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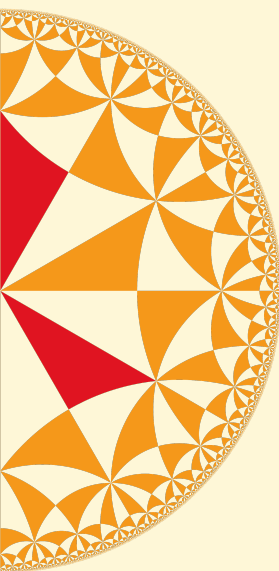
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# SEMICLASSICAL GEVREY OPERATORS IN THE COMPLEX DOMAIN

by Michael HITRIK, Richard LASCAR,  
Johannes SJÖSTRAND & Maher ZERZERI

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ABSTRACT. — We study semiclassical Gevrey pseudodifferential operators, acting on exponentially weighted spaces of entire holomorphic functions. The symbols of such operators are Gevrey functions defined on suitable I-Lagrangian submanifolds of the complexified phase space, which are extended almost holomorphically in the same Gevrey class, or in some larger space, to complex neighborhoods of these submanifolds. Using almost holomorphic extensions, we obtain uniformly bounded realizations of such operators on a natural scale of exponentially weighted spaces of holomorphic functions for all Gevrey indices, with remainders that are optimally small, provided that the Gevrey index is less than or equal to 2.

RÉSUMÉ. — Nous étudions les opérateurs pseudo-différentiels de classe Gevrey dans le cadre semi-classique, agissant sur des sous-espaces  $L^2$  à poids exponentiels de fonctions holomorphes. Les symboles de tels opérateurs sont des fonctions de classe Gevrey définies sur des sous-variétés I-lagrangiennes appropriées du complexifié de l'espace des phases, qui sont prolongés par des extensions presque-analytiques dans la même classe de Gevrey, ou dans un espace fonctionnel plus large, définies sur des voisinages complexes de ces sous-variétés. En utilisant les extensions presque-analytiques et la méthode de déformation de contours dans le plan complexe, nous obtenons – avec des choix de contours adéquats « bon contours » – des réalisations uniformément bornées de ces opérateurs sur une famille « naturelle » de sous-espaces  $L^2$  à poids exponentiels de fonctions holomorphes pour tous les indices de Gevrey, avec des estimées optimales des restes sous la condition que l'indice de Gevrey est inférieur ou égal à 2.

## 1. Introduction and statement of results

Starting with the pioneering work [24], the study of (pseudo)differential operators with Gevrey coefficients has had a long tradition in the PDE theory, see also [15, 16, 17, 25]. The work [16], in particular, develops the

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*Keywords:* Semiclassical Gevrey pseudodifferential operator, almost holomorphic extension, FBI transform, Bargmann space, strictly plurisubharmonic weight function.

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semiclassical Weyl calculus of pseudodifferential operators on  $\mathbf{R}^n$ , with symbols having Gevrey regularity in both the position and momentum variables. Now it is well known that the Weyl quantization assumes a particularly simple and convenient form when passing from the Schrödinger representation in the real setting to the FBI–Bargmann representation in the complex domain, conjugating the operators by a suitable globally defined metaplectic FBI transformation, see [12, 33]. Once transported to the FBI transform side, pseudodifferential operators in the Weyl quantization act on exponentially weighted spaces of entire holomorphic functions of the form

$$(1.1) \quad H_{\Phi_0}(\mathbf{C}^n) = \text{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),$$

where the weight function  $\Phi_0$  is quadratic strictly plurisubharmonic, and  $L(dx)$  is the Lebesgue measure on  $\mathbf{C}^n$ . The purpose of the present work is to apply some of the  $H_{\Phi}$ -techniques to the study of semiclassical Weyl pseudodifferential operators with Gevrey symbols in the complex domain, showing a number of fairly general results concerning their symbolic and mapping properties. While the present work does not contain any applications to the study of propagation of singularities in Gevrey spaces, we expect the results established here to be useful in this respect and plan to return to these aspects in the near future. Let us now proceed to describe the precise assumptions and state the main results.

Let  $s > 1$ . The (global) Gevrey class  $\mathcal{G}_b^s(\mathbf{R}^m)$  consists of all functions  $u \in C^\infty(\mathbf{R}^m)$  such that there exist  $A > 0$ ,  $C > 0$  such that for all  $\alpha \in \mathbf{N}^m$ , we have

$$(1.2) \quad |\partial^\alpha u(x)| \leq AC^{|\alpha|}(\alpha!)^s, \quad x \in \mathbf{R}^m.$$

Let us also set  $\mathcal{G}_0^s(\mathbf{R}^m) = \mathcal{G}_b^s(\mathbf{R}^m) \cap C_0^\infty(\mathbf{R}^m)$ .

Associated to the quadratic form  $\Phi_0$  in (1.1) is the real linear subspace

$$(1.3) \quad \Lambda_{\Phi_0} = \left\{ \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right), x \in \mathbf{C}^n \right\} \subset \mathbf{C}^{2n} = \mathbf{C}_x^n \times \mathbf{C}_\xi^n,$$

which can be viewed as the image of the real phase space  $\mathbf{R}^{2n}$  under a complex linear canonical transformation, see [12, 31, 33]. Identifying  $\Lambda_{\Phi_0}$  linearly with  $\mathbf{C}_x^n$ , via the projection map  $\pi_x : \Lambda_{\Phi_0} \ni (x, \xi) \mapsto x \in \mathbf{C}_x^n$ , we may define the Gevrey spaces  $\mathcal{G}_b^s(\Lambda_{\Phi_0})$ ,  $\mathcal{G}_0^s(\Lambda_{\Phi_0})$ . Given  $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ , for some  $s > 1$ , and  $u \in \text{Hol}(\mathbf{C}^n)$  such that  $u(x) = \mathcal{O}_{h,N}(1)\langle x \rangle^{-N} e^{\Phi_0(x)/h}$  for all  $N \in \mathbf{N}$ , let us introduce the semiclassical Weyl quantization of  $a$  acting

on  $u$ ,

$$(1.4) \quad a_{\Gamma}^w(x, hD_x)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a\left(\frac{x+y}{2}, \theta\right) u(y) \, dy \wedge d\theta.$$

Here  $0 < h \leq 1$  is the semiclassical parameter and  $\Gamma(x) \subset \mathbf{C}_{y,\theta}^{2n}$  is the natural integration contour given by

$$(1.5) \quad \theta = \frac{2}{i} \frac{\partial\Phi_0}{\partial x} \left( \frac{x+y}{2} \right).$$

In this work, we shall consider deformations of the standard weight  $\Phi_0$ , and to this end let  $\Phi_h \in C^{1,1}(\mathbf{C}^n; \mathbf{R})$  be such that

$$(1.6) \quad \|\nabla^k(\Phi_h - \Phi_0)\|_{L^\infty(\mathbf{C}^n)} \leq \frac{1}{C} h^{1-\frac{1}{s}}, \quad k = 0, 1, 2,$$

for some  $C > 0$  sufficiently large. Our first main result is as follows – see also Theorem 3.1 and Theorem 3.3 below for a slightly more general statement.

**THEOREM 1.1.** — *Let  $\omega = h^{1-\frac{1}{s}}$  and introduce the following  $2n$ -dimensional Lipschitz contour and  $x \in \mathbf{C}^n$ ,*

$$(1.7) \quad \Gamma_{\omega}^{\Phi_h}(x) : \quad \theta = \frac{2}{i} \frac{\partial\Phi_h}{\partial x} \left( \frac{x+y}{2} \right) + i f_{\omega}(x-y), \quad y \in \mathbf{C}^n,$$

where

$$(1.8) \quad f_{\omega}(z) = \begin{cases} \bar{z}, & |z| \leq \omega, \\ \omega \frac{\bar{z}}{|z|}, & |z| > \omega. \end{cases}$$

Let  $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ , for some  $1 < s \leq 2$ , and let  $\tilde{a} \in \mathcal{G}_b^s(\mathbf{C}^{2n})$  be an almost holomorphic extension of  $a$  such that  $\text{supp } \tilde{a} \subset \Lambda_{\Phi_0} + B_{\mathbf{C}^{2n}}(0, C_0)$ , for some  $C_0 > 0$ . We have

$$(1.9) \quad a_{\Gamma}^w(x, hD_x) - \tilde{a}_{\Gamma_{\omega}^{\Phi_h}}^w(x, hD_x) = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\right) : \\ L^2(\mathbf{C}^n, e^{-2\Phi_h/h} L(dx)) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_h/h} L(dx)),$$

where the realization

$$\tilde{a}_{\Gamma_{\omega}^{\Phi_h}}^w(x, hD_x)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma_{\omega}^{\Phi_h}(x)} e^{\frac{i}{h}(x-y)\cdot\theta} \tilde{a}\left(\frac{x+y}{2}, \theta\right) u(y) \, dy \wedge d\theta$$

satisfies

$$(1.10) \quad \tilde{a}_{\Gamma_{\omega}^{\Phi_h}}^w(x, hD_x) = \mathcal{O}(1) : H_{\Phi_h}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_h/h} L(dx)).$$

Here we have set, similarly to (1.1),

$$H_{\Phi_h}(\mathbf{C}^n) = \text{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, e^{-2\Phi_h/h} L(dx)).$$

In the range of Gevrey indices  $s > 2$ , it turns out that we have to accept remainders that are larger than the ones in (1.9), when obtaining uniformly bounded realizations of the operator  $a_{\Gamma}^w(x, hD_x)$  on the weighted spaces  $H_{\Phi_0}(\mathbf{C}^n)$ ,  $H_{\Phi_h}(\mathbf{C}^n)$ . The following is the second main result of this work.

THEOREM 1.2. — *Let  $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ , for some  $s > 2$ , and let  $\tilde{a} \in \mathcal{G}_b^s(\mathbf{C}^{2n})$  be an almost holomorphic extension of  $a$  as in Theorem 1.1. Let  $\Gamma_{h^{1/2}}^{\Phi_h}(x)$  be the  $2n$ -dimensional Lipschitz contour defined as in (1.7), (1.8), with  $\omega$  replaced by  $h^{1/2} \geq \omega$ . We have*

$$(1.11) \quad a_{\Gamma}^w(x, hD_x) - \tilde{a}_{\Gamma_{h^{1/2}}^{\Phi_h}}^w(x, hD_x) = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{2s-2}}\right) : L^2(\mathbf{C}^n, e^{-2\Phi_h/h} L(dx)) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_h/h} L(dx)),$$

where

$$(1.12) \quad \tilde{a}_{\Gamma_{h^{1/2}}^{\Phi_h}}^w(x, hD_x) = \mathcal{O}(1) : H_{\Phi_h}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_h/h} L(dx)).$$

As is seen in the statements of Theorem 1.1 and Theorem 1.2, a crucial role in this work is played by the existence of a Gevrey almost holomorphic extension of a symbol  $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ , off the maximally totally real linear subspace  $\Lambda_{\Phi_0} \subset \mathbf{C}^{2n}$  to a complex neighborhood. As discussed in Section 2, the existence of such an extension may be obtained by solving a Borel problem in the Gevrey space, relying on the work [1] by Carleson. Alternatively, the existence of an extension  $\tilde{a} \in C_b^\infty(\mathbf{C}^{2n})$  of  $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$  such that

$$(1.13) \quad |\bar{\partial}\tilde{a}(\rho)| \leq \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} \text{dist}(\rho, \Lambda_{\Phi_0})^{-\frac{1}{s-1}}\right), \quad \rho \in \mathbf{C}^{2n},$$

may be obtained by adapting a construction of Mather [21], see also [3], working with the Fourier inversion formula with a cutoff.

We would like to emphasize that, as explained in Section 3, replacing the Lipschitz contour  $\Gamma_{\omega}^{\Phi_0}(x)$  in Theorem 1.1 by a contour of the form

$$\theta = \frac{2}{i} \frac{\partial\Phi_0}{\partial x} \left(\frac{x+y}{2}\right) + \frac{i}{C} \overline{(x-y)}, \quad C > 0,$$

natural in the holomorphic category [12, 31, 33], leads only to remainder estimates of the form

$$R = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{2s-1}}\right) : H_{\Phi_0}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),$$

and thus, working with a “mixed” contour such as  $\Gamma_{\omega}^{\Phi_0}(x)$ , staying closer to  $\Lambda_{\Phi_0}$ , seems essential when obtaining accurate remainder estimates. The price that we have to pay for working with the contour  $\Gamma_{\omega}^{\Phi_h}(x)$  in (1.7),

(1.8) is that the realizations of our Gevrey pseudodifferential operators along such contours are uniformly bounded in the range  $1 < s \leq 2$  only. Closely related to this is the well known observation [16, 17] that while the class of operators of the form  $a_{\Gamma}^w(x, hD_x)$ , with  $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ , is stable under the composition, the standard asymptotic Weyl calculus does not lead to some sharp control of the remainders in the semiclassical expansions – see also (3.151) below.

Let us conclude the introduction by mentioning several works where the Gevrey regularity questions were studied, which have provided some of the motivation for the present paper. The recent work [9] gives a detailed treatment of the Gevrey regularity framework on arbitrary real analytic compact manifolds, motivated by the microlocal study of dynamical zeta functions and trace formulas for Anosov flows. In the context of scattering theory, Gevrey regularity questions were considered in [7, 26]. We would finally like to refer to [28, 29, 30] and to [15, 18] for results on the propagation of analytic and Gevrey singularities for boundary value problems, see also [32]. To the best of our knowledge, the question whether the result of [18] on the non-diffraction of Gevrey 3 singularities holds true in the complement of a Gevrey 3 obstacle is still open, see [17].

The plan of the paper is as follows. Section 2 is devoted to the discussion of almost holomorphic extensions of Gevrey symbols. We also establish an approximate uniqueness of almost holomorphic extensions satisfying (1.13). In Section 3 we study semiclassical Gevrey pseudodifferential operators acting on  $H_{\Phi}$ -spaces, establishing Theorem 1.1 and Theorem 1.2, using almost holomorphic extensions and contour deformations. The section is concluded by the discussion of the composition of semiclassical Gevrey operators, by the methods of phase symmetries and contour deformations.

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## 2. Gevrey spaces and almost holomorphic extensions

In this section we shall recall some well known facts concerning almost holomorphic extensions. One can consult [6] for a recent very general treatment with plenty of references, in particular to the pioneering work of Dyn'kin [4]. See also [5, 17, 22], and [3].

Let  $\Omega \subset \mathbf{C}^d$  be open and put  $\Omega_R = \Omega \cap \mathbf{R}^d$ . A function  $\tilde{u} \in C^\infty(\Omega)$  is called an almost holomorphic extension of  $u \in C^\infty(\Omega_R)$  if

- (i)  $\tilde{u}|_{y=0} = u$ , and
- (ii)  $\bar{\partial}\tilde{u} = \sum_{j=1}^d \partial_{\bar{z}_j} \tilde{u} d\bar{z}_j$  is flat on  $y = 0$ .

Here, we identify  $\mathbf{C}^d$  with  $\mathbf{R}^{2d}$  in the usual way:  $\mathbf{C}^d \ni z = x + iy$ ,  $(x, y) \in \mathbf{R}^{2d}$ . Recall also that for  $j = 1, \dots, d$ , we have  $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ ,  $\partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ ,  $\partial_{z_j} - \partial_{\bar{z}_j} = -i\partial_{y_j}$  and  $\partial_{z_j} + \partial_{\bar{z}_j} = \partial_{x_j}$ . If  $\tilde{u}$  is an almost holomorphic extension of  $u$ , the conditions (i) and (ii) above determine the asymptotic (Taylor) expansion of  $\tilde{u}$  on  $y = 0$ , i.e.,

$$\tilde{u}(x + iy) = \sum_{|\alpha| < N} \frac{i^{|\alpha|}}{\alpha!} u^{(\alpha)}(x) y^\alpha + \mathcal{O}(|y|^N),$$

locally uniformly on  $\text{neigh}(\Omega_R, \Omega)$  for every  $N \geq 1$ . Here we write

$$\text{neigh}(A, B)$$

as an abbreviation for “some neighborhood of  $A$  in  $B$ ”, and  $u^{(\alpha)} = \partial^\alpha u$ .

The function  $\tilde{u}$  is an almost holomorphic extension of  $u$  if and only if  $\tilde{u}$  solves the Borel problem:

$$(2.1) \quad (\partial_y^\alpha \tilde{u})|_{y=0} = i^{|\alpha|} u^{(\alpha)}, \quad \text{for all } \alpha \in \mathbf{N}^d.$$

Indeed, we have already checked the necessity of (2.1), and if (2.1) is satisfied by  $\tilde{u} \in C^\infty(\mathbf{C}^d)$  then  $\tilde{u}|_{y=0} = u$  and more generally  $(\partial_x^\gamma \partial_y^\beta \tilde{u})|_{y=0} = i^{|\beta|} u^{(\beta+\gamma)}$  for all  $\beta, \gamma \in \mathbf{N}^d$ . It follows that  $\partial_{\bar{z}_j} \tilde{u}$  is flat on  $y = 0$  as

$$(\partial_x^\gamma \partial_y^\beta \partial_{\bar{z}_j} \tilde{u})|_{y=0} = \frac{i^{|\beta|}}{2} (u^{(\beta+\gamma+e_j)} + i^2 u^{(\beta+\gamma+e_j)}) = 0.$$

Here  $e_j$  denotes the multi-index  $(\delta_k^j)_{1 \leq k \leq d} \in \mathbf{N}^d$ , where  $\delta_k^j$  is the Kronecker delta.

Let  $\mathcal{U}$  be an open subset of  $\mathbf{R}^d$ , and let  $s \geq 1$ . The Gevrey space  $\mathcal{G}^s(\mathcal{U})$  is the space of functions  $u \in C^\infty(\mathcal{U})$  such that for every  $K \Subset \mathcal{U}$ , there exist  $A > 0$ ,  $C > 0$  such that

$$(2.2) \quad |\partial^\alpha u(x)| \leq A C^{|\alpha|} (\alpha!)^s, \quad \text{for all } x \in K, \alpha \in \mathbf{N}^d.$$

The class  $\mathcal{G}^1(\mathcal{U})$  is the space of real analytic functions in  $\mathcal{U}$ , while for  $s > 1$ , we have  $\mathcal{G}_0^s(\mathcal{U}) := \mathcal{G}^s(\mathcal{U}) \cap C_0^\infty(\mathcal{U}) \neq \{0\}$ , see [13, Theorem 1.3.5]. We also let  $\mathcal{G}_b^s(\mathbf{R}^d) \subset \mathcal{G}^s(\mathbf{R}^d)$  be the space of functions  $u \in C^\infty(\mathbf{R}^d)$  satisfying



the Gevrey condition (2.2) uniformly on all of  $\mathbf{R}^d$ : we have  $u \in \mathcal{G}_b^s(\mathbf{R}^d)$  precisely when there exist  $A > 0, C > 0$  such that for all  $\alpha \in \mathbf{N}^d$ , we have

$$(2.3) \quad |\partial^\alpha u(x)| \leq AC^{|\alpha|}(\alpha!)^s, \quad x \in \mathbf{R}^d.$$

**2.1. Almost holomorphic extensions via a result of Carleson**

We assume now that  $u \in \mathcal{G}_0^s(\mathbf{R}^d)$  with  $s > 1$ . In view of the above remark, one has to solve the Borel problem (2.1) in  $\mathcal{G}^s(\mathbf{R}^{2d})$  in order to obtain an almost holomorphic extension  $\tilde{u}$  of  $u$  in the same Gevrey class.

This may be achieved through the Carleson theorem with a suitable choice of the weight function, see [1, Theorem 2 and Example 2]. This theorem is a corollary of a more general result which has allowed to resolve the issue known as a “universal moment problem”. See [1, Theorem 1]. We get

PROPOSITION 2.1. — *Let  $u \in \mathcal{G}_0^s(\mathbf{R}^d)$ . Then  $u$  has an almost holomorphic extension  $\tilde{u} \in \mathcal{G}^s(\mathbf{C}^d)$ .*

Clearly,  $\tilde{u}$  vanishes to infinite order on  $\mathbf{R}^d \setminus \text{supp}(u)$ . Let  $\chi \in \mathcal{G}_0^s(\mathbf{C}^d)$  be equal to 1 near  $\text{supp}(u)$ . Then  $\chi\tilde{u} \in \mathcal{G}_0^s(\mathbf{C}^d)$  is also an almost holomorphic extension of  $u$ . This gives the following variant:

PROPOSITION 2.2. — *Let  $\Omega_R, \Omega$  be as above and let  $u \in \mathcal{G}_0^s(\Omega_R)$ . Then  $u$  has an almost holomorphic extension  $\tilde{u} \in \mathcal{G}_0^s(\Omega)$ .*

With the help of Gevrey cutoffs and partitions of unity, we get the following variant:

PROPOSITION 2.3. — *Let  $\Omega_R, \Omega$  be as above and let  $u \in \mathcal{G}^s(\Omega_R)$ . Then  $u$  has an almost holomorphic extension  $\tilde{u} \in \mathcal{G}^s(\Omega)$ .*

In the case when  $u \in \mathcal{G}_b^s(\mathbf{R}^d)$ , we obtain that there exists an almost holomorphic extension  $\tilde{u} \in \mathcal{G}_b^s(\mathbf{C}^d)$ , which is supported in a bounded tubular neighborhood of  $\mathbf{R}^d \subset \mathbf{C}^d$ .

Let  $\tilde{u} \in \mathcal{G}_0^s(\mathbf{C}^d)$  be an almost holomorphic extension of  $u \in \mathcal{G}_0^s(\mathbf{R}^d)$ . In view of Taylor’s formula, there exist  $C > 0, A > 0$  such that

$$(2.4) \quad \left| (\bar{\partial}\tilde{u})(z) \right| \leq CA^N (N!)^{s-1} |\text{Im}(z)|^N, \quad \text{for all } N \in \mathbf{N}.$$

Here,  $\text{Im}(z)$  denotes the imaginary part of  $z \in \mathbf{C}^d$ . Taking the infimum over  $N$  one gets

$$(2.5) \quad \left| (\bar{\partial}\tilde{u})(z) \right| = \mathcal{O}(1) \exp \left( -\frac{1}{\mathcal{O}(1)} |\text{Im}(z)|^{-\frac{1}{s-1}} \right),$$

where  $\mathcal{O}(1)$  denotes a number whose modulus is bounded by some large positive constant, positive when appearing in a denominator.

Now if  $\tilde{u}_1, \tilde{u}_2$  are two almost holomorphic extensions of  $u$  as above, we have in view of (2.1) and of Taylor’s formula

$$|(\tilde{u}_1 - \tilde{u}_2)(z)| \leq CA^N (N!)^{s-1} |\operatorname{Im}(z)|^N,$$

for all  $N \in \mathbf{N}$ . Taking the infimum over  $N$ , we get,

$$(2.6) \quad |(\tilde{u}_1 - \tilde{u}_2)(z)| = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} |\operatorname{Im}(z)|^{-\frac{1}{s-1}}\right).$$

In the analogous context of Proposition 2.3, we get (2.6) locally uniformly on  $\Omega$ .

### 2.2. Fourier transforms

The Fourier transform of a function  $u \in \mathcal{S}(\mathbf{R}^d)$  is given by

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbf{R}^d} e^{-iy \cdot \xi} u(y) dy, \text{ where } y \cdot \xi = \sum_{j=1}^d y_j \xi_j,$$

and we have the Fourier inversion formula,

$$u(x) = (2\pi)^{-d} \int_{\mathbf{R}^d} e^{ix \cdot \xi} \hat{u}(\xi) d\xi.$$

If  $s > 1$  and  $u \in \mathcal{G}_0^s(\mathbf{R}^d)$ , there exists a constant  $C > 0$  such that

$$(2.7) \quad |\hat{u}(\xi)| \leq C \exp\left(-\frac{1}{C} |\xi|^{\frac{1}{s}}\right), \text{ for every } \xi \in \mathbf{R}^d.$$

Indeed, if  $N \in 2\mathbf{N}$ , we have  $\mathcal{F}\left((1 - \Delta)^{\frac{N}{2}} u\right)(\xi) = (1 + |\xi|^2)^{\frac{N}{2}} \hat{u}(\xi)$ , and it follows that  $|\hat{u}(\xi)| \leq A^{N+1} (1 + |\xi|^2)^{-\frac{N}{2}} (N!)^s$ . It then suffices to choose  $N \sim \left(\frac{|\xi|}{A}\right)^{\frac{1}{s}}$  and apply Stirling’s formula.

Conversely, from the Fourier inversion formula, we see that if  $u \in \mathcal{S}(\mathbf{R}^d)$  and (2.7) holds, then  $u \in \mathcal{G}_b^s(\mathbf{R}^d)$ .

### 2.3. Almost holomorphic extensions in the spirit of Mather

Let  $u \in \mathcal{G}_0^s(\mathbf{R}^d)$  and let  $\tilde{u}$  be an almost holomorphic extension of  $u$  satisfying (2.5). If  $\xi \in \mathbf{R}^d$ ,  $|\xi| \geq 1$ , we may assume after an orthogonal

change of coordinates, that  $\xi/|\xi| = e_d = (0, \dots, 0, 1) \in \mathbf{R}^d$  and in view of Stokes' formula we have

$$\widehat{u}(\xi) = \int_{\mathbf{\Pi}_d^-} e^{-iz \cdot \xi} (\partial_{\bar{z}_d} \tilde{u})(x', z) dx' \wedge d\bar{z}_d \wedge dz_d,$$

where  $\mathbf{\Pi}_d^- = \{(x', z_d) \in \mathbf{R}^{d-1} \times \mathbf{C}; \text{Im}(z_d) \leq 0\}$ .

From (2.5) we get  $|\partial_{\bar{z}_d} \tilde{u}(z)| = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} |\text{Im}(z)|^{-\frac{1}{s-1}}\right)$  for all  $z \in \mathbf{\Pi}_d^-$ . Then,

$$|\widehat{u}(\xi)| = \mathcal{O}(1) \exp\left(-\inf_{0 \leq t < \infty} (t|\xi| + \tilde{C}t^{-\frac{1}{s-1}})\right),$$

for some  $\tilde{C} > 0$ . A straightforward calculation shows that the infimum over the positive half axis is attained at

$$t_\xi = \left(\frac{\tilde{C}}{s-1}\right)^{\frac{s-1}{s}} |\xi|^{-\frac{s-1}{s}},$$

and the corresponding value of the infimum is equal to

$$\tilde{C}^{\frac{s-1}{s}} \frac{s}{(s-1)^{\frac{s-1}{s}}} |\xi|^{\frac{1}{s}},$$

which implies that

$$(2.8) \quad |\widehat{u}(\xi)| = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} |\xi|^{\frac{1}{s}}\right), \quad \text{for all } \xi \in \mathbf{R}^d.$$

Conversely, assume that (2.8) holds (which is the case when  $u \in \mathcal{G}_0^s(\mathbf{R}^d)$ ). Following [21], we look for an extension  $\tilde{u}(z) = \tilde{u}(x + iy)$  by truncation in Fourier's inversion formula. Let us start with the formal identity,

$$\tilde{u}(z) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{i\xi \cdot x - \xi \cdot y} \widehat{u}(\xi) d\xi = \mathcal{O}(1) \int_{\mathbf{R}^d} e^{|\xi||y| - \frac{|\xi|^{\frac{1}{s}}}{C}} d\xi,$$

where  $C$  is a positive constant. For  $|y| \leq |\xi|^{-(s-1)/s}/(2C)$ ,  $|\xi| \geq 1$ , the integrand in the last integral is  $\leq \exp(-|\xi|^{1/s}/(2C))$ . Let  $\psi \in C_0^\infty([0, 1/(2C)))$  be equal to 1 near 0 and set

$$(2.9) \quad \tilde{u}(z) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \psi\left(|y| |\xi|^{\frac{s-1}{s}}\right) \widehat{u}(\xi) e^{i(x+iy) \cdot \xi} d\xi$$

so that  $\tilde{u} \in C^\infty(\mathbf{C}^d)$ . We have

$$(2.10) \quad \bar{\partial} \tilde{u}(z) = \frac{1}{(2\pi)^d} \int (\partial_r \psi)\left(|y| |\xi|^{\frac{s-1}{s}}\right) |\xi|^{\frac{s-1}{s}} \partial_{\bar{z}}(|y|) \widehat{u}(\xi) e^{i(x+iy) \cdot \xi} d\xi.$$

Here,  $\partial_{\bar{z}}(|y|) = \mathcal{O}(1)$  and on the support of  $(\partial_r \psi)(|y| |\xi|^{(s-1)/s})$  we have for some constant  $\tilde{C} > 2C$ ,

$$\frac{1}{\tilde{C}} \leq |y| |\xi|^{\frac{s-1}{s}} \leq \frac{1}{2C},$$

i.e.

$$(2.11) \quad \left( \frac{1}{\tilde{C}|y|} \right)^{\frac{1}{s-1}} \leq |\xi|^{\frac{1}{s}} \leq \left( \frac{1}{2C|y|} \right)^{\frac{1}{s-1}}.$$

We conclude that for  $\tilde{u}$  given in (2.9),

$$(2.12) \quad |\bar{\partial} \tilde{u}(z)| \leq \mathcal{O}(1) \exp\left(-|\operatorname{Im} z|^{-\frac{1}{s-1}} / \mathcal{O}(1)\right), \quad |\operatorname{Im} z| \leq \mathcal{O}(1).$$

We get the same estimates for  $\partial_z^\alpha \partial_{\bar{z}}^\beta \bar{\partial} \tilde{u}$ , for all  $\alpha, \beta \in \mathbf{N}^d$ .

### 2.4. Approximate uniqueness via a Carleman estimate

The existence of almost holomorphic extensions was established in the previous subsections by appealing to the result of Carleson [1] and also by Mather’s method [21]. We shall consider here almost holomorphic extensions which are not necessarily Gevrey and we shall get approximate uniqueness through a Carleman estimate of Hörmander type.

To get the desired uniqueness estimate at a given point  $z \in \mathbf{C}^d$  with  $\operatorname{Im} z \neq 0$ , we can restrict the attention to the complex line

$$(2.13) \quad \operatorname{Re} z + \mathbf{C} \operatorname{Im} z,$$

so it will suffice to consider the one-dimensional case.

Let  $\Omega \Subset \mathbf{C}$  be open connected with smooth boundary. Let  $\phi \in C^\infty(\bar{\Omega}; \mathbf{R})$  be strictly subharmonic,

$$(2.14) \quad \partial_z^2 \bar{\partial} \phi > 0 \quad \text{on } \bar{\Omega}.$$

To the operators

$$(2.15) \quad \begin{aligned} P &:= D_{\bar{z}} = \frac{1}{i} \partial_{\bar{z}} = \frac{1}{2} (D_x + iD_y) \\ \text{and } P^* &:= D_z = \frac{1}{i} \partial_z = \frac{1}{2} (D_x - iD_y), \end{aligned}$$

we associate the symbols

$$\bar{\zeta} = \frac{1}{2}(\xi + i\eta) \quad \text{and} \quad \zeta = \frac{1}{2}(\xi - i\eta),$$

respectively. Here  $z = x + iy$  with  $(x, y) \in \mathbf{R}^2$ . We introduce the corresponding conjugated operators:

$$P_\phi = e^\phi P e^{-\phi} \quad \text{and} \quad P_\phi^* = e^{-\phi} P^* e^\phi .$$

More explicitly,

$$(2.16) \quad \begin{cases} P_\phi = \frac{1}{i}(\partial_{\bar{z}} - \partial_z\phi), \\ P_\phi^* = \frac{1}{i}(\partial_z + \partial_{\bar{z}}\phi). \end{cases}$$

We think of  $P_\phi$  as  $P$ , acting on  $e^{-\phi} L^2(\Omega) = L^2(\Omega; e^{2\phi(z)} L(dz))$ , where  $L(dz)$  is the Lebesgue measure on  $\mathbf{C}$ . Formally, we have

$$(2.17) \quad [P_\phi^*, P_\phi] = -[\partial_z + \partial_z\phi, \partial_{\bar{z}} - \partial_{\bar{z}}\phi] = 2\partial_z^2\phi = \frac{1}{2}\Delta_{x,y}(\phi).$$

For  $v \in H_0^1(\Omega)$  (i.e. of class  $H^1(\Omega)$ , vanishing on the boundary), we have, using the  $L^2(\Omega)$ -norm and the scalar product,

$$(2.18) \quad \|P_\phi v\|^2 = (P_\phi^* P_\phi v | v) = ([P_\phi^*, P_\phi] v | v) + \|P_\phi^* v\|^2,$$

and (2.17) leads to the Carleman estimate

$$(2.19) \quad 2(\partial_z^2\phi v | v) \leq \|P_\phi v\|^2,$$

i.e.

$$(2.20) \quad \sqrt{2} \left\| (\partial_z^2\phi)^{\frac{1}{2}} v \right\| \leq \|P_\phi v\|,$$

or after removing the conjugation,

$$(2.21) \quad \sqrt{2} \left\| e^\phi (\partial_z^2\phi)^{\frac{1}{2}} v \right\| \leq \left\| e^\phi \partial_z v \right\|, \quad \text{for } v \in H_0^1(\Omega).$$

We may notice here that in Hörmander’s approach to  $\bar{\partial}$ , we work in the weighted space  $e^\phi L^2(\Omega)$  and get a priori estimates for  $P_\phi^*$ , leading to existence results for the operator  $P_\phi$ , see [14, Chapter 4].

We shall next discuss the choice of  $\phi$ . Assume that  $\Omega$  is contained in the open upper half-plane with  $\bar{\Omega} \cap \mathbf{R} = [-1, 1]$ . We know that a  $\mathcal{G}^s$  function  $u$ , defined near  $[-1, 1]$ , has an extension  $\tilde{u} \in C^\infty(\bar{\Omega})$ , which satisfies

$$(2.22) \quad \bar{\partial}\tilde{u}(z) = \mathcal{O}(1) \exp\left(-\frac{1}{C_0}(\text{Im } z)^{-\frac{1}{s-1}}\right),$$

for some  $C_0 > 0$ . In the following we shall assume that  $C_0 = 1$  for simplicity. In order to apply (2.20), (2.21), we would like to have a suitable modification of

$$(2.23) \quad \phi(z) = (\text{Im } z)^{-\frac{1}{s-1}} .$$

Recalling that  $\partial_z^2 = \frac{1}{4} \Delta_{\text{Re } z, \text{Im } z}$ , we compute

$$\partial_{\text{Im } z} \phi = -(s-1)^{-1} (\text{Im } z)^{-(1+\frac{1}{s-1})},$$

and then

$$(2.24) \quad \Delta_{\text{Re } z, \text{Im } z} \phi = \partial_{\text{Im } z, \text{Im } z}^2 \phi = \frac{s}{(s-1)^2} (\text{Im } z)^{-(2+\frac{1}{s-1})} > 0.$$

We would like to apply (2.21) to the difference  $\tilde{u}_1 - \tilde{u}_2$  of two almost holomorphic extensions of the same function  $u$ , both satisfying (2.22) and run into two technical difficulties:

- (i) the function  $\phi$  in (2.23) is not smooth up to the real part of the boundary of  $\Omega$ ,
- (ii) the difference  $(\tilde{u}_1 - \tilde{u}_2)$  does not vanish on all of  $\partial\Omega$ , but only on  $\partial\Omega \cap \mathbf{R}$ .

The first difficulty is easy to resolve by replacing  $\phi$  by  $\phi_\varepsilon(z) = \phi(z + i\varepsilon)$  and letting  $\varepsilon$  tend to 0. To resolve the second difficulty, one can multiply  $(\tilde{u}_1 - \tilde{u}_2)$  by a cutoff function that vanishes near  $\partial\Omega \setminus ]-1, 1[$  and we then need to modify  $\phi$  in this region.

In general, let  $f(z)$  be smooth and real valued, defined near some point  $z_0 \in \mathbf{C}$  where  $f(z_0) = 0$  and  $df(z_0) \neq 0$ . Consider

$$(2.25) \quad \phi(z) = f(z)^{-\frac{1}{s-1}}$$

in  $\{z \in \text{neigh}(z_0, \mathbf{C}); f(z) > 0\}$ . Then

$$(2.26) \quad \begin{aligned} \partial_z \phi(z) &= -(s-1)^{-1} f(z)^{-(1+\frac{1}{s-1})} \partial_z f(z), \\ \partial_z^2 \phi(z) &= f(z)^{-(2+\frac{1}{s-1})} \frac{s}{(s-1)^2} \left( |\partial_z f|^2 - \frac{s-1}{s} f(z) \partial_z \bar{z} f(z) \right), \end{aligned}$$

generalizing (2.24), where  $f(z)$  was equal to  $\text{Im } z$ .

Let  $\psi \in C_0^\infty ]-1, 1[; [0, 1]$  be equal to one on  $[-\frac{1}{2}, \frac{1}{2}]$  and  $< 1$  outside that interval, let  $g(t) = 1 - \psi(t)$ , and put

$$(2.27) \quad f_\varepsilon(z) = \text{Im } z + ag(\text{Re } z) + \varepsilon,$$

$$(2.28) \quad \phi_\varepsilon(z) = f_\varepsilon(z)^{-\frac{1}{s-1}}.$$

Here  $0 < a \ll 1$  is fixed and  $\varepsilon \geq 0$  is a small parameter. We notice that

- (1)  $\phi_\varepsilon \in C^\infty(\Omega)$  when  $\varepsilon \geq 0$ ,  $\phi_\varepsilon \in C^\infty(\bar{\Omega})$  when  $\varepsilon > 0$ ,
- (2) the function  $\varepsilon \mapsto f_\varepsilon$  is increasing, while  $\varepsilon \mapsto \phi_\varepsilon$  is decreasing.
- (3) We have by (2.26) that

$$(2.29) \quad \partial_z^2 \phi_\varepsilon(z) \asymp f_\varepsilon(z)^{-(2+\frac{1}{s-1})},$$

uniformly for  $(z, \varepsilon) \in \Omega \times ]0, \varepsilon_0]$ , for some  $\varepsilon_0 > 0$ .

By (2), we have

$$\phi_\varepsilon \leq \phi_0 = \left( \operatorname{Im} z + ag(\operatorname{Re} z) \right)^{-\frac{1}{s-1}},$$

and  $\phi_0 \in C^\infty(\bar{\Omega} \setminus [-\frac{1}{2}, \frac{1}{2}])$ . Let  $\chi \in C^\infty(\bar{\Omega})$  be equal to one near  $\operatorname{supp} \psi$  and vanish near  $\partial\Omega \setminus ]-1, 1[$ .

Let  $\tilde{u}_1, \tilde{u}_2 \in C^\infty(\bar{\Omega})$  be two almost holomorphic extensions of the same  $\mathcal{G}^s$  function  $u$ , defined near  $\bar{\Omega} \cap \mathbf{R}$ , which satisfy (2.22) (with  $C_0 = 1$  for simplicity), so that

$$(2.30) \quad \bar{\partial}(\tilde{u}_1 - \tilde{u}_2) = \mathcal{O}(1) \exp\left(-(\operatorname{Im} z)^{-\frac{1}{s-1}}\right) \text{ in } \bar{\Omega},$$

$$(2.31) \quad \tilde{u}_1 - \tilde{u}_2 = 0 \text{ on } [-1, 1].$$

With  $\chi$  as above, let

$$(2.32) \quad v := \chi(\tilde{u}_1 - \tilde{u}_2) \in C^\infty(\bar{\Omega}).$$

Then

$$(2.33) \quad v|_{\partial\Omega} = 0,$$

$$(2.34) \quad \partial_{\bar{z}} v = (\tilde{u}_1 - \tilde{u}_2) \partial_{\bar{z}} \chi + \chi \partial_{\bar{z}}(\tilde{u}_1 - \tilde{u}_2) = \mathcal{O}(1) e^{-\phi_0} \text{ in } \Omega.$$

Combining this with (2.21), (2.29), (2.30) and letting  $\varepsilon \rightarrow 0$ , we get

$$(2.35) \quad \left\| (f_0(z))^{-\frac{1}{2}(2+\frac{1}{s-1})} e^{\phi_0} v \right\|_{L^2(\Omega)} \leq \mathcal{O}(1) \left\| e^{\phi_0} \partial_{\bar{z}} v \right\|_{L^2(\Omega)}.$$

Let  $W$  be a small complex neighborhood of  $[-1/2, 1/2]$  so that

$$\phi_0(z) = (\operatorname{Im} z)^{-\frac{1}{s-1}} \text{ and } \chi = 1 \text{ in } W.$$

Then

$$(2.36) \quad \left\| (\operatorname{Im} z)^{-\frac{1}{2}(2+\frac{1}{s-1})} \exp\left((\operatorname{Im} z)^{-\frac{1}{s-1}}\right) (\tilde{u}_1 - \tilde{u}_2) \right\|_{L^2(W)} \leq \mathcal{O}(1) \left\| e^{\phi_0} \partial_{\bar{z}} v \right\|_{L^2(\Omega)}.$$

By (2.34), the right hand side of (2.36) is  $\mathcal{O}(1)$ . More explicitly, from (2.36), (2.34), and the fact that  $\phi_0$  is bounded on  $\operatorname{supp}(\bar{\partial}\chi)$  and  $\leq \phi$ , we have

$$\begin{aligned} & \left\| (\operatorname{Im} z)^{-\frac{1}{2}(2+\frac{1}{s-1})} \exp\left((\operatorname{Im} z)^{-\frac{1}{s-1}}\right) (\tilde{u}_1 - \tilde{u}_2) \right\|_{L^2(W)} \\ & \leq \mathcal{O}(1) \left( \left\| \exp\left((\operatorname{Im} z)^{-\frac{1}{s-1}}\right) \bar{\partial}(\tilde{u}_1 - \tilde{u}_2) \right\|_{L^2(\Omega)} + \|\tilde{u}_1 - \tilde{u}_2\|_{L^2(\operatorname{supp} \bar{\partial}\chi)} \right), \end{aligned}$$

and we have proved the following slightly more general statement:

LEMMA 2.4. — *Let  $\Omega \Subset \mathbf{C}$  be open with smooth boundary, contained in the open upper half plane, with  $\bar{\Omega} \cap \mathbf{R} = [-1, 1]$ . Let  $\phi$  be given by (2.23). Then there exists an open neighborhood  $W$  of  $[-1/2, 1/2]$  in  $\bar{\Omega}$  such that*

$$(2.37) \quad \left\| (\text{Im } z)^{-\frac{1}{2}(2+\frac{1}{s-1})} e^\phi(\tilde{u}_1 - \tilde{u}_2) \right\|_{L^2(W)} \leq \mathcal{O}(1) \left( \|e^\phi \bar{\partial}(\tilde{u}_1 - \tilde{u}_2)\|_{L^2(\Omega)} + \|\tilde{u}_1 - \tilde{u}_2\|_{L^2(\Omega)} \right),$$

for all  $(\tilde{u}_1, \tilde{u}_2) \in (H^1(\Omega))^2$  with  $\tilde{u}_1 = \tilde{u}_2$  on  $\bar{\Omega} \cap \mathbf{R}$  and  $\bar{\partial}(\tilde{u}_1 - \tilde{u}_2) \in e^{-\phi} L^2(\Omega)$ .

This applies to the case when  $\tilde{u}_1, \tilde{u}_2$  are two almost holomorphic extensions of the same function  $u \in \mathcal{G}^s(\text{neigh}([-1, 1], \mathbf{R}))$ , satisfying (2.22) with  $C_0 = 1$ .

We observed after (2.12) that if  $u \in \mathcal{G}_0^s(\mathbf{R}^d)$ , and  $\tilde{u}$  is given in (2.9), then for all  $\alpha, \beta \in \mathbf{N}^d$ , there exists  $C_{\alpha, \beta} > 0$  such that

$$(2.38) \quad |\bar{\partial} \partial_z^\alpha \partial_{\bar{z}}^\beta \tilde{u}(z)| \leq C_{\alpha, \beta} \exp\left(-|\text{Im } z|^{\frac{1}{s-1}}/C_0\right),$$

where  $C_0 > 0$  is independent of  $\alpha, \beta$ .

In the one-dimensional case, if  $\tilde{u}_1, \tilde{u}_2$  are two almost holomorphic extensions of the same function  $u \in \mathcal{G}_0^s$ , satisfying (2.38) (assuming for simplicity that  $C_0 = 1$ ), we can apply (2.37) with  $\tilde{u}_j$  replaced by  $\partial_z^\alpha \partial_{\bar{z}}^\beta \tilde{u}_j$  and see that

$$(2.39) \quad \left\| (\text{Im } z)^{-\frac{1}{2}(2+\frac{1}{s-1})} e^\phi \partial_z^\alpha \partial_{\bar{z}}^\beta (\tilde{u}_1 - \tilde{u}_2) \right\|_{L^2(W)} \leq \mathcal{O}_{\alpha, \beta}(1).$$

Remaining in the one-dimensional case, we shall next show how to get from (2.39) an estimate of a weighted  $L^\infty$ -norm, having in mind that if  $u \in H^2(D(0, 1))$ , then in view of the Sobolev embedding theorem, we have

$$(2.40) \quad \|u\|_{L^\infty(D(0, 1/2))} \leq \mathcal{O}(1) \|u\|_{H^2(D(0, 1))}.$$

Because of the presence of exponential weights in (2.39), we shall work in very small discs  $D(z, r)$  with the property that  $e^{\phi(\zeta)} \asymp e^{\phi(z)}$  for  $\zeta \in D(z, r)$ . We have  $\nabla \phi(z) = \mathcal{O}(1)(\text{Im } z)^{-1-1/(s-1)}$  and  $\phi(\zeta) - \phi(z) = \mathcal{O}(1)$  if  $|\zeta - z| \leq (\text{Im } z)^{1+1/(s-1)}$  and  $0 < \text{Im } z \ll 1$ . Thus, with

$$(2.41) \quad r = r(z) = (\text{Im } z)^{1+\frac{1}{s-1}},$$

we have

$$(2.42) \quad e^{\phi(\zeta)} \asymp e^{\phi(z)} \quad \text{when } 0 < \text{Im } z \ll 1, \zeta \in D(z, r),$$

and we have

$$(\text{Im } \zeta)^{-\frac{1}{2}(2+\frac{1}{s-1})} \asymp (\text{Im } z)^{-\frac{1}{2}(2+\frac{1}{s-1})},$$



in the same set. For  $\zeta \in D(z, r)$ , write  $\zeta = z + rw$ ,  $w \in D(0, 1)$ . Using that

$$\partial_\zeta = r^{-1}\partial_w, \partial_{\bar{\zeta}} = r^{-1}\partial_{\bar{w}}, L(d\zeta) = r^2L(dw),$$

and replacing  $W$  in (2.39) by the smaller set  $D(z, r)$  (so we have to take  $z$  in a slightly shrunk copy of  $W$ ) we get from (2.39),

$$(2.43) \quad (\text{Im } z)^{-\frac{1}{2}(2+\frac{1}{s-1})} e^{\phi(z)} r(z)^{1-\alpha-\beta} \left\| \partial_w^\alpha \partial_{\bar{w}}^\beta (\tilde{u}_1 - \tilde{u}_2) \right\|_{L^2(D(0,1))} \leq \mathcal{O}_{\alpha,\beta}(1),$$

where  $\tilde{u}_1 - \tilde{u}_2 = (\tilde{u}_1 - \tilde{u}_2)(z + rw)$  is viewed as a function of  $w$ . Using (2.43) for  $\alpha + \beta \leq 2$ , we get a bound for  $\|\tilde{u}_1 - \tilde{u}_2\|_{H^2(D(0,1))}$  (in the  $w$ -variable) and with (2.40), we get

$$(\text{Im } z)^{-\frac{1}{2}(2+\frac{1}{s-1})} e^{\phi(z)} r(z) \|\tilde{u}_1 - \tilde{u}_2\|_{L^\infty(D(0,1/2))} \leq \mathcal{O}(1).$$

With  $w = 0$ , we obtain

$$(2.44) \quad (\text{Im } z)^{-\frac{1}{2}(2+\frac{1}{s-1})} e^{\phi(z)} r(z) |\tilde{u}_1(z) - \tilde{u}_2(z)| \leq \mathcal{O}(1),$$

uniformly on  $W$  (after a slight decrease of  $W$  or a slight increase of the original  $W$ ). Using (2.39) for higher derivatives, gives an extension,

$$(2.45) \quad (\text{Im } z)^{-\frac{1}{2}(2+\frac{1}{s-1})} e^{\phi(z)} r(z) \left| \partial_z^\alpha \partial_{\bar{z}}^\beta (\tilde{u}_1 - \tilde{u}_2)(z) \right| \leq \mathcal{O}(1),$$

uniformly on  $W$ , for every  $(\alpha, \beta) \in \mathbf{N}^2$ . Here we notice that by (2.41),

$$(2.46) \quad (\text{Im } z)^{-\frac{1}{2}(2+\frac{1}{s-1})} r(z) = (\text{Im } z)^{\frac{1}{2(s-1)}}.$$

We now return to the  $d$ -dimensional case and apply the observation around (2.13) about the reduction to the one-dimensional case. From this and (2.45), (2.46), we get the main result of this subsection,

**THEOREM 2.5.** — *Let  $u \in \mathcal{G}_0^s(\mathbf{R}^d)$ , let  $N_0 \in \mathbf{N}$ , and let  $\tilde{u}_1, \tilde{u}_2$  be two almost holomorphic extensions of  $u$  such that (cf. (2.12) and the subsequent remark) for all  $\alpha, \beta \in \mathbf{N}^d$  with  $|\alpha| + |\beta| \leq N_0 + 2$ , we have for  $j = 1, 2$ ,*

$$(2.47) \quad \left| \partial_z^\alpha \partial_{\bar{z}}^\beta \bar{\partial} \tilde{u}_j(z) \right| \leq \mathcal{O}(1) \exp\left(-|\text{Im } z|^{\frac{1}{s-1}}/C_0\right), \quad |\text{Im } z| \leq 1/\mathcal{O}(1),$$

where  $C_0 > 0$ . Then,

$$(2.48) \quad \left| \partial_z^\alpha \partial_{\bar{z}}^\beta (\tilde{u}_1(z) - \tilde{u}_2(z)) \right| \leq \mathcal{O}(1) |\text{Im } z|^{-\frac{1}{2(s-1)}} \exp\left(-|\text{Im } z|^{\frac{1}{s-1}}/C_0\right), \quad |\text{Im } z| \leq 1/\mathcal{O}(1),$$

for all  $\alpha, \beta \in \mathbf{N}^d$  with  $|\alpha| + |\beta| \leq N_0$ . We get the same conclusion for  $u \in \mathcal{G}_b^s(\mathbf{R}^d)$ .

### 3. Pseudodifferential operators with Gevrey symbols in the complex domain

#### 3.1. Almost holomorphic extensions and contour deformations

In the beginning of this subsection, we shall recall, following [12, 33], some basic facts concerning semiclassical pseudodifferential operators in the Weyl quantization, acting on quadratic exponentially weighted spaces of holomorphic functions (Bargmann spaces).

Let  $\Phi_0$  be a strictly plurisubharmonic quadratic form on  $\mathbf{C}^n$ . Associated to  $\Phi_0$  we introduce the real  $2n$ -dimensional linear subspace

$$(3.1) \quad \Lambda_{\Phi_0} = \left\{ \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right), x \in \mathbf{C}^n \right\} \subset \mathbf{C}^{2n} = \mathbf{C}_x^n \times \mathbf{C}_\xi^n.$$

The linear subspace  $\Lambda_{\Phi_0}$  is I-Lagrangian and R-symplectic, in the sense that the restriction of the complex symplectic  $(2,0)$  form

$$(3.2) \quad \sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$$

on  $\mathbf{C}_x^n \times \mathbf{C}_\xi^n$  to  $\Lambda_{\Phi_0}$  is real and non-degenerate. In particular,  $\Lambda_{\Phi_0}$  is maximally totally real. Let

$$(3.3) \quad S(\Lambda_{\Phi_0}) = \{ a \in C^\infty(\Lambda_{\Phi_0}); \partial^\alpha a = \mathcal{O}_\alpha(1) \quad \forall \alpha \in \mathbf{N}^{2n} \}.$$

We shall let symbols  $a \in S(\Lambda_{\Phi_0})$  depend on the semiclassical parameter  $h \in (0, 1]$ , provided that  $a(\cdot; h)$  belongs to a bounded set in  $S(\Lambda_{\Phi_0})$ , when  $h$  varies in  $(0, 1]$ .

Let  $a \in S(\Lambda_{\Phi_0})$  and let  $u \in \text{Hol}(\mathbf{C}^n)$  be such that

$$u(x) = \mathcal{O}_{h,N}(1) \langle x \rangle^{-N} e^{\frac{\Phi_0(x)}{h}},$$

for all  $N \geq 0$ . We set

$$(3.4) \quad a_\Gamma^w(x, hD_x)u(x) := \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a\left(\frac{x+y}{2}, \theta; h\right) u(y) dy \wedge d\theta,$$

where  $\Gamma(x) \subset \mathbf{C}_{y,\theta}^n$  is the natural  $2n$ -dimensional contour of integration given by

$$(3.5) \quad \theta = \frac{2}{i} \frac{\partial \Phi_0}{\partial x} \left( \frac{x+y}{2} \right), \quad y \in \mathbf{C}^n.$$

The following consequence of Taylor’s formula,

$$(3.6) \quad \operatorname{Re} \left( 2\partial_x \Phi_0 \left( \frac{x+y}{2} \right) \cdot (x-y) \right) = \Phi_0(x) - \Phi_0(y),$$

valid for the real valued quadratic form  $\Phi_0$ , assures that the integral in (3.4) converges absolutely. Let us also recall from [12, 33] that  $a_\Gamma^w(x, hD_x)u \in \operatorname{Hol}(\mathbf{C}^n)$ .

It is established in [12, 33] that the operator  $a_\Gamma^w(x, hD_x)$  extends to a uniformly bounded map

$$(3.7) \quad a_\Gamma^w(x, hD_x) = \mathcal{O}(1) : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\Phi_0}(\mathbf{C}^n).$$

Here  $H_{\Phi_0}(\mathbf{C}^n)$  is the Bargmann space defined by

$$(3.8) \quad H_{\Phi_0}(\mathbf{C}^n) = \operatorname{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),$$

with  $L(dx)$  being the Lebesgue measure on  $\mathbf{C}^n$ . The proof of the mapping property (3.7) given in [12, 33] proceeds by introducing an almost holomorphic extension  $\tilde{a} \in C^\infty(\mathbf{C}^{2n})$  of  $a \in S(\Lambda_{\Phi_0})$ , such that  $\partial^\alpha \tilde{a} \in L^\infty(\mathbf{C}^{2n})$  for all  $\alpha$ , and with the property  $\operatorname{supp} \tilde{a} \subset \Lambda_{\Phi_0} + B_{\mathbf{C}^{2n}}(0, \tilde{C})$ , for some  $\tilde{C} > 0$ . One then performs a contour deformation argument, letting  $\Gamma^t(x) \subset \mathbf{C}_{y,\theta}^{2n}$ ,  $t \in [0, 1]$ , be the contour given by

$$(3.9) \quad \theta = \frac{2}{i} \frac{\partial \Phi_0}{\partial x} \left( \frac{x+y}{2} \right) + i\overline{(x-y)}, \quad y \in \mathbf{C}^n,$$

and using Stokes’ formula to get

$$(3.10) \quad a_\Gamma^w(x, hD_x)u = \tilde{a}_{\Gamma^1}^w(x, hD_x)u + Ru.$$

Here

$$\tilde{a}_{\Gamma^1}^w(x, hD_x)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma^1(x)} e^{\frac{i}{h}(x-y)\cdot\theta} \tilde{a} \left( \frac{x+y}{2}, \theta; h \right) u(y) dy \wedge d\theta,$$

and writing

$$(3.11) \quad \tilde{a}_{\Gamma^1}^w(x, hD_x)u(x) = \int k_{\Gamma^1}(x, y; h)u(y) L(dy),$$

we see, using (3.6) and (3.9), that the effective kernel

$$e^{-\frac{\Phi_0(x)}{h}} k_{\Gamma^1}(x, y; h) e^{\frac{\Phi_0(y)}{h}}$$

of the operator  $\tilde{a}_{\Gamma^1}^w(x, hD_x)$  satisfies

$$(3.12) \quad e^{-\frac{\Phi_0(x)}{h}} k_{\Gamma^1}(x, y; h) e^{\frac{\Phi_0(y)}{h}} = \mathcal{O}(1)h^{-n} e^{-\frac{1}{h}|x-y|^2}.$$

Therefore, by Schur’s lemma, we get

$$(3.13) \quad \tilde{a}_{\Gamma^1}^w(x, hD_x) = \mathcal{O}(1) : L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),$$

and it is shown in [33, Proposition 1.2], [12, Section 1.4] that the remainder  $R$  in (3.10) satisfies

$$(3.14) \quad R = \mathcal{O}(h^\infty) : H_{\Phi_0}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)).$$

We are now ready to start the discussion of the Gevrey case. When doing so, let us notice first that identifying  $\Lambda_{\Phi_0}$  linearly with  $\mathbf{C}_x^n$ , via the projection map  $\Lambda_{\Phi_0} \ni (x, \xi) \mapsto x \in \mathbf{C}_x^n$ , we may define the Gevrey spaces  $\mathcal{G}^s(\Lambda_{\Phi_0})$ ,  $\mathcal{G}_0^s(\Lambda_{\Phi_0})$ , and  $\mathcal{G}_b^s(\Lambda_{\Phi_0})$ , for  $s > 1$ .

Given  $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0}) \subset S(\Lambda_{\Phi_0})$ , for some  $s > 1$ , we would like to establish an analogue of (3.10), (3.13), (3.14), replacing the deformed contour  $\Gamma^1(x)$  in (3.9) by another one, if necessary, where we expect the Gevrey smoothness of  $a$  to allow us to strengthen (3.14) to the estimate

$$(3.15) \quad R = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\right) : H_{\Phi_0}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)).$$

Specifically, we would like the effective kernel of the remainder to be

$$(3.16) \quad \mathcal{O}(h^{-n}) \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\right).$$

The motivation for such a decay estimate, as  $h \rightarrow 0^+$ , comes from the characterization of the space  $\mathcal{G}_0^s(\mathbf{R}^d)$  via decay properties of the Fourier transforms, see Subsection 2.2.

Let  $\tilde{a} \in \mathcal{G}_b^s(\mathbf{C}^{2n})$  be an almost holomorphic extension of  $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ , such that  $\text{supp } \tilde{a} \subset \Lambda_{\Phi_0} + B_{\mathbf{C}^{2n}}(0, \tilde{C})$ , for some  $\tilde{C} > 0$ . The existence of such an extension has been established in Section 2, and we have the following natural analogue of (2.5),

$$(3.17) \quad |\bar{\partial}\tilde{a}(x, \xi)| \leq \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} \text{dist}((x, \xi), \Lambda_{\Phi_0})^{-\frac{1}{s-1}}\right), \quad (x, \xi) \in \mathbf{C}^{2n}.$$

Proceeding similarly to the  $C^\infty$  case, let us first perform a contour deformation to the contour  $\Gamma^1(x)$  given in (3.9). We then still have (3.10), (3.13), and we only need to take a closer look at the remainder  $R$  in (3.10), making use of the full strength of (3.17).

Stokes' formula gives that

$$(3.18) \quad Ru(x) = \frac{1}{(2\pi h)^n} \iiint_{G_{[0,1]}(x)} e^{\frac{i}{h}(x-y)\cdot\theta} u(y) \bar{\partial}\left(\tilde{a}\left(\frac{x+y}{2}, \theta\right)\right) \wedge dy \wedge d\theta,$$

where  $G_{[0,1]}(x) \subset \mathbf{C}_y^n \times \mathbf{C}_\theta^n$  is the  $(2n + 1)$ -dimensional contour, given by (3.9), parametrized by  $(t, y) \in [0, 1] \times \mathbf{C}^n$ . Along  $G_{[0,1]}(x)$ , we have for

$1 \leq j \leq n,$

$$\begin{aligned}
 d\theta_j &= \frac{2}{i} \sum_{k=1}^n \Phi''_{0,x_j x_k} \frac{1}{2} dy_k + \frac{2}{i} \sum_{k=1}^n \Phi''_{0,x_j \bar{x}_k} \frac{1}{2} d\bar{y}_k - it d\bar{y}_j + i(\overline{x_j - y_j}) dt \\
 (3.19) \quad &= \sum_{k=1}^n \mathcal{O}(1) dy_k + \sum_{k=1}^n \mathcal{O}(1) d\bar{y}_k + i(\overline{x_j - y_j}) dt,
 \end{aligned}$$

and when computing  $\bar{\partial}(\tilde{a}(\frac{x+y}{2}, \theta)) \wedge dy \wedge d\theta$ , all the terms have to contain precisely one factor of  $dt$ . This form can therefore be expressed as  $|x - y|\mathcal{O}(1)L(dy) dt$ , and using (3.6), (3.9), and (3.17), we see that the expression in (3.18) takes the form

$$(3.20) \quad Ru(x) = \frac{\mathcal{O}(1)}{h^n} \int_0^1 dt \int_{\mathbf{C}^n} e^{\frac{1}{h}(\Phi_0(x) - \Phi_0(y) - t|x-y|^2)} e^{-C_1(t|x-y|)^{-\frac{1}{s-1}}} |x - y|u(y) L(dy),$$

for some  $C_1 > 0$ . Here we have also used that along  $G_{[0,1]}(x)$ , we have

$$\text{dist} \left( \left( \frac{x+y}{2}, \theta \right), \Lambda_{\Phi_0} \right) = \mathcal{O}(1)t|x-y|.$$

Writing

$$Ru(x) = \int r(x, y; h)u(y) L(dy),$$

we obtain that the effective kernel  $e^{-\frac{\Phi_0(x)}{h}} r(x, y; h) e^{\frac{\Phi_0(y)}{h}}$  of the operator  $R$  in (3.20) satisfies

$$\begin{aligned}
 (3.21) \quad &e^{-\frac{\Phi_0(x)}{h}} r(x, y; h) e^{\frac{\Phi_0(y)}{h}} \\
 &= \mathcal{O}(1) h^{-n} \int_0^1 e^{-C_1(t|x-y|)^{-\frac{1}{s-1}}} e^{-\frac{t}{h}|x-y|^2} |x-y| dt \\
 &\leq \mathcal{O}(1) h^{-n} \int_0^1 t^{-1/2} e^{-C_1(t|x-y|)^{-\frac{1}{s-1}}} e^{-\frac{t}{2h}|x-y|^2} dt \\
 &\leq \mathcal{O}(1) h^{-n} \sup_{t \in [0,1]} \left( \exp \left( -\frac{C}{h}(t|x-y|)^2 - C_1(t|x-y|)^{-\frac{1}{s-1}} \right) \right).
 \end{aligned}$$

Here  $C, C_1 > 0$ .

Setting

$$(3.22) \quad g(\sigma) = \frac{C}{h}\sigma^2 + C_1\sigma^{-\frac{1}{s-1}}, \quad \text{for } \sigma > 0,$$

we can rewrite (3.21) as follows,

$$(3.23) \quad e^{-\frac{\Phi_0(x)}{h}} r(x, y; h) e^{\frac{\Phi_0(y)}{h}} \leq \mathcal{O}(1) h^{-n} \exp \left( -\inf_{\sigma > 0} g(\sigma) \right).$$

A straightforward computation shows that the infimum of  $g$  over the positive half axis is attained at the unique point

$$(3.24) \quad \sigma_{\min} = \left( \frac{C_1 h}{2C(s-1)} \right)^{\frac{s-1}{2s-1}},$$

and the corresponding value of the infimum is equal to

$$(3.25) \quad \inf_{\sigma>0} g(\sigma) = \frac{C}{h} \sigma_{\min}^2 + C_1 \sigma_{\min}^{-\frac{1}{s-1}} = \frac{1}{\mathcal{O}(1)} h^{-\frac{1}{2s-1}}.$$

We get, using (3.23) and (3.25),

$$(3.26) \quad e^{-\frac{\Phi_0(x)}{h}} r(x, y; h) e^{\frac{\Phi_0(y)}{h}} \leq \mathcal{O}(h^{-n}) \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{2s-1}}\right),$$

which is a strictly larger upper bound than the desired one in (3.16), for all  $s > 1$ . We may therefore regard the discussion above as an indication of the fact that the deformed contour  $\Gamma^1(x)$  in (3.9), natural in the analytic case [12, 31, 33], is not quite adapted to the Gevrey theory.

As a new attempt, we shall now consider the following piecewise smooth Lipschitz “mixed” contour  $\Gamma_\omega(x) \subset \mathbf{C}_{y,\theta}^{2n}$ , defined as follows,

$$(3.27) \quad \Gamma_\omega(x) : \quad \theta = \frac{2}{i} \frac{\partial \Phi_0}{\partial x} \left( \frac{x+y}{2} \right) + i f_\omega(x-y), \quad y \in \mathbf{C}^n,$$

with

$$(3.28) \quad f_\omega(z) = \begin{cases} \bar{z}, & |z| \leq \omega, \\ \omega \frac{\bar{z}}{|z|}, & |z| > \omega. \end{cases}$$

Here  $0 < \omega < \sigma_{\min}$  is to be chosen, with  $\sigma_{\min}$  given in (3.24).

In view of (3.6), (3.27), and (3.28), we have along  $\Gamma_\omega(x)$ ,

$$(3.29) \quad \begin{aligned} \operatorname{Re}(i(x-y) \cdot \theta) + \Phi_0(y) - \Phi_0(x) &= -\operatorname{Re}((x-y) \cdot f_\omega(x-y)) \\ &= -F_\omega(x-y), \end{aligned}$$

where

$$(3.30) \quad 0 \leq F_\omega(z) = \begin{cases} |z|^2, & |z| \leq \omega, \\ \omega|z|, & |z| > \omega. \end{cases}$$

The  $2n$ -dimensional contours  $\Gamma(x)$  in (3.5) and  $\Gamma_\omega(x)$  in (3.27) are homotopic, with the homotopy given by the family of contours,

$$(3.31) \quad \Gamma_\omega(x, t) : \quad \theta = \frac{2}{i} \frac{\partial \Phi_0}{\partial x} \left( \frac{x+y}{2} \right) + it f_\omega(x-y), \quad y \in \mathbf{C}^n,$$

for  $t \in [0, 1]$ . Let also  $G_{[0,1],\omega}(x) \subset \mathbf{C}_{y,\theta}^{2n}$  be the  $(2n+1)$ -dimensional contour given by (3.31), parametrized by  $(t, y) \in [0, 1] \times \mathbf{C}^n$ . When  $u \in \operatorname{Hol}(\mathbf{C}^n)$

is such that  $u(x) = \mathcal{O}_{h,N}(1)\langle x \rangle^{-N} e^{\frac{\Phi_0(x)}{h}}$ , for all  $N \geq 0$ , we have, similarly to (3.10), by an application of Stokes' formula,

$$(3.32) \quad a_\Gamma^w(x, hD_x)u = \tilde{a}_{\Gamma_\omega}^w(x, hD_x)u + Ru.$$

Here

$$(3.33) \quad \begin{aligned} &\tilde{a}_{\Gamma_\omega}^w(x, hD_x)u(x) \\ &= \frac{1}{(2\pi h)^n} \iint_{\Gamma_\omega(x)} e^{\frac{i}{h}(x-y)\cdot\theta} \tilde{a}\left(\frac{x+y}{2}, \theta; h\right) u(y) \, dy \wedge d\theta, \end{aligned}$$

and

$$(3.34) \quad \begin{aligned} &Ru(x) \\ &= \frac{1}{(2\pi h)^n} \iiint_{G_{[0,1],\omega}(x)} e^{\frac{i}{h}(x-y)\cdot\theta} u(y) \bar{\partial}\left(\tilde{a}\left(\frac{x+y}{2}, \theta\right)\right) \wedge dy \wedge d\theta. \end{aligned}$$

We shall now estimate the effective kernel of the operator  $R$  in (3.34). When doing so, we notice that along  $G_{[0,1],\omega}(x)$ , we have in view of (3.31),

$$(3.35) \quad \text{dist}\left(\left(\frac{x+y}{2}, \theta\right), \Lambda_{\Phi_0}\right) \leq \mathcal{O}(1)t|f_\omega(x-y)|,$$

and using also (3.6), (3.29), (3.17), and (3.35), we conclude that, similarly to (3.20), we can write

$$(3.36) \quad \begin{aligned} &Ru(x) \\ &= \frac{\mathcal{O}(1)}{h^n} \int_0^1 dt \int_{\mathbb{C}^n} e^{\frac{i}{h}(\Phi_0(x)-\Phi_0(y)-tF_\omega(x-y))} e^{-C_1(t|f_\omega(x-y)|)^{-\frac{1}{s-1}}} u(y) L(dy), \end{aligned}$$

for some  $C_1 > 0$ . Setting

$$Ru(x) = \int r(x, y; h)u(y) L(dy),$$

we obtain from (3.36), (3.28), and (3.30) that the absolute value of the effective kernel  $e^{-\Phi_0(x)/h} r(x, y; h) e^{\Phi_0(y)/h}$  of the operator  $R$  in (3.34) does not exceed

$$(3.37) \quad \frac{\mathcal{O}(1)}{h^n} \sup_{t \in [0,1]} \begin{cases} \exp\left(-\frac{C}{h}(t|x-y|)^2 - C_1(t|x-y|)^{-\frac{1}{s-1}}\right), & |x-y| \leq \omega, \\ \exp\left(-\frac{C}{h}\omega t|x-y| - C_1(t\omega)^{-\frac{1}{s-1}}\right), & |x-y| > \omega. \end{cases}$$

Here  $C, C_1 > 0$ .

We shall now discuss the choice of the parameter  $0 < \omega < \sigma_{\min}$  in (3.27), (3.28), and here our goal is to achieve an upper bound of the form (3.16)

for (3.37). First, in the region  $|x - y| \leq \omega$ , we have in view of (3.37),

$$e^{-\Phi_0(x)/h} r(x, y; h) e^{\Phi_0(y)/h} \leq \mathcal{O}(1)h^{-n} \exp\left(-\inf_{t \in [0,1]} g(t|x-y|)\right),$$

where the function  $g$  has been defined in (3.22). Since  $g$  is decreasing on the interval  $(0, \sigma_{\min})$ , it suffices to choose  $0 < \omega < \sigma_{\min}$  so that

$$(3.38) \quad g(\omega) \geq \frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}.$$

To this end, recalling (3.22), let us choose  $0 < \omega$  such that  $\omega^{-\frac{1}{s-1}} = \frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}$ , i.e.,

$$(3.39) \quad \omega = \frac{1}{C_0} h^{1-\frac{1}{s}} \ll \sigma_{\min}.$$

The choice (3.39) assures that (3.38) holds, and let us also notice that the first term in the expression for  $g(\omega)$  satisfies

$$\frac{C}{h} \omega^2 \ll h^{-\frac{1}{s}},$$

so that

$$g(\omega) \asymp h^{-\frac{1}{s}}.$$

Here and in what follows we write  $A \asymp B$  for  $A, B \in \mathbf{R}$  if  $A, B$  have the same sign (or vanish), and we have  $A = \mathcal{O}(B)$  and  $B = \mathcal{O}(A)$ .

We conclude therefore that in the region  $|x - y| \leq \omega$ , with  $\omega$  given in (3.39), we have

$$(3.40) \quad \begin{aligned} e^{-\Phi_0(x)/h} r(x, y; h) e^{\Phi_0(y)/h} &\leq \mathcal{O}(1)h^{-n} \exp(-g(|x-y|)) \\ &\leq \mathcal{O}(1)h^{-n} \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\right). \end{aligned}$$

Next, a straightforward computation shows that in the region

$$(3.41) \quad \omega < |x - y| \leq \frac{C_1}{C(s-1)} C_0^{\frac{s}{s-1}},$$

the function

$$(3.42) \quad [0, 1] \ni t \mapsto \frac{C}{h} \omega t |x - y| + C_1 (t\omega)^{-\frac{1}{s-1}}$$



is decreasing, and therefore using (3.37) we obtain in the region (3.41),

$$\begin{aligned}
 (3.43) \quad e^{-\Phi_0(x)/h} r(x, y; h) e^{\Phi_0(y)/h} & \leq \mathcal{O}(1) h^{-n} \exp\left(-\frac{C}{h} \omega |x - y| - C_1 \omega^{-\frac{1}{s-1}}\right) \\
 & \leq \mathcal{O}(1) h^{-n} \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\right).
 \end{aligned}$$

Here we have also used (3.39). Finally, in the exterior region

$$(3.44) \quad \frac{C_1}{C(s-1)} C_0^{\frac{s}{s-1}} < |x - y|,$$

the function in (3.42) achieves its infimum at the unique critical point

$$t_{\min} = \left(\frac{C_1}{C(s-1)}\right)^{\frac{s-1}{s}} \frac{C_0}{|x - y|^{(s-1)/s}} \in (0, 1),$$

and the corresponding critical value is of the form

$$(3.45) \quad \frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}} |x - y|^{1/s}.$$

We get therefore in the region (3.44),

$$\begin{aligned}
 (3.46) \quad e^{-\Phi_0(x)/h} r(x, y; h) e^{\Phi_0(y)/h} & \leq \mathcal{O}(1) h^{-n} \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}} |x - y|^{1/s}\right) \\
 & \leq \mathcal{O}(1) h^{-n} \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\right).
 \end{aligned}$$

Combining (3.40), (3.43), and (3.46), we conclude that the effective kernel of the operator  $R$  in (3.34) obeys an upper bound of the form (3.16), provided that  $\omega$  is chosen as in (3.39).

It is now easy to derive precise bounds on the operator norms of the operators in (3.33) and (3.34), viewed as linear continuous maps on the  $L^2$ -space  $L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx))$ . Indeed, an application of Schur’s lemma together with (3.40), (3.43), and (3.46), shows first that the operator norm

of  $R$  in (3.34) does not exceed

$$\begin{aligned}
 (3.47) \quad & \mathcal{O}(1)h^{-n} \int_{|x| \leq \omega} \exp(-g(|x|)) L(dx) \\
 & + \mathcal{O}(1)h^{-n} \int_{\omega \leq |x| \leq \mathcal{O}(1)} \exp\left(-C_1\omega^{-\frac{1}{s-1}}\right) L(dx) \\
 & + \mathcal{O}(1)h^{-n} \int_{\mathcal{O}(1) \leq |x|} \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}|x|^{1/s}\right) L(dx) \\
 & = I_1 + I_2 + I_3,
 \end{aligned}$$

with the function  $g$  defined in (3.22). Here we clearly have

$$I_j = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\right), \quad j = 2, 3,$$

in view of (3.39), and when estimating the first contribution in (3.47), we obtain in view of (3.40),

$$I_1 \leq \mathcal{O}(1)h^{-n} \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\right) \leq \mathcal{O}(1) \exp\left(-\frac{1}{2\mathcal{O}(1)}h^{-\frac{1}{s}}\right).$$

We get therefore,

$$\begin{aligned}
 (3.48) \quad R &= \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\right) : \\
 & L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)).
 \end{aligned}$$

Next, turning the attention to the operator  $\tilde{a}_{\Gamma_\omega}^w(x, hD_x)$  in (3.33), and writing

$$\tilde{a}_{\Gamma_\omega}^w(x, hD_x)u(x) = \int k_{\Gamma_\omega}(x, y; h) L(dx),$$

we get in view of (3.29),

$$(3.49) \quad e^{-\Phi_0(x)/h} k_{\Gamma_\omega}(x, y; h) e^{\Phi_0(y)/h} \leq \mathcal{O}(1)h^{-n} \exp(-F_\omega(x - y)/h).$$

Recalling (3.30), in view of Schur’s lemma, we only have to control the  $L^1$  norm

$$\begin{aligned}
 (3.50) \quad \mathcal{O}(1)h^{-n} \int \exp\left(-\frac{F_\omega(x)}{h}\right) L(dx) \\
 \leq \mathcal{O}(1)h^{-n} \int_{|x| \leq \omega} \exp\left(-\frac{|x|^2}{h}\right) L(dx) \\
 + \mathcal{O}(1)h^{-n} \int_{|x| \geq \omega} \exp\left(-\frac{\omega|x|}{h}\right) L(dx) \\
 = \mathcal{O}(1) + \mathcal{O}(1)\frac{h^n}{\omega^{2n}} = \mathcal{O}(1) + \mathcal{O}(1)h^{-n(1-\frac{2}{s})}.
 \end{aligned}$$

We conclude that

$$\begin{aligned}
 (3.51) \quad \tilde{a}_{\Gamma_\omega}^w(x, hD_x) = \mathcal{O}(1) \max\left(1, h^{-n(1-\frac{2}{s})}\right) : \\
 L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),
 \end{aligned}$$

and in particular, this operator is  $\mathcal{O}(1)$  precisely when  $1 < s \leq 2$ .

We may summarize the discussion above in the following theorem, which is the main result of this subsection.

**THEOREM 3.1.** — *Let  $\Phi_0$  be a strictly plurisubharmonic quadratic form on  $\mathbf{C}^n$  and let  $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ , for some  $s > 1$ . Let  $\tilde{a} \in \mathcal{G}_b^s(\mathbf{C}^{2n})$  be an almost holomorphic extension of  $a$  such that  $\text{supp } \tilde{a} \subset \Lambda_{\Phi_0} + B_{\mathbf{C}^{2n}}(0, C)$ , for some  $C > 0$ , or more generally, let  $\tilde{a} \in C_b^1(\mathbf{C}^{2n})$  be an extension of  $a$  with the same support properties, such that (3.17) holds. Let furthermore  $\Gamma_\omega(x) \subset \mathbf{C}_{y,\theta}^{2n}$  be the piecewise smooth Lipschitz contour given in (3.27), (3.28), where  $0 < \omega$  satisfies (3.39). We have*

$$(3.52) \quad a_\Gamma^w(x, hD_x) = \tilde{a}_{\Gamma_\omega}^w(x, hD_x) + R,$$

where the operator  $\tilde{a}_{\Gamma_\omega}^w(x, hD_x)$  in (3.33) satisfies

$$\begin{aligned}
 (3.53) \quad \tilde{a}_{\Gamma_\omega}^w(x, hD_x) = \mathcal{O}(1) \max\left(1, h^{-n(1-\frac{2}{s})}\right) : \\
 H_{\Phi_0}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),
 \end{aligned}$$

and

$$\begin{aligned}
 (3.54) \quad R = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\right) : \\
 L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)).
 \end{aligned}$$

Recalling the approximate uniqueness of almost holomorphic extensions, see (2.6), (2.48), we also get the following result.

COROLLARY 3.2 (Dependence on the choice of an almost holomorphic extension). — Let  $\Phi_0$  be a strictly plurisubharmonic quadratic form on  $\mathbf{C}^n$  and let  $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ , for some  $s > 1$ . Let  $\tilde{a}_1, \tilde{a}_2 \in C_b^\infty(\mathbf{C}^{2n})$  be two almost holomorphic extensions of  $a$ , such that for  $j = 1, 2$ , we have  $\text{supp } \tilde{a}_j \subset \Lambda_{\Phi_0} + B_{\mathbf{C}^{2n}}(0, C)$ , for some  $C > 0$ , and also,

$$(3.55) \quad |\partial^\alpha \bar{\partial}^\beta \tilde{a}_j(\rho)| \leq \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} \text{dist}(\rho, \Lambda_{\Phi_0})^{-\frac{1}{s-1}}\right), \quad \rho \in \mathbf{C}^{2n},$$

for  $|\alpha| + |\beta| \leq 2$ . Then we have

$$(3.56) \quad (\tilde{a}_1)_{\Gamma_\omega}^w(x, hD_x) = (\tilde{a}_2)_{\Gamma_\omega}^w(x, hD_x) + R,$$

where

$$\begin{aligned} & (\tilde{a}_j)_{\Gamma_\omega}^w(x, hD_x) \\ &= \mathcal{O}(1) \max\left(1, h^{-n(1-\frac{2}{s})}\right) : H_{\Phi_0}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)), \end{aligned}$$

for  $j = 1, 2$ , and

$$\begin{aligned} R &= \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\right) : \\ & L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)). \end{aligned}$$

Remark. — We have, for  $(y, \theta) \in \Gamma_\omega(x)$ , in view of (3.27), (3.28),

$$\text{dist}\left(\left(\frac{x+y}{2}, \theta\right), \Lambda_{\Phi_0}\right) \leq \omega = \frac{1}{C_0} h^{1-\frac{1}{s}}.$$

When working in the Gevrey category, we should therefore stay closer to the real domain  $\Lambda_{\Phi_0}$ , than in the analytic case, see (3.9).

Remark. — As we saw in the beginning of this subsection, using analytic contours, such as  $\Gamma^1(x)$  in (3.9), leads to estimates of the form (3.26) for the effective kernels of the remainders, that are not quite precise. Closely related to this observation is the phenomenon of the loss of Gevrey smoothness in stationary phase expansions, see [17, 26]. To illustrate it in a simple setting, let  $a \in \mathcal{G}_0^s(\mathbf{R}^d)$ , for some  $s > 1$ . Arguing as in [8, Exercise 2.4], we see that there exists  $C > 0$  such that for all  $N \in \mathbf{N}$ , we have

$$(3.57) \quad \frac{1}{(2\pi h)^{d/2}} \int e^{-|x|^2/2h} a(x) dx = \sum_{j=0}^{N-1} \frac{h^j}{j!} \left(\frac{\Delta}{2}\right)^j a(0) + R_N(h),$$

where

$$(3.58) \quad |R_N(h)| \leq C^{N+1} (N!)^{2s-1} h^N.$$

Choosing  $N \sim (1/Ch)^{1/(2s-1)}$  leads to the remainder estimate of the form

$$\mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{2s-1}}\right).$$

We shall finish this subsection by discussing the dependence of the realization  $\tilde{a}_{\Gamma_\omega}^w$  on the choice of the parameter  $\omega$ , such that  $\omega \asymp h^{1-\frac{1}{s}}$ . To this end, let  $0 < \omega_j, j = 1, 2$ , be such that

$$(3.59) \quad \omega_j \asymp h^{1-\frac{1}{s}}, \quad j = 1, 2,$$

and let us introduce the natural homotopy between the contours  $\Gamma_{\omega_1}$  and  $\Gamma_{\omega_2}$ , given by

$$\Gamma_{\omega_t}(x) : \quad \theta = \frac{2}{i} \frac{\partial \Phi_0}{\partial x} \left( \frac{x+y}{2} \right) + i f_{\omega_t}(x-y), \quad y \in \mathbf{C}^n,$$

with  $\omega_t = (1-t)\omega_1 + t\omega_2, t \in [0, 1]$ . Introducing also the  $(2n+1)$ -dimensional contour  $\bigcup_{t \in [0,1]} \Gamma_{\omega_t}(x)$  and applying Stokes' formula, we get

$$(3.60) \quad \tilde{a}_{\Gamma_{\omega_1}}^w(x, hD_x)u = \tilde{a}_{\Gamma_{\omega_2}}^w(x, hD_x)u + Ru.$$

Here, similarly to (3.36), the remainder  $R$  takes the form

$$(3.61) \quad Ru(x) = \frac{\mathcal{O}(1)}{h^n} \int_0^1 dt \int_{\mathbf{C}^n} e^{\frac{1}{h}(\Phi_0(x)-\Phi_0(y)-F_{\omega_t}(x-y))} e^{-C_1|f_{\omega_t}(x-y)|^{-\frac{1}{s-1}}} \times u(y) L(dy),$$

for some  $C_1 > 0$ . To control the norm of the operator

$$R : L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),$$

in (3.61), it suffices, in view of Schur's lemma, to estimate the  $L^1$  norm

$$(3.62) \quad \mathcal{O}(1) h^{-n} \int_0^1 dt \int e^{-\frac{F_{\omega_t}(x)}{h} - C_1|f_{\omega_t}(x)|^{-\frac{1}{s-1}}} L(dx) = I_1 + I_2.$$

Here

$$(3.63) \quad I_1 = \mathcal{O}(1) h^{-n} \int_0^1 dt \int_{|x| \leq \omega_t} \exp\left(-\frac{|x|^2}{h} - C_1|x|^{-\frac{1}{s-1}}\right) L(dx),$$

and

$$(3.64) \quad I_2 = \mathcal{O}(1) h^{-n} \int_0^1 dt \int_{\omega_t < |x|} \exp\left(-\frac{|x|\omega_t}{h} - C_1\omega_t^{-\frac{1}{s-1}}\right) L(dx).$$

We have

$$\begin{aligned}
 I_1 &= \mathcal{O}(1)h^{-n} \int_0^1 dt \exp\left(-C_1\omega_t^{-\frac{1}{s-1}}\right) \int_{|x|\leq\omega_t} \exp\left(-\frac{|x|^2}{h}\right) L(dx) \\
 &\leq \mathcal{O}(1) \int_0^1 \exp\left(-C_1\omega_t^{-\frac{1}{s-1}}\right) dt \leq \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\right),
 \end{aligned}$$

since  $\omega_t^{-1/(s-1)} \geq \frac{1}{\mathcal{O}(1)}h^{-1/s}$ , for  $0 \leq t \leq 1$ . Furthermore, making the change of variables  $y = x\omega_t/h$  in (3.64), we get

$$\begin{aligned}
 (3.65) \quad I_2 &\leq \mathcal{O}(1)h^n \int_0^1 \omega_t^{-2n} \exp\left(-C_1\omega_t^{-\frac{1}{s-1}}\right) dt \\
 &\leq \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\right).
 \end{aligned}$$

We conclude that

$$\begin{aligned}
 (3.66) \quad \tilde{a}_{\Gamma_{\omega_1}}^w(x, hD_x) - \tilde{a}_{\Gamma_{\omega_2}}^w(x, hD_x) &= \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\right) : \\
 &L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),
 \end{aligned}$$

provided that  $0 < \omega_j$  are such that (3.59) holds.

### 3.2. Deformations of exponential weights

Let  $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ , where  $s > 1$ , and let  $\tilde{a} \in \mathcal{G}_b^s(\mathbf{C}^{2n})$  be an almost holomorphic extension of  $a$  such that  $\text{supp } \tilde{a} \subset \Lambda_{\Phi_0} + B_{\mathbf{C}^{2n}}(0, C)$ , for some  $C > 0$ . In Theorem 3.1, it was established that

$$\begin{aligned}
 (3.67) \quad \tilde{a}_{\Gamma_\omega}^w(x, hD_x) \\
 = \mathcal{O}(1) \max\left(1, h^{-n(1-\frac{2}{s})}\right) : H_{\Phi_0}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)).
 \end{aligned}$$

Here the  $\Gamma_\omega^{\Phi_0}(x) := \Gamma_\omega(x) \subset \mathbf{C}_{y,\theta}^{2n}$  has been introduced in (3.27), (3.28), with the parameter  $0 < \omega$  given in (3.39). The Gevrey smoothness of  $a$  allows us to consider deformations of the quadratic weight function  $\Phi_0$  – see [23, 31, 33] for this idea in the analytic case, where  $\frac{1}{\mathcal{O}(1)}$ -perturbations of  $\Phi_0$  are allowed, and [2, 10] for the  $C^\infty$ -theory, where deformations should be  $\mathcal{O}(h|\log h|)$ -close to  $\Phi_0$ . See also [20, Chapter 3] for the Gevrey case.

Let  $\Phi_h = \Phi_0 + \psi \in C^{1,1}(\mathbf{C}^n; \mathbf{R})$  be such that

$$(3.68) \quad \|\nabla^k \psi\|_{L^\infty(\mathbf{C}^n)} \leq \frac{\omega}{\mathcal{O}(1)}, \quad k = 0, 1, 2,$$

where the implicit constant in (3.68) is large enough, and let  $\Gamma_{\omega}^{\Phi_h}(x) \subset \mathbf{C}_{y,\theta}^{2n}$  be the following Lipschitz contour adapted to the weight  $\Phi_h$ , defined analogously to (3.27),

$$(3.69) \quad \Gamma_{\omega}^{\Phi_h}(x) : \quad \theta = \frac{2}{i} \frac{\partial \Phi_h}{\partial x} \left( \frac{x+y}{2} \right) + i f_{\omega}(x-y), \quad y \in \mathbf{C}^n.$$

Here  $f_{\omega}$  has been defined in (3.28), and  $0 < \omega$  satisfies (3.39). We would like to replace the contour  $\Gamma_{\omega}^{\Phi_0}(x)$  in (3.33) by  $\Gamma_{\omega}^{\Phi_h}(x)$ , and to this end we introduce the natural intermediate family of contours,

$$(3.70) \quad \Gamma_{\omega}^{\Phi_{t,h}}(x) : \quad \theta = \frac{2}{i} \frac{\partial \Phi_{t,h}}{\partial x} \left( \frac{x+y}{2} \right) + i f_{\omega}(x-y), \quad y \in \mathbf{C}^n,$$

where  $\Phi_{t,h} := (1-t)\Phi_0 + t\Phi_h$ ,  $t \in [0, 1]$ . Let  $G_{[0,1]}(x) \subset \mathbf{C}_{y,\theta}^{2n}$  be the  $(2n+1)$ -dimensional contour given by (3.70), parametrized by  $(t, y) \in [0, 1] \times \mathbf{C}^n$ , and write, by an application of Stokes' formula,

$$(3.71) \quad \begin{aligned} & \tilde{a}_{\Gamma_{\omega}^{\Phi_0}}^w(x, hD_x)u(x) - \tilde{a}_{\Gamma_{\omega}^{\Phi_h}}^w(x, hD_x)u(x) \\ &= \frac{1}{(2\pi h)^n} \iiint_{G_{[0,1]}(x)} e^{\frac{i}{h}(x-y)\cdot\theta} u(y) \bar{\partial} \left( \tilde{\alpha} \left( \frac{x+y}{2}, \theta \right) \right) \wedge dy \wedge d\theta \\ &=: Ru(x). \end{aligned}$$

Here  $u \in H_{\Phi_0}(\mathbf{C}^n)$ . Along  $G_{[0,1]}(x)$ , we have in view of (3.68), (3.70),

$$(3.72) \quad \begin{aligned} & \text{dist} \left( \left( \frac{x+y}{2}, \theta \right), \Lambda_{\Phi_0} \right) \\ & \leq \mathcal{O}(1)t \left| \nabla \psi \left( \frac{x+y}{2} \right) \right| + |f_{\omega}(x-y)| \leq \mathcal{O}(1)\omega, \end{aligned}$$

and combining (3.72) with (3.17), (3.29), (3.68), and (3.70), we get that  $Ru(x)$  is equal to

$$(3.73) \quad \begin{aligned} & \frac{\mathcal{O}(1)}{h^n} \int_0^1 dt \int e^{\frac{1}{h}(\Phi_0(x) - \Phi_0(y) - F_{\omega}(x-y) + \frac{t\omega|x-y|}{\mathcal{O}(1)})} \\ & \quad \times \exp \left( -\frac{h^{-\frac{1}{s}}}{\mathcal{O}(1)} \right) u(y) L(dy). \end{aligned}$$

The absolute value of the effective kernel of the operator  $R$  in (3.71) does not exceed therefore

$$\mathcal{O}(1) h^{-n} \exp \left( -\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}} \right) \exp \left( \frac{1}{h} \left( -F_{\omega}(x-y) + \frac{\omega|x-y|}{\mathcal{O}(1)} \right) \right).$$

Recalling (3.30) and making use of Schur’s lemma, we conclude, in view of (3.71), that

$$(3.74) \quad \tilde{a}_{\Gamma_{\Phi_0}}^w(x, hD_x) - \tilde{a}_{\Gamma_{\omega}}^w(x, hD_x) = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\right) : \\ H_{\Phi_0}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)).$$

In view of (3.67), we have now established the first part of the following result.

**THEOREM 3.3.** — *Let  $\Phi_h = \Phi_0 + \psi \in C^{1,1}(\mathbf{C}^n, \mathbf{R})$  be such that (3.68) holds and let us introduce the contour  $\Gamma_{\omega}^{\Phi_h}(x) \subset \mathbf{C}_{y,\theta}^{2n}$ , defined in (3.69). The realization*

$$\tilde{a}_{\Gamma_{\omega}^{\Phi_h}}^w(x, hD_x)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma_{\omega}^{\Phi_h}(x)} e^{\frac{i}{h}(x-y)\cdot\theta} \tilde{a}\left(\frac{x+y}{2}, \theta; h\right) u(y) dy \wedge d\theta$$

enjoys the following mapping properties:

(i) We have

$$(3.75) \quad \tilde{a}_{\Gamma_{\omega}^{\Phi_h}}^w(x, hD_x) = \mathcal{O}(1) \max\left(1, h^{-n(1-\frac{2}{s})}\right) : \\ H_{\Phi_0}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)).$$

(ii) We have

$$(3.76) \quad \tilde{a}_{\Gamma_{\omega}^{\Phi_h}}^w(x, hD_x) = \mathcal{O}(1) \max\left(1, h^{-n(1-\frac{2}{s})}\right) : \\ H_{\Phi_h}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_h/h} L(dx)).$$

Here we have set  $H_{\Phi_h}(\mathbf{C}^n) = \text{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, e^{-2\Phi_h/h} L(dx))$ .

*Proof.* — We only need to check the validity of the second statement, and when doing so, let us consider along  $\Gamma_{\omega}^{\Phi_h}(x)$ ,

$$(3.77) \quad -\Phi_h(x) + \text{Re}(i(x-y)\cdot\theta) + \Phi_h(y) \\ = -\Phi_h(x) + \text{Re}\left(2\frac{\partial\Phi_h}{\partial x}\left(\frac{x+y}{2}\right)\cdot(x-y)\right) + \Phi_h(y) - F_{\omega}(x-y) \\ = -\psi(x) + \left\langle \nabla\psi\left(\frac{x+y}{2}\right), x-y \right\rangle_{\mathbf{R}^{2n}} + \psi(y) - F_{\omega}(x-y).$$

Here we have used (3.6) on the last line. We have

$$(3.78) \quad -\psi(x) + \left\langle \nabla\psi\left(\frac{x+y}{2}\right), x-y \right\rangle_{\mathbf{R}^{2n}} + \psi(y) \leq 2\|\nabla\psi\|_{L^\infty(\mathbf{C}^n)}|x-y|,$$



and an application of Taylor’s formula gives that

$$\begin{aligned}
 (3.79) \quad & -\psi(x) + \left\langle \nabla\psi\left(\frac{x+y}{2}\right), x-y \right\rangle_{\mathbf{R}^{2n}} + \psi(y) \\
 &= \int_0^1 (1-t)\psi''\left(\frac{x+y}{2} - t\left(\frac{x-y}{2}\right)\right) \frac{(x-y)}{2} \cdot \frac{(x-y)}{2} dt \\
 &\quad - \int_0^1 (1-t)\psi''\left(\frac{x+y}{2} + t\left(\frac{x-y}{2}\right)\right) \frac{(x-y)}{2} \cdot \frac{(x-y)}{2} dt \\
 &\leq \frac{1}{4} \|\nabla^2\psi\|_{L^\infty(\mathbf{C}^n)} |x-y|^2.
 \end{aligned}$$

Here the Hessian and the scalar product are taken in the sense of  $\mathbf{R}^{2n}$ . Writing

$$\tilde{a}_{\Gamma_\omega^{\Phi_h}}^w(x, hD_x)u(x) = \int k_{\Gamma_\omega^{\Phi_h}}(x, y; h)u(y) L(dy),$$

we obtain, in view of (3.68), (3.77), (3.78), and (3.79), that the effective kernel of  $\tilde{a}_{\Gamma_\omega^{\Phi_h}}^w(x, hD_x)$  satisfies

$$\begin{aligned}
 (3.80) \quad & e^{-\frac{\Phi_h(x)}{h}} k_{\Gamma_\omega^{\Phi_h}}(x, y; h) e^{\frac{\Phi_h(y)}{h}} \\
 &\leq \mathcal{O}(1)h^{-n} \exp\left(\frac{1}{h}\left(-F_\omega(x-y) + \frac{\omega|x-y|}{\mathcal{O}(1)}\min(1, |x-y|)\right)\right) \\
 &\leq \mathcal{O}(1)h^{-n} \exp\left(-\frac{1}{2h}F_\omega(x-y)\right),
 \end{aligned}$$

provided that the implicit constant in (3.68) is sufficiently large. The pointwise estimate (3.80), on the level of effective kernels, is therefore of the same kind as (3.49), and arguing as in the proof of Theorem 3.1, we get the operator norm bound (3.76). □

Combining Theorem 3.1 and Theorem 3.3, we get

$$(3.81) \quad a_\Gamma^w(x, hD_x) = \tilde{a}_{\Gamma_\omega^{\Phi_h}}^w(x, hD_x) + R,$$

where

$$R = \mathcal{O}(1) \exp\left(-\frac{1}{C}h^{-\frac{1}{s}}\right) : H_{\Phi_0}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),$$

for some  $C > 0$ , and therefore

$$(3.82) \quad R = \mathcal{O}(1) \exp\left(-\frac{1}{2C}h^{-\frac{1}{s}}\right) : H_{\Phi_h}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_h/h} L(dx)),$$

provided that the implicit constant in (3.68) is large enough. Another application of Theorem 3.3 together with (3.81), (3.82) allows us to conclude

that the operator  $a_\Gamma^w(x, hD_x)$  extends to a uniformly bounded map

$$(3.83) \quad a_\Gamma^w(x, hD_x) = \mathcal{O}(1) : H_{\Phi_h}(\mathbf{C}^n) \rightarrow H_{\Phi_h}(\mathbf{C}^n),$$

for  $1 < s \leq 2$ , and we can view the operator  $\tilde{a}_{\Gamma_\omega^h}^w(x, hD_x)$  as the corresponding uniformly bounded realization.

In the remainder of this subsection, we shall be concerned with the problem of finding uniformly bounded realizations of the operator  $a_\Gamma^w(x, hD_x)$  in the region  $s > 2$ . As we shall see, we shall then have to accept a remainder which is larger than the one in (3.82). Let us start with the following largely heuristic remark.

*Remark.* — In Theorem 3.1, we have established that

$$a_\Gamma^w(x, hD_x) - \tilde{a}_{\Gamma_\omega}^w(x, hD_x) = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\right) : L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),$$

where the realization  $\tilde{a}_{\Gamma_\omega}^w(x, hD_x)$  is uniformly bounded on the weighted space  $L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx))$  in the range  $1 < s \leq 2$ , while we only have

$$\tilde{a}_{\Gamma_\omega}^w(x, hD_x) = \mathcal{O}(1)h^{-n(1-\frac{2}{s})} : L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),$$

for  $s > 2$ . Our purpose here is to address the question whether there exists a (Lipschitz) contour  $\tilde{\Gamma}(x) \subset \mathbf{C}_{y,\theta}^{2n}$  of dimension  $2n$ , such that the following two properties,

$$(3.84) \quad a_\Gamma^w(x, hD_x) - \tilde{a}_{\tilde{\Gamma}}^w(x, hD_x) = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\right) : L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),$$

and

$$(3.85) \quad \tilde{a}_{\tilde{\Gamma}}^w(x, hD_x) = \mathcal{O}(1) : L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),$$

hold, for  $s > 2$ . Indeed, let us pose the following question.

QUESTION. — *Let  $s > 2$ . Is there a (Lipschitz) function  $f : \mathbf{C}^n \rightarrow \mathbf{C}^n$  such that with the choice*

$$\tilde{\Gamma}(x) : \quad \theta = \frac{2}{i} \frac{\partial \Phi_0}{\partial x} \left( \frac{x+y}{2} \right) + if(x-y), \quad y \in \mathbf{C}^n,$$

*the properties (3.84), (3.85) hold?*

The following discussion seems to indicate that the answer to the question is likely to be negative. Let us try  $f$  of the form

$$f(z) = \widehat{f}(|z|) \frac{\bar{z}}{|z|},$$

for a suitable continuous  $\widehat{f} \geq 0$  on  $[0, \infty)$ . The absolute value of the effective kernel of the realization  $\widetilde{a}_{\Gamma}^w(x, hD_x)$  then does not exceed

$$(3.86) \quad \mathcal{O}(1)h^{-n} \exp\left(-\frac{1}{h} |x - y| \widehat{f}(|x - y|)\right),$$

and in view of Schur's lemma, the property (3.85) holds provided that

$$(3.87) \quad h^{-n} \int e^{-\frac{1}{h}|x|\widehat{f}(|x|)} L(dx) \leq \mathcal{O}(1).$$

On the other hand, introducing the intermediate contours given by

$$\widetilde{\Gamma}_t(x) : \quad \theta = \frac{2}{i} \frac{\partial \Phi_0}{\partial x} \left( \frac{x + y}{2} \right) + itf(x - y), \quad y \in \mathbf{C}^n,$$

for  $t \in [0, 1]$ , and applying Stokes' formula, we see that the effective kernel of the contribution coming from the region  $\bigcup_{0 \leq t \leq 1} \widetilde{\Gamma}_t(x)$  has the form

$$(3.88) \quad \mathcal{O}(1)h^{-n} \int_0^1 \exp\left(-\frac{1}{h} t \widehat{f}(|x - y|) |x - y| - C_1 \left(t \widehat{f}(|x - y|)\right)^{-\frac{1}{s-1}}\right) dt,$$

for some  $C_1 > 0$ . Here we have ignored the possible trouble coming from the Jacobian  $\det(\partial_{\bar{y}}\theta)$ . Recalling (3.84) and taking  $C_1 = 1$  in (3.88) for simplicity, we are led to the following pointwise condition on  $\widehat{f}$ ,

$$\frac{1}{h} \widehat{f}(r)r + \widehat{f}(r)^{-\frac{1}{s-1}} \geq \frac{h^{-\frac{1}{s}}}{\mathcal{O}(1)}, \quad r \geq 0.$$

In particular, we need that for each  $r \geq 0$ , uniformly,

$$\frac{1}{h} \widehat{f}(r)r \geq \frac{h^{-\frac{1}{s}}}{\mathcal{O}(1)} \quad \text{or} \quad \widehat{f}(r)^{-\frac{1}{s-1}} \geq \frac{h^{-\frac{1}{s}}}{\mathcal{O}(1)},$$

or equivalently,

$$\widehat{f}(r) \geq \frac{h^{1-\frac{1}{s}}}{\mathcal{O}(1)r} \quad \text{or} \quad \widehat{f}(r) \leq \frac{h^{1-\frac{1}{s}}}{\mathcal{O}(1)}.$$

Using that  $\widehat{f}$  is bounded near 0, we conclude that  $\widehat{f}(r) \leq \frac{h^{1-\frac{1}{s}}}{\mathcal{O}(1)}$  on some non-trivial interval of the form  $[0, \frac{1}{\mathcal{O}(1)}]$ . In other words,

$$\widehat{f}(r) \leq \mathcal{O}(1)\omega, \quad \text{for} \quad 0 \leq r \leq \frac{1}{\mathcal{O}(1)},$$

and therefore we get

$$h^{-n} \int e^{-\frac{1}{h}|x|\widehat{f}(|x|)} L(dx) \geq h^{-n} \int_{|x| \leq \frac{1}{\mathcal{O}(1)}} e^{-\frac{\mathcal{O}(1)}{h}\omega|x|} L(dx) \asymp h^{-n(1-\frac{2}{s})}.$$

Here the expression in the right hand side is unbounded as  $h \rightarrow 0^+$ , for  $s > 2$ , which is incompatible with (3.87).

When finding a uniformly bounded realization of the operator  $a_{\Gamma}^w(x, hD_x)$  on the space  $H_{\Phi_h}(\mathbf{C}^n)$ , for  $s > 2$ , we are going to perform an additional contour deformation, starting from the unbounded realization given by  $\widetilde{a}_{\Gamma_{\omega_h}}^w(x, hD_x)$ . The price that we have to pay is that we should then allow for a remainder that is larger than the one in (3.82), and is only moderately smaller than the remainder naturally associated to the contour (3.9) used in the analytic theory, see (3.26).

**THEOREM 3.4.** — *Assume that  $s > 2$ , and let  $\Phi_h = \Phi_0 + \psi \in C^{1,1}(\mathbf{C}^n)$  be such that (3.68) holds. Let  $\Gamma_{h^{1/2}}^{\Phi_h}(x) \subset \mathbf{C}_{y,\theta}^{2n}$  be the contour, defined as in (3.69), with  $\omega$  replaced by  $h^{1/2}$ . We have*

$$(3.89) \quad \widetilde{a}_{\Gamma_{\omega_h}}^w(x, hD_x) = \widetilde{a}_{\Gamma_{h^{1/2}}^{\Phi_h}}^w(x, hD_x) + R,$$

where the realization

$$\begin{aligned} \widetilde{a}_{\Gamma_{h^{1/2}}^{\Phi_h}}^w(x, hD_x)u(x) &= \frac{1}{(2\pi h)^n} \iint_{\Gamma_{h^{1/2}}^{\Phi_h}(x)} e^{\frac{i}{h}(x-y)\cdot\theta} \widetilde{a}\left(\frac{x+y}{2}, \theta; h\right) u(y) dy \wedge d\theta \end{aligned}$$

satisfies

$$(3.90) \quad \widetilde{a}_{\Gamma_{h^{1/2}}^{\Phi_h}}^w(x, hD_x) = \mathcal{O}(1) : H_{\Phi_h}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_h/h} L(dx)).$$

Furthermore,

$$(3.91) \quad R = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{2s-2}}\right) : H_{\Phi_h}(\mathbf{C}^n) \rightarrow L^2(\mathbf{C}^n, e^{-2\Phi_h/h} L(dx)).$$

*Proof.* — Let us note, first of all, that

$$(3.92) \quad \omega = \frac{1}{\mathcal{O}(1)}h^{1-\frac{1}{s}} \leq h^{1/2}, \quad \text{for all } s > 2.$$

With this in mind, we shall adapt the approach used at the end of subsection 3.1. Let us introduce the natural family of intermediate contours,

$$(3.93) \quad \Gamma_{\omega_t}^{\Phi_h}(x) : \quad \theta = \frac{2}{i} \frac{\partial \Phi_h}{\partial x} \left(\frac{x+y}{2}\right) + i f_{\omega_t}(x-y), \quad y \in \mathbf{C}^n,$$

where  $\omega_t = (1 - t)\omega + th^{1/2}$ ,  $t \in [0, 1]$ . An application of Stokes' formula gives that

$$\begin{aligned}
 (3.94) \quad & \tilde{a}_{\Gamma_{\omega_t}^{\Phi_h}}^w(x, hD_x)u(x) - \tilde{a}_{\Gamma_{h^{1/2}}^{\Phi_h}}^w(x, hD_x)u(x) \\
 &= \frac{1}{(2\pi h)^n} \iiint_{\bigcup_{t \in [0,1]} \Gamma_{\omega_t}^{\Phi_h}(x)} e^{\frac{i}{h}(x-y) \cdot \theta} u(y) \bar{\partial} \left( \tilde{a} \left( \frac{x+y}{2}, \theta \right) \right) \wedge dy \wedge d\theta \\
 &=: Ru(x).
 \end{aligned}$$

As usual, let us now proceed to estimate the effective kernel of the operator  $R$  in (3.94). To this end, we notice that along the contour  $\bigcup_{t \in [0,1]} \Gamma_{\omega_t}^{\Phi_h}(x)$  we have, similarly to (3.77),

$$\begin{aligned}
 (3.95) \quad & -\Phi_h(x) + \operatorname{Re}(i(x-y) \cdot \theta) + \Phi_h(y) \\
 &= -\psi(x) + \left\langle \nabla \psi \left( \frac{x+y}{2} \right), x-y \right\rangle_{\mathbf{R}^{2n}} + \psi(y) - F_{\omega_t}(x-y) \\
 &\leq -F_{\omega_t}(x-y) + \frac{\omega_t |x-y|}{\mathcal{O}(1)} \min(1, |x-y|) \leq -\frac{1}{2} F_{\omega_t}(x-y).
 \end{aligned}$$

Here we have also used (3.78), (3.79), as well as the fact that  $\omega \leq \omega_t$ , for  $t \in [0, 1]$ . Noticing also that along the contour  $\bigcup_{t \in [0,1]} \Gamma_{\omega_t}^{\Phi_h}(x)$  we have, in view of (3.92) and (3.93),

$$\operatorname{dist} \left( \left( \frac{x+y}{2}, \theta \right), \Lambda_{\Phi_0} \right) \leq 2 \left| \nabla \psi \left( \frac{x+y}{2} \right) \right| + \omega_t \leq \frac{\omega}{\mathcal{O}(1)} + \omega_t \leq 2h^{1/2},$$

we conclude that the absolute value of the effective kernel of the operator  $R$  in (3.94), for the boundedness on  $L^2(\mathbf{C}^n, e^{-2\Phi_h/h} L(dx))$ , does not exceed

$$(3.96) \quad \mathcal{O}(1) h^{-n} \int_0^1 e^{-\frac{1}{2h} F_{\omega_t}(x-y)} \exp \left( -\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{2(s-1)}} \right) dt.$$

In view of Schur's lemma and (3.96), to estimate the operator norm of  $R$ , we have to control the  $L^1$ -norm

$$(3.97) \quad \mathcal{O}(1) h^{-n} \int_0^1 dt \int e^{-\frac{1}{2h} F_{\omega_t}(x)} \exp \left( -\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{2(s-1)}} \right) L(dx) = I_1 + I_2,$$

where

$$\begin{aligned}
 (3.98) \quad I_1 &= \mathcal{O}(1) h^{-n} \int_0^1 dt \int_{|x| \leq \omega_t} \exp \left( -\frac{|x|^2}{\mathcal{O}(1)h} \right) \\
 &\quad \times \exp \left( -\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{2(s-1)}} \right) L(dx),
 \end{aligned}$$

and

$$(3.99) \quad I_2 = \mathcal{O}(1)h^{-n} \int_0^1 dt \int_{\omega_t < |x|} \exp\left(-\frac{|x|\omega_t}{\mathcal{O}(1)h}\right) \times \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{2(s-1)}}\right) L(dx).$$

We have

$$(3.100) \quad I_1 = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{2(s-1)}}\right),$$

and

$$(3.101) \quad \begin{aligned} I_2 &= \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{2(s-1)}}\right) \int_0^1 \frac{h^n}{\omega_t^{2n}} dt \\ &\leq \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{2(s-1)}}\right) \frac{h^n}{\omega^{2n}} \\ &\leq \mathcal{O}(1) \exp\left(-\frac{1}{2\mathcal{O}(1)}h^{-\frac{1}{2(s-1)}}\right). \end{aligned}$$

The estimate (3.91) follows, in view of (3.94), (3.97), (3.100), and (3.101).

We shall finally verify the uniform boundedness property (3.90) for the realization  $\tilde{a}_{\Gamma_{h^{1/2}}}^w(x, hD_x)$ . To this end, let us observe that along the contour  $\Gamma_{h^{1/2}}^{\Phi_h}(x)$ , we have, similarly to (3.95),

$$(3.102) \quad -\Phi_h(x) + \operatorname{Re}(i(x - y) \cdot \theta) + \Phi_h(y) \leq -\frac{1}{2}F_{h^{1/2}}(x - y).$$

Writing

$$\tilde{a}_{\Gamma_{h^{1/2}}}^w(x, hD_x)u(x) = \int k_{\Gamma_{h^{1/2}}}^{\Phi_h}(x, y; h)u(y) L(dy),$$

we obtain therefore, in view of (3.102),

$$(3.103) \quad e^{-\frac{\Phi_h(x)}{h}} k_{\Gamma_{h^{1/2}}}^{\Phi_h}(x, y; h) e^{\frac{\Phi_h(y)}{h}} \leq \mathcal{O}(1) h^{-n} \exp\left(-\frac{1}{2}F_{h^{1/2}}(x - y)\right).$$

The pointwise bound (3.103), on the level of effective kernels, is therefore of the same kind as (3.49), with the only difference that the small parameter  $\omega$  has been replaced by  $h^{1/2} \geq \omega$ . An application of Schur’s lemma gives therefore immediately (3.90). This completes the proof.  $\square$

Combining (3.81), (3.82), and Theorem 3.4, we get in the region  $s > 2$ ,

$$(3.104) \quad a_{\Gamma}^w(x, hD_x) = \tilde{a}_{\Gamma_{h^{1/2}}}^w(x, hD_x) + R,$$

where (3.90) holds, and the remainder  $R$  satisfies (3.91). We conclude in particular that the operator  $a_\Gamma^w(x, hD_x)$  extends to a uniformly bounded map

$$(3.105) \quad a_\Gamma^w(x, hD_x) = \mathcal{O}(1) : H_{\Phi_h}(\mathbf{C}^n) \rightarrow H_{\Phi_h}(\mathbf{C}^n),$$

for  $s > 2$ , and we can view the operator  $\tilde{a}_{\Gamma_{h^{1/2}}}^w(x, hD_x)$  as the corresponding uniformly bounded realization.

Theorem 1.1 and Theorem 1.2 in the introduction now follow from Theorem 3.1, Theorem 3.3, and Theorem 3.4.

*Remark.* — In the work [11], prepared simultaneously with the present one, the mapping property (3.105) in the range  $s \geq 2$  is established using alternative methods, not relying upon the contour deformations techniques.

The discussion in this section gives, in particular, the following result.

**COROLLARY 3.5.** — *Let  $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ ,  $s > 1$ , and let  $\Phi_h = \Phi_0 + \psi \in C^{1,1}(\mathbf{C}^n; \mathbf{R})$  be such that (3.68) holds. The operator  $\text{Op}_h^w(a)$  extends to a uniformly bounded map*

$$\text{Op}_h^w(a) = \mathcal{O}(1) : H_{\Phi_h}(\mathbf{C}^n) \rightarrow H_{\Phi_h}(\mathbf{C}^n).$$

### 3.3. Phase symmetries and composition of Gevrey operators

In the first part of this subsection, we shall develop an approach to the composition of semiclassical Weyl quantizations in the complex domain, based on the representation of the operators as superpositions of suitable phase symmetries [27]. Such an approach is carried out in [19] in the real setting, and here we shall adapt it to the present complex environment.

Let  $\Phi_0$  be a strictly plurisubharmonic quadratic form on  $\mathbf{C}^n$  and let the I-Lagrangian R-symplectic linear subspace  $\Lambda_{\Phi_0} \subset \mathbf{C}_x^n \times \mathbf{C}_\xi^n$  be given by (3.1). Given  $a \in \mathcal{S}(\Lambda_{\Phi_0})$ , let us consider following (3.4),

$$(3.106) \quad a_\Gamma^w(x, hD_x)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a\left(\frac{x+y}{2}, \theta\right) u(y) dy \wedge d\theta.$$

Here  $u \in H_{\Phi_0}(\mathbf{C}^n)$  and  $\Gamma(x) \subset \mathbf{C}_{y,\theta}^{2n}$  is the contour given by (3.5). Setting

$$(3.107) \quad a_{\Phi_0}(x) = a\left(x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x)\right), \quad x \in \mathbf{C}^n,$$

and recalling (3.6), we can write in view of (3.106),

$$(3.108) \quad a_{\Gamma}^w(x, hD_x)u(x) = \frac{2^n \det(\Phi''_{0,\bar{x}x})}{(2\pi h)^n} \times \int e^{\frac{1}{h}(\Phi_0(x) - \Phi_0(y) + 2i \operatorname{Im}((x-y) \cdot \partial_x \Phi_0(\frac{x+y}{2})))} a_{\Phi_0}\left(\frac{x+y}{2}\right) u(y) L(dy).$$

Here we have also used that along  $\Gamma(x)$ , we have

$$dy \wedge d\theta = 2^n \det(\Phi''_{0,\bar{x}x})L(dy),$$

provided that the orientation has been chosen suitably.

Let  $u, v \in H_{\Phi_0}(\mathbf{C}^n)$ , and let us set  $U = e^{-\Phi_0/h} u \in L^2(\mathbf{C}^n)$ ,  $V = e^{-\Phi_0/h} v \in L^2(\mathbf{C}^n)$ . We get, using (3.108),

$$(3.109) \quad (a_{\Gamma}^w(x, hD_x)u, v)_{H_{\Phi_0}} = \int a_{\Gamma}^w(x, hD)u(x) \overline{v(x)} e^{-\frac{2}{h}\Phi_0(x)} L(dx) = \frac{2^n \det(\Phi''_{0,\bar{x}x})}{(2\pi h)^n} \times \iint e^{\frac{2i}{h} \operatorname{Im}((x-y) \cdot \partial_x \Phi_0(\frac{x+y}{2}))} a_{\Phi_0}\left(\frac{x+y}{2}\right) U(y) \overline{V(x)} L(dy) L(dx).$$

Making the linear change of variables in (3.109),

$$x' = \frac{x+y}{2}, \quad y' = x-y,$$

where the absolute value of the Jacobian is 1, we obtain after dropping the primes,

$$(3.110) \quad (a_{\Gamma}^w(x, hD_x)u, v)_{H_{\Phi_0}} = \frac{2^n \det(\Phi''_{0,\bar{x}x})}{(2\pi h)^n} \times \iint e^{\frac{2i}{h} \operatorname{Im}(y \cdot \partial_x \Phi_0(x))} a_{\Phi_0}(x) U\left(x - \frac{1}{2}y\right) \overline{V\left(x + \frac{1}{2}y\right)} L(dy) L(dx), = \frac{2^n \det(\Phi''_{0,\bar{x}x})}{(2\pi h)^n} \int a_{\Phi_0}(x) \mathcal{K}(U, V)(x) L(dx).$$

Here  $\mathcal{K}(U, V)(x)$  is “the Wigner function” given by

$$(3.111) \quad \mathcal{K}(U, V)(x) = \int e^{\frac{2i}{h} \operatorname{Im}(y \cdot \partial_x \Phi_0(x))} U\left(x - \frac{1}{2}y\right) \overline{V\left(x + \frac{1}{2}y\right)} L(dy).$$



Performing the change of variables  $y \mapsto \tilde{y} = x + \frac{1}{2}y$  in (3.111), we get after dropping the tilde,

$$(3.112) \quad \begin{aligned} \mathcal{K}(U, V)(x) &= 2^{2n} \int e^{\frac{4i}{h} \operatorname{Im}((y-x) \cdot \partial_x \Phi_0(x))} U(2x - y) \overline{V(y)} L(dy) \\ &= 2^{2n} (\Sigma_x U, V)_{L^2(\mathbf{C}^n)}, \end{aligned}$$

where  $\Sigma_x, x \in \mathbf{C}^n$ , is the unitary map on  $L^2(\mathbf{C}^n)$  given by

$$(3.113) \quad (\Sigma_x U)(y) = e^{\frac{4i}{h} \operatorname{Im}((y-x) \cdot \partial_x \Phi_0(x))} U(2x - y).$$

We obtain, combining (3.110) and (3.112),

$$(3.114) \quad \begin{aligned} (a_\Gamma^w(x, hD_x)u, v)_{H_{\Phi_0}} &= \frac{2^n \det(\Phi''_{0, \bar{x}x})}{(2\pi h)^n} \int a_{\Phi_0}(x) 2^{2n} (\Sigma_x U, V)_{L^2(\mathbf{C}^n)} L(dx), \end{aligned}$$

and therefore,

$$(3.115) \quad e^{-\frac{\Phi_0}{h}} a_\Gamma^w(x, hD_x) e^{\frac{\Phi_0}{h}} = \frac{2^n \det(\Phi''_{0, \bar{x}x})}{(2\pi h)^n} \int a_{\Phi_0}(x) 2^{2n} \Sigma_x L(dx).$$

Here we may notice that the realization  $a_\Gamma^w(x, hD_x)$  of the Weyl quantization  $a^w(x, hD_x)$  in (3.106) acts on the weighted  $L^2$ -space

$$L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)),$$

whereas  $a^w(x, hD_x)$  is defined on the holomorphic subspace only. The decomposition (3.115) can be regarded as the complex analogue of the corresponding representation obtained in [19, Chapter 2] in the real domain. When deriving an explicit formula for the composition  $a^w(x, hD_x) \circ b^w(x, hD_x)$ , for  $a, b \in \mathcal{S}(\Lambda_{\Phi_0})$ , we shall proceed by computing first the composition  $\Sigma_y \circ \Sigma_z$  for  $y, z \in \mathbf{C}^n$ .

When doing so, let us consider the decomposition

$$(3.116) \quad \Phi_0 = \Phi_{\text{herm}} + \Phi_{\text{plh}},$$

where  $\Phi_{\text{herm}}(x) = \Phi''_{0, \bar{x}x} x \cdot \bar{x}$  is positive definite Hermitian and  $\Phi_{\text{plh}}(x) = \operatorname{Re}(\Phi''_{0, xx} x \cdot x)$  is pluriharmonic. Let

$$A = \frac{2}{i} (\Phi_{\text{plh}})''_{xx} = \frac{2}{i} \Phi''_{0, xx}.$$

The complex linear canonical transformation

$$(3.117) \quad \mathbf{C}^{2n} \ni (y, \eta) \mapsto \kappa_A(y, \eta) = (y, \eta - Ay) \in \mathbf{C}^{2n}$$

satisfies

$$\kappa_A(\Lambda_{\Phi_0}) = \Lambda_{\Phi_{\text{herm}}},$$

and associated to  $\kappa_A$  is the metaplectic Fourier integral operator

$$(3.118) \quad \mathcal{U}u = ue^{-f}, \quad f(x) = \Phi''_{0,xx}x \cdot x,$$

which maps  $H_{\Phi_0}(\mathbf{C}^n)$  unitarily onto  $H_{\Phi_{\text{herm}}}(\mathbf{C}^n)$ . By an application of the exact Egorov theorem we get

$$\mathcal{U} \circ a^w(x, hD_x) \circ \mathcal{U}^{-1} = b^w(x, hD_x),$$

where  $b \in \mathcal{S}(\Lambda_{\Phi_{\text{herm}}})$  is given by  $b = a \circ \kappa_A^{-1}$ . Conjugating  $a^w(x, hD_x)$  by the operator  $\mathcal{U}$  in (3.118), we obtain a reduction to the case when the pluriharmonic part of  $\Phi_0$  vanishes, and in what follows, we shall therefore make this assumption.

The unitary map  $\Sigma_x$  in (3.113) takes the form

$$(3.119) \quad \begin{aligned} (\Sigma_x U)(y) &= e^{\frac{4i}{h} \text{Im}(\Phi''_{0,\bar{x}x}(y-x)\cdot\bar{x})} U(2x - y) \\ &= e^{\frac{4i}{h} \text{Im}(\Phi''_{0,\bar{x}x}y\cdot\bar{x})} U(2x - y) = e^{\frac{4i}{h} \text{Im} \Psi_0(y,\bar{x})} U(2x - y), \end{aligned}$$

where  $\Psi_0$  is the polarization of  $\Phi_0$ , i.e., the unique holomorphic quadratic form on  $\mathbf{C}^n_x \times \mathbf{C}^n_y$  such that  $\Psi_0(x, \bar{x}) = \Phi_0(x)$ .

LEMMA 3.6. — *Let  $\Phi_0$  be a strictly plurisubharmonic quadratic form on  $\mathbf{C}^n$  with vanishing pluriharmonic part. We have for  $y, z \in \mathbf{C}^n$ ,*

$$(3.120) \quad \Sigma_y \circ \Sigma_z = \frac{2^n \det(\Phi''_{0,\bar{x}x})}{(2\pi h)^n} \int e^{\frac{8i}{h} \text{Im} \Psi_0(x-y,\bar{x}-\bar{z})} 2^{2n} \Sigma_x L(dx).$$

Here  $\Psi_0$  is the polarization of  $\Phi_0$ .

*Proof.* — By a direct computation, using (3.119), we get

$$(3.121) \quad (\Sigma_y \circ \Sigma_z U)(y') = e^{\frac{4i}{h} \text{Im} \Psi_0(y',\bar{y}-\bar{z})} e^{\frac{8i}{h} \text{Im} \Psi_0(y,\bar{z})} U(y' - 2y + 2z).$$

On the other hand, the operator in the right hand side of (3.120) acting on  $U$ , is given by

$$(3.122) \quad \begin{aligned} (\mathbb{L}U)(y') &:= \frac{2^n \det(\Phi''_{0,\bar{x}x})}{(2\pi h)^n} \int e^{\frac{8i}{h} \text{Im} \Psi_0(x-y,\bar{x}-\bar{z})} 2^{2n} (\Sigma_x U)(y') L(dx) \\ &= \frac{2^n \det(\Phi''_{0,\bar{x}x})}{(2\pi h)^n} e^{\frac{8i}{h} \text{Im} \Psi_0(y,\bar{z})} \\ &\quad \times \int 2^{2n} e^{\frac{4i}{h} \text{Im} \Psi_0(y'-2y+2z,\bar{x})} U(2x - y') L(dx). \end{aligned}$$

Here we have also used the skew-symmetry property

$$\text{Im} \Psi_0(x, \bar{z}) = -\text{Im} \Psi_0(z, \bar{x}).$$

Making the change of variables  $\zeta = 2x - y'$  in (3.122), we get

$$\begin{aligned}
 (3.123) \quad (\mathbb{L}U)(y') &= \frac{2^n \det(\Phi''_{0,\bar{x}x})}{(2\pi h)^n} e^{\frac{8i}{h} \operatorname{Im} \Psi_0(y,\bar{z})} \\
 &\quad \times \int e^{\frac{4i}{h} \operatorname{Im} \Psi_0(y'-2y+2z, \frac{y'+\bar{\zeta}}{2})} U(\zeta) L(d\zeta) \\
 &= \frac{2^n \det(\Phi''_{0,\bar{x}x})}{(2\pi h)^n} e^{\frac{8i}{h} \operatorname{Im} \Psi_0(y,\bar{z})} e^{\frac{4i}{h} \operatorname{Im} \Psi_0(y',\bar{y}-\bar{z})} \\
 &\quad \times \int e^{\frac{2i}{h} \operatorname{Im} \Psi_0(y'-2y+2z,\bar{\zeta})} U(\zeta) L(d\zeta).
 \end{aligned}$$

On the other hand, taking  $a = 1$  in (3.115), we obtain for  $W \in L^2(\mathbf{C}^n)$  such that  $e^{\Phi_0/h} W \in H_{\Phi_0}(\mathbf{C}^n)$ ,

$$\begin{aligned}
 (3.124) \quad W(y') &= \frac{2^n \det(\Phi''_{0,\bar{x}x})}{(2\pi h)^n} \int 2^{2n} e^{\frac{4i}{h} \operatorname{Im} \Psi_0(y',\bar{x})} W(2x - y') L(dx) \\
 &= \frac{2^n \det(\Phi''_{0,\bar{x}x})}{(2\pi h)^n} \int W(\zeta) e^{\frac{2i}{h} \operatorname{Im} \Psi_0(y',\bar{\zeta})} L(d\zeta).
 \end{aligned}$$

Here on the second line we have again made the change of variables  $\zeta = 2x - y'$ . Using (3.124) we conclude that the expression in the right hand side of (3.123) becomes

$$e^{\frac{8i}{h} \operatorname{Im} \Psi_0(y,\bar{z})} e^{\frac{4i}{h} \operatorname{Im} \Psi_0(y',\bar{y}-\bar{z})} U(y' - 2y + 2z),$$

which agrees with  $(\Sigma_y \circ \Sigma_z U)(y')$ , in view of (3.121). The proof is complete. □

We are now ready to compute the composition of two Weyl quantizations. Let  $a, b \in \mathcal{S}(\Lambda_{\Lambda_{\Phi_0}})$ , and let us write following (3.115),

$$\begin{aligned}
 e^{-\frac{\Phi_0}{h}} a_{\Gamma}^w(x, hD_x) e^{\frac{\Phi_0}{h}} &= \frac{\det(\Phi''_{0,\bar{x}x})}{(\pi h)^n} \int a_{\Phi_0}(y) 2^{2n} \Sigma_y L(dy), \\
 e^{-\frac{\Phi_0}{h}} b_{\Gamma}^w(x, hD_x) e^{\frac{\Phi_0}{h}} &= \frac{\det(\Phi''_{0,\bar{x}x})}{(\pi h)^n} \int b_{\Phi_0}(z) 2^{2n} \Sigma_z L(dz).
 \end{aligned}$$

Using (3.120), we get

$$\begin{aligned}
 (3.125) \quad & e^{-\frac{\Phi_0}{h}} a_\Gamma^w(x, hD_x) \circ b_\Gamma^w(x, hD_x) e^{\frac{\Phi_0}{h}} \\
 &= \left( \frac{\det(\Phi''_{0,\bar{x}x})}{(\pi h)^n} \right)^2 \iint a_{\Phi_0}(y) b_{\Phi_0}(z) 2^{4n} \Sigma_y \circ \Sigma_z L(dy) L(dz) \\
 &= \left( \frac{\det(\Phi''_{0,\bar{x}x})}{(\pi h)^n} \right)^3 \\
 &\quad \times \iiint a_{\Phi_0}(y) b_{\Phi_0}(z) 2^{4n} e^{\frac{8i}{h} \operatorname{Im} \Psi_0(x-y, \bar{x}-\bar{z})} 2^{2n} \Sigma_x L(dx) L(dy) L(dz),
 \end{aligned}$$

and therefore the operator  $c^w(x, hD_x) = a^w(x, hD_x) \circ b^w(x, hD_x)$  satisfies

$$(3.126) \quad e^{-\frac{\Phi_0}{h}} c_\Gamma^w(x, hD_x) e^{\frac{\Phi_0}{h}} = \frac{\det(\Phi''_{0,\bar{x}x})}{(\pi h)^n} \int c_{\Phi_0}(x) 2^{2n} \Sigma_x L(dx),$$

where

$$\begin{aligned}
 (3.127) \quad c_{\Phi_0}(x) &= \left( \frac{\det(\Phi''_{0,\bar{x}x})}{(\pi h)^n} \right)^2 \\
 &\quad \times \iint a_{\Phi_0}(y) b_{\Phi_0}(z) 2^{4n} e^{\frac{8i}{h} \operatorname{Im} \Psi_0(x-y, \bar{x}-\bar{z})} L(dy) L(dz) \\
 &= \left( \frac{\det(\Phi''_{0,\bar{x}x})}{(\pi h)^n} \right)^2 \\
 &\quad \times \iint a_{\Phi_0}(x+y) b_{\Phi_0}(x+z) 2^{4n} e^{\frac{8i}{h} \operatorname{Im} \Psi_0(y, \bar{z})} L(dy) L(dz).
 \end{aligned}$$

Let us rewrite (3.127) in more invariant terms. When doing so, we make the following two observations.

- (i) The restriction of the complex symplectic  $(2, 0)$ -form  $\sigma$  on  $\mathbf{C}^{2n}$  to  $\Lambda_{\Phi_0}$  is given by

$$\sigma(Y, Z) = -4 \operatorname{Im} (\Phi''_{0,\bar{x}x} y \cdot \bar{z}) = -4 \operatorname{Im} \Psi_0(y, \bar{z}),$$

where  $Y, Z \in \Lambda_{\Phi_0}$  are the points in  $\Lambda_{\Phi_0}$  above  $y, z \in \mathbf{C}^n$ , respectively.

- (ii) The symplectic volume form on  $\Lambda_{\Phi_0}$ ,  $\frac{\sigma^n}{n!}|_{\Lambda_{\Phi_0}}$ , is equal to

$$dx = 2^{2n} \det(\Phi''_{0,\bar{x}x}) L(dx), \quad X = \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right) \in \Lambda_{\Phi_0},$$

see also (3.108) and the following comment.

We get therefore from (3.127),

$$(3.128) \quad c(X) = (a\#b)(X) = \frac{1}{(\pi h)^{2n}} \iint_{\Lambda_{\Phi_0} \times \Lambda_{\Phi_0}} e^{-2i\sigma(Y,Z)/h} a(X+Y)b(X+Z) dy dz.$$

*Remark.* — The integral representation formula (3.128) can also be obtained directly from the corresponding formula in the real domain [13], [34, Chapter 4], thanks to the metaplectic invariance of the Weyl calculus [12, 33].

We would next like to rewrite the expression (3.127) for  $c_{\Phi_0}$  in terms of a suitable Gaussian Fourier multiplier on  $\mathbf{C}^{2n}$ , acting on  $a_{\Phi_0} \otimes b_{\Phi_0}$ , similarly to the Weyl composition formula in the real domain [3]. To this end, introducing the positive definite Hermitian matrix  $B = \Phi''_{0,\bar{x}x}$  and performing the change of variables

$$Y = 2B^{1/2}y, \quad Z = 2B^{1/2}z,$$

in (3.127), we obtain

$$(3.129) \quad c_{\Phi_0}(x) = \frac{1}{(\pi h)^{2n}} \iint a_{\Phi_0}\left(x + \frac{B^{-1/2}Y}{2}\right) b_{\Phi_0}\left(x + \frac{B^{-1/2}Z}{2}\right) \times e^{\frac{2i}{h} \operatorname{Im}(Y \cdot \bar{Z})} L(dy) L(dz) = \frac{1}{(\pi h)^{2n}} \iint a_{\Phi_0}\left(x + \frac{1}{2}B^{-1/2}Y\right) b_{\Phi_0}\left(x + \frac{1}{2}B^{-1/2}Z\right) \times e^{-\frac{2i}{h} \sigma_{\mathbf{R}}(Z,Y)} L(dy) L(dz).$$

Here we have noticed that

$$(3.130) \quad \operatorname{Im}(Y \cdot \bar{Z}) = \sigma_{\mathbf{R}}(Y, Z) = -\sigma_{\mathbf{R}}(Z, Y),$$

where  $\sigma_{\mathbf{R}}$  is the standard symplectic form on  $\mathbf{R}^{2n}$ , when identifying this space with  $\mathbf{C}^n$  with the help of the map  $\mathbf{C}^n \ni Y = y + i\eta \mapsto (y, \eta) \in \mathbf{R}^{2n}$ . Recall next that if  $A$  is an  $N \times N$  real symmetric non-degenerate matrix, we have for  $u \in \mathcal{S}(\mathbf{R}^N)$ ,

$$(3.131) \quad e^{\frac{ih}{2}AD \cdot D} u(x) = \frac{1}{(2\pi h)^{N/2}} \frac{e^{\frac{i\pi}{4} \operatorname{sgn} A}}{|\det A|^{1/2}} \int e^{-\frac{i}{2h} A^{-1}y \cdot y} u(x+y) dy.$$

Applying (3.131) with  $\mathbf{R}^N = \mathbf{R}^{2n}_{z,\zeta} \times \mathbf{R}^{2n}_{y,\eta}$ , and

$$AD \cdot D = \sigma_{\mathbf{R}}(D_z, D_\zeta; D_y, D_\eta) = D_\zeta \cdot D_y - D_\eta \cdot D_z,$$

we get using the complex notation  $Y = y + i\eta$ ,  $Z = z + i\zeta$ ,

$$(3.132) \quad \left( e^{\frac{i\hbar}{2} \sigma_{\mathbf{R}}(D_z, D_\zeta; D_y, D_\eta)} u(Z, Y) \right) \Big|_{Z=Y=0} \\ = \frac{1}{(\pi\hbar)^{2n}} \iint e^{-\frac{2i}{\hbar} \sigma_{\mathbf{R}}(Z, Y)} u(Z, Y) L(dy) L(dz).$$

Here we also have

$$(3.133) \quad \sigma_{\mathbf{R}}(D_z, D_\zeta; D_y, D_\eta) = \frac{2}{i} (D_{\bar{z}} \cdot D_Y - D_Z \cdot D_{\bar{y}}) \\ = \frac{2}{i} \sigma_{\mathbf{R}}(D_Z, D_{\bar{z}}; D_Y, D_{\bar{y}}),$$

where

$$D_Y = \frac{1}{2} (D_y - iD_\eta), \quad D_{\bar{y}} = \frac{1}{2} (D_y + iD_\eta),$$

with  $D_Z, D_{\bar{z}}$  being defined similarly, see also (2.15). Combining (3.129), (3.132), and (3.133), we get

$$(3.134) \quad c_{\Phi_0}(x) \\ = e^{h\sigma_{\mathbf{R}}(D_z, D_{\bar{z}}; D_Y, D_{\bar{y}})} \left( a_{\Phi_0} \left( x + \frac{B^{-1/2}Y}{2} \right) b_{\Phi_0} \left( x + \frac{B^{-1/2}Z}{2} \right) \right) \Big|_{Y=Z=0} \\ = \exp \left( \frac{i\hbar}{2} \frac{({}^t B^{-1} D_x \cdot D_{\bar{y}} - {}^t B^{-1} D_y \cdot D_{\bar{x}})}{2i} \right) (a_{\Phi_0}(x) b_{\Phi_0}(y)) \Big|_{y=x}.$$

Here the symbol of the second order constant coefficient differential operator on  $\mathbf{C}_{x,y}^{2n}$ ,

$$\frac{1}{2i} ({}^t B^{-1} D_x \cdot D_{\bar{y}} - {}^t B^{-1} D_y \cdot D_{\bar{x}})$$

is a quadratic form on  $\mathbf{C}_{\xi,\eta}^{2n}$  given by

$$(3.135) \quad \frac{1}{8i} ({}^t B^{-1} \bar{\xi} \cdot \eta - {}^t B^{-1} \bar{\eta} \cdot \xi) = -\frac{1}{4} \text{Im} (B^{-1} \xi \cdot \bar{\eta}).$$

Letting

$$(3.136) \quad \sigma_{\Phi_0} = \frac{2}{i} \sum_{j,k=1}^n \frac{\partial^2 \Phi_0}{\partial \bar{x}_j \partial x_k} d\bar{x}_j \wedge dx_k$$

be the pullback of the complex symplectic form  $\sigma$  on  $\mathbf{C}^{2n}$  under the map

$$\mathbf{C}^n \ni x \mapsto \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right) \in \Lambda_{\Phi_0} \subset \mathbf{C}^{2n},$$

we see that

$$\sigma_{\Phi_0}(\xi, \eta) = -4 \text{Im} (B\xi \cdot \bar{\eta}), \quad \xi, \eta \in \mathbf{C}^n,$$

and therefore the quadratic form in (3.135) can be regarded as the dual to  $\sigma_{\Phi_0}$ , when the latter is viewed as a quadratic form on  $\mathbf{C}_{\xi,\eta}^{2n}$ . Setting

$$\sigma_{\Phi_0}^{-1}(\xi, \eta) = -\frac{1}{4} \operatorname{Im} (B^{-1}\xi \cdot \bar{\eta}),$$

we may summarize the discussion above in the following result.

PROPOSITION 3.7. — *Let  $a, b \in \mathcal{S}(\Lambda_{\Phi_0})$  and let*

$$c^w(x, hD_x) = a^w(x, hD_x) \circ b^w(x, hD_x).$$

The symbol  $c \in \mathcal{S}(\Lambda_{\Phi_0})$  is given by

$$(3.137) \quad c(X) = (a\#b)(X) \\ = \frac{1}{(\pi h)^{2n}} \iint_{\Lambda_{\Phi_0} \times \Lambda_{\Phi_0}} e^{-2i\sigma(Y,Z)/h} a(X+Y)b(X+Z) \, dy \, dz.$$

We also have

$$(3.138) \quad c_{\Phi_0}(x) = \exp\left(\frac{i h}{2} \sigma_{\Phi_0}^{-1}(D_{x,\bar{x}}, D_{y,\bar{y}})\right) (a_{\Phi_0}(x)b_{\Phi_0}(y))|_{y=x}.$$

*Remark.* — We refer to the recent work [11] for an alternative approach to the composition formulas for the semiclassical Weyl calculus in the complex domain, based on the Fourier inversion formula on  $\Lambda_{\Phi_0}$  and the method of magnetic translations.

We shall finish this subsection by discussing the formula for the composition (3.137) in the case when  $a, b \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ , for some  $s > 1$ . It has been established in [16, 17], working in the real domain, that we then have  $c = a\#b \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ . The argument in [16] proceeds by repeated partial integrations and suitable quasinorm estimates, and our purpose here is to provide an alternative approach to the proof of this result, making use of the method of contour deformations. When doing so, rather than working with Gevrey symbols of  $\Lambda_{\Phi_0}$ , in view of the metaplectic invariance of the Weyl calculus [12, 33], it will be sufficient for us to work on  $\mathbf{R}^m \simeq T^*\mathbf{R}^n$ , where  $m = 2n$ .

Let  $a, b \in \mathcal{G}_b^s(\mathbf{R}^m)$  and let us set following (3.137),

$$(3.139) \quad c(X) = (a\#b)(X) \\ = \frac{1}{(\pi h)^m} \iint_{\mathbf{R}^m \times \mathbf{R}^m} e^{-2i\sigma(Y,Z)/h} a(X+Y)b(X+Z) \, dy \, dz.$$

Here  $\sigma$  is the standard symplectic form on  $\mathbf{R}^m$ , and the integral in (3.139) is an oscillatory one. Let  $\chi \in \mathcal{G}_0^s(\mathbf{R}^{2m})$  be such that  $\chi(Y, Z) = 1$  for  $|(Y, Z)| \leq$

1, with  $\text{supp } \chi \subset B(0, 2)$ , and define also

$$(3.140) \quad r_\chi(X) = \frac{1}{(\pi h)^m} \iint_{\mathbf{R}^m \times \mathbf{R}^m} e^{-2i\sigma(Y,Z)/h} (1 - \chi(Y, Z)) a(X + Y)b(X + Z) dy dz.$$

The standard semiclassical calculus [3] gives that

$$\|\partial^\alpha r_\chi\|_{L^\infty(\mathbf{R}^m)} = \mathcal{O}_\alpha(h^\infty),$$

for all  $\alpha \in \mathbf{N}^m$ , and we would like to sharpen these asymptotic bounds, thanks to the Gevrey smoothness of the symbols  $a, b$ . To this end, we have the following result, due to [16, 17].

PROPOSITION 3.8. — *Let  $a, b \in \mathcal{G}_b^s(\mathbf{R}^m)$ , for some  $s > 1$ , and let us define  $r_\chi \in C^\infty(\mathbf{R}^m)$  as in (3.140). There exists  $C > 0$  such that for all  $\alpha \in \mathbf{N}^m$  and  $h \in (0, 1]$ , we have*

$$(3.141) \quad |\partial^\alpha r_\chi(X)| \leq C^{1+|\alpha|} (\alpha!)^s \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\right), \quad X \in \mathbf{R}^m,$$

*Proof.* — We shall prove the following more general statement, implying (3.141): let  $q(x)$  be a real valued non-degenerate quadratic form on  $\mathbf{R}^N$ , let  $a \in \mathcal{G}_b^s(\mathbf{R}^N)$ , for some  $s > 1$ , and let  $\chi \in \mathcal{G}_0^s(\mathbf{R}^N)$  be such that  $\chi(x) = 1$  for  $|x| \leq 1$ ,  $\text{supp } \chi \subset B_{\mathbf{R}^N}(0, 2)$ . Setting

$$(3.142) \quad r_\chi(x) = h^{-N/2} \int e^{iq(y)/h} (1 - \chi(y)) a(x + y) dy,$$

we shall prove that  $r_\chi$  enjoys the same estimates as in (3.141). To this end, let  $\tilde{a} \in \mathcal{G}_b^s(\mathbf{C}^N)$  be an almost holomorphic extension of  $a$  such that  $\text{supp } \tilde{a} \subset \mathbf{R}^N + iB_{\mathbf{R}^N}(0, C)$ , for some  $C > 0$ , and let  $\tilde{\chi} \in \mathcal{G}_0^s(\mathbf{C}^N)$  be an almost holomorphic extension of  $\chi$ , with  $\text{supp } \tilde{\chi}$  close to that of  $\chi$ . We shall replace the integration in (3.142) along  $\mathbf{R}^N$  by the integration along the contour

$$(3.143) \quad \Gamma_{\theta_0} : \quad \mathbf{R}^N \ni y \mapsto y + i\theta_0 \frac{q'(y)}{|q'(y)|} \in \mathbf{C}^N, \quad |y| \geq \frac{1}{2},$$

for some  $\theta_0 > 0$  small enough, where we notice that along  $\Gamma_{\theta_0}$ , we have

$$(3.144) \quad \text{Im } q(z) = \theta_0 |q'(y)| \asymp \theta_0 |y|, \quad z \in \Gamma_{\theta_0},$$



since  $q$  is non-degenerate. Introducing also the damping factor  $e^{-\frac{\varepsilon y^2}{2}}$ ,  $\varepsilon > 0$ , in (3.142), we get by an application of Stokes formula,

$$\begin{aligned}
 (3.145) \quad & \int_{\mathbf{R}^N} e^{iq(y)/h} e^{-\varepsilon y^2/2} (1 - \chi(y)) a(x + y) dy \\
 &= \int_{\Gamma_{\theta_0}} e^{iq(z)/h} e^{-\varepsilon z^2/2} (1 - \tilde{\chi}(z)) \tilde{a}(x + z) dz \\
 &+ \iint_{G_{[0, \theta_0]}} e^{iq(z)/h} e^{-\varepsilon z^2/2} \bar{\partial} ((1 - \tilde{\chi}(z)) \tilde{a}(x + z)) \wedge dz.
 \end{aligned}$$

Here  $G_{[0, \theta_0]} \subset \mathbf{C}^N$  is the  $(n + 1)$ -dimensional contour given by

$$G_{[0, \theta_0]} = \bigcup_{\theta \in [0, \theta_0]} \Gamma_{\theta},$$

with  $\Gamma_{\theta}$  defined similarly to (3.143). Taking  $\theta_0 = \frac{1}{C_0} h^{1-\frac{1}{s}}$ , for some constant  $C_0 > 0$  large enough, we obtain in view of (3.144),

$$\begin{aligned}
 (3.146) \quad & \int_{\Gamma_{\theta_0}} e^{iq(z)/h} e^{-\varepsilon z^2/2} (1 - \tilde{\chi}(z)) \tilde{a}(x + z) dz \\
 &= \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\right),
 \end{aligned}$$

uniformly in  $\varepsilon > 0$ . Furthermore, using (2.5) and (3.144), we see that for some  $C_1 > 0$ , the second term in the right hand side of (3.145) is of the form

$$\begin{aligned}
 (3.147) \quad & \mathcal{O}(1) \int_0^{\theta_0} d\theta \int_{|y| \geq 1/2} e^{-\frac{\theta |y|}{\mathcal{O}(1)h}} \exp\left(-\frac{1}{C_1} \theta^{-\frac{1}{s-1}}\right) dy \\
 &\leq \mathcal{O}(1) \int_0^{\theta_0} \theta^{-N} \exp\left(-\frac{1}{C_1} \theta^{-\frac{1}{s-1}}\right) d\theta \leq \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\right),
 \end{aligned}$$

uniformly in  $\varepsilon > 0$ . Here we have also used the fact that the function

$$t \rightarrow \exp\left(-\frac{1}{2C_1} t^{-\frac{1}{s-1}}\right)$$

is increasing on  $[0, \theta_0]$ . Combining (3.145), (3.146), and (3.147) and letting  $\varepsilon \rightarrow 0^+$ , we get that the oscillatory integral in (3.142) satisfies,

$$(3.148) \quad r_{\chi}(x) = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\right), \quad x \in \mathbf{R}^N.$$

Considering the derivatives of  $r_{\chi}$  in (3.142) and using the fact that for each  $\alpha \in \mathbf{N}^N$ , the function  $\partial_x^{\alpha} \tilde{a} \in \mathcal{G}_b^s(\mathbf{C}^N)$  is an almost holomorphic extension

of  $\partial^\alpha a \in \mathcal{G}_b^s(\mathbf{R}^N)$ , we obtain, arguing as above,

$$(3.149) \quad |\partial^\alpha r_\chi(x)| \leq C^{1+|\alpha|} (\alpha!)^s \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\right), \quad x \in \mathbf{R}^N.$$

The proof is complete.  $\square$

Continuing to use the notation in the proof of Proposition 3.8, let us also consider

$$(3.150) \quad \ell_\chi(x) = h^{-N/2} \int e^{iq(y)/h} \chi(y) a(x+y) dy,$$

where we write  $q(y) = \frac{1}{2} Ay \cdot y$ . Letting

$$C_A = \frac{(2\pi)^{N/2} e^{i\frac{\pi}{4} \operatorname{sgn}(A)}}{|\det A|^{1/2}},$$

where  $\operatorname{sgn}(A)$  is the signature of  $A$ , and

$$\ell_{\chi,K}(x) = \ell_\chi(x) - C_A \sum_{k=0}^{K-1} \frac{h^k}{k!} \left(\frac{1}{2i} A^{-1} D \cdot D\right)^k a(x), \quad K = 1, 2, \dots,$$

we conclude by quadratic stationary phase and the fact that  $\chi_0 \in \mathcal{G}_0^s(\mathbf{R}^N)$ ,  $a \in \mathcal{G}_b^s(\mathbf{R}^N)$ , that there exists  $C > 0$  such that for all  $\alpha \in \mathbf{N}^N$ ,  $K \in \mathbf{N}$ , we have

$$(3.151) \quad |\partial_x^\alpha \ell_{\chi,K}(x)| \leq C^{1+K+|\alpha|} (K!)^{2s-1} (\alpha!)^s h^K.$$

In particular, we have  $\ell_\chi \in \mathcal{G}_b^s(\mathbf{R}^N)$ , and here once again we encounter the phenomenon of the loss of Gevrey smoothness in stationary phase expansions, see also (3.57), (3.58).

The discussion above gives, in particular, an alternative proof of the following result due to [16, 17].

**COROLLARY 3.9.** — *Let  $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ ,  $b \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ , for some  $s > 1$ . Then the symbol  $c = a \# b$ , defined in (3.137), satisfies  $c \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ .*

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