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Limit of the environment viewed from Sinai's walk

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Abstract

For Sinai's walk (X_k) we show that the empirical measure of the environment seen from the particle $(\tilde{\omega}_k) = (\omega(X_k + x), x \in \mathbb{Z})$ converges in law to some random measure \mathcal{S}_∞ , explicitly given in terms of the infinite valley \tilde{V} , defined in Golosov (1984). As a consequence an 'in law' ergodic theorem holds:

$$\frac{1}{n} \sum_{k=1}^n F(\tilde{\omega}_k) \xrightarrow{\mathcal{L}} \int_{\Omega} F d\mathcal{S}_\infty.$$

This allows some extensions to the recurrent case of the 'environment method' of Kozlov and Molchanov (1984). In particular, we show the LLN and the CLT with the convergence to a mixture of Gaussians for $\sum_{k=1}^n f(\omega(X_k), X_{k+1} - X_k)$.

The new ingredient is the convergence in distribution in ℓ^1 of the sequence $(\exp - (V(b_n + x) - V(b_n))); x \in \mathbb{Z})_n$ where b_n is the bottom of the main valley of the potential V , to $(\exp - \tilde{V}(x), x \in \mathbb{Z})$. Whereas the finite dimensional convergence here is known since Golosov (1984) and Bertoin (1993), the tightness result is new and relies on new estimations of the growth of the random walk conditioned to stay positive.

Keywords: Random walk in random environment, Sinai's localization, Environment seen from the particle, Random walk conditioned to stay positive

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1. Introduction, assumptions and main results

1.1. Model

Let $\omega = \{\omega(x); x \in \mathbb{Z}\}$ be a collection of i.i.d. random variables taking values in $[0, 1]$. Denote $\Omega := [0, 1]^{\mathbb{Z}}$, \mathbb{P} the distribution of ω on $(\Omega, \mathcal{B}(\Omega))$ and \mathbb{E} the expectation under this law. For fixed $\omega \in \Omega$, let $X = (X_k)_{k \in \mathbb{N}}$ be the time-homogeneous Markov chain on \mathbb{Z}_+ with transition function p^ω given by $p^\omega(0, 1) = 1$, and for all $x \in \mathbb{Z}_+^*$,

$$p^\omega(x, y) = \begin{cases} \omega(x) & \text{if } y = x + 1, \\ 1 - \omega(x) & \text{if } y = x - 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in \mathbb{Z}_+$ and fixed $\omega \in \Omega$, we denote by P_x^ω the law on $(\mathbb{Z}_+^{\mathbb{N}}, \mathcal{B}(\mathbb{Z}_+^{\mathbb{N}}))$ of the Markov chain X starting from x . This is the *quenched* law of X . The *annealed* law of the couple (ω, X) is the probability measure $\mathbf{P} := \mathbf{P}_x$ on $(\Omega \times \mathbb{Z}_+^{\mathbb{N}}, \mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbb{Z}_+^{\mathbb{N}}))$ defined for all $x \in \mathbb{Z}_+$ and all $F \in \mathcal{B}(\Omega)$ and $G \in \mathcal{B}(\mathbb{Z}_+^{\mathbb{N}})$ by

$$\mathbf{P}_x(F \times G) = \int_F P_x^\omega(G) \mathbb{P}(d\omega).$$

We write E_x^ω and \mathbf{E} for the corresponding quenched and annealed expectations, respectively. For simplicity and following Golosov (1984) and Gantert et al. (2010), we consider the walk on the positive integers reflected at 0, but we need the environment to be defined on \mathbb{Z} to introduce later the infinite valley of the potential.

Denote, for $x \in \mathbb{Z}$,

$$\rho_x = \frac{1 - \omega(x)}{\omega(x)}. \tag{1}$$

It was shown in Solomon (1975) that when

$$\mathbb{E} \log \rho_0 = 0, \tag{2}$$

for \mathbb{P} -almost all ω the Markov chain X is recurrent, otherwise the walk is transient. This paper focuses on the recurrent case, hence (2) will be in force for all our results.

1.2. *Motivation: Method of the environment viewed from the particle.*

For $\omega \in \Omega$ and $x \in \mathbb{Z}$, denote by T_x the shift operator $T_x : \Omega \rightarrow \Omega$, which translates the environment by the vector x , i.e.

$$\forall y \in \mathbb{Z}, \quad (T_x \omega)(y) = \omega(x + y).$$

The *environment seen from the particle* is the Ω -valued process $(\bar{\omega}_k)$ given by

$$\bar{\omega}_k = T_{X_k} \omega, \quad k \in \mathbb{N}.$$

It is well known since Kozlov and Molchanov (1984) that $(\bar{\omega}_k, k \geq 0)$ is a Markov chain (with respect to both \mathbf{P} and P_0^ω), with the transition kernel

$$R(\omega, d\omega') = \omega(0)\delta_{T_1\omega}(d\omega') + (1 - \omega(0))\delta_{T_{-1}\omega}(d\omega'). \quad (3)$$

The state space of this Markov chain is very complex, however, in the transient ballistic case, which is characterized in (Solomon (1975)) by the linear speed of escape of the walk to infinity, namely

$$X_n/n \rightarrow \nu = (1 - \mathbb{E}\rho_0)/(1 + \mathbb{E}\rho_0), \quad (4)$$

Kozlov and Molchanov (1984) showed that there exists a unique invariant probability \mathbb{Q} for the kernel R , which is absolutely continuous with respect to \mathbb{P} , with an explicit density

$$p = d\mathbb{Q}/d\mathbb{P} = \nu(1 + \rho_0) \left(1 + \sum_{x=1}^{\infty} \prod_{j=1}^x \rho_j \right) \quad (5)$$

(see Molchanov (1994) p.273 or Theorem 1.2 in Sznitman (2002)). In particular, Birkhoff's a.s. ergodic theorem applies and gives for all \mathbb{Q} -integrable F

$$\frac{1}{n} \sum_{k=1}^n F(\bar{\omega}_k) \longrightarrow \int_{\Omega} F(\omega) p(\omega) \mathbb{P}(d\omega) \quad \mathbf{P} - a.s. \quad (6)$$

This constitutes the basis of the 'method of the environment viewed from the particle'. To recall it briefly, let us sketch the proof of Solomon's result (4) on the asymptotic velocity for the ballistic random walk. Let $\Delta X_n := X_{n+1} - X_n$, $\mathcal{F}_n = \sigma\{\Delta X_0, \dots, \Delta X_n, \omega(X_0), \dots, \omega(X_{n+1})\}$. We can write the classical martingale differences decomposition:

$$X_n/n = \frac{1}{n} \sum_{k=1}^n [\Delta X_k - \mathbf{E}(\Delta X_k | \mathcal{F}_{k-1})] + 1/n \sum_{k=1}^n \mathbf{E}(\Delta X_k | \mathcal{F}_{k-1}). \quad (7)$$

The first sum in (7) is composed of centered, uncorrelated terms, hence it tends to zero in \mathbb{L}^2 and, using the martingale's convergence, even a.s. Moreover, since

$$\mathbf{E}(\Delta X_k | \mathcal{F}_{k-1}) = \omega(X_k) - 1(1 - \omega(X_k)) = 2\bar{\omega}_k(0) - 1,$$

for the second term of (7) we can apply Birkhoff's theorem (6) and get

$$\frac{1}{n} \sum_{k=1}^n \mathbf{E}(\Delta X_k | \mathcal{F}_{k-1}) = 1/n \sum_{k=1}^n (2\bar{\omega}_k(0) - 1) \longrightarrow \int_{\Omega} (2\omega(0) - 1) p(\omega) \mathbb{P}(d\omega) = \nu \quad a.s.$$

For further illustration of this method see Sznitman (2002), Zeitouni (2004) and L.V.Bogachev (2006).

In this work we are interested in the limits of additive functionals of the environment's chain in the recurrent case and our main result, Theorem(1.1), shows that under (2)

$$\frac{1}{n} \sum_{k=1}^n F(\bar{\omega}_k) \xrightarrow{\mathcal{L}} \int_{\Omega} F d\mathcal{S}_{\infty}, \quad (8)$$

where \mathcal{S}_{∞} is a random probability measure on Ω , defined precisely in (20). This 'in law' ergodic theorem allows us to extend in a certain sense the environment's method to the recurrent case. In particular, let f be bounded and

$$S_n^f := \sum_{k=1}^n f(\omega(X_k), \Delta X_k).$$

As we have

$$\mathbf{E}[f(\omega(X_k), \Delta X_k) | \mathcal{F}_{k-1}] = f(\omega(X_k), 1)\omega(X_k) + f(\omega(X_k), -1)(1 - \omega(X_k))$$

we can represent S_n^f as in (7):

$$S_n^f = M_n + A_n,$$

where

$$M_n = \sum_{k=1}^n (f(\omega(X_k), \Delta X_k) - \mathbf{E}[f(\omega(X_k), \Delta X_k) | \mathcal{F}_{k-1}])$$

is a martingale and A_n is an additive functional of the environment seen from the particle, to which we can apply the convergence (8). In Section (5), using this method, we show a LLN and a CLT with the convergence to a mixture of normal laws for S_n^f . The sums of the form S_n^f arise in particular in the study of

the limit distribution of the Maximum Likelihood Estimator (MLE) of the law of the environment. The consistency and the asymptotic normality of MLE in the ballistic and sub-ballistic cases were studied in Comets et al. (2014) and Falconnet et al. (2013) and Falconnet et al. (2014). The consistency in the recurrent case, when the law of the environment has finite support, was shown in Comets et al. (2016). Finding the limit distribution of the MLE in this last case requires the knowledge of the limit behaviour of the score function, i.e. the derivative of the likelihood, and this last function depends on the functionals of the form

$$\sum_{k=1}^n 1(\omega(X_k) = a)1(\Delta X_k = 1), \quad (9)$$

where a is a given point of the support of ω_x . The functional (9) represents the total number of steps to the right from the sites where the environment is equal to a . In Section 5, relying on our result (8), we give the limit behavior and the fluctuations of such functionals. Note also that despite the fact that (8) gives only an ‘in law’ version of the ergodic theorem, in many examples the limit in (8) is deterministic, hence the convergence (8) actually holds in probability. We give such examples in Section (5).

To define precisely the limit random measure \mathcal{S}_∞ we need to introduce the notion of the potential and of the infinite valley.

1.3. Potential and infinite valley

Let $\rho_x, x \in \mathbb{Z}$ be given by (1) and define the potential $V = \{V(x) : x \in \mathbb{Z}\}$ by

$$V(x) = \begin{cases} \sum_{y=1}^x \log \rho_y & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\sum_{y=x+1}^0 \log \rho_y & \text{if } x < 0. \end{cases} \quad (10)$$

Then, V is a (double-sided) random walk, an example of a realisation of V can be seen on Figure 1. It is easy to see that the measure μ defined as

$$\mu(0) = 1, \quad \mu(x) = \exp[-V(x-1)] + \exp[-V(x)], \quad x \in \mathbb{Z}_+^*, \quad (11)$$

is a reversible and invariant measure for the Markov chain X under quenched law. Define the right border c_n of the ‘valley’ with depth $\log n + (\log n)^{1/2}$ as the random variable

$$c_n = \min \left\{ x \geq 0 : V(x) - \min_{0 \leq y \leq x} V(y) \geq \log n + (\log n)^{1/2} \right\}, \quad (12)$$

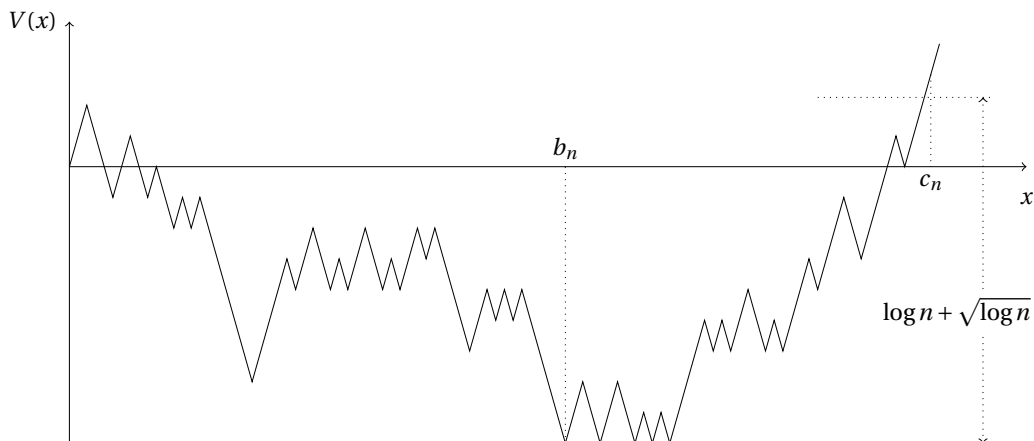


Figure 1: Example of potential derived from a Temkin random environment with parameter $a = 0.3$. Simulation with $n = 1000$.

and the bottom b_n of the valley as

$$b_n = \min \left\{ x \geq 0 : V(x) = \min_{0 \leq y \leq c_n} V(y) \right\}. \quad (13)$$

Figure 1 shows a representation of b_n and c_n for recurrent Temkin random environment with parameter $a = 0.3$. Namely, the environment's law here is $\mathbb{P}(\omega_x = a) = \mathbb{P}(\omega_x = 1 - a) = 1/2$. The salient probabilistic feature of the recurrent RWRE is the strong localization revealed by Sinai (1982). Considered on the spatial scale $\ln^2 n$ the RWRE becomes localized near b_n . We are interested in the shape of the valley $(0, b_n, c_n)$ when n tends to infinity and we recall the concept of infinite valley introduced by Golosov (1984).

Let $\tilde{V} = \{\tilde{V}(x) : x \in \mathbb{Z}\}$ be a collection of random variables distributed as V conditioned to stay positive for any negative x , and non-negative for any non negative x . Such events having probability zero, a formal definition uses Doob's h -transform (see Golosov (1984)[Lemma 4], Bertoin (1993)). It is known (Golosov (1984), pp. 494-495) that

$$\sum_{x \in \mathbb{Z}} \exp(-\tilde{V}(x)) < \infty. \quad (14)$$

$(\tilde{V}(x), x \in \mathbb{Z})$ is the so-called infinite valley, introduced in Golosov (1984).

Let $\tilde{\omega} = \{\tilde{\omega}(x), x \in \mathbb{Z}\}$ be the *environment of the walk in the infinite valley*.

$$\tilde{\omega}(x) = \frac{\exp[-\tilde{V}(x)]}{\exp[-\tilde{V}(x)] + \exp[-\tilde{V}(x-1)]}, \quad x \in \mathbb{Z}. \quad (15)$$

Let $\tilde{\nu}$ be a probability measure on \mathbb{Z} defined by

$$\tilde{\nu}(x) = \frac{\exp[-\tilde{V}(x-1)] + \exp[-\tilde{V}(x)]}{2 \sum_{z \in \mathbb{Z}} \exp[-\tilde{V}(z)]}, \quad x \in \mathbb{Z}. \quad (16)$$

Thanks to (14) the probability measure (16) is well defined, and is a stationary (and reversible) distribution of the random walk in in the ‘infinite valley’, i.e. the walk governed by the environment $\tilde{\omega}$.

1.4. Assumptions and main results

Assumption I. $\mathbb{E} \log \rho_0 = 0$ and $\mathbb{V}(\log \rho_0) > 0$.

Under Assumption (I) for \mathbb{P} -almost ω the Markov chain X is recurrent.

Assumption II. $\mathbb{P}(\delta_0 \leq \omega(0) \leq 1 - \delta_0) = 1$ for some $\delta_0 \in (0, 1)$.

The assumption (II) is technical and commonly admitted.

We are interested in the limit behavior of additive functionals of the environment’s chain $(\tilde{\omega}_k)$. The empirical law \mathcal{S}_n of this chain, defined as

$$\mathcal{S}_n = \frac{1}{n} \sum_{k=1}^n \delta_{\tilde{\omega}_k}, \quad (17)$$

allows to represent Birkhoff sums of $F : \Omega \rightarrow \mathbb{R}$ along the chain as an integral

$$\frac{1}{n} \sum_{k=1}^n F(\tilde{\omega}_k) = \int_{\Omega} F d\mathcal{S}_n.$$

Define for $n \in \mathbb{N}$ and $x \in \mathbb{Z}$, the local time of the walk at the position x as

$$\xi(n, x) = \sum_{k=1}^n 1\{X_k = x\}. \quad (18)$$

Note that the empirical law (17) of the environment seen from the walker can be expressed using the local times as

$$\mathcal{S}_n = \sum_{x \in \mathbb{Z}} \frac{\xi(n, x)}{n} \delta_{T_x \tilde{\omega}}. \quad (19)$$

Let \mathcal{S}_{∞} be a random measure on Ω given by

$$\mathcal{S}_{\infty} := \sum_{x \in \mathbb{Z}} \tilde{\nu}(x) \delta_{T_x \tilde{\omega}}, \quad (20)$$

where $\tilde{v}(x)$ and $\tilde{\omega}$ are given respectively by (16) and (15) and denote the stationary distribution of the walk and the environment in the infinite valley. Provide $\Omega := [0, 1]^{\mathbb{Z}}$ with the distance of the infinite product given by

$$d(\omega, \omega') = \sum_{x \in \mathbb{Z}} 2^{-|x|} |\omega(x) - \omega'(x)|. \quad (21)$$

Theorem 1.1. *Under Assumptions I and II the empirical law of the environment seen from the walker converges in distribution, as $n \rightarrow \infty$:*

$$\mathcal{S}_n \xrightarrow{\mathcal{L}} \mathcal{S}_\infty \quad (22)$$

in the space $\mathcal{P}(\Omega)$ equipped with the topology of the weak convergence of probability measures. As a consequence, for every bounded and continuous $F : \Omega \rightarrow \mathbb{R}$,

$$\frac{1}{n} \sum_{k=1}^n F(\tilde{\omega}_k) \xrightarrow{\mathcal{L}} \int_{\Omega} F d\mathcal{S}_\infty.$$

Note that in particular, for every $m \in \mathbb{N}$ and $g : [0, 1]^{2m+1} \rightarrow \mathbb{R}$, continuous and bounded,

$$\frac{1}{n} \sum_{k=1}^n g(\omega(X_k - m), \dots, \omega(X_k + m)) \xrightarrow{\mathcal{L}} \sum_{x \in \mathbb{Z}} g(\tilde{\omega}(x - m), \dots, \tilde{\omega}(x + m)) \tilde{v}(x).$$

The simplest example is given by $f : [0, 1] \rightarrow \mathbb{R}$, continuous and bounded:

$$\frac{1}{n} \sum_{k=1}^n f(\omega(X_k)) \xrightarrow{\mathcal{L}} \sum_{x \in \mathbb{Z}} f(\tilde{\omega}(x)) \tilde{v}(x).$$

Denote by \mathcal{E} the expectation with respect to the law of $\tilde{V} = (\tilde{V}(x))_{x \in \mathbb{Z}}$ and let us define $\mathbb{Q} \in \mathcal{P}(\Omega)$ by

$$\int_{\Omega} F d\mathbb{Q} = \mathcal{E} \left[\int_{\Omega} F d\mathcal{S}_\infty \right] = \sum_{x \in \mathbb{Z}} \mathcal{E} [\tilde{v}(x) F(T_x \tilde{\omega})], \quad (23)$$

for bounded $F : \Omega \rightarrow \mathbb{R}$. We can view \mathbb{Q} as the \mathcal{E} -expectation of \mathcal{S}_∞ .

Proposition 1.2. *The probability \mathbb{Q} is invariant and reversible for the Markov chain $(\tilde{\omega}_k, k \geq 0)$ in Ω . The measures \mathbb{P} and \mathbb{Q} are mutually singular.*

As we mentioned in Section (1.2), the invariant probability of the environment's chain in the ballistic case (5) is absolutely continuous with respect to the law of the environment \mathbb{P} , see (Molchanov, 1994, P. 273). To the best of our knowledge, the one we find here is the first one to be obtained as a limit in the case of zero velocity, and it is singular with respect to \mathbb{P} .

The proof of Theorem (1.1) is partially inspired by the paper of Gantert et al. (2010) concerning the convergence in distribution of the centered local times $(\frac{\xi(n, b_n+x)}{n}, x \in \mathbb{Z})$ to $(\tilde{v}(x), x \in \mathbb{Z})$. Based on the observation (see Sinaï (1982), Zeitouni (2004)), that at each single site around the bottom b_n the walk spends an asymptotically positive amount of time proportional to the weight of the site with respect to the invariant measure, these local times, similarly to Gantert et al. (2010), can be approximated in probability by the normalized invariant measure of the walk μ , given by (11). So finally the question of the convergence of the empirical measure \mathcal{S}_n (19) can be reduced to the question of the convergence in distribution to $(\exp - \tilde{V}(x); x \in \mathbb{Z})$ of the sequence of random vectors (Ξ_n) given by

$$\Xi_n = (\exp[-(V(b_n+x) - V(b_n))] 1_{\{-b_n, \dots, c_n - b_n - 1\}}(x); x \in \mathbb{Z}) \quad (24)$$

in the space $\ell^1 = \{(\ell(x), x \in \mathbb{Z}; \sum_{x \in \mathbb{Z}} |\ell(x)| < \infty)\}$. Note that the fidi convergence here is known since Golosov (1984), Bertoin (1993), but the fact that the sequence (Ξ_n) is tight in ℓ^1 , proved in Theorem 1.4, is new. Showing this tightness boils down to showing that the potential's increments grow fast enough when $x \rightarrow \pm\infty$. This is done in Proposition 1.3 below. This result continues a series of papers about the random walk conditioned to stay positive, as the increments on the right of b_n have the same law that V , conditioned to reach $\log n + \sqrt{\log n}$ before 0_- . These subjects were actively studied in the nineties, see Bertoin (1993), Bertoin and Donney (1994), Tanaka (1989), Ritter (1981). Most of these papers deal with the convergence of finite dimensional distributions, or study the law of the limit walk, but as the size of the main valley tends to the infinity, the fidi convergence is not sufficient here. Proposition 1.3 is a key new technical result of the paper. Note that the tightness of (Ξ_n) is also needed in Gantert et al. (2010), but is missing there. Hence together with Proposition 1.3 Theorem 1.4 completes the proof of Theorem 1.2 of Gantert et al. (2010).

Proposition 1.3. *Suppose that Assumptions (I) and (II) are satisfied. Then the following holds:*

i) For all $\eta \in]0, 1/2[$ and $\delta > 0$,

$$\lim_{K \rightarrow +\infty} \liminf_{n \rightarrow \infty} \mathbb{P}(V(b_n+x) - V(b_n) \geq \delta x^\eta, \forall x \in \llbracket K, c_n - b_n \rrbracket) = 1.$$

ii) Moreover,

$$\lim_{K \rightarrow +\infty} \liminf_{n \rightarrow \infty} \mathbb{P}(V(b_n - x) - V(b_n) \geq 2 \log x, \forall x \in \llbracket K, b_n \rrbracket) = 1.$$

This proposition is proven in Section (3). Denote by P_{Ξ_n} the law of Ξ_n on $(\ell^1, \|\cdot\|_1)$.

Theorem 1.4. *Suppose that Assumptions (I) and (II) are satisfied. Then the sequence P_{Ξ_n} is relatively compact in $(\ell^1, \|\cdot\|_1)$. As a consequence, the sequence P_{Ξ_n} converges weakly to the law of $\{\exp[-\tilde{V}(x)]; x \in \mathbb{Z}\}$ in $(\ell^1, \mathcal{B}(\ell^1))$.*

1.5. Structure of the paper

Section 2 is devoted to the proof of Theorem 1.1. Section 3 deals with the properties of the random walk conditioned to stay positive. Here we prove the main new technical ingredient, Proposition 1.3, and the tightness result, Theorem 1.4. Section 4 deals with the properties of the invariant measure of the environment's chain, Proposition 1.2. In Section 5 we show how the environment's method can be deduced from Theorem 1.1. There we prove the LLN, Proposition (5.1), and the convergence to the mixture of Gaussian laws, Proposition 5.2, for normalized sums $\sum_{k=1}^n f(\omega(X_k), \Delta X_k)$. Finally Supplementary material, Section 6 contains the proof of an important auxiliary result, Proposition 2.1, which gives an approximation in probability of an additive functional of the environment's chain by a functional depending only on the environment.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. The proof is based on the approximation in probability of an additive functional of the environment's chain by a functional depending on the environment only, which is the content of Subsection 6, Proposition 2.1. Then in Subsection 2.2 we give the proof of Theorem 1.1, explaining its main steps.

2.1. Approximation in probability of $\int_{\Omega} F d\mathcal{S}_n$.

Let $\tilde{X}^n = (\tilde{X}_t^n)_{t \in \mathbb{N}}$ be the Markov chain with value in $\{0, \dots, c_n\}$, reflected in 0 and c_n and with transition given by

$$\tilde{p}^{\omega, n}(x, x+1) = \omega_x, \quad \tilde{p}^{\omega, n}(x, x-1) = 1 - \omega_x; \quad \forall x \in \{1, \dots, c_n - 1\}.$$

For $x \in \{0, \dots, c_n\}$ we denote by $\tilde{P}_x^{\omega, n}$ the law of this chain starting from x . The idea of the proof of Theorem 1.1 consists in considering the original walk X only after it reaches b_n and before it exits $[0, c_n]$, i.e. to replace X with \tilde{X}^n , since on the time interval $[0, n]$ it turns out to be a good approximation in probability. The advantage to work with \tilde{X}^n is in the fact that it is irreducible, positively recurrent with the invariant probability μ_n defined by (29). As a consequence the decomposition on the i.i.d. cycles between successive visits of a given state can be used to obtain the deviation of an additive functional, as well as some well known formulas linking the invariant probability and the expectation of hitting times. The idea to replace X by \tilde{X}^n is borrowed from Gantert et al. (2010), but the use of this idea, namely through the life-cycles decomposition, is different.

Proposition 2.1. *Let \mathcal{S}_n be given by (19) and Σ_n by (30). For any continuous $F : [0, 1]^{2m+1} \rightarrow \mathbb{R}$ and all $\varepsilon > 0$ we have*

$$\mathbf{P}(|\mathcal{S}_n(F) - \Sigma_n(F)| > \varepsilon) \rightarrow 0.$$

The proof of Proposition 2.1 is given in Supplementary material, Section 6.

2.2. Proof of Theorem 1.1

Proof. Recall that \mathcal{E} denotes the expectation with respect to the law of \tilde{V} . By definition, claim (22) is equivalent to

$$\lim_{n \rightarrow \infty} \mathbf{E}G(\mathcal{S}_n) = \mathcal{E}G(\mathcal{S}_\infty) \quad (25)$$

for all bounded continuous $G : \mathcal{P}(\Omega) \rightarrow \mathbb{R}$. We first observe that it is sufficient to prove (25) for all G of the form

$$G(\mathcal{S}) = \sum_{l=1}^n \left(\int_{\Omega} F_1 d\mathcal{S} \times \dots \times \int_{\Omega} F_l d\mathcal{S} \right), \quad \mathcal{S} \in \mathcal{P}(\Omega) \quad (26)$$

with arbitrary integers n, m, l and continuous $F_k : [0, 1]^{2m+1} \rightarrow \mathbb{R}$ ($1 \leq k \leq l$). Indeed, with d defined by (21) (Ω, d) is a compact separable metric space and hence $(\mathcal{P}(\Omega), \rho)$, endowed with the Prohorov metric ρ , is a compact separable metric space too. The set \mathcal{G} of functions G of the form (26) is an algebra of continuous functions on the compact metric space $\mathcal{P}(\Omega)$ which contains constant functions and separates the points. By Stone-Weierstrass Theorem, this set is dense in the space $\mathcal{C}(\mathcal{P}(\Omega); \mathbb{R})$ for the supremum norm, and then it suffices to

prove (25) for such G . This, in turn, is equivalent to prove the following convergence in distribution:

$$\left(\int_{\Omega} F_1 d\mathcal{S}_n, \dots, \int_{\Omega} F_l d\mathcal{S}_n \right) \xrightarrow{\mathcal{L}} \left(\int_{\Omega} F_1 d\mathcal{S}_{\infty}, \dots, \int_{\Omega} F_l d\mathcal{S}_{\infty} \right). \quad (27)$$

Using the Cramer-Wold device (27) is equivalent to

$$\forall (t_1, \dots, t_l) \in \mathbb{R}^l, \quad \sum_{i=1}^l t_i \int_{\Omega} F_i d\mathcal{S}_n \xrightarrow{\mathcal{L}} \sum_{i=1}^l t_i \int_{\Omega} F_i d\mathcal{S}_{\infty},$$

and finally, as $\sum_{i=1}^l t_i F_i$ is a continuous function on Ω , depending only on a finite number of coordinates, we only need to prove that

$$\forall m \in \mathbb{N}^*, \forall F \in \mathcal{C}_b([0, 1]^{2m+1}), \quad \int_{\Omega} F d\mathcal{S}_n \xrightarrow{\text{law}} \int_{\Omega} F d\mathcal{S}_{\infty}. \quad (28)$$

Below we give the proof of (28), which is separated in 3 main steps.

Step 1. *Approximation in probability of $\int_{\Omega} F d\mathcal{S}_n$.*

For F as in (28), using (18) and (19) we can write $\mathcal{S}_n(F)$ in the ‘spatial’ form

$$\mathcal{S}_n(F) = \sum_{x \in \mathbb{Z}} F(T_x \omega) \frac{\xi(n, x)}{n} = \sum_{x \in \mathbb{Z}} F(\omega(-m+x), \dots, \omega(m+x)) \frac{\xi(n, x)}{n}.$$

Fix $n \in \mathbb{N}^*$ and denote $\mu_n = \mu_n^{\omega}$ the random probability measure on \mathbb{Z}_+ , s.t.

$$\mu_n(x) := \begin{cases} \frac{1}{Z_n} (e^{-V(x)} + e^{-V(x-1)}) & \text{if } 0 < x < c_n, \\ \frac{1}{Z_n} & \text{if } x = 0, \\ \frac{1}{Z_n} e^{-V(c_n-1)} & \text{if } x = c_n, \\ 0 & \text{if } x \notin \{0, \dots, c_n\}, \end{cases} \quad (29)$$

where $Z_n = 2 \sum_{x=0}^{c_n-1} e^{-V(x)}$, and c_n and V are respectively defined by (12) and (10). The local times $\frac{\xi(n, x)}{n}$, $x \in \mathbb{Z}_+$ can be approached in probability by the quantities $\mu_n(x)$, $x \in \mathbb{Z}_+$. This argument was found by Gantert et al. (2010). Here we show that more generally, the additive functional $\mathcal{S}_n(F)$ can be approached in probability by $\Sigma_n(F) = \int_{\Omega} F d\Sigma_n$ with

$$\Sigma_n = \sum_{x \in \mathbb{Z}} \mu_n(x) \delta_{T_x \omega}. \quad (30)$$

Proposition 2.1 states that $\forall \varepsilon > 0$,

$$\mathbf{P}(|\mathcal{S}_n(F) - \Sigma_n(F)| > \varepsilon) \rightarrow 0. \quad (31)$$

Note that \mathcal{S}_n depends on the walk and on the environment, whereas Σ_n depends only on the environment. The next two steps allow to show the convergence in law: $\Sigma_n(F) \xrightarrow{\mathcal{L}} \mathcal{S}_\infty(F)$.

Step 2. *Expressing $\Sigma_n(F)$ as a continuous function of a weakly convergent sequence.*

Note that

$$\Sigma_n(F) = \sum_{x \in \mathbb{Z}} F(T_x \omega) \mu_n(x) = \sum_{x \in \mathbb{Z}} F(T_{b_n+x} \omega) \mu_n(b_n+x).$$

Remember that we denoted Ξ_n the random element in ℓ^1 given by:

$$\Xi_n := \{\exp[-(V(b_n+x) - V(b_n))] 1_{\{-b_n, \dots, c_n - b_n - 1\}}(x); x \in \mathbb{Z}\}.$$

Both $\mu_n(b_n+x)$ and $\omega(b_n+x) : x = -b_n, \dots, c_n - b_n - 1$ can be expressed in terms of Ξ_n :

$$\mu_n(b_n+x) = \frac{\Xi_n(x) + \Xi_n(x-1)}{2 \sum_{y \in \mathbb{Z}} \Xi_n(y)},$$

and

$$\omega(b_n+x) = \frac{\Xi_n(x)}{\Xi_n(x) + \Xi_n(x-1)}.$$

Thus, $\Sigma_n(F) = H_F(\Xi_n)$ where $H_F : \ell^1 \rightarrow \mathbb{R}$ is continuous. In Theorem 1.4 we showed that the distribution of Ξ_n converges weakly to that of $\{\exp[-\tilde{V}(x)]; x \in \mathbb{Z}\}$ in this space. Together with the continuity on ℓ^1 of $\Sigma_n(F) = H_F(\Xi_n)$ that gives the convergence in law

$$\Sigma_n(F) \xrightarrow{\mathcal{L}} \mathcal{S}_\infty(F). \quad (32)$$

Step 3. Conclusion: Using (31) and (32) we conclude that $\mathcal{S}_n(F) \xrightarrow{\mathcal{L}} \mathcal{S}_\infty(F)$. This ends the proof of Theorem 1.1. \square

3. Random walk conditioned to stay positive and weak convergence to the infinite valley

In this section we prove Proposition 1.3 and Theorem 1.4. Note that the idea of the proof of *ii*) is the decomposition of the dynamic on the i.i.d. cycles corresponding to the strict descending ladders, whereas the proof of *i*) is partially inspired by Ritter (1981).

3.1. Proof of Proposition 1.3

Proof. We start by proving *ii*). Let $T_0 := 0$ and for all $\ell \in \mathbb{N}^*$, put

$$T_{\ell+1} := \inf\{y > T_\ell, V(y) < V(T_\ell)\}.$$

The sequence $(T_\ell)_{\ell \in \mathbb{N}}$ is the sequence of the strict descending ladder epochs of V . Let

$$e_\ell = ((V(z) - V(T_{\ell-1})), T_{\ell-1} \leq z < T_\ell), \quad \ell \in \mathbb{N}^*.$$

Using the strong Markov property of V , the sequence of excursions (e_ℓ) , $\ell \in \mathbb{N}^*$ is an i.i.d. sequence. Let $N(n)$ be a random time, such that b_n defined by (13) satisfies $b_n = T_{N(n)}$. Namely, setting as previously $L_n := \log n + \sqrt{\log n}$, we have following Golosov (1984) p.492,

$$N(n) := \inf\{\ell \in \mathbb{N}^*; \max\{V(z) - V(T_{\ell-1}); T_{\ell-1} \leq z < T_\ell\} \geq L_n\}.$$

Due to the independence and the equidistribution of the excursions (e_ℓ) , $\ell \in \mathbb{N}^*$, the random variable $N(n)$ is geometrically distributed with the parameter

$$p_n := \mathbb{P}(\tau_{L_n} < \tau_{0-}), \quad \text{where} \tag{33}$$

$$\tau_{L_n} := \inf\{z > 0, V(z) \geq L_n\} \quad \text{and} \quad \tau_{0-} := \inf\{z > 0; V(z) < 0\}.$$

Denote also

$$\tau_{0-}^\ell := \tau_{0-} \circ \theta_{T_\ell} \quad \text{and} \quad \tau_{L_n}^\ell := \tau_{L_n} \circ \theta_{T_\ell}.$$

With this notation $\tau_{0-}^0 := \tau_{0-}$ and $\tau_{L_n}^0 := \tau_{L_n}$. We also denote $\sigma_\ell := T_\ell - T_{\ell-1}$ and $H_\ell := V(T_\ell) - V(T_{\ell-1})$ respectively the length and the high of the ℓ -th ladder. Let $K \in \mathbb{N}^*$ and

$$\begin{aligned} C(K, n) &:= \{\forall x = K, \dots, b_n, V(b_n - x) - V(b_n) \geq 2 \log x\} \\ &= \{\forall y = 0, \dots, b_n - K, V(y) - V(b_n) \geq 2 \log |y - b_n|\}. \end{aligned}$$

Keeping in mind the relation $b_n = T_{N(n)}$, we can observe that

$$C(K, n) = \{\forall \ell = 0, \dots, N(n) - 1; |T_\ell - T_{N(n)}| \geq K; V(T_\ell) - V(T_{N(n)}) \geq 2 \log |T_\ell - T_{N(n)}|\} \quad (34)$$

Indeed, let $\ell(y) \in \mathbb{N}$ be the number of ladder excursions containing y , i.e. $T_{\ell(y)} \leq y < T_{\ell(y)+1}$, then using the fact that the function $x \rightarrow 2 \log x$ is increasing on \mathbb{R}_+ , $V(y) \geq V(T_{\ell(y)})$ and $V(T_\ell) - V(T_{N(n)}) \geq 2 \log |T_\ell - T_{N(n)}|$ for all $\ell \in \mathbb{N}$, we have

$$\begin{aligned} V(y) - V(T_{N(n)}) &= V(y) - V(T_{\ell(y)}) + V(T_{\ell(y)}) - V(T_{N(n)}) \geq \\ &2 \log |T_{\ell(y)} - T_{N(n)}| \geq 2 \log |y - T_{N(n)}|. \end{aligned}$$

Which proves (34).

Now we can write:

$$\begin{aligned} \mathbb{P}(C^c(K, n)) &= \\ &\sum_{N=1}^{\infty} \mathbb{P}(\exists \ell = 0, \dots, N-1, T_N - T_\ell \geq K, V(T_\ell) - V(T_N) < 2 \log |T_\ell - T_N|; N(n) = N) \leq \\ &\sum_{N=1}^M \mathbb{P}(N(n) = N) + \sum_{N=M+1}^{\infty} \mathbb{P}(\exists m = 1, \dots, M, T_N - T_{N-m} \geq K; N(n) = N) + \\ &\sum_{N=M+1}^{\infty} \mathbb{P}(\exists m = M+1, \dots, N, T_N - T_{N-m} \geq \exp \frac{1}{2} (V(T_N) - V(T_{N-m}))); N(n) = N := \\ &S_1(n, M) + S_2(K, n, M) + S_3(n, M). \end{aligned}$$

Here in the third line we denoted $m = N - \ell$ the number of 'ladder' between T_N and T_ℓ , and the auxiliary $M \in \mathbb{N}^*$ will be chosen later. We obviously have

$$S_1(n, M) = \sum_{N=1}^M (1 - p_n)^{N-1} p_n \sim M p_n \rightarrow 0 \quad \text{if } n \rightarrow \infty. \quad (35)$$

For the second sum we can write

$$\begin{aligned} S_2(K, n, M) &:= \sum_{N=M+1}^{\infty} \mathbb{P}(\exists m = 1, \dots, M, T_N - T_{N-m} \geq K; N(n) = N) \leq \\ &\sum_{N=M+1}^{\infty} \mathbb{P}(T_N - T_{N-M} \geq K; N(n) = N) = \sum_{N=M+1}^{\infty} \mathbb{P}(\sigma_{N-M+1} + \dots + \sigma_N \geq K; N(n) = N) \leq \\ &\sum_{N=M+1}^{\infty} \sum_{\ell=N-M+1}^N \mathbb{P}(\sigma_\ell \geq K/M; N(n) = N). \end{aligned}$$

The event $\{N(n) = N\}$ can be written as

$$\{N(n) = N\} = \{\tau_{0-}^1 < \tau_{L_n}^1; \dots; \tau_{0-}^{N-1} < \tau_{L_n}^{N-1}; \tau_{0-}^N > \tau_{L_n}^N\}.$$

Let $c_{Sp} > 0$ be such that $\forall a > c_{Sp}$, $\mathbb{P}(\tau_{0-} > a) \leq Ca^{-1/2}$, where C is a positive constant. Following Spitzer (1960) we can choose such a constant c_{Sp} . Since $K \rightarrow \infty$, for all $M > 0$, $K/M > c_{Sp}$ will be satisfied. Then, using the independence of the ladder excursions, together with the definition (33), we can write

$$\mathbb{P}(\sigma_\ell \geq \frac{K}{M}; N(n) = N) \leq \mathbb{P}(\sigma_\ell \geq \frac{K}{M} \cap \tau_{0-}^\ell < \tau_{L_n}^\ell) p_n (1-p_n)^{N-2} \leq C \sqrt{\frac{M}{K}} p_n (1-p_n)^{N-2}.$$

Using this bound we see that

$$S_2(K, n, M) \leq \sum_{N=M+1}^{\infty} \frac{C(M)^{3/2} p_n (1-p_n)^{N-2}}{\sqrt{K}} = \frac{C(M)^{3/2} (1-p_n)^{M-1}}{\sqrt{K}} \leq \frac{C(M)^{3/2}}{\sqrt{K}}. \quad (36)$$

Finally

$$\begin{aligned} S_3(n, M) &= \quad (37) \\ &\sum_{N=M+1}^{\infty} \mathbb{P}(\exists m = M+1, \dots, N, T_N - T_{N-m} \geq \exp \frac{1}{2} (V(T_N) - V(T_{N-m}))); N(n) = N) \leq \\ &\sum_{N=M+1}^{\infty} \sum_{m=M+1}^N \mathbb{P}(\sigma_{N-m+1} + \dots + \sigma_N > \exp \frac{1}{2} (H_{N-m+1} + \dots + H_N); N(n) = N) \leq \\ &\leq \sum_{N=M+1}^{\infty} \sum_{m=M+1}^N \sum_{\ell=N-m+1}^N (1-p_n)^{N-m} p_n \mathbb{P}(m\sigma_\ell > \exp \frac{1}{2} \sum_{k=N-m+1; k \neq \ell}^N H_\ell) \quad (38) \end{aligned}$$

Where in the last line we used the independence of the excursions and the fact that $H_k > 0$. The term $(1-p_n)^{N-m} p_n$ comes from $N(n) = N$, where the $N-m$ first excursion are independent of the m last, and hence of the event $\sigma_{N-m+1} + \dots + \sigma_N > \exp \frac{1}{2} (H_{N-m+1} + \dots + H_N)$, whereas p_n appears because the excursion corresponding to the ‘success’ is after $T_{N(n)}$. Note that we used $H_\ell > 0$ to get rid of H_ℓ in the exponential. In this way σ_ℓ is independent of the sum

$\sum_{k=N-m+1; k \neq \ell}^N H_k$. Now using Markov inequality,

$$\begin{aligned}
S_3(n, M) &\leq \tag{39} \\
&\sum_{N=M+1}^{\infty} \sum_{m=M+1}^N \sum_{\ell=N-m+1}^N (1-p_n)^{N-m} p_n \mathbb{P}(m\sigma_\ell > \exp \frac{1}{2} \sum_{k=N-m+1; k \neq \ell}^N H_k) \leq \\
&\sum_{N=M+1}^{\infty} \sum_{m=M+1}^N (1-p_n)^{N-m} p_n m^2 \mathbb{E}\sigma_1 (\mathbb{E}e^{-1/2H_1})^{m-1} \leq \\
&\sum_{N=M+1}^{\infty} (1-p_n)^{N-M} p_n \sum_{m=M+1}^N m^2 e^{-mc} \leq \sum_{N=M+1}^{\infty} (1-p_n)^{N-M} p_n e^{-Mc'} \leq e^{-Mc'}
\end{aligned}$$

And finally putting together (35), (36) and (39)

$$\begin{aligned}
\mathbb{P}(C^c(K, n)) &= \\
S_1(n, M) + S_2(K, n, M) + S_3(n, M) &\leq Mp_n + \frac{M^{3/2}}{\sqrt{K}} + e^{-Mc'}.
\end{aligned}$$

We first choose M large enough, such that $e^{-Mc'} \leq \varepsilon$. Then we get

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(C^c(K, n)) \leq \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this concludes the proof of *ii*).

Now we prove *i*).

Recall that $V(x) = \sum_{y=1}^x \log \rho_y$ if $x > 0$, and $(\log \rho_y)_{y \in \mathbb{Z}}$ are i.i.d. and centered. Denote

$$\tau_{\ln} = \inf \{x \in \mathbb{Z}_+, V(x) \geq \log n + \sqrt{\log n}\}, \quad \tau_{0-} = \inf \{x \in \mathbb{Z}_+, V(x) < 0\}.$$

Using the strong Markov property,

$$\mathcal{L}(\{V(b_n+x) - V(b_n), x = K - b_n, \dots, c_n - b_n\}) = \mathcal{L}(\{V(x), x = K, \dots, \tau_{\ln} | \tau_{\ln} < \tau_{0-}\}).$$

Hence we have to prove

$$\lim_{K \rightarrow +\infty} \inf_n \mathbb{P}(V(x) \geq \delta x^\eta, \forall x = K, \dots, \tau_{\ln} \wedge \tau_{0-} | \tau_{\ln} < \tau_{0-}) = 1.$$

or equivalently

$$\lim_{K \rightarrow +\infty} \sup_n \frac{\mathbb{P}(\exists x \geq K, x < \tau_{\ln} \wedge \tau_{0-}, V(x) < \delta x^\eta, \tau_{\ln} < \tau_{0-})}{\mathbb{P}(\tau_{\ln} < \tau_{0-})} = 0. \tag{40}$$

Let $c > 1$ be an integer such that $\forall a > c, \mathbb{P}(\tau_{0-} > a) \leq Ca^{-1/2}$, where C is a positive constant. Following Spitzer (1960) we can choose such a constant c . For $r \in \mathbb{N}^*$, denote \mathcal{C}_r the following event

$$\mathcal{C}_r := \{\exists x \in [c^{r-1}, c^r[; x < \tau_{\ln} \wedge \tau_{0-}; V(x) < \delta x^\eta; \tau_{\ln} < \tau_{0-}\}$$

and denote for $x \in [c^{r-1}, c^r[$

$$\mathcal{A}_x := \{\forall z \in [c^{r-1}, x[; V_z \geq \delta z^\eta; V(x) < \delta x^\eta; x < \tau_n \wedge \tau_{0-}\}$$

Note that \mathcal{A}_x are disjoint for $x \in [c^{r-1}, c^r[$ and that

$$\mathcal{C}_r := \bigcup_{x=c^{r-1}}^{c^r-1} \{\mathcal{A}_x \cap \{\tau_{\ln} < \tau_{0-}\}\}.$$

For all $x \in \mathbb{N}$, denote $\mathbb{P} := \mathbb{P}_0$, $\mathcal{F}_x = \sigma\{V_0, \dots, V_x\}$. Denote $L_n = \log n + \sqrt{\log n}$. It is easy to see that

$$\forall 0 \leq y < L_n, \quad \mathbb{P}_y(\tau_{\ln} < \tau_{0-}) \leq \frac{y + c_0}{L_n}$$

for some positive constant $c_0 = \sup_{n,y \geq 0} \mathbb{E}_y(-V(\tau_{0-}) \mid \tau_{0-} < \tau_{\ln})$. Indeed, using Doob stopping theorem, and the fact that $V_{\tau_L} > L$,

$$\mathbb{P}_y(\tau_{\ln} < \tau_{0-}) = \frac{y - \mathbb{E}_y[V_{\tau_{0-}} \mid \tau_{0-} < \tau_{\ln}]}{\mathbb{E}_y[V_{\tau_L} \mid \tau_L < \tau_{0-}] - \mathbb{E}_y[V_{\tau_{0-}} \mid \tau_{0-} < \tau_{\ln}]}.$$

Since the event $\{\tau_{\ln} < \tau_{0-}\}$ is invariant under shift, using Markov property we can write:

$$\begin{aligned} \mathbb{P}_0(\mathcal{A}_x \cap \{\tau_{\ln} < \tau_{0-}\}) &= \mathbb{E}_0[\mathbb{P}_0(\mathcal{A}_x \cap \{\tau_{\ln} < \tau_{0-}\} \mid \mathcal{F}_x)] = \\ &= \mathbb{E}_0(\forall z \in [c^{r-1}, x[; V_z \geq \delta z^\eta; V(x) < \delta x^\eta; x < \tau_{\ln} \wedge \tau_{0-}; \mathbb{P}_{V_x}(\tau_{\ln} < \tau_{0-})) \leq \\ &= \mathbb{P}_0(\forall z \in [c^{r-1}, x[; V_z \geq \delta z^\eta; V(x) < \delta x^\eta; x < \tau_{\ln} \wedge \tau_{0-})(\delta x^\eta + c_0)(L_n)^{-1} = \\ &= \mathbb{P}(\mathcal{A}_x)(\delta x^\eta + c_0)(L_n)^{-1}. \end{aligned}$$

Hence, using the fact that \mathcal{A}_x are disjoint and that

$$\forall x = c^{r-1}, \dots, c^r - 1; \quad \mathcal{A}_x \subset \{\tau_{0-} > x\} \subset \{\tau_{0-} > c^{r-1}\},$$

with our choice of c , we have

$$\begin{aligned} \mathbb{P}(\mathcal{C}_r) &= \sum_{x=c^{r-1}}^{c^r-1} \mathbb{P}(\mathcal{A}_x \cap \{\tau_{\ln} < \tau_{0-}\}) \leq \frac{\delta c^{r\eta} + c_0}{L_n} \sum_{x=c^{r-1}}^{c^r-1} \mathbb{P}(\mathcal{A}_x) = \\ &= \frac{\delta c^{r\eta} + c_0}{L_n} \mathbb{P}\left(\bigcup_{x=c^{r-1}}^{c^r-1} \mathcal{A}_x\right) \leq \frac{\delta c^{r\eta} + c_0}{L_n} \mathbb{P}(\tau_{0-} > c^{r-1}) \leq \\ &= C\sqrt{c}(\delta c^{r(\eta-1/2)} + c_0 c^{-r/2})(L_n)^{-1}. \end{aligned}$$

Finally, for any n and $R \geq 2$,

$$\begin{aligned} \mathbb{P}(\exists x > c^{R-1}; x < \tau_{\ln} \wedge \tau_{0-}; V(x) < \delta x^\eta; \tau_{\ln} < \tau_{0-}) \leq \\ \sum_{r=R}^{\infty} \mathbb{P}(\mathcal{C}_r) \leq C_1(\delta + c_0 c^{-R\eta})c^{R(\eta-1/2)}(L_n)^{-1}, \end{aligned}$$

and (40) follows from $\mathbb{P}_0(\tau_{\ln} < \tau_{0-}) \sim (L_n)^{-1}$. □

3.2. Proof of Theorem 1.4

Using Proposition 1.3 we prove the tightness result for the potential increments (Ξ_n) given by (24).

Proof. Recall that $(\ell^1, \|\cdot\|_1)$ is a complete separable metric space and hence, using Prohorov's theorem (Billingsley (1999)), the sequence of distributions P_{Ξ_n} is relatively compact if and only if it is tight. Recall also the characterization of the compacts in $(\ell^1, \|\cdot\|_1)$:

$$\mathcal{K} \subset \ell^1 \text{ is compact} \iff \sup_{l \in \mathcal{K}} \|l\|_1 < \infty \text{ and } \limsup_{N \rightarrow \infty} \sup_{l \in \mathcal{K}} \sum_{|x| \geq N} |l_x| = 0.$$

Let $\eta \in (0, 1/2)$, $K > 0$. Denote

$$\mathcal{K}(\eta, K) := \{l \in \ell^1; |l_x| \leq 1; l_x \leq \frac{1}{x^2}, \forall x \in \mathbb{Z}_- \cap \{|x| \geq K\}; l_x \leq e^{-|x|^\eta}, \forall x \in \mathbb{Z}_+ \cap \{|x| \geq K\}\}.$$

$\mathcal{K}(\eta, K)$ is a compact in ℓ^1 . As a consequence of Proposition 1.3, for n large enough $\mathbb{P}(\Xi_n \in \mathcal{K}(\eta, K)) \geq 1 - \varepsilon$ and the sequence P_{Ξ_n} is tight.

From Golosov (1984) the following convergence of finite dimensional distributions (fidi) holds:

$$\Xi_n = \{\exp[-(V(b_n + x) - V(b_n))]; x \in \mathbb{Z}\} \xrightarrow{\text{fidi}} \{\exp[-\tilde{V}(x)]; x \in \mathbb{Z}\}.$$

Using $b_n \rightarrow \infty$ a.s. and $c_n - b_n \rightarrow \infty$ a.s., the finite dimensional distributions of Ξ_n converge weakly to those of $\{\exp[-\tilde{V}(x)]; x \in \mathbb{Z}\}$. Denote by \mathcal{M} the class of continuous, bounded, finite-dimensional functions:

$$\mathcal{M} := \bigcup_{k \in \mathbb{N}^*} \{f \in \mathcal{C}_b(\ell^1) : f(l_i) = f(l'_i) \forall i \in \llbracket -k, k \rrbracket \implies f(l) = f(l')\}.$$

It is clear that \mathcal{M} separates points: if $l \in \ell^1$, $l' \in \ell^1$, $l \neq l'$, that there exists $f \in \mathcal{M}$, such that $f(l) \neq f(l')$. Since (ℓ^1, d) is separable and complete, and \mathcal{M} separates points, using Theorem 4.5 from Stewart N. Ethier (2005) \mathcal{M} is separating. Now the claim follows directly from proposition (1.4) and lemma 4.3 of Stewart N. Ethier (2005). □

4. Proof of Proposition 1.2

Proof. To show reversibility, we need to prove that for the chain starting from $\tilde{\omega}_0 \sim \mathbb{Q}$, defined by (23),

$$\mathbf{E}F(\tilde{\omega}_0)G(\tilde{\omega}_1) = \mathbf{E}F(\tilde{\omega}_1)G(\tilde{\omega}_0) \quad (41)$$

for F, G continuous and bounded on Ω . Let $\tilde{\mathbb{P}}$ denote the law of $\tilde{\omega}$. By definition (3) of the transition R , the LHS in (41) is equal to

$$\begin{aligned} & \int_{\Omega} d\tilde{\mathbb{P}}(\tilde{\omega}) \sum_{x \in \mathbb{Z}} \tilde{v}(x) F(T_x \tilde{\omega}) [\tilde{\omega}(x) G(T_{x+1} \tilde{\omega}) + (1 - \tilde{\omega}(x)) G(T_{x-1} \tilde{\omega})] \\ &= \int_{\Omega} d\tilde{\mathbb{P}}(\tilde{\omega}) \left[\sum_{x \in \mathbb{Z}} \tilde{v}(x) F(T_x \tilde{\omega}) \tilde{\omega}(x) G(T_{x+1} \tilde{\omega}) + \sum_{x \in \mathbb{Z}} \tilde{v}(x) F(T_x \tilde{\omega}) (1 - \tilde{\omega}(x)) G(T_{x-1} \tilde{\omega}) \right] \\ &= \int_{\Omega} d\tilde{\mathbb{P}}(\tilde{\omega}) \left[\sum_{x \in \mathbb{Z}} \tilde{v}(x+1) F(T_x \tilde{\omega}) (1 - \tilde{\omega}(x+1)) G(T_{x+1} \tilde{\omega}) + \sum_{x \in \mathbb{Z}} \tilde{v}(x-1) F(T_x \tilde{\omega}) \tilde{\omega}(x-1) G(T_{x-1} \tilde{\omega}) \right] \\ &= \int_{\Omega} d\tilde{\mathbb{P}}(\tilde{\omega}) \left[\sum_{y \in \mathbb{Z}} \tilde{v}(y) F(T_{y-1} \tilde{\omega}) (1 - \tilde{\omega}(y)) G(T_y \tilde{\omega}) + \sum_{z \in \mathbb{Z}} \tilde{v}(z) F(T_{z+1} \tilde{\omega}) \tilde{\omega}(z) G(T_z \tilde{\omega}) \right], \end{aligned}$$

where in the third line we used the relation

$$\tilde{v}(x-1) \tilde{\omega}(x-1) = \tilde{v}(x) (1 - \tilde{\omega}(x)) \quad (42)$$

and we changed the variables $y = x+1$, $z = x-1$ in the last one. The last expression being the RHS of (41), we obtain reversibility of \mathbb{Q} , which implies invariance by taking $G = 1$. Finally consider the event

$$\Omega_+ = \left\{ \omega \in \Omega : \sum_{y=1}^x \log \frac{1 - \omega(y)}{\omega(y)} \geq 0 \quad \forall x \geq 1 \right\}.$$

By construction of \tilde{V} and since V is a mean-zero random walk under \mathbb{P} , we have

$$\mathbb{Q}(\Omega_+) = 1, \quad \mathbb{P}(\Omega_+) = 0,$$

so the two measures are mutually singular. \square

Remark 4.1. *From the proof of reversibility we see that there exist many invariant measures by the transition R . Indeed, the proof works thanks to the relation (42), (which means the reversibility of the measure $\tilde{\nu}$ with respect to the walk in the environment $\tilde{\omega}$) and thanks to the fact that $\sum_{x \in \mathbb{Z}} \exp[-\tilde{V}(x)]$ finite a.s. . As a consequence for any \hat{V} (in the place of \tilde{V}), $\hat{\omega}$ and $\hat{\nu}$ defined with the help of \hat{V} using (15),(16) the measure on Ω given by $E \sum_{x \in \mathbb{Z}} \hat{\nu}(x) \delta_{T_x \hat{\omega}}$, where the expectation is taken w.r.t. to the law of \hat{V} , is reversible for R .*

5. Examples

In this section we give some examples of application of Theorem 1.1. All these applications deal with the limit behavior of properly normalized sums $S_n^f = \sum_{k=1}^n f(\omega(X_k), \Delta X_k)$, for which we perform a decomposition, representing S_n^f as a sum of martingale and an additive functional of the environment's chain. For such a functional we can apply our ergodic theorem (Theorem 1.1). In this sense, this is an extension of the environments method of Kozlov and Molchanov (1984) to the recurrent RWRE.

5.1. Law of large numbers for functions of the environment seen from the particle and steps

Proposition 5.1. *Let $f : \Omega \times \{-1; 1\} \rightarrow \mathbb{R}$, bounded. Denote $\Delta X_k = X_{k+1} - X_k$, $k \in \mathbb{N}$. Then the following convergence in distribution under annealed probability holds :*

$$\frac{1}{n} \sum_{k=1}^n f(\omega(X_k), \Delta X_k) \xrightarrow{\mathcal{L}} \sum_{x \in \mathbb{Z}} (f(\tilde{\omega}_x, 1) \tilde{\omega}_x + f(\tilde{\omega}_x, -1)(1 - \tilde{\omega}_x)) \tilde{\nu}(x), \quad n \rightarrow \infty.$$

In particular, if f does not depend on ω , $f : \{-1; 1\} \rightarrow \mathbb{R}$, then the following convergence in annealed probability holds :

$$\frac{1}{n} \sum_{k=1}^n f(\Delta X_k) \xrightarrow{\mathbf{P}} \frac{f(1) + f(-1)}{2}, \quad n \rightarrow \infty.$$

Proof. Denote

$$\mathcal{F}_n = \sigma\{\Delta X_0, \dots, \Delta X_n, \omega(X_0), \dots, \omega(X_{n+1})\}$$

and let's write the martingale difference decomposition :

$$\frac{1}{n} \sum_{k=1}^n f(\omega(X_k), \Delta X_k) = \frac{1}{n} \sum_{k=1}^n D_k + \frac{1}{n} \sum_{k=1}^n \mathbf{E}(f(\omega(X_k), \Delta X_k) | \mathcal{F}_{k-1}). \quad (43)$$

Where

$$D_k := f(\omega(X_k), \Delta X_k) - \mathbf{E}(f(\omega(X_k), \Delta X_k) | \mathcal{F}_{k-1}); \quad k \in \mathbb{N}$$

is centered conditionally to \mathcal{F}_{k-1} , measurable with respect to \mathcal{F}_k , uniformly bounded random variable. As a consequence $(D_k); k \in \mathbb{N}$, are bounded, non-correlated and

$$\frac{1}{n} \sum_{k=1}^n D_k \xrightarrow{\mathbb{L}^2} 0. \quad (44)$$

Remark that

$$\mathbf{E}(f(\omega(X_k), \Delta X_k) | \mathcal{F}_{k-1}) = f(\omega(X_k), 1)\omega(X_k) + f(\omega(X_k), -1)(1 - \omega(X_k)).$$

Theorem 1.1 gives the following convergence in distribution :

$$\frac{1}{n} \sum_{k=1}^n \mathbf{E}(f(\omega(X_k), \Delta X_k) | \mathcal{F}_{k-1}) \xrightarrow{\mathcal{L}} \sum_{x \in \mathbb{Z}} (f(\tilde{\omega}_x, 1)\tilde{\omega}(x) + f(\tilde{\omega}_x, -1)(1 - \tilde{\omega}(x))) \tilde{\nu}(x). \quad (45)$$

When f does not depend on ω , the previous limit becomes

$$\frac{1}{n} \sum_{k=1}^n \mathbf{E}(f(\Delta X_k) | \mathcal{F}_{k-1}) \xrightarrow{\mathcal{L}} \sum_{x \in \mathbb{Z}} (f(1)\tilde{\omega}(x) + f(-1)(1 - \tilde{\omega}(x))) \tilde{\nu}(x) = \frac{f(1) + f(-1)}{2}.$$

Indeed, using the definitions (16) and (15) ,

$$\sum_{x \in \mathbb{Z}} \tilde{\omega}(x) \tilde{\nu}(x) = \sum_{x \in \mathbb{Z}} (1 - \tilde{\omega}(x)) \tilde{\nu}(x) = \frac{1}{2}.$$

Using (44) and (45) together in (43) this completes the proof. \square

Proposition 5.2. *Let $f : \Omega \times \{-1; 1\} \rightarrow \mathbb{R}$, bounded. Then the following convergence in distribution holds:*

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n (f(\omega(X_k), \Delta X_k) - f(\omega(X_k), 1)\omega(X_k) - f(\omega(X_k), -1)(1 - \omega(X_k))) \xrightarrow{\mathcal{L}} Z,$$

where Z is a random variable with the characteristic function

$$\phi_Z(t) = \mathbf{E}(\exp(-\frac{1}{2}\eta^2 t^2)),$$

and η is a random variable distributed as:

$$\eta^2 \stackrel{\mathcal{L}}{=} \sum_{x \in \mathbb{Z}} (f(\tilde{\omega}_x, 1) - f(\tilde{\omega}_x, -1))^2 \tilde{\omega}(x)(1 - \tilde{\omega}(x)) \tilde{\nu}(x).$$

That is $Z \stackrel{\mathcal{L}}{=} \eta U$ where η and U are independent and $U \sim \mathcal{N}(0, 1)$.

In particular, if f does not depend on ω , $f : \{-1; 1\} \rightarrow \mathbb{R}$, then the following convergence in distribution holds:

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n (f(\Delta X_k) - f(1)\omega(X_k) - f(-1)(1 - \omega(X_k))) \xrightarrow{\mathcal{L}} Z,$$

where Z is a random variable with the characteristic function $\phi_Z(t) = \mathbf{E}(\exp(-\frac{1}{2}\eta^2 t^2))$, and η is a random variable defined by: $\eta^2 \stackrel{\mathcal{L}}{=} (f(1) - f(-1))^2 \sum_{x \in \mathbb{Z}} \tilde{\omega}(x)(1 - \tilde{\omega}(x)) \tilde{\nu}(x)$.

That is $Z \stackrel{\mathcal{L}}{=} \eta U$ where η and U are independent and $U \sim \mathcal{N}(0, 1)$.

Proof. In this proof we will rely on Theorem 3.4 from Hall and Heyde (1980) and use their notation. Define

$$D_{nk} := \frac{1}{\sqrt{n}} (f(\omega(X_k), \Delta X_k) - \mathbf{E}(f(\omega(X_k), \Delta X_k) | \mathcal{F}_{k-1})).$$

Let $S_{n0} = 0$, $S_{ni} = \sum_{k=1}^i D_{nk}$ and put for $i = 1, \dots, n$, $\mathcal{F}_{ni} := \mathcal{F}_i$. Then S_{ni} is adapted to \mathcal{F}_i . Let $U_{ni}^2 = \sum_{k=1}^i D_{nk}^2$. Denote

$$\mathcal{G}_n = \sigma\{\omega(X_0), \dots, \omega(X_{n+1})\},$$

Clearly $\mathcal{G}_n \subset \mathcal{F}_n$. Let for $i = 1, \dots, n$,

$$\mathcal{G}_{n,i} := \mathcal{F}_{ni} \vee \mathcal{G}_n = \mathcal{F}_i \vee \mathcal{G}_n = \sigma(\Delta X_1, \dots, \Delta X_i, \omega(X_0), \dots, \omega(X_{n+1})).$$

Moreover,

$$\max_{i=1 \dots n} |D_{ni}| \leq \frac{2\|f\|}{\sqrt{n}} \longrightarrow 0 \quad \text{and} \quad \mathbf{E} \left(\max_{i=1 \dots n} |D_{ni}|^2 \right) \leq \frac{2\|f\|^2}{n}. \quad (46)$$

Define a random sequence (u_n^2) by

$$u_n^2 = \sum_{k=1}^n \mathbf{E}(D_{nk}^2 | \mathcal{F}_{k-1}).$$

It is easy to see that

$$\mathbf{E}(D_{nk}^2 | \mathcal{F}_{k-1}) = \frac{1}{n} (f(\omega(X_k), 1) - f(\omega(X_k), -1))^2 \omega(X_k) (1 - \omega(X_k)),$$

hence the sequence (u_n^2) is (\mathcal{G}_n) -adapted. In order to show the following convergence in probability (condition (3.28) of Theorem 3.4 from Hall and Heyde (1980)):

$$U_{nn}^2 - u_n^2 = \sum_{k=1}^n (D_{nk}^2 - \mathbf{E}(D_{nk}^2 | \mathcal{F}_{k-1})) \xrightarrow{\mathbb{P}} 0, \quad (47)$$

we remark that $(D_{nk}^2 - \mathbf{E}(D_{nk}^2 | \mathcal{F}_{k-1}))$; $k = 1, \dots, n$, are bounded, centered conditionally on \mathcal{F}_{k-1} , (\mathcal{F}_k) -adapted and hence non correlated. Hence, using (46), $U_{nn}^2 - u_n^2$ converges to 0 in \mathbb{L}^2 :

$$\mathbf{E} \left(\sum_{k=1}^n (D_{nk}^2 - \mathbf{E}(D_{nk}^2 | \mathcal{F}_{k-1})) \right)^2 = \sum_{k=1}^n \mathbf{E} (D_{nk}^2 - \mathbf{E}(D_{nk}^2 | \mathcal{F}_{k-1}))^2 \leq \frac{2n \|f\|^4}{n^2} \rightarrow 0.$$

Then for all $i = 1 \dots n$, $\mathbf{E}(D_{ni} | \mathcal{G}_{n,i-1}) = 0$, and the condition (3.29) of Theorem 3.4 from Hall and Heyde (1980) is satisfied. Applying Theorem (1.1) we see that

$$u_n^2 = \frac{1}{n} \sum_{k=1}^n (f(\omega(X_k), 1) - f(\omega(X_k), -1))^2 \omega(X_k) (1 - \omega(X_k)) \xrightarrow{\mathcal{L}} \sum_{x \in \mathbb{Z}} (f(\tilde{\omega}_x, 1) - f(\tilde{\omega}_x, -1))^2 \tilde{\omega}(x) (1 - \tilde{\omega}(x)) \tilde{\nu}(x).$$

Using (47),

$$U_{nn}^2 = (U_{nn}^2 - u_n^2) + u_n^2 \xrightarrow{\mathcal{L}} \sum_{x \in \mathbb{Z}} (f(\tilde{\omega}_x, 1) - f(\tilde{\omega}_x, -1))^2 \tilde{\omega}(x) (1 - \tilde{\omega}(x)) \tilde{\nu}(x).$$

and the theorem follows. \square

Example I. Suppose that $\omega(x) \in A$, where A is countable and let $a \in A$. Then using Proposition (5.1) we obtain

$$\frac{1}{n} \sum_{k=1}^n \mathbf{1}(\omega(X_k) = a) \mathbf{1}(\Delta(X_k) = 1) \xrightarrow{\mathcal{L}} a \sum_{x \in \mathbb{Z}} \mathbf{1}(\tilde{\omega}_x = a) \tilde{\nu}(x) = a \tilde{\nu}(x : \tilde{\omega}_x = a)$$

Moreover, using Proposition (5.2),

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n \mathbf{1}(\omega(X_k) = a) [1(\Delta(X_k) = 1) - a] \xrightarrow{\mathcal{L}} \sqrt{a(1-a)} \tilde{v}(x : \tilde{\omega}_x = a) Z;$$

where $Z \sim \mathcal{N}(0, 1)$ is independent of $\tilde{v}(x : \tilde{\omega}_x = a)$.

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SUPPLEMENTARY MATERIAL

The supplement contains the additional Section 6. We prove there the crucial Proposition 2.1 used in the proof of Theorem 1.1. This proposition allows to approximate an additive functional of the environment's chain by a functional depending only on the environment.

6. Proof of the approximation in probability of $\int_{\Omega} F d\mathcal{S}_n$.

Proof. Let $n \in \mathbb{N}^*$ and $\omega \in \Omega$ be fixed. Denote $T^0 = 0$. For $y \in \mathbb{Z}^+$, $\forall k \geq 1$, denote

$$T_y^k := \inf\{t > T_y^{k-1}, X_t = y\}$$

We shorten $T_y := T_y^1$. Using the recurrence of X , $T_y^k < \infty$ $\mathbf{P} - a.s.$ Denote k_n the number of visits of b_n by the walk before the time n :

$$k_n := \xi(n, b_n) = \sum_{t=0}^n 1(X_t = b_n).$$

As μ_n is an invariant probability of (\tilde{X}^n) , k_n/n can be compared with μ_n :

$$\begin{aligned} P_0^\omega \left(\left| \frac{k_n}{n} - \mu_n(b_n) \right| > \varepsilon \right) &\leq \\ P_0^\omega \left(\left| \frac{k_n}{n} - \mu_n(b_n) \right| > \varepsilon, T_{b_n} < n\varepsilon/2, T_{c_n} > n \right) &+ P_0^\omega (T_{b_n} \geq n\varepsilon/2) + P_0^\omega (T_{c_n} \leq n). \end{aligned} \quad (48)$$

For the first term of the inequality (48) we can write:

$$P_0^\omega \left(\left| \frac{k_n}{n} - \mu_n(b_n) \right| > \varepsilon, T_{b_n} < n\varepsilon/2, T_{c_n} > n \right) \leq P_0^\omega(B_1) + P_0^\omega(B_2); \quad (49)$$

Where we have denoted

$$\begin{aligned} B_1 &:= \{k_n \geq [n(\mu_n(b_n) + \varepsilon)] + 1, T_{b_n} < n\varepsilon/2, T_{c_n} > n\}; \\ B_2 &:= \{k_n \leq [n(\mu_n(b_n) - \varepsilon)], T_{b_n} < n\varepsilon/2, T_{c_n} > n\}. \end{aligned}$$

Both events B_1 and B_2 concern with the part of the trajectory X_0, \dots, X_n , where the value c_n did not occur. Hence $P_0^\omega(B_i) = \tilde{P}_0^{\omega, n}(B_i)$, $i = 1, 2$. Then, using the

definition of $(T_{b_n}^k)$, strong Markov property and Markov inequality, for $n \geq 2/\varepsilon$

$$\begin{aligned}
P_0^\omega(B_1) &= \tilde{P}_0^{\omega,n}(B_1) \leq \tilde{P}_0^\omega\left(T_{b_n}^{\lfloor n(\mu_n(b_n)+\varepsilon) \rfloor+1} \leq n\right) \leq \\
&\tilde{P}_0^{\omega,n}\left(\sum_{k=1}^{\lfloor n(\mu_n(b_n)+\varepsilon) \rfloor} (T_{b_n}^{k+1} - T_{b_n}^k) \leq n\right) \leq \tilde{P}_{b_n}^{\omega,n}\left(\sum_{k=1}^{\lfloor n(\mu_n(b_n)+\varepsilon) \rfloor} \eta_k \leq -\frac{n\varepsilon-1}{\mu_n(b_n)}\right) \\
&\leq \frac{n(\mu_n(b_n)+\varepsilon)\mu_n^2(b_n)\tilde{V}_{b_n}^{\omega,n}\eta_1}{(n\varepsilon-1)^2} \leq 4\frac{(1+\varepsilon)\tilde{V}_{b_n}^{\omega,n}\eta_1}{\varepsilon^2 n}, \quad (50)
\end{aligned}$$

where

$$\eta_k = T_{b_n}^{k+1} - T_{b_n}^k - 1/\mu_n(b_n)$$

are i.i.d. and centered under $\tilde{P}_{b_n}^{\omega,n}$, since μ_n is the invariant probability for the irreducible positively recurrent chain \tilde{X}^n and $\tilde{E}_{b_n}^{\omega,n}T_{b_n}^1 = \frac{1}{\mu_n(b_n)}$. Similar arguments give

$$\begin{aligned}
P_0^\omega(B_2) &= \tilde{P}_0^{\omega,n}(B_2) \leq \tilde{P}_0^\omega\left(T_{b_n}^{\lfloor n(\mu_n(b_n)-\varepsilon) \rfloor} \geq n, T_{b_n} < n\varepsilon/2\right) \leq \\
&\tilde{P}_0^{\omega,n}\left(\sum_{k=1}^{\lfloor n(\mu_n(b_n)-\varepsilon) \rfloor-1} (T_{b_n}^{k+1} - T_{b_n}^k) \geq n(1-\varepsilon/2)\right) \leq \tilde{P}_{b_n}^{\omega,n}\left(\sum_{k=1}^{\lfloor n(\mu_n(b_n)-\varepsilon) \rfloor-1} \eta_k \geq \frac{n\varepsilon}{2}\right) \\
&\leq 4\frac{n(\mu_n(b_n)-\varepsilon)\tilde{V}_{b_n}^{\omega,n}\eta_1}{n^2\varepsilon^2} \leq 4\frac{(1-\varepsilon)\tilde{V}_{b_n}^{\omega,n}\eta_1}{\varepsilon^2 n}. \quad (51)
\end{aligned}$$

Finally, putting together (48), (49), (50) and (51) we get:

$$P_0^\omega\left(\left|\frac{k_n}{n} - \mu_n(b_n)\right| > \varepsilon\right) \leq \frac{8}{\varepsilon^2 n}\tilde{V}_{b_n}^{\omega,n}T_{b_n} + P_0^\omega(T_{b_n} \geq n\varepsilon/2) + P_0^\omega(T_{c_n} \leq n). \quad (52)$$

Note that for a fixed $\omega \in \Omega$, $\Sigma_n(F) = \sum_{x \in \mathbb{Z}} F(T_x \omega) \mu_n(x)$ is fixed. Denote for $x \in \mathbb{Z}_+$

$$F_\omega(x) := F(T_x \omega), \quad \bar{F}_\omega(x) := F_\omega(x) - \Sigma_n(F) \quad \text{and} \quad \tilde{\varepsilon} = \varepsilon/3\|F\|_\infty.$$

We will first obtain a non-asymptotic bound on the quenched probability of the deviation $|\mathcal{S}_n(F) - \Sigma_n(F)| = \left|\frac{1}{n}\sum_{k=0}^n F(T_{X_k} \omega) - \Sigma_n(F)\right| = \left|\frac{1}{n}\sum_{k=0}^n F_\omega(X_k) - \Sigma_n(F)\right|$.

$$\begin{aligned}
P_0^\omega \left(\left| \frac{1}{n} \sum_{k=0}^n F_\omega(X_k) - \Sigma_n(F) \right| > \varepsilon \right) &\leq P_0^\omega \left(\left| \frac{k_n}{n} - \mu_n(b_n) \right| > \varepsilon \right) + P_0^\omega (T_{c_n} < n) + \\
P_0^\omega \left(\sum_{k=0}^{T_{b_n}^{k_n}-1} |\bar{F}_\omega(X_k)| > n\varepsilon/3 \right) &+ P_0^\omega \left(\left| \sum_{k=T_{b_n}^{k_n}}^{T_{b_n}^{k_n}-1} \bar{F}_\omega(X_k) \right| > n\varepsilon/3, \left| \frac{k_n}{n} - \mu_n(b_n) \right| \leq \varepsilon, T_{c_n} > n \right) + \\
P_0^\omega \left(\sum_{k=T_{b_n}^{k_n}}^n |\bar{F}_\omega(X_k)| > n\varepsilon/3, T_{c_n} > n \right) & \tag{53}
\end{aligned}$$

Using the definition of $\tilde{\varepsilon}$ we see that

$$P_0^\omega \left(\sum_{k=0}^{T_{b_n}^{k_n}-1} |\bar{F}_\omega(X_k)| > n\varepsilon/3 \right) \leq P_0^\omega (T_{b_n} > \tilde{\varepsilon}n). \tag{54}$$

The law of $T_{b_n}^{k_n+1} - T_{b_n}^{k_n}$ conditionally on $\mathcal{F}_{T_{b_n}^{k_n}}$ is that of T_{b_n} . Also, using the definition (29) we can see that $\mu_n(b_n) \geq 1/2c_n$. Hence, using again $\tilde{E}_{b_n}^{\omega,n} T_{b_n}^1 = \frac{1}{\mu_n(b_n)}$,

$$\begin{aligned}
P_0^\omega \left(\sum_{k=T_{b_n}^{k_n}}^n |\bar{F}_\omega(X_k)| > n\varepsilon/3, T_{c_n} > n \right) &= \tilde{P}_0^\omega \left(\sum_{k=T_{b_n}^{k_n}}^n |\bar{F}_\omega(X_k)| > n\varepsilon/3, T_{c_n} > n \right) \leq \\
\tilde{P}_0^\omega \left(\sum_{k=T_{b_n}^{k_n}}^{T_{b_n}^{k_n+1}} |\bar{F}_\omega(X_k)| > n\varepsilon/3 \right) &\leq \tilde{P}_{b_n}^{\omega,n} (T_{b_n} > n\tilde{\varepsilon}) \leq \frac{1}{n\mu_n(b_n)\tilde{\varepsilon}} \leq \frac{2c_n}{n\tilde{\varepsilon}}. \tag{55}
\end{aligned}$$

Now we obtain a bound for the main term of the decomposition (53). Denote for $k \in \mathbb{N}^*$,

$$\xi_k := \sum_{l=T_{b_n}^k}^{T_{b_n}^{k+1}-1} \bar{F}_\omega(X_l).$$

Under $\tilde{P}_0^{\omega,n}$ the random variables $\xi_k, k \in \mathbb{N}^*$ are i.i.d. Their law is that of $\sum_{l=0}^{T_{b_n}^{k+1}-1} \bar{F}_\omega(X_l)$ under $\tilde{P}_{b_n}^{\omega,n}$ and they are centered. Indeed, for an irreducible positively recurrent chain (Y_k) with an invariant probability μ , and f defined on the state space,

$$\mathbf{E}_y \sum_{k=0}^{T_y-1} f(Y_k) = \mu(F) \mathbf{E}_y T_y,$$

see again Norris (1997) Theorems 1.7.6 and 1.7.7, which become here

$$E_{b_n}^{\omega, n} \sum_{l=0}^{T_{b_n}-1} F_\omega(X_l) = \mu_n(F_\omega(\cdot)) E_{b_n}^{\omega, n} T_{b_n} = \Sigma_n(F) E_{b_n}^{\omega, n} T_{b_n}.$$

Hence $M_m := \sum_{k=1}^m \xi_k$; $m \in \mathbb{N}^*$ is a square-integrable martingale under $\tilde{P}_0^{\omega, n}$. Using Kolmogorov inequality we get:

$$\begin{aligned} P_0^\omega \left(\left| \sum_{k=T_{b_n}}^{T_{b_n}^{k_n+1}-1} \bar{F}_\omega(X_k) \right| > n\varepsilon/3, \left| \frac{k_n}{n} - \mu_n(b_n) \right| \leq \varepsilon, T_{b_n} < \varepsilon n, T_{c_n} > n \right) &\leq \quad (56) \\ \tilde{P}_0^{\omega, n} \left(\left| \sum_{k=1}^{k_n} \xi_k \right| > n\varepsilon/3, \left| \frac{k_n}{n} - \mu_n(b_n) \right| \leq \varepsilon \right) &\leq \\ \tilde{P}_0^{\omega, n} \left(\sup_{m=1, \dots, \lfloor n(\mu_n(b_n) + \varepsilon) \rfloor} \left| \sum_{k=1}^m \xi_k \right| > n\varepsilon/3 \right) &\leq \frac{9(\mu_n(b_n) + \varepsilon) \tilde{V}_{b_n}^{\omega, n}(\xi_1)}{n\varepsilon^2} \end{aligned}$$

Plugging in (53) the bounds (52), (54), (55) and (56) we obtain:

$$\begin{aligned} P_0^\omega (|\mathcal{L}_n(F) - \Sigma_n(F)| > \varepsilon) &\leq \\ &P_0^\omega (T_{b_n} \geq n\varepsilon/2) + P_0^\omega (T_{b_n} \geq n\tilde{\varepsilon}) + 2P_0^\omega (T_{c_n} \leq n) + \\ &\frac{8}{n\varepsilon^2} \tilde{V}_{b_n}^{\omega, n} T_{b_n} + \frac{9(1 + \varepsilon) \tilde{V}_{b_n}^{\omega, n}(\xi_1)}{n\varepsilon^2} + \frac{2c_n}{n\tilde{\varepsilon}}. \end{aligned}$$

To conclude the proof we need to estimate $\tilde{V}_{b_n}^{\omega, n}(\xi_1)$. For $x \in Z_+$ introduce

$$Y_x = \sum_{j=0}^{T_{b_n}} 1_{\{x\}}(X_j)$$

the local time in x during 1-th excursion from b_n to b_n . Note that under $\tilde{P}_{b_n}^n$,

$$\xi_1 = \sum_{x=0, \dots, c_n} \bar{F}_\omega(x) Y_x \quad (57)$$

and

$$\tilde{V}_{b_n}^{\omega, n}(\xi_1) \leq (c_n + 1) \|\bar{F}_\omega\|_\infty^2 \sum_{x=0, \dots, c_n} \tilde{V}_{b_n}^{\omega, n}(Y_x).$$

Taking $\bar{F}_\omega = 1$ in (57) we get:

$$\tilde{V}_{b_n}^{\omega, n}(T_{b_n}) \leq (c_n + 1) \sum_{x=0, \dots, c_n} \tilde{V}_{b_n}^{\omega, n}(Y_x).$$

Using Lemma (6.1), which is given in Appendix, for all $\eta > 0$, there exists $\delta > 0$ and an event $\Omega_{\eta, \delta} \subset \Omega$ with $\mathbb{P}(\Omega_{\eta, \delta}) > 1 - \eta$ such that : $\forall \omega \in \Omega_{\eta, \delta}, \forall x \in [0, c_n]$,

$$\tilde{V}_{b_n}^{\omega, n}(Y_x) \leq n^{1-\delta}.$$

The proof of Lemma 6.1 is given below. Note that the bound (2.13) on the variance of the similar quantity in Gantert et al. (2010) is not sufficient. As a consequence, for all $\eta > 0$, there exists $1 > \delta > 0$, and a set $\Omega_{\eta, \delta}$ with $\mathbb{P}(\Omega_{\eta, \delta}) > 1 - \eta$, such that for all $\omega \in \Omega_{\eta, \delta}$ it holds

$$P_0^\omega (|\mathcal{S}_n(F) - \Sigma_n(F)| > \varepsilon) \leq C \frac{c_n^2 n^{1-\delta}}{n} + 2P_0^\omega (T_{b_n} \geq (\varepsilon + \tilde{\varepsilon})n) + 2P_0^\omega (T_{c_n} \leq n).$$

The last bound tends to zero. Indeed, from Golosov (1984), Lemma 1, $P_0^\omega (T_{b_n} > n\varepsilon') \rightarrow 0$ for all $\omega \in \Omega$ and $\varepsilon' > 0$. And from Golosov (1984), Lemma 7, for all $\omega \in \Omega$, $P_0^\omega (T(c_n) \leq n) \rightarrow 0$. \square

In our proof we correct the crucial estimation of the variance of the number of visits of a given state during one life cycle, formula (2.13) in Gantert et al. (2010). It is done in following Lemma (6.1).

Recall the definition:

$$Y_x = \sum_{j=0}^{T_{b_n}} 1_{\{x\}}(X_j)$$

Lemma 6.1. *For all $\eta > 0$ there exists $\delta > 0$ and an event $\Omega_{\eta, \delta} \subset \Omega$ with*

$$\mathbb{P}(\Omega_{\eta, \delta}) > 1 - \eta$$

such that for all $\omega \in \Omega_{\eta, \delta}$, for all $x \in [0, c_n]$,

$$\tilde{V}_{b_n}^{\omega, n}(Y_x) \leq n^{1-\delta}.$$

Proof. Note that $Y_{b_n} = 1$. Using (2.10) of Gantert et al. (2010), for $x \neq b_n$ we have $\tilde{V}^\omega(Y_x) \leq \frac{4}{\beta(x)}$, where

$$\beta(x) = (1 - \omega_x) \tilde{P}_{x-1}^{\omega, n}(T(b_n) < T(x)), \quad x = b_n + 1, \dots, c_n;$$

$$\beta(x) = \omega_x \tilde{P}_{x+1}^{\omega, n}(T(b_n) < T(x)), \quad x = 0, \dots, b_n - 1.$$

Then using Assumption II, (2.8) and (2.11) of Gantert et al. (2010) we can find a constant C_0 such that for $y = 1, \dots, c_n - b_n$,

$$\tilde{\mathbb{V}}^\omega(Y_{b_n+y}) \leq C_0 \sum_{j=b_n}^{b_n+y-1} e^{V(j)-V(b_n+y-1)},$$

and for $y = -b_n, \dots, -1$,

$$\tilde{\mathbb{V}}^\omega(Y_{b_n+y}) \leq C_0 \sum_{j=b_n+y}^{b_n-1} e^{V(j)-V(b_n+y)}.$$

However the exponential is missing in the bound (2.13) of Gantert et al. (2010), hence, to complete the proof we have to handle the term

$$A_n^\omega(x) := \begin{cases} \sum_{j=b_n}^{b_n+x-1} e^{V(j)-V(b_n+x-1)} & \text{if } x = 1, \dots, c_n - b_n, \\ \sum_{j=b_n+y}^{b_n-1} e^{V(j)-V(b_n+y)} & \text{if } x = -b_n, \dots, -1 \end{cases}$$

more carefully. Using Komlos-Mayor-Tusnady theorem we can construct a probability space (Ω, \mathcal{A}, P) on which are defined the environment ω and a Brownian motion W s.t. a.s.

$$\sup_{0 \leq s \leq t} |V(s) - \sigma W_s| \leq C \ln t,$$

where $\sigma^2 = \mathbb{E}(\log \rho)^2$ and $V(s)$, $s \geq 0$, is defined equal to $V(j)$ on $[j, j+1[$, $j \in \mathbb{Z}_+$. Let $t = c_n$, $W^{(n)}(s) = \frac{1}{\sigma \ln n} W(\sigma^2 \ln^2 n)$, $\bar{b}_n = \frac{b_n}{\sigma^2 \ln^2 n}$, $\bar{c}_n = \frac{c_n}{\sigma^2 \ln^2 n}$ and $\ln_2 := \ln \ln$. Then we can write, with $z = \sigma^2 \ln^2 n \times u$,

$$A_n^\omega(x) = \int_{b_n}^{b_n+x-1} \exp\{\sigma |W(z) - W(b_n + x - 1)| + O(\ln c_n)\} dz = \sigma^2 \ln^2 n \int_{\bar{b}_n}^{\bar{b}_n + \frac{x-1}{\sigma^2 \ln^2 n}} e^{\{\sigma^2 \ln n [W^{(n)}(u) - W^{(n)}(\bar{b}_n + \frac{x-1}{\sigma^2 \ln^2 n})]\}} du \times e^{O(\ln c_n)}.$$

Denote

$$\Delta_n := \max\{W^{(n)}(u) - W^{(n)}(v); \bar{b}_n \leq u \leq v \leq \bar{c}_n\}$$

and

$$\Delta'_n := \max\{W^{(n)}(u) - W^{(n)}(v); 0 \leq v \leq u \leq \bar{b}_n\}$$

It follows that

$$\max\{A_n^\omega(x), x = 0, \dots, c_n - b_n\} \leq c_n \exp\{\sigma^2 \ln n \times \Delta_n + O(\ln c_n)\}.$$

Similarly, for $x \in [-b_n, \dots, 0]$,

$$\max\{A_n^\omega(x), x = -b_n, \dots, 0\} \leq c_n \exp\{\sigma^2 \ln n \times \Delta'_n + O(\ln c_n)\}.$$

And hence,

$$\max\left\{\frac{\ln A_n^\omega(x)}{\sigma^2 \ln n}; x \in [-b_n; c_n - b_n]\right\} \leq \Delta_n \vee \Delta'_n + O\left(\frac{\ln c_n}{\ln n}\right). \quad (58)$$

By Donsker theorem, $(\bar{b}_n, \bar{c}_n, W^{(n)})$ converges in distribution to $(\bar{b}, \bar{c}, \bar{W})$, where \bar{W} is a Brownian motion,

$$\bar{c} = \inf\{s \geq 0, \bar{W}(s) - \min_{0 \leq t \leq s} \bar{W}(t) \geq 1\}$$

and

$$\bar{b} := \inf\{u \geq 0, \bar{W}(u) = \min_{0 \leq t \leq \bar{c}} \bar{W}(t)\}.$$

Therefore, (Δ_n, Δ'_n) converges in distribution to (Δ, Δ') , with

$$\Delta := \max\{\bar{W}(u) - \bar{W}(v); \bar{b} \leq u \leq v \leq \bar{c}\}$$

and

$$\Delta' := \max\{\bar{W}(u) - \bar{W}(v); 0 \leq v \leq u \leq \bar{b}\}$$

Both $\Delta < 1$ and $\Delta' < 1$ a.s., so $\Delta \vee \Delta' < 1$ a.s. Using the monotonicity and again Donsker theorem it follows that $\forall \eta > 0, \exists \delta > 0$, such that

$$\liminf_{n \rightarrow \infty} \mathbb{P}(\Delta_n \vee \Delta'_n < 1 - \delta) > 1 - \eta \quad (59)$$

It follows from (59) and (58) that $\forall \eta > 0, \exists \delta > 0$, s.t.

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(\max_{[-b_n, c_n - b_n]} A_n^\omega(x) \leq n^{1-\delta}\right) \geq 1 - \eta$$

Denote $\Omega_{\eta, \delta} := \{\omega \in \Omega; \max_{[-b_n, c_n - b_n]} A_n^\omega(x) \leq n^{1-\delta}\}$ such that $\mathbb{P}(\Omega_{\eta, \delta}) > 1 - \eta$. Suppose that $\omega \in \Omega_{\eta, \delta}$. Then for all $x \in [-b_n, c_n - b_n]$, $\tilde{V}_\omega Y_n(x) \leq n^{1-\delta}$. \square

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