

# CONGRUENCES CONCERNING LEGENDRE POLYNOMIALS MODULO $p^2$

AERAN KIM

ABSTRACT. In this article, we extend Z. H. Sun's congruences concerning Legendre polynomials  $P_{\frac{p-1}{2}}(x)$  to  $P_{\frac{p+1}{2}}(x)$  for odd prime  $p$ , which enables us to

deduce some congruences resembling  $\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \pmod{p^2}$ .

## 1. INTRODUCTION

Let  $\{P_n(x)\}$  be the Legendre polynomials given by

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1-2xt+t^2}} \quad \text{for } |t| < 1.$$

It is well known [1, 2, 3] that

$$(1) \quad P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k} = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n,$$

where  $\lfloor x \rfloor$  is the greatest integer not exceeding  $x$  and

$$(2) \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

Moreover,

$$(3) \quad P_n(x) = (-1)^n P_n(-x) \quad \text{and} \quad P_n(0) = \begin{cases} 0 & \text{for odd } n, \\ \frac{(-1)^{\frac{n}{2}}}{2^n} \binom{n}{\frac{n}{2}} & \text{for even } n. \end{cases}$$

Using the property of Legendre polynomials in Eq. (1), Z. H. Sun obtained various congruences for  $P_{\frac{p-1}{2}}(x)$  modulo  $p^2$ , where  $p$  is an odd prime and  $x$  is a rational  $p$ -integer. For example, he showed that

$$\sum_{k=0}^{p-1} \frac{1}{16^k} \binom{2k}{k}^2 \left( x^k - (-1)^{\frac{p-1}{2}} (1-x)^k \right) \equiv 0 \pmod{p^2}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{1}{16^k} \binom{2k}{k}^2 \left( x^k - \left( \frac{x}{p} \right) x^{-k} \right) \equiv 0 \pmod{p} \quad \text{for } x \not\equiv 0 \pmod{p},$$

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2010 *Mathematics Subject Classification.* 11A07, 33C45.

*Key words and phrases.* Legendre polynomial, congruence.

where  $\left(\frac{x}{p}\right)$  is the Jacobi symbol. Inspired by these results, we consider  $P_{\frac{p+1}{2}}(x)$

and get some congruences modulo  $p^2$  resembling  $\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2$  with odd prime  $p$ . Especially using the relation  $P_{\frac{p-1}{2}}(x)$ ,  $P_{\frac{p+1}{2}}(x)$  and (2), we construct a congruence  $P_{\frac{p-3}{2}}(x)$  modulo  $p^2$ .

## 2. MAIN RESULTS

**Proposition 2.1.** *For  $n \in \mathbb{N}$  we have*

$$P_n(x) = \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} \left(\frac{x-1}{2}\right)^k.$$

*Proof.* Refer to [3]. □

**Corollary 2.2.** *Let  $p$  be an odd prime and  $k \in \{1, 2, \dots, \frac{p+1}{2}\}$ . Then*

$$\binom{\frac{p+1}{2} + k}{2k} \equiv \frac{(p+2k+1)(p+2k-1)}{(-16)^k (2k-1)^2} \binom{2k}{k} \left( p^2 \sum_{i=1}^{k-1} \frac{1}{(2i-1)^2} - 1 \right) \pmod{p^4}.$$

*Proof.* We can easily know that

$$\begin{aligned} & \binom{\frac{p+1}{2} + k}{2k} \\ &= \frac{(\frac{p+1}{2} + k)(\frac{p+1}{2} + k - 1)(\frac{p+1}{2} + k - 2) \cdots (\frac{p+1}{2} - k + 1)}{(2k)!} \\ &= \frac{(p+2k+1)(p+2k-1)(p+2k-3) \cdots (p-2k+3)}{2^{2k} \cdot (2k)!} \\ &= \frac{(p+2k+1)(p+2k-1) \cdot (p^2-1^2)(p^2-3^2) \cdots (p^2-(2k-3)^2)}{2^{2k} \cdot (2k)!} \\ &\equiv \frac{(p+2k+1)(p+2k-1)}{2^{2k} \cdot (2k)!} \left[ (-1)^{k-2} p^2 \left\{ 1^2 \cdot 3^2 \cdots (2k-7)^2 (2k-5)^2 \right. \right. \\ &\quad \left. \left. + 1^2 \cdot 3^2 \cdots (2k-7)^2 (2k-3)^2 + \cdots + 3^2 \cdots (2k-5)^2 (2k-3)^2 \right\} \right. \\ &\quad \left. + (-1)^{k-1} \cdot 1^2 \cdot 3^2 \cdots (2k-5)^2 (2k-3)^2 \right] \pmod{p^4} \\ &= \frac{(p+2k+1)(p+2k-1)}{2^{2k} \cdot (2k)!} (-1)^k \cdot 1^2 \cdot 3^2 \cdots (2k-5)^2 (2k-3)^2 \\ &\quad \times \left( p^2 \sum_{i=1}^{k-1} \frac{1}{(2i-1)^2} - 1 \right) \pmod{p^4} \end{aligned}$$

then since

$$\begin{aligned}
1^2 \cdot 3^2 \cdots (2k-5)^2 (2k-3)^2 &= \frac{(2k-3)!^2}{2^2 \cdot 4^2 \cdots (2k-6)^2 (2k-4)^2} \\
&= \frac{(2k-3)!^2}{(2^{k-2})^2 \cdot (k-2)!^2} \\
&= \frac{(2k-3)!}{2^{2k-4}} \cdot \frac{k^2 (k-1)^2}{(2k-2)(2k-1)(2k)} \cdot \frac{(2k)!}{k!^2} \\
&= \frac{(2k-3)!}{2^{2k-2}} \cdot \frac{k(k-1)}{2k-1} \cdot \binom{2k}{k},
\end{aligned}$$

the above identity can be written as

$$\begin{aligned}
\binom{\frac{p+1}{2} + k}{2k} &\equiv \frac{(p+2k+1)(p+2k-1)}{2^{2k} \cdot (2k)!} (-1)^k \cdot \frac{(2k-3)!}{2^{2k-2}} \cdot \frac{k(k-1)}{2k-1} \cdot \binom{2k}{k} \\
&\quad \times \left( p^2 \sum_{i=1}^{k-1} \frac{1}{(2i-1)^2} - 1 \right) \pmod{p^4} \\
&= \frac{(p+2k+1)(p+2k-1)}{(-16)^k (2k-1)^2} \binom{2k}{k} \left( p^2 \sum_{i=1}^{k-1} \frac{1}{(2i-1)^2} - 1 \right) \pmod{p^4}.
\end{aligned}$$

□

**Lemma 2.3.** *Let  $p$  be an odd prime and let  $x$  be a variable. Then*

$$P_{\frac{p+1}{2}}(x) \equiv - \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left( \frac{x-1}{2} \right)^k \pmod{p^2}.$$

*Proof.* From Proposition 2.1 and Corollary 2.2 we observe that

$$\begin{aligned}
P_{\frac{p+1}{2}}(x) &= \sum_{k=0}^{\frac{p+1}{2}} \binom{\frac{p+1}{2} + k}{2k} \binom{2k}{k} \left( \frac{x-1}{2} \right)^k \\
&= 1 + \sum_{k=1}^{\frac{p+1}{2}} \binom{\frac{p+1}{2} + k}{2k} \binom{2k}{k} \left( \frac{x-1}{2} \right)^k \\
&\equiv 1 + \sum_{k=1}^{\frac{p+1}{2}} \frac{(p+2k+1)(p+2k-1)}{(-16)^k (2k-1)^2} \binom{2k}{k} \left( p^2 \sum_{i=1}^{k-1} \frac{1}{(2i-1)^2} - 1 \right) \\
&\quad \times \binom{2k}{k} \left( \frac{x-1}{2} \right)^k \pmod{p^4} \\
&\equiv 1 - \sum_{k=1}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left( \frac{x-1}{2} \right)^k \pmod{p^2} \\
&= - \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left( \frac{x-1}{2} \right)^k,
\end{aligned}$$

where we use  $(p+2k+1)(p+2k-1) = (p+2k)^2 - 1 \equiv 4pk + 4k^2 - 1 \pmod{p^2}$ . □

**Proposition 2.4.** For  $n \in \mathbb{N}$  we have

$$P_{\frac{p-1}{2}}(x) \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{1}{16^k} \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k \pmod{p^2}.$$

*Proof.* See [3]. □

**Lemma 2.5.** Let  $p$  be an odd prime and let  $x$  be a variable. Then

$$\begin{aligned} & P_{\frac{p-3}{2}}(x) \\ & \equiv - \sum_{k=0}^{\frac{p-3}{2}} \frac{4(2px + 2p + 1)k^2 - 4p(2x - 1)k + 2px - 2p - 1}{16^k (2k - 1)^2} \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k \\ & \quad - p \binom{p-1}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \pmod{p^2}. \end{aligned}$$

In particular, we have  $P_{\frac{p-3}{2}}(1) \equiv 1 \pmod{p^2}$ .

*Proof.* Replacing  $n = \frac{p-1}{2}$  in (2) and multiplying  $2(p+1)$  on both sides, we obtain

$$P_{\frac{p-3}{2}}(x) \equiv (2p+1)P_{\frac{p+1}{2}}(x) - 2pxP_{\frac{p-1}{2}}(x) \pmod{p^2}.$$

Then applying Lemma 2.3 and Proposition 2.4 to this identity, we have

$$\begin{aligned} & P_{\frac{p-3}{2}}(x) \\ & \equiv (2p+1) \left[ - \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{x-1}{2}\right)^k \right] \\ & \quad - 2px \cdot \sum_{k=0}^{\frac{p-1}{2}} \frac{1}{16^k} \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k \\ & = (2p+1) \left[ - \sum_{k=0}^{\frac{p-3}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{x-1}{2}\right)^k \right. \\ & \quad - \frac{4p \cdot \frac{p-1}{2} + 4 \left(\frac{p-1}{2}\right)^2 - 1}{(-16)^{\frac{p-1}{2}} \left(2 \cdot \frac{p-1}{2} - 1\right)^2} \left(2 \cdot \frac{p-1}{2}\right)^2 \left(\frac{x-1}{2}\right)^{\frac{p-1}{2}} \\ & \quad \left. - \frac{4p \cdot \frac{p+1}{2} + 4 \left(\frac{p+1}{2}\right)^2 - 1}{(-16)^{\frac{p+1}{2}} \left(2 \cdot \frac{p+1}{2} - 1\right)^2} \left(2 \cdot \frac{p+1}{2}\right)^2 \left(\frac{x-1}{2}\right)^{\frac{p+1}{2}} \right] \\ & \quad - 2px \left[ \sum_{k=0}^{\frac{p-3}{2}} \frac{1}{16^k} \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k \right. \\ & \quad \left. + \frac{1}{16^{\frac{p-1}{2}}} \binom{2 \cdot \frac{p-1}{2}}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \right] \pmod{p^2}. \end{aligned} \tag{4}$$

Here the 2nd term is

$$\begin{aligned}
& \frac{4p \cdot \frac{p-1}{2} + 4 \left(\frac{p-1}{2}\right)^2 - 1 \left(2 \cdot \frac{p-1}{2}\right)^2}{(-16)^{\frac{p-1}{2}} \left(2 \cdot \frac{p-1}{2} - 1\right)^2} \left(\frac{p-1}{2}\right)^2 \left(\frac{x-1}{2}\right)^{\frac{p-1}{2}} \\
& \equiv \frac{-4p}{16^{\frac{p-1}{2}} (-4p+4)} \left(\frac{p-1}{2}\right)^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \\
& \equiv \frac{-p}{16^{\frac{p-1}{2}}} \left(\frac{p-1}{2}\right)^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \pmod{p^2}
\end{aligned}$$

and the 3rd term is

$$\begin{aligned}
& \frac{4p \cdot \frac{p+1}{2} + 4 \left(\frac{p+1}{2}\right)^2 - 1 \left(2 \cdot \frac{p+1}{2}\right)^2}{(-16)^{\frac{p+1}{2}} \left(2 \cdot \frac{p+1}{2} - 1\right)^2} \left(\frac{p+1}{2}\right)^2 \left(\frac{x-1}{2}\right)^{\frac{p+1}{2}} \\
& = \frac{2p(p+1) + (p+1)^2 - 1}{16^{\frac{p+1}{2}} p^2} \left(\frac{p+1}{2}\right)^2 \left(\frac{1-x}{2}\right)^{\frac{p+1}{2}} \\
& \equiv \frac{4p}{16^{\frac{p-1}{2}} \cdot 16p^2} \cdot \left(\frac{(p+1)p}{\frac{p+1}{2} \cdot \frac{p+1}{2}}\right)^2 \left(\frac{p-1}{2}\right)^2 \left(\frac{1-x}{2}\right)^{\frac{p+1}{2}} \\
& \equiv \frac{4p}{16^{\frac{p-1}{2}}} \left(\frac{p-1}{2}\right)^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \left(\frac{1-x}{2}\right) \\
& \equiv \frac{2p(1-x)}{16^{\frac{p-1}{2}}} \left(\frac{p-1}{2}\right)^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \pmod{p^2}
\end{aligned}$$

so that we can write Eq. (4) as

$$\begin{aligned}
& P_{\frac{p-3}{2}}(x) \\
& \equiv (2p+1) \left[ - \sum_{k=0}^{\frac{p-3}{2}} \frac{4pk+4k^2-1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{x-1}{2}\right)^k \right. \\
& \quad \left. + \frac{p}{16^{\frac{p-1}{2}}} \binom{p-1}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} - \frac{2p(1-x)}{16^{\frac{p-1}{2}}} \binom{p-1}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \right] \\
& \quad - 2px \left[ \sum_{k=0}^{\frac{p-3}{2}} \frac{1}{16^k} \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k + \frac{1}{16^{\frac{p-1}{2}}} \binom{p-1}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \right] \\
& \equiv - \sum_{k=0}^{\frac{p-3}{2}} \frac{(2p+1)(4pk+4k^2-1)}{16^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k \\
& \quad + \frac{p}{16^{\frac{p-1}{2}}} \binom{p-1}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} - \frac{2p(1-x)}{16^{\frac{p-1}{2}}} \binom{p-1}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \\
& \quad - \sum_{k=0}^{\frac{p-3}{2}} \frac{2px}{16^k} \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k - \frac{2px}{16^{\frac{p-1}{2}}} \binom{p-1}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \\
& = - \sum_{k=0}^{\frac{p-3}{2}} \frac{1}{16^k} \left[ \frac{(2p+1)(4pk+4k^2-1)}{(2k-1)^2} + 2px \right] \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k \\
& \quad - \frac{p}{16^{\frac{p-1}{2}}} \binom{p-1}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \\
& \equiv - \sum_{k=0}^{\frac{p-3}{2}} \frac{4(2px+2p+1)k^2 - 4p(2x-1)k + 2px - 2p - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k \\
& \quad - p \binom{p-1}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \pmod{p^2},
\end{aligned}$$

where we utilize  $16^{\frac{p-1}{2}} \equiv \left(\frac{16}{p}\right) = 1 \pmod{p}$  by Legendre's original definition.  $\square$

**Theorem 2.6.** *Let  $p$  be an odd prime and let  $x$  be a variable. Then*

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk+4k^2-1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left(x^k - (-1)^{\frac{p+1}{2}} (1-x)^k\right) \equiv 0 \pmod{p^2}.$$

*Proof.* By (3) and Lemma 2.3 we note that

$$\begin{aligned}
 & \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k \\
 &= \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{x-1}{2}\right)^k \\
 &\equiv -P_{\frac{p+1}{2}}(x) \\
 &= -(-1)^{\frac{p+1}{2}} P_{\frac{p+1}{2}}(-x) \\
 &\equiv -(-1)^{\frac{p+1}{2}} \left[ -\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{-x-1}{2}\right)^k \right] \\
 &= (-1)^{\frac{p+1}{2}} \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{x+1}{2}\right)^k \pmod{p^2}
 \end{aligned}$$

and so

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left[ \left(\frac{1-x}{2}\right)^k - (-1)^{\frac{p+1}{2}} \left(\frac{x+1}{2}\right)^k \right] \equiv 0 \pmod{p^2}.$$

Now putting  $x := 1 - 2y$  in the above identity, we conclude that

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left( y^k - (-1)^{\frac{p+1}{2}} (1-y)^k \right) \equiv 0 \pmod{p^2}.$$

□

**Theorem 2.7.** *Let  $p$  be an odd prime. Then*

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{(-1)^{\frac{p-3}{4}}}{2^{\frac{p+1}{2}}} \binom{\frac{p+1}{2}}{\frac{p+1}{4}} \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

*Proof.* When  $p \equiv 1 \pmod{4}$ , taking  $x = \frac{1}{2}$  in Theorem 2.6 we obtain

$$\begin{aligned}
 0 &\equiv \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left( \left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^k \right) \\
 &= 2 \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 \pmod{p^2}
 \end{aligned}$$

then since  $2 \nmid p^2$ , we have  $\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 \equiv 0 \pmod{p^2}$ . Next when  $p \equiv 3 \pmod{4}$ , setting  $x = 0$  in Lemma 2.3 and applying Eq. (3) we get

$$\begin{aligned} \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 &= \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(-\frac{1}{2}\right)^k \\ &\equiv -P_{\frac{p+1}{2}}(0) \\ &= -\frac{(-1)^{\frac{p+1}{4}}}{2^{\frac{p+1}{2}}} \binom{\frac{p+1}{2}}{\frac{p+1}{4}} \\ &= \frac{(-1)^{\frac{p-3}{4}}}{2^{\frac{p+1}{2}}} \binom{\frac{p+1}{2}}{\frac{p+1}{4}} \pmod{p^2} \end{aligned}$$

so the proof is complete.  $\square$

**Theorem 2.8.** *Let  $p$  be an odd prime and  $r \in \{1, 2, \dots, \frac{p+1}{2}\}$ . Then*

$$\begin{aligned} &\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 k(k-1) \cdots (k-r+1) \\ &\equiv \begin{cases} 0 \pmod{p^2} & \text{if } 4 \mid (p+3-2r), \\ (-1)^{\frac{p-3+2r}{4}} 2^{-\frac{p+1}{2}} \frac{(\frac{p+1}{2} + r)!}{\frac{p+1-2r}{4}! \frac{p+1+2r}{4}!} \pmod{p^2} & \text{if } 4 \mid (p+1-2r). \end{cases} \end{aligned}$$

*Proof.* First, according to Lemma 2.3 we can see that

$$\begin{aligned} &\left. \frac{d^r}{dx^r} P_{\frac{p+1}{2}}(x) \right|_{x=0} \\ &\equiv - \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-32)^k (2k-1)^2} \binom{2k}{k}^2 \cdot \left. \frac{d^r}{dx^r} (x-1)^k \right|_{x=0} \\ (5) \quad &= - \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-32)^k (2k-1)^2} \binom{2k}{k}^2 \cdot k(k-1) \cdots (k-r+1) (x-1)^{k-r} \Big|_{x=0} \\ &= - \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-32)^k (2k-1)^2} \binom{2k}{k}^2 k(k-1) \cdots (k-r+1) (-1)^{k-r} \\ &= (-1)^{-r+1} \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 k(k-1) \cdots (k-r+1) \pmod{p^2}. \end{aligned}$$



Second by (1), we have

$$\begin{aligned}
 & \left. \frac{d^r}{dx^r} P_{\frac{p+1}{2}}(x) \right|_{x=0} \\
 &= \frac{1}{2^{\frac{p+1}{2}}} \sum_{k=0}^{\lfloor \frac{p+1}{4} \rfloor} \frac{(-1)^k (p+1-2k)!}{k! \left(\frac{p+1}{2}-k\right)! \left(\frac{p+1}{2}-2k\right)!} \cdot \left. \frac{d^r}{dx^r} x^{\frac{p+1}{2}-2k} \right|_{x=0} \\
 &= \frac{1}{2^{\frac{p+1}{2}}} \sum_{k=0}^{\lfloor \frac{p+1}{4} \rfloor} \frac{(-1)^k (p+1-2k)!}{k! \left(\frac{p+1}{2}-k\right)! \left(\frac{p+1}{2}-2k\right)!} \cdot \left(\frac{p+1}{2}-2k\right) \left(\frac{p+1}{2}-2k-1\right) \\
 & \quad \times \cdots \left(\frac{p+1}{2}-2k-r+1\right) \cdot \begin{cases} 0 & \text{if } \frac{p+1}{2}-2k-r \neq 0, \\ \delta_{\frac{p+1}{2}-2k-r} & \text{if } \frac{p+1}{2}-2k-r = 0 \end{cases} \\
 (6) \quad &= \frac{1}{2^{\frac{p+1}{2}}} \sum_{k=0}^{\lfloor \frac{p+1}{4} \rfloor} \frac{(-1)^k (p+1-2k)!}{k! \left(\frac{p+1}{2}-k\right)! \left(\frac{p+1}{2}-2k-r\right)!} \\
 & \quad \times \begin{cases} 0 & \text{if } r \not\equiv \frac{p+1}{2} \pmod{2}, \\ \delta_{\frac{p+1}{2}-2k-r} & \text{if } r = \frac{p+1}{2} - 2k \end{cases} \\
 &= \begin{cases} 0 & \text{if } r \not\equiv \frac{p+1}{2} \pmod{2}, \\ \frac{1}{2^{\frac{p+1}{2}}} \cdot \frac{(-1)^k (p+1-2k)!}{k! \left(\frac{p+1}{2}-k\right)!} & \text{if } r = \frac{p+1}{2} - 2k. \end{cases}
 \end{aligned}$$

Therefore equating (5) with (6), we deduce that

$$\begin{aligned}
 & \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 k(k-1) \cdots (k-r+1) \\
 & \equiv \begin{cases} 0 \pmod{p^2} & \text{if } r \not\equiv \frac{p+1}{2} \pmod{2}, \\ \frac{(-1)^{r-1+k} (p+1-2k)!}{2^{\frac{p+1}{2}} k! \left(\frac{p+1}{2}-k\right)!} \pmod{p^2} & \text{if } r = \frac{p+1}{2} - 2k \end{cases} \\
 & = \begin{cases} 0 \pmod{p^2} & \text{if } 4 \mid (p+3-2r), \\ (-1)^{\frac{p-3+2r}{4}} 2^{-\frac{p+1}{2}} \frac{\left(\frac{p+1}{2}+r\right)!}{\frac{p+1-2r}{4}! \frac{p+1+2r}{4}!} \pmod{p^2} & \text{if } 4 \mid (p+1-2r). \end{cases}
 \end{aligned}$$

□

**Corollary 2.9.** *Let  $p$  be an odd prime. Then*

(a)

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{k(4pk + 4k^2 - 1)}{32^k (2k-1)^2} \binom{2k}{k}^2 \equiv \begin{cases} (-1)^{\frac{p-1}{4}} 2^{-\frac{p+1}{2}} \frac{\frac{p+3}{2}!}{\frac{p-1}{4}! \frac{p+3}{4}!} \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

(b)

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{k^2(4pk + 4k^2 - 1)}{32^k (2k-1)^2} \binom{2k}{k}^2 \equiv \begin{cases} (-1)^{\frac{p-1}{4}} 2^{-\frac{p+1}{2}} \frac{\frac{p+3}{2}!}{\frac{p-1}{4}! \frac{p+3}{4}!} \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{\frac{p+1}{4}} 2^{-\frac{p+1}{2}} \frac{\frac{p+5}{2}!}{\frac{p-3}{4}! \frac{p+5}{4}!} \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

*Proof.* The proof of part (a) is obvious by letting  $r = 1$  in Theorem 2.8. In a similar manner,  $r = 2$  in Theorem 2.8 shows that

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{(4pk + 4k^2 - 1)}{32^k (2k-1)^2} \binom{2k}{k}^2 k(k-1) \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{\frac{p+1}{4}} 2^{-\frac{p+1}{2}} \frac{\frac{p+5}{2}!}{\frac{p-3}{4}! \frac{p+5}{4}!} \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Using this fact, the result of part (a), and  $k^2 = k(k-1) + k$  we can deduce part (b).  $\square$

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A PRIVATE MATHEMATICS ACADEMY, 1-#406, 23, MAEBONG 5-GIL, DEOKJIN-GU, CHONJU, CHONBUK, 54921, REPUBLIC OF KOREA  
 Email address: ae\_ran\_kim@hotmail.com