfies all the condition of the previous theorem.

Here we have, if we replace the condition $u_i \geq 0$ by $u_i \geq -M$:

Corollary 1.2 *For sequences $(u_i)_i$ and $(V_i)_i$ of the first equation of (P), for all compact subset K of Ω we have:*

$$\|u_i\|_{L^\infty(K)} \leq c(b, \beta, C, M, K, \Omega).$$

Remarks: 1) In general, the solutions of the previous equation are not C^2 , but we can add assumptions on V to have C^2 solutions and C^2 compactness result. Indeed, if we add the conditions, $V_i \in C^{0,\epsilon}$, $\epsilon > 0$ and $V_i \rightarrow V$ in the space $C^{0,\epsilon}$, $\epsilon > 0$, one can reduce the problem to the Newtonian potential of radial distributions $f(x) = f(|x|) = \frac{V_i(0)e^{u_i(0)}}{\left(-\log \frac{|x|}{2d}\right)^\beta} \eta(|x|)$ with η a radial cutoff

function with compact support and $\eta \equiv 1$ in a neighborhood of 0, see the book of Dautray-Lions, [16].

2) To have an example of (P) , we can add the Dirichlet condition, but we must assume the domain regular and the solutions $u \in W_0^{1,1}(\Omega)$ and apply the maximum principle. Indeed, to apply the maximum principle, we need Agmon regularity theorem, the C^2 regularity of the domain is necessary in this case, or we apply the weak version of the maximum principle by mean of Kato's inequality, in this case the smoothness of the domain is needed. In all cases, we must assume that the domain is at least a C^2 domain or smooth domain and the solutions more regular.

Here we consider a bounded domain not necessarily regular. And we assume that $\inf_\Omega u > -\infty$ or we add the assumption that $u \in C^0(\bar{\Omega})$. Also, one can consider the problem in a smaller domain $\Omega' \subset\subset \Omega$.

In the previous corollary we have:

$$\sup_K u \leq c(b, \beta, C, K, \Omega, \inf_\Omega u).$$

It is an estimate of the maximum on each compact subset of Ω of the solutions by mean of the infimum on Ω and b, β, C, K and Ω . (Also we have an a priori estimate).

Now, we assume $\Omega = B_1(0)$ the unit disk. Here, we have for any open domain $\Omega' \subset\subset \Omega = B_1(0)$ with $0 \in \Omega'$:

Corollary 1.3 *For u solution of the first equation of (P) relative to V , for all compact subset K of Ω' we have:*

$$\sup_K u \leq c(b, \beta, C, K, \Omega', \inf_{\Omega'} u).$$

2 Proof of the Theorem

We have:

$$u_i \in L_{loc}^1(\Omega), \text{ and } \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta e^{u_i} \in L^1(\Omega).$$

By solving a Dirichlet problem and if use Weyl Theorem and the Corollary 1 of Brezis and Merle paper, see [12] we have:

$$e^{|u_i|} \in L_{loc}^k(\Omega), \forall k \geq 1,$$

By the local elliptic estimates of Agmon, see [1], we have:

$$u_i \in W_{loc}^{2,k}(\Omega) \cap C^{1,\epsilon}(\Omega), \quad k > 2.$$

We write:

$$u_i = \sqrt{\left(-\log \frac{|x|}{2d}\right)^\beta} \times \sqrt{\left(\frac{1}{-\log \frac{|x|}{2d}}\right)^\beta} u_i,$$

We use the Holder inequality to have:

$$\|u_i\|_{L^1(B(x_0, r_0))} \leq \|(\sqrt{-\log \frac{|x|}{2d}})^\beta\|_{L^2(B(x_0, r_0))} \times \left\| \left(\sqrt{\frac{1}{-\log \frac{|x|}{2d}}} \right)^\beta u_i \right\|_{L^2(B(x_0, r_0))},$$

Thus,

$$\begin{aligned} \|u_i\|_{L^1(B(x_0, r_0))} &\leq \left(\|(-\log \frac{|x|}{2d})^\beta\|_{L^1(B(x_0, r_0))} \right)^{1/2} \times \left(\left\| \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta u_i^2 \right\|_{L^1(B(x_0, r_0))} \right)^{1/2} \leq \\ &\leq C \int_{B(x_0, r)} \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta e^{u_i} dx \leq C. \end{aligned}$$

Thus,

$$\|u_i\|_{L^1(B(x_0, r_0))} \leq C.$$

Thus we proved that u_i are locally uniformly bounded in L^1 .

Since,

$$\int_{\Omega} \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta V_i e^{u_i} dx \leq C,$$

We have a convergence to a nonnegative measure μ :

$$\int_{\Omega} \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta V_i e^{u_i} \phi dx \rightarrow \int_{\Omega} \phi d\mu, \quad \forall \phi \in C_c(\Omega).$$

We set S the following set:

$$S = \{x \in \Omega, \exists (x_i) \in \Omega, x_i \rightarrow x, u_i(x_i) \rightarrow +\infty\}.$$

We say that x_0 is a regular point of μ if there function $\psi \in C_c(\Omega)$, $0 \leq \psi \leq 1$, with $\psi = 1$ in a neighborhood of x_0 such that:

$$\int \psi d\mu < 4\pi. \quad (1)$$

We can deduce that a point x_0 is non-regular if and only if $\mu(x_0) \geq 4\pi$.

A consequence of this fact is that if x_0 is a regular point then:

$$\exists R_0 > 0 \text{ such that one can bound } (u_i) = (u_i^+) \text{ in } L^\infty(B_{R_0}(x_0)). \quad (2)$$

We deduce (2) from corollary 4 of Brezis-Merle paper, because we have proved that locally we have:

$$\|u_i^+\|_1 = \|u_i\|_1 \leq c.$$

We denote by Σ the set of non-regular points.

Step 1: $S = \Sigma$.

We have $S \subset \Sigma$. Let's consider $x_0 \in \Sigma$. Then we have:

$$\forall R > 0, \lim \|u_i^+\|_{L^\infty(B_R(x_0))} = +\infty. \quad (3)$$

Suppose contrary that:

$$\|u_i^+\|_{L^\infty(B_{R_0}(x_0))} \leq C.$$

Then:

$$\|e^{u_{i_k}}\|_{L^\infty(B_{R_0}(x_0))} \leq C, \text{ and}$$

$$\int_{B_R(x_0)} \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta V_{i_k} e^{u_{i_k}} = o(1).$$

For R small enough, which imply (1) for a function ψ and x_0 will be regular, contradiction. Then we have (3). We choose $R_0 > 0$ small such that $B_{R_0}(x_0)$ contain only x_0 as non-regular point. Σ . Let's $x_i \in B_{R_0}(x_0)$ such that:

$$u_i^+(x_i) = \max_{B_{R_0}(x_0)} u_i^+ \rightarrow +\infty.$$

We have $x_i \rightarrow x_0$. Else, there exists $x_{i_k} \rightarrow \bar{x} \neq x_0$ and $\bar{x} \notin \Sigma$, i.e. \bar{x} is a regular point. It is impossible because we would have (2).

Since the measure is finite, if there are blow-up points, or non-regular points, $S = \Sigma$ is finite.

Step 2: $\Sigma = \{\emptyset\}$.

Now: suppose contrary that there exists a non-regular point x_0 . We choose a radius $R > 0$ such that $B_R(x_0)$ contain only x_0 as non-regular point. Thus outside Σ we have local uniform boundedness of u_i , also in C^1 norm. Also, we have weak *-convergence of V_i to $V \geq 0$ with $V \leq b$.

Let's consider (by a variational method):

$$z_i \in W_0^{1,2}(B_R(x_0)),$$

$$-\Delta z_i = f_i = \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta V_i e^{u_i} \text{ in } B_R(x_0), \text{ et } z_i = 0 \text{ on } \partial B_R(x_0).$$

By a duality theorem:

$$z_i \in W_0^{1,q}(B_R), \|\nabla z_i\|_q \leq C_q.$$

By the maximum principle, $u_i \geq z_i$ in $B_R(x_0)$.

$$\int \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta e^{z_i} \leq \int \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta e^{u_i} \leq C. \quad (4)$$

On the other hand, $z_i \rightarrow z$ a.e. (uniformly on compact sets of $B_R(x_0) - \{x_0\}$) with z solution of :

$$-\Delta z = \mu \text{ in } B_R(x_0), \text{ et } z = 0 \text{ on } \partial B_R(x_0).$$

Also, we have up to a subsequence, $z_i \rightarrow z$ in $W_0^{1,q}(B_R(x_0))$, $1 \leq q < 2$ weakly, and thus $z \in W_0^{1,q}(B_R(x_0))$.

Then by Fatou lemma:

$$\int \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta e^z \leq C. \quad (5)$$

As $x_0 \in S$ is not regular point we have $\mu(\{x_0\}) \geq 4\pi$, which imply that, $\mu \geq 4\pi\delta_{x_0}$ and by the maximum principle in $W_0^{1,1}(B_R(x_0))$ (obtained by Kato's inequality)

$$z(x) \geq 2 \log \frac{1}{|x - x_0|} + O(1) \text{ if } x \rightarrow x_0.$$

Because,

$$z_1 \equiv 2 \log \frac{1}{|x - x_0|} + 2 \log R \in W_0^{1,s}(B_R(x_0)), \quad 1 \leq s < 2.$$

Thus,

$$\left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta e^z \geq \frac{C}{|x - x_0|^2 \left(-\log \frac{|x|}{2d} \right)^\beta}, \quad C > 0.$$

Both in the cases $x_0 = 0$ and $x_0 \neq 0$ we have because $\beta \in (0, 1)$:

$$\int_{B_R(x_0)} \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta e^z = \infty.$$

But, by (5):

$$\int \left(\frac{1}{-\log \frac{|x|}{2d}} \right)^\beta e^z \leq C.$$

which is a contradiction.

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