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Convex Asset Pricing

Emy Lécuyer* V. Filipe Martins-da-Rocha†

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Abstract

In order to encompass general financial frictions, we generalize the fundamental theorem of asset pricing to convex price functionals. We identify a new arbitrage condition, called robust no-arbitrage, that characterizes viability and generalizes the well-known no-arbitrage condition used in models with a linear pricing.

1 Introduction

We consider a standard two-period financial markets economy under uncertainty, where an investor purchases an optimal portfolio to allocate consumption between the two dates and across states of nature. The trading possibilities are described by a price functional that defines the cost of holding a portfolio after all possible trades at $t = 0$. An equilibrium price functional necessarily satisfies *viability* in the sense that an optimal trade exists for at least one trader with convex, continuous, and strictly increasing preferences. In their seminal contributions, Harrison and Kreps (1979) and Kreps (1981) show that a linear price functional is viable if, and only if, it does not allow for an *arbitrage opportunity*. A portfolio is an arbitrage opportunity when it is a claim to non-negative consumption tomorrow available for nothing or less today. The Fundamental Theorem of Asset Pricing (FTAP henceforth) states that a linear price functional admits no arbitrage opportunities if, and only if, assets are

*Université Paris-Dauphine, Université PSL, LEDA, CNRS, IRD, 75016 Paris, France

†Université Paris-Dauphine, Université PSL, LEDA, CNRS, IRD, 75016 Paris, France and Sao Paulo School Economics–FGV

linearly priced using strictly positive states prices or equivalently strictly positive stochastic discount factors (see, for instance, Magill and Quinzii (1996) and LeRoy and Werner (2014)).

The linearity of a price functional is not consistent with frictions and transaction costs in financial markets. If transaction costs are proportional to the volume dealt with, then the pricing rules are sublinear (positively homogeneous and subadditive) but not necessarily linear.[1] The extension of the FTAP for sublinear pricing rules has been analyzed by Jouini and Kallal (1995) and Luttmer (1996). Recently, Cerreia-Vioglio, Maccheroni and Marinacci (2015) generalized the FTAP to price functionals that are positively homogeneous but not necessarily subadditive. Positive homogeneity implies that the ratio of the order's price to the order's size, the unitary price, is a constant function of the size. A proxy for the unitary price is the temporary market impact. It represents the average price change conditioned on the size and the nature of an order placed in the markets. Large institutions needing to place large orders in the markets are especially attentive to this effect. To reduce their costs associated with short-running (the cost associated with finding a counterparty immediately), the imperfect substitution effects (the cost associated with the absence of a perfect substitute for the traded asset), and the information effect (the cost associated with agents believing that the asset is mispriced), they split their orders and keep secret their true size. Several empirical studies succeeded in reconstituting the total order, called meta order, placed on the markets by large institutions (Almgren, Thum, Hauptmann and Li (2005), Moro, Vicente, Moyano, Gerig, Farmer, Vaglica, Lillo and Mantegna (2009), Tóth, Lemperiere, Deremble, Lataillade, Kockelkoren and Bouchaud (2011), Bacry, Iuga, Lasnier and Lehalle (2015), and Donier and Bonart (2015)). These studies differ significantly in many points: the analyzed database, the way the price impact is defined and measured, the way different assets and periods are collated together in the analysis, and how the fit is performed. Nevertheless, they have consistently shown that the temporary price impact is approximately increasing as the square root of the order size. We show in section 2.2 that the resulting price functional,

[1] A price functional is positively homogeneous if the price $p(\lambda\theta)$ of $\lambda \geqslant 0$ units of some portfolio θ is $\lambda p(\theta)$. It is subadditive if $p(\theta+\eta) \leqslant p(\theta)+p(\eta)$ for any pair θ, η of portfolios.

which measures the total price paid for an order, is convex in quantity and not positively homogeneous.

In addition to the large and well-established empirical literature that studies market impact, several security trading models obtain a convex price functional in different settings. The most well-known is probably Kyle (1985). In his simple one-shot trading model with a noise trader, an insider, and a market maker trading a single risky asset, Kyle (1985) shows that the equilibrium unitary price increases linearly in quantity. It implies a convex price functional. Based on this model, Carvajal and Weretka (2012) and Carvajal (2018) also obtain convex price functionals when they explicitly model the existence of market impact. Biais, Martimort and Rochet (2000) prove the existence of a unique equilibrium with a convex price schedule, or put more simply, with convex prices in a financial market with many traders, an insider, and mamy market makers. In another setup, Glosten (1994) analyzes idealized electronic limit order books. He shows that the possibility of information-motivated trade implies convex prices in his setup. More recently, Beissner and Riedel (2019) introduced the concept of Knight-Walras equilibrium where "the market" has imprecise probabilistic information about states of nature and values contingent claims using a sublinear expectation operator.

The objective of this paper is to extend the existing literature on asset pricing by no-arbitrage and provide a new version of the FTAP that accommodates convex financial frictions. We start by discussing the appropriate arbitrage concept when the price functional is convex but not necessarily linear. The standard notion of an arbitrage opportunity is related to the strict monotonicity of the investor's preference relation. Formally, recall that a portfolio η is an arbitrage opportunity at some investor's position θ when, replacing the current position θ by the new position $\theta + \eta$, the associated intertemporal incremental consumption is positive, meaning that it is non-negative in any contingency (including the first period) and strictly positive in at least one contingency (including the first period). If we replace the property that "the associated intertemporal incremental consumption is increased" with the weaker property that "the associated intertemporal incremental consumption belongs to an open and convex set Γ containing the positive cone", we then obtain the concept of a

weak arbitrage opportunity at the position θ for the set Γ. The motivation for this definition comes from the fact that preferences are assumed to be strictly increasing and continuous. Indeed, if the incremented consumption is positive, then, by strict monotonicity, the investor prefers the new position $\theta + \eta$. Since his preferences are continuous, he also strictly prefers the new position if the corresponding incremented consumption belongs to an open and convex set close enough to the positive cone. The new consumption may decrease in some contingencies, but it should sufficiently increase in others to get an overall compensation. We say that a price functional is a *robust no-arbitrage price* when there exists a position θ and an open and convex set Γ containing the positive cone such that there are no weak arbitrage opportunities at the position θ for the set Γ. When the price functional is linear, our notion of robust no-arbitrage reduces to the standard concept of no-arbitrage price.

Our first contribution is to show that a convex price functional is viable if, and only if, it is a robust no-arbitrage price. We then generalize the FTAP by showing that a price functional is a robust no-arbitrage price if, and only if, it is supported by strictly positive state prices in the sense that there exists a portfolio $\theta^\star$ such that any incremental price $p(\theta) - p(\theta^\star)$ associated with a different position θ is at least as large as the present value (with respect to the strictly positive state prices) of the incremental payoff.

As a first step, we modeled trading opportunities by considering primitive assets and portfolios. The market structure is then described by the price of a portfolio of primitive assets and the associated payoff. In the financial economics literature, there is an alternative approach. Instead of analyzing the price and the payoff of portfolios, we may directly consider the function defining the cost at the first period to implement some random consumption at the second period. This function is called a *pricing rule* and the notions of viability and no-arbitrage can be adapted to this modeling. This approach is more general since a pricing rule can be derived from a price functional and a payoff mapping by super-replication.

We start by analyzing the properties of a super-replication pricing rule that is derived from a price functional. Formally, we show that the super-replication pricing rule associated with any convex price functional is naturally convex but also monotone. If the price functional is viable, then the super-replication pricing rule is also viable. Furthermore, we show

that an abstract pricing rule is viable if, and only if, it is a robust no-arbitrage price. Then, we extend the FTAP to pricing rules by showing that a pricing rule is a robust no-arbitrage price if, and only if, it is supported by strictly positive state prices. Finally, we show that when every payoff can be super-replicated (it is for example the case if there is a riskless asset), a pricing rule is a robust no-arbitrage if, and only if, it is *eventually strictly increasing*.

2 The Model

We start with standard notations and definitions. Given a non-empty finite set K, we denote by $\mathbb{R}^K$ the set of functions $x : K \to \mathbb{R}$. An element $x \in \mathbb{R}^K$ is also called a vector and denoted by $x = (x(k))_{k\in K}$ or $x = (x_k)_{k\in K}$. If L is a non-empty subset of K, then $\mathbf{1}_L$ denotes the vector in $\mathbb{R}^K$ defined by $\mathbf{1}_L(k) = 1$ if $k \in L$ and $\mathbf{1}_L(k) = 0$ if $k \notin L$. When $L = \{\ell\}$ is a singleton, we slightly abuse notation and prefer the notation $\mathbf{1}_\ell$. A vector $x \in \mathbb{R}^K$ is nonnegative (strictly positive) when $x(k) \geqslant 0$ (resp. $x(k) > 0$) for all $k \in K$. The set of nonnegative (strictly positive) vectors is denoted by $\mathbb{R}^K_+$ (resp. $\mathbb{R}^K_{++}$). A vector A function $f : \mathbb{R}^K \to \mathbb{R}$ is convex when $f(\lambda x + \mu y) \leqslant \lambda f(x) + \mu f(y)$ for any $x, y \in \mathbb{R}^K$ and any pair $(\lambda, \mu) \in \mathbb{R}^2_+$ of non-negative numbers satisfying $\lambda + \mu = 1$. A function $f : \mathbb{R}^K \to \mathbb{R}$ is sub-additive when $f(x + y) \leqslant f(x) + f(y)$ for every $x, y \in \mathbb{R}^K$, and positively homogeneous when $f(\lambda x) = \lambda f(x)$ for any $\lambda > 0$ and $x \in \mathbb{R}^K$.[2] When f is sub-additive and positively homogeneous, we say that f is sublinear.[3]

2.1 Primitives

There are two periods $t \in \{0, 1\}$. There is a single perishable good at each period which can be consumed. Uncertainty is represented by a finite set Ω of states of nature that can occur at $t = 1$. An agent is represented by a triple (e_0, e_1, U) where $e_0 > 0$ represents the agent's non-financial income (such as labor income) at $t = 0$, $e_1 \in \mathbb{R}^\Omega_{++}$ is a random variable

[2]The function f is super-additive when $-f$ is sub-additive.

[3]A sublinear function is necessarily convex, but the converse is not always true.

representing future non-financial income, and $U : C \to \mathbb{R}$ is a utility function defined on the consumption set $C := \mathbb{R}_+ \times \mathbb{R}_+^\Omega$ representing the agent's preference relation over consumption plans $c = (c_0, c_1) \in C$. We denote by $\mathcal{A}$ the set of triples (e_0, e_1, U) such that the utility function $U : C \to \mathbb{R}$ is strictly increasing on the interior of C, lower semi-continuous on the interior of C, strictly quasi-concave on the interior of C, and satisfies Inada's condition at the origin.[4] We assume that $\mathcal{A}$ represents the agents populating the economy.

To simplify the presentation, we consider the following notations. Without any loss of generality, we can assume that $0 \notin \Omega$. We let $\Sigma := \{0\} \cup \Omega$. Slightly abusing notations, we identify the sets $\mathbb{R} \times \mathbb{R}^\Omega$ and $\mathbb{R}^\Sigma$ and shall use the notation $h = (h_0, h_1)$ to describe a vector $h \in \mathbb{R}^\Sigma$, with $h_0 \in \mathbb{R}$ describing quantities at the first period, and $h_1 \in \mathbb{R}^\Omega$ describing a random vector of quantities at the second period. We denoted by Δ the set of probability measures on Ω.[5]

2.2 Markets

Trade occurs at period $t = 0$ and consumption occurs in both periods $t = 0$ and $t = 1$. There is a finite set J of primary assets. In this paper, we exclude any kind of portfolio restrictions (like short sales constraints or leverage limitations). Therefore, a **portfolio of primary assets** is a vector $\eta = (\eta_j)_{j \in J} \in \mathbb{R}^J$ where η_j represents the units of asset j in the portfolio. We do not model the microstructure of the markets. Agents may trade over the counter or in centralized markets. They can trade with a single intermediary or split their trade among several intermediaries or platforms. Our convention is that the vector $\eta \in \mathbb{R}^J$ represents the agent's aggregate net holdings of primary assets at the end of the trading period $t = 0$. The object of study is the lowest cost, denoted by $p(\eta)$, the agent has to pay (in units of consumption) to hold the portfolio η at the end of $t = 0$. The value $p(\eta)$ internalizes the possible gains in splitting the trade η into several smaller trades. The

[4] We refer to Appendix 8.4 for a formal definition of these standard properties.

[5] A probability measure P on Ω is an element of $\mathbb{R}_+^\Omega$ such that $P \cdot \mathbf{1}_\Omega = 1$. The expectation under $P \in \Delta$ of a random vector $h_1 \in \mathbb{R}^\Omega$ is denoted by $\mathbb{E}_P[h_1] := P \cdot h_1$.

function

$$p : \mathbb{R}^J \to \mathbb{R}$$

is called the **price schedule**. We analyze the decision of an agent trading only at $t = 0$. Therefore, he has no initial portfolio holding inherited from un-modeled past transactions and will liquidate all his positions at the end of period $t = 1$. This implies that for this agent, $p(\eta)$ is also the cost, paid at $t = 0$, for holding the portfolio η at the beginning of the period $t = 1$.

For each possible state $\omega \in \Omega$ at $t = 1$, we denote by $G(\eta, \omega) \in \mathbb{R}$ the **payoff** (in units of consumption) of the portfolio η contingent on the state ω. We interchangeably use the notation $G_\omega(\eta)$. Let $G : \mathbb{R}^J \to \mathbb{R}^\Omega$ be the payoff mapping defined by

$$\forall \eta \in \mathbb{R}^J, \quad G(\eta) := (G_\omega(\eta))_{\omega \in \Omega}.$$

Consider an agent holding the portfolio η at the beginning of period $t = 1$, i.e., after trading in period $t = 0$ and before liquidating all his position at the end of period $t = 1$. If $G(\eta, \omega) > 0$, then the agent is entitled to $G(\eta, \omega)$ units of the consumption good in state ω. If $G(\eta, \omega) < 0$, then the agent is supposed to deliver the amount $|G(\eta, \omega)|$ in state ω. The pair (p, G) is called the **market structure**.

Example 2.1. If dividends are proportional, there are no taxes, and every asset j is short-lived, then

$$G(\eta, \omega) = \sum_{j \in J} \eta_j g_j(\omega)$$

where $g_j(\omega) \in \mathbb{R}$ is the unitary dividend of asset j. In that case, the mapping G is a linear operator. Following Prisman (1986) and Ross (1987), we may allow for taxes by considering the mapping

$$G(\eta, \omega) = \sum_{j \in J} \eta_j g_j(\omega) - T\left(\sum_{j \in J} \eta_j g_j(\omega)\right)$$

where $T : \mathbb{R} \to \mathbb{R}$ is the tax function. Observe that if T is a convex (thus progressive) tax function, then each function $G_\omega : \mathbb{R}^J \to \mathbb{R}$ is concave.

Example 2.2. One may also encompass the case where there are no taxes, no dividends at $t = 1$ but assets are long-lived. In that situation, we have

$$G(\eta, \omega) = p_1(-\eta, \omega)$$

where $p_1(\cdot, \omega) : \mathbb{R}^J \to \mathbb{R}$ is the price schedule at date 1 and state ω. Observe that if $p_1(\cdot, \omega)$ is a convex function, then $G_\omega : \mathbb{R}^J \to \mathbb{R}$ is concave.

2.3 Convex Market Structure

We assume that there is no liabilities and no gains if the agent does not trade in the financial markets.

Assumption 2.1. We have $p(0) = 0$ and $G_\omega(0) = 0$ for every $\omega \in \Omega$.

We also impose the following convexity properties.

Assumption 2.2. The **market structure** (p, G) **is convex** in the sense that the price schedule $p : \mathbb{R}^J \to \mathbb{R}$ is convex and, for every state ω, the payoff function $G_\omega : \mathbb{R}^J \to \mathbb{R}$ is concave.[6]

A price schedule $p : \mathbb{R}^J \to \mathbb{R}$ that is convex and satisfies $p(0) = 0$ is called a **price functional**. Convexity of p implies that the bid-ask spread of any portfolio is non-negative, i.e., $p(\theta) \geqslant -p(-\theta)$ for any portfolio $\theta \in \mathbb{R}^J$.[7]

Example 2.1. Assume that there is a transaction cost $c_j(|\theta_j|)$ for trading θ_j units of security j. Each function $c_j : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be convex and to satisfy $c_j(0) = 0$. Consider the following price functional

$$p(\theta) = \sum_{j \in J} p_j \theta_j + c_j(|\theta_j|) \tag{1}$$

[6]See the appendix for standard definitions and properties of convex analysis.

[7]Indeed, convexity implies $0 = p((1/2)\theta + (1/2)(-\theta)) \leqslant (1/2)p(\theta) + (1/2)p(-\theta)$.

where p_j is the (cost-free) unitary price for purchasing or selling each unit of asset j. Observe that when each function c_j is linear, i.e., $c_j(z) = \bar{c}_j z$ for some $\bar{c}_j > 0$, then we get the standard price functional with proportional transaction costs:

$$p(\theta) = \sum_{j \in J} p_j^b [\theta_j]^+ + \sum_{j \in J} p_j^a [\theta_j]^- \tag{2}$$

where $p_j^b := p_j + \bar{c}_j$ and $p_j^a := p_j - \bar{c}_j$.[8]

A market structure (p, G) is said to be **subadditive** when the function $p : \mathbb{R}^J \to \mathbb{R}$ is subadditive and each function $G_\omega : \mathbb{R}^J \to \mathbb{R}$ is superadditive. It is said to be **positively homogeneous** when $(p(\lambda\theta), G(\lambda\theta)) = \lambda(p(\theta), G(\theta))$ for any positive scalar $\lambda > 0$ and vector $\theta \in \mathbb{R}^J$. If the function p is subadditive and positively homogeneous, then it is said to be **sublinear**. If a price schedule $p : \mathbb{R}^J \to \mathbb{R}$ is sublinear, then it is a price functional (i.e., p is convex and $p(0) = 0$). Observe that the functions defined by (2) are sublinear.

Example 2.2. Following the discussion in the Introduction, a large and well established empirical literature has demonstrated that the temporary market impact on unitary prices increases approximately as the square root of the quantity traded. We claim that convexity allows accounting for this effect of quantity on price. Indeed, assume markets are segmented in the sense that the price schedule satisfies $p(\theta) = \sum_{j \in J} p_j(\theta_j)$.[9] We let $p_j^u : \mathbb{R} \to \mathbb{R}$ represent the unitary price of asset j depending on the quantity traded, that is, $p_j(x) = p_j^u(x)x$ for every $x \in \mathbb{R}$. If we assume that the temporary market impact on the unitary price is exactly the square root of the quantity traded, then we obtain the following expression for p_j^u:[10]

$$p_j^u(x) = \begin{cases} \overline{p}_j^u + \beta_j \sqrt{x}, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ \underline{p}_j^u - \gamma_j \sqrt{-x}, & \text{if } x < 0, \end{cases}$$

[8] Equivalently, we have $p(\theta) = \sum_{j \in J} p_j \theta_j + \sum_{j \in J} c_j |\theta_j|$. In that case, the bid-ask spreads are proportional since $p(\theta) + p(-\theta) = 2 \sum_{j \in J} c_j |\theta_j|$.

[9] That is, there is no cross-price correlations or cross-impact on the total price.

[10] Observe that the model with proportional bid-ask spreads assumes $p_j^u(x) = \overline{p}_j^u$ if $x > 0$ and $p_j^u(x) = \underline{p}_j^u$ if $x < 0$. Only the sign of x matters to determine the unitary price $p_j^u(x)$.

where $\overline{p}_j^u$ and $\underline{p}_j^u$ and price-impact coefficients β_j and γ_j are positive for all asset $j \in J$.[11] The higher the quantity traded, the higher is the market impact on unitary price. This implies that the more you purchase, the higher is the unitary price and the more you sell, the lower is the unitary price. If we assume that $\overline{p}_j^u \geqslant \underline{p}_j^u$, we can easily verify that the corresponding price p is a price functional. We plot an example in Figure 1.

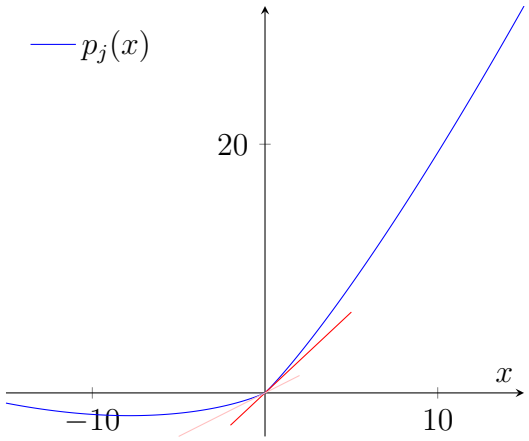

Figure 1: Price functional with market impact where $\overline{p}_j^u = 1.3$, $\underline{p}_j^u = 0,7$, $\beta_j = 1/5$, and $\gamma_j = 1/6$.

Example 2.3. Following Beissner and Riedel (2019), consider the market structure (p, G) with linear payoffs (i.e., the mapping G is linear) and Knightian uncertainty about state prices. Formally, there exists a convex and compact set $\mathbb{P} \subseteq \Delta$ of probability measures on Ω such that

$$p(\theta) = \max_{P \in \mathbb{P}} \mathbb{E}_P[\psi G(\theta)] \tag{3}$$

where $\psi(\omega) > 0$ is the price of state ω. A possible interpretation suggested by Beissner and Riedel (2019) is that cautious financial intermediaries (market makers) have imprecise probabilistic information about the states of the world and compute the maximal expected present value over the set $\mathbb{P}$ of models to hedge Knightian uncertainty. Observe that $p(\theta) =$

[11] Since we do not model the trade microstructure, there is no reason a priori to assume that the price impact is symmetric in the sense that $\beta_j = \gamma_j$.

$-\min_{P\in\mathbb{P}} \mathbb{E}_P[\psi(-G(\theta))]$ where $-G(\theta)$ is the contingent claim received by market makers at $t=1$. This means that market makers' valuation of some contingent claim $g_1 \in \mathbb{R}^\Omega$ is

$$V(g_1) = \min_{P\in\mathbb{P}} \mathbb{E}_P[\psi g_1].$$

This corresponds to Gilboa and Schmeidler (1989) maxmin expected utility model. The approach proposed by Beissner and Riedel (2019) can be extended to other ambiguity models, like the variational preferences of Maccheroni, Marinacci and Rustichini (2006). In that case, the price functional takes the following form

$$p(\theta) = \max_{P\in\Delta} \left(\mathbb{E}_P[\psi G(\theta)] - c(P)\right) \tag{4}$$

where $c : \Delta \to [0,\infty]$ is convex, lower semicontinuous function such that $c(P) = 0$ for at least one $P \in \Delta$. The price functional defined by (3) is sub-linear, but the price functional defined by (4) is neither sub-additive, nor positively homogeneous.

3 Viability

An agent $(e_0, e_1, U) \in \mathcal{A}$ chooses a consumption plan $c = (c_0, c_1) \in C$ that satisfies the flow budget constraints at $t=0$ and at $t=1$ for any contingency. This means that there exists a portfolio $\theta \in \mathbb{R}^J$ such that $c - e \leqslant (-p(\theta), G(\theta))$, or, equivalently,

$$c_0 + p(\theta) \leqslant e_0 \quad \text{and} \quad c_1 \leqslant G(\theta) + e_1.$$

Since the utility function U is strictly increasing, optimal consumption plans satisfy the flow budget constraints with equality. This implies that the agent's maximization problem is equivalent to solving

$$\max\{v(\theta) \ : \ \theta \in \Theta\}$$

where Θ is the set of budget feasible portfolios, i.e.,

$$\Theta := \{\theta \in \mathbb{R}^J \ : \ p(\theta) \leqslant e_0 \quad \text{and} \quad G(\theta) \geqslant -e_1\}$$

and $v(\theta) := U(e_0 - p(\theta), e_1 + G(\theta))$ is the indirect utility.

Remark 3.1. Since U is strictly increasing, we have the following important property: for any budget feasible portfolio $\theta \in \Theta$ and any incremental portfolio $\eta \in \mathbb{R}^J$, if

$$p(\theta + \eta) \leqslant p(\theta) \quad \text{and} \quad G(\theta + \eta) \geqslant G(\theta), \tag{5}$$

then the new portfolio $\theta + \eta$ is also budget feasible, i.e., $\theta + \eta \in \Theta$. Moreover, since U is strictly increasing, we have $v(\theta + \eta) \geqslant v(\theta)$ with a strict inequality if there is a strict inequality in one of the inequalities in (5). This simple argument is at origin of the central notion of arbitrage.

We recall the concept of viability introduced by Harrison and Kreps (1979) (see also Kreps (1981)).

Definition 3.1. A market structure (p, G) is said to be **viable** when it is compatible with utility maximization for at least one agent. Formally, there exists an agent $(e_0, e_1, U) \in \mathcal{A}$ such that his maximization problem has a solution, i.e., there exists a portfolio $\theta^\star \in \mathbb{R}^J$ satisfying

$$\theta^\star \in \operatorname{argmax}\{v(\theta) \ : \ \theta \in \Theta\}. \tag{6}$$

As far as we know, all published papers dealing with viability assume that $\theta^\star = 0$. This assumption is without any loss of generality if we impose the following restrictions on the market structure.

Proposition 3.1. Consider a market structure (p, G) such that the price functional p is subadditive and, for each state ω, the payoff function G_ω is superadditive. If p is viable, then there exists an agent who finds it optimal not to trade.

Proof. Viability of p implies that there exists an agent $a := (e_0, e_1, U) \in \mathcal{A}$ and a portfolio $\theta^\star \in \operatorname{argmax}\{v(\theta) \ : \ \theta \in \Theta\}$. Denote by $c^\star = e + (-p(\theta^\star), G(\theta^\star))$ the associated optimal consumption plan, i.e., $c_0^\star := e_0 - p(\theta^\star)$ and $c_1^\star := e_1 + G(\theta^\star)$. Inada's conditions at the origin implies that $c^\star$ belongs to the interior $\operatorname{int}(C)$ of C, i.e., $c_0^\star > 0$ and $c_1^\star(\omega) > 0$ for every state ω. Consider the agent $\tilde{a} := (\tilde{e}_0, \tilde{e}_1, U)$ whose initial endowments are defined by

$\tilde{e}_0 := c_0^\star$ and $\tilde{e}_1 := c_1^\star$. We claim that no trade is optimal for agent $\tilde{a}$. Assume, by way of contradiction, that there exists a portfolio θ satisfying the budget restrictions

$$p(\theta) \leqslant \tilde{e}_0 = e_0 - p(\theta^\star) \quad \text{and} \quad 0 \leqslant G(\theta) + \tilde{e}_1 = G(\theta) + G(\theta^\star) + e_1$$

such that $U(\tilde{e}_0 - p(\theta), \tilde{e}_1 + G(\theta)) > U(\tilde{e}_0, \tilde{e}_1)$. Sub-additivity of p and super-additivity of each G_ω imply that the portfolio $\theta + \theta^\star$ is budget feasible for agent a since we have

$$p(\theta + \theta^\star) \leqslant p(\theta) + p(\theta^\star) \leqslant e_0 \quad \text{and} \quad e_1 + G(\theta + \theta^\star) \geqslant e_1 + G(\theta) + G(\theta^\star) \geqslant 0.$$

This contradicts the optimality of $\theta^\star$ for agent a since we have

$$U(e_0 - p(\theta + \theta^\star), e_1 + G(\theta + \theta^\star)) > U(\tilde{e}_0, \tilde{e}_1) = U(e_0 - p(\theta^\star), e_1 + G(\theta^\star)).$$

We have thus proved that no-trade is the optimal decision for agent $\tilde{a}$. □

We do not want to assume subadditivity of the price functional since we typically want to address non-proportional frictions as the one of Example 2.2 (see also Figure 1) and Example 2.3. This is the reason why our concept of viability is necessarily more general that the one considered in the literature and does not restrict attention to no-trade as the possible optimal portfolio for an agent.

4 No-Arbitrage Conditions

Assume that a market structure (p, G) is viable. This means that there exists an agent (e_0, e_1, U) and a portfolio $\theta^\star \in \mathbb{R}^J$ such that

$$\theta^\star \in \operatorname{argmax}\{U(e_0 - p(\theta), e_1 + G(\theta)) \,:\, p(\theta) \leqslant e_0 \quad \text{and} \quad e_1 + G(\theta) \geqslant 0\}.$$

We let $c^\star$ denote the optimal consumption plan defined by $c_0^\star := e_0 - p(\theta^\star)$ and $c_1^\star := e_1 + G(\theta^\star)$. Let $H := \mathbb{R}^\Sigma$ and denote by[12]

$$H^+ := \mathbb{R}^\Sigma_+ \setminus \{0\}$$

[12]Recall that $\Sigma = \{0\} \cup \Omega$ and the set $\mathbb{R}^\Sigma$ is identified with $\mathbb{R} \times \mathbb{R}^S$.

the set of positive consumption increments. Recall that U is strictly increasing in the interior of C. This implies that $U(c^\star + h) > U(c^\star)$ for any $h \in H^+$.[13] Let's denote by $\mathrm{FI}(\theta^\star)$ the set of **financially feasible consumption increments** from portfolio $\theta^\star$ defined as the set of all vectors $h = (h_0, h_1) \in H$ such that there exists an incremental portfolio $\eta \in \mathbb{R}^J$ satisfying

$$h_0 \leqslant p(\theta^\star) - p(\theta^\star + \eta) \quad \text{and} \quad h_1 \leqslant G(\theta^\star + \eta) - G(\theta^\star).$$

The new consumption plan $c^\star + h$ can be financed by the new portfolio $\theta^\star + \eta$. The optimality of $c^\star$ then implies that

$$\mathrm{FI}(\theta^\star) \cap H^+ = \emptyset.$$

Therefore, in our context where the price schedule can be non-linear, the suitable no-arbitrage condition associated to the assumption that preferences are strictly increasing is the following one.

Definition 4.1. The pair (p, G) is a **no-arbitrage market structure** when there exists a portfolio $\theta^0 \in \mathbb{R}^J$ such that

$$\mathrm{FI}(\theta^0) \cap H^+ = \emptyset \tag{7}$$

or, equivalently, for every portfolio increment $\eta \in \mathbb{R}^J$, the conditions

$$G(\theta^0 + \eta) \geqslant G(\theta^0) \quad \text{and} \quad p(\theta^0 + \eta) \leqslant p(\theta^0)$$

imply

$$G(\theta^0 + \eta) = G(\eta) \quad \text{and} \quad p(\theta^0 + \eta) = p(\theta^0).$$

Property (7) can also be characterized by the following implications:

$$\forall \theta \in \mathbb{R}^J, \quad \left[G(\theta) > G(\theta^0) \Longrightarrow p(\theta) > p(\theta^0)\right] \quad \text{and} \quad \left[G(\theta) = G(\theta^0) \Longrightarrow p(\theta) \geqslant p(\theta^0)\right].$$

When the market structure is linear, the above definition coincides with the standard definition of no-arbitrage: for any portfolio $\eta \in \mathbb{R}^J$,

$$\left[G(\eta) \geqslant 0 \quad \text{and} \quad p(\eta) \leqslant 0\right] \Longrightarrow \left[G(\eta) = 0 \quad \text{and} \quad p(\eta) = 0\right].$$

[13] Inada's conditions at the origin guarantee that $c^\star$ belongs to the interior of C.

The no-arbitrage condition (7) is necessary for viability and is built upon the assumption that agents have strictly increasing preferences. We introduce the following strengthening of no-arbitrage that also exploits continuity of preferences.

Definition 4.2. The pair (p, G) is a **robust no-arbitrage market structure** when there exist a portfolio θ^0 and an open and convex set $\Gamma \subseteq \mathbb{R}^\Sigma$ such that

$$\mathrm{FI}(\theta^0) \cap \Gamma = \emptyset \quad \text{and} \quad \Gamma \supseteq H^+. \tag{8}$$

Our definition of robust no-arbitrage is motivated by the following straightforward result.

Proposition 4.1. If a market structure is viable, then it is a robust no-arbitrage market structure.

Proof. Consider a market structure (p, G) that is viable. This means that there exist an agent $(e_0, e_1, U) \in \mathcal{A}$ and an optimal portfolio $\theta^\star \in \operatorname{argmax}\{v(\theta) : \theta \in \Theta\}$. We claim that the robust no-arbitrage condition (15) is satisfied for $\theta^0 = \theta^\star$. Recall that $v(\theta) = U(e_0 - p(\theta), e_1 + G(\theta))$. We pose $c^\star = e + (-p(\theta^\star), G(\theta^\star))$. Since U satisfies the Inada's condition at the origin, we must have $c^\star \in \mathbb{R}^\Omega_{++}$. Let Γ be the set consumption increments that increase utility starting form $c^\star$:

$$\Gamma := \{h \in \mathbb{R}^\Sigma : \quad c^\star + h \in \mathbb{R}^\Omega_{++} \quad \text{and} \quad U(c^\star + h) > U(c^\star)\}.$$

Since U is lower semi-continuous, the set Γ is open in $\mathbb{R}^\Sigma$. Since U is strictly quasi-concave, the set Γ is convex. Since U is strictly increasing, the set Γ contains $c^\star$. Since $c^\star$ is optimal, for any utility-improving increment $h \in \Gamma$, we cannot find a change of portfolio η that finances the incremental consumption h. In other words, we must have $\mathrm{FI}(\theta^\star) \cap \Gamma = \emptyset$. This means that (p, G) is a robust no-arbitrage market structure. □

The definition of robust no-arbitrage is the natural strengthening of the standard no-arbitrage condition when, in addition to the strict monotonicity, we incorporate the continuity property of agents' utility function when analyzing the set of utility-improving consumption increments. Comparing (7) and (15), we see that the concept of robust no-arbitrage is

stronger than the concept of no-arbitrage. The converse is valid when the market structure is linear.

Proposition 4.2. A linear market structure is a no-arbtirage market structure if, and only if it is a robust no-arbitrage market structure.

Proof. Under linearity, we know from the Fundamental Theorem of Finance (Harrison and Kreps (1979)) that if (p, G) is a no-arbitrage market structure, then there exists a strictly positive vector $\mu \in \mathbb{R}^{\Omega}_{++}$ of state prices such that $p = \mu \cdot G := \sum_{\omega \in \Omega} \mu(\omega) G_{\omega}$.[14] Choosing $\Gamma := \{h = (h_0, h_1) \in \mathbb{R}^{\Sigma} : \ h_0 > -\mu \cdot h_1\}$, we get that $\mathrm{FI}(\theta^0) \cap \Gamma = \emptyset$ for any θ^0. Since μ is strictly positive, we also have $H \subseteq \Gamma$. This shows that (p, G) is a robust no-arbitrage market structure. □

To illustrate the difference between these two concepts when the market structure is non-linear, we consider a simple market model without uncertainty where agents trade a single asset. The asset's payoff at date 1 is linear with $G(\eta) = \eta$. In Figures 2 and 3, the h_0 axis represents the increments in consumption at date 0, and the h_1 axis represents the increments in consumption at date 1. In Figure 2, we illustrate the geometry of a price functional p satisfying robust no-arbitrage at $\theta^0 = 0$. Formally, we have $\mathrm{FI}(\theta^0) \cap H^+ = \emptyset$ and $p'(\theta^0) > 0$. These properties are sufficient to construct a convex and open set Γ such that $H^+ \subseteq \Gamma$ and $\mathrm{FI}(\theta^0) \cap \Gamma = \emptyset$.

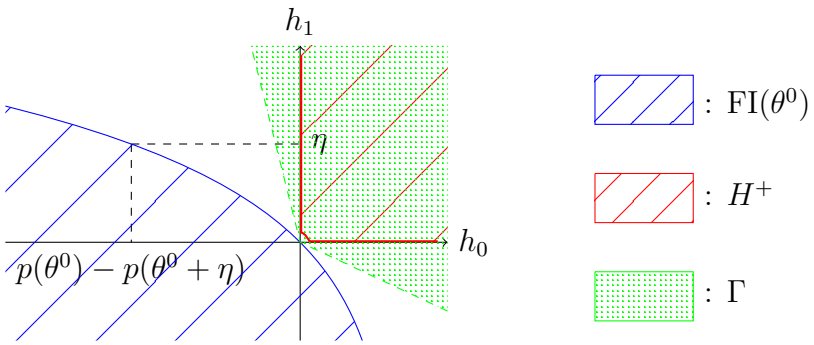

Figure 2: Geometry of robust no-arbitrage

[14] This means that for every portfolio $\theta \in \mathbb{R}^J$, we have $p(\theta) = \sum_{\omega \in \Omega} \mu(\omega) G(\theta, \omega)$.

In Figure 3, we represent a different market structure such that the no-arbitrage condition (7) is satisfied for the portfolio $\theta^0 = 0$. The price functional satisfies $p'(\theta^0) = 0$.[15] This implies that it is not possible to find an open and convex set Γ such that $H^+ \subseteq \Gamma$ and $\text{FI}(\theta^0) \cap \Gamma = \emptyset$. In other words, the robust no-arbitrage condition (15) is not satisfied at the portfolio $\theta^0 = 0$. However, this is a robust no-arbitrage market structure since condition (15) is satisfied by another portfolio θ.[16]

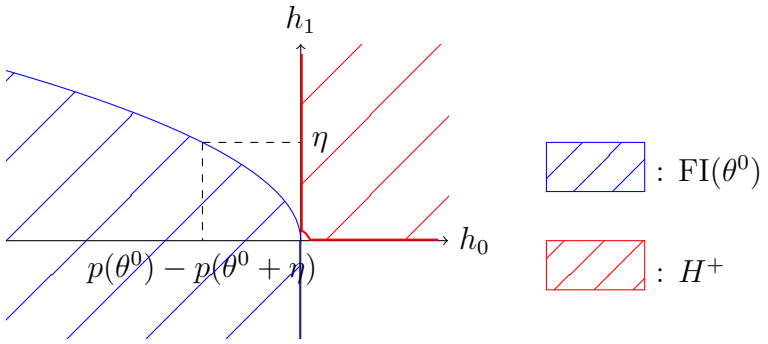

Figure 3: Geometry of no-arbitrage

It is straightforward to verify if a market structure satisfies the robust no-arbitrage condition (15) at some portfolio θ^0, then it satisfies the no-arbitrage condition (7) at the exact same portfolio θ^0. The example represented in Figure 3, illustrate that the converse is not always true. However, in this same example, the robust no-arbitrage condition is satisfied for another portfolio. This turns out to be a quite general property. Indeed, we show below that if the set of financial feasible consumption increments is closed, then the concepts of robust no-arbitrage and no-arbitrage market structures are equivalent.

Theorem 4.1. *Let (p, G) be a market structure such that the set of financial feasible con-*

[15] Formally, we have $p(\eta) := 0$ if $\eta \leqslant 0$ and $p(\eta) = \eta^2$ if $\eta \geqslant 0$.

[16] Recall that we have a robust no-arbitrage market structure when there exists at least one portfolio θ such that condition (15) is satisfied. This is in sharp contrast with the existing literature that restrict attention to the specific portfolio $\theta^0 = 0$.

sumption increments[17]

$$\mathrm{FI} := \{h \in \mathbb{R}^\Sigma \ : \ \exists \eta \in \mathbb{R}^J, \quad h \leqslant (-p(\eta), G(\eta))\}$$

is closed. If (p, G) *is a no-arbitrage market structure, then it is also a robust no-arbitrage market structure.*

Proof. Assume that (p, G) is a no-arbitrage market structure. This means that there exists a portfolio $\theta^0 \in \mathbb{R}^J$ such that

$$\mathrm{FI}(\theta^0) \cap H^+ = \emptyset.$$

To simplify the presentation, we shall assume that $\theta^0 = 0$. This implies that $\mathrm{FI}(\theta^0) = \mathrm{FI}$. Observe that the lower contour set FI is convex. Let $0^+ \mathrm{FI}$ denote the asymptotic (or recession) cone of FI defined as the set of all directions $\xi \in \mathbb{R}^\Sigma$ such that

$$\forall h \in \mathrm{FI}, \quad \forall t \geqslant 0, \quad h + t\xi \in \mathrm{FI}\,.$$

This is a closed convex cone. Moreover, we can verify that a direction ξ belongs to $0^+ \mathrm{FI}$ if, and only if, there exists a sequence (h^n) of vectors in FI and a decreasing sequence (λ^n) of positive numbers converging to 0 such that $\xi = \lim \lambda^n h^n$.

Recall that the negative polar cone $[0^+ \mathrm{FI}]^0$ is the set of all $\mu \in \mathbb{R}^\Sigma$ such that $\mu \cdot h \leqslant 0$ for every $h \in 0^+ \mathrm{FI}$.

Lemma 4.1. The negative polar cone $[0^+ \mathrm{FI}]^0$ contains a strictly positive vector $\mu \in \mathbb{R}^\Sigma_{++}$.

Proof. Since $0 \in \mathrm{FI}$, the asymptotic cone $0^+ \mathrm{FI}$ is a subset of FI. We have assumed that (p, G) satisfies the no-arbitrage price condition (7) for the portfolio $\theta^0 = 0$. This means that $\mathrm{FI} \cap H^+ = \emptyset$. Therefore, we get that $0^+ \mathrm{FI} \cap H^+ = \emptyset$. Denote by Δ the simplex of $\mathbb{R}^\Sigma$.[18] Since $\Delta \subseteq H^+$, we deduce that the two closed convex sets $0^+ \mathrm{FI}$ and Δ have an empty

[17] Observe that $\mathrm{FI} = \mathrm{FI}(0)$. Since $\mathrm{FI}(\theta^0) = \{(p(\theta^0), -G(\theta^0))\} + \mathrm{FI}$, assuming that FI is closed is equivalent to assuming that $\mathrm{FI}(\theta^0)$ is closed for any other portfolio θ^0.

[18] Δ is the set of vectors $h \in \mathbb{R}^\Sigma_+$ such that $\mathbf{1}_\Sigma \cdot h = 1$.

intersection. Since Δ is also compact, we can apply the Strict Convex Separation Theorem to deduce the existence of $\mu \in \mathbb{R}^\Sigma$, $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$\forall \xi \in 0^+\,\mathrm{FI}, \quad \mu \cdot \xi \leqslant \alpha < \alpha + \varepsilon \leqslant \mu \cdot h, \quad \forall h \in \Delta.$$

Since $0 \in 0^+\,\mathrm{FI}$, we deduce that $\mu \in \mathbb{R}^\Sigma_{++}$. Since $0^+\,\mathrm{FI}$ is a cone, we have $t(\mu \cdot \xi) \leqslant \alpha$ for every $t > 0$ and every $\xi \in 0^+\,\mathrm{FI}$. This is sufficient to conclude that μ belongs to the negative polar cone $[0^+\,\mathrm{FI}]^0$. □

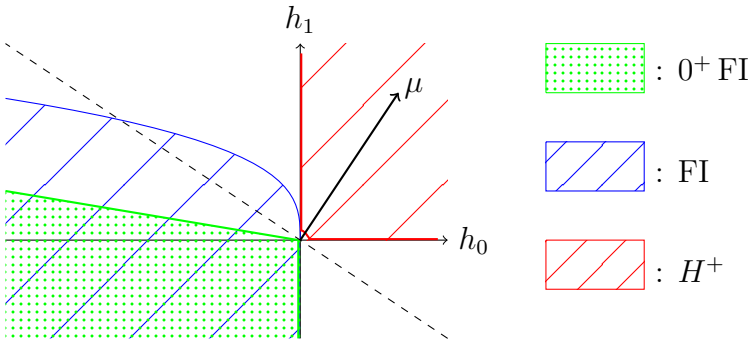

Figure 4: Geometry of no-arbitrage

Without any loss of generality, we can assume that $\mu \in \Delta$, that is, $\mu \cdot \mathbf{1}_\Sigma = 1$. Since μ is strictly positive, it belongs to $\mathrm{ri}(\Delta)$, the relative interior of the simplex Δ. This implies that there exists $\varepsilon > 0$ small enough such that[19]

$$K(\mu) := \left[\mu + \varepsilon \overline{B}\right] \cap \Delta \subseteq \mathrm{ri}(\Delta).$$

Observe that $K(\mu)$ is a compact subset of $\mathrm{ri}(\Delta)$ that contains μ in its interior relative to Δ.

[19]The closed unit ball is $\overline{B} := \{h \in \mathbb{R}^\Sigma \ : \ \|h\| = 1\}$.

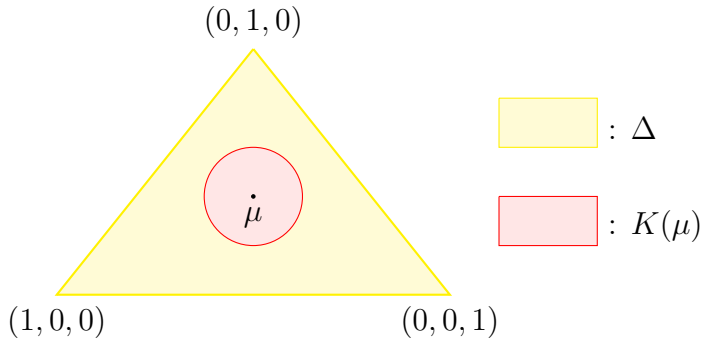

Figure 5: The set $K(\mu)$ with two states of nature.

Fix arbitrary strictly positive initial endowments $e = (e_0, e_1) \in \mathbb{R}^{\Sigma}_{++}$ and consider the utility function $U : C \longmapsto \mathbb{R}$ defined by $U(c) := V(c - e)$ where $V : \mathbb{R}^{\Sigma} \longmapsto \mathbb{R}$ is defined by

$$V(h) := \min\{\tilde{\mu} \cdot h \ : \ \tilde{\mu} \in K(\mu)\}.$$

The utility function U is continuous, concave and strictly increasing. Let a be the agent (e_0, e_1, U). By construction, we have

$$U(e_0 - p(\theta), e_1 + G(\theta)) = V(-p(\theta), G(\theta)).$$

This implies that the maximization problem of the agent a admits a solution if, and only if, the set $\operatorname{argmax}\{V(h) \ : \ h \in \mathrm{FI}\}$ is non-empty.

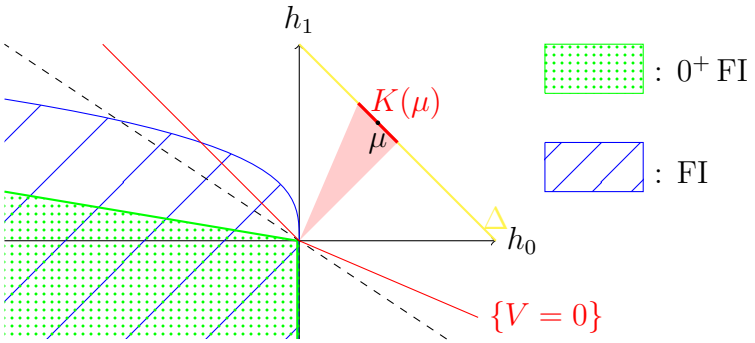

Figure 6: Indifference curve of V

Lemma 4.2. The maximization problem of the agent a has a solution, i.e.,

$$\operatorname{argmax}\{V(h) \,:\, h \in \mathrm{FI}\} \neq \emptyset.$$

Proof. Observe that $V(h) \leqslant \mu \cdot h$ for every $h \in \mathrm{FI}$. Let (h^n) be a sequence in FI such that

$$\lim V(h^n) = \sup\{V(h) \,:\, h \in \mathrm{FI}\}.$$

If (h^n) is bounded, we get the desired result from the continuity of V and the fact that FI is closed. Assume, by way of contradiction, that (h^n) is unbounded. Passing to a subsequence if necessary, we can assume that $(\|h^n\|)$ is an increasing sequence of positive numbers that converges to infinite. Let $\xi^n := (1/\|h^n\|)h^n$. We have $\|\xi^n\| = 1$. Passing to a subsequence if necessary, we can assume that there exists $\xi \in \mathbb{R}^\Sigma$ such that $\xi = \lim \xi^n$. By construction, we have $\xi \in 0^+ \mathrm{FI}$ and $\|\xi\| = 1$. Since the function V is continuous and positive homogeneous, we have[20]

$$V(\xi) = \lim V(\xi^n) = \lim V((1/\|h^n\|)h^n) = \lim \frac{1}{\|h^n\|} V(h^n) \geqslant 0.$$

Recall from Lemma 4.1 that $\mu \cdot \xi \leqslant 0$. Since $V(\xi) \geqslant 0$, we also have that

$$\mu \cdot \xi = 0 \quad \text{and} \quad \tilde{\mu} \cdot \xi \geqslant 0, \quad \text{for all } \tilde{\mu} \in K(\mu).$$

Since μ belongs to the relative interior of $K(\mu)$, we deduce that $\xi = 0$. Indeed, since $\mu \in \mathbb{R}^\Sigma_{++}$, we can choose $\alpha > 0$ small enough such that $\mu - \alpha\xi \in \mathbb{R}^\Sigma_{++}$. We can also find $\lambda > 0$ such that $\tilde{\mu} := \lambda(\mu - \alpha\xi) \in \Delta$. This implies that $\tilde{\mu} \in K(\mu)$. However, we have

$$\tilde{\mu} \cdot \xi = \lambda\,(\mu \cdot \xi - \alpha\xi \cdot \xi) = -\lambda\alpha\xi \cdot \xi < 0.$$

We cannot have $\xi = 0$ since $\|\xi\| = 1$. Therefore, the sequence (h^n) is bounded. □

Lemma 4.2 proves that the market structure (p, G) is viable. Applying Proposition 4.1, we deduce that (p, G) is a robust no-arbitrage market structure. □

When the set of financially feasible consumption increments is not closed, the two notions of no-arbitrage market structure may not coincide.

[20] Since $0 \in \mathrm{FI}$, we can assume that $V(h^n) \geqslant 0$ for every $n \in \mathbb{N}$.

Proposition 4.3. There exists a no-arbitrage market structure that is not a robust no-arbitrage.

Proof. To prove this result, we consider the following market structure. There are two primitive assets $J = \{a, b\}$ and a single state of nature. The payoff mapping $G : \mathbb{R}^J \to \mathbb{R}$ is given by $G(\theta) = G_a(\theta_a) + G_b(\theta_b)$ where the functions G_a and G_b are given by

$$G_a(\theta_a) := \begin{cases} \theta_a & \text{if } \theta_a \leqslant 0, \\ 0 & \text{if } \theta_a \geqslant 0, \end{cases} \quad \text{and} \quad G_b(\theta_b) := \theta_b, \quad \text{for all } \theta_b \in \mathbb{R}.$$

The mapping G is concave and satisfies $G(0) = 0$. Observe, moreover, that $G(\theta) > 0$ if, and only if,

$$[\theta_a \geqslant 0 \quad \text{and} \quad \theta_b > 0] \quad \text{or} \quad [\theta_a < 0 \quad \text{and} \quad \theta_b > -\theta_a]\,. \tag{9}$$

We plot the function G in Figure 7.

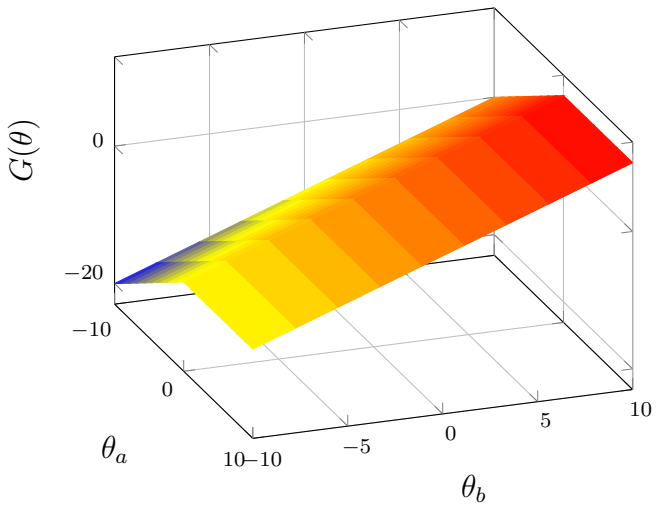

Figure 7: The payoff mapping $\theta \mapsto G(\theta)$

The price functional is given by

$$p(\theta_a, \theta_b) := \begin{cases} \theta_b^2/(\theta_a + \theta_b) & \text{if } (\theta_a, \theta_b) \in [0, \infty) \times (0, \infty), \\ -\theta_a + \theta_b & \text{if } (\theta_a, \theta_b) \in (-\infty, 0] \times [0, \infty), \\ -\theta_a & \text{if } (\theta_a, \theta_b) \in (-\infty, 0) \times (-\infty, 0], \\ 0 & \text{if } (\theta_a, \theta_b) \in (0, \infty) \times (-\infty, 0). \end{cases}$$

It is straightforward to verify that the function p is a price functional (convex and $p(0) = 0$). We plot the function p in Figure 8.

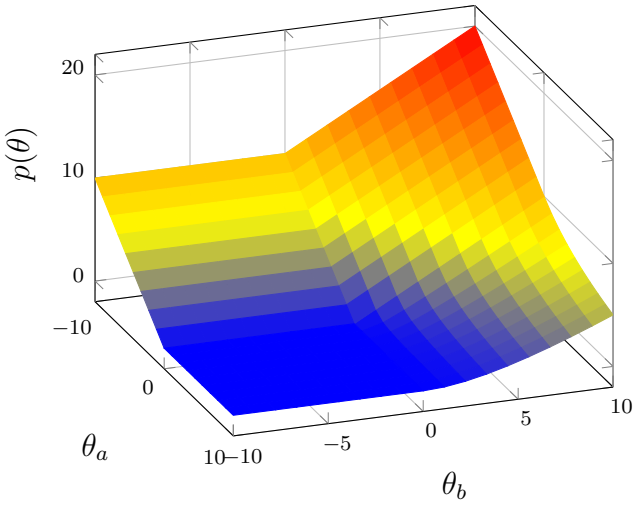

Figure 8: The price functional $\theta \mapsto p(\theta)$

Lemma 4.3. The price functional p satisfies no-arbitrage at the portfolio $\theta^0 = (0, 0)$.

Proof of Lemma 4.3. We have $G(\theta^0) = 0$ and $p(\theta^0) = 0$. Since the function p is always non-negative, we cannot have $p(\theta) < 0$ and $G(\theta) \geqslant 0$. Fix an arbitrary portfolio θ such that $G(\theta) > 0$. We shall prove that $p(\theta) > 0$. According to (9), we have two cases to analyze. First, assume that $\theta_a \geqslant 0$ and $\theta_b > 0$. In that case, we have $p(\theta) = \theta_b^2/(\theta_a + \theta_b) > 0$. Second, assume that $\theta_a < 0$ and $\theta_b > -\theta_a$. In that case, we have $p(\theta) \geqslant -\theta_a > 0$. □

Lemma 4.4. The set FI of financially feasible consumption increments is not closed.

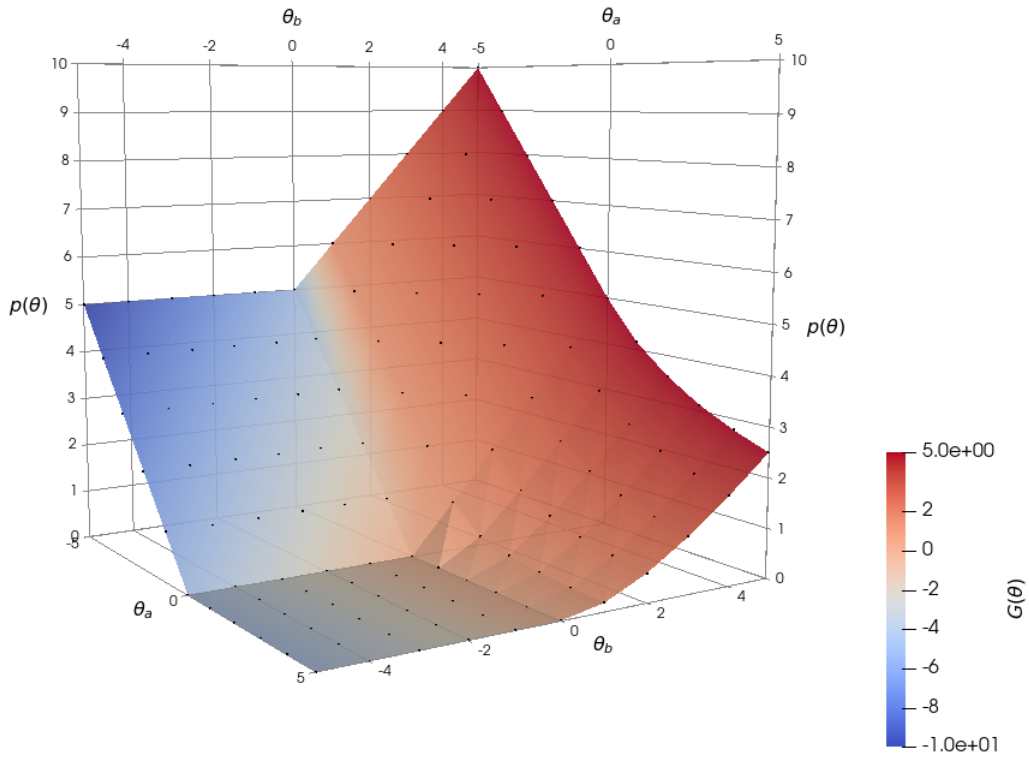

Figure 9: The price functional $\theta \mapsto p(\theta)$ and the payoff mapping $\theta \mapsto G(\theta)$

Proof of Lemma 4.4. Fix an arbitrary $h_1 > 0$ and consider the sequence $(\theta^n)_{n\in\mathbb{N}}$ defined by $\theta^n_a = n$ and $\theta^b_n = h_1$. For every $n \in \mathbb{N}$, we have

$$G(\theta^n) = h_1 \quad \text{and} \quad p(\theta^n) = \frac{x^2}{n + h_1}.$$

For every $n \in \mathbb{N}$, the pair $(-p(\theta^n), G(\theta^n)) = (-h_1^2/(n + h_1), h_1)$ belongs to the set FI. Since $\lim p(\theta_n) = 0$, we have that the pair $(0, x)$ belongs to the closure of FI. However, the pair $(0, h_1)$ does not belong to FI. Indeed, if we had that $(0, h_1) \in$ FI, then we should be able to find a portfolio θ such that $G(\theta) \geqslant h_1$ and $p(\theta) \leqslant 0$. However, this would contradict the fact that p satisfies no-arbitrage at 0. □

We plot in Figure 10 the set FI of financially feasible consumption increments.

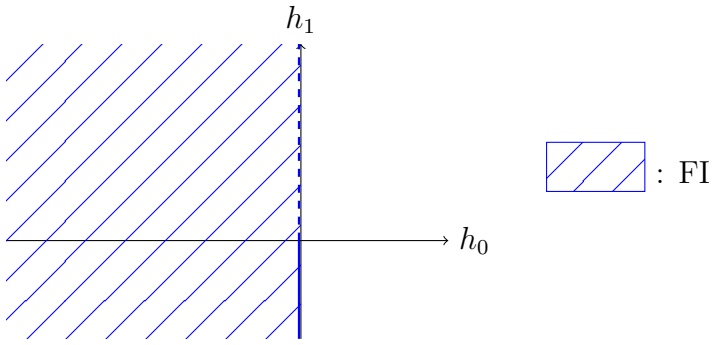

Figure 10: The set of financially feasible consumption increments is not closed.

Lemma 4.5. The pair (p, G) is not a robust no-arbitrage market structure.

Proof of Lemma 4.5. Assume, by way of contradiction, that (p, G) is a robust no-arbitrage market structure. We show in Proposition 5.1 that p is supported by strictly positive state prices. This means that there exists $\theta^1 \in \mathbb{R}^J$ and $\mu > 0$ such that $p(\theta) - p(\theta^1) \geqslant \mu[G(\theta) - G(\theta^1)]$ for every portfolio $\theta \in \mathbb{R}^J$. We analyze three cases. First, assume that $G(\theta^1) < 0$. Since p is always non-negative, we get the following contradiction:

$$-p(\theta^1) = p(0) - p(\theta^1) \geqslant \mu[G(0) - G(\theta^1)] = -\mu G(\theta^1) > 0.$$

Second, assume that $G(\theta^1) > 0$. This implies that $\theta_b^1 > 0$. Observe that, when $\theta_b^1 > 0$, the function

$$\theta_a \mapsto p(\theta_a, \theta_b^1) = \begin{cases} \dfrac{(\theta_b^1)^2}{\theta_a + \theta_b^1} & \text{if } \theta_a \geqslant 0, \\ -\theta_a + \theta_b^1 & \text{if } \theta_a \leqslant 0, \end{cases}$$

is strictly decreasing. Therefore, for any $\varepsilon > 0$, we have $p(\theta^1 + (\varepsilon, 0)) < p(\theta^1)$. Since G is increasing, we also have $G(\theta^1 + (\varepsilon, 0)) \geqslant G(\theta^1)$. This leads to the following contradiction:

$$0 > p(\theta^1 + (\varepsilon, 0)) - p(\theta^1) \geqslant \mu[G(\theta^1 + (\varepsilon, 0)) - G(\theta^1)] \geqslant 0.$$

Third, assume that $G(\theta^1) = 0$. This implies either $0 > \theta_a^1 = -\theta_b^1$ or $(\theta_a^1, \theta_b^1) = (0, 0)$. If $0 > \theta_a^1 = -\theta_b^1$ then $p(\theta^1) = 2\theta_b^1 > 0$. This leads to the following contradiction

$$0 > -p(\theta^1) \geqslant \mu G(\theta^1).$$

If $(\theta_a^1, \theta_b^1) = (0, 0)$, then $p(\theta^1) = 0$ and $G(\theta^1) = 0$. This means that $p(\theta) \geqslant \mu G(\theta)$ for every $\theta \in \mathbb{R}^J$. In particular, for every $\theta_a > 0$, we have

$$p((\theta_a, 1)) \geqslant \mu G((\theta_a, 1))$$

that is,

$$\frac{1}{\theta_a + 1} \geqslant \mu 1 > 0$$

which is impossible. We have thus proved that p cannot be supported by strictly positive state prices. □

The pair (p, G) is a no-arbitrage market structure (Lemma 4.3) that is not a robust no-arbitrage market structure (Lemma 4.5). □

5 Fundamental Theorem of Finance

In the line of the standard Fundamental Theorem of Finance in frictionless models, we would like to identify a tractable and simple characterization of robust no-arbitrage by means of positive state-prices. This is the purpose of the following definition.

Definition 5.1. A market structure (p, G) is **supported by strictly positive state prices** when there exists a strictly positive vector $\mu \in \mathbb{R}^{\Omega}_{++}$ and a portfolio $\theta^0 \in \mathbb{R}^J$ such that

$$p(\theta) - p(\theta^0) \geqslant \mu \cdot \left[G(\theta) - G(\theta^0)\right], \quad \text{for all } \theta \in \mathbb{R}^J. \tag{10}$$

Remark 5.1. Assume there is a frictionless riskless market in the sense that there exists a riskless asset $j_0 \in J$ satisfying the following properties: for every portfolio $\theta \in \mathbb{R}^J$ and holding $\lambda \in \mathbb{R}$, we have $G(\theta + \lambda \mathbf{1}_{j_0}) = G(\theta) + \lambda$ and $p(\theta + \lambda \mathbf{1}_{j_0}) = p(\theta) + \lambda/(1+r)$ where $r > -1$ is the riskless interest rate. For any vector μ of strictly positive state prices supporting the price functional p, we must have $\sum_{\omega \in \Omega} \mu(\omega) = 1/(1+r)$ and the vector $((1+r)\mu(\omega))_{\omega \in \Omega}$ can be interpreted as a probability measure.

Choosing $\theta = \theta^0 + t\eta$ for any $t > 0$ and incremental portfolio $\eta \in \mathbb{R}^J$, we get that Equation (10) is equivalent to

$$\frac{1}{t}\left[p(\theta^0 + t\eta) - p(\theta^0)\right] \geqslant \mu \cdot \frac{1}{t}\left[G(\theta^0 + t\eta) - G(\theta^0)\right], \quad \text{for all } t > 0 \text{ and } \eta \in \mathbb{R}^J.$$

The LHS increases as t increases, while the RHS decreases as t increases. Therefore, Equation (10) is equivalent to

$$p'(\theta^0; \eta) \geqslant \mu \cdot G'(\theta^0; \eta), \quad \text{for all } \eta \in \mathbb{R}^J, \tag{11}$$

where $p'(\theta^0; \eta)$ is the derivative of p at θ^0 in the direction η and $G'(\theta^0; \eta)$ is the vector $(G'_\omega(\theta^0; \eta))_{\omega \in \Omega}$ of directional derivatives.[21] If p and G are differentiable at θ^0, then we necessarily have

$$\nabla p(\theta^0) = \mu \cdot \nabla G(\theta^0) \tag{12}$$

where $\nabla p(\theta^0)$ is the gradient of p at θ^0 and $\nabla G(\theta^0)$ is the vector of gradients $(\nabla G_\omega(\theta^0))_{\omega \in \Omega}$. In particular, if the mapping

$$\eta \mapsto (\nabla G_\omega(\theta^0) \cdot \eta)_{\omega \in \Omega}$$

[21] We refer to Appendix 8.2 for the definition and straightforward properties of the directional derivatives of a convex (ir concave) function.

from $\mathbb{R}^J$ to $\mathbb{R}^\Omega$ is surjective, then there exists at most one vector μ of state prices satisfying the asset pricing equation (12).

If a price functional p satisfies the asset pricing inequality (10), then

$$(p0^+)(\eta) \geqslant p'(\theta^0;\eta) \geqslant \mu \cdot G'(\theta^0;\eta) \geqslant \mu \cdot (G0^+)(\eta), \quad \text{for all } \eta \in \mathbb{R}^J$$

where $(p0^+)(\eta)$ is the recession function of p at η and $G0^+(\eta)$ is the vector of recession functions $(G0^+(\eta,\omega))_{\omega\in\Omega}$.[22] If p is sub-additive and $G(\cdot,\omega)$ is super-additive for every ω, then $p0^+ = p$ and $G0^+ = G$. In that case, p is supported by strictly positive state prices if, and only if,

$$p(\eta) \geqslant \mu \cdot G(\eta), \quad \text{for all } \eta \in \mathbb{R}^J.$$

The above property is consistent with Proposition 3.1 since it does not depend on a specific portfolio θ^0. Moreover, if there are no frictions in the sense that p and G are linear, then (10) is equivalent to the standard fundamental asset pricing equation

$$p(\eta) = \mu \cdot G(\eta), \quad \text{for all } \eta \in \mathbb{R}^J.$$

When the market structure is convex, the existence of strictly positive supporting state prices is a necessary condition for robust no-arbitrage.

Proposition 5.1. A robust no-arbitrage market structure is supported by strictly positive state prices.

Proof. Consider a robust no-arbitrage market structure (p, G). There exists a portfolio θ^0 and an open and convex set Γ of $\mathbb{R}^\Sigma$ such that $\mathrm{FI}(\theta^0) \cap \Gamma = \emptyset$ and Γ contains H^+. Since p is convex, the set $\mathrm{FI}(\theta^0)$ is convex. Applying the standard Convex Separation Theorem, there exists a non-zero $(\xi_0, \xi_1) \in \mathbb{R} \times \mathbb{R}^\Omega$ such that

$$\xi_0 x_0 + \xi_1 \cdot x_1 \leqslant \xi_0 \gamma_0 + \xi_1 \cdot \gamma_1,$$

[22] We refer to Appendix 8.2 for the definition and straightforward properties of the recession function of a convex function.

for all $x = (x_0, x_1) \in A$ and all $\gamma = (\gamma_0, \gamma_1) \in \Gamma$. Choosing $x = (0, 0)$, we derive that $\xi_0 \geqslant 0$ and $\xi_1 \geqslant 0$. Actually, since Γ is open and contains H^+, we deduce that $\xi_0 > 0$ and $\xi_1 \gg 0$. Fix now an arbitrary $\eta \in \mathbb{R}^J$. Choosing $x_0 = p(\theta^0) - p(\theta^0 + \eta)$ and $x_1 = G(\theta^0 + \eta) - G(\theta^0)$, we deduce that

$$p(\theta^0 + \eta) - p(\theta^0) \geqslant \mu \cdot [G(\theta^0 + \eta) - G(\theta^0)]$$

where $\mu := (1/\xi_0)\xi_1$. □

Once we have strictly positive supporting state prices, we can follow standard argument to get viability.

Proposition 5.2. If a market structure is supported by strictly positive state prices, then it is viable.

Proof. Let (p, G) be a market structure satisfying (10). Fix an arbitrary strictly concave, strictly increasing and differentiable function $u : [0, \infty) \to \mathbb{R}$ such that

$$\lim_{x \to 0} \frac{u(x) - u(0)}{x} = \infty \quad \text{and} \quad \lim_{x \to \infty} u'(x) = 0.$$

We can take, for instance, $u(x) := \sqrt{x}$. Fix an arbitrary $\beta \in (0, 1)$ and an arbitrary strictly positive vector $P \in \mathbb{R}^{\Omega}_{++}$ satisfying $\sum_{\omega \in \Omega} P(\omega) = 1$ (i.e., P is a probability measure on Ω with full support). Choose $e_0 > \max\{0, p(\theta^0)\}$ and $e_1 \gg 0$ such that

$$\frac{\beta P(\omega) u'(c_1^\star(\omega))}{u'(c_0^\star)} = \mu(\omega), \quad \text{for all } \omega \in \Omega \tag{13}$$

where

$$c_0^\star := e_0 - p(\theta^0) > 0 \quad \text{and} \quad c_1^\star := e_1 + G(\theta^0) \gg 0.$$

The existence of $e_1 \gg 0$ satisfying the above conditions follows from the fact $u' : (0, \infty) \to (0, \infty)$ is a one-to-one function. We pose

$$U(c_0, c_1) := u(c_0) + \beta \sum_{\omega \in \Omega} P(\omega) u(c_1(\omega)).$$

Observe that $\theta^0 \in \Theta$. We claim that θ^0 is optimal. Indeed, let $\theta \in \Theta$. By concavity of u, we have

$$v(\theta) - v(\theta^0) \leqslant u'(c_0^\star)\left[-p(\theta) + p(\theta^0)\right] + \beta \sum_{\omega\in\Omega} P(\omega)u'(c_1^\star(\omega))\left[G(\theta,\omega) - G(\theta^0,\omega)\right].$$

Inequality (10) combined with (13) then implies the desired result. □

Combining the above results, we get the following general version of the Fundamental Theorem of Finance.

Theorem 5.1. *For any market structure (p, g), the following properties are equivalent:*

(i) *(P, G) is viable;*

(ii) *(p, g) is a robust no-arbitrage market structure;*

(iii) *(p, G) is supported by strictly positive state prices.*

6 Viability and the Law of One Price

In this section, we show that without any loss of generality, we can assume that a viable market structure satisfies the standard law of one price as defined below.

Definition 6.1. A market structure (p, G) satisfies the **law of one price** when two portfolios having the same payoff also have the same cost: for any $\theta, \eta \in \mathbb{R}^J$, the condition $G(\eta) = G(\theta)$ implies $p(\eta) = p(\theta)$.

Recall that the agent's maximization problem consists in finding a portfolio $\theta^\star \in \Theta(p)$ such that

$$\theta^\star \in \operatorname{argmax}\{v(\theta|p) \ : \ \theta \in \Theta(p)\}.$$

If $\theta \in \Theta(p)$, then, by strict monotonicity of preferences, the agent prefers any portfolio η satisfying $G(\eta) \geqslant G(\theta)$ and $p(\eta) < p(\theta)$. This naturally leads to the following definition

$$\forall \theta \in \mathbb{R}^J, \quad \tilde{p}(\theta) := \inf\{p(\eta) \ : \ \eta \in \mathbb{R}^J \quad \text{and} \quad G(\eta) \geqslant G(\theta)\}.$$

The function $\tilde{p}$ may a priori take the value $-\infty$. Nonetheless, it is still convex in the sense that its epigraph $\{(\theta, \mu) \in \mathbb{R}^J \times \mathbb{R} : \tilde{p}(\theta) \leqslant \mu\}$ is a convex subset of $\mathbb{R}^J \times \mathbb{R}$.

Since p never takes the value $+\infty$, convexity of p implies that either $\tilde{p}(\mathbb{R}^J) \subseteq \mathbb{R}$ or $\tilde{p}(\mathbb{R}^J) = \{-\infty\}$. We then obtain the following result.

Proposition 6.1. If the market structure (p, G) is viable, then the associated market structure $(\tilde{p}, G)$ satisfies the law of one price.

Proof. Assume that the market structure (p, G) is viable. This means that there exists an optimal portfolio $\theta^\star \in \operatorname{argmax}\{v(\theta|p) : \theta \in \Theta(p)\}$. Observe that we must have $\tilde{p}(\theta^\star) = p(\theta^\star)$ by optimality of $\theta^\star$. This implies that $\tilde{p}(\theta^\star) > -\infty$. We then deduce that $\tilde{p}(\mathbb{R}^J) \subseteq \mathbb{R}$. Since we already know that $\tilde{p}$ is convex, we conclude that $\tilde{p}$ is a price functional. To prove that $\tilde{p}$ satisfies the law of one price, we fix two portfolios $\theta, \theta' \in \mathbb{R}^J$ satisfying $G(\theta) = G(\theta')$. Observe that for any $\eta \in \mathbb{R}^J$, the conditions $G(\eta) \geqslant G(\theta)$ and $G(\eta) \geqslant G(\theta')$ are equivalent. We then get that $\tilde{p}(\theta) = \tilde{p}(\theta')$. □

Actually, a solution to the agent's problem for the market structure (p, G) is also a solution to the agent's problem under the associated market structure $(\tilde{p}, G)$.

Proposition 6.2. If the market structure (p, G) is viable, then the associated market structure $(\tilde{p}, G)$ is also viable. More precisely, if $\theta^\star$ solves the agent's maximization problem for the market structure (p, G), then the same portfolio solves the agent's maximization problem for the associated market structure $(\tilde{p}, G)$, i.e.,

$$\operatorname{argmax}\{v(\theta|p) : \theta \in \Theta(p)\} \subseteq \operatorname{argmax}\{v(\theta|\tilde{p}) : \theta \in \Theta(\tilde{p})\}.$$

Moreover, we have $p(\theta^\star) = \tilde{p}(\theta^\star)$.

Proof. Let $\theta^\star \in \operatorname{argmax}\{v(\theta|p) : \theta \in \Theta(p)\}$. Since $\tilde{p} \leqslant p$, we have $\Theta(p) \subseteq \Theta(\tilde{p})$ and $v(\theta|p) \leqslant v(\theta|\tilde{p})$ for any $\theta \in \Theta(p)$. In particular, we have $\theta^\star \in \Theta(\tilde{p})$. To prove that $\theta^\star \in \operatorname{argmax}\{v(\theta|\tilde{p}) : \theta \in \Theta(\tilde{p})\}$, we fix an arbitrary $\theta \in \Theta(\tilde{p})$ and show that $v(\theta|\tilde{p}) \leqslant v(\theta^\star|\tilde{p})$. Actually, we have $v(\theta|\tilde{p}) \leqslant v(\theta^\star|p)$. Indeed, fix an arbitrarily small $\varepsilon > 0$. Continuity of U

implies that there exists $\eta \in \mathbb{R}^J$ with $G(\eta) \geqslant G(\theta)$ such that

$$v(\theta|\tilde{p}) - \varepsilon \leqslant U(e_0 - p(\eta), e_1 + G(\theta)) \leqslant v(\eta|p) \leqslant v(\theta^\star|p).$$

Since this inequality is valid for any $\varepsilon > 0$, we deduce that $v(\theta|\tilde{p}) \leqslant v(\theta^\star|p)$. Since $\tilde{p} \leqslant p$, we also have $v(\theta^\star|p) \leqslant v(\theta^\star|\tilde{p})$. We have thus proved that

$$\forall \theta \in \Theta(\tilde{p}), \quad v(\theta|\tilde{p}) \leqslant v(\theta^\star|p) \leqslant v(\theta^\star|\tilde{p})$$

which implies that $\theta^\star$ is optimal under the price function $\tilde{p}$. Moreover, replacing θ by $\theta^\star$ in the above inequality, we deduce that $v(\theta^\star|\tilde{p}) = v(\theta^\star|p)$, and therefore $\tilde{p}(\theta^\star) = p(\theta^\star)$. □

Combining Proposition 6.1 and Proposition 6.2, we obtain the following result.

Corollary 6.1. *If a market structure (p, G) is viable, then the associated market structure $(\tilde{p}, G)$ is also viable and satisfies the law of one price.*

7 Super-Replication and Pricing Rules

Instead of analyzing the agent's decision problem in terms of portfolios of primitive assets, we may focus directly on the cost at $t = 0$ to implement a specific consumption at $t = 1$. Formally, what is the amount of resources the agent should spend at $t = 0$ in order to implement a specific random consumption plan c_1 at $t = 1$ by trading portfolios? A natural answer is to look for the cheapest portfolio θ satisfying $G(\theta) \geqslant c_1 - e_1$. This leads to the notion of super replication price defined below.

7.1 Super-Replication Price

The set of payoffs (vectors in $\mathbb{R}^\Omega$) which can be **super replicated** by trading portfolios is

$$X^G := \{x \in \mathbb{R}^\Omega : \exists \theta \in \mathbb{R}^J \text{ such that } x \leqslant G(\theta)\}.$$

The set X^G satisfies free-disposal in the sense that $X^G - \mathbb{R}^\Omega_+ \subseteq X^G$. Concavity of G implies that X^G is convex.

Remark 7.1. We say that there exists a **riskless asset** when $\mathbb{R}\mathbf{1}_\Omega := \{\lambda \mathbf{1}_\Omega : \lambda \in \mathbb{R}\}$ belongs to the marketed set $\text{Im}(G) := \{x \in \mathbb{R}^\Omega : \exists \theta \in \mathbb{R}^J, \quad x = G(\theta)\}$ defined as the set of payoffs which can be exactly obtained by trading portfolios.[23] When there exists a riskless asset, we have $X^G = \mathbb{R}^\Omega$.

Definition 7.1. The **super-replication price** associated to the market structure (p, G) is the function $\pi^p : X^G \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\forall x \in X^G, \quad \pi^p(x) := \inf\{p(\theta) : \theta \in \mathbb{R}^J \quad \text{and} \quad x \leqslant G(\theta)\}. \tag{14}$$

We start by analyzing straightforward properties. Observe first that a super-replication price π^p may take a priori the value $-\infty$.[24] Nonetheless, convexity of the super-replication price follows from the convexity of the price functional.

Proposition 7.1. The super-replication price is convex.

Proof. Fix a market structure (p, G). To prove that the super-replication price $\pi^p : X^G \to \mathbb{R} \cup \{-\infty\}$ is a convex function, we show that $\text{epi}(\pi^p)$ is a convex subset of $\mathbb{R}^\Omega$. We have

$$\text{epi}(\pi^p) = \{(x, \mu) \in \Phi \times \mathbb{R} : \mu \geqslant \pi^p(x)\}.$$

Let $((x, \mu), (x', \mu')) \in \text{epi}(\pi^p) \times \text{epi}(\pi^p)$, then there exist $\theta \in \mathbb{R}^J$ such that $G(\theta) \geqslant \pi^p(x)$ and $p(\theta) \leqslant \mu$ and $\theta' \in \mathbb{R}^J$ such that $G(\theta') \geqslant \pi^p(x')$ and $p(\theta') \leqslant \mu'$. Convexity of p implies that for all $\lambda \in (0, 1)$,

$$p(\lambda\theta + (1-\lambda)\theta') \leqslant \lambda p(\theta) + (1-\lambda)p(\theta') \leqslant \lambda\mu + (1-\lambda)\mu'.$$

Since $G(\lambda\theta + (1-\lambda)\theta') \geqslant \lambda\mu + (1-\lambda)\mu'$, we have

$$\pi^p(\lambda x + (1-\lambda)x') \leqslant p(\lambda\theta + (1-\lambda)\theta') \leqslant \lambda\mu + (1-\lambda)\mu'.$$

Hence $\text{epi}(\pi^p)$ is a convex subset of $\mathbb{R}^\Omega$. □

[23] This property is satisfied, for instance, if there is a primitive asset $j_0 \in J$ such that $G(\lambda \mathbf{1}_{j_0}) = \lambda \mathbf{1}_\Omega$.

[24] We show in Section 7.2 that if (p, G) is a viable price functional, then the super-replication price π^p only takes finite values.

As a direct consequence of the definition, we get that π^p is increasing.

Proposition 7.2. The super-replication price is an increasing function.[25]

Proof. Fix a price functional $p : \mathbb{R}^J \to \mathbb{R}$. Let x, y be two payoffs in X^G such that $x \geqslant y$. Fix an arbitrary portfolio $\theta \in \mathbb{R}^J$ such that $G(\theta) \geqslant x$. We also have $G(\theta) \geqslant y$ and we deduce that $\pi^p(y) \leqslant p(\theta)$. We have thus proved that $\pi^p(y)$ is a lower bound of the set $\{p(\theta) : G(\theta) \geqslant x\}$. This implies that $\pi^p(y) \leqslant \pi^p(x)$. □

The super-replication price associated with a market structure (p, G) coincides with the super-replication price associated with the market structure $(\tilde{p}, G)$.

Lemma 7.1. Given a market structure (p, G), replacing the price functional p by its associated function $\tilde{p}$ leads to the same super-replication price, i.e., $\pi^p = \pi^{\tilde{p}}$.

Proof. Since $\tilde{p} \leqslant p$, we have $\pi^{\tilde{p}} \leqslant \pi^p$. To prove the converse inequality, we fix $x \in X^G$ and start by analyzing the case where $\pi^{\tilde{p}}(x) \in \mathbb{R}$. Fix some arbitrary $\varepsilon > 0$. By definition of $\pi^{\tilde{p}}(x)$, there must exist $\theta \in \mathbb{R}^J$ with $G(\theta) \geqslant x$ such that $\pi^{\tilde{p}}(x) + \varepsilon \geqslant \pi^{\tilde{p}}(\theta)$. By definition of $\tilde{p}(\theta)$, there exists $\eta \in \mathbb{R}^J$ with $G(\eta) \geqslant G(\theta)$ such that $\tilde{p}(\theta) + \varepsilon \geqslant p(\eta)$.We have thus proved that $\pi^{\tilde{p}}(x) + 2\varepsilon \geqslant p(\eta)$ where $G(\eta) \geqslant x$. It then follows that $\pi^{\tilde{p}}(x) + 2\varepsilon \geqslant \pi^p(x)$. Since this is true for any $\varepsilon > 0$, passing to the limit when ε tends to zero, we get the desired result. Now, we analyze the case where $\pi^{\tilde{p}}(x) = -\infty$. Fix an arbitrary $M > 0$. There must exist $\theta \in \mathbb{R}^J$ with $G(\theta) \geqslant x$ such that $\tilde{p}(\theta) \leqslant -M$. By definition of $\tilde{p}(\theta)$, there must exist $\eta \in \mathbb{R}^J$ with $G(\eta) \geqslant G(\theta)$ such that $\tilde{p}(\theta) \leqslant p(\eta) \leqslant -M/2$. We have thus proved that for any $M > 0$, there exists $\eta \in \mathbb{R}^J$ such that $G(\eta) \geqslant x$ and $p(\eta) \leqslant -M/2$. This necessarily implies that $\pi^p(x) = -\infty$. □

Remark 7.2. By definition, a price functional p satisfies $p(0) = 0$. This property implies that $\pi^p(0) \leqslant 0$. However, we do not necessarily have $\pi^p(0) = 0$, even if (p, G) is a no-arbitrage market structure. Indeed, consider the following illustrative example. There is a single state

[25] A function $\pi : X^G \to \mathbb{R} \cup \{-\infty\}$ is increasing when $\pi(x) \geqslant \pi(y)$, for any $x \in X^G$ and any $y \leqslant x$. We take the standard convention that $-\infty + z = -\infty$ for any $z \in \mathbb{R} \cup \{-\infty\}$.

of nature and a single asset. The mapping $G : \mathbb{R} \to \mathbb{R}$ is linear and satisfies $G(\theta) = \theta$, for any $\theta \in \mathbb{R}$. Consider the price functional $p : \mathbb{R} \to \mathbb{R}$ defined by $p(\theta) := |\theta - \xi| - \xi$ for some exogenous $\xi > 0$. The pair (p, G) is a robust no-arbitrage market structure since we have $p(\theta^0 + \eta) - p(\theta^0) \geqslant \eta$ for any $\eta \geqslant 0$ and any $\theta^0 \geqslant \xi$. Moreover, we have

$$\pi^p(\theta) = \begin{cases} -\xi & \text{if } \theta \leqslant \xi, \\ \theta - 2\xi & \text{if } \theta > \xi. \end{cases}$$

In particular, we have $\pi^p(0) = -\xi < 0$ and the bid-ask spread is not necessarily non-negative since $\pi^p(2\xi) = 0 < -\pi^p(-2\xi) = \xi$.

We say that a market structure is **complete** when $\mathrm{Im}(G) = \mathbb{R}^\Omega$.[26] If a market structure is complete and satisfies the law of one price, we can follow Cerreia-Vioglio, Maccheroni and Marinacci (2015) and consider the **replication pricing rule** $\tilde{\pi}^p : \mathbb{R}^\Omega \to \mathbb{R}$ associated with the price functional p defined by $\tilde{\pi}^p(x) := p(\theta)$ for any portfolio θ satisfying $G(\theta) = x$.[27] It turns out that the super-replication price π^p corresponds to an inf-convolution of the replication pricing rule $\tilde{\pi}^p$.

Proposition 7.3. Assume the market structure (p, G) is complete and satisfies the law of one price. The super-replication price π^p associated with p satisfies

$$\pi^p(x) = \inf\{\tilde{\pi}^p(z) \ : \ z \in \mathbb{R}^\Omega, \quad x \leqslant z\}.$$

Proof. Fix $z \in \mathbb{R}^\Omega$ such that $x \leqslant z$. Since the market structure is complete, there exists a portfolio $\theta \in \mathbb{R}^J$ such that $z = G(\theta)$. By definition of $\tilde{\pi}^p$, we have $\tilde{\pi}^p(z) = p(\theta)$. This implies that $\pi^p(x) \leqslant \tilde{\pi}^p(z)$ and we proved that

$$\pi^p(x) \leqslant \rho(x) := \inf\{\tilde{\pi}^p(z) \ : \ z \in \mathbb{R}^\Omega \quad \text{and} \quad x \leqslant z\}.$$

Reciprocally, let $\theta \in \mathbb{R}^J$ such that $G(\theta) \geqslant x$. By posing $z := G(\theta)$, we get that $p(\theta) = \tilde{\pi}^p(z)$, and consequently, $\rho(x) \leqslant p(\theta)$. We have thus proved that $\rho(x)$ is a lower bound of the set $\{p(\theta) \ : \ x \leqslant G(\theta)\}$. This, in turn, implies that $\rho(x) \leqslant \pi^p(x)$. □

[26] Recall that $\mathrm{Im}(G) := \{x \in \mathbb{R}^\Omega \ : \ \exists \theta \in \mathbb{R}^J, \quad x = G(\theta)\}$.

[27] The law of one price guarantees that $\tilde{\pi}^p$ is well-defined.

Viability of the market structure (p, G) is a sufficient condition to guarantee that the super-replication price only take real values.

Proposition 7.4. If the market structure (p, G) is viable, then the super-replication price π^p is a function from X^G to $\mathbb{R}$.

Proof. Assume that the market structure (p, G) is viable. There exists an optimal portfolio $\theta^\star \in \operatorname{argmax}\{v(\theta|p) : \theta \in \Theta(p)\}$. Denote by $c^\star$ the associated optimal consumption plan defined by $c_0^\star := e_0 - p(\theta^\star)$ and $c_1^\star := e_1 + G(\theta^\star)$. Recall that Inada's conditions at the origin imply that $c^\star \in \mathbb{R}^\Sigma_{++}$. Assume, by way of contradiction, that there exists $x^0 \in X^G$ such that $\pi(x^0) = -\infty$. This means that there exists a sequence $\theta^{0,n} \in \mathbb{R}^J$ such that $x^0 \leqslant G(\theta^{0,n})$ and $\lim p(\theta^{0,n}) = -\infty$. Fix some arbitrary $\alpha \in (0, 1)$ and consider the alternative portfolio $\theta^n := (1-\alpha)\theta^\star + \alpha\theta^{0,n}$. Concavity of G implies that $G(\theta^n) \geqslant (1-\alpha)G(\theta^\star) + \alpha x^0$, and convexity of p implies that $p(\theta^n) \leqslant (1-\alpha)p(\theta^\star) + \alpha p(\theta^{0,n})$. Denote by c^n the consumption plan implemented by θ^n, i.e., $c_0^n := e_0 - p(\theta^n)$ and $c_1^n = e_1 + G(\theta^n)$. Observe that $c_1^n \geqslant (1-\alpha)c_1^\star + \alpha(e_1 + x^0)$. Choosing $\alpha > 0$ small enough, we have $c_1^n \in \mathbb{R}^\Omega_+$. Since $\lim c_0^n = \infty$, we can use the continuity of U to find $\alpha > 0$ close enough to 0 and n large enough such that $U(c^n) > U(c^\star)$: contradiction. □

7.2 General Pricing Rules and Viability

Inspired by the analysis of the super-replication price $\pi^p : X^G \to \mathbb{R} \cup \{-\infty\}$ derived from the market structure (p, G), we consider a general formulation described by a pair (π, X) where $X \subseteq \mathbb{R}^\Omega$ represents the set of attainable payoffs and $\pi : X \to \mathbb{R}$ is a function which associates a cost $\pi(x)$ to every attainable payoffs $x \in X$. We impose the following structural assumptions.

Definition 7.2. Fix a nonempty set $X \subseteq \mathbb{R}^\Omega$ and a function $\pi : X \to \mathbb{R}$. The pair (π, X) is called a **pricing rule** when X is a convex and comprehensive set, and π is increasing and convex.[28]

[28]The set X is comprehensive means that $X - \mathbb{R}^\Omega \subseteq X$. In other words, if a payoff is attainable, then

It follows from Proposition 7.4 that if the market structure (p, G) is viable, then the pair (π^p, X^G) is a pricing rule. The definition of viability stated for the market structure (p, G) can be extended to pricing rules.

Definition 7.3. A pricing rule (π, X) is said to be **viable** when there exists an agent $(e_0, e_1, U) \in \mathcal{A}$ and an attainable payoff $x^\star \in X$ satisfying

$$x^\star \in \operatorname{argmax}\{U(e_0 - \pi(x), e_1 + x) \ : \ x \in X(\pi)\},$$

where $X(\pi)$ is the set of all attainable payoffs $x \in X$ satisfying the flow budget constraints: $\pi(x) \leqslant e_0$ and $x \geqslant -e_1$.

Viability of the super-replication pricing (π^p, X^G) follows from the viability of the market structure (p, G).

Proposition 7.5. If a market structure (p, G) is viable, then the super-replication pricing rule (π^p, X^G) is also viable. More precisely, if $\theta^\star$ is optimal for (p, G), i.e.,

$$\theta^\star \in \operatorname{argmax}\{U(e_0 - p(\theta), e_1 + G(\theta)) \ : \ \theta \in \Theta(p)\},$$

then the associated payoff $G(\theta^\star)$ is optimal for (π^p, X^G) , i.e.,

$$G(\theta^\star) \in \operatorname{argmax}\{U(e_0 - \pi(x), e_1 + x) \ : \ x \in X(\pi^p)\}.$$

Proof. Fix a viable market structure (p, G). Let $\theta^\star \in \operatorname{argmax}\{v(\theta|p) \ : \ \theta \in \Theta(p)\}$ be an optimal portfolio for some agent $(e_0, e_1, U) \in \mathcal{A}$. Let $x^\star := G(\theta^\star)$. We shall prove that $\pi^p(x^\star) = p(\theta^\star)$. By construction, we have $\pi^p(G(\theta^\star)) \leqslant p(\theta^\star)$. Assume, by way of contradiction, that $\pi^p(G(\theta^\star)) < p(\theta^\star)$. Then $p(\theta^\star)$ is not a lower bound of the set $\{p(\theta) \ : \ \theta \in \mathbb{R}^J \quad \text{and} \quad G(\theta^\star) \leqslant G(\theta)\}$. Therefore, there exists $\theta \in \mathbb{R}^J$ such that $G(\theta) \geqslant G(\theta^\star)$ and $p(\theta) < p(\theta^\star)$. This contradicts the optimality of $\theta^\star$. Hence, $\pi^p(x^\star) = p(\theta^\star)$.

Fix an arbitrary payoff $x \in X^G$ satisfying the budget restrictions: $\pi^p(x) \leqslant e_0$ and $x \geqslant -e_1$. We first assume that $\pi^p(x) < e_0$. Choose $\varepsilon > 0$ small enough such that $\pi^p(x) + \varepsilon \leqslant e_0$.

any lower payoff is also attainable. The function π is increasing means that for every $x \in X$, if $y \leqslant x$, then $\pi(y) \leqslant \pi(x)$.

By definition of $\pi^p(x)$, there must exist some portfolio $\theta \in \mathbb{R}^J$ with $x \leqslant G(\theta)$ such that $p(\theta) \leqslant \pi(x) + \varepsilon$. It then follows that $\theta \in \Theta(p)$. Since $\theta^\star$ is optimal, we deduce that

$$U(e_0 - p(\theta), e_1 + G(\theta)) \leqslant U(e_0 - p(\theta^\star), e_1 + G(\theta^\star)) = U(e_0 - \pi(x^\star), e_1 + x^\star).$$

This implies that

$$U(e_0 - \pi(x) - \varepsilon, e_1 + x) \leqslant U(e_0 - \pi(x^\star), e_1 + x^\star).$$

Passing to the limit when ε tends to zero, we get the desired result.

We now assume that $\pi^p(x) = e_0$. Convexity of π^p implies that for any $\lambda \in [0, 1)$, we have

$$\pi^p(\lambda x) \leqslant \lambda \pi^p(x) + (1 - \lambda)\pi^p(0) \leqslant \lambda \pi^p(x) < e_0.$$

We can then apply the previous argument to show that

$$U(e_0 - \pi^p(\lambda x), e_1 + \lambda x) \leqslant U(e_0 - \pi^p(x^\star), e_1 + x^\star).$$

Passing to the limit when λ tends to 1, we get the desired result. □

7.3 No-Arbitrage and Monotonicity

We can adapt the concepts of no-arbitrage and robust no-arbitrage to pricing rule. For a given payoff $x^0 \in X$, we denote by $\mathrm{FI}(x^0)$ the set of financially feasible consumption increments defined as the set of all vectors $(v_0, v_1) \in \mathbb{R} \times \mathbb{R}^\Omega$ such that there exists another attainable payoff $x \in X$ satisfying

$$v_0 \leqslant \pi(x^0) - \pi(x) \quad \text{and} \quad v_1 \leqslant x - x^0.$$

If a consumption plan $(c_0, c_1) \in C$ is financed by the payoff x^0 in the sense that $c_0 + \pi(x^0) \leqslant e_0$ and $c_1 \leqslant e_1 + x^0$, then the alternative consumption plan $(c_0 + v_0, c_1 + v_1)$ can be financed by the alternative payoff x.

Definition 7.4. A pricing rule (π, X) satisfies **no-arbitrage** when there exists an attainable payoff $x^0 \in X$ such that

$$\mathrm{FI}(x^0) \cap H^+ = \emptyset,$$

or, equivalently, for every alternative attainable payoff $x \in X$, the following property is satisfied

$$\big[x \geqslant x^0 \quad \text{and} \quad \pi(x) \leqslant \pi(x^0)\big] \Longrightarrow x = x^0.$$

The pricing rule (π, X) satisfies **robust no-arbitrage** when there exist an attainable payoff $x^0 \in X$ and an open and convex set $\Gamma \subseteq \mathbb{R}^\Sigma$ such that

$$\mathrm{FI}(x^0) \cap \Gamma = \emptyset \quad \text{and} \quad \Gamma \supseteq H^+. \tag{15}$$

When dealing with pricing rules, the concepts of no-arbitrage and robust no-arbitrage coincide if X is closed.

Proposition 7.6. Consider a **closed pricing rule** (π, X) in the sense that X is closed and π is lower semicontinuous on X. The pricing rule (π, X) satisfies no-arbitrage if, and only if, it satisfies robust no-arbitrage.

If all payoffs are attainable, i.e., $X = \mathbb{R}^\Omega$, then the pricing rule (π, X) is continuous since π is convex. Recall that if a market structure (p, G) admits a riskless asset, then the super-replication pricing rule (π^p, X^G) is closed since $X^G = \mathbb{R}^\Omega$.

Proof of Proposition 7.6. This result follows directly from the arguments in Proposition 4.1 if we show that the set

$$\mathrm{FI} = \{v = (v_0, v_1) \in \mathbb{R}^\Sigma \ : \ \exists x \in X, \quad v_0 \leqslant -\pi(x) \quad \text{and} \quad v_1 \leqslant x\}$$

is closed. Let (v^n) be a sequence in FI that converges to v. There exists a sequence (x^n) of attainable payoffs $x^n \in X$ such that $v^n \leqslant (-\pi(x^n), x^n)$. Since X is comprehensive, we have $v_1^n \in X$. Since π is increasing, we have $v_0^n \leqslant -\pi(v_1^n)$. In other words, we can assume, without any loss of generality, that $x^n = v_1^n$. Since (v^n) converges to v, we have (v_1^n) converges to v_1 and (v_0^n) converges to v_0. Since X is closed, we deduce that $v_1 \in X$. Since π is lower semicontinuous, we have $v_0 \leqslant -\pi(v_1)$. This proves that $v \in \mathrm{FI}$. □

Making straightforward adjustments to the proofs of Proposition 4.1, Proposition 5.1 and Proposition 5.2, we derive the following equivalence results.

Corollary 7.1. *For any pricing rule (π, X), the following properties are equivalent:*

(i) (π, X) *is viable;*

(ii) (π, X) *satisfies robust no-arbitrage;*

(iii) (π, X) *is* ***supported by strictly positive state prices*** *in the sense that there exists a payoff $x^0 \in X$ and a vector $\mu \in \mathbb{R}^\Omega_{++}$ of strictly positive state prices such that*

$$\pi(x) - \pi(x^0) \geqslant \mu \cdot (x - x^0)$$

for all payoffs $x \in X$.

We propose to characterize no-arbitrage and robust no-arbitrage when $X = \mathbb{R}^\Omega$ by strengthening monotonicity.

Definition 7.5. A function $\pi : X \to \mathbb{R}$ is **eventually strictly increasing** when, for any asymptotic direction $y \in 0^+ X$, we have

$$y > 0 \Longrightarrow (\pi 0^+)(y) > 0.$$

The above property can be expressed in two different (but equivalent) ways. Recall that for any $x \in X$, we have

$$(\pi 0^+)(y) = \lim_{\lambda \to \infty} \frac{\pi(x + \lambda y) - \pi(x)}{\lambda}.$$

We then deduce that π is eventually strictly increasing if, and only if, for every $y \in 0^+ X$ and for every $x \in X$, we have

$$y > 0 \Longrightarrow \exists \lambda^\star > 0, \quad \pi(x + \lambda^\star y) > \pi(x).$$

Since the function $\lambda \mapsto \pi(x + \lambda y)$ is convex, we also have that it is strictly increasing on $[\lambda^\star, \infty)$. Recall moreover that

$$(\pi 0^+)(y) = \sup\{\pi(x + y) - \pi(x) \ : \ x \in \mathbb{R}^\Omega\}.$$

This implies that π is eventually strictly increasing if, and only if, for all $y \in 0^+ X$, we have

$$y > 0 \Longrightarrow \exists x \in \mathbb{R}^\Omega, \quad \pi(x + y) > \pi(x).$$

It turns out that eventual strict monotonicity characterizes robust no-arbitrage when X is closed.

Theorem 7.1. *If a pricing rule (π, X) satisfies robust no-arbitrage price, then it is eventually strictly increasing. The converse is valid if (π, X) is closed.*

Proof. Fix a pricing rule (π, X). We first assume that (π, X) satisfies robust no-arbitrage. It follows from Corollary 7.1 that there exist a payoff $x^0 \in X$ and a strictly positive vector $\mu \in \mathbb{R}^\Omega_{++}$ satisfying

$$\pi(x) - \pi(x^0) \geqslant \mu \cdot (x - x^0), \quad \text{for all } x \in X.$$

Fix an arbitrary $y \in 0^+ X$ with $y > 0$. For any $\lambda > 0$, we have

$$\pi(x^0 + \lambda y) - \pi(x^0) \geqslant \lambda \mu \cdot y.$$

Passing to the limit when λ tends to infinite, we get that $(\pi 0^+)(y) \geqslant \mu \cdot y$. Since $y > 0$ and $\mu \gg 0$, we deduce the desired result: $(\pi 0^+)(y) > 0$.

Now, we assume that (π, X) is eventually strictly increasing and closed. To prove that (π, X) satisfies robust no-arbitrage, it is sufficient to show that (π, X) is viable. Fix an arbitrary agent $(e_0, e_1, U) \in \mathcal{A}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of payoffs such that $x_n \in X(\pi)$ for each n and

$$\lim \pi(x_n) = \sup\{U(e_0 - \pi(x), e_1 + x) \ : \ x \in X(\pi)\},$$

where we recall that $X(\pi)$ is the set of all payoffs $x \in X$ satisfying the flow budget constraints: $\pi(x) \leqslant e_0$ and $x \geqslant -e_1$. We claim that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. Assume, by way of contradiction, that $\lim \|x_n\| = \infty$. We pose

$$y_n := \frac{1}{\|x_n\|} x_n.$$

Observe that $\|y_n\| = 1$ for each n. Passing to a subsequence if necessary, we can assume that $(y_n)_{n\in\mathbb{N}}$ converges to some y. By construction, $y \in 0^+ X$. Recall that $y_n \geqslant -e_1$ for each n. Passing to the limit, we deduce that $y \geqslant 0$. Since $\|y\| = 1$, we actually have that $y > 0$. Moreover, $x_n \in \Gamma := \{x \in \mathbb{R}^\Omega \ : \ \pi(x) \leqslant e_0\}$. We then deduce that y belongs to the

recession cone $0^+\Gamma$. If follows from Proposition 8.2 that y belongs to the recession cone of π, i.e., $(\pi 0^+)(y) \leqslant 0$. This contradicts the fact that (π, X) is eventually strictly increasing.

We have thus proved that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. Passing to a subsequence if necessary, we can assume that there exists $x^\star \in \mathbb{R}^\Omega$ such that $\lim x_n = x^\star$. Since X is closed, we have $x^\star \in X$. Lower semicontinuity of the function π and continuity of the utility function U imply that $x^\star$ solves the agent's maximization problem. In particular, (π, X) is viable. □

Combining Theorem 5.1, Proposition 7.5 and Theorem 7.1, we get the following result.

Corollary 7.2. *Consider a market structure (p, G), the associated super-replication pricing rule (π^p, X^G) and an arbitrary pricing rule (π, X). The following properties are satisfied:*

(i) (p, G) is viable $\iff$ (p, G) satisfies robust no-arbitrage $\iff$ (p, G) is supported by strictly positive states prices.

(ii) (π, X) is viable $\iff$ (π, X) satisfies robust no-arbitrage $\iff$ (π, X) is supported by strictly positive states prices. If (π, X) is closed, then all these properties are equivalent to (π, X) is eventually strictly increasing.

(iii) If (p, G) is viable, then (π^p, X^G) is viable.

(iv) If (p, G) satisfies robust no-arbitrage, then (π^p, X^G) satisfies robust no-arbitrage.

(v) Assume that $X^G = \mathbb{R}^\Omega$. The super-replication pricing rule (π^p, X^G) is viable if, and only if, it is eventually strictly increasing.

When $X^G = \mathbb{R}^\Omega$, we have the converse of properties (iii) and (iv) if we replace p by the associated $\tilde{p}$ where we recall that $\tilde{p}(\theta) = \inf\{p(\theta + \eta) \,:\, \eta \in \mathrm{Ker}(G)\}$.

Proposition 7.7. Fix a market structure (p, G) such that $X^G = \mathbb{R}^\Omega$. If (π^p, X^G) is an eventually strictly increasing pricing rule, then $(\tilde{p}, G)$ is a robust no-arbitrage market structure. Equivalently, if (π^p, X^G) is a viable pricing rule, then $(\tilde{p}, G)$ is viable.

Proof. Fix a market structure (p, G) such that $X^G = \mathbb{R}^\Omega$. Recall that the super-replication price $\pi^p : \mathbb{R}^\Omega \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$\pi^p(x) := \inf\{p(\theta) \ : \ \theta \in \mathbb{R}^J \quad \text{and} \quad G(\theta) \geqslant x\}.$$

We assume that $(\pi^p, \mathbb{R}^\Omega)$ is a pricing rule (i.e., $\pi^p(\mathbb{R}^\Omega) \subseteq \mathbb{R}$) and eventually strictly increasing. It follows from Lemma 7.1 that

$$\pi^p(x) = \inf\{\tilde{p}(\theta) \ : \ x \leqslant G(\theta)\}.$$

In particular, we have $\tilde{p}(\theta) \neq -\infty$ for all θ and $\tilde{p}$ is a price functional. To prove that $(\tilde{p}, G)$ is a robust no-arbitrage market structure, we fix a portfolio $\theta^0 \in \mathbb{R}^J$ and let $\eta \in \mathbb{R}^J$ be such that

$$\tilde{p}'(\theta; \eta) \leqslant 0 \quad \text{and} \quad G(\eta) \geqslant 0.$$

We let $y := G(\eta)$ and claim that $y = 0$. To prove this, we fix an arbitrary payoff $x \in \mathbb{R}^\Omega$ and let $\theta \in \mathbb{R}^J$ be a portfolio satisfying $G(\theta) \geqslant x$. We have

$$\pi^p(x + y) \leqslant \tilde{p}(\theta + \eta) \leqslant \tilde{p}(\theta)$$

where the first inequality follows from the definition of $\pi^p(x + y)$ and the second inequality follows from the property: $\tilde{p}'(\theta; \eta) \leqslant 0$. Since $\pi^p(x + y) \leqslant \tilde{p}(\theta)$ for any $\theta \in \mathbb{R}^J$ satisfying $G(\theta) \geqslant x$, we deduce that $\pi^p(x + y) \leqslant \pi^p(x)$. This inequality is valid for any $x \in \mathbb{R}^\Omega$. We then deduce that $([\pi^p]0^+)(y) \leqslant 0$. Since $(\pi^p, \mathbb{R}^\Omega)$ is eventually strictly increasing, we must have $y = G(\eta) = 0$. Since $(\tilde{p}, G)$ satisfies the law of one price, we deduce that $\tilde{p}'(\theta; \eta) = 0$. □

8 Appendix

Fix some arbitrary integer $n \in \mathbb{N}$. We recall the properties of convex functions $f : X \to \mathbb{R}$ defined on a nonempty and convex set $X \subseteq \mathbb{R}^n$. We follow the notations of Rockafellar (1970).

8.1 Recession Cone

Fix some non-empty convex set $C \subseteq \mathbb{R}^n$. A vector $v \in \mathbb{R}^n$ is a **direction of recession** of C when

$$\forall c \in C, \quad \forall \lambda \geqslant 0, \quad c + \lambda v \in C.$$

The set of all directions of recession of C is a convex cone containing the origin, called the **recession cone** of C, and is denoted by 0^+C. Convexity of C implies that

$$0^+C = \{v \in \mathbb{R}^n \ : \ C + v \subseteq C\}.$$

If C is a non-empty convex and closed subset of $\mathbb{R}^n$, then 0^+C is also closed. Moreover, we have

$$0^+C = \{v \in \mathbb{R}^n \ : \ \exists c_0 \in C, \quad \forall \lambda \geqslant 0, \quad c_0 + \lambda v \in C\}$$

and v is a direction of recession if, and only if, there exists a sequence $(c_n)_{n\in\mathbb{N}}$ of C and a decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to 0 such that

$$v = \lim \lambda_n c_n.$$

A non-empty closed convex set is bounded if, and only if, its recession cone consists of the zero vector alone.

The vector $v \in \mathbb{R}^n$ is a **direction in which the set C is linear** when $c + \lambda v \in C$ for every $c \in C$ and every $\lambda \in \mathbb{R}$. The set of all directions in which C is linear is called the **lineality space** of C and is denoted by $\mathrm{Li}(C)$. We have

$$\mathrm{Li}(C) = 0^+C \cap (-0^+C).$$

8.2 Convex Functions with Finite Values

Fix a convex function $f : X \to \mathbb{R}$ where X is a non-empty convex subset of $\mathbb{R}^n$. The function f is necessarily continuous on the interior $\mathrm{int}(C)$ of C. The convex function f is said **closed** when X is closed and f is lower semi-continuous on X.[29]

[29] Observe that when $X = \mathbb{R}^n$, then any convex function $f : X \to \mathbb{R}$ is closed.

Given a vector $x \in \mathbb{R}^n$ and a direction $v \in 0^+ X$, the function

$$\lambda \longmapsto \frac{f(x+\lambda v) - f(x)}{\lambda}$$

is increasing and we pose

$$f'(x;v) := \inf_{\lambda > 0} \frac{f(x+\lambda v) - f(x)}{\lambda} = \lim_{\lambda \downarrow 0} \frac{f(x+\lambda v) - f(x)}{\lambda}.$$

The number $f'(x;v)$ is called the derivative of f at x in the direction v. The function $v \mapsto f'(x;v)$ is convex and positively homogeneous.

The recession function $f0^+ : 0^+(X) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$(f0^+)(v) := \sup\{f(x+v) - f(x) \ : x \in \mathbb{R}^n\}.$$

The function $f0^+$ is positively homogeneous and convex. Moreover, it satisfies

$$(f0^+)(v) = \sup_{\lambda > 0} \frac{f(x+\lambda v) - f(x)}{\lambda} = \lim_{\lambda \uparrow \infty} \frac{f(x+\lambda v) - f(x)}{\lambda}$$

for any $x \in 0^+(X)$. We also have that

$$f'(x,v) \leqslant \frac{f(x+\lambda v) - f(x)}{\lambda} \leqslant f0^+(v)$$

for any $\lambda > 0$.

Remark 8.1. If $X = 0^+ X$ and f is subadditive, then $f0^+ = f$.[30] Indeed, by subadditivity, we have $f(x+v) - f(x) \leqslant f(v)$ with an equality when $x = 0$.

The following proposition corresponds to Theorem 8.6 in Rockafellar (1970).

Proposition 8.1. Let $f : X \to \mathbb{R}$ be a closed convex function defined on a nonempty closed convex set $X \subseteq \mathbb{R}^n$. The following properties are equivalent: for every direction of recession $v \in 0^+ X$,

(i) $(f0^+)(v) \leqslant 0$;

[30] Recall that f is subadditive when $X + X \subset X$ (this last property is equivalent to $X = 0^+(X)$) and $f(x+y) \leqslant f(x) + f(y)$ for every $x, y \in X$.

(ii) for every $x \in \mathbb{R}^n$, the function $\lambda \mapsto f(x + \lambda v)$ is non-increasing on $\mathbb{R}$;

(iii) there exists $x \in \mathbb{R}^n$ such that function $\lambda \mapsto f(x + \lambda v)$ is non-increasing on $\mathbb{R}$.

The set of all vectors v such that $(f0^+)(v) \leqslant 0$ is called the **recession cone of** f. This is a convex closed cone containing 0. A vector in the recession cone of f is called a **direction in which** f **recedes.**

The following proposition corresponds to Corollary 8.6.1 in Rockafellar (1970).

Proposition 8.2. Let $f : X \to \mathbb{R}$ be a closed convex function defined on a nonempty closed convex set $X \subseteq \mathbb{R}^n$. The following properties are equivalent: for every direction of recession $v \in 0^+ X$,

(i) for every $x \in \mathbb{R}^n$, the function $\lambda \mapsto f(x + \lambda v)$ is constant;

(ii) there exists $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $f(x + \lambda v) \leqslant \alpha$ for every $\lambda \in \mathbb{R}$;

(iii) $(f0^+)(v) \leqslant 0$ and $(f0^+)(-v) \leqslant 0$.

The set of vectors v such that $(f0^+)(v) \leqslant 0$ and $(f0^+)(-v) \leqslant 0$ is the largest linear subspace contained in the recession cone of f. It is called the **constancy space of** f and vectors in this space are called **directions in which** f **is constant**.

The following proposition corresponds to Theorem 8.7 in Rockafellar (1970).

Proposition 8.3. Let $f : X \to \mathbb{R}$ be a closed convex function defined on a nonempty closed convex set $X \subseteq \mathbb{R}^n$. All the non-empty level sets of the form $\{x \in \mathbb{R}^n : f(x) \leqslant \alpha\}$ with $\alpha \in \mathbb{R}$, have the same recession cone and the same lineality space, namely the recession cone and the constancy cone of f, respectively.

The following proposition corresponds to Corollary 8.6.1 in Rockafellar (1970).

Proposition 8.4. Let $f : X \to \mathbb{R}$ be a convex function defined on a nonempty convex set $X \subseteq \mathbb{R}^n$. If it is bounded on an affine set, then it is constant on this set.

8.3 Convex Functions with Infinite Values

Fix a convex function $f : X \to \mathbb{R} \cup \{-\infty\}$ where X is a non-empty convex subset of $\mathbb{R}^n$. The epigraph of the function f is the set

$$\{(x, \gamma) \in X \times \mathbb{R} : \ f(x) \leqslant \gamma\}.$$

We say that **the function f is convex** when its epigraph is a convex subset of $X \times \mathbb{R}$. We can verify that f is convex if, and only if, $f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha)f(y)$ for every $x, y \in X$ and $\alpha \in (0, 1)$, where we take the convention that $\alpha(-\infty) = -\infty$ and $-\infty + \gamma = -\infty$ for any $\alpha > 0$ and $\gamma \in \mathbb{R}$.

Lemma 8.1. If $f : X \to \mathbb{R} \cup \{-\infty\}$ is a convex function defined on a nonempty linear subspace $X \subseteq \mathbb{R}^n$, then either $f(\mathbb{R}^n) \subseteq \mathbb{R}$ or $f(\mathbb{R}^n) = \{-\infty\}$.

Proof. Assume that there exists $x_0 \in X$ such that $f(x_0) = -\infty$. Fix an arbitrary $x \in X$ and let $x_1 := 2x - x_0$. Convexity implies that

$$f(x) \leqslant (1/2)f(x_0) + (1/2)f(x_1) \leqslant -\infty.$$

We have thus proved that $f(x) = -\infty$ for every $x \in X$. □

8.4 Utility Functions

Recall that an agent's consumption set is $C := \mathbb{R}_+ \times \mathbb{R}_+^\Omega$. The interior of C, denoted by $\text{int}(C)$, is the set $\mathbb{R}_{++} \times \mathbb{R}_{++}^\Omega$ of all strictly positive vectors. Fix a utility function $U : C \to \mathbb{R}$. It is **strictly increasing** when for any consumption plans $\tilde{c}, c \in C$,

$$\tilde{c} > c \Longrightarrow U(\tilde{c}) > U(c)$$

with the standard convention that $\tilde{c} > c$ means $\tilde{c} \geqslant c$ and $\tilde{c} \neq c$. The function U satisfies **Inada's condition at the origin** when, for any $c \in C$,

$$\lim_{\varepsilon \to 0} \frac{U(\varepsilon, c_1) - U(0, c_1)}{\varepsilon} = \infty$$

and for any $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} \frac{U(c_0, (\varepsilon, c_1(-\omega))) - U(c_0, (0, c_1(-\omega)))}{\varepsilon} = \infty$$

where $(x, c_1(-\omega))$ is the vector $(y(s))_{s\in\Omega}$ defined by $y(\omega) := x$ and $y(s) = c_1(s)$ for any $s \neq \omega$. The function U is **lower semi-continuous** on the interior of C when, for any possible utility level $u \in \mathbb{R}$, the strict upper contour set $\{x \in \text{int}(C) : U(x) > u\}$ is open in $\text{int}(C)$. The function U is **strictly quasi-concave** on the interior of C when, for any consumption bundles $c, \tilde{c} \in \text{int}(C)$ and for any $\alpha \in (0, 1)$, we have $U(\alpha c + (1 - \alpha)\tilde{c}) > \min\{U(c), U(\tilde{c})\}$.

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