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A numerical technique for solving variable order time fractional differential-integro equations

M. H. Derakhshan^{a*}

^a *Zand institute of higher education, faculty of technology and engineering, Shiraz, Iran*

Abstract

In this manuscripts, we consider the coupled differential-integral equations including the variable-order Caputo fractional operator. To solve numerically these type of equations, we apply the shifted Jacobi–Gauss collocation scheme. Using this numerical method a system of algebraic equations is constructed. We solve this system with a recursive method in the nonlinear case and we solve it in linear case with algebraic formulas. Finally, for the high performance of the suggested method three Examples are illustrated.

Keywords: Coupled differential-integral equation; Caputo fractional operator; Shifted fractional Jacobi collocation method; Variable-order.

Mathematics Subject Classification: 26A33, 34A08.

1 Introduction

A coupled differential-integral equation including the Caputo fractional operator is introduced as follows:

$$\underbrace{CaputoD_{\varrho_1(x)}[p_1(x)] + p_1'(x) + \int_0^x p_1(v)dv}_{\text{the initial condition for this types of equation is } p_1(0)=p_1'(0)=0} = \theta_1(x) \quad , \quad (1)$$

the initial condition for this types of equation is $p_1(0)=p_1'(0)=0$,

$$\underbrace{CaputoD_{\varrho_2(x)}[p_2(x)] + p_2'(x) + \int_0^x p_2(v)dv}_{\text{the initial condition for this types of equation is } p_2(0)=p_2'(0)=0} = \theta_2(x) \quad , \quad (2)$$

the initial condition for this types of equation is $p_2(0)=p_2'(0)=0$,

where $CaputoD_{\varrho_i(x)}$, $i = 1, 2$ are the Caputo fractional operators of variable orders $0 < \varrho_i(x) \leq 1$, $i = 1, 2$ which are given by:

$$CaputoD_{\varrho_i(x)}p(x) =_{i=1,2} \begin{cases} p'(x), & \varrho_i(x) = 1, \\ \frac{1}{\Gamma(1-\varrho_i(x))} \int_0^x (x-v)^{-\varrho_i(x)} p'(v) dv, & 0 < \varrho_i(x) \leq 1. \end{cases} \quad (3)$$

In the coupled differential-integral equations (1), (2), $p_1(x), p_2(x)$ and $\theta_1(x), \theta_2(x)$ are indistincted and defined respectively. The main reason for choosing these kind of differential-integral equations is because of their role and applications in various fields as computational and mathematical sciences[14, 2, 16], engineering sciences which are modeled using these types of equations[12, 23, 24], physics[6, 11, 17] and mathematical models in chemistry[18, 26].

The main purpose of this paper is to obtain the solution of the coupled differential-integral equations are introduced in (1),(2) by using a numerical method that this numerical method is called a

*Corresponding Author: M.h.derakhshan.20@gmail.com

shifted Jacobi–Gauss collocation algorithm. This numerical method contains a shifted Jacobi polynomials. Obtaining a solution by using different numerical methods was studied by some authors in the mathematical field. , that in this paper a few numerical methods are mentioned, for Example, in [13] was studied a numerical method based on Chebyshev cardinal wavelets, the new algorithm base on the discretization method to obtain solution of the fractional functional integral equations of variable-order in [15] was discussed, the explicit and implicit Euler methods[28], the Legendre wavelets method[4], the numerical method base on Laplace and Sumudu transform methods[8], the Bezier curve method[9], the Adams-Bashforth-Moulton scheme[5], the optimization method [25] and other ways[19, 20, 10, 21, 3].

So, this article is divided into four sections as follows. In Section 2, Lemma and definitions of Jacobi polynomials, shifted Jacobi polynomials and their properties are introduced. In Section 3, approximation function and the numerical method to obtaining solution of the coupled differential-integral equation are described. In order to accuracy of the presented method four examples in Section 4 are described.

2 Some main lemma and definitions about the Jacobi polynomials and shifted Jacobi polynomials

This section deals with some definitions about the Jacobi polynomials and shifted Jacobi polynomials and Lemma which will be applied in next section.

Definition 1 Let $\mu, \nu > -1, x \in [-1, 1]$. Then the Jacobi polynomial of degree n is defined by[7]:

$$\mathfrak{P}_{n+1}^{\mu, \nu}(x) = (a_n^{\mu, \nu} x - b_n^{\mu, \nu})\mathfrak{P}_n^{\mu, \nu}(x) - c_n^{\mu, \nu}\mathfrak{P}_{n-1}^{\mu, \nu}(x), \quad (4)$$

where for $n = 0, 1$ the functions $\mathfrak{P}_0^{\mu, \nu}(x)$ and $\mathfrak{P}_1^{\mu, \nu}(x)$ are defined by:

$$\mathfrak{P}_0^{\mu, \nu}(x) = 1, \quad \mathfrak{P}_1^{\mu, \nu}(x) = \frac{(\mu + \nu + 2)x + \mu - \nu}{2}, \quad (5)$$

and the coefficients $a_n^{\mu, \nu}, b_n^{\mu, \nu}, c_n^{\mu, \nu}$ are given by:

$$\begin{aligned} a_n^{\mu, \nu} &= \frac{(2n + \mu + \nu + 1)(2n + \mu + \nu + 2)}{(2n + 2)(n + \mu + \nu + 1)}, \\ b_n^{\mu, \nu} &= \frac{(\nu^2 - \mu^2)(2n + \mu + \nu + 1)}{(2n + 2)(n + \mu + \nu + 1)(2n + \mu + \nu)}, \\ c_n^{\mu, \nu} &= \frac{(n + \mu)(n + \nu)(2n + \mu + \nu + 2)}{(n + 1)(n + \mu + \nu + 1)(2n + \mu + \nu)}. \end{aligned} \quad (6)$$

Also, the Jacobi polynomial has a finite series as follows:

$$\mathfrak{P}_n^{\mu, \nu}(x) = \frac{\Gamma(\mu + n + 1)}{n!\Gamma(\mu + \nu + n + 1)} \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(\mu + \nu + k + n + 1)}{\Gamma(\mu + k + 1)} \left(\frac{x-1}{2}\right)^k, \quad (7)$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2 Let $\mu, \nu > -1$. Then the shifted Jacobi polynomial of degree n is defined by[7]:

$$\begin{aligned} \mathbb{P}_n^{\mu, \nu}(x) &= \mathfrak{P}_n^{\mu, \nu}\left(\underbrace{2x-1}_{2x-1 \in [-1, 1]}\right) = \frac{\Gamma(\mu + n + 1)}{n!\Gamma(\mu + \nu + n + 1)} \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(\mu + \nu + k + n + 1)}{\Gamma(\mu + k + 1)} (x-1)^k \\ &= \frac{\Gamma(\nu + n + 1)}{\Gamma(\mu + \nu + n + 1)} \sum_{k=0}^n (-1)^{n-k} \frac{\Gamma(\mu + \nu + k + n + 1)}{k!(n-k)!\Gamma(\nu + k + 1)} x^k. \end{aligned} \quad (8)$$

Also, the shifted Jacobi polynomial respect to the weight function $\omega^{\mu, \nu}(x) = (1-x)^\mu x^\nu$ is orthogonal, that the orthogonality condition is given as:

$$\langle \mathbb{P}_n^{\mu, \nu}(x), \mathbb{P}_m^{\mu, \nu}(x) \rangle_{\omega^{\mu, \nu}(x)} = \int_0^1 \omega^{\mu, \nu}(x) \mathbb{P}_n^{\mu, \nu}(x) \mathbb{P}_m^{\mu, \nu}(x) dx = \begin{cases} 0, & n \neq m, \\ \frac{\Gamma(n+\mu+\nu+1)}{n!\Gamma(2n+\mu+\nu+1)}, & n = m, \end{cases} \quad (9)$$

where $\langle \cdot, \cdot \rangle$ is the inner product.

Lemma 1 Let $0 < \varrho_i \leq 1, i = 1, 2$. Then for any $\zeta \geq -1$ the Caputo fractional operator of variable order $\varrho_i, i = 1, 2$ is given by[27]:

$$\text{Caputo}D_{\varrho_i(x)}x^\zeta = \begin{cases} 0, & \zeta = \text{constant}, \\ \frac{\Gamma(\zeta+1)}{\Gamma(\zeta-\varrho_i(x)+1)}x^{\zeta-\varrho_i(x)}, & \zeta \geq -1, \\ \frac{d^m}{dx^m}(x^\zeta), & \varrho_i(x) = m, m \in \mathbb{N}. \end{cases} \quad (10)$$

So, applying the relation (10) on (8), for $i = 1, 2$ we obtain:

$$\begin{aligned} \text{Caputo}D_{\varrho_i(x)}\mathbb{P}_n^{\mu,\nu}(x) &= \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)} \sum_{k=0}^n (-1)^{n-k} \frac{\Gamma(k+1)}{\Gamma(k-\varrho_i(x)+1)} \times \frac{\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)} x^{k-\varrho_i(x)} \\ &= \sum_{k=0}^n \Omega_{k,n}^{\mu,\nu} x^{k-\varrho_i(x)}, \end{aligned} \quad (11)$$

$$\begin{aligned} \underbrace{\text{Caputo}D_{\varrho_i(x)}\mathbb{P}_n^{\mu,\nu}(x)}_{\text{for } \varrho_i(x)=1, m=1} &= \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)} \sum_{k=1}^n (-1)^{n-k} \frac{k\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)} x^{k-1} \\ &= \sum_{k=1}^n \Psi_{k,n}^{\mu,\nu} x^{k-1}, \end{aligned} \quad (12)$$

where $\Omega_{k,n}^{\mu,\nu} = \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)} \times (-1)^{n-k} \frac{\Gamma(k+1)}{\Gamma(k-\varrho_i(x)+1)} \times \frac{\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)}$ and $\Psi_{k,n}^{\mu,\nu} = \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)} \times (-1)^{n-k} \frac{k\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)}$.

3 The function approximation and proposed method algorithm

Due to the orthogonality condition is presented in (9) about the shifted Jacobi polynomial, any $p_1(x), p_2(x) \in L^2[0, 1]$ can be approximated in terms of the shifted Jacobi polynomial as follows:

$$p_1(x) = \sum_{k=0}^{\infty} a_k \mathbb{P}_k^{\mu,\nu}(x), \quad (13)$$

$$p_2(x) = \sum_{k=0}^{\infty} b_k \mathbb{P}_k^{\mu,\nu}(x), \quad (14)$$

considering the first n sentence of the relations (13), (14), we have:

$$p_1(x) \simeq p_{1,n}(x) = \sum_{k=0}^n a_k \mathbb{P}_k^{\mu,\nu}(x), \quad (15)$$

$$p_2(x) \simeq p_{2,n}(x) = \sum_{k=0}^n b_k \mathbb{P}_k^{\mu,\nu}(x), \quad (16)$$

where the coefficients a_j, b_j are calculated as:

$$a_j = \langle \mathbb{P}_n^{\mu,\nu}(x), p_1(x) \rangle_{\omega^{\mu,\nu}(x)} = \frac{\int_0^1 \omega^{\mu,\nu}(x) \mathbb{P}_n^{\mu,\nu}(x) p_1(x) dx}{\| \mathbb{P}_n^{\mu,\nu}(x) \|^2}, \quad (17)$$

$$b_j = \langle \mathbb{P}_n^{\mu,\nu}(x), p_2(x) \rangle_{\omega^{\mu,\nu}(x)} = \frac{\int_0^1 \omega^{\mu,\nu}(x) \mathbb{P}_n^{\mu,\nu}(x) p_2(x) dx}{\| \mathbb{P}_n^{\mu,\nu}(x) \|^2}. \quad (18)$$

Theorem 2 Let $p_{1,n}$ be as an approximation of the function $p_1(x)$, that it satisfy in the equation(1) and the following conditions for $\lambda_1, \lambda_2 \in (0, 1)$ hold:

$$\| p_{1,n}(x) - p_{1,n-1}(x) \| \leq \lambda_1 \| p_{1,n-1}(x) - p_{1,n-2}(x) \| \quad (19)$$

$$\| p'_{1,n}(x) - p'_{1,n-1}(x) \| \leq \lambda_2 \| p'_{1,n-1}(x) - p'_{1,n-2}(x) \| . \quad (20)$$

Then the sequence $p_{1,n}$ to $p_1(x)$ converges, that $p_1(x)$ is the exact solution of the equation(1),

Proof. Suppose $\mathbf{R}_n(x)$ be a residual function, which is defined as:

$$\begin{aligned}\mathbf{R}_n(x) &= {}^{Caputo}D_{\varrho_1(x)}[p_{1,n-1}(x)] + p'_{1,n-1}(x) + \int_0^x p_{1,n-1}(v)dv - \theta_1(x), \\ \mathbf{R}_{n-1}(x) &= {}^{Caputo}D_{\varrho_1(x)}[p_{1,n-2}(x)] + p'_{1,n-2}(x) + \int_0^x p_{1,n-2}(v)dv - \theta_1(x).\end{aligned}\quad (21)$$

Then, we have:

$$\begin{aligned}\|\mathbf{R}_n(x) - \mathbf{R}_{n-1}(x)\| &= \| {}^{Caputo}D_{\varrho_1(x)}[p_{1,n-1}(x) - p_{1,n-2}(x)] + (p'_{1,n-1}(x) - p'_{1,n-2}(x)) + \int_0^x (p_{1,n-1}(v) - p_{1,n-2}(v))dv \\ &= \left\| \frac{1}{\Gamma(1 - \varrho_1(x))} \int_0^x (x-v)^{-\varrho_1(x)} [p'_{1,n-1}(v) - p'_{1,n-2}(v)] dv + (p'_{1,n-1}(x) - p'_{1,n-2}(x)) \right. \\ &\quad \left. + \int_0^x (p_{1,n-1}(v) - p_{1,n-2}(v)) dv \right\| \\ &\leq \frac{1}{\Gamma(1 - \varrho_1(x))} \int_0^x (x-v)^{-\varrho_1(x)} \|p'_{1,n-1}(v) - p'_{1,n-2}(v)\| dv + \|p'_{1,n-1}(x) - p'_{1,n-2}(x)\| \\ &\quad + \int_0^x \|p_{1,n-1}(v) - p_{1,n-2}(v)\| dv \\ &= \frac{x^{1-\varrho_1(x)}}{\Gamma(2 - \varrho_1(x))} \int_0^x \|p'_{1,n-1}(v) - p'_{1,n-2}(v)\| dv + \|p'_{1,n-1}(x) - p'_{1,n-2}(x)\| \\ &\quad + \int_0^x \|p_{1,n-1}(v) - p_{1,n-2}(v)\| dv.\end{aligned}$$

Using (19), (20), we get:

$$\begin{aligned}\|\mathbf{R}_n(x) - \mathbf{R}_{n-1}(x)\| &\leq \frac{\lambda_2 x^{1-\varrho_1(x)}}{\Gamma(2 - \varrho_1(x))} \int_0^x \|p'_{1,n-2}(v) - p'_{1,n-3}(v)\| dv + \lambda_2 \|p'_{1,n-2}(x) - p'_{1,n-3}(x)\| \\ &\quad + \lambda_1 \int_0^x \|p_{1,n-2}(v) - p_{1,n-3}(v)\| dv \\ &\quad \vdots \\ &\leq \frac{\lambda_2^{n-2} x^{1-\varrho_1(x)}}{\Gamma(2 - \varrho_1(x))} \int_0^x \|p'_{1,1}(v) - p'_{1,0}(v)\| dv + \lambda_2^{n-2} \|p'_{1,1}(x) - p'_{1,0}(x)\| \\ &\quad + \lambda_1^{n-2} \int_0^x \|p_{1,1}(v) - p_{1,0}(v)\| dv \\ &\leq \frac{\lambda_2^{n-2} x^{2-\varrho_1(x)}}{\Gamma(2 - \varrho_1(x))} \max_{x \in [0,1]} \|p'_{1,1}(x) - p'_{1,0}(x)\| + \lambda_2^{n-2} \max_{x \in [0,1]} \|p'_{1,1}(x) - p'_{1,0}(x)\| \\ &\quad + \lambda_1^{n-2} \max_{x \in [0,1]} \|p_{1,1}(x) - p_{1,0}(x)\|.\end{aligned}\quad (23)$$

So, $\|\mathbf{R}_n(x) - \mathbf{R}_{n-1}(x)\| \rightarrow 0$ when $n \rightarrow \infty$, then $\mathbf{R}_n(x)$ a Cauchy sequence in $L^2[0, 1]$ and $L^2[0, 1]$ is complete, therefore the sequence $\mathbf{R}_n(x)$ is convergent that is

$$\lim_{n \rightarrow \infty} \mathbf{R}_n(x) = \mathbf{R}(x), \quad (24)$$

where

$$\begin{aligned}\mathbf{R}_n(x) &= {}^{Caputo}D_{\varrho_1(x)}[p_{1,n-1}(x)] + p'_{1,n-1}(x) + \int_0^x p_{1,n-1}(v)dv - \theta_1(x), \\ \mathbf{R}(x) &= {}^{Caputo}D_{\varrho_1(x)}[p_1(x)] + p'_1(x) + \int_0^x p_1(v)dv - \theta_1(x).\end{aligned}\quad (25)$$

The proof is complete. \square Like the Theorem2 can be considered a similar process for the equation (2).

Theorem 3 [22] Suppose $f(x)$ belongs to the space $\mathbf{H}^\sigma(0, 1)$, $\sigma \geq 0$ that $\mathbf{H}^\sigma(0, 1)$ is defined by:

$$\mathbf{H}^\sigma(0, 1) = \left\{ f \in L^2(0, 1) : \|f\|_{\mathbf{H}^\sigma(0,1)} = \left(\sum_{k=0}^{\sigma} \|f^{(k)}\|_{L^2(0,1)}^2 \right)^{\frac{1}{2}} < \infty \right\}, \quad (26)$$

and $P_{k,M}$ as the best approximation of f is considered. Then the following inequality is hold:

$$\|f - P_{k,M}\|_{L^2(0,1)} \leq cM^{-\sigma} 2^{-k\sigma} \|f^{(\sigma)}\|_{L^2(0,1)}^2, \quad (27)$$

$$\|f - P_{k,M}\|_{\mathbf{H}^q(0,1)} \leq cM^{2q-\frac{1}{2}-\sigma} 2^{k(q-\sigma)} \|f^{(\sigma)}\|_{L^2(0,1)}^2, \quad q \geq 1. \quad (28)$$

Theorem 4 Suppose $p_1(x)$ belongs to the space $\mathbf{H}^\sigma(0, 1)$, $\sigma \geq 0$ and p_N is the best approximation of $p_1(x)$. Then we have:

$$\|\mathbb{E}\|_{L^2(0,1)} \leq \frac{cN^{-\sigma} \|p_1^{(\sigma)}\|_{L^2(0,1)}^2}{\Gamma(2 - \varrho_1(x))} + 2cN^{-\sigma} \|p_1^{(\sigma)}\|_{L^2(0,1)}^2, \quad (29)$$

$$\|\mathbb{E}\|_{L^2(0,1)} \leq \frac{cN^{2q-\frac{1}{2}-\sigma} \|p_1^{(\sigma)}\|_{L^2(0,1)}^2}{\Gamma(2 - \varrho_1(x))} + 2cN^{2q-\frac{1}{2}-\sigma} \|p_1^{(\sigma)}\|_{L^2(0,1)}^2, \quad q \geq 1, \quad (30)$$

here $\mathbb{E} = p_1(x) - p_N(x)$ is considered.

Proof. Using (??), we obtain

$$\begin{aligned} \|\mathbb{E}\|_{L^2(0,1)} &= \|{}^{Caputo}D_{\varrho_1(x)}[p_1(x) - p_N(x)] + (p_1'(x) - p_N'(x)) + \int_0^x (p_1(v) - p_N(v))dv\|_{L^2(0,1)} \\ &= \left\| \frac{1}{\Gamma(1 - \varrho_1(x))} \int_0^x (x-v)^{-\varrho_1(x)} [p_1'(v) - p_N'(v)]dv + (p_1'(x) - p_N'(x)) + \int_0^x (p_1(v) - p_N(v))dv \right\|_{L^2(0,1)} \\ &\leq \frac{1}{\Gamma(1 - \varrho_1(x))} \left\| \int_0^x (x-v)^{-\varrho_1(x)} [p_1'(v) - p_N'(v)]dv \right\|_{L^2(0,1)} + \|p_1'(x) - p_N'(x)\|_{L^2(0,1)} \\ &+ \left\| \int_0^x (p_1(v) - p_N(v))dv \right\|_{L^2(0,1)}. \end{aligned} \quad (31)$$

we use $\|f * g\|_{L^2(0,1)} \leq \|f\|_1 \|g\|_{L^2(0,1)}$ for Eq.(31), we have:

$$\begin{aligned} \|\mathbb{E}\|_{L^2(0,1)} &\leq \frac{1}{\Gamma(1 - \varrho_1(x))} \left\| \int_0^x (x-v)^{-\varrho_1(x)} [p_1'(v) - p_N'(v)]dv \right\|_{L^2(0,1)} + \|p_1'(x) - p_N'(x)\|_{L^2(0,1)} \\ &+ \left\| \int_0^x (p_1(v) - p_N(v))dv \right\|_{L^2(0,1)} \\ &\leq \frac{x^{1-\varrho_1(x)}}{\Gamma(2 - \varrho_1(x))} \|p_1'(x) - p_N'(x)\|_{L^2(0,1)} + \|p_1'(x) - p_N'(x)\|_{L^2(0,1)} + \left\| \int_0^x (p_1(v) - p_N(v))dv \right\|_{L^2(0,1)} \\ &\leq \frac{x^{1-\varrho_1(x)}}{\Gamma(2 - \varrho_1(x))} \|p_1(x) - p_N(x)\|_{L^2(0,1)} + \|p_1(x) - p_N(x)\|_{L^2(0,1)} + x \|p_1(x) - p_N(x)\|_{L^2(0,1)} \\ &\leq \frac{\|p_1(x) - p_N(x)\|_{L^2(0,1)}}{\Gamma(2 - \varrho_1(x))} + 2 \|p_1(x) - p_N(x)\|_{L^2(0,1)}, \end{aligned} \quad (32)$$

so, we employ Eqs.(27),(28) on the relation (32), we get:

$$\|\mathbb{E}\|_{L^2(0,1)} \leq \frac{cN^{-\sigma} \|p_1^{(\sigma)}\|_{L^2(0,1)}^2}{\Gamma(2 - \varrho_1(x))} + 2cN^{-\sigma} \|p_1^{(\sigma)}\|_{L^2(0,1)}^2, \quad (33)$$

$$\|\mathbb{E}\|_{L^2(0,1)} \leq \frac{cN^{2q-\frac{1}{2}-\sigma} \|p_1^{(\sigma)}\|_{L^2(0,1)}^2}{\Gamma(2 - \varrho_1(x))} + 2cN^{2q-\frac{1}{2}-\sigma} \|p_1^{(\sigma)}\|_{L^2(0,1)}^2, \quad q \geq 1. \quad (34)$$

The desired result is achieved. \square

Theorem 5 Suppose $p_2(x)$ is belongs to the space $\mathbf{H}^\sigma(0,1)$, $\sigma \geq 0$ and $p_{N'}$ is the best approximation of $p_1(x)$. Then we have:

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN'^{-\sigma} \| p_1^{(\sigma)} \|_{L^2(0,1)}^2}{\Gamma(2 - \varrho_2(x))} + 2cN'^{-\sigma} \| p_2^{(\sigma)} \|_{L^2(0,1)}^2, \quad (35)$$

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN'^{2q-\frac{1}{2}-\sigma} \| p_2^{(\sigma)} \|_{L^2(0,1)}^2}{\Gamma(2 - \varrho_2(x))} + 2cN'^{2q-\frac{1}{2}-\sigma} \| p_2^{(\sigma)} \|_{L^2(0,1)}^2, \quad q \geq 1. \quad (36)$$

Proof. The proof of this Theorem is similar to the proof of the Theorem 4. \square

3.1 The shifted Jacobi-Gauss collocation algorithm

This subsection describes the proposed method for solving Eqs.(1), (2). By substituting the relations (15) and (16) in Eqs.(1), (2), we obtain

$$\underbrace{Caputo D_{\varrho_1(x)} \left[\sum_{k=0}^n a_k \mathbb{P}_k^{\mu,\nu}(x) \right] + \frac{d}{dx} \left(\sum_{k=0}^n a_k \mathbb{P}_k^{\mu,\nu}(x) \right) + \int_0^x \sum_{k=0}^n a_k \mathbb{P}_k^{\mu,\nu}(v) dv}_{\text{the initial condition for this types of equation is } \sum_{k=0}^n a_k \mathbb{P}_k^{\mu,\nu}(0) = \sum_{k=0}^n a_k \mathbb{P}'_k{}^{\mu,\nu}(0) = 0}, = \theta_1(x), \quad (37)$$

$$\underbrace{Caputo D_{\varrho_2(x)} \left[\sum_{k=0}^n b_k \mathbb{P}_k^{\mu,\nu}(x) \right] + \sum_{k=0}^n b_k \mathbb{P}'_k{}^{\mu,\nu}(x) + \int_0^x \sum_{k=0}^n b_k \mathbb{P}_k^{\mu,\nu}(v) dv}_{\text{the initial condition for this types of equation is } \sum_{k=0}^n b_k \mathbb{P}_k^{\mu,\nu}(0) = \sum_{k=0}^n b_k \mathbb{P}'_k{}^{\mu,\nu}(0) = 0}, = \theta_2(x), \quad (38)$$

by applying Eqs.(11),(12) on the equations (37) and (38), we have:

$$\sum_{k=0}^n \sum_{m=0}^k a_k \Omega_{m,k}^{\mu,\nu} x^{m-\varrho_1(x)} + \sum_{k=0}^n \sum_{m=1}^{k+1} a_k \Psi_{m,k}^{\mu,\nu} x^{m-1} + \sum_{k=0}^n \sum_{m=0}^k a_k \Upsilon_{m,k}^{\mu,\nu} x^{m+1} = \theta_1(x), \quad (39)$$

$$\sum_{k=0}^n \sum_{m=0}^k b_k \Omega_{m,k}^{\mu,\nu} x^{m-\varrho_2(x)} + \sum_{k=0}^n \sum_{m=1}^{k+1} b_k \Psi_{m,k}^{\mu,\nu} x^{m-1} + \sum_{k=0}^n \sum_{m=0}^k b_k \Upsilon_{m,k}^{\mu,\nu} x^{m+1} = \theta_2(x), \quad (40)$$

where $\Upsilon_{m,k}^{\mu,\nu} = (-1)^{k-m} \frac{\Gamma(\nu+k+1)\Gamma(\mu+\nu+m+k+1)}{(m+1)!(k-m)!\Gamma(\nu+m+1)}$. Now we calculate the equations (39) and (40) in points $x_{\mu,\nu,i} = \frac{x_{\mu,\nu,i+1} + x_{\mu,\nu,i-1}}{2}$ which is the nodes of the standard Jacobi-Gauss interpolation in the interval $[-1, 1]$ and their introduced in [7]. Then for $i = 1, 2, \dots, n$ we get:

$$\sum_{k=0}^n \sum_{m=0}^k a_k \Omega_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-\varrho_1(x)} + \sum_{k=0}^n \sum_{m=1}^{k+1} a_k \Psi_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-1} + \sum_{k=0}^n \sum_{m=0}^k a_k \Upsilon_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m+1} = \theta_1(x_{\mu,\nu,i}),$$

$$\sum_{k=0}^n a_k \mathbb{P}_k^{\mu,\nu}(0) = \sum_{k=0}^n a_k \mathbb{P}'_k{}^{\mu,\nu}(0) = 0. \quad (41)$$

$$\sum_{k=0}^n \sum_{m=0}^k b_k \Omega_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-\varrho_2(x)} + \sum_{k=0}^n \sum_{m=1}^{k+1} b_k \Psi_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-1} + \sum_{k=0}^n \sum_{m=0}^k b_k \Upsilon_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m+1} = \theta_2(x_{\mu,\nu,i}),$$

$$\sum_{k=0}^n b_k \mathbb{P}_k^{\mu,\nu}(0) = \sum_{k=0}^n b_k \mathbb{P}'_k{}^{\mu,\nu}(0) = 0. \quad (42)$$

By solving these equations together with the initial conditions, the uncertain coefficients a_k, b_k are obtained. When the functions $\theta_1(x), \theta_2(x)$ are nonlinear, in this case the equations (41) and (42) changes

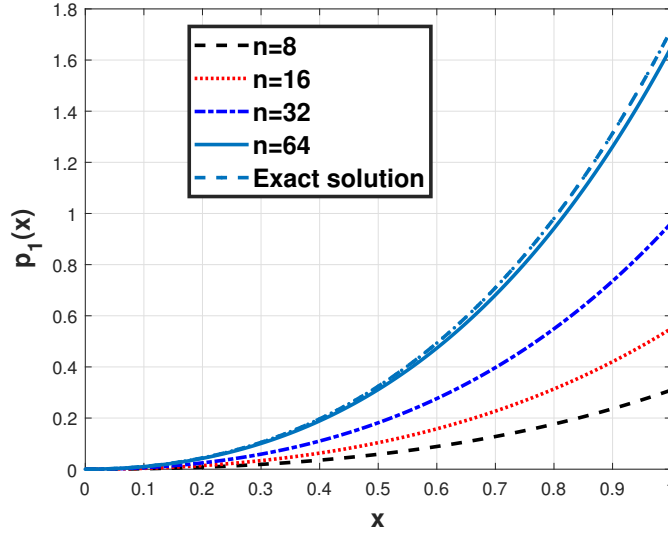


Figure 1: Numerical and exact solutions of the Example 1 with $\varrho_1(x) = 1 - 0.001x$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$.

to the following equations:

$$\sum_{k=0}^n \sum_{m=0}^k a_k \Omega_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-\varrho_1(x)} + \sum_{k=0}^n \sum_{m=1}^{k+1} a_k \Psi_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-1} + \sum_{k=0}^n \sum_{m=0}^k a_k \Upsilon_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m+1} = \theta_1(x_{\mu,\nu,i}, p_1(x_{\mu,\nu,i}), p_1'(x_{\mu,\nu,i})),$$

$$\sum_{k=0}^n a_k \mathbb{P}_k^{\mu,\nu}(0) = \sum_{k=0}^n a_k \mathbb{P}'_k{}^{\mu,\nu}(0) = 0. \quad (43)$$

$$\sum_{k=0}^n \sum_{m=0}^k b_k \Omega_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-\varrho_2(x)} + \sum_{k=0}^n \sum_{m=1}^{k+1} b_k \Psi_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-1} + \sum_{k=0}^n \sum_{m=0}^k b_k \Upsilon_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m+1} = \theta_2(x_{\mu,\nu,i}, p_2(x_{\mu,\nu,i}), p_2'(x_{\mu,\nu,i})),$$

$$\sum_{k=0}^n b_k \mathbb{P}_k^{\mu,\nu}(0) = \sum_{k=0}^n b_k \mathbb{P}'_k{}^{\mu,\nu}(0) = 0. \quad (44)$$

By applying a recursive method on the relations (43) and (44) can be obtained the uncertain coefficients a_k, b_k .

4 Numerical Examples

This section examines four examples by using the proposed method to show its performance and accuracy.

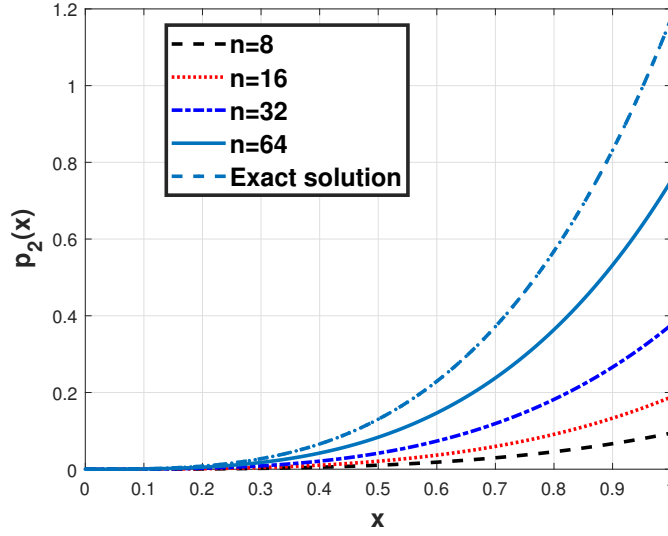


Figure 2: Numerical and exact solutions of the Example 1 with $\varrho_2(x) = 1 - \sin^2(x)$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$.

Table 1: The absolute error for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$ and $\varrho_1(x) = 1 - 0.001x$ for Example 1.

2-34-5 x	$ p_1(x) - p_{1,n}(x) $			$ p_1(x) - p_{1,n}(x) $	
	$n = 8$	$n = 16$	$n = 32$	$n = 32$	$n = 64$
0.1	$0.96485e - 15$	$0.99077e - 16$	$1.01719e - 17$	$1.01719e - 17$	$1.04410e - 18$
0.2	$0.86589e - 15$	$0.88993e - 16$	$0.91443e - 17$	$0.91443e - 17$	$0.93940e - 18$
0.3	$0.77425e - 15$	$0.79650e - 16$	$0.81918e - 17$	$0.81918e - 17$	$0.84231e - 18$
0.4	$0.68954e - 15$	$0.71009e - 16$	$0.73105e - 17$	$0.73105e - 17$	$0.75244e - 18$
0.5	$0.61139e - 15$	$0.63033e - 16$	$0.64967e - 17$	$0.64967e - 17$	$0.66940e - 18$
0.6	$0.53947e - 15$	$0.55688e - 16$	$0.57467e - 17$	$0.57467e - 17$	$0.59284e - 18$
0.7	$0.47343e - 15$	$0.48940e - 16$	$0.50573e - 17$	$0.50573e - 17$	$0.52242e - 18$
0.8	$0.41297e - 15$	$0.42757e - 16$	$0.44252e - 17$	$0.44252e - 17$	$0.45780e - 18$
0.9	$0.35778e - 15$	$0.37110e - 16$	$0.38473e - 17$	$0.38473e - 17$	$0.39869e - 18$

4.1 Example 1.

We consider the following coupled differential-integral equation of order $\varrho_1(x) = 1 - 0.001x, \varrho_2(x) = 1 - \sin^2(x)$:

$$\begin{aligned} {}^{Caputo}D_{\varrho_1(x)}[p_1(x)] + p_1'(x) + \int_0^x p_2(v)dv &= \theta_1(x), \\ p_1(0) &= p_1'(0) = 0, \end{aligned} \quad (45)$$

$$\begin{aligned} {}^{Caputo}D_{\varrho_2(x)}[p_2(x)] + p_2'(x) + \int_0^x p_1(v)dv &= \theta_2(x), \\ p_2(0) &= p_2'(0) = 0, \end{aligned} \quad (46)$$

where $\theta_1(x) = x^2 \cosh(x) - 2t \sinh(x) + 2 \cosh(x) + 2(e^x + 2xe^x - 1)$ and $\theta_2(x) = 4x \cosh(x) + 2x^2 \cosh(x) + e^x(x - 1)$. The analytical solutions for these questions are $p_1(x) = xe^x - x, p_2(x) = x^2 \sinh(x)$. This Example is solved by the proposed method and the results of the approximate and exact solutions are shown in the Figures 1 and 2. In the Tables 1 and 2, the absolute error between the approximate and analytical solutions are displayed.

Table 2: The absolute error for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$ and $\varrho_2(x) = 1 - \sin^2(x)$ for Example 1.

2-34-5 x	$ p_2(x) - p_{2,n}(x) $		$ p_2(x) - p_{2,n}(x) $	
	n=8	n=16	n=32	n=64
0.1	$1.02256e - 15$	$1.12266e - 17$	$1.22921e - 19$	$1.34242e - 21$
0.2	$1.46252e - 17$	$1.58973e - 19$	$1.72428e - 21$	$1.86639e - 23$
0.3	$0.68232e - 17$	$0.75878e - 19$	$0.84084e - 21$	$0.92869e - 23$
0.4	$0.42820e - 17$	$0.48438e - 19$	$0.54532e - 21$	$0.61123e - 23$
0.5	$0.24707e - 17$	$0.28622e - 19$	$0.32932e - 21$	$0.37658e - 23$
0.6	$0.12620e - 17$	$0.15145e - 19$	$0.17989e - 21$	$0.21170e - 23$
0.7	$0.05314e - 17$	$0.06759e - 19$	$0.08446e - 21$	$0.10393e - 23$
0.8	$0.01572e - 17$	$0.02239e - 19$	$0.03073e - 21$	$0.04091e - 23$
0.9	$0.00196e - 17$	$0.00383e - 19$	$0.00663e - 21$	$0.01053e - 23$

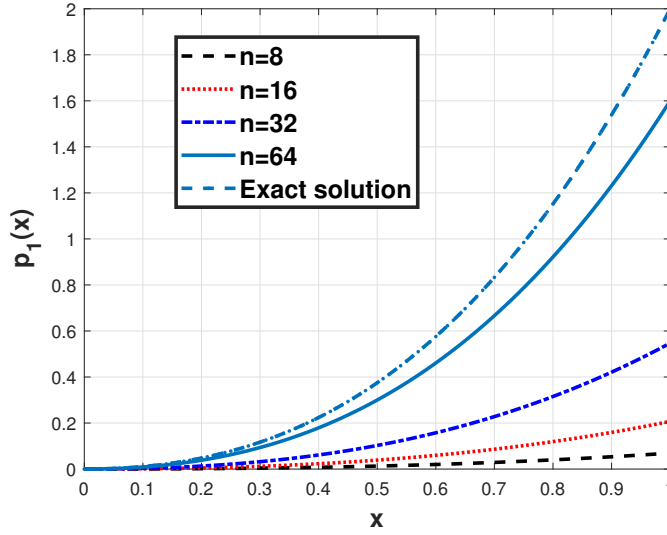


Figure 3: Numerical and exact solutions of the Example 2 with $\varrho_1(x) = \sin(\frac{x}{3})$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$.

4.2 Example 2.

Consider the following coupled differential-integral equation with $\varrho_1(x) = \sin(\frac{x}{3}), \varrho_2(x) = \frac{x}{3}$:

$$\begin{aligned} {}^{Caputo}D_{\varrho_1(x)}[p_1(x)] + p_1'(x) + \int_0^x p_1(v)dv &= 4x(x+1) + 2x^2 + \frac{x^4}{4} + \frac{x^3}{3}, \\ p_1(0) &= p_1'(0) = 0, \end{aligned} \quad (47)$$

$$\begin{aligned} {}^{Caputo}D_{\varrho_2(x)}[p_2(x)] + p_2'(x) + \int_0^x p_2(v)dv &= 4x(x-1) + 2x^x + \frac{x^4}{4} - \frac{x^3}{3}, \\ p_2(0) &= p_2'(0) = 0, \end{aligned} \quad (48)$$

the exact solutions of Eqs.(47),(48) are $p_1(x) = x^2(x+1), p_2(x) = x^2(x-1)$. This Example is solved by the proposed method and the results of the approximate and exact solutions are shown in the Figures 3 and 4. In the Tables 3 and 4, the absolute error between the approximate and analytical solutions are displayed.

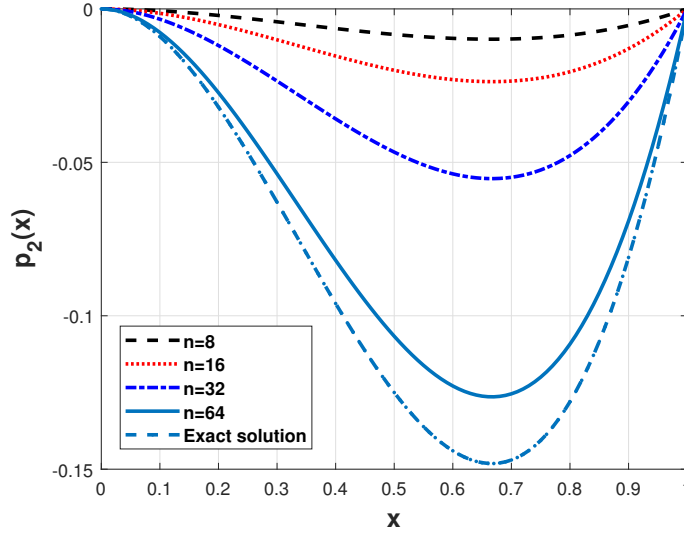


Figure 4: Numerical and exact solutions of the Example 2 with $\varrho_2(x) = \frac{x}{3}$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$.

Table 3: The absolute error for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$ and $\varrho_1(x) = \sin(\frac{x}{3})$ for Example 2.

2-34-5 x	$ p_1(x) - p_{1,n}(x) $		$ p_1(x) - p_{1,n}(x) $	
	n=8	n=16	n=32	n=64
0.1	$0.20113e - 15$	$0.20639e - 16$	$0.21173e - 17$	$0.21717e - 18$
0.2	$0.18095e - 15$	$0.18586e - 16$	$0.19087e - 17$	$0.19595e - 18$
0.3	$0.16210e - 15$	$0.16669e - 16$	$0.17136e - 17$	$0.17611e - 18$
0.4	$0.14456e - 15$	$0.14883e - 16$	$0.15317e - 17$	$0.15760e - 18$
0.5	$0.12827e - 15$	$0.13223e - 16$	$0.13626e - 17$	$0.14037e - 18$
0.6	$0.11319e - 15$	$0.11685e - 16$	$0.12058e - 17$	$0.12439e - 18$
0.7	$0.09928e - 15$	$0.10265e - 16$	$0.10609e - 17$	$0.10960e - 18$
0.8	$0.08649e - 15$	$0.08959e - 16$	$0.09275e - 17$	$0.09598e - 18$
0.9	$0.07479e - 15$	$0.07762e - 16$	$0.08051e - 17$	$0.08347e - 18$

4.3 Example 3.

We describe the following nonlinear coupled differential-integral equation with $\varrho_1(x) = e^x - x^2, \varrho_2(x) = \sin^2(x)$:

$$\begin{aligned}
 {}^{Caputo}D_{\varrho_1(x)} [p_1(x)] + p_1'(x) + \int_0^x p_1(v)dv &= 2x^2 + \frac{x^4}{12} + \frac{x^9}{27} - p_1^3(x), \\
 p_1(0) &= p_1'(0) = 0,
 \end{aligned} \tag{49}$$

$$\begin{aligned}
 {}^{Caputo}D_{\varrho_2(x)} [p_2(x)] + p_2'(x) + \int_0^x p_2(v)dv &= 4x + \frac{7x^3}{3} - p_2'(x)p_2(x), \\
 p_2(0) &= p_2'(0) = 0,
 \end{aligned} \tag{50}$$

the exact solutions of Eqs.(49),(50) are $p_1(x) = \frac{1}{3}x^3, p_2(x) = x^2$. This Example is solved by the proposed method and the results of the approximate and exact solutions are shown in the Figures 5 and 6. In the Tables 5 and 6, the absolute error between the approximate and analytical solutions are displayed.

Table 4: The absolute error for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$ and $\varrho_2(x) = \frac{x}{3}$ for Example 2

2-34-5 x	$ p_2(x) - p_{2,n}(x) $		$ p_2(x) - p_{2,n}(x) $	
	n=8	n=16	n=32	n=64
0.1	$2.18198e - 15$	$2.26887e - 16$	$2.35518e - 17$	$2.44075e - 18$
0.2	$1.83178e - 15$	$1.91941e - 16$	$2.00709e - 17$	$2.09466e - 18$
0.3	$1.48495e - 15$	$1.57079e - 16$	$1.65732e - 17$	$1.74436e - 18$
0.4	$1.15160e - 15$	$1.23312e - 16$	$1.31596e - 17$	$1.39996e - 18$
0.5	$0.84181e - 15$	$0.91649e - 16$	$0.99312e - 17$	$1.07154e - 18$
0.6	$0.56569e - 15$	$0.63101e - 16$	$0.69891e - 17$	$0.76923e - 18$
0.7	$0.33335e - 15$	$0.38678e - 16$	$0.44342e - 17$	$0.50311e - 18$
0.8	$0.15489e - 15$	$0.19390e - 16$	$0.23676e - 17$	$0.28329e - 18$
0.9	$0.04040e - 15$	$0.06247e - 16$	$0.08902e - 17$	$0.11987e - 18$

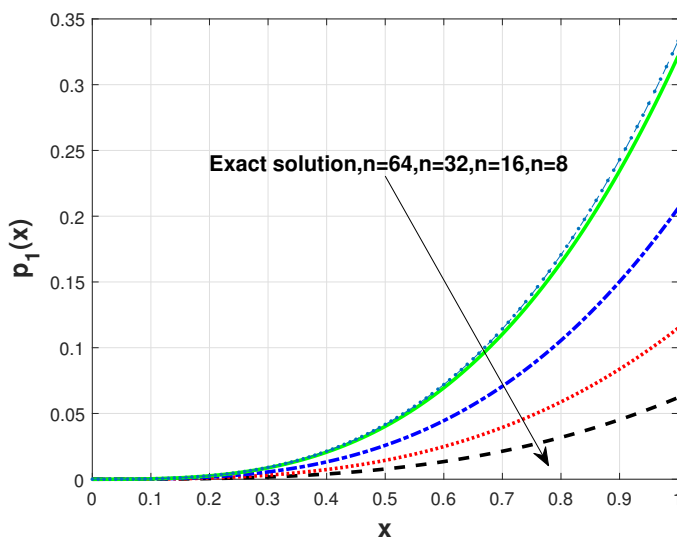


Figure 5: Numerical and exact solutions of the Example 3 with $\varrho_1(x) = e^x - x^2$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$.

5 Conclusion

This article is focused on the coupled differential-integral equation including the Caputo fractional operator of variable-orders and it is proposed a numerical method based on the shifted Jacobi–Gauss collocation scheme to obtain the solution of the coupled differential-integral equation. Also, in this paper about the convergence and an upper bound on the error are discussed. Some numerical Examples have been showed in order to display the high exactness of the suggested method.

References

- [1] Amiri S, Hajipour M, Baleanu D. A spectral collocation method with piecewise trigonometric basis functions for nonlinear Volterra–Fredholm integral equations. Applied Mathematics and Computation. 2020 Apr 1;370:124915.
- [2] Abdou MA. Fredholm–Volterra integral equation of the first kind and contact problem. Applied Mathematics and Computation. 2002 Jan 25;125(2-3):177-93.

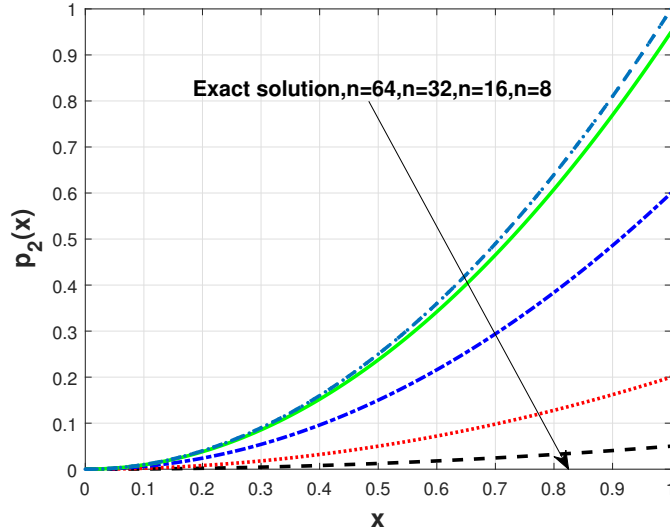


Figure 6: Numerical and exact solutions of the Example 3 with $\varrho_2(x) = \sin^2(x)$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$.

Table 5: The absolute error for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$ and $\varrho_1(x) = e^x - x^2$ for Example 3 .

2-34-5 x	$ p_1(x) - p_{1,n}(x) $			$ p_1(x) - p_{1,n}(x) $	
	n=8	n=16	n=32	n=64	n=64
0.1	$1.35242e - 15$	$1.42118e - 16$	$1.49223e - 17$	$1.56560e - 18$	$1.56560e - 18$
0.2	$1.09957e - 15$	$1.15953e - 16$	$1.22164e - 17$	$1.28592e - 18$	$1.28592e - 18$
0.3	$0.88038e - 15$	$0.93215e - 16$	$0.98592e - 17$	$1.04171e - 18$	$1.04171e - 18$
0.4	$0.69244e - 15$	$0.73662e - 16$	$0.78265e - 17$	$0.83056e - 18$	$0.83056e - 18$
0.5	$0.53335e - 15$	$0.57055e - 16$	$0.60944e - 17$	$0.65006e - 18$	$0.65006e - 18$
0.6	$0.40071e - 15$	$0.43153e - 16$	$0.46388e - 17$	$0.49781e - 18$	$0.49781e - 18$
0.7	$0.29212e - 15$	$0.31715e - 16$	$0.34356e - 17$	$0.37141e - 18$	$0.37141e - 18$
0.8	$0.20516e - 15$	$0.22501e - 16$	$0.24609e - 17$	$0.26845e - 18$	$0.26845e - 18$
0.9	$0.13744e - 15$	$0.15270e - 16$	$0.16905e - 17$	$0.18652e - 18$	$0.18652e - 18$

[3] Alshammari S, Al-Smadi M, Hashim I, Alias MA. Applications of fractional power series approach in solving fractional Volterra integro-differential equations. In AIP Conference Proceedings 2019 Jun 27 (Vol. 2111, No. 1, p. 020003). AIP Publishing LLC.

[4] Chen YM, Wei YQ, Liu DY, Yu H. Numerical solution for a class of nonlinear variable order fractional differential equations with Legendre wavelets. Applied Mathematics Letters. 2015 Aug 1;46:83–8.

[5] Coronel-Escamilla A, Gómez-Aguilar JF. A novel predictor-corrector scheme for solving variable-order fractional delay differential equations involving operators with Mittag-Leffler kernel. Discrete & Continuous Dynamical Systems-S. 2020;13(3):561.

[6] Dabiri A, Butcher EA, Nazari M. Coefficient of restitution in fractional viscoelastic compliant impacts using fractional Chebyshev collocation. Journal of Sound and Vibration. 2017 Feb 3;388:230–44.

[7] Doha EH, Abdelkawy MA, Amin AZ, Lopes AM. On spectral methods for solving variable-order fractional integro-differential equations. Computational and Applied Mathematics. 2018 Jul 1;37(3):3937–50.

[8] Gómez-Aguilar JF. Analytical and Numerical solutions of a nonlinear alcoholism model via variable-order fractional differential equations. Physica A: Statistical Mechanics and its Applications. 2018 Mar 15;494:52–75.

Table 6: The absolute error for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$ and $\varrho_2(x) = \sin^2(x)$ for Example 3 .

2-34-5 x	$ p_2(x) - p_{2,n}(x) $		$ p_2(x) - p_{2,n}(x) $	
	n=8	n=16	n=32	n=64
0.1	$1.96991e - 15$	$2.02502e - 16$	$2.08087e - 17$	$2.13749e - 18$
0.2	$1.75711e - 15$	$1.80917e - 16$	$1.86199e - 17$	$1.91558e - 18$
0.3	$1.55647e - 15$	$1.60549e - 16$	$1.65527e - 17$	$1.70581e - 18$
0.4	$1.36799e - 15$	$1.41398e - 16$	$1.46072e - 17$	$1.50822e - 18$
0.5	$1.19168e - 15$	$1.23462e - 16$	$1.27832e - 17$	$1.32277e - 18$
0.6	$1.02752e - 15$	$1.06742e - 16$	$1.10808e - 17$	$1.14950e - 18$
0.7	$0.87552e - 15$	$0.91238e - 16$	$0.95000e - 17$	$0.98838e - 18$
0.8	$0.73568e - 15$	$0.76950e - 16$	$0.80408e - 17$	$0.83942e - 18$
0.9	$0.60800e - 15$	$0.63878e - 16$	$0.67032e - 17$	$0.70262e - 18$

- [9] Ghomanjani F. A new approach for solving linear fractional integro-differential equations and multi variable order fractional differential equations. *Proyecciones (Antofagasta)*. 2020 Feb;39(1):199–218.
- [10] Ganji RM, Jafari H. Numerical solution of variable order integro-differential equations. *Advanced Mathematical Models & Applications*. 2019;4(1).
- [11] Holm aker K. Global asymptotic stability for a stationary solution of a system of integro-differential equations describing the formation of liver zones. *SIAM Journal on Mathematical Analysis*. 1993 Jan;24(1):116–28.
- [12] Hariharan G, Kannan K. Review of wavelet methods for the solution of reaction-diffusion problems in science and engineering. *Applied Mathematical Modelling*. 2014 Feb 1;38(3):799–813.
- [13] Heydari MH. Chebyshev cardinal wavelets for nonlinear variable-order fractional quadratic integral equations. *Applied Numerical Mathematics*. 2019 Oct 1;144:190–203.
- [14] Kaucher EW, Miranker WL. Self-validating numerics for function space problems: Computation with guarantees for differential and integral equations. Elsevier; 2014 Jun 20.
- [15] Keshi FK, Moghaddam BP, Aghili A. A numerical approach for solving a class of variable-order fractional functional integral equations. *Computational and Applied Mathematics*. 2018 Sep 1;37(4):4821–34.
- [16] Li P. Singular integral equations of convolution type with Cauchy kernel in the class of exponentially increasing functions. *Applied Mathematics and Computation*. 2019 Mar 1;344:116–27.
- [17] Li Y, Jiang M, Liu F. Time fractional super-diffusion model and its application in peak-preserving smoothing. *Chemometrics and Intelligent Laboratory Systems*. 2018 Apr 15;175:13–9.
- [18] Mvogo A, Ben-Bolie GH, Kofan  TC. Fractional nonlinear dynamics of DNA breathing. *Communications in Nonlinear Science and Numerical Simulation*. 2017 Jul 1;48:258–69.
- [19] Mohebbi A, Saffarian M. Implicit RBF Meshless Method for the Solution of Two-dimensional Variable Order Fractional Cable Equation. *Journal of Applied and Computational Mechanics*. 2020 Apr 1;6(2):235–47.
- [20] Moghaddam BP, Machado JA. A computational approach for the solution of a class of variable-order fractional integro-differential equations with weakly singular kernels. *Fractional Calculus and Applied Analysis*. 2017 Aug 28;20(4):1023–42.
- [21] Mousavi BK, Heydari MH. Wilson wavelets method for solving nonlinear fractional Fredholm-Hammerstein integro-differential equations. *International Journal of Computer Mathematics*. 2019 Nov 7:1–3.

- [22] Mashayekhi S, Ordokhani Y, Razzaghi M. A hybrid functions approach for the Duffing equation. *Physica Scripta*. 2013 Jul 15;88(2):025002.
- [23] Rong SI. *Theory of stochastic differential equations with jumps and applications: mathematical and analytical techniques with applications to engineering*. Springer Science & Business Media; 2006 May 6.
- [24] Radid A, Rhofir K. *Mathematical Modelling of Engineering Problems*. Journal homepage: <http://iieta.org/Journals/MMEP>. 2019 Jun;6(2):235–40.
- [25] Soradi-Zeid S, Jahanshahi H, Yousefpour A, Bekiros S. King algorithm: A novel optimization approach based on variable-order fractional calculus with application in chaotic financial systems. *Chaos, Solitons & Fractals*. 2020 Mar 1;132:109569.
- [26] Xu K, Wang J. Weighted fractional permutation entropy and fractional sample entropy for nonlinear Potts financial dynamics. *Physics Letters A*. 2017 Feb 26;381(8):767–79.
- [27] Xie J, Wang T, Ren Z, Zhang J, Quan L. Haar wavelet method for approximating the solution of a coupled system of fractional-order integral-differential equations. *Mathematics and Computers in Simulation*. 2019 Sep 1;163:80–9.
- [28] Zhuang P, Liu F, Anh V, Turner I. Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term. *SIAM Journal on Numerical Analysis*. 2009;47(3):1760–81.