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FLUCTUATIONS FOR MEAN FIELD LIMITS OF INTERACTING SYSTEMS OF SPIKING NEURONS

EVA LÖCHERBACH

ABSTRACT. We consider a system of N neurons, each spiking randomly with rate depending on its membrane potential. When a neuron spikes, its potential is reset to 0 and all other neurons receive an additional amount h/N of potential, where $h > 0$ is some fixed parameter. In between successive spikes, each neuron's potential follows a deterministic flow with drift b expressing both the attraction to an equilibrium potential and some leakage factors. While the propagation of chaos of the system, as $N \rightarrow \infty$, to a limit nonlinear jumping stochastic differential equation has already been established in a series of papers, see [7], [13], [17], the present paper is devoted to the associated central limit theorem. More precisely we study the measure valued process of fluctuations at scale $N^{-1/2}$ of the empirical measures of the membrane potentials, centered around the associated limit. We show that this fluctuation process, interpreted as càdlàg process taking values in a suitable weighted Sobolev space, converges in law to a limit process characterized by a system of stochastic differential equations driven by Gaussian white noise. We complete this picture by studying the fluctuations, at scale $N^{-1/2}$, of the membrane potential processes around their associated limit quantities, giving rise to a mesoscopic approximation of the membrane potentials that take into account the correlations within the finite system.

Keywords: Convergence of fluctuations, weighted Sobolev spaces, systems of interacting neurons, Piecewise deterministic Markov processes, Mean field interactions.

AMS Classification 2010: 60G55; 60F05; 60G57; 92B20

1. INTRODUCTION

In the present paper we study the fluctuations for the mean field limits of systems of interacting and spiking neurons as the number of neurons tends to infinity. For any fixed size N , the system is characterized by the vector of potential values of the N neurons, $X^N = (X_t^N)_{t \geq 0}$. Here, for any time $t \geq 0$, $X_t^N = (X_t^{N,1}, \dots, X_t^{N,N})$ and $X_t^{N,i} \geq 0$ denotes the membrane potential of neuron i at time t . The process X^N is a Markov process having generator L^N given by

$$(1) \quad L^N \varphi(x) = \sum_{i=1}^N b(x^i) \partial_{x^i} \varphi(x) + \sum_{i=1}^N f(x^i) \left(\varphi(x + \sum_{j \neq i} \frac{h}{N} e_j - x^i e_i) - \varphi(x) \right),$$

for any smooth test function φ . In the above equation, $x = (x^1, \dots, x^N) \in \mathbb{R}_+^N$, and $e_i, 1 \leq i \leq N$, denotes the i -th unit vector in \mathbb{R}^N . $h > 0$ is a positive constant, the synaptic weight, and $b : \mathbb{R}_+ \rightarrow \mathbb{R}$, $b(0) \geq 0$, is the drift function defining the deterministic evolution of the membrane potential in between successive spikes of the system: the attraction to an equilibrium potential value right after the spike and some general leakage phenomena. The function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the jump rate function. Since $h > 0$, we are working in the purely excitatory case, such that all membrane potentials take values in \mathbb{R}_+ .

The above system of interacting neurons (or slight variations of it) and its mean field limits have been studied in a series of papers, starting with [7], [13] and [18], followed by [5]–[6] which are

devoted to the longtime behavior of the associated nonlinear limit process. Spatially structured versions of these convergence results have moreover been obtained in [8] and [4]. All these papers establish the propagation of chaos property implying that, in the limit model, different neurons are independent. The present paper completes this study by presenting the associated central limit theorem. In particular we will be able to present a mesoscopic approximation for each neuron's potential that takes care of the correlations between different neurons within finite, but large, systems, giving a precise form of the factor of common noise.

Up to our knowledge, this is the first time the precise fluctuations for such systems of interacting and spiking neurons are studied. Let us mention however the recent paper [19] proposing an adhoc mesoscopic model to describe finite size neuronal population equations, taking into account the finite size fluctuations. While the model proposed in [19], which is based on [20], is shown to give an accurate numerical approximation to the dynamics of finite-size networks of spiking neurons, this model is however not a precise extension of the original model around its large population limit, in the sense of a precise limit theorem.

Finally the two recent papers [9] and [10] shed a different light on this topic, by studying a related problem in which the spiking neuron distributes a centered random synaptic weight to its postsynaptic partners, which is renormalized by $1/\sqrt{N}$. In this way this model automatically leads to the study of fluctuations, the associated limit process is close to the one found in the present paper, but with the notable difference that it is driven by a single Brownian motion and that the fluctuations of the spiking rate do not have to be taken into account.

1.1. The model. To introduce the precise model, consider a family of i.i.d. Poisson measures $(\pi^i(ds, dz))_{i \geq 1}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ having intensity measure $dsdz$ each, as well as an i.i.d. family $(X_0^i)_{i \geq 1}$ of \mathbb{R}_+ -valued random variables, independent of the Poisson measures, distributed according to some probability measure g_0 on \mathbb{R} . Then we may represent each neuron's potential as

$$(2) \quad X_t^{N,i} = X_0^i + \int_0^t b(X_s^{N,i}) ds + \frac{h}{N} \sum_{j=1, j \neq i}^N \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(X_{s-}^{N,j})\}} \pi^j(ds, dz) \\ - \int_{[0,t] \times \mathbb{R}_+} X_{s-}^{N,i} \mathbf{1}_{\{z \leq f(X_{s-}^{N,i})\}} \pi^i(ds, dz), 1 \leq i \leq N.$$

It has been shown in [7], [13] and [18] that under appropriate assumptions on b , f and g_0 , the asymptotic evolution, as $N \rightarrow \infty$, of the membrane potential processes can be described as solution of the following infinite i.i.d. system of non-linear stochastic differential equations

$$(3) \quad \bar{X}_t^i = X_0^i + \int_0^t b(\bar{X}_s^i) ds + h \int_0^t \mathbb{E}(f(\bar{X}_s^i)) ds - \int_{[0,t] \times \mathbb{R}_+} \bar{X}_{s-}^i \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz), i \geq 1.$$

In this paper, we rely mainly on the approach proposed in [13] to prove and quantify the convergence of the finite system (2) to the limit system (3). We think of spiking rate functions of the form $f(x) = (x/x_0)^\alpha$ for some (possibly large) $\alpha > 0$ and some fixed value $x_0 > 0$. This means that whenever the membrane potential x of a neuron is below x_0 , the spiking rate $f(x)$ is very low, while for values $x > x_0$, the rate is very large, such that x_0 can be interpreted as soft threshold.

Throughout this paper we strengthen and adapt the conditions of [13] to the present frame and impose the following conditions.

- Assumption 1.** (1) $f \in C^4(\mathbb{R}_+, \mathbb{R}_+)$ is convex and non-decreasing such that $f(x) > 0$ for all $x > 0$ and $f(0) = 0$.
- (2) There exists some $\alpha \geq 1$ such that for all $0 \leq k \leq 4$, $\sup_{x \in \mathbb{R}_+} \frac{|f^{(k)}(x)|}{(1+|x|^\alpha)} < \infty$.
- (3) $\sup_{x \geq 1} [f'/f + f''/f'](x) < \infty$.
- (4) Moreover, there exists a constant C_f such that $f(x+y) \leq C_f(1+f(x)+f(y))$ for all $x, y \geq 0$.

Concerning the drift function, we think of functions of the type $b(x) = b - \lambda x$ for $b \geq 0, \lambda > 0$, expressing the attraction to some equilibrium potential value b/λ in absence of any spike of the system. More precisely, we impose the following condition.

- Assumption 2.** (1) $b \in C^4(\mathbb{R}_+, \mathbb{R}_+)$ is of linear growth, bounded from above by a positive constant and satisfies $b(0) \geq 0$.
- (2) For all $1 \leq k \leq 4$, $\sup_{x \in \mathbb{R}_+} |b^{(k)}(x)| < \infty$.

Let us mention that Items (1) and (2) of Assumptions 1 and 2 are natural in this context. We will be obliged to work with test functions being four times continuously differentiable, and we need some a priori controls on the membrane potential values. Items (3) and (4) of Assumption 1 are technical. They are used to obtain a quantified propagation of chaos result, see (7) below, the first main building block upon which we construct our fluctuation result.

Concerning the distribution of the initial potential values we impose

- Assumption 3.** We suppose that g_0 is compactly supported and possesses a probability density $g_0(x)$ which belongs to $C^1(\mathbb{R}_+, \mathbb{R}_+)$.

Then by [7] and [13], there exists a unique strong solution both for (2) and for (3). Moreover, by Theorem 7 of [13], constructing $X^{N,i}$ and \bar{X}^i using the same underlying Poisson random measure π^i , we have for any $T > 0$ and for any $1 \leq i \leq N$,

$$(4) \quad \sup_{t \leq T} \mathbb{E}(|X_t^{N,i} - \bar{X}_t^i|) \leq \frac{C_T}{\sqrt{N}},$$

where the constant C_T does not depend on N . Introducing

$$(5) \quad g_t = \mathcal{L}(\bar{X}_t^1),$$

and the empirical measure of the finite system together with the associated projection onto time t ,

$$(6) \quad \mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{X^{N,i}}, \quad \mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{N,i}},$$

we also have, due to a generalization of Theorem 7 of [13] to our frame (see the Appendix for details),

$$(7) \quad \sup_{t \leq T} \mathbb{E}(\mathcal{W}_1(\mu_t^N, g_t)) \leq \frac{C_T}{\sqrt{N}}.$$

Here, the Monge-Kantorovich-Wasserstein distance $\mathcal{W}_1(\mu, \nu)$ between two probability measures μ and ν on \mathbb{R}_+ with finite expectations is defined by $\mathcal{W}_1(\mu, \nu) = \inf\{\mathbb{E}[|U - V|], \mathcal{L}(U) = \mu \text{ and } \mathcal{L}(V) = \nu\}$.

Finally, let us mention that $g_t = \mathcal{L}(\bar{X}_t^i)$ is solution of a nonlinear PDE which in its strong form reads as

$$\partial_t g_t(x) = [-b(x) - h g_t(f)] \partial_x g_t(x) - [b'(x) + f(x)] g_t(x), \quad t \geq 0, x > 0,$$

where we note $g_t(f) = \int_0^\infty f(x) g_t(dx)$. The above PDE starts from the initial value g_0 at time $t = 0$, and we have the boundary condition $g_t(0) = \frac{g_t(f)}{b(0) + h g_t(f)}$ for all $t > 0$.

As a consequence of (7), interpreting μ_t^N as random variable in the space $\mathcal{P}(\mathbb{R}_+)$ of all probability measures on \mathbb{R}_+ , we have convergence in probability $\mu_t^N \rightarrow g_t$, as $N \rightarrow \infty$, and the rate of convergence is at least $N^{-1/2}$. It is therefore natural to study the associated process of fluctuations, given by

$$(8) \quad \eta_t^N = \sqrt{N}(\mu_t^N - g_t),$$

together with the fluctuations of the processes of membrane potentials

$$(9) \quad U_t^{N,i} = \sqrt{N}(X_t^{N,i} - \bar{X}_t^i), i \geq 1,$$

where we put $U^{N,i} = 0$ for any $i \geq N+1$. In (9), the processes $X^{N,i}$ and \bar{X}^i are constructed according to the so-called Sznitman coupling (see [21]): They are defined on the same probability space, starting from the same initial condition X_0^i and using the same underlying Poisson random measure π^i , for each $1 \leq i \leq N$. In the sequel, we write for short $U^N = (U^{N,i})_{i \geq 1}$.

In the present paper we prove convergence in law of the sequence of processes (U^N, η^N) to a limit process $(\bar{U}, \bar{\eta})$, as $N \rightarrow \infty$, where $\bar{U} = (\bar{U}^i)_{i \geq 1}$. The limit process $\bar{\eta}$, interpreted as distribution acting on appropriate test functions, follows an infinite dimensional differential equation stated precisely in (15) below. Moreover, for each $i \geq 1$, the limit process \bar{U}^i follows an Ornstein-Uhlenbeck type dynamic with variable length memory, that is, for any $t \geq 0$,

$$(10) \quad \bar{U}_t^i = \int_0^t b'(\bar{X}_s^i) \bar{U}_s^i ds + h \int_0^t \bar{\eta}_s(f) ds - \int_{[0,t] \times \mathbb{R}_+} \bar{U}_{s-}^i \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz) + hM_t.$$

Here, $(M_t)_t$ is a Gaussian martingale having quadratic variation

$$\langle M \rangle_t = \int_0^t g_s(f) ds = \mathbb{E} \int_0^t f(\bar{X}_s^i) ds.$$

In (10), the presence both of this Gaussian martingale and of the integral of the fluctuations of the spiking rate, $\int_0^t \bar{\eta}_s(f) ds$, induces a factor of common noise explaining the correlations between different neurons in the finite system.

As a consequence, we obtain the following second order error correction to the mean field approximation

$$(11) \quad X_t^{N,i} = \bar{X}_t^i + \frac{1}{\sqrt{N}} \bar{U}_t^i, \text{ where } \bar{U}_t^i = h \int_{\bar{L}_t^i}^t e^{\int_s^t b'(\bar{X}_u^i) du} \bar{\eta}_s(f) ds + h \int_{\bar{L}_t^i}^t e^{\int_s^t b'(\bar{X}_u^i) du} dM_s,$$

with $\bar{L}_t^i = \sup\{s \leq t : \Delta \bar{X}_s^i \neq 0\}$ the last spiking time of neuron i in the limit process, before time t , with $\sup \emptyset := 0$.

While in (10) above the convergence of $(U^{N,i})_{i \geq 1}$ has to be understood as convergence of stochastic processes with càdlàg trajectories, that is, of random variables taking values in $D(\mathbb{R}_+, \mathbb{R})^{\mathbb{N}^*}$, we did not specify so far in which space the convergence of the rescaled empirical measures η^N takes place. Following the Hilbertian approach introduced in [11] and [12] and then applied to the framework of point processes in [3], throughout this paper we interpret η^N as a stochastic process taking values in a suitable distributional space which is the dual of some weighted Sobolev space of test functions. The regularity of test functions we need to impose is related to the order up to which we have to develop the error terms that appear when replacing the contribution of small jumps (i.e., the last term appearing in (1)) by the associated limit drift. Moreover, since the finite size process does not take values in a compact set, we need to work with a Sobolev space supported by \mathbb{R}_+ . Finally, it turns out that we have to include constant functions into our class of admissible test functions, as well as the firing rate function f which is of polynomial growth. Therefore we are led to work with weighted

Sobolev spaces, where the weights are chosen to be polynomial, of power $p > \alpha + \frac{1}{2}$, where α is the growth rate of f and its derivatives (see Assumption 1).

The approach used in this article follows closely the study of fluctuations for McKean-Vlasov diffusions in [12] and the adaptation of this work to the framework of age-dependent Hawkes process proposed in [3]. The main difference with respect to [12] is that, as in [3], the limit processes \bar{X}^i and \bar{U}^i remain jump processes; the big jumps induced by spikes survive also in the limit process. The main difference with respect to [3] is the following. Being interested in age-dependent Hawkes processes, in [3], the limit process undergoes a deterministic drift given by $b(x) = 1$. This trivially implies good coupling properties. In our model however, the time dependent drift of the limit process is given by $b(x) + hg_t(f)$ at time t , where $b(x)$ is e.g. a function of the type $b(x) = b - \lambda x$. This depends both on the position x , but also on the average spiking rate of the system. This makes the study of coupling more complicated, which is one of the main reasons why it is more difficult to prove the uniqueness of the limit equation in the present frame. In particular, to prove the uniqueness, we do also have to establish regularity properties of the time inhomogeneous semigroup associated to the limit process (3) which is non-diffusive and associated to a transport equation. We rely on Girsanov's theorem for jump processes to tackle this problem, see Proposition 13 below.

1.2. General notation. The space of bounded functions of class C^k , defined on \mathbb{R}_+ , with bounded derivatives of each order up to order k , is denoted by C_b^k . C_c^∞ denotes the space of infinitely differentiable functions defined on \mathbb{R}_+ , having compact support. The space of càdlàg functions defined on \mathbb{R}_+ and taking values in some Polish space E is denoted by $D(\mathbb{R}_+, E)$. If μ is a measure on E and $\varphi : E \rightarrow \mathbb{R}$ measurable and integrable, we write $\mu(\varphi) := \int_E \varphi d\mu$. C denotes a constant that may change from one occurrence to another, even within one line.

Throughout this paper we work with the canonical filtration $(\mathcal{F}_t)_{t \geq 0}$ where $\mathcal{F}_0 = \sigma\{X_0^i, i \geq 1\}$ and $\mathcal{F}_t = \sigma\{X_0^i, i \geq 1, \pi^j(A) : A \subset [0, t] \times \mathbb{R}_+, j \geq 1\}$.

2. MAIN RESULTS

The aim of this section is to state the convergence in law of the sequence of processes $(U^N, \eta^N)_N$, $U^N = (U^{N,i})_{i \geq 1}$, defined by

$$(12) \quad U_t^{N,i} := \sqrt{N}(X_t^{N,i} - \bar{X}_t^i) \text{ for any } 1 \leq i \leq N, \text{ and } \eta_t^N = \sqrt{N}(\mu_t^N - g_t),$$

where we interpret η^N as stochastic process with values in a suitable space of distributions. Here we put $U^{N,i} = 0$ for all $i \geq N + 1$. In the above definition, $X^{N,i}$ and \bar{X}^i are constructed according to the Sznitman coupling (see [21]), that is, using the same initial value X_0^i and driven by the same underlying Poisson random measure π^i .

We start gathering some basic definitions and results on weighted Sobolev spaces.

2.1. Weighted Sobolev spaces. Since we are working in the purely excitatory case and the membrane potentials take values in \mathbb{R}_+ , in what follows, all test functions that we consider are defined on \mathbb{R}_+ . Fixing an integer k and a positive real number $p \geq 0$, we introduce the norm $\|\psi\|_{k,p}$ for all functions $\psi \in C_c^\infty$ given by

$$\|\psi\|_{k,p} := \left(\sum_{l=0}^k \int_0^\infty \frac{|\psi^{(l)}(x)|^2}{1 + |x|^{2p}} dx \right)^{1/2}$$

and define the space $\mathcal{W}_0^{k,p}$ to be the completion of C_c^∞ with respect to this norm.

The space $\mathcal{W}_0^{k,p}$ is a separable Hilbert space, and we denote $\mathcal{W}_0^{-k,p}$ its dual space, equipped with the norm $\|\cdot\|_{-k,p}$ defined for any $\eta \in \mathcal{W}_0^{-k,p}$ by

$$\|\eta\|_{-k,p} = \sup\{|\langle \eta, \psi \rangle| : \psi \in \mathcal{W}_0^{k,p}, \|\psi\|_{k,p} = 1\}.$$

Finally, $C^{k,p}$ is the space of all C^k -functions such that for all $l \leq k$, $\sup_{x \in \mathbb{R}_+} |\psi^{(l)}(x)|/(1+|x|^p) < \infty$. This space is equipped with the norm

$$\|\psi\|_{C^{k,p}} := \sum_{l=0}^k \sup_{x \in \mathbb{R}_+} \frac{|\psi^{(l)}(x)|}{1+|x|^p} < \infty.$$

The most important facts about Sobolev spaces that we use throughout this paper are collected in the Appendix Section 7.1.

2.2. Weak convergence of the fluctuation process. Given the knowledge of the function $t \mapsto g_t(f)$, the non-Markovian limit process (3) is described by the time dependent infinitesimal generator given by

$$(13) \quad L_s \varphi(x) = b(x)\varphi'(x) + hg_s(f)\varphi'(x) + f(x)S\varphi(x), \quad S\varphi(x) = \varphi(0) - \varphi(x),$$

for all $s \geq 0$ and $\varphi \in C_b^1$, where g_s is given by (5). Our main result reads as follows.

Theorem 4. *Under Assumptions 1, 2 and 3, for any $p > \alpha + \frac{1}{2}$, we have convergence in law of (U^N, η^N) in $D(\mathbb{R}_+, \mathbb{R})^{\mathbb{N}^*} \times D(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$ to the limit process $(\bar{U}, \bar{\eta})$ taking values in $D(\mathbb{R}_+, \mathbb{R})^{\mathbb{N}^*} \times C(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$, which is solution of the system of stochastic differential equations*

$$(14) \quad \bar{U}_t^i = \int_0^t b'(\bar{X}_s^i) \bar{U}_s^i ds + h \int_0^t \bar{\eta}_s(f) ds - \int_{[0,t] \times \mathbb{R}_+} \bar{U}_{s-}^i \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz) + hW_t(1), i \geq 1,$$

and

$$(15) \quad \bar{\eta}_t(\varphi) = \bar{\eta}_0(\varphi) + \int_0^t \bar{\eta}_s(L_s \varphi) ds + h \int_0^t g_s(\varphi') \bar{\eta}_s(f) ds + W_t(S\varphi) + h \int_0^t g_s(\varphi') dW_s(1),$$

where the above equation holds for all $\varphi \in \mathcal{W}_0^{5,p}$, and where, for any $\psi \in \mathcal{W}_0^{4,p}$,

$$(16) \quad W_t(\psi) = \int_0^t \int_{\mathbb{R}} \sqrt{f(x)} \psi(x) dM(s, x),$$

with $M(dt, dx)$ an orthogonal martingale measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $dtg_t(dx)$.

If in addition to the above assumptions we have $p > 2\alpha + \frac{1}{2}$ and we suppose moreover that $f \in C^{6,\alpha}$, $b \in C^6$ having all derivatives up to order 6 bounded and $b(x) \geq -\lambda x$ for all $x \geq 0$, where $\lambda > 0$, then the limit process solving (14) and (15) is unique.

Remark 5. Notice that by construction,

$$\langle W(\varphi), W(\psi) \rangle_t = \int_0^t \int_{\mathbb{R}} g_s(dx) \varphi(x) \psi(x) f(x) ds = \int_0^t g_s(f\varphi\psi) ds = \mathbb{E} \int_0^t (f\varphi\psi)(\bar{X}_s^i) ds.$$

2.3. Plan of the paper. The remainder of this paper is devoted to the proof of Theorem 4. Section 3 starts with useful a priori bounds on the finite size process and its limit, before establishing the uniqueness of any solution of (14) and (15) in Theorem 10. We then continue, in Section 4 by establishing a decomposition of the finite size fluctuations in Proposition 23 which is the starting point of the proof of our main result. We prove the tightness of (U^N, η^N) in Theorem 25 of Section 5. Theorem 31 in Section 6 then states that any possible limit $(\bar{U}, \bar{\eta})$ of (U^N, η^N) is necessarily solution of the system of differential equations of Theorem 4. The Appendix section collects some useful results about the limit process together with some technical results.

3. UNIQUENESS OF THE LIMIT EQUATION

3.1. Preliminaries. We first investigate the mappings that appear in the generator of the limit process. These are the linear mapping S associated to the spikes of a given neuron and defined by $S\varphi := \varphi(0) - \varphi$, the mapping $bD : \varphi \mapsto [x \mapsto b(x)\varphi'(x)]$ and the mapping $D : \varphi \mapsto \varphi'$.

Lemma 6. *S is a continuous mapping from $\mathcal{W}_0^{k,p}$ to itself, for any $k \geq 1$ and $p > \frac{1}{2}$. If we suppose moreover that Assumptions 1 and 2 hold, then for any $p > \alpha + \frac{1}{2}$, and any $k \leq 4$,*

$$\|fS\varphi\|_{k,p} \leq C\|f\|_{C^{k,\alpha}}\|S\varphi\|_{k,p-\alpha} \leq C\|f\|_{C^{k,\alpha}}\|\varphi\|_{k,p-\alpha}.$$

Finally, for any $k \geq 2$, $D : \mathcal{W}_0^{k,p} \rightarrow \mathcal{W}_0^{k-1,p}$ and $bD : \mathcal{W}_0^{k,p} \rightarrow \mathcal{W}_0^{k-1,p+1}$ are continuous mappings satisfying $\|D\varphi\|_{k-1,p} \leq C\|\varphi\|_{k,p}$ and $\|bD\varphi\|_{k-1,p+1} \leq C\|\varphi\|_{k,p}$.

If in addition to the above assumptions we suppose moreover that $f \in C^{6,\alpha}$ and $b \in C^6$ such that all derivatives up to order 6 are bounded, recalling moreover that $\alpha \geq 1$, then the application L_s introduced in (13) is a linear continuous mapping from $\mathcal{W}_0^{k,p}$ to $\mathcal{W}_0^{k-1,p+\alpha}$, for any $p > \frac{1}{2}$, $k \leq 6$, and for all $\psi \in \mathcal{W}_0^{k,p}$,

$$\sup_{s \leq T} \frac{\|L_s \psi\|_{k-1,p+\alpha}^2}{\|\psi\|_{k,p}^2} < \infty.$$

The proof of the above lemma is given in the Appendix. In the sequel we shall also rely on the following result.

Lemma 7. *For any $x \in \mathbb{R}_+$ and any $p > 0$, the mapping $\delta_x : \mathcal{W}_0^{1,p} \rightarrow \mathbb{R}, \psi \mapsto \psi(x)$ is continuous. Moreover there exists a constant C not depending on x such that*

$$(17) \quad \|\delta_x\|_{-1,p} \leq C(1 + |x|^p).$$

Similarly, $D^*\delta_x : \mathcal{W}_0^{2,p} \rightarrow \mathbb{R}, \psi \mapsto \psi'(x)$ is continuous and there exists a constant C not depending on x such that

$$\|D^*\delta_x\|_{-2,p} \leq C(1 + |x|^p).$$

Proof. We only show the second assertion. We have for any $\psi \in \mathcal{W}_0^{2,p}$,

$$|\langle D^*\delta_x, \psi \rangle| = |\psi'(x)| \leq \|\psi\|_{C^{1,p}}(1 + |x|^p).$$

Moreover, using the Sobolev embedding theorem (see (62) below), there exists a constant C not depending on x , such that $\|\psi\|_{C^{1,p}} \leq C\|\psi\|_{2,p}$. This implies the assertion. ■

The following a priori bounds on (2) and (3) will be used throughout this paper.

Lemma 8. *Under Assumptions 1, 2 and 3, for any $1 \leq i \leq N$ and $N \geq 1$, there exists a constant c_0 only depending on g_0 such that*

$$(18) \quad X_t^{N,i} \leq c_0 + 4\bar{b}t + 4hN_t^N,$$

where $N_t^N := N^{-1} \sum_{j=1}^N \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(2h)\}} \pi^j(ds, dz)$ and where $\bar{b} > 0$ is such that $b(x) \leq \bar{b}$ for all $x \geq 0$. In particular, for any $T, p > 0$,

$$(19) \quad \sup_N \mathbb{E}(\sup_{t \leq T} |X_t^{N,i}|^p) \leq C_T(p)$$

for a constant depending only on T, p, b and g_0 . Introducing the set

$$(20) \quad G_T^N = \left\{ \sum_{i=1}^N \int_{[0,T] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(2h)\}} \pi^i(ds, dz) \leq 2f(2h)NT \right\},$$

we also have the upper bound

$$(21) \quad \mathbf{1}_{G_T^N} \left(\sup_{t \leq T} \sup_{1 \leq i \leq N} |X_t^{N,i}| \right) \leq c_0 + 4\bar{b}T + 8hf(2h)T \text{ and the control } \mathbb{P}((G_T^N)^c) \leq c_1 e^{-c_2 NT},$$

for some constants c_1, c_2 . Finally there exists a constant \bar{C}_T only depending on g_0 and on \bar{b} , such that

$$(22) \quad \sup_{t \leq T} |\bar{X}_t^i| \leq \bar{C}_T.$$

The proof of this lemma is given in the Appendix. We immediately state a useful corollary. Introducing

$$(23) \quad Z_t^{N,i} = \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(X_s^{N,i})\}} \pi^i(ds, dz) \text{ and } \bar{Z}_t^i = \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(\bar{X}_s^i)\}} \pi^i(ds, dz)$$

and the total variation distance

$$\|Z^{N,i} - \bar{Z}^i\|_{TV,[0,T]} := \#\{t \leq T : t \text{ is a jump of } Z^{N,i} \text{ or of } \bar{Z}^i \text{ but not of both}\},$$

we have that

Corollary 9. *Under Assumptions 1, 2 and 3,*

$$(24) \quad \mathbb{E}\|Z^{N,i} - \bar{Z}^i\|_{TV,[0,T]} = \mathbb{E} \int_0^T |f(X_s^{N,i}) - f(\bar{X}_s^i)| ds \leq C_T N^{-1/2},$$

for a constant C_T only depending on T , but not on N .

Proof. Clearly,

$$\begin{aligned} & \sqrt{N} \mathbb{E} \|Z^{N,i} - \bar{Z}^i\|_{TV,[0,T]} \\ &= \sqrt{N} \mathbb{E} \int_{[0,T] \times \mathbb{R}_+} \left| \mathbf{1}_{\{z \leq f(X_s^{N,i})\}} - \mathbf{1}_{\{z \leq f(\bar{X}_s^i)\}} \right| \pi^i(ds, dz) = \sqrt{N} \mathbb{E} \int_0^T |f(X_s^{N,i}) - f(\bar{X}_s^i)| ds \\ &\leq \sqrt{N} \int_0^T \mathbb{E}[|f(X_s^{N,i}) - f(\bar{X}_s^i)| \mathbf{1}_{G_T^N}] ds + \sqrt{N} T f(\bar{C}_T) \mathbb{P}((G_T^N)^c) \\ &\quad + C \sqrt{N} T \mathbb{E} \left(\mathbf{1}_{(G_T^N)^c} [1 + (c_0 + 4\bar{b}t + 4hN_T^N)^\alpha] \right), \end{aligned}$$

where we have used that by (18) and since f is non-decreasing,

$$f(X_s^{N,i}) \leq C(1 + (c_0 + 4\bar{b}t + 4hN_T^N)^\alpha) \text{ and } f(\bar{X}_s^i) \leq f(\bar{C}_T)$$

for all $s \leq T$.

Due to (21), on G_T^N , $X_t^{N,i} \leq c_0 + 4\bar{b}t + 8hf(2h)T$ for all i , and all $t \leq T$. Therefore, using the Lipschitz continuity of f on $[0, c_0 + 4\bar{b}t + 8hf(2h)T] \vee \bar{C}_T$ and (4), we have

$$\sup_N \sqrt{N} \int_0^T \mathbb{E}[|f(X_s^{N,i}) - f(\bar{X}_s^i)| \mathbf{1}_{G_T^N}] ds \leq C_T.$$

Using the deviation estimate on $\mathbb{P}((G_T^N)^c)$ together with Hölder's inequality implies moreover that

$$\sup_N \left(\sqrt{NT} f(\bar{C}_T) \mathbb{P}((G_T^N)^c) + C\sqrt{NT} \mathbb{E} \left(\mathbf{1}_{(G_T^N)^c} [1 + (c_0 + 4\bar{b}t + 4hN_T^N)^\alpha] \right) \right) \leq C_T$$

such that

$$\sup_N \sqrt{N} \mathbb{E} \|Z^{N,i} - \bar{Z}^i\|_{TV,[0,T]} = \sup_N \sqrt{N} \mathbb{E} \int_0^T |f(X_s^{N,i}) - f(\bar{X}_s^i)| ds \leq C_T < \infty,$$

implying the assertion. ■

After these preliminary results, we now turn to the proof of our first main result which is the uniqueness of the limit equation.

3.2. Uniqueness. This section is devoted to the proof of the following

Theorem 10. *Grant Assumptions 1, 2 and 3 and suppose moreover that $f \in C^{6,\alpha}$, $b \in C^6$ having all derivatives up to order 6 bounded and $b(x) \geq -\lambda x$ for all $x \geq 0$, where $\lambda > 0$. Then for any fixed initial condition $(\bar{\eta}_0)$ and driving underlying noise $\pi^i, i \geq 1$, and W , the system (14)–(15) has at most one solution in $D(\mathbb{R}_+, \mathbb{R})^{N^*} \times C(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$, for any $p > 2\alpha + \frac{1}{2}$.*

Since given $\bar{X}^i, \bar{\eta}$ and W , the equation for \bar{U}^i is linear, it is sufficient to prove uniqueness for $\bar{\eta}$.

Suppose $\bar{\eta}$ and $\hat{\eta}$ are both solution of (15), driven by the same underlying W and starting from the same initial condition. Then $\tilde{\eta}_t := \bar{\eta}_t - \hat{\eta}_t$ satisfies

$$(25) \quad \langle \tilde{\eta}_t, \varphi \rangle = \int_0^t \langle \tilde{\eta}_s, L_s \varphi \rangle ds + h \int_0^t g_s(\varphi') \tilde{\eta}_s(f) ds,$$

where we recall that

$$L_s \varphi(x) = b(x) \varphi'(x) + h g_s(f) \varphi'(x) + f(x) S \varphi(x).$$

Traditionally, to prove uniqueness we have to deduce from (25) that $\tilde{\eta} = 0$, that is, $\|\tilde{\eta}_t\|_{-k,p} = 0$ for suitable k and p . However, when applying $\|\cdot\|_{-k,p}$ to (25), we have to treat the term $\int_0^t \langle \tilde{\eta}_s, L_s \varphi \rangle ds$ which involves a derivative and multiplication with f and therefore gives rise to $\|\tilde{\eta}_s\|_{-k+1,p+\alpha}$ which cannot be compared to the norm $\|\tilde{\eta}_s\|_{-k,p}$ since it is greater. The same problem arises when treating the last term $\int_0^t g_s(\varphi') \tilde{\eta}_s(f) ds$.

Of course, this problem has already appeared – and solved – both in [12] and [3], yet in a simpler framework, since in [12], the underlying diffusion generates regularity of the associated semigroup, while in [3] the underlying flow is particularly simple, having drift $\equiv 1$. In what follows we show how to adapt these ideas to the present frame and propose several tricks to get rid of the above derivatives by using integration by parts or by solving directly the flow associated to L_s .

We start gathering known results about the marginal law g_s of the limit process \bar{X}_s^1 of (3), starting from $\bar{X}_0^1 \sim g_0$, that is, $g_s = \mathcal{L}(\bar{X}_s^1)$. We introduce the associated flow defined by

$$(26) \quad \varphi_{s,t}(x) = x + \int_s^t b(\varphi_{s,u}(x)) du + h \int_s^t g_u(f) du,$$

for any $s \leq t$, representing the evolution of \bar{X}^1 in between the successive jumps.

Proposition 11. *Under Assumptions 1, 2 and 3, for all $t \geq 0$, $g_t(dy) = g_t(y)dy$ is absolutely continuous having Lebesgue density $g_t(y)$, and for all $t \leq T$, g_t is compactly supported, that is, $g_t(y) = 0$ for all $y \geq \bar{C}_T$, where \bar{C}_T is as in (22). Moreover, for all $y \neq \varphi_{0,t}(0)$, g_t is differentiable in y having derivative g'_t which is continuous on $(0, \varphi_{0,t}(0)) \cup (\varphi_{0,t}(0), \infty)$. Finally,*

$$t \mapsto \int_0^\infty (1 + |x|^p) |g'_t(x)| dx \text{ is locally bounded,}$$

for all $p \geq 0$.

The proof of the above result is postponed to the Appendix.

Using integration by parts this implies that we may rewrite the last term appearing in (25) as follows.

$$\begin{aligned} g_s(\varphi') &= \int_0^{\varphi_{0,s}(0)} \varphi'(x) g_s(x) dx + \int_{\varphi_{0,s}(0)}^\infty \varphi'(x) g_s(x) dx \\ &= g_s(\varphi_{0,s}(0)-) \varphi(\varphi_{0,s}(0)) - g_s(0) \varphi(0) - g_s(\varphi_{0,s}(0)+) \varphi(\varphi_{0,s}(0)) - \int_0^\infty \varphi(x) g'_s(x) dx \\ &= -\frac{g_s(f)}{b(0) + hg_s(f)} \varphi(0) - \Delta g_s(\varphi_{0,s}(0)) \varphi(\varphi_{0,s}(0)) - \int_0^\infty \varphi(x) g'_s(x) dx, \end{aligned}$$

for any $s > 0$, where we used the identity $g_s(0) = \frac{g_s(f)}{b(0) + hg_s(f)}$ which follows from (67) stated in the Appendix below, and where

$$\Delta g_s(\varphi_{0,s}(0)) = g_s(\varphi_{0,s}(0)+) - g_s(\varphi_{0,s}(0)-) = e^{-\int_0^s (f(\varphi_{0,u}(0)) + b'(\varphi_{0,u}(0))) du} [g_0(0) - \frac{g_s(f)}{b(0) + hg_s(f)}],$$

such that

$$h \int_0^t g_s(\varphi') \tilde{\eta}_s(f) ds = \int_0^t h_s(\varphi) ds,$$

where

$$h_s(\varphi) = -\frac{hg_s(f)}{b(0) + hg_s(f)} \varphi(0) \tilde{\eta}_s(f) - h\varphi(\varphi_{0,s}(0)) \Delta g_s(\varphi_{0,s}(0)) \tilde{\eta}_s(f) - h \left(\int_0^\infty \varphi(x) g'_s(x) dx \right) \tilde{\eta}_s(f).$$

Relying on Proposition 11, we deduce

Proposition 12. *Let $\psi \in \mathcal{W}_0^{k,q}$, for some $k \geq 1, q \geq 0$. Fix $T > 0$. Then for all $0 \leq t \leq T$, and for all $p > \alpha + \frac{1}{2}$,*

$$|h_t(\psi)| \leq C_T \|\psi\|_{k,q} \|\tilde{\eta}_t\|_{-4,p} \|f\|_{4,p}.$$

Proof. We use that by the Sobolev embedding,

$$|\psi(x)| \leq \|\psi\|_{C^{0,q}} (1 + |x|^q) \leq C \|\psi\|_{k,q} (1 + |x|^q),$$

since $k \geq 1$, such that

$$|\psi(0)| \leq C \|\psi\|_{k,q} \text{ and } |\psi(\varphi_{0,t}(0)) \Delta g_t(\varphi_{0,t}(0))| \leq C_T \|\psi\|_{k,q}$$

and moreover

$$\int_0^\infty |\psi(x) g'_s(x)| dx \leq C \|\psi\|_{k,q} \int_0^\infty (1 + |x|^q) |g'_s(x)| dx \leq C_T \|\psi\|_{k,q},$$

where we have used the bound of Proposition 11. The conclusion follows from

$$|\tilde{\eta}_t(f)| \leq \|\tilde{\eta}_t\|_{-4,p} \|f\|_{4,p},$$

since $f \in \mathcal{W}_0^{4,p}$ by Assumption 1, due to the fact that $C^{4,\alpha} \subset \mathcal{W}_0^{4,p}$ for any $p > \alpha + \frac{1}{2}$. ■

We now turn to the study of the action of L_s . Given the fixed function $t \mapsto g_t(f)$, $t \geq 0$, we introduce the time inhomogeneous Markov process $Y_{s,t}(x)$, for any $0 \leq s \leq t$ and $x \in \mathbb{R}_+$, which is solution of

$$Y_{s,t}(x) = x + \int_s^t (hg_u(f) + b(Y_{s,u}(x))) du - \int_{]s,t] \times \mathbb{R}_+} Y_{s,u-}(x) \mathbf{1}_{\{z \leq f(Y_{s,u-}(x))\}} \pi^1(du, dz).$$

Clearly, since $h \int_0^t g_s(f) ds \leq \bar{C}_T$ for all $t \leq T$, by (22), and since b is bounded by a positive constant, $Y_{s,t}(x) \leq x + \bar{C}_T$, for all $s \leq t \leq T$, such that the above process is well-defined. We denote $P_{s,t}$ the associated semigroup, that is, $P_{s,t}\psi(x) = \mathbb{E}\psi(Y_{s,t}(x))$, for any measurable test function.

Proposition 13. *Under the assumptions of Theorem 10, we have that for any $0 \leq s \leq t$ and any $p \geq 0$, $P_{s,t}$ is a continuous mapping from $\mathcal{W}_0^{6,p} \rightarrow \mathcal{W}_0^{6,p}$, and*

$$\|P_{s,t}\psi\|_{k,p} \leq C_T \|\psi\|_{k,p},$$

for all $k \leq 6$, $s \leq t \leq T$.

Moreover, for any $\psi \in C_c^\infty$, $P_{s,t}\psi$ belongs to C_b^6 and is rapidly decreasing, that is, for all $\gamma > 0$ and all $k \leq 6$,

$$(27) \quad \lim_{x \rightarrow \infty} x^\gamma |(P_{s,t}\psi)^{(k)}(x)| = 0.$$

The proof of this result is also postponed to the Appendix.

We notice that L_s is the time dependent infinitesimal generator associated to the time inhomogeneous semigroup $P_{s,t}$, that is,

$$\frac{d}{ds} P_{s,t}\psi = -L_s P_{s,t}\psi \text{ and } \frac{d}{dt} P_{s,t}\psi = P_{s,t} L_t \psi,$$

whenever the above quantities are well-defined.

Now we proceed further with our proof. Let $0 \leq s \leq t \leq T$ be fixed. Consider a test function $\psi \in C_c^\infty$. Then we have that

$$P_{s,t}\psi(x) = \psi(x) - \int_s^t \frac{\partial}{\partial v} P_{v,t}\psi(x) dv = \psi(x) + \int_s^t L_v P_{v,t}\psi(x) dv$$

such that

$$(28) \quad P_{s,t}\psi = \psi + \int_s^t L_v P_{v,t}\psi dv.$$

Plugging this into (25) and observing that ψ and $P_{s,t}\psi$, and thus, a posteriori, also $\int_s^t L_v P_{v,t}\psi dv$ are valid test functions, we obtain

$$(29) \quad \int_0^t \langle \tilde{\eta}_s, L_s \psi \rangle ds = \int_0^t \langle \tilde{\eta}_s, L_s P_{s,t}\psi \rangle ds - \int_0^t \int_s^t \langle \tilde{\eta}_s, L_s L_u P_{u,t}\psi \rangle duds.$$

Let us consider the double integral appearing in the above expression. By the definition of L_s and using equation (27) of Proposition 13, we know that

$$(30) \quad \Psi_{s,u,t} := L_s L_u P_{u,t}\psi \in C^{4,2\alpha}.$$

This implies that for all $p > 2\alpha + \frac{1}{2}$,

$$\sup_{s \leq u \leq t \leq T} \|L_s L_u P_{u,t} \psi\|_{4,p} = C_T < \infty$$

such that

$$| \langle \tilde{\eta}_s, L_s L_u P_{u,t} \psi \rangle | \leq \|\tilde{\eta}_s\|_{-4,p} \|L_s L_u P_{u,t} \psi\|_{4,p} \leq C_T \|\tilde{\eta}_s\|_{-4,p}.$$

Since $\tilde{\eta}$ takes values in $C(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$, $\sup_{s \leq t} \|\tilde{\eta}_s\|_{-4,p} < \infty$, and therefore we may use Fubini's theorem and obtain

$$(31) \quad \int_0^t \int_s^t \langle \tilde{\eta}_s, L_s L_u P_{u,t} \psi \rangle duds = \int_0^t \int_0^u \langle \tilde{\eta}_s, L_s L_u P_{u,t} \psi \rangle dsdu.$$

Now we apply (25) to the admissible test function $\varphi := L_u P_{u,t} \psi$, for fixed $u < t$, at time u . Then

$$\langle \tilde{\eta}_u, \varphi \rangle = \int_0^u \langle \tilde{\eta}_s, L_s \varphi \rangle ds + H_u(L_u P_{u,t} \psi),$$

where we write for short

$$(32) \quad H_u(\cdot) := \int_0^u h_s(\cdot) ds.$$

We deduce that the double integral in (31) can be rewritten as

$$(33) \quad \int_0^t \int_0^u \langle \tilde{\eta}_s, L_s L_u P_{u,t} \psi \rangle dsdu = \int_0^t \langle \tilde{\eta}_u, L_u P_{u,t} \psi \rangle du - \int_0^t H_u(L_u P_{u,t} \psi) du.$$

(29) together with (33) now implies that

$$\int_0^t \langle \tilde{\eta}_s, L_s \psi \rangle ds = \int_0^t H_u(L_u P_{u,t} \psi) du.$$

Using the same trick as above,

$$H_t(\psi) = \int_0^t h_s(\psi) ds = \int_0^t h_s(P_{s,t} \psi) ds - \int_0^t \int_s^t h_s(L_v P_{v,t} \psi) dv ds.$$

Proposition 12, with $k = 1$, $q = 2\alpha$ together with Lemma 6 implies

$$|h_s(L_v P_{v,t} \psi)| \leq C_t \|L_v P_{v,t} \psi\|_{1,2\alpha} \|\tilde{\eta}_s\|_{-4,p} \|f\|_{4,p} \leq C_t \|P_{v,t} \psi\|_{2,\alpha} \|\tilde{\eta}_s\|_{-4,p} \|f\|_{4,p},$$

which is bounded uniformly in $0 \leq s \leq v \leq t$, due to Proposition 13, since $\psi \in C_c^\infty$. Therefore, we may use Fubini's theorem once more to deduce that

$$\int_0^t \int_s^t h_s(L_v P_{v,t} \psi) dv ds = \int_0^t H_v(L_v P_{v,t} \psi) dv.$$

Gathering all these terms, we end up with

$$(34) \quad \langle \tilde{\eta}_t, \psi \rangle = \int_0^t h_s(P_{s,t} \psi) ds.$$

We are now ready to finish this proof. Equality (34) together with Proposition 12 applied with $k = 4$ and $q = p$ and Proposition 13 imply that for all $t \leq T$ and all $\psi \in C_c^\infty$,

$$| \langle \tilde{\eta}_t, \psi \rangle | \leq C_T \|\psi\|_{4,p} \|f\|_{4,p} \int_0^t \|\tilde{\eta}_s\|_{-4,p} ds.$$

Since C_c^∞ is dense in $\mathcal{W}_0^{4,p}$, this implies $\|\tilde{\eta}_t\|_{-4,p} \leq C_T \int_0^t \|\tilde{\eta}_s\|_{-4,p} ds$, and Gronwall's lemma implies $\|\tilde{\eta}_t\|_{-4,p} = 0$ for all $t \geq 0$. \blacksquare

4. DECOMPOSITION OF THE FLUCTUATIONS

We now turn to the second main part of this paper and propose a first decomposition of the fluctuation measure η^N for a fixed system size N . The following purely discontinuous martingale, defined for any measurable bounded test function φ , will play a key role in our study.

$$(35) \quad W_t^N(\varphi) = \frac{1}{\sqrt{N}} \sum_{i=1}^N \int_{[0,t] \times \mathbb{R}_+} \varphi(X_{s-}^{N,i}) \mathbf{1}_{\{z \leq f(X_{s-}^{N,i})\}} \tilde{\pi}^i(ds, dz),$$

where $\tilde{\pi}^i(ds, dz) = \pi^i(ds, dz) - dsdz$ is the compensated Poisson random measure. Clearly, $(W_t^N(\varphi))_{t \geq 0}$ is a real valued martingale with angle bracket given by

$$(36) \quad \langle W^N(\varphi) \rangle_t = \int_0^t \mu_s^N(f\varphi^2) ds.$$

We obtain the following first decomposition of $\eta_t^N(\varphi)$, for sufficiently smooth test functions φ .

Proposition 14. *Grant Assumptions 1, 2 and 3. Then for any test function $\varphi \in C_b^2$ and $t \geq 0$,*

$$(37) \quad \eta_t^N(\varphi) = \eta_0^N(\varphi) + \int_0^t \eta_s^N(L_s\varphi) ds + W_t^N(S\varphi) \\ + h \int_0^t \mu_{s-}^N(\varphi') dW_s^N(1) + h \int_0^t \eta_s^N(f)\mu_s^N(\varphi') ds + R_t^{N,1}(\varphi),$$

where the remainder term is given by

$$(38) \quad R_t^{N,1}(\varphi) = \frac{h}{N^{3/2}} \\ \sum_{i=1}^N \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(X_{s-}^{N,i})\}} \left(\left[\sum_{j=1, j \neq i}^N \int_0^1 (\varphi'(X_{s-}^{N,j} + \vartheta \frac{h}{N}) - \varphi'(X_{s-}^{N,j})) d\vartheta \right] - \varphi'(X_{s-}^{N,i}) \right) \pi^i(ds, dz).$$

Proof. Using Taylor's formula at order two, we obtain for any $\varphi \in C_b^2$,

$$\begin{aligned} \mu_t^N(\varphi) &= \mu_0^N(\varphi) - \alpha \int_0^t \mu_s^N(\varphi' \cdot x) ds + \frac{1}{\sqrt{N}} W_t^N(S\varphi) + \int_0^t \mu_s^N(fS\varphi) ds \\ &\quad + \frac{h}{\sqrt{N}} \int_{[0,t]} \mu_{s-}^N(\varphi') dW_s^N(1) + \int_0^t \mu_s^N(f) h \mu_s^N(\varphi') ds + \frac{1}{\sqrt{N}} R_t^{N,1}(\varphi) \\ &= \mu_0^N(\varphi) + \int_0^t \mu_s^N(L_s\varphi) ds + \frac{1}{\sqrt{N}} W_t^N(S\varphi) \\ &\quad + \frac{h}{\sqrt{N}} \int_0^t \mu_{s-}^N(\varphi') dW_s^N(1) + h \int_0^t (\mu_s^N(f) - g_s(f)) \mu_s^N(\varphi') ds + \frac{1}{\sqrt{N}} R_t^{N,1}(\varphi). \end{aligned}$$

In the above development we have used that

$$\frac{1}{N} \sum_{i=1}^N \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(X_{s-}^{N,i})\}} \left(\frac{h}{N} \sum_{j=1}^N \varphi'(X_{s-}^{N,j}) \right) \tilde{\pi}^i(ds, dz) = \frac{h}{\sqrt{N}} \int_0^t \mu_{s-}^N(\varphi') dW_s^N(1).$$

Using that $g_t(\varphi) = g_0(\varphi) + \int_0^t g_s(L_s\varphi) ds$, we obtain the result. ■

We now give estimates of the terms $\eta^N, W^N, R^{N,1}$ appearing in (37) above, interpreted as elements of $\mathcal{W}_0^{-k,p}$, for the smallest possible k, p . This will be useful later to deduce the tightness of these processes.

Proposition 15. *Grant Assumptions 1, 2 and 3. Then for any $p > 1/2$ and any $T > 0$,*

$$\sup_{t \leq T} \sup_N \mathbb{E}(\|\eta_t^N\|_{-2,p}) < \infty.$$

Remark 16. *We stress that we obtain a weaker result than the corresponding Proposition 3.5 in [12] or Proposition 4.7 in [3] since we are not able to control the expectation of the square of the norm $\mathbb{E}(\|\eta_t^N\|_{-2,p}^2)$. This is due to two facts.*

Fact 1. *We are working in the framework of point processes, not of diffusions. Therefore, the control*

$$\mathbb{E}|\bar{X}_t^i - X_t^{N,i}| \leq C_T N^{-1/2}$$

given in (4) cannot be improved to higher order moments of the strong error as in [12]. This intrinsic difficulty is common to any study of point processes.

Fact 2. *Julien Chevallier in [3] proposes to remediate this difficulty by considering rather higher order moments of the total variation distance; that is, proving and exploiting the fact that*

$$\mathbb{P}(\|Z^{N,i} - \bar{Z}^i\|_{TV,[0,T]} \neq 0 \text{ for all } 1 \leq i \leq k) \leq C_T N^{-k/2}.$$

However, in our model, even on $\{\|Z^{N,i} - \bar{Z}^i\|_{TV,[0,T]} = 0\}$, the two processes do not couple since they are driven by two different drift terms. This is a crucial difference with the age-structured Hawkes process where the drift is always $\equiv 1$, independently of anything else (compare more precisely to (A.10) of [3]). It is for the same reason that we have to take test functions that are twice continuously differentiable, such that we work in $\mathcal{W}_0^{-2,p}$.

Corollary 17. *Since $f \in C^{2,\alpha} \subset \mathcal{W}_0^{2,p}$ for any $p > \alpha + \frac{1}{2}$, such that $|\eta_t^N(f)| \leq \|\eta_t^N\|_{-2,p} \|f\|_{2,p}$, we deduce from Proposition 15 the useful upper bound*

$$(39) \quad \sup_{t \leq T} \sup_N \mathbb{E}(|\eta_t^N(f)|) < \infty.$$

Proof of Proposition 15. Let $\bar{X}^i, 1 \leq i \leq N$, be independent copies of the limit system (3), driven by the same Poisson random measures as $X^{N,i}, 1 \leq i \leq N$, and starting from the same initial positions $X_0^i, 1 \leq i \leq N$, as the finite system. We decompose, for any $\psi \in \mathcal{W}_0^{2,p}$,

$$\eta_t^N(\psi) = \sqrt{N} \left(\frac{1}{N} \sum_{i=1}^N [\psi(X_t^{N,i}) - \psi(\bar{X}_t^i)] + [\psi(\bar{X}_t^i) - \mathbb{E}(\psi(\bar{X}_t^i))] \right) =: \eta_t^{N,1}(\psi) + \eta_t^{N,2}(\psi),$$

such that $\|\eta_t^N\|_{-2,p} \leq \|\eta_t^{N,1}\|_{-2,p} + \|\eta_t^{N,2}\|_{-2,p}$.

Step 1. We take an orthonormal basis $(\psi_k)_k$ composed of C_c^∞ -functions of $\mathcal{W}_0^{2,p}$ such that

$$\|\eta_t^{N,2}\|_{-2,p}^2 = \sum_k \langle \eta_t^{N,2}, \psi_k \rangle^2.$$

Using the independence of the $\bar{X}^i, i \geq 1$, we have

$$\mathbb{E}(\langle \eta_t^{N,2}, \psi_k \rangle^2) = \mathbb{E}\left(\frac{1}{N} \sum_{i=1}^N [\psi_k(\bar{X}_t^i) - \mathbb{E}(\psi_k(\bar{X}_t^i))]^2\right) = \mathbb{E}([\psi_k(\bar{X}_t^1) - \mathbb{E}(\psi_k(\bar{X}_t^1))]^2) \leq \mathbb{E}(\psi_k^2(\bar{X}_t^1)).$$

Observing that $\mathbb{E}(\psi_k^2(\bar{X}_t^1)) = \mathbb{E}(\langle \delta_{\bar{X}_t^1}, \psi_k \rangle^2)$, we obtain by monotone convergence

$$\begin{aligned} \mathbb{E}\|\eta_t^{N,2}\|_{-2,p}^2 &= \mathbb{E}\left(\sum_k \langle \eta_t^{N,2}, \psi_k \rangle^2\right) = \sum_k \mathbb{E}(\langle \eta_t^{N,2}, \psi_k \rangle^2) \leq \sum_k \mathbb{E}(\psi_k^2(\bar{X}_t^1)) \\ &= \sum_k \mathbb{E}(\langle \delta_{\bar{X}_t^1}, \psi_k \rangle^2) = \mathbb{E}\left(\sum_k \langle \delta_{\bar{X}_t^1}, \psi_k \rangle^2\right) = \mathbb{E}\|\delta_{\bar{X}_t^1}\|_{-2,p}^2. \end{aligned}$$

Thanks to (17) together with (61), we have that $\|\delta_{\bar{X}_t^1}\|_{-2,p} \leq C(1 + |\bar{X}_t^1|^p) \leq \bar{C}_T$, where we have used the a priori estimate (22). As a consequence,

$$\sup_{t \leq T} \sup_N \mathbb{E}\|\eta_t^{N,2}\|_{-2,p} \leq C_T \|\psi\|_{2,p}.$$

Step 2. We now study the first term. For any $\psi \in \mathcal{W}_0^{2,p}$, using that for any $x, y \geq 0$, by Taylor's formula and the Sobolev embedding,

$$|\psi(x) - \psi(y)| \leq C \|\psi\|_{C^{1,p}} (1 + |x|^p + |y|^p) |x - y| \leq C \|\psi\|_{2,p} (1 + |x|^p + |y|^p) |x - y|,$$

we obtain, using the upper bound (22),

$$|\psi(X_t^{N,i}) - \psi(\bar{X}_t^i)| \leq \bar{C}_T \|\psi\|_{2,p} (1 + |X_t^{N,i}|^p) |X_t^{N,i} - \bar{X}_t^i|,$$

such that

$$\|\eta_t^{N,1}\|_{-2,p} = \sup_{\psi: \|\psi\|_{2,p}=1} |\eta_t^{N,1}(\psi)| \leq \bar{C}_T \frac{1}{\sqrt{N}} \sum_{i=1}^N (1 + |X_t^{N,i}|^p) |X_t^{N,i} - \bar{X}_t^i|.$$

Recall the set G_T^N introduced in (20) above. On the set G_T^N , using (21), we have that $\sup_{t \leq T} |X_t^{N,i}|^p \leq C_T$, whence

$$\mathbb{E}(\|\eta_t^{N,1}\|_{-2,p} \mathbf{1}_{G_T^N}) \leq \frac{C_T}{\sqrt{N}} \sum_{i=1}^N \mathbb{E}(|X_t^{N,i} - \bar{X}_t^i|).$$

We then deduce from (4) that

$$\mathbb{E}(\|\eta_t^{N,1}\|_{-2,p} \mathbf{1}_{G_T^N}) \leq C_T.$$

Moreover, on $(G_T^N)^c$, we simply upper bound, using once more (22),

$$\|\eta_t^{N,1}\|_{-2,p} \leq \bar{C}_T \frac{1}{\sqrt{N}} \sum_{i=1}^N (1 + |X_t^{N,i}|^p) (|X_t^{N,i} - \bar{X}_t^i|) \leq C_T \frac{1}{\sqrt{N}} \sum_{i=1}^N (1 + |X_t^{N,i}|^{p+1}),$$

where we recall that constants may change from one appearance to another and where we have used that $(1 + x^p)(1 + x) \leq C(1 + x^{p+1})$, for a suitable constant. Therefore,

$$\mathbb{E}(\|\eta_t^{N,1}\|_{-2,p} \mathbf{1}_{(G_T^N)^c}) \leq C_T \sqrt{N} \mathbb{E}((1 + |X_t^{N,i}|^{p+1}) \mathbf{1}_{(G_T^N)^c}).$$

Using (19) with $2(p+1)$ and the Cauchy-Schwarz inequality together with $\mathbb{P}((G_T^N)^c) \leq ac_1 e^{-c_2 NT}$, this gives

$$\mathbb{E}(\|\eta_t^{N,1}\|_{-2,p}; (G_T^N)^c) \leq C_T \sqrt{N} e^{-(c_2/2)NT}.$$

All in all we therefore get

$$\sup_N \sup_{t \leq T} \mathbb{E}\|\eta_t^{N,1}\|_{-2,p} \leq C_T,$$

and this concludes the proof. \blacksquare

Proposition 18. *Under Assumptions 1, 2 and 3, for any $p > 1/2$, the process W_t^N is a $(\mathcal{F}_t)_{t \geq 0}$ -martingale with paths in $D(\mathbb{R}_+, \mathcal{W}_0^{-1,p})$ almost surely. Furthermore,*

$$(40) \quad \sup_N \mathbb{E} \left(\sup_{t \leq T} \|W_t^N\|_{-1,p}^2 \right) < \infty.$$

Proof. Take an orthonormal basis $(\psi_k)_{k \geq 1}$ of $\mathcal{W}_0^{1,p}$, composed of C_c^∞ -functions, and use that

$$\sup_{t \leq T} \|W_t^N\|_{-1,p}^2 = \sup_{t \leq T} \sum_{k \geq 1} (W_t^N(\psi_k))^2 \leq \sum_k \sup_{t \leq T} (W_t^N(\psi_k))^2.$$

As a consequence, by Doob's inequality and monotone convergence, and relying on (36),

$$\begin{aligned} \mathbb{E}(\sup_{t \leq T} \|W_t^N\|_{-1,p}^2) &\leq 4 \sum_k \mathbb{E}(W_T^N(\psi_k))^2 = 4 \sum_k \mathbb{E} \int_0^T \mu_s^N(f\psi_k^2) ds = 4 \sum_k \mathbb{E} \int_0^T f(X_s^{N,1}) \psi_k^2(X_s^{N,1}) ds \\ &= 4 \mathbb{E} \int_0^T f(X_s^{N,1}) \sum_k \psi_k^2(X_s^{N,1}) ds = 4 \mathbb{E} \int_0^T f(X_s^{N,1}) \|\delta_{X_s^{N,1}}\|_{-1,p}^2 ds, \end{aligned}$$

where we have used the exchangeability of the finite system to obtain the last term of the first line.

By Lemma 7, there exists a constant not depending on $X_s^{N,1}$ such that $\|\delta_{X_s^{N,1}}\|_{-1,p}^2 \leq C(1 + |X_s^{N,1}|^{2p})$. Moreover, $f(X_s^{N,1}) \leq C(1 + |X_s^{N,1}|^\alpha)$. Using (19) with $2p + \alpha$, this implies (40).

Once (40) is checked, the remainder of the proof follows the lines of the proof of Proposition 4.7, item (ii) of [3]. ■

We now check that

Proposition 19. *Grant Assumptions 1, 2 and 3. For all $p > 1/2$ we have*

$$\sup_N \sqrt{N} \mathbb{E}(\sup_{t \leq T} \|R_t^{N,1}\|_{-3,p}) < \infty.$$

Proof. Let $\psi \in \mathcal{W}_0^{3,p}$ and recall that, by Taylor's formula and the Sobolev embedding, for all $\vartheta \in [0, 1]$,

$$|\psi'(x + \vartheta \frac{h}{N}) - \psi'(x)| \leq C \|\psi\|_{C^{2,p}} (1 + x^p) \frac{h}{N} \leq C \|\psi\|_{3,p} (1 + x^p) \frac{h}{N}.$$

Therefore,

$$\sqrt{N} |R_t^{N,1}(\psi)| \leq C \|\psi\|_{3,p} \frac{h}{N} \sum_{i=1}^N \int_{[0,T] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(X_{s-}^{N,i})\}} \left(\frac{h}{N} \sum_{j=1}^N (1 + |X_{s-}^{N,j}|^p) + |X_{s-}^{N,i}|^p \right),$$

where we have also used that $|\psi'(x)| \leq C \|\psi\|_{C^{1,b}} (1 + |x|^p) \leq C \|\psi\|_{3,p} (1 + |x|^p)$. This implies

$$\begin{aligned} \sqrt{N} \sup_{t \leq T} \|R_t^{N,1}\|_{-3,p} &= \sup_{t \leq T} \sup_{\psi: \|\psi\|_{3,p}=1} \sqrt{N} |R_t^{N,1}(\psi)| \leq \\ &C \frac{h}{N} \sum_{i=1}^N \int_{[0,T] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(X_{s-}^{N,i})\}} \left(\frac{h}{N} \sum_{j=1}^N (1 + |X_{s-}^{N,j}|^p) + |X_{s-}^{N,i}|^p \right). \end{aligned}$$

Taking expectation and using the a priori bound (19) together with $f(x) \leq C(1 + x^\alpha)$ yields the result. ■

Finally, recall that $D : \psi \rightarrow \psi'$ denotes the differential operator, and D^* the associated dual.

Lemma 20. *Fix $p > 1/2$. Then under Assumptions 1, 2 and 3, the mapping defined by $\psi \mapsto \eta_s^N(f) D^* \mu_s^N(\psi)$ is almost surely continuous from $\mathcal{W}_0^{2,p} \rightarrow \mathbb{R}$ and satisfies*

$$\sup_N \sup_{t \leq T} \mathbb{E}(\|\eta_t^N(f) D^* \mu_t^N\|_{-2,p}) < \infty.$$

Proof. The result follows from

$$|\eta_t^N(f) D^* \mu_t^N(\psi)| \leq |\eta_t^N(f)| \frac{1}{N} \sum_{i=1}^N |\psi'(X_t^{N,i})| \leq C \|\psi\|_{2,p} |\eta_t^N(f)| \frac{1}{N} \sum_{i=1}^N (1 + |X_t^{N,i}|^p).$$

The conclusion is then similar as in the proof of Proposition 15.

■

We now turn to the study of the last stochastic integral appearing in (37).

Proposition 21. *For any $p > 1/2$, the process $\int_0^t D^* \mu_{s-}^N dW_s^N(1)$ is a $(\mathcal{F}_t)_{t \geq 0}$ -martingale with paths in $D(\mathbb{R}_+, \mathcal{W}_0^{-2,p})$ almost surely. Furthermore,*

$$(41) \quad \sup_N \mathbb{E} \left(\sup_{t \leq T} \left\| \int_0^t D^* \mu_{s-}^N dW_s^N(1) \right\|_{-2,p}^2 \right) < \infty.$$

Proof. The proof is similar to the proof of Proposition 15. As there, we take an orthonormal basis $(\psi_k)_{k \geq 1}$, now of $\mathcal{W}_0^{2,p}$, composed of C_c^∞ -functions. We have

$$\sup_{t \leq T} \left\| \int_0^t D^* \mu_{s-}^N dW_s^N(1) \right\|_{-2,p}^2 = \sup_{t \leq T} \sum_{k \geq 1} \left(\int_0^t \mu_s^N(\psi'_k) dW_s^N(1) \right)^2 \leq \sum_{k \geq 1} \sup_{t \leq T} \left(\int_0^t \mu_s^N(\psi'_k) dW_s^N(1) \right)^2.$$

Applying first Doob's and then Jensen's inequality and finally monotone convergence,

$$\begin{aligned} \mathbb{E} \left(\sup_{t \leq T} \left\| \int_0^t D^* \mu_{s-}^N dW_s^N(1) \right\|_{-2,p}^2 \right) &\leq 4 \sum_k \mathbb{E} \int_0^T \mu_s^N(f) (\mu_s^N(\psi'_k))^2 ds \leq 4 \sum_k \mathbb{E} \int_0^T \mu_s^N(f) \mu_s^N((\psi'_k)^2) ds \\ &= 4 \mathbb{E} \int_0^T \mu_s^N(f) \left[\frac{1}{N} \sum_{i=1}^N \sum_k (\psi'_k)^2(X_s^{N,i}) \right] ds. \end{aligned}$$

Now we rely on Lemma 7 and use that

$$\|\delta_x \circ D\|_{-2,p}^2 = \sum_k (\delta_x \circ D(\psi_k))^2 = \sum_k (\psi'_k(x))^2$$

to identify

$$\sum_k (\psi'_k)^2(X_s^{N,i}) = \|\delta_{X_s^{N,i}} \circ D\|_{-2,p}^2 \leq C(1 + |X_s^{N,i}|^{2p})$$

such that

$$\mathbb{E} \left(\sup_{t \leq T} \left\| \int_0^t D^* \mu_{s-}^N dW_s^N(1) \right\|_{-2,p}^2 \right) \leq C \mathbb{E} \int_0^T \mu_s^N(f) (1 + \mu_s^N(|\cdot|^{2p})) ds,$$

which, together with our a priori estimate (19), using similar arguments as in the end of the proof of Proposition 18, allows to conclude that

$$\mathbb{E} \left(\sup_{t \leq T} \left\| \int_0^t D^* \mu_{s-}^N dW_s^N(1) \right\|_{-2,p}^2 \right) \leq C_T.$$

The remainder of the assertion follows once more along the lines of the proof of item (ii) of Proposition 4.7 in [3] ■

To close this section, we state the following

Lemma 22. *Under Assumptions 1, 2 and 3, for any $p > \frac{1}{2}$, the integrals*

$$\int_0^t L_s^* \eta_s^N ds \text{ and } \int_0^t \eta_s^N(f) D^* \mu_s^N ds$$

(where L_s^* and D^* denote the dual operators of L_s and of D) are almost surely well defined as Bochner integrals in $\mathcal{W}_0^{-3,p}$. Furthermore, $t \mapsto \int_0^t L_s^* \eta_s^N ds$ and $t \mapsto \int_0^t \eta_s^N(f) D^* \mu_s^N ds$ are almost surely strongly continuous in $\mathcal{W}_0^{-3,p}$.

The proof of the above lemma is sketched in the beginning of the proof of Proposition 3.5 in [12].

Resuming what we have done so far, we conclude that

Proposition 23. *Grant Assumptions 1, 2 and 3. Then for any $p > \alpha + \frac{1}{2}$, we have the decomposition in $\mathcal{W}_0^{-3,p}$*

$$(42) \quad \eta_t^N = \eta_0^N + \int_0^t L_s^* \eta_s^N ds + S^* W_t^N + h \int_0^t D^* \mu_{s-}^N dW_s^N(1) + h \int_0^t \eta_s^N(f) D^* \mu_s^N ds + R_t^{N,1},$$

where S^* denotes the dual operator of $S : \psi \mapsto \psi(0) - \psi(\cdot)$ and where $R_t^{N,1}$ is given in (38).

Moreover,

$$(43) \quad \sup_N \mathbb{E}(\sup_{t \leq T} \|\eta_t^N\|_{-3,p}) < \infty$$

and $t \mapsto \eta_t^N$ belongs to $D(\mathbb{R}_+, \mathcal{W}_0^{-3,p})$ almost surely. In particular,

$$(44) \quad \sup_N \mathbb{E}(\sup_{t \leq T} |\eta_t^N(f)|) < \infty.$$

Remark 24. *The above decomposition is stated in $\mathcal{W}_0^{-3,p}$ for any $p > \alpha + \frac{1}{2}$. This lower bound comes from the fact that we have to apply η_t^N to the jump rate function f which belongs to $C^{4,\alpha}$, hence to $C^{3,\alpha} \subset \mathcal{W}_0^{3,p}$ under the condition $p > \alpha + \frac{1}{2}$.*

Proof. Decomposition (42) follows from our previous results Proposition 14–21. It implies that

$$\begin{aligned} \sup_{t \leq T} \|\eta_t^N\|_{-3,p} &\leq \|\eta_0^N\|_{-3,p} + \int_0^T \|L_s^* \eta_s^N\|_{-3,p} ds + |h| \int_0^T \|\eta_s^N(f) D^* \mu_s^N\|_{-3,p} ds + \sup_{t \leq T} \|S^* W_t^N\|_{-3,p} \\ &\quad + |h| \sup_{t \leq T} \left\| \int_0^t D^* \mu_{s-}^N dW_s^N(1) \right\|_{-3,p} + \sup_{t \leq T} \|R_t^{N,1}\|_{-3,p}. \end{aligned}$$

We know by Lemma 6 that

$$\mathbb{E} \int_0^T \|L_s^* \eta_s^N\|_{-3,p} ds \leq C_T \sup_{s \leq T} \mathbb{E} \|\eta_s^N\|_{-2,p+\alpha}$$

which is finite by Proposition 15. Moreover, by Lemma 20,

$$\mathbb{E} \int_0^T \|\eta_s^N(f) D^* \mu_s^N\|_{-3,p} ds < \infty.$$

By continuity of the application S , the stochastic integral terms have already been treated in Propositions 18 and 21 and the remainder term in Proposition 19 such that the conclusion follows. The proof of the fact that almost surely $t \mapsto \eta_t^N$ belongs to $D(\mathbb{R}_+, \mathcal{W}_0^{-3,p})$ is analogous to the proof of Proposition 4.10 in [3]. Finally we use that $|\eta_t^N(f)| \leq \|\eta_t^N\|_{-3,p} \|f\|_{3,p}$ to deduce (44). ■

5. TIGHTNESS

This section is devoted to the proof of the tightness of the laws of η^N interpreted as stochastic processes with càdlàg paths taking values in $\mathcal{W}_0^{-4,p}$, for some $p > \alpha + \frac{1}{2}$. Although the above decomposition (42) is stated in $\mathcal{W}_0^{-3,p}$, we shall see in Remark 26 below why we have to add one degree of regularity and consider the process as process taking values in the bigger space $\mathcal{W}_0^{-4,p}$.

As it is classically done, we rely on the tightness criterion of Aldous for Hilbert space valued stochastic processes that we quote from [16]. This criterion reads as follows. A sequence $(X^N)_{N \geq 1}$ of processes in $D(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$, defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, is tight if

- (1) For every $t \geq 0$ and every $\varepsilon > 0$ there exists a Hilbert space H_0 such that the embedding $H_0 \hookrightarrow \mathcal{W}_0^{-4,p}$ is Hilbert-Schmidt and such that for all $t \geq 0$,

$$(45) \quad \sup_N \mathbb{E}(\|X_t^N\|_{H_0}) < \infty.$$

- (2) For all $\varepsilon_1, \varepsilon_2 > 0$ and $T \geq 0$ there exist $\delta^* > 0$ and N_0 such that for all $(\mathcal{F}_t)_{t \geq 0}$ -stopping times $\tau_N \leq T$,

$$(46) \quad \sup_{N \geq N_0} \sup_{\delta \leq \delta^*} \mathbb{P}(\|X_{\tau_N + \delta}^N - X_{\tau_N}^N\|_{-4,p} \geq \varepsilon_1) \leq \varepsilon_2.$$

Theorem 25. *Grant Assumptions 1, 2 and 3. Then the sequences of laws of η^N , of W^N and of $\int_0^\cdot D^* \mu_{s-}^N dW_s^N(1)$ are tight in $D(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$, for any $p > \alpha + \frac{1}{2}$.*

Proof. Step 1. We start studying Condition (45). It is satisfied with $H_0 = \mathcal{W}_0^{-2,p+1}$ for η^N as a consequence of Proposition 15 since the embedding $\mathcal{W}_0^{-2,p+1} \hookrightarrow \mathcal{W}_0^{-4,p}$ is of Hilbert-Schmidt type, by Maurin's theorem, see Section 7.1 in the Appendix. For W^N it even holds with $H_0 = \mathcal{W}_0^{-1,p+1}$, by Proposition 18, and for $\int_0^\cdot D^* \mu_{s-}^N dW_s^N(1)$, it follows from Proposition 21, with $H_0 = \mathcal{W}_0^{-2,p+1}$.

Step 2. We now check Condition (46) for W^N . By Rebolledo's theorem (see [16], page 40), it is sufficient to show that it holds for the trace of the processes $\ll W^N \gg$ where each $\ll W^N \gg$ is the linear continuous mapping from $\mathcal{W}_0^{4,p}$ to $\mathcal{W}_0^{-4,p}$ given for all $\varphi_1, \varphi_2 \in \mathcal{W}_0^{4,p}$ by

$$\langle \ll W^N \gg_t(\varphi_1), \varphi_2 \rangle = \int_0^t \mu_s^N(f \varphi_1 \varphi_2) ds.$$

We take an orthonormal basis $(\psi_k)_k$ of $\mathcal{W}_0^{4,p}$. Fix some $\delta^* > 0$. Then for all $\delta \leq \delta^*$,

$$\begin{aligned} & |Tr \ll W^N \gg_{\tau_N+\delta} - Tr \ll W^N \gg_{\tau_N} | \\ &= \left| \sum_k \langle \ll W^N \gg_{\tau_N+\delta} (\psi_k), \psi_k \rangle - \langle \ll W^N \gg_{\tau_N} (\psi_k), \psi_k \rangle \right| \\ &= \sum_k \int_{\tau_N}^{\tau_N+\delta} \mu_s^N(f\psi_k^2) ds = \int_{\tau_N}^{\tau_N+\delta} \sum_k \mu_s^N(f\psi_k^2) ds \\ &= \int_{\tau_N}^{\tau_N+\delta} \frac{1}{N} \sum_{i=1}^N f(X_s^{N,i}) \sum_k \psi_k^2(X_s^{N,i}) ds = \int_{\tau_N}^{\tau_N+\delta} \frac{1}{N} \sum_{i=1}^N f(X_s^{N,i}) \|\delta_{X_s^{N,i}}\|_{-4,p}^2 ds. \end{aligned}$$

By Lemma 7, there exists a constant C with $\|\delta_{X_s^{N,i}}\|_{-4,p}^2 \leq C(1 + |X_s^{N,i}|^{2p})$ such that we may upper bound the above expression by

$$C\delta \frac{1}{N} \sum_{i=1}^N \sup_{s \leq T+\delta} (1 + |X_s^{N,i}|^{2p})(1 + |X_s^{N,i}|^\alpha)$$

having expectation which is upper bounded uniformly in N by $C_T\delta^*$ thanks to our a priori estimates (19). This implies (46) for W^N .

We now turn to the study of Condition (46) for the martingale $M^N := \int_0^\cdot D^* \mu_s^N dW_s^N(1)$. We have

$$\langle \ll M^N \gg_t (\psi_1), \psi_2 \rangle = \int_0^t \mu_s^N(f) \mu_s^N(\psi_1') \mu_s^N(\psi_2') ds$$

such that, using Jensen's inequality,

$$\begin{aligned} & |Tr \ll M^N \gg_{\tau_N+\delta} - Tr \ll M^N \gg_{\tau_N} | \\ &= \left| \sum_k \langle \ll M^N \gg_{\tau_N+\delta} (\psi_k), \psi_k \rangle - \langle \ll M^N \gg_{\tau_N} (\psi_k), \psi_k \rangle \right| \\ &= \sum_k \int_{\tau_N}^{\tau_N+\delta} \mu_s^N(f) (\mu_s^N(\psi_k'))^2 ds \leq \int_{\tau_N}^{\tau_N+\delta} \mu_s^N(f) \sum_k \mu_s^N((\psi_k')^2) ds, \end{aligned}$$

and the conclusion follows similarly.

Finally, using decomposition (42) and the fact that the sequence of laws of S^*W^N (by continuity of S) and of M^N have already been shown to be tight, to show the tightness of η^N , it suffices to check condition (46) for the remaining terms

$$\mathcal{T}_t^N = \eta_0^N + \int_0^t L_s^* \eta_s^N ds + h \int_0^t \eta_s^N(f) D^* \mu_s^N ds + R_t^{N,1}.$$

We have

$$\|\mathcal{T}_{\tau_N+\delta}^N - \mathcal{T}_{\tau_N}^N\|_{-4,p} \leq \delta \sup_{s \leq T+\delta^*} (\|L_s^* \eta_s^N\|_{-4,p} + |h| \|\eta_s^N(f) D^* \mu_s^N\|_{-4,p}) + \frac{1}{\sqrt{N}} \sup_{t \leq T+\delta^*} \|\sqrt{N} R_t^{N,1}\|_{-4,p}.$$

The last term above is controlled thanks to Proposition 19 above, choosing $N \geq N_0$ for N_0 sufficiently large. Using the same arguments as in the proof of Proposition 15, we have

$$\begin{aligned} \sup_{s \leq T+\delta^*} \|\eta_s^N(f) D^* \mu_s^N\|_{-4,p} &\leq C_{T+\delta^*} \sup_{t \leq T+\delta^*} |\eta_t^N(f)| \\ &+ C_T \sqrt{N} \left(\frac{1}{N} \sum_{i=1}^N (1 + \sup_{t \leq T+\delta^*} |X_t^{N,i}|^p) \right) \left(\frac{1}{N} \sum_{i=1}^N (1 + \sup_{t \leq T+\delta^*} |X_t^{N,i}|^\alpha) \right) \mathbf{1}_{(G_T^N)^c}. \end{aligned}$$

Recalling that by (44), $\sup_N \mathbb{E} \sup_{t \leq T+\delta^*} |\eta_t^N(f)| < \infty$, we deduce that

$$\sup_N \mathbb{E} \sup_{s \leq T+\delta^*} \|\eta_s(f) D^* \mu_s^N\|_{-4,p} \leq C_{T+\delta^*}.$$

We conclude the proof recalling that by Lemma 6,

$$(47) \quad \|L_s^* \eta_s^N\|_{-4,p} \leq C_{T+\delta^*} \sup_{s \leq T+\delta^*} \|\eta_s^N\|_{-3,p+\alpha},$$

which, together with (43) implies the assertion. ■

Remark 26. *The above proof relies on the decomposition (42) and on the uniform in time upper bound (43) which have been stated in $\mathcal{W}_0^{-3,p}$. However, the presence of the integral $\int_0^t L_s^* \eta_s^N ds$, canceling one order of derivative and the fact that (43) does only hold in $\mathcal{W}_0^{-3,p}$ imply that we have to work in $\mathcal{W}_0^{-4,p}$ to be able to obtain the tightness of all terms. This is a crucial difference with respect to [12] and [3]. They both use the upper bound*

$$\mathbb{E} \int_{\tau_N}^{\tau_N+\delta} \|L_s^* \eta_s^N\|_{-k,p} ds \leq \delta \mathbb{E} \int_{\tau_N}^{\tau_N+\delta} \|L_s^* \eta_s^N\|_{-k,p}^2 ds$$

and are hence able to use a non-uniform in time bound on the expectation of the square of the operator norm of η_t^N . Since we are not able to control the square of the operator norm within our framework, see Remark 16 above, the prize to pay is to impose one degree of regularity more, as we did here.

Proposition 27. *Under Assumptions 1, 2 and 3, for any $p > 1/2$, the limit laws of η^N , of W^N and of $\int_0^\cdot D^* \mu_{s-}^N dW_s^N(1)$ are supported in $C(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$.*

Proof. Following [2], Theorem 13.4, it suffices to show that the maximal jump size within a fixed time interval converges to 0 almost surely. Let us check this for W_t^N , for $t \in [0, T]$. We have for any $\psi \in \mathcal{W}_0^{4,p}$,

$$(48) \quad \Delta W_t^N(\psi) = \frac{1}{\sqrt{N}} \sum_{i=1}^N \psi(X_{t-}^{N,i}) \Delta Z_t^{N,i}, \quad \text{where } Z_t^{N,i} = \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(X_{s-}^{N,i})\}} \pi^i(ds, dz).$$

As a consequence, for all $\psi \in \mathcal{W}_0^{4,p}$ with $\|\psi\|_{4,p} = 1$, since at each jump time t , only one of the processes $Z^{N,i}$ has a jump,

$$\sup_{t \leq T} |\Delta W_t^N(\psi)| \leq (C/\sqrt{N}) (1 + \sup_{1 \leq i \leq N} \sup_{t \leq T} |X_t^{N,i}|^p).$$

Thanks to Assumption 3, using (18), $\sup_{1 \leq i \leq N} \sup_{t \leq T} |X_t^{N,i}|^p \leq C(1 + |N_T^N|^p)$. By the strong law of large numbers, almost surely,

$$\lim_{N \rightarrow \infty} |N_T^N|^p = T^p f(2h)^p < \infty$$

such that almost surely,

$$\lim_{N \rightarrow \infty} \sup_{t \leq T} |\Delta W_t^N(\psi)| = 0.$$

Similarly, since $\Delta \eta_t^N = \sqrt{N} \Delta \mu_t^N$,

$$\Delta \eta_t^N(\psi) = \frac{1}{\sqrt{N}} \sum_{i=1}^N \left(S \psi(X_{t^-}^{N,i}) + \frac{h}{N} \sum_{j \neq i} \int_0^1 \psi'(X_{t^-}^{N,j} + \theta \frac{h}{N}) d\theta \right) \Delta Z_t^{N,i}$$

and

$$\Delta \int_0^t \mu_{s^-}^N(\psi') dW_s^N(1) = \frac{1}{\sqrt{N}} \sum_{i=1}^N \left(\frac{1}{N} \sum_j \psi'(X_{t^-}^{N,j}) \right) \Delta Z_t^{N,i},$$

and these terms are treated analogously. ■

We close this section with the following

Theorem 28. *Grant Assumptions 1, 2 and 3. Then the sequence of laws of $(U^{N,i})_{i \geq 1}$ is tight in $D(\mathbb{R}_+, \mathbb{R})^{\mathbb{N}^*}$.*

Proof. Step 1. We first prove that for any fixed $n \geq 1$, the sequence of laws of $(U^{N,i})_{1 \leq i \leq n}$ is tight in $D(\mathbb{R}_+, \mathbb{R}^n)$. To do so, we rely once more on the criterion of Aldous, now stated for \mathbb{R}^n -valued processes having càdlàg paths, see Jacod and Shiryaev [15, Theorem VI. 4.5 page 356]. More precisely, writing for short $U^N = (U^{N,1}, \dots, U^{N,n})$ and $\|u\| = \sum_{i=1}^n |u^i|$ for the L^1 -norm on \mathbb{R}^n , we shall prove that

(a) for all $T > 0$, all $\varepsilon > 0$, $\lim_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \sup_{(S,S') \in A_{\delta,T}} \mathbb{P}(\|U_{S'}^N - U_S^N\| > \varepsilon) = 0$, where $A_{\delta,T}$ is the set of all pairs of stopping times (S, S') such that $0 \leq S \leq S' \leq S + \delta \leq T$ a.s.,

(b) for all $T > 0$, $\lim_{K \uparrow \infty} \sup_N \mathbb{P}(\sup_{t \in [0,T]} \|U_t^N\| \geq K) = 0$.

To show (b), we start with the decomposition

$$(49) \quad U_t^{N,i} = \int_0^t b'(\bar{X}_s^i) U_s^{N,i} ds + h W_t^N(1) + h \int_0^t \eta_s^N(f) ds \\ - \int_{[0,t] \times \mathbb{R}_+} U_{s^-}^{N,i} \mathbf{1}_{\{z \leq f(\bar{X}_{s^-}^i)\}} \pi^i(ds, dz) + R_t^{N,2,i},$$

for all $1 \leq i \leq n$, where

$$(50) \quad R_t^{N,2,i} = \int_0^t \left(\int_0^1 b'(\bar{X}_s^i + \frac{\vartheta}{\sqrt{N}} U_s^{N,i}) - b'(\bar{X}_s^i) d\vartheta \right) U_s^{N,i} ds - \frac{h}{\sqrt{N}} \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(\bar{X}_{s^-}^i)\}} \tilde{\pi}^i(ds, dz) \\ - \sqrt{N} \int_{[0,t] \times \mathbb{R}_+} X_{s^-}^{N,i} \left(\mathbf{1}_{\{z \leq f(X_{s^-}^{N,i})\}} - \mathbf{1}_{\{z \leq f(\bar{X}_{s^-}^i)\}} \right) \pi^i(ds, dz).$$

We first show that (49) implies

$$(51) \quad \sup_N \mathbb{E}(\sup_{s \leq T} \|U_s^N\|) < \infty;$$

once (51) is shown, (b) follows immediately.

To prove (51), notice that (49) implies

$$\begin{aligned} \sup_{s \leq T} \|U_s^N\| &\leq \|b'\|_\infty \int_0^T \|U_s^N\| ds + h n \sup_{s \leq T} |W_s^N(1)| + h n \int_0^T |\eta_s^N(f)| ds \\ &\quad + \sum_{i=1}^n \int_{[0,T] \times \mathbb{R}_+} |U_{s-}^{N,i}| \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz) + \sup_{s \leq T} \sum_{i=1}^n |R_s^{N,2,i}|. \end{aligned}$$

By (4),

$$(52) \quad \sup_N \mathbb{E} \int_0^T \|U_s^N\| ds \leq nTC_T.$$

Moreover, by Burkholder-Davis-Gundy's inequality for discontinuous martingales and (19) once more,

$$\mathbb{E} \sup_{s \leq T} |W_s^N(1)|^2 \leq C \mathbb{E} \int_0^T \mu_s^N(f) ds \leq C_T, \text{ such that } \sup_N \mathbb{E} \sup_{s \leq T} |W_s^N(1)| \leq \sqrt{C_T}.$$

We also use the upper bound $|\eta_s(f)| \leq \|\eta_s^N\|_{-3, \alpha+1} \|f\|_{3, \alpha+1}$ and (44) to deal with the term $hn \int_0^T |\eta_s^N(f)| ds$. Moreover, since for all $s \leq T$ and for all $1 \leq i \leq n$, $\bar{X}_{s-}^i \leq \bar{C}_T$ by (22) such that $f(\bar{X}_{s-}^i) \leq f(\bar{C}_T)$,

$$\sum_{i=1}^n \mathbb{E} \int_{[0,T] \times \mathbb{R}_+} |U_{s-}^{N,i}| \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz) \leq f(\bar{C}_T) \mathbb{E} \int_0^T |U_s^N| ds$$

which is treated using (52).

Finally to deal with $\sup_{t \leq T} |R_t^{N,2,i}|$, the first term appearing in the decomposition of $R^{N,2,i}$ is upper bounded by

$$\|b''\|_\infty N^{-1/2} \int_0^T |U_s^{N,i}| ds,$$

having expectation bounded by C_T/\sqrt{N} .

Concerning the second term appearing in the decomposition of $R^{N,2,i}$, using once more that $f(\bar{X}_{s-}^i) \leq f(\bar{C}_T)$, for all $s \leq T$, we have that

$$\mathbb{E} \left| \frac{h}{\sqrt{N}} \int_{[0,T] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \tilde{\pi}^i(ds, dz) \right| \leq T f(\bar{C}_T) 2h/\sqrt{N}.$$

Moreover, using the set G_T^N introduced in (20) above and the fact that $\sup_{s \leq T} |X_s^{N,i}| \leq C_T$ on G_T^N , we have the upper bound for the last term appearing in the decomposition of $R^{N,2,i}$

$$\begin{aligned} &\sup_{t \leq T} |\sqrt{N} \int_{[0,T] \times \mathbb{R}_+} X_{s-}^{N,i} \left(\mathbf{1}_{\{z \leq f(X_{s-}^{N,i})\}} - \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \right) \pi^i(ds, dz)| \\ &\leq C_T \sqrt{N} \int_{[0,T] \times \mathbb{R}_+} \left| \mathbf{1}_{\{z \leq f(X_{s-}^{N,i})\}} - \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \right| \pi^i(ds, dz) + \\ &\quad \mathbf{1}_{(G_T^N)^c} \sup_{s \leq T} |X_s^{N,i}| \sqrt{N} \int_{[0,T] \times \mathbb{R}_+} \left[\mathbf{1}_{\{z \leq f(X_{s-}^{N,i})\}} + \mathbf{1}_{\{z \leq f(\bar{C}_T)\}} \right] \pi^i(ds, dz). \end{aligned}$$

The first line of the rhs is treated using (24), the second using the a priori estimates (19) and the deviation estimate on $\mathbb{P}((G_T^N)^c)$. All in all this implies

$$\sum_{i=1}^n \sup_N \mathbb{E} \sup_{t \leq T} |\sqrt{N} \int_{[0,t] \times \mathbb{R}_+} X_{s-}^{N,i} \left(\mathbf{1}_{\{z \leq f(X_{s-}^{N,i})\}} - \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \right) \pi^i(ds, dz)| \leq C_T < \infty,$$

and we have just finished the proof of (51).

We finish this step with the observation that $\sup_{t \leq T} |R_t^{N,2,i}|$ converges to 0 in probability, as $N \rightarrow \infty$, for any $1 \leq i \leq n$. We only need to consider

$$\begin{aligned} \sup_{t \leq T} |\sqrt{N} \int_{[0,t] \times \mathbb{R}_+} X_{s-}^{N,i} (\mathbf{1}_{\{z \leq f(X_{s-}^{N,i})\}} - \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}}) \pi^i(ds, dz)| \\ \leq \sqrt{N} \sup_{t \leq T} |X_t^{N,i}| \|Z^{N,i} - \bar{Z}^i\|_{TV,[0,T]}, \end{aligned}$$

such that for any $\varepsilon > 0$,

$$(53) \quad \mathbb{P}(\sup_{t \leq T} |\sqrt{N} \int_{[0,t] \times \mathbb{R}_+} X_{s-}^{N,i} (\mathbf{1}_{\{z \leq f(X_{s-}^{N,i})\}} - \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}}) \pi^i(ds, dz)| \geq \varepsilon) \\ \leq \mathbb{P}(\|Z^{N,i} - \bar{Z}^i\|_{TV,[0,T]} \geq 1) \leq \mathbb{E}\|Z^{N,i} - \bar{Z}^i\|_{TV,[0,T]} \leq C_T N^{-1/2} \rightarrow 0$$

as $N \rightarrow \infty$, where we have used once more (24).

Finally, (a) follows from the fact that (49) implies

$$\begin{aligned} \|U_{S'}^N - U_S^N\| \leq C\delta \left(\sup_{s \leq T} \|U_s^N\| + n \sup_{s \leq T} |\eta_s(f)| \right) \\ + \sum_{i=1}^n \int_{[S,S'] \times \mathbb{R}_+} |U_{s-}^{N,i} \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz) + \sup_{t \leq T} \sum_{i=1}^n |R_t^{N,2,i}| \\ + hn |W_{S'}^N(1) - W_S^N(1)|. \end{aligned}$$

The first line of the rhs is treated using (51) and (44). To deal with the second line we use that

$$\sum_{i=1}^n \mathbb{E} \int_{[S,S'] \times \mathbb{R}_+} |U_{s-}^{N,i} \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz) \leq f(\bar{C}_T) \mathbb{E} \int_S^{S'} \|U_s^N\| ds \leq f(\bar{C}_T) \delta \mathbb{E} \sup_{s \leq T} \|U_s^N\|$$

and that $\sup_{t \leq T} \sum_{i=1}^n |R_t^{N,2,i}|$ converges to 0 in probability. Finally,

$$\mathbb{E} |W_{S'}^N(1) - W_S^N(1)|^2 = \mathbb{E} \int_S^{S'} \mu_s^N(f) ds \leq \delta \mathbb{E} \sup_{s \leq T} f(X_s^{N,1}) = C_T \delta.$$

This concludes the proof of the fact that the sequence of laws of $(U^{N,i})_{1 \leq i \leq n}$ is tight in $D(\mathbb{R}_+, \mathbb{R}^n)$.

Step 2. As a consequence of the first step we obtain the weaker result that for all $n \geq 1$, the sequence of laws of $(U^{N,i})_{1 \leq i \leq n}$ is tight in $D(\mathbb{R}_+, \mathbb{R})^n$. As a consequence, the sequence of laws of $(U^{N,i})_{i \geq 1}$ is tight in $D(\mathbb{R}_+, \mathbb{R})^{\mathbb{N}^*}$. ■

6. CHARACTERIZATION OF THE LIMIT

In this section we study the possible limits of the sequence of η^N . Recall the definition of W in (16). We start with the following preliminary result.

Proposition 29. *Grant Assumptions 1, 2 and 3. Then for any $p > \alpha + \frac{1}{2}$, the sequence of processes W^N converges in $D(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$ to W .*

Proof. We have already proven the tightness of W^N . To identify any possible limit, consider, for any $\psi_1, \psi_2 \in \mathcal{W}_0^{4,p}$, the difference

$$\langle \ll W^N \gg_t(\psi_1), \psi_2 \rangle - \int_0^t \langle g_s, \psi_1 \psi_2 f \rangle ds = \int_0^t \langle \mu_s^N - g_s, \psi_1 \psi_2 f \rangle ds.$$

We have that

$$\mathbb{E} \left| \int_0^t \langle \mu_s^N - g_s, \psi_1 \psi_2 f \rangle ds \right| = \frac{1}{\sqrt{N}} \mathbb{E} \left| \int_0^t \langle \eta_s^N, \psi_1 \psi_2 f \rangle ds \right| \rightarrow 0$$

as $N \rightarrow \infty$, where this last convergence follows from Proposition 23 and Sobolev's embedding theorem. More precisely, since $f \in C^{3,\alpha}$ and $\psi_1 \in \mathcal{W}_0^{4,p} \subset C^{3,p}$, we have $\psi_1 \psi_2 f \in \mathcal{W}_0^{3,2p+\alpha}$ such that

$$|\langle \eta_s^N, \psi_1 \psi_2 f \rangle| \leq \sup_{s \leq t} \|\eta_s^N\|_{3,2p+\alpha} \|\psi_1 \psi_2 f\|_{3,2p+\alpha}.$$

Moreover we have already shown that the maximal jump size of W^N converges to 0 almost surely. Then the result follows from Rebolledo's central limit theorem for local martingales, following the lines of the proof of Prop. 5.3 in [3]. ■

Coming back to the decomposition of η^N in (42), we see that we need to consider the joint convergence of S^*W^N and $\int_0^\cdot \mu_{s-}^N dW_s^N(1)$ since both martingales depend on the same underlying Poisson noise.

Proposition 30. *Grant Assumptions 1, 2 and 3 and fix $p > \alpha + \frac{1}{2}$. Then we have convergence in law in $D(\mathbb{R}_+, \mathcal{W}_0^{-4,p} \times \mathcal{W}_0^{4,p})$ of $(S^*W^N, \int_0^\cdot D^* \mu_{s-}^N dW_s^N(1))$ to the limit process*

$$(S^*W, \int_0^\cdot D^* g_s dW_s(1)).$$

Proof. We have already shown the tightness of $(S^*W^N, \int_0^\cdot D^* \mu_{s-}^N dW_s^N(1))$, and we know that we have convergence in law $W^N \rightarrow W$. To prove the above convergence we first decompose

$$\int_0^\cdot D^* \mu_{s-}^N dW_s^N(1) = \int_0^\cdot D^* g_s dW_s^N(1) + R^{N,3},$$

where

$$(54) \quad R^{N,3} = \int_0^\cdot D^*(\mu_{s-}^N - g_s) dW_s^N(1).$$

Step 1. We show that $\mathbb{E}[\sup_{t \leq T} \|R_t^{N,3}\|_{-4,p}^2] \rightarrow 0$ as $N \rightarrow \infty$. For that sake, let $(\psi_k)_k$ be an orthonormal basis of $\mathcal{W}_0^{4,p}$, composed of C_c^∞ -functions. We have that

$$\sup_{t \leq T} \|R_t^{N,3}\|_{-4,p}^2 \leq \sum_k \sup_{t \leq T} \left(\int_0^t (\mu_{s-}^N - g_s)(\psi_k') dW_s^N(1) \right)^2,$$

such that

$$\mathbb{E} \sup_{t \leq T} \|R_t^{N,3}\|_{-4,p}^2 \leq 4 \sum_k \mathbb{E} \int_0^T \mu_s^N(f) [(\mu_s^N - g_s)(\psi_k')]^2 ds \leq C \mathbb{E} \int_0^T \mu_s^N(f) \|D^*(\mu_s^N - g_s)\|_{-4,p}^2 ds.$$

On G_T^N , we upper bound

$$\|D^*(\mu_s^N - g_s)\|_{-4,p}^2 \leq C \|\mu_s^N - g_s\|_{-3,p}^2 \leq C \|\mu_s^N - g_s\|_{-3,p} \sup_{s \leq T} (\|\mu_s^N\|_{-3,p} + \|g_s\|_{-3,p})$$

and use that on G_T^N , $\sup_{s \leq T} \mu_s^N(f)$ ($\|\mu_s^N\|_{-3,p} + \|g_s\|_{-3,p}$) $\leq C_T$ such that

$$\mathbb{E} \left(\sup_{t \leq T} \|R_t^{N,3}\|_{-4,p}^2 \mathbf{1}_{G_T^N} \right) \leq C_T \mathbb{E} \int_0^T \|\mu_s^N - g_s\|_{-3,p} ds = \frac{C_T}{\sqrt{N}} \mathbb{E} \int_0^T \|\eta_s^N\|_{-3,p} ds,$$

which converges to 0 as $N \rightarrow \infty$ thanks to Proposition 15. Moreover, on $(G_T^N)^c$, we upper bound

$$\|D^*(\mu_s^N - g_s)\|_{-4,p}^2 \leq C \|\mu_s^N - g_s\|_{-3,p}^2 \leq C_T(1 + \|\mu_s^N\|_{-3,p}^2).$$

Using (18), we have

$$\|\mu_s^N\|_{-3,p}^2 \leq C \left(1 + (N_T^N)^{2p} \right).$$

Together with a similar bound for $\mu_s^N(f)$, we obtain

$$\mathbb{E} \left(\sup_{t \leq T} \|R_t^{N,3}\|_{-4,p}^2 \mathbf{1}_{(G_T^N)^c} \right) \leq C_T e^{-(c_2/2)NT} \rightarrow 0$$

as $N \rightarrow \infty$.

As a consequence of this step, it suffices to prove the convergence in law

$$(S^*W^N, \int_0^\cdot D^*g_s dW_s^N(1)) \rightarrow (S^*W, \int_0^\cdot D^*g_s dW_s(1)),$$

as $N \rightarrow \infty$.

Step 2. We now replace the process D^*g_s serving as integrand by a process which is piecewise constant over time intervals of step-size $\varepsilon > 0$. We put

$$g_s^\varepsilon := g_{\delta(s)}, \quad \delta(s) = k\varepsilon \text{ for all } k\varepsilon \leq s < (k+1)\varepsilon, k \geq 0,$$

and let

$$M^\varepsilon := \int_0^\cdot D^*(g_s^\varepsilon - g_s) dW_s(1).$$

Using similar arguments as in Step 1, we have

$$\mathbb{E} \sup_{s \leq T} \|M_s^\varepsilon\|_{-4,p}^2 \leq C \sum_k \int_0^T g_s(f) [(g_s^\varepsilon - g_s)(\psi'_k)]^2 ds \leq C_T \int_0^T \|g_s^\varepsilon - g_s\|_{-3,p}^2 ds.$$

Using Lemma 34 stated in the Appendix below, this last expression is upper bounded by $C_T \varepsilon^2$.

We introduce similarly

$$M^{N,\varepsilon} := \int_0^\cdot D^*(g_s^\varepsilon - g_s) dW_s^N(1)$$

and have, since $\sup_N \mathbb{E} \sup_{s \leq T} \mu_s^N(f) \leq C_T$,

$$\mathbb{E} \sup_{s \leq T} \|M_s^{N,\varepsilon}\|_{-4,p}^2 \leq C \sum_k \mathbb{E} \int_0^T \mu_s^N(f) [(g_s^\varepsilon - g_s)(\psi'_k)]^2 ds \leq C_T \int_0^T \|g_s^\varepsilon - g_s\|_{-3,p}^2 ds \leq C_T \varepsilon^2,$$

where the constant C_T does not depend on N . As a consequence of this step, it suffices to prove the joint convergence of

$$(S^*W^N, \int_0^\cdot D^*g_s^\varepsilon dW_s^N(1)) \rightarrow (S^*W, \int_0^\cdot D^*g_s^\varepsilon dW_s(1)),$$

as $N \rightarrow \infty$, for each fixed $\varepsilon > 0$.

Step 3. To do so, it suffices to prove convergence of the marginal laws

$$(55) \quad ((S^*W_{t_1}^N, \int_0^{t_1} D^*g_s^\varepsilon dW_s^N(1)), \dots, (S^*W_{t_k}^N, \int_0^{t_k} D^*g_s^\varepsilon dW_s^N(1)))$$

to the associated limit

$$(56) \quad ((S^*W_{t_1}, \int_0^{t_1} D^*g_s^\varepsilon dW_s(1)), \dots, (S^*W_{t_k}, \int_0^{t_k} D^*g_s^\varepsilon dW_s(1))),$$

for any $k \geq 1, t_1 \leq t_2 \leq \dots \leq t_k \leq T$. Note that we can rewrite

$$\int_0^t D^*g_s^\varepsilon dW_s^N(1) = \sum_{k:k\varepsilon \leq t} D^*g_{k\varepsilon}^\varepsilon (W_{(k+1)\varepsilon \wedge t}^N(1) - W_{k\varepsilon}^N(1)).$$

Since we have convergence in law in $D(\mathbb{R}_+, \mathbb{R})$ of $W^N(1)$ to the limit process $W(1)$ which is continuous, the above expression converges in law to

$$\sum_{k:k\varepsilon \leq t} D^*g_{k\varepsilon}^\varepsilon (W_{(k+1)\varepsilon \wedge t}(1) - W_{k\varepsilon}(1)) = \int_0^t D^*g_s^\varepsilon dW_s(1),$$

such that the convergence in law of (55) to (56) is indeed implied. Finally, letting $\varepsilon \rightarrow 0$, the convergence of the finite dimensional distributions

$$((S^*W_{t_1}^N, \int_0^{t_1} D^*g_s dW_s^N(1)), \dots, (S^*W_{t_k}^N, \int_0^{t_k} D^*g_s dW_s^N(1)))$$

to the associated limit

$$((S^*W_{t_1}, \int_0^{t_1} D^*g_s dW_s(1)), \dots, (S^*W_{t_k}, \int_0^{t_k} D^*g_s dW_s(1)))$$

follows, and this concludes the proof. \blacksquare

We close this section with the following partial result.

Theorem 31. *For any fixed $n \geq 1$, under Assumptions 1, 2 and 3 and for any $p > \alpha + \frac{1}{2}$, any limit $((\bar{U}^i)_{1 \leq i \leq n}, \bar{\eta})$ of $((U^{N,i})_{1 \leq i \leq n}, \eta^N)$ is solution in $D(\mathbb{R}_+, \mathbb{R}^n) \times C(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$ of the following system of stochastic differential equations*

$$(57) \quad \bar{U}_t^i = \int_0^t b'(\bar{X}_s^i) \bar{U}_s^i ds + h \int_0^t \bar{\eta}_s(f) ds - \int_{[0,t] \times \mathbb{R}_+} \bar{U}_{s-}^i \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz) + hW_t(1), t \geq 0, 1 \leq i \leq n,$$

and for any $\varphi \in \mathcal{W}_0^{5,p}$,

$$(58) \quad \bar{\eta}_t(\varphi) = \bar{\eta}_0(\varphi) + \int_0^t \bar{\eta}_s(L_s \varphi) ds + h \int_0^t g_s(\varphi') \bar{\eta}_s(f) ds + W_t(S\varphi) + h \int_0^t g_s(\varphi') dW_s(1), t \geq 0.$$

Remark 32. *We have stated the decomposition (42) in $\mathcal{W}_0^{-4,p}$. However, the operator L_s appearing in (58) above reduces regularity by one, such that test functions $\varphi \in \mathcal{W}_0^{4,p}$ are reduced to test functions in $\mathcal{W}_0^{3,p+\alpha}$. Yet, we have proven tightness of $(\eta^N)_N$ only in $\mathcal{W}_0^{-4,p}$, such that we have to state the above decomposition in $\mathcal{W}_0^{-5,p}$, although the limit process $\bar{\eta}$ takes values in the smaller space $\mathcal{W}_0^{4,p}$. This is analogous to Remark 5.7 in [3].*

Proof. Step 1. We have already proven the tightness of the sequence of laws of $(U^{N,i})_{1 \leq i \leq n}$. To characterize the dependencies between \bar{U}^i, \bar{X}^i and π^i in the limit, we now consider the càdlàg process

$$\mathcal{Y}^N := (U^{N,i}, \bar{X}^i, \int_{[0, \cdot] \times \mathbb{R}_+} U_{s-}^{N,i} \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz))_{1 \leq i \leq n}$$

which belongs to $D(\mathbb{R}_+, \mathbb{R}^{3n})$.

Using analogous arguments as those in the proof of Theorem 28 allows to deduce the tightness of the sequence of processes \mathcal{Y}^N in $D(\mathbb{R}_+, \mathbb{R}^{3n})$. Details of the proof are omitted.

Step 2. Due to the previous step and the continuity of any limit law of $(\eta^N, W^N, \int_0^\cdot D^* g_s dW_s^N(1))$, we know that

$$\left(\left(U^{N,i}, \bar{X}^i, \int_{[0, \cdot] \times \mathbb{R}_+} U_{s-}^{N,i} \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz) \right)_{1 \leq i \leq n}, \eta^N, W^N, \int_0^\cdot D^* g_s dW_s^N(1) \right)$$

is tight in

$$D(\mathbb{R}_+, \mathbb{R}^{3n} \times \mathcal{W}_0^{-4,p} \times \mathcal{W}_0^{-4,p} \times \mathcal{W}_0^{-4,p}).$$

In what follows we assume without loss of generality that the above sequence converges to some limit

$$\left((\bar{U}^i, \bar{X}^i, \bar{V}^i)_{1 \leq i \leq n}, \bar{\eta}, W, \int_0^\cdot D^* g_s dW_s(1) \right),$$

where, to simplify notation, we use the same letter \bar{X}^i to denote the limit process as well as the one defining the second coordinates of \mathcal{Y}^N .

To identify the limit, let $(\psi_k)_k$ be an orthonormal basis of $\mathcal{W}_0^{4,p}$, composed of C_c^∞ -functions. Define for any k the functional $F_k : D(\mathbb{R}_+, \mathcal{W}_0^{-4,p} \times \mathcal{W}_0^{-4,p} \times \mathcal{W}_0^{-4,p}) \rightarrow D(\mathbb{R}_+, \mathbb{R})$ by

$$\begin{aligned} F_k(f^1, f^2, f^3)_t = & \langle f_t^1, \psi_k \rangle - \langle f_0^1, \psi_k \rangle - \int_0^t \langle f_s^1, L_s \psi_k \rangle ds - h \int_0^t \langle f_s^1, f \rangle \langle g_s, \psi_k' \rangle ds \\ & - \langle f_t^2, S \psi_k \rangle - h \langle f_t^3, \psi_k \rangle \end{aligned}$$

and $G : D(\mathbb{R}_+, \mathbb{R} \times \mathbb{R} \times \mathcal{W}_0^{-4,p} \times \mathcal{W}_0^{-4,p}) \rightarrow D(\mathbb{R}_+, \mathbb{R})$ by

$$G(g^1, g^2, f^1, f^2)_t = g_t^1 - g_0^1 - \int_0^t b'(g_s^2) g_s^1 ds - h \int_0^t \langle f_s^1, f \rangle ds - h \langle f_t^2, 1 \rangle.$$

Then the system (57)–(58) is equivalent to

$$F_k(\bar{\eta}, W, h \int_0^\cdot D^* g_s dW_s(1)) = 0, \quad G(\bar{U}^i, \bar{X}^i, \bar{\eta}, W) = - \int_{[0, \cdot] \times \mathbb{R}_+} \bar{U}_{s-}^i \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz),$$

for all $k \geq 1$ and for all $i \geq 1$.

Step 2.1. In this step we show that $F_k(\bar{\eta}, W, h \int_0^\cdot D^* g_s dW_s(1)) = 0$. We first prove that F_k is continuous at every point $(f^1, f^2, f^3) \in C(\mathbb{R}_+, \mathcal{W}_0^{-4,p} \times \mathcal{W}_0^{-4,p} \times \mathcal{W}_0^{-4,p})$. Indeed, the continuity of

$$f^1 \mapsto \left(t \mapsto \langle f_t^1, \psi_k \rangle - \langle f_0^1, \psi_k \rangle - \int_0^t \langle f_s^1, L_s \psi_k \rangle ds - h \int_0^t \langle f_s^1, f \rangle \langle g_s, \psi_k' \rangle ds \right)$$

at every point $f^1 \in C(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$ follows as in the proof of Theorem 5.6 of [3]. Similarly, the functionals

$$f^2 \mapsto (t \mapsto \langle f_t^2, S \psi_k \rangle) \quad \text{and} \quad f^3 \mapsto (t \mapsto \langle f_t^3, \psi_k \rangle)$$

are continuous as well at every point f^2 and f^3 belonging to $C(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$.

Step 2.2. Before proceeding further, we rewrite (42) as follows

$$(59) \quad \eta_t^N = \eta_0^N + \int_0^t L_s^* \eta_s^N ds + S^* W_t^N + h \int_0^t D^* g_s dW_s^N(1) + h \int_0^t \eta_s^N(f) D^* g_s ds + R_t^{N,4},$$

where, recalling (54), $R_t^{N,4} = R_t^{N,1} + hR_t^{N,3} + h \int_0^t \eta_s^N(f) D^*(\mu_s^N - g_s) ds$ has already been shown to converge to 0 in $\mathcal{W}_0^{-4,p}$; that is,

$$\lim_{N \rightarrow \infty} \mathbb{E} \sup_{t \leq T} \|R_t^{N,4}\|_{-4,p} = 0.$$

Writing for short $M^N = \int_0^\cdot D^* g_s dW_s^N(1)$, (59) implies that for all $t \geq 0$,

$$F_k(\eta^N, W^N, M^N)(t) = R_t^{N,4}(\psi_k) \quad \text{and} \quad \mathbb{E}(\sup_{t \leq T} |R_t^{N,4}(\psi_k)|^2) \rightarrow 0$$

as $N \rightarrow \infty$. Therefore, we have convergence in probability,

$$\sup_{t \leq T} |F_k(\eta^N, W^N, M^N)(t)| \rightarrow 0.$$

On the other hand, by the continuous mapping theorem, we have convergence in law $F_k(\eta^N, W^N, M^N) \rightarrow F_k(\bar{\eta}, W, \int_0^\cdot D^* g_s dW_s(1))$; this allows to identify the limit which has to equal the zero-process.

Step 2.3. We now turn to the study of G . Firstly, G is continuous at every point $(g^1, g^2, f^1, f^2) \in D(\mathbb{R}_+, \mathbb{R}^2) \times C(\mathbb{R}_+, \mathcal{W}_0^{-4,p} \times \mathcal{W}_0^{-4,p})$. Indeed we only need to check the continuity of

$$(g^1, g^2) \mapsto \left(t \mapsto \int_0^t b'(g_s^2) g_s^1 ds \right)$$

at every point $(g^1, g^2) \in D(\mathbb{R}_+, \mathbb{R}^2)$, which follows from the basic properties of the Skorokhod topology. From (49) we have that

$$G(U^{N,i}, \bar{X}^i, \eta^N, W^N) = R^{N,2,i} - \int_{[0,\cdot] \times \mathbb{R}_+} U_{s-}^{N,i} \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz),$$

and by (53), $R^{N,2,i}$ converges to 0 in probability, for the uniform convergence on finite time intervals, for any fixed i . This implies that

$$G(\bar{U}^i, \bar{X}^i, \bar{\eta}, W) = -\bar{V}^i.$$

It remains to identify

$$(60) \quad \bar{V}^i = \int_{[0,\cdot] \times \mathbb{R}_+} \bar{U}_{s-}^i \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz),$$

for each $1 \leq i \leq n$.

In what follows, we write for short $V^{N,i} = \int_{[0,\cdot] \times \mathbb{R}_+} U_{s-}^{N,i} \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz)$. We already know that $(U^{N,i}, V^{N,i}, \bar{X}^i)_{1 \leq i \leq n}$ converges in law to $(\bar{U}^i, \bar{V}^i, \bar{X}^i)_{1 \leq i \leq n}$, where once more, by abuse of notation, we use the same letter \bar{X}^i for the limit process of the third coordinate. Moreover, $(V^{N,i}, \bar{X}^i)_{1 \leq i \leq n}$ is a semimartingale taking values in \mathbb{R}^{2n} with characteristics

$$B^{N,i} = \left(\int_0^\cdot U_s^{N,i} f(\bar{X}_s^i) ds, \int_0^\cdot [b(\bar{X}_s^i) - \bar{X}_s^i f(\bar{X}_s^i) + h g_s(f)] ds \right), 1 \leq i \leq n,$$

$$C^N = 0, \nu^N(dt, dv, dx) = \sum_{i=1}^n f(\bar{X}_{t-}^i) dt \left(\prod_{j=1, j \neq i}^n \delta_{(0,0)}(dv^j, dx^j) \otimes \delta_{(-U_{t-}^{N,i}, -\bar{X}_{t-}^i)}(dv^i, dx^i) \right).$$

Clearly we have weak convergence $(V^{N,i}, \bar{X}^i, B^{N,i})_{1 \leq i \leq n} \rightarrow (\bar{V}^i, \bar{X}^i, \bar{B}^i)_{1 \leq i \leq n}$ where

$$\bar{B}^i = \left(\int_0^\cdot \bar{U}_s^i f(\bar{X}_s^i) ds, \int_0^\cdot [b(\bar{X}_s^i) - \bar{X}_s^i f(\bar{X}_s^i) + hg_s(f)] ds \right),$$

by the continuity properties of the Skorokhod topology and since $(U^{N,i}, \bar{X}^i)_{1 \leq i \leq n} \rightarrow (\bar{U}^i, \bar{X}^i)_{1 \leq i \leq n}$.

It is shown analogously that we have weak convergence $g * \nu^N \rightarrow g * \bar{\nu}$, for any continuous and bounded test function g , where

$$\bar{\nu} = \sum_{i=1}^n f(\bar{X}_{t-}^i) dt \left(\prod_{j=1, j \neq i}^n \delta_{(0,0)}(dv^j, dx^j) \otimes \delta_{(-\bar{U}_{t-}^i, -\bar{X}_{t-}^i)}(dv^i, dx^i) \right).$$

Then Jacod and Shiryaev [15, Theorem 2.4 page 528] implies that necessarily $(\bar{V}^i, \bar{X}^i)_{1 \leq i \leq n}$ is a semimartingale with characteristics $(\bar{B}, 0, \bar{\nu})$. Finally, the representation theorem [15, Theorem III.2.26 page 157] implies that there exist n independent Poisson random measures which, by abuse of notation, we still denote $\pi^i(ds, dz)$, having Lebesgue intensity, such that

$$\bar{X}_t^i = \bar{X}_0^i + \int_0^t b(\bar{X}_s^i) ds + \int_0^t hg_s(f) ds - \int_{[0, \cdot] \times \mathbb{R}_+} \bar{X}_{s-}^i \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz)$$

and

$$\bar{V}_t^i = \int_{[0, \cdot] \times \mathbb{R}_+} \bar{U}_{s-}^i \mathbf{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} \pi^i(ds, dz).$$

This gives the desired identity (60) and thus finishes our proof. ■

We close this section with the

Proof of Theorem 4. Theorem 25 implies the tightness of (η^N) and Theorem 28 the tightness of $(U^{N,i})_{1 \leq i \leq n}$ for any fixed $n \geq 1$. Moreover, Theorem 31 implies that any limit $((\bar{U}^i)_{1 \leq i \leq n}, \bar{\eta})$ of $((U^{N,i})_{1 \leq i \leq n}, \eta^N)$ is solution of the system of differential equations (57)–(58). Finally, under the additional assumption $p > 2\alpha + \frac{1}{2}$, Theorem 10 implies pathwise uniqueness for this limit system, and the Yamada-Watanabe theorem allows to deduce weak uniqueness and thus the uniqueness of the limit law implying the weak convergence of $((U^{N,i})_{1 \leq i \leq n}, \eta^N)$ in $D(\mathbb{R}_+, \mathbb{R}^n) \times D(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$. This implies the weaker convergence of $((U^{N,i})_{1 \leq i \leq n}, \eta^N)$ in $D(\mathbb{R}_+, \mathbb{R})^n \times D(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$ and thus the convergence of the infinite sequence $((U^{N,i})_{i \geq 1}, \eta^N)$ in $D(\mathbb{R}_+, \mathbb{R})^{\mathbb{N}^*} \times D(\mathbb{R}_+, \mathcal{W}_0^{-4,p})$.
■

7. APPENDIX

7.1. Useful results on weighted Sobolev spaces. In what follows we collect the most important facts about weighted Sobolev spaces, see [1] and also Section 2.1 of [12]. First of all, obviously, for all $k \leq k'$,

$$(61) \quad \|\cdot\|_{k,p} \leq \|\cdot\|_{k',p}, \text{ implying that } \|\cdot\|_{-k',p} \leq \|\cdot\|_{-k,p}.$$

We also have that for all $p \leq p'$,

$$\|\cdot\|_{k,p'} \leq C \|\cdot\|_{k,p}, \text{ implying that } \|\cdot\|_{-k,p} \leq C \|\cdot\|_{-k,p'}.$$

Finally, $C^{k,\alpha} \subset \mathcal{W}_0^{k,p}$ for any $p > \alpha + \frac{1}{2}$. In particular, constant functions belong to $\mathcal{W}_0^{k,p}$ for any $p > \frac{1}{2}$.

The following embeddings have been used throughout this paper.

(1) **Sobolev embedding.** There exists a constant C such that for all $m \geq 1, k \geq 0$ and $p \geq 0$,

$$(62) \quad \|\psi\|_{C^{k,p}} \leq C \|\psi\|_{m+k,p}.$$

(2) **Maurin's theorem.** The embedding $\mathcal{W}_0^{m+k,p} \hookrightarrow \mathcal{W}_0^{k,p+p'}$ is of Hilbert-Schmidt type for any $m \geq 1, k \geq 0$ and $p \geq 0, p' > 1/2$. This implies that the embedding is compact and that there exists a constant C such that

$$\|\psi\|_{k,p+p'} \leq C \|\psi\|_{k+m,p}.$$

(3) The dual embedding $\mathcal{W}_0^{-k,p+p'} \hookrightarrow \mathcal{W}_0^{-(k+m),p}$ is of Hilbert-Schmidt type.

We have also used several times that for any $k, p \geq 0$, there exists an orthonormal basis composed of C_c^∞ – functions $(\psi_i)_i$ of $\mathcal{W}_0^{k,p}$ such that for any element $w \in \mathcal{W}_0^{-k,p}$,

$$\|w\|_{-k,p}^2 = \sum_i \langle w, \psi_i \rangle^2.$$

We now give the proof of some of the results stated in Lemma 6. It follows the arguments given in the proof of Lemma 4.1 and Lemma 4.2 in [3].

Proof of Lemma 6. We quickly show that S is a continuous mapping from $\mathcal{W}_0^{k,p}$ to itself, following the arguments of [3]. Since $(S\varphi)(x) = \varphi(0) - \varphi(x)$, we have that

$$\|S\varphi\|_{k,p}^2 \leq 2 \int_0^\infty \frac{|\varphi(0)|^2}{1+|x|^{2p}} dx + 2\|\varphi\|_{k,p}^2 = C|\varphi(0)|^2 + 2\|\varphi\|_{k,p}^2,$$

where $C = 2 \int_0^\infty \frac{1}{1+|x|^{2p}} dx < \infty$ since $p > 1/2$. The conclusion then follows from $|\varphi(0)| \leq \|\varphi\|_{C^0,p} \leq C\|\varphi\|_{k,p}$ for any $k \geq 1$, by the Sobolev embedding theorem. The other points of the lemma follow similarly. ■

7.2. Proof of Lemma 8. A straightforward adaptation of [13, Prop.15] yields

Proposition 33. *Grant Assumptions 1 and 2. Then for all $t \geq 0$, all $i = 1, \dots, N$,*

$$(63) \quad X_t^{N,i} \leq X_0^{N,i} + 3\bar{X}_0^N + 4\bar{b}t + 4hN_t^N,$$

$$(64) \quad \frac{1}{N} \sum_{j=1}^N \int_0^t \int_0^\infty (h + X_{s^-}^{N,j}) \mathbf{1}_{\{z \leq f(X_{s^-}^{N,j})\}} \pi^j(ds, dz) \leq 3\bar{X}_0^N + 3\bar{b}t + 4hN_t^N,$$

where $N_t^N := N^{-1} \sum_{j=1}^N \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(2h)\}} \pi^j(ds, dz)$ and where $\bar{b} > 0$ is such that $b(x) \leq \bar{b}$ for all $x \geq 0$.

Proof. For the convenience of the reader we briefly sketch how to adapt the proof of [13] to the present frame. Taking the (empirical) mean of (2) and using that b is upper bounded by a positive constant, say \bar{b} , we find

$$(65) \quad \bar{X}_t^N \leq \bar{X}_0^N + \bar{b}t + \frac{1}{N} \sum_{i=1}^N \int_{[0,t] \times \mathbb{R}_+} \left(h \frac{N-1}{N} - X_{s^-}^{N,i} \right) \mathbf{1}_{\{z \leq f(X_{s^-}^{N,i})\}} \pi^i(ds, dz)$$

which implies

$$\frac{1}{N} \sum_{i=1}^N \int_{[0,t] \times \mathbb{R}_+} (X_{s^-}^{N,i} - h) \mathbf{1}_{\{z \leq f(X_{s^-}^{N,i})\}} \pi^i(ds, dz) \leq \bar{X}_0^N + \bar{b}t.$$

Using that $x - h \geq (x + h)/3 - (4/3)h\mathbf{1}_{\{x \leq 2h\}}$ for all $x \geq 0$ and that f is non-decreasing, we deduce that

$$\begin{aligned} & \frac{1}{N} \sum_{i=1}^N \int_{[0,t] \times \mathbb{R}_+} (h + X_{s^-}^{N,i}) \mathbf{1}_{\{z \leq f(X_{s^-}^{N,i})\}} \pi^i(ds, dz) \\ & \leq 3\bar{X}_0^N + 3\bar{b}t + \frac{4h}{N} \sum_{i=1}^N \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{X_{s^-}^{N,i} \leq 2h\}} \mathbf{1}_{\{z \leq f(X_{s^-}^{N,i})\}} \pi^i(ds, dz) \\ & \leq 3\bar{X}_0^N + 3\bar{b}t + \frac{4h}{N} \sum_{i=1}^N \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(2h)\}} \pi^i(ds, dz). \end{aligned}$$

Now, for all $1 \leq i \leq N$, starting from (2),

$$X_t^{N,i} \leq X_0^{N,i} + \bar{b}t + \frac{h}{N} \sum_{j=1}^N \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq f(X_{s^-}^{N,j})\}} \pi^j(ds, dz) \leq X_0^{N,i} + 3\bar{X}_0^N + 4\bar{b}t + 4hN_t^N,$$

which concludes. ■

The proof of (19) then follows from the fact that $N_t^N = U/N$ where $U \sim \text{Pois}(Ntf(2h))$ and that g_0 is of compact support.

We finally give the

Proof of (22). We adapt the proof of Proposition 14 of [13] to the present frame. Since f is non-decreasing, we have that $f(x)(h-x) = -xf(x)/2 + f(x)(h-x/2) \leq -xf(x)/2 + f(2h)h$ for all $x \geq 0$. Taking expectation in (3), we therefore obtain that

$$\mathbb{E}(\bar{X}_t^i) \leq \mathbb{E}(\bar{X}_0^i) + \bar{b}t + \int_0^t \mathbb{E}(f(\bar{X}_s^i)(h - \bar{X}_s^i))ds \leq \mathbb{E}(\bar{X}_0^i) + [\bar{b} + f(2h)h]t - \frac{1}{2} \int_0^t \mathbb{E}(\bar{X}_s^i f(\bar{X}_s^i))ds.$$

Since $\mathbb{E}(\bar{X}_t^i) \geq 0$, this implies

$$\int_0^t \mathbb{E}(\bar{X}_s^i f(\bar{X}_s^i))ds \leq 2\mathbb{E}(\bar{X}_0^i) + 2[\bar{b} + f(2h)h]t.$$

We conclude using that $f(x) \leq C(1 + xf(x))$ and observing that

$$\bar{X}_t^i \leq \bar{X}_0^i + \bar{b}t + h \int_0^t \mathbb{E}(f(\bar{X}_s^i))ds \leq \bar{X}_0^i + \bar{b}t + h \int_0^t \mathbb{E}(C(1 + \bar{X}_s^i f(\bar{X}_s^i)))ds.$$

This finishes the proof of Lemma 8. ■

7.3. Useful properties of the limit process.

Lemma 34. *For any $p \geq 0$, g_t is continuous in $\mathcal{W}_0^{-2,p}$, and for all $t, t+h \leq T$, we have $\|g_{t+h} - g_t\|_{-2,p} \leq C_T h$.*

Proof. We have for all $\psi \in \mathcal{W}_0^{2,p}$,

$$g_{t+h}(\psi) - g_t(\psi) = \mathbb{E}[\psi(\bar{X}_{t+h}^1) - \psi(\bar{X}_t^1)] = \int_t^{t+h} \mathbb{E}L_s \psi(\bar{X}_s^1) ds.$$

Using the Sobolev embedding theorem and Lemma 6, since $|\bar{X}_s| \leq \bar{C}_T$ for all $s \leq T$,

$$|L_s \psi(\bar{X}_s)| \leq \|L_s \psi\|_{C^{0,p+\alpha}} (1 + |\bar{X}_s|^{p+\alpha}) \leq C_T \|L_s \psi\|_{1,p+\alpha} \leq C_T \|\psi\|_{2,p}$$

implying that

$$|g_{t+h}(\psi) - g_t(\psi)| \leq C_T h \|\psi\|_{2,p},$$

which concludes the proof. ■

We continue this section with the

Proof of Proposition 11. Step 1. Firstly, since g_0 is of compact support, g_t is of compact support as well, for any fixed $t \geq 0$, due to the a priori upper bound (22).

Moreover, we obtain similarly to Lemma 24 in [13] the representation

$$(66) \quad \bar{X}_t^i = \varphi_{0,t}(\bar{X}_0^i) \mathbf{1}_{\{\tau_t=0\}} + \varphi_{\tau_t,t}(0) \mathbf{1}_{\{\tau_t>0\}},$$

where $\tau_t = \sup\{s \leq t : \Delta \bar{X}_s^i \neq 0\} = t - \bar{L}_t^i$ is the last jump time of neuron i , before time t . Here we put $\sup \emptyset := 0$ if no such jump has happened. Using similar arguments as those in Proposition 25 in [13], the law of τ_t is given by

$$\mathcal{L}(\tau_t)(ds) = \mathbb{E}(e^{-\int_0^t f(\varphi_{0,u}(X_0^i)) du}) \delta_0(ds) + g_s(f) e^{-\int_s^t f(\varphi_{s,u}(0)) du} ds.$$

Since $\bar{X}_0^i \sim g_0(x) dx$, clearly $\bar{X}_0^i > 0$ almost surely, which implies that $\varphi_{0,t}(\bar{X}_0^i) > 0$ almost surely as well, by the properties of the deterministic flow. Since $\mathbb{P}(\tau_t = 0) > 0$, this implies, using (66), that $g_t(f) \geq \mathbb{E}(f(\varphi_{0,t}(\bar{X}_0^i)) \mathbf{1}_{\{\tau_t=0\}}) > 0$ for all $t \geq 0$ since $f(x) > 0$ for all $x > 0$.

Having this established, using a change of variables relying on the regularity of the initial law g_0 for the first term of (66) and a change of variables relying on the density of τ_t on $(0, t)$ within the second term of (66), we deduce the explicit form

$$(67) \quad g_t(y) = \frac{g_t(f)}{b(0) + hg_t(f)} e^{-\int_{\beta_t(y)}^t (b'(\varphi_{\beta_t(y),s}(0)) + f(\varphi_{\beta_t(y),s}(0))) ds} \mathbf{1}_{\{y < \varphi_{0,t}(0)\}} \\ + e^{-\int_0^t (b'(\varphi_{s,t}^{-1}(y)) + f(\varphi_{s,t}^{-1}(y))) ds} g_0 \circ \varphi_{0,t}^{-1}(y) \mathbf{1}_{\{y \geq \varphi_{0,t}(0)\}},$$

where $\varphi_{0,t}^{-1}(y)$ denotes the inverse flow satisfying $\varphi_{0,t}^{-1}(\varphi_{0,t}(y)) = y$ and where $\varphi_{s,t}^{-1}(y) = \varphi_{0,s} \circ \varphi_{0,t}^{-1}(y)$. In the above formula, $\beta_t(y)$ denotes the unique real in $]0, t]$ satisfying

$$(68) \quad \varphi_{\beta_t(y),t}(0) = y,$$

for any $y < \varphi_{0,t}(0)$.

Step 2. Standard arguments show that

$$\frac{\partial \varphi_{s,t}(0)}{\partial s} = -(hg_s(f) + b(0)) e^{\int_s^t b'(\varphi_{s,u}(0)) du} < 0$$

since $g_s(f) > 0$ and $b(0) \geq 0$. Thus, the function $[0, t] \ni s \mapsto \varphi_{s,t}(0)$ is strictly decreasing. The function $s \mapsto g_s(f)$ being continuous, the function $[0, t] \ni s \mapsto \varphi_{s,t}(0)$ also differentiable. As a consequence, its inverse function β_t is differentiable as well.

Therefore, for any fixed $t > 0$, the Lebesgue density $g_t(y)$ is differentiable at every point $y \neq \varphi_{0,t}(0)$. The fact that

$$s \mapsto \int_0^\infty (1+x^p) |g'_s(x)| dx \text{ is locally bounded}$$

follows easily from the above explicit representation. ■

We finally give the

Proof of Proposition 13. Step 1. Before starting the proof, let us first mention that a simple change of variables formula implies that for any fixed $s < t$, the mapping $\psi \mapsto [x \mapsto \psi \circ \varphi_{s,t}(x)]$ is continuous from $\mathcal{W}_0^{6,p} \rightarrow \mathcal{W}_0^{6,p}$ for any $p \geq 0$, where we recall that $\varphi_{s,t}(x) = x + \int_s^t b(\varphi_{s,u}(x))du + h \int_s^t g_u(f)du$. This follows from the fact that b is bounded by a positive constant and that all derivatives of b up to order 6 are bounded. Moreover we have

$$\|\psi \circ \varphi_{s,t}\|_{6,p} \leq C_T \|\psi\|_{6,p},$$

for all $s \leq t \leq T$.

Step 2. Introduce now for any $0 \leq s \leq t$ and $x \geq 0$ the process

$$(69) \quad \bar{Y}_{s,t}(x) = x + \int_s^t (hg_u(f) + b(\bar{Y}_{s,u}(x))) du - \int_{]s,t] \times \mathbb{R}_+} Y_{s,u-}(x) \mathbf{1}_{\{z \leq 1\}} \pi^1(du, dz);$$

that is, $\bar{Y}_{s,t}(x)$ follows the same dynamic as $Y_{s,t}(x)$, but jumps occur at constant rate 1. We still have the upper bound

$$(70) \quad \bar{Y}_{s,t}(x) \leq x + \bar{C}_T, \text{ for all } s \leq t \leq T.$$

Let us write for short

$$\Pi_t = \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq 1\}} \pi^1(du, dz),$$

that is, $(\Pi_t)_t$ is the Poisson process having intensity 1 governing the jumps of \bar{Y} . Write $T_1 < T_2 < \dots < T_n < \dots$ for the successive jumps of $(\Pi_t)_t$. Then Girsanov's theorem for jump processes, see [14], implies that

$$P_{s,t}\psi(x) = \mathbb{E}\psi(Y_{s,t}(x)) = \mathbb{E} \left(\psi(\bar{Y}_{s,t}(x)) \prod_{n: T_n \in]s,t]} f(\bar{Y}_{s,T_n-}(x)) e^{-\int_s^t [f(\bar{Y}_{s,u}(x)) - 1] du} \right).$$

We notice that for all $t < T_1(s) := \inf\{T_n : T_n > s\}$, $\bar{Y}_{s,t}(x) = \varphi_{s,t}(x)$. Therefore,

$$P_{s,t}\psi(x) = \psi(\varphi_{s,t}(x)) e^{-\int_s^t (f(\varphi_{s,u}(x)) - 1) du} \mathbb{P}(t < T_1(s)) + e^{t-s} \mathbb{E} \left(f(\varphi_{s,T_1(s)}(x)) e^{-\int_s^{T_1(s)} f(\varphi_{s,u}(x)) du} Q_{T_1(s),t}^1(\psi); t \geq T_1(s) \right),$$

where

$$Q_{T_1(s),t}^1(\psi) = \psi(\bar{Y}_{s,t}(x)) \prod_{n: T_n \in]T_1(s),t]} f(\bar{Y}_{s,T_n-}(x)) e^{-\int_{T_1(s)}^t f(\bar{Y}_{s,u}(x)) du}.$$

Using the strong Markov property at time $T_1(s)$ and the fact that at time $T_1(s)$, $\bar{Y}_{s,T_1(s)}(x) = 0$ is reset to 0 and thus forgets its starting position x at this time, we obtain

$$\begin{aligned} \mathbb{E} \left(f(\varphi_{s,T_1(s)}(x)) e^{-\int_s^{T_1(s)} f(\varphi_{s,u}(x)) du} Q_{T_1(s),t}^1(\psi); t \geq T_1(s) \right) \\ = \int_0^{t-s} e^{-v} f(\varphi_{s,s+v}(x)) e^{-\int_s^{s+v} f(\varphi_{s,u}(x)) du} Q_{s,v,t}^2(\psi) dv, \end{aligned}$$

where

$$Q_{s,v,t}^2(\psi) = \mathbb{E} \left(\psi(\bar{Y}_{s+v,t}(0)) \prod_{n: T_n \in]s+v,t]} f(\bar{Y}_{s+v,T_n-}(0)) e^{-\int_{s+v}^t f(\bar{Y}_{s+v,u}(0)) du} \right).$$

Summarizing, we have

$$\begin{aligned} P_{s,t}\psi(x) &= \psi(\varphi_{s,t}(x))e^{-\int_s^t f(\varphi_{s,u}(x))du} + e^{t-s} \int_0^{t-s} e^{-v} f(\varphi_{s,s+v}(x))e^{-\int_s^{s+v} f(\varphi_{s,u}(x))du} Q_{s,v,t}^2(\psi)dv \\ &:= P_{s,t}^1\psi(x) + P_{s,t}^2\psi(x), \end{aligned}$$

where, using (70),

$$(71) \quad \sup_{v \leq t-s} |Q_{s,v,t}^2(\psi)| \leq C_t \|\psi\|_{1,p}.$$

Clearly, $\psi \in C^6$ implies that $P_{s,t}\psi \in C^6$ as well, since $f \in C^6$. It is also clear at this stage that $\psi \in C_c^6$ implies $P_{s,t}^1\psi \in C_c^6$, having a support that depends on s and t .

Step 3. We now investigate the dependence on x of the first six derivatives of $P_{s,t}\psi(x)$ with respect to x . Firstly, recalling (71) and observing that $\int_0^{t-s} e^{-v} f(\varphi_{s,s+v}(x))e^{-\int_s^{s+v} f(\varphi_{s,u}(x))du} \leq 1$,

$$\|P_{s,t}\psi\|_{0,p} \leq C_t \|\psi\|_{1,p}.$$

Let us now study the successive derivatives of $P_{s,t}^1\psi$. We have

$$\begin{aligned} \frac{\partial}{\partial x} P_{s,t}^1\psi(x) &= \\ &= \left[\psi'(\varphi_{s,t}(x))e^{\int_s^t b'(\varphi_{s,u}(x))du} - \psi(\varphi_{s,t}(x)) \left(\int_s^t f'(\varphi_{s,u}(x))e^{\int_s^u b'(\varphi_{s,r}(x))dr} \right) \right] e^{-\int_s^t f(\varphi_{s,u}(x))du}. \end{aligned}$$

Since by Step 1, $\|\psi \circ \varphi_{s,t}\|_{1,p} \leq C_T \|\psi\|_{1,p}$, we only have to investigate the second term of the above expression. We use the following facts. The function f is non-decreasing and we have $f'(x) \leq C(1+x^\alpha)$, $f(x) \geq cx\mathbf{1}_{\{x \geq 1\}}$ by convexity, since $f(0) = 0$. Moreover, $b(x) \geq -\lambda x$ implies that for all $s \leq u \leq t \leq T$,

$$\varphi_{s,u}(x) \geq e^{-\lambda T} x \text{ such that } f(\varphi_{s,u}(x)) \geq f(e^{-\lambda T} x) \geq ce^{-\lambda T} x \mathbf{1}_{\{x \geq e^{\lambda T}\}}.$$

Therefore,

$$e^{-\int_s^t f(\varphi_{s,u}(x))du} \leq e^{-ce^{-\lambda T}(t-s)x} \mathbf{1}_{\{x \geq e^{\lambda T}\}} + \mathbf{1}_{\{x < e^{\lambda T}\}},$$

implying

$$\sup_x \left(\left(\int_s^t |f'(\varphi_{s,u}(x))| e^{\int_s^u b'(\varphi_{s,r}(x))dr} du \right) e^{-\int_s^t f(\varphi_{s,u}(x))du} \right) = C_T < \infty$$

and thus

$$\|P_{s,t}^1\psi\|_{1,p} \leq C_T \|\psi\|_{1,p}.$$

Similar arguments give $\|P_{s,t}^1\psi\|_{6,p} \leq C_T \|\psi\|_{6,p}$.

Finally, the same arguments as above give that for all $s \leq t$,

$$x \mapsto \int_0^{t-s} e^{-v} f(\varphi_{s,s+v}(x))e^{-\int_s^{s+v} f(\varphi_{s,u}(x))du} =: F_{s,t}(x) \in C_b^6$$

and for all $\gamma > 0$ and all $0 \leq k \leq 6$,

$$\lim_{x \rightarrow \infty} x^\gamma |F_{s,t}^{(k)}(x)| = 0$$

such that

$$\|P_{s,t}^2\psi\|_{6,p} \leq C_t \|\psi\|_{6,p} \text{ and even } \|x^\gamma P_{s,t}^2\psi\|_{6,p} \leq C_t(\gamma) \|\psi\|_{6,p},$$

where $x^\gamma P_{s,t}^2\psi$ denotes the function $x \mapsto x^\gamma P_{s,t}^2\psi(x)$. This concludes the proof. ■

7.4. Some hints on the proof of (7). The proof of the quantified propagation of chaos result (7) relies mainly on the introduction of a convenient distance function $H(x)$ that allows to compare neurons in the finite system and the associated ones in the limit system. As in [13], we take $H(x) = f(x) + \arctan(x)$. The main goal of this distance function is to be able to deal with the big jump terms (the reset terms $-xf(x)$ appearing when a neuron having potential value x spikes—these terms are not Lipschitz since f is of polynomial growth) and to control both the usual L^1 -distance $|x - y|$ and the distance $|f(x) - f(y)|$ which naturally appears when controlling the asynchronous jumps of two neurons, one having potential x and the other having potential y .

The function $H(x) = f(x) + \arctan(x)$ is a sort of Lyapunov-function in the sense that the following properties hold.

Proposition 35 (Proposition 18 of [13]). *Grant Assumptions 1, 2 and 3. Then there exists a constant C such that for all $x, y \in \mathbb{R}_+$, we have*

$$(0) |H''(x)| \leq CH'(x),$$

$$(i) x + H'(x) \leq C(1 + f(x)),$$

$$(ii) |x - y| + |H'(x) - H'(y)| + |f(x) - f(y)| \leq C|H(x) - H(y)|,$$

$$(iii) \text{sign}(x - y)(b(x)H'(x) - b(y)H'(y)) \leq C|H(x) - H(y)|,$$

$$(iv) -(f(x) \wedge f(y))|H(x) - H(y)| + |f(x) - f(y)|(H(x) \wedge H(y) - |H(x) - H(y)|) \leq C|H(x) - H(y)|.$$

Proof. We only have to check point (iii), all other points have already been proven in Proposition 18 of [13]. To do so, let $y \leq x$ such that the left hand side of (iii) is given by

$$b(x)f'(x) - b(y)f'(y) + \frac{b(x)}{1+x^2} - \frac{b(y)}{1+y^2} := T_1 + T_2.$$

Since $b(x)/1+x^2$ is Lipschitz by the properties of the function b , clearly $|T_2| \leq C|x - y| \leq C|H(x) - H(y)|$ by item (ii). Moreover,

$$T_1 = \int_y^x b'(z)f'(z)dz + \int_y^x b(z)f''(z)dz =: T_{11} + T_{12}.$$

Let us start dealing with T_{12} . Since f'' is positive and b is upper bounded by a positive constant \bar{b} , we have that $b(z)f''(z) \leq \bar{b}f''(z) \leq C(1 + f'(z)) \leq C(1 + H'(z))$ which is then controlled thanks to item (ii). Moreover, $|b'(z)f'(z)| \leq Cf'(z)$ such that $|T_{11}| \leq C \int_y^x f'(z)dz$, which is once more controlled using (ii). ■

Once these properties fixed, Theorem 7 of [13] gives that

$$\sup_{t \leq T} \mathbb{E}(|H(X_t^{N,i}) - H(\bar{X}_t^i)|) \leq C_T/\sqrt{N},$$

implying, together with item (ii) of the above proposition that

$$\sup_{t \leq T} \mathbb{E}(|X_t^{N,i} - \bar{X}_t^i|) \leq C_T/\sqrt{N},$$

from which it is easy to conclude that (7) holds as well.

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