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Semiring Algebraic Structure for Metarouting with Automatic Tunneling

Noureddine Mouhoub

LaBRI - CNRS

Université de Bordeaux

Talence, France

noureddine.mouhoub@u-bordeaux.fr

Mohamed Lamine Lamali

LaBRI - CNRS

Université de Bordeaux

Talence, France

mohamed-lamine.lamali@u-bordeaux.fr

Damien Magoni

LaBRI - CNRS

Université de Bordeaux

Talence, France

damien.magoni@u-bordeaux.fr

Abstract—Metarouting models routing protocols in the form of an algebraic structure called routing algebra. It aims to help designing or validating routing protocols. Most research work on routing algebras have been applied to routing protocols used in networks having a single addressing and forwarding protocol. In this context, some of the basic algebraic structures used are semirings. In this paper, we define a new algebraic structure for dealing with networks containing multiple forwarding protocols, which may induce many (and possibly nested) tunnels. We widely generalize the semiring structure for modeling the routing problem with automatic tunneling. We define a new model of routing algebra with tunneling. It is defined as a semi-direct product of two structures, the well-known shortest paths algebra and a new proposed valid paths algebra. We show that it has a fixed point and we prove the iterative convergence to the optimal solution of the valid shortest paths problem.

Index Terms—Metarouting, tunneling, multilayer network, path computation.

I. INTRODUCTION

Routing algebras for classical dynamic routing protocols have been proposed in 1971 [1]. However, they were studied in the context of Internet routing protocols (such as BGP) only as 2003 [2]. The problem of automatic tunneling management for multilayer network routing was itself only tackled since 2009 [3]. It has gotten an increasing attention due to the advent of IPv6 and VPN deployments, where tunneling is pervasive. In this context, leveraging routing algebras to derive properties of multilayer networks is compelling. Our aim in this paper is to define such an algebra and to give some properties related to network routing with automatic tunneling. Our contributions are summarized below:

- We present recent related work (Section II), algebraic elements (Section III) and network model (Section IV).
- We define a routing algebra as a semiring for path computation with tunnels (Section V).
- We show that this routing algebra is isotonic and non-monotonic with a partial order (Section VI.A).
- We propose a fixed point for this new algebra and we prove the iterative convergence to the optimal solution of the valid shortest paths problem *vsp* (Section VI.B).

II. RELATED WORK

In [4], the authors proposed an algebraic approach called metarouting in order to define routing protocols in a declarative

and high-level way. For example, this approach allowed to define a “scoped” product to model the combined metric of iBGP and eBGP. In [4], [5], [6], the authors proposed a *lexicographic* product to combine several Quality of Service metrics to obtain a single composite metric. They also showed which properties are required to guarantee optimal global and local solutions when using the lexicographic products. Another so-called *functional product* was used in [7] to model EIGRP non-lexical metrics. More recently, the authors of [8] showed that a small set of conditions (called *Sobrinho’s conditions*) is sufficient for a routing protocol to converge to a single solution. They illustrate this result by the case of the asynchronous Bellman-Ford algorithm. Recently, the authors of [9] studied the problem of routing without an important metric condition called *isotonicity* (see Section VI-A for a formal definition). To preserve isotonicity (and thus convergence), they proposed new routing protocols based on partial order over the routing protocol metrics.

On the side of routing in multilayer networks, the authors of [10] proposed the first distributed algorithm to compute the shortest paths while automatically establishing tunnels. It is widely inspired by Bellman-Ford algorithm. However, as tunneling induces a stack of headers (and thus of protocols), the algorithm propagates *stack vectors* together with distance vectors. It allows to compute routing tables where the packets are forwarded according to their stack vector and to the distance to the destination. Another approach (inspired by the Floyd-Warshall algorithm) was proposed in [11]. The subpaths are concatenated to form a path under some conditions, the most important one being the compatibility of their stack vectors. In both works, the notion of *valid (sub)path* is crucial. Informally, a path is valid if each node along it is able to deal with a packet it receives having some protocol stack. Figure 1 illustrates a multilayer network with a valid and an invalid paths. x, y, \dots are protocols (for example IPv4 and IPv6), and nodes capabilities (encapsulation, etc.), are drawn near each node. The notations and formal definitions are given in Section IV.

In order to define these new structures, we aim to construct two operations on the valid shortest paths set. The first operation is used to concatenate valid shortest paths and the second one is used to choose between them. The concatenation

operation must make possible to calculate the weight sum of a path under the condition of preserving its validity. However, the second operation must keep all valid paths, and in equality cases, *i.e.*, valid paths having the same protocol stack (tunnel), must keep the shortest one. The proposed algebraic structures can be used to prove some convergence properties of the stack-vector algorithm proposed in [10].

III. ALGEBRAIC FOUNDATIONS

We use the same algebraic structures as the ones defined in [12], [6].

Definition 1. A semiring is a structure $SM = (S, \oplus, \otimes, \bar{0}, \bar{1})$ where:

- \oplus and \otimes are associative binary operations over S
- \oplus is commutative
- $\bar{0}$ is an identity element for \oplus and an annihilator for \otimes
- $\bar{1}$ is an identity for \otimes
- \otimes is left and right distributive on \oplus

The generalized shortest paths problem is modeled by the semiring, $SM_{sp} = (\mathbb{N}^\infty, \min, +, +\infty, 0)$.

In the idempotent semiring (\oplus is idempotent) the relation \leq_\oplus is a partial order over S if:

$$\begin{aligned}(a \leq_\oplus b) &\equiv (a = a \oplus b) \\ (a <_\oplus b) &\equiv (a = a \oplus b \neq b)\end{aligned}$$

This order is total if the operation \oplus is selective.

Definition 2. Given a semiring $(S, \oplus, \otimes, \bar{0}, \bar{1})$, the semiring of $n \times n$ matrix is $(\mathbf{M}_n(S), \oplus, \otimes, \mathbf{N}, \mathbf{I})$, where:

- all the elements of the matrix \mathbf{N} are equals to $\bar{0}$
- $\mathbf{I}_{i,j} = \begin{cases} \bar{1} & \text{if } (i = j) \\ \bar{0} & \text{otherwise} \end{cases}$

And for any two matrices $\mathbf{X}, \mathbf{Y} \in \mathbf{M}_n(S)$, the two operations \oplus and \otimes are defined as follow:

- $(\mathbf{X} \oplus \mathbf{Y})_{i,j} = \sum_n \mathbf{X}_{i,n} \oplus \mathbf{Y}_{n,j}$
- $(\mathbf{X} \otimes \mathbf{Y})_{i,j} = \bigoplus_{k=1}^n \mathbf{X}_{i,k} \otimes \mathbf{Y}_{k,j}$

Given a semiring $(S, \oplus, \otimes, \bar{0}, \bar{1})$ and a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with n nodes. We can define a weight function over the links of \mathcal{G} , $\omega : \mathcal{E} \rightarrow S$ and we denote by ω_{ij} the weight of the link $e_{ij} = v_i v_j$.

Let $\mathbf{A} \in \mathbf{M}_n(S)$ the $n \times n$ weighted adjacency matrix of G such that,

$$\mathbf{A}_{i,j} = \begin{cases} \omega_{i,j} & \text{if } e_{i,j} \in \mathcal{E} \\ \bar{0} & \text{otherwise}\end{cases}$$

For a path $p = v_0, v_1, v_2, \dots, v_{k-1}, v_k$, the weight $\omega(p)$ is,

$$\begin{aligned}\omega(p) &= \mathbf{A}_{0,1} \otimes \mathbf{A}_{1,2} \otimes \dots \otimes \mathbf{A}_{k-1,k} \\ &= \omega_{0,1} \otimes \omega_{1,2} \otimes \dots \otimes \omega_{k-1,k} = \bigotimes_{i=0}^{k-1} \omega_{i,i+1}\end{aligned}$$

We define recursively the power of a matrix $\mathbf{A} \in \mathbf{M}_n(S)$ with,

$$\mathbf{A}^k = \begin{cases} \mathbf{I} & \text{if } k = 0 \\ \mathbf{A} \otimes \mathbf{A}^{k-1} & \text{otherwise}\end{cases}$$

Let $\mathcal{P}_{i,j}^k$ be the set of all paths from node v_i to node v_j of size k . We denote by $\mathcal{P}_{i,j}^{(k)}$ the set of all paths from v_i to v_j of size at most k . We define the weight of an optimal path from v_i to v_j of size at most k as follows,

$$\mathbf{A}_{i,j}^{(k)} = \omega_{i,j}^{(k)} = \bigoplus_{p \in \mathcal{P}_{i,j}^{(k)}} \omega(p)$$

Where:

$$\mathbf{A}^{(k)} = \bigoplus_{p \in \mathcal{P}^{(k)}} \omega(p)$$

The global optimal solution for the generalized path problem consists in finding (if it any) the matrix \mathbf{A}^* defined as:

$$\mathbf{A}^* = \bigoplus_{k \geq 0} \mathbf{A}^{(k)} = \bigoplus_{p \in \mathcal{P}^{(\infty)}} \omega(p)$$

IV. MULTILAYER NETWORK MODEL

We use the same model, definitions and notations as in [10].

Definition 3. A multilayer network is modeled by 4-tuple $\mathcal{N} = (\mathcal{G}, \mathcal{A}, \mathcal{F}, \omega)$ where:

- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a directed graph modeling the network topology. The set of nodes \mathcal{V} models the routers and the set of links \mathcal{E} models the physical links. The number of nodes (resp. links) is denoted by $|\mathcal{V}| = n$ (resp. $|\mathcal{E}| = m$.)
- $\mathcal{A} = \{x, y, \dots\}$ is the set of protocols available in the network (not necessarily at each router). The number of protocols is denoted by $|\mathcal{A}| = \lambda$. The set of protocols that a node v can receive (resp. send) is denoted by $In(v)$ (resp. $Out(v)$).
- \mathcal{F} is the set of adaptation functions available in the network. For each node $v \in \mathcal{V}$, $\mathcal{F}(v)$ denotes the set of adaptation functions available on node v . An adaptation function may be:
 - *A Conversion*: the header of a packet of protocol x is transformed into the header of another protocol y . It is denoted by $(x \rightarrow y)$. If $x = y$, then it is a classical transmission;
 - *Encapsulation*: the whole packet of protocol x is encapsulated in the data field of a packet of protocol y . It is denoted by $(x \rightarrow xy)$. Note that a packet can be encapsulated in another one of the same protocol, *e.g.*, *IP-in-IP*;
 - *Decapsulation*: a packet of protocol x is extracted from the data field of a packet of protocol y , this is the reverse operation of $(x \rightarrow xy)$, and is denoted by $(xy \rightarrow x)$. Note that this operation can be performed only if the received packet of protocol y contains a packet of protocol x in its data field.
- $\omega : \mathcal{V} \times \mathcal{F} \times \mathcal{V} \rightarrow \mathbb{N}$ is a weight function modeling any additive metric. $\omega(v_i, f, v_j)$ is the cost of performing the adaptation function f on v_i then transmitting the packet

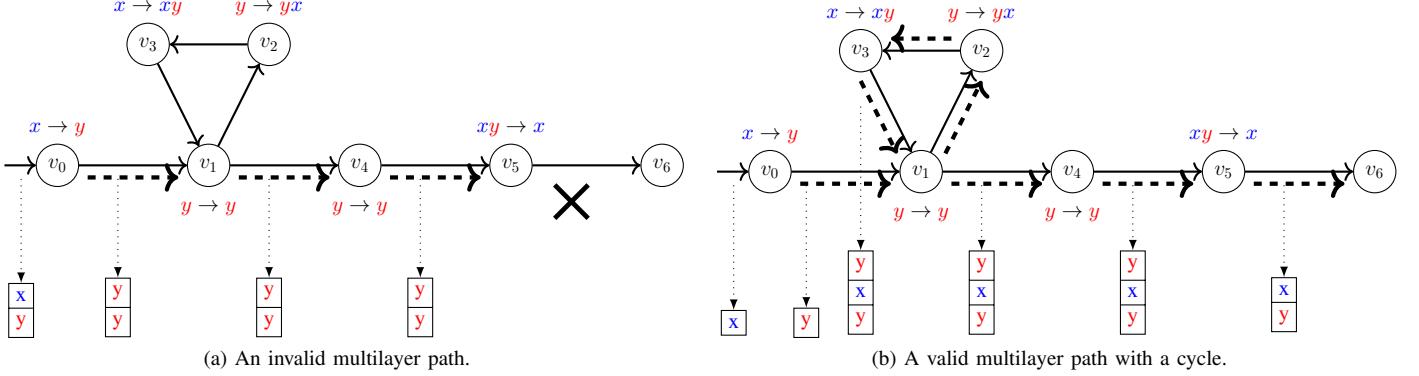


Fig. 1: Example of an invalid path and a valid path with a cycle.

on the link (v_i, v_j) . Its sum over a path is the cost to be minimized. It is very generic and can model: the number of hops ($\omega(v_i, f, v_j) = 1$ for each (v_i, f, v_j)), the number of encapsulations ($\omega(v_i, f, v_j) = 1$ only when f is an encapsulation and 0 otherwise), etc.

A sequence of adaptation functions induces a *protocol stack*. For example, the sequence:

$$(x \rightarrow y)(y \rightarrow yx)(x \rightarrow x)(x \rightarrow xx)(x \rightarrow xy)$$

induces the stack $yxxxy$ (top on the right). The protocol at the top of the stack h is denoted by $Top(h)$.

Let f be an adaptation function and h a protocol stack. We denote by $f(h)$ the application of the adaptation function f to the stack h . For example, if f is the decapsulation ($xy \rightarrow x$) and $h = yxy$, then $f(h) = yx$. The outer protocol y is removed and the inner one (protocol x) is extracted, without any change to the underlying protocols that are still nested (here the bottom y). Applying an adaptation function to a protocol stack is not always possible. For example, it is impossible to apply $f = (yx \rightarrow y)$ to stack $h = x$ since there is no nested (packet of) protocol y in h . In such cases, we denote the resulting stack by $f(h) = \phi$, where ϕ is called the *forbidden stack*. For any adaptation function f , we have $f(\phi) = \phi$. This means that we cannot apply an adaptation function at some point of a path and thus there is no way to continue through this path.

Definition 4. Let $f_i f_{i+1} \dots f_{j-1} f_j$ be a sequence of adaptation functions and h_i an initial protocol stack. Then

$$h_k = f_{k-1}(h_{k-1}), \quad i+1 \leq k \leq j+1$$

More formally, let $\mathcal{H} = \{\phi, x, y, \dots, xx, yy, xy, yx, \dots\}$ be the set of all possible protocol stacks. Note that this set is finite if we consider only the stacks induced par the shortest paths, since these stacks are bounded by λn^2 (see [10] for the proof). We can define an elementary adaptation function as a function f from $H_a \rightarrow H_b$ where H_a (resp. H_b) is a set of protocol stacks in \mathcal{H} representing the domain (resp. co-domain) of f . For example, if the function is the decapsulation $xy \rightarrow x$, then $f : H_{xy} \rightarrow H_x$ where H_{xy} (resp. H_x) is the set

of all protocol stacks starting with the sub protocol stack xy (resp. the protocol x). We define the application of a sequence of adaptation functions $f_i f_{i+1} \dots f_{j-1} f_j$ on a given protocol stack h by a composition of these adaptation functions.

Definition 5. Let $f : H_a \rightarrow H_b$ and $f' : H'_a \rightarrow H'_b$ be two elementary adaptation functions in \mathcal{F} . We define the composition by the new function $f'' = f' \odot f$ as follows:

$$f'' = \begin{cases} H''_a \rightarrow H''_b & \text{if } (H''_a \neq \emptyset) \wedge (H''_b \neq \emptyset) \\ \{\phi\} \rightarrow \{\phi\} & \text{otherwise} \end{cases}$$

Where:

- $H''_a = \{h \in H_a \mid f(h) \in H'_a\}$
- $H''_b = \{f'(h) \mid h \in (H_b \cap H'_a)\}$

Note that this composition \odot is associative and not commutative. The function $\{\phi\} \rightarrow \{\phi\}$ is denoted by the *forbidden function* $\phi \rightarrow \phi$. It is clear that the set of elementary adaptation functions \mathcal{F} is not closed under composition. For example, if the composition is $(x \rightarrow xy) \odot (y \rightarrow yx)$ then the composed function is $(y \rightarrow yxy)$. This new function is not in the set \mathcal{F} . Thus, we define a new set $\hat{\mathcal{F}}$ of all adaptation functions closed under composition. Each composition of elementary adaptation functions can be represented by a new composed adaptation function. The impossibility of composition is represented by the *forbidden function* $\phi \rightarrow \phi$. Let $\hat{\mathcal{F}} = \{\phi \rightarrow \phi, x \rightarrow x, x \rightarrow y, x \rightarrow xxx, yxy \rightarrow x, \dots\}$ be the set of all adaptation functions closed under composition. We say that two functions f' and f of $\hat{\mathcal{F}}$ are equal if and only if they have the same domain and co-domain. For example, the two functions $(x \rightarrow xx)$ and $(y \rightarrow x) \odot (x \rightarrow xy)$ are equal and they can apply on protocol stacks starting with x and give a protocol stacks starting with xx . In the rest of the paper, we will denote by \mathcal{F}_{id} the set of all passive functions, *i.e.*, classical transmissions $\{x \rightarrow x, y \rightarrow y, \dots\}$ called *identity adaptation functions*.

A *multilayer path* is represented by a directed path in the underlined graph \mathcal{G} by taking into account the adaptation function involved at each node in the path. Thus, a path from node v_i to node v_j in \mathcal{N} is a mixed sequence of nodes and

adaptation functions with starting and ending protocol stacks $h_i v_i f_i v_{i+1} f_{i+1} \dots v_{j-1} f_{j-1} h_j v_j$ where each v_k , $i \leq k \leq j$ is a node, and each f_k is an adaptation function. The starting stack is h_i and the ending stack is h_j .

Definition 6. A path $p = h_i v_i f_i v_{i+1} f_{i+1} \dots v_{j-1} f_{j-1} h_j v_j$ from v_i to v_j is *valid* if and only if:

- The sequence $v_i v_{i+1} \dots v_{j-1} v_j$ is a classical path in \mathcal{G} and each $f_k \in \mathcal{F}(v_k)$;
- $h_i \neq \phi$ and $\text{Top}(h_i) \in \text{In}(v_i)$;
- $h_j \neq \phi$ and $\text{Top}(h_j) \in \text{In}(v_j)$;
- $h_j = f_{j-1} \left(f_{j-2} \left(\dots f_{i+1} \left(f_i(h_i) \right) \right) \right)$.

Note that a valid path $p = h_i v_i f_i v_{i+1} f_{i+1} \dots v_{j-1} f_{j-1} h_j v_j$ from v_i to v_j can be represented by the valid composition of the elementary adaptation functions $f_{j-1} \odot \dots \odot f_{i+1} \odot f_i$. This latter composition is defined by the function $h_i \rightarrow h_j$ of the set $\hat{\mathcal{F}}$.

In Figure 1a, the depicted path from v_0 to v_6 with the starting protocol stack yx is:

$$v_0(x \rightarrow y) v_1(y \rightarrow y) v_4(y \rightarrow y) v_5(xy \rightarrow x) v_6$$

It cannot be valid because, the node v_5 receives the protocol stack yy and it cannot decapsulate the protocol x from y , i.e., it cannot perform the function $xy \rightarrow x$ on the stack yy . The composed function of this path is $\phi \rightarrow \phi$. In contrast, Figure 1b depicts a valid path v_0 to v_6 with the starting stack x and the arrival stack xy :

$$\begin{aligned} &v_0(x \rightarrow y) v_1(y \rightarrow y) v_2(y \rightarrow yx) v_3(x \rightarrow xy) \\ &v_1(y \rightarrow y) v_4(y \rightarrow y) v_5(xy \rightarrow x) v_6 \end{aligned}$$

The composed function of this valid path is $x \rightarrow xy$. The used links in the path are in bold and the corresponding protocol stacks are below the links in Figure 1.

The weight of a valid path¹ p from node v_i to node v_j with stacks h_i and h_j , $p = h_i v_i f_i v_{i+1} f_{i+1} \dots v_{j-1} f_{j-1} h_j v_j$ is the sum of the weights of its links and its adaptation functions. It is denoted by

$$\omega(p) \stackrel{\text{def}}{=} \sum_{k=i}^{j-1} \omega(v_k, f_k, v_{k+1})$$

V. SEMIRING WITH TUNNELS

In this section, we define a semiring algebraic structure for the valid path problem vp . Then, we study the semi-direct product of the valid paths structures vp with the shortest paths structures sp in order to model the valid shortest paths vsp .

In order to enumerate the set of all valid shortest paths between each pair of nodes in a multilayer network, we introduce a semi-direct product over a set of weighted elements², in which the concatenation operation is a direct product and

¹The weight of an invalid path is not defined and can be set to ∞ value.

²These elements represent the compositions of adaptation functions and they can represent other depending on the nature of the algebraic structure.

the choice operation is based on a new binary operation that combines the *union* and the *min* operations.

Let $S = \{a, b, c, \dots\}$ be a finite set of elements. We define the positive weighted set of S as the set product $S \times \mathbb{N}^\infty$. We denote $\mathcal{P}(S \times \mathbb{N}^\infty)$, the power set in which each subset contains a unique pair for each element. We define the *union-min* operation over subsets of weighted elements as follows:

Definition 7. Let S_1 and S_2 be two subsets of $\mathcal{P}(S \times \mathbb{N}^\infty)$. We define the *union-min* operation of S_1 and S_2 as follows:

$$S_1 \underset{\text{min}}{\cup} S_2 = \left\{ (a, \omega_a) \mid (1) \vee (2) \right\}$$

$$(a, \omega_a) \in S_1 \wedge \forall (b, \omega_b) \in S_2, (a = b) \Rightarrow \omega_a = \min[\omega_a, \omega_b] \quad (1)$$

$$(a, \omega_a) \in S_2 \wedge \forall (b, \omega_b) \in S_1, (a = b) \Rightarrow \omega_a = \min[\omega_a, \omega_b] \quad (2)$$

This new operation, i.e., union-min, introduces the idea of enumerating different weighted paths from a source to a destination. And in the case of equality of paths (with the same elements), it keeps the path with the smallest weight. It is easy to check that the *union-min* operation is idempotent, associative and commutative, but not selective (it can return a new subset different from the two initial subsets). The identity element of this operation is the empty subset and the annihilator element is the subset of unique elements with weight 0, i.e., $\{(a, 0), (b, 0), (c, 0), \dots\}$.

Definition 8. Let S_1 and S_2 be two subsets of $\mathcal{P}(S \times \mathbb{N}^\infty)$. We define the order relation as follows:

$$S_1 \subseteq S_2 \equiv \forall (a, \omega_a) \in S_1 \Rightarrow \exists (b, \omega_b) \in S_2, (a = b) \wedge (\omega_a \leq \omega_b)$$

It is clear that this order is a partial order, i.e., there exists some incomparable subsets. For example, the two subsets $\{(a, 1), (b, 5)\}$ and $\{(a, 4), (b, 2)\}$ are incomparable. We use this operation to define a semi-direct product for semirings.

Recall that a valid multilayer path is represented by a valid composition of adaptation functions. In order to compute the set of all valid paths, i.e., valid compositions, we extend the definition of adaptation function composition on sets of compositions.

Let be $\hat{\mathcal{F}}$ the set of all adaptation functions closed under composition, and $\mathcal{P}(\hat{\mathcal{F}})$ its power set. If \hat{F}_1 and \hat{F}_2 are two subsets of $\mathcal{P}(\hat{\mathcal{F}})$, then we define the set of pair-wise compositions,

$$\hat{F}_1 \odot \hat{F}_2 = \{\hat{f}_1 \odot \hat{f}_2 \mid \hat{f}_1 \in \hat{F}_1 \text{ and } \hat{f}_2 \in \hat{F}_2\}$$

Note that in the case of invalid composition, the resulting forbidden function will be removed from the result set. The composition with an empty set is always an empty set. Based on this operation, we define our semiring valid paths SM_{vp} that enumerates all valid paths as follows,

$$SM_{vp} = \left(\mathcal{P}(\hat{\mathcal{F}}), \cup, \odot, \emptyset, \mathcal{F}_{id} \right)$$

Where, \emptyset is the empty set composition and \mathcal{F}_{id} is the set of identity adaptation functions $\{x \rightarrow x, y \rightarrow y, \dots\}$. It is easy to check that the composition \odot is associative and non-commutative with the \emptyset as annihilator. We check the identity set compositions \mathcal{F}_{id} and the distributivity of \odot over \cup ,

Identity of \odot : Let $\hat{F} = \{\hat{f}_i, \hat{f}_{i+1}, \dots, \hat{f}_{j-1}, \hat{f}_j\}$ be a set of composed functions in $\mathcal{P}(\hat{\mathcal{F}})$. We want to show that,

$$\hat{F} \odot \mathcal{F}_{id} = \mathcal{F}_{id} \odot \hat{F} = \hat{F} \quad \forall \hat{F} \in \mathcal{P}(\hat{\mathcal{F}})$$

Let $f_k, i \leq k \leq j, f_k : H_a \rightarrow H_b$ be a function in \hat{F} . There are two possible situations:

- 1) H_a is a set of protocol stacks starting with x . In this case, $f_k \odot (x \rightarrow x) = f_k$ and for all $f \in \mathcal{F}_{id} - \{(x \rightarrow x)\}$ we have $f_k \odot f = (\phi \rightarrow \phi)$.
- 2) H_b is a set of protocol stacks starting with x . In this case, $(x \rightarrow x) \odot f_k = f_k$ and for all $f \in \mathcal{F}_{id} - \{(x \rightarrow x)\}$ we have $f \odot f_k = (\phi \rightarrow \phi)$.

Note that, by the definition of \odot , the forbidden functions are removed from the result set.

Distributivity of \odot over \cup : Let $\hat{F}_1, \hat{F}_2, \hat{F}_3$ be three sets of composed functions in $\mathcal{P}(\hat{\mathcal{F}})$. By the definition of \odot over sets of composed functions, which compute the set of pair-wise compositions, we can see that,

$$\hat{F}_1 \odot (\hat{F}_2 \cup \hat{F}_3) = (\hat{F}_1 \odot \hat{F}_2) \cup (\hat{F}_1 \odot \hat{F}_3)$$

And,

$$(\hat{F}_2 \cup \hat{F}_3) \odot \hat{F}_1 = (\hat{F}_2 \odot \hat{F}_1) \cup (\hat{F}_3 \odot \hat{F}_1)$$

Now, we can define the valid shortest paths semiring SM_{vsp} by the following semi-direct product of SM_{vp} and SM_{sp} ,

$$SM_{vsp} = SM_{vp} \rtimes SM_{sp}$$

$$SM_{vsp} = \left(\mathcal{P}(\hat{\mathcal{F}} \times \mathbb{N}^\infty), \cup_{min}, (\odot \times +), \emptyset, (\mathcal{F}_{id} \times 0) \right)$$

It is well known that the direct product of semigroups preserves the associativity property, the identity elements and the annihilators [13]. In most situations where algebraic structures violate some axioms of semirings, they do not generally satisfy the distributivity of \otimes over \oplus . We check this property in order to ensure that our structure defines a semiring.

Distributivity of $(\odot \times +)$ over \cup_{min} : We want to show that for all $\hat{F}_1, \hat{F}_2, \hat{F}_3$ sets of weighted composed functions in $\mathcal{P}(\hat{\mathcal{F}} \times \mathbb{N}^\infty)$ we have $lhs = rhs$ where,

$$lhs = \hat{F}_1 (\odot \times +) (\hat{F}_2 \cup_{min} \hat{F}_3)$$

$$rhs = (\hat{F}_1 (\odot \times +) \hat{F}_2) \cup_{min} (\hat{F}_1 (\odot \times +) \hat{F}_3)$$

Based on the definition of the union-min operation, we distinguish between the two following cases:

Case 1: The two sets \hat{F}_2 and \hat{F}_3 are strictly different and don't have any common composed function,

$$\hat{F}_2 \cup_{min} \hat{F}_3 = \hat{F}_2 \cup \hat{F}_3$$

In this case, we can see that,

$$lhs = rhs = (F_1 (\odot \times +) \hat{F}_2) \cup (F_1 (\odot \times +) \hat{F}_3)$$

Case 2: The two sets have some common composed functions,

$$\hat{F}_2 \cup_{min} \hat{F}_3 = \hat{F}_2^* \cup \hat{F}_3^*$$

Where: $\hat{F}_2 = \hat{F}_2^* \cup \hat{F}$ and $\hat{F}_3 = \hat{F}_3^* \cup \hat{F}'$ and \hat{F} (resp. \hat{F}') is the non empty set of the common non optimal composed functions of \hat{F}_2 (resp. \hat{F}_3). In this situation, we have,

$$\hat{F}_1 (\odot \times +) \hat{F}_2 = (\hat{F}_1 (\odot \times +) \hat{F}_2^*) \cup (\hat{F}_1 (\odot \times +) \hat{F})$$

$$\hat{F}_1 (\odot \times +) \hat{F}_3 = (\hat{F}_1 (\odot \times +) \hat{F}_3^*) \cup (\hat{F}_1 (\odot \times +) \hat{F}')$$

And,

$$(\hat{F}_1 (\odot \times +) \hat{F}_2^*) \cup_{min} (\hat{F}_1 (\odot \times +) \hat{F}') = (\hat{F}_1 (\odot \times +) \hat{F}_2^*)$$

$$(\hat{F}_1 (\odot \times +) \hat{F}_3^*) \cup_{min} (\hat{F}_1 (\odot \times +) \hat{F}) = (\hat{F}_1 (\odot \times +) \hat{F}_3^*)$$

As we can see, the two sets \hat{F}_2^* and \hat{F}_3^* are different. So we can apply the case 1,

$$lhs = rhs = (F_1 (\odot \times +) \hat{F}_2^*) \cup (F_1 (\odot \times +) \hat{F}_3^*)$$

We can check the right distributivity in the same way.

Using this semiring, we can define the adjacency matrix \mathbf{A} of all valid paths where $\mathbf{A}_{i,j}$ is the set of valid paths from node i to node j represented by the corresponding compositions of adaptation functions and its weights involved in each path. Missing paths are simply represented by the empty set of compositions.

VI. CONVERGENCE PROPERTIES

A. Monotonicity and Isotonicity

In general, the convergence of routing protocols, *e.g.*, distance-vector and path-vector protocols, is based on the two important properties of the corresponding routing algebra.

The first property, *i.e.*, monotonicity, guarantees that the routing protocol converges in any network, but not necessarily to a global optimal solution [14].

Definition 9. Let \otimes be the extension operation and the order relation \leq_{\oplus} . The operation \otimes is monotonic if and only if,

$$a \leq_{\oplus} a \otimes b \equiv a = a \oplus (a \otimes b) \quad \forall a, b \in S$$

We can see that the proposed algebra in this paper does not satisfy this property. The composition of two sets can yield a new set (possibly the empty set) which is incomparable to the first sets. And the application of a set of adaptation functions on a set of protocol stacks can give a new set of protocol stacks (possibly the empty set) which is incomparable to the first set. For this, we prove the following proposition,

Proposition 1. The direct product operator $(\odot \times +)$ over the power set $\mathcal{P}(\hat{\mathcal{F}} \times \mathbb{N}^\infty)$ is not monotonic.

The second property, *i.e.*, isotonicity, guarantees that the routing protocol converges to a global optimal solution [14].

Definition 10. Let \otimes be the extension operation and the order relation \leq_{\oplus} . The operation \otimes is isotonic if and only if,

$$a \leq_{\oplus} b \implies a \otimes c \leq_{\oplus} b \otimes c \quad \forall a, b, c \in S$$

In order to show that our algebras are isotonic, we prove the following proposition for semirings with composition.

Proposition 2. The direct product operator $(\odot \times +)$ over the power set $\mathcal{P}(\hat{\mathcal{F}} \times \mathbb{N}^\infty)$ is isotonic.

B. Iterative Convergence

It is known that in classical networks without absorbing cycles (with only positive weights) called free networks, the global optimal solution \mathbf{A}^* converges to the matrix $\mathbf{A}^{(n-1)}$ where $n-1$ is the maximum length of an elementary path [1], [15]. This means that cycles only increase the weight of paths and therefore will be ignored by the computation. In our case, cycles are allowed and sometimes necessary for some paths *i.e.*, constructing the necessary protocol stack for the path to be valid. For this reason, we firstly define the multilayer cycle and the multilayer elementary path in the multilayer network model in order to generalize the convergence theorem cited above.

Definition 11. A multilayer path $p = h_i v_i f_i \dots v_{j-1} f_{j-1} h_j v_j$ is a multilayer cycle if and only if:

- The node v_i is the same node v_j , *i.e.*, $v_i = v_j$
- The two stacks h_i and h_j received by v_i (or v_j) are the same, *i.e.*, $h_i = h_j$

Using the above definition, we now define the multilayer elementary paths.

Definition 12. A multilayer elementary path is a multilayer path in which its cycles (if any) are non multilayer cycles.

Definition 13. A free multilayer network is a network in which all of its multilayer cycles have positive weights.

In [16], the authors proved some bounds on the valid shortest paths problem in a multilayer network. We leverage on the following proposition whose proof can be found in [16]:

Proposition 3. For any multilayer network \mathcal{N} , the valid shortest path (if any) between two nodes is upper bounded by

$$2^{(\lambda+1)\lambda^2 n^2}.$$

Using this proposition, we can prove the following theorem;

Theorem 1. In a free multilayer network \mathcal{N} we have:

$$\mathbf{A}^* = \mathbf{A}^{(k)} = \mathbf{I} \oplus \mathbf{A} \oplus \mathbf{A}^2 \oplus \dots \oplus \mathbf{A}^k$$

Where k is the maximum length of the multilayer elementary paths in \mathcal{N} , and it is equal to $2^{(\lambda+1)\lambda^2 n^2} - 1$.

VII. CONCLUSION AND FUTURE WORK

Routing algebras are powerful abstractions for studying complex structures such as multilayer networks. In this paper, we have defined a new routing algebra based on semirings for path computation with tunnels. We have shown that this routing algebra is isotonic and non-monotonic with a partial order. We have proposed a fixed point for this algebra and we have proved the iterative convergence to the optimal solution of the valid shortest paths problem *vsp*.

For future work, we plan to adapt these modifications to other existing algebraic structures (algebra of endomorphisms and Sobrinho's algebra). Using these algebras, we aim to study the asynchronous convergence of the stack-vector protocol by

applying, if possible, the asynchronous convergence [8] of [10] and implementing it in AGDA [17].

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REFERENCES

- [1] B. A. Carré, “An Algebra for Network Routing Problems,” *IMA Journal of Applied Mathematics*, vol. 7, no. 3, pp. 273–294, 1971.
- [2] J. a. L. Sobrinho, “Network routing with path vector protocols: Theory and applications,” in *Proceedings of the 2003 Conference on Applications, Technologies, Architectures, and Protocols for Computer Communications*, 2003, p. 49–60.
- [3] F. Kuipers and F. Dijkstra, “Path selection in multi-layer networks,” *Comput. Commun.*, vol. 32, no. 1, pp. 78–85, 2009.
- [4] T. G. Griffin and J. L. Sobrinho, “Metarouting,” in *Proc. of ACM SIGCOMM*, 2005, p. 1–12.
- [5] A. J. T. Gurney and T. G. Griffin, “Lexicographic products in metarouting,” in *Proc. of IEEE ICNP*, 2007, pp. 113–122.
- [6] S. Dynerowicz and T. G. Griffin, “On the forwarding paths produced by internet routing algorithms,” in *Proc. of IEEE ICNP*, 2013, pp. 1–10.
- [7] H. Khayou and B. Sarakbi, “A validation model for non-lexical routing protocols,” *J. Netw. Comput. Appl.*, vol. 98, no. C, p. 58–64, 2017.
- [8] M. L. Daggitt, A. J. T. Gurney, and T. G. Griffin, “Asynchronous convergence of policy-rich distributed bellman-ford routing protocols,” in *Proc. of ACM SIGCOMM*, 2018, p. 103–116.
- [9] J. L. Sobrinho and M. A. Ferreira, “Routing on multiple optimality criteria,” in *Proc. of ACM SIGCOMM*, 2020, p. 211–225.
- [10] M. L. Lamali, S. Lassourreille, S. Kunne, and J. Cohen, “A stack-vector routing protocol for automatic tunneling,” in *Proc. of IEEE INFOCOM*, 2019, pp. 1675–1683.
- [11] N. Mouhoub, M. L. Lamali, and D. Magoni, “A highly parallelizable algorithm for routing with automatic tunneling,” in *2022 IFIP Networking Conference (IFIP Networking)*, 2022.
- [12] M. Mohri, “Semiring frameworks and algorithms for shortest-distance problems,” *J. Autom. Lang. Comb.*, vol. 7, no. 3, p. 321–350, 2002.
- [13] J. C. Harden, “Direct and semidirect products of semigroups,” 1949.
- [14] J. Sobrinho, “An algebraic theory of dynamic network routing,” *IEEE/ACM Trans. on Networking*, vol. 13, no. 5, pp. 1160–1173, 2005.
- [15] M. Gondran, “Algèbre linéaire et cheminement dans un graphe,” *RAIRO Recherche Opérationnelle*, vol. 9, no. VI, pp. 77–99, 1975.
- [16] M. L. Lamali, N. Fergani, and J. Cohen, “Algorithmic and complexity aspects of path computation in multi-layer networks,” *IEEE/ACM Trans. on Networking*, vol. 26, no. 6, pp. 2787–2800, 2018.
- [17] M. L. Daggitt, R. Zimrod, and T. G. Griffin, “AGDA-routing for SIGCOMM 2018.”