

Congruences concerning Legendre polynomials modulo p^2

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Abstract. In this article, we extend Z. H. Sun’s congruences concerning Legendre polynomials $P_{\frac{p-1}{2}}(x)$ to $P_{\frac{p+1}{2}}(x)$ for odd prime p , which enables us to deduce some congruences resembling

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k - 1)^2} \binom{2k}{k}^2 \pmod{p^2}.$$

1 Introduction

Let $\{P_n(x)\}$ be the Legendre polynomials given by

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1 - 2xt + t^2}} \quad \text{for } |t| < 1.$$

It is well known [1], [2], [3] that

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2n - 2k)!}{k! (n - k)! (n - 2k)!} x^{n-2k} = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2 - 1)^n, \quad (1)$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x and

$$(n + 1) P_{n+1}(x) = (2n + 1) x P_n(x) - n P_{n-1}(x). \quad (2)$$

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Moreover,

$$P_n(x) = (-1)^n P_n(-x) \quad \text{and} \quad P_n(0) = \begin{cases} 0 & \text{for odd } n, \\ \frac{(-1)^{\frac{n}{2}}}{2^n} \binom{n}{\frac{n}{2}} & \text{for even } n. \end{cases} \quad (3)$$

Using the property of Legendre polynomials in Eq. (1), Z. H. Sun obtained various congruences for $P_{\frac{p-1}{2}}(x)$ modulo p^2 , where p is an odd prime and x is a rational p -integer. For example, he showed that

$$\sum_{k=0}^{p-1} \frac{1}{16^k} \binom{2k}{k}^2 \left(x^k - (-1)^{\frac{p-1}{2}} (1-x)^k \right) \equiv 0 \pmod{p^2}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{1}{16^k} \binom{2k}{k}^2 \left(x^k - \left(\frac{x}{p} \right) x^{-k} \right) \equiv 0 \pmod{p} \quad \text{for } x \not\equiv 0 \pmod{p},$$

where $\left(\frac{x}{p} \right)$ is the Jacobi symbol. Inspired by these results, we consider $P_{\frac{p+1}{2}}(x)$ and

get some congruences modulo p^2 resembling $\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2$ with odd prime p .

Especially using the relation $P_{\frac{p-1}{2}}(x)$, $P_{\frac{p+1}{2}}(x)$ and (2), we construct a congruence $P_{\frac{p-3}{2}}(x)$ modulo p^2 .

2 Main results

Proposition 2.1. *For $n \in \mathbb{N}$ we have*

$$P_n(x) = \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} \left(\frac{x-1}{2} \right)^k.$$

Proof. Refer to [1]. □

Corollary 2.2. *Let p be an odd prime and $k \in \{1, 2, \dots, \frac{p+1}{2}\}$. Then*

$$\binom{\frac{p+1}{2} + k}{2k} \equiv \frac{(p+2k+1)(p+2k-1)}{(-16)^k (2k-1)^2} \binom{2k}{k} \left(p^2 \sum_{i=1}^{k-1} \frac{1}{(2i-1)^2} - 1 \right) \pmod{p^4}.$$

Proof. We can easily know that

$$\begin{aligned}
 & \binom{\frac{p+1}{2} + k}{2k} \\
 &= \frac{\left(\frac{p+1}{2} + k\right) \left(\frac{p+1}{2} + k - 1\right) \left(\frac{p+1}{2} + k - 2\right) \cdots \left(\frac{p+1}{2} - k + 1\right)}{(2k)!} \\
 &= \frac{(p + 2k + 1)(p + 2k - 1)(p + 2k - 3) \cdots (p - 2k + 3)}{2^{2k} \cdot (2k)!} \\
 &= \frac{(p + 2k + 1)(p + 2k - 1) \cdot (p^2 - 1^2)(p^2 - 3^2) \cdots (p^2 - (2k - 3)^2)}{2^{2k} \cdot (2k)!} \\
 &\equiv \frac{(p + 2k + 1)(p + 2k - 1)}{2^{2k} \cdot (2k)!} \left[(-1)^{k-2} p^2 \left\{ 1^2 \cdot 3^2 \cdots (2k - 7)^2 (2k - 5)^2 \right. \right. \\
 &\quad \left. \left. + 1^2 \cdot 3^2 \cdots (2k - 7)^2 (2k - 3)^2 + \cdots + 3^2 \cdots (2k - 5)^2 (2k - 3)^2 \right\} \right. \\
 &\quad \left. + (-1)^{k-1} \cdot 1^2 \cdot 3^2 \cdots (2k - 5)^2 (2k - 3)^2 \right] \pmod{p^4} \\
 &= \frac{(p + 2k + 1)(p + 2k - 1)}{2^{2k} \cdot (2k)!} (-1)^k \cdot 1^2 \cdot 3^2 \cdots (2k - 5)^2 (2k - 3)^2 \\
 &\quad \times \left(p^2 \sum_{i=1}^{k-1} \frac{1}{(2i - 1)^2} - 1 \right) \pmod{p^4}
 \end{aligned}$$

then since

$$\begin{aligned}
 1^2 \cdot 3^2 \cdots (2k - 5)^2 (2k - 3)^2 &= \frac{(2k - 3)!^2}{2^2 \cdot 4^2 \cdots (2k - 6)^2 (2k - 4)^2} \\
 &= \frac{(2k - 3)!^2}{(2^{k-2})^2 \cdot (k - 2)!^2} \\
 &= \frac{(2k - 3)!}{2^{2k-4}} \cdot \frac{k^2 (k - 1)^2}{(2k - 2)(2k - 1)(2k)} \cdot \frac{(2k)!}{k!^2} \\
 &= \frac{(2k - 3)!}{2^{2k-2}} \cdot \frac{k(k - 1)}{2k - 1} \cdot \binom{2k}{k},
 \end{aligned}$$

the above identity can be written as

$$\begin{aligned}
 \binom{\frac{p+1}{2} + k}{2k} &\equiv \frac{(p + 2k + 1)(p + 2k - 1)}{2^{2k} \cdot (2k)!} (-1)^k \cdot \frac{(2k - 3)!}{2^{2k-2}} \cdot \frac{k(k - 1)}{2k - 1} \cdot \binom{2k}{k} \\
 &\quad \times \left(p^2 \sum_{i=1}^{k-1} \frac{1}{(2i - 1)^2} - 1 \right) \pmod{p^4} \\
 &= \frac{(p + 2k + 1)(p + 2k - 1)}{(-16)^k (2k - 1)^2} \binom{2k}{k} \left(p^2 \sum_{i=1}^{k-1} \frac{1}{(2i - 1)^2} - 1 \right) \pmod{p^4}. \quad \square
 \end{aligned}$$

Lemma 2.3. *Let p be an odd prime and let x be a variable. Then*

$$P_{\frac{p+1}{2}}(x) \equiv - \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{x-1}{2}\right)^k \pmod{p^2}.$$

Proof. From Proposition 2.1 and Corollary 2.2 we observe that

$$\begin{aligned} P_{\frac{p+1}{2}}(x) &= \sum_{k=0}^{\frac{p+1}{2}} \binom{\frac{p+1}{2} + k}{2k} \binom{2k}{k} \left(\frac{x-1}{2}\right)^k \\ &= 1 + \sum_{k=1}^{\frac{p+1}{2}} \binom{\frac{p+1}{2} + k}{2k} \binom{2k}{k} \left(\frac{x-1}{2}\right)^k \\ &\equiv 1 + \sum_{k=1}^{\frac{p+1}{2}} \frac{(p+2k+1)(p+2k-1)}{(-16)^k (2k-1)^2} \binom{2k}{k} \left(p^2 \sum_{i=1}^{k-1} \frac{1}{(2i-1)^2} - 1\right) \\ &\quad \times \binom{2k}{k} \left(\frac{x-1}{2}\right)^k \pmod{p^4} \\ &\equiv 1 - \sum_{k=1}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{x-1}{2}\right)^k \pmod{p^2} \\ &= - \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{x-1}{2}\right)^k, \end{aligned}$$

where we use $(p+2k+1)(p+2k-1) = (p+2k)^2 - 1 \equiv 4pk + 4k^2 - 1 \pmod{p^2}$. \square

Proposition 2.4. *For $n \in \mathbb{N}$ we have*

$$P_{\frac{p-1}{2}}(x) \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{1}{16^k} \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k \pmod{p^2}.$$

Proof. See [1]. \square

Lemma 2.5. *Let p be an odd prime and let x be a variable. Then*

$$\begin{aligned} P_{\frac{p-3}{2}}(x) &\equiv - \sum_{k=0}^{\frac{p-3}{2}} \frac{4(2px + 2p + 1)k^2 - 4p(2x-1)k + 2px - 2p - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k \\ &\quad - p \binom{p-1}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \pmod{p^2}. \end{aligned}$$

In particular, we have $P_{\frac{p-3}{2}}(1) \equiv 1 \pmod{p^2}$.

Proof. Replacing $n = \frac{p-1}{2}$ in (2) and multiplying $2(p+1)$ on both sides, we obtain

$$P_{\frac{p-3}{2}}(x) \equiv (2p+1)P_{\frac{p+1}{2}}(x) - 2pxP_{\frac{p-1}{2}}(x) \pmod{p^2}.$$

Then applying Lemma 2.3 and Proposition 2.4 to this identity, we have

$$\begin{aligned} P_{\frac{p-3}{2}}(x) &\equiv (2p+1) \left[- \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{x-1}{2}\right)^k \right] \\ &\quad - 2px \cdot \sum_{k=0}^{\frac{p-1}{2}} \frac{1}{16^k} \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k \\ &= (2p+1) \left[- \sum_{k=0}^{\frac{p-3}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{x-1}{2}\right)^k \right. \\ &\quad - \frac{4p \cdot \frac{p-1}{2} + 4 \left(\frac{p-1}{2}\right)^2 - 1}{(-16)^{\frac{p-1}{2}} \left(2 \cdot \frac{p-1}{2} - 1\right)^2} \binom{2 \cdot \frac{p-1}{2}}{\frac{p-1}{2}}^2 \left(\frac{x-1}{2}\right)^{\frac{p-1}{2}} \\ &\quad - \frac{4p \cdot \frac{p+1}{2} + 4 \left(\frac{p+1}{2}\right)^2 - 1}{(-16)^{\frac{p+1}{2}} \left(2 \cdot \frac{p+1}{2} - 1\right)^2} \binom{2 \cdot \frac{p+1}{2}}{\frac{p+1}{2}}^2 \left(\frac{x-1}{2}\right)^{\frac{p+1}{2}} \left. \right] \\ &\quad - 2px \left[\sum_{k=0}^{\frac{p-3}{2}} \frac{1}{16^k} \binom{2k}{k}^2 \left(\frac{1-x}{2}\right)^k \right. \\ &\quad \left. + \frac{1}{16^{\frac{p-1}{2}}} \binom{2 \cdot \frac{p-1}{2}}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \right] \pmod{p^2}. \end{aligned} \tag{4}$$

Here the 2nd term is

$$\begin{aligned} &\frac{4p \cdot \frac{p-1}{2} + 4 \left(\frac{p-1}{2}\right)^2 - 1}{(-16)^{\frac{p-1}{2}} \left(2 \cdot \frac{p-1}{2} - 1\right)^2} \binom{2 \cdot \frac{p-1}{2}}{\frac{p-1}{2}}^2 \left(\frac{x-1}{2}\right)^{\frac{p-1}{2}} \\ &\equiv \frac{-4p}{16^{\frac{p-1}{2}} (-4p+4)} \binom{p-1}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \\ &\equiv \frac{-p}{16^{\frac{p-1}{2}}} \binom{p-1}{\frac{p-1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p-1}{2}} \pmod{p^2} \end{aligned}$$

and the 3rd term is

$$\begin{aligned} &\frac{4p \cdot \frac{p+1}{2} + 4 \left(\frac{p+1}{2}\right)^2 - 1}{(-16)^{\frac{p+1}{2}} \left(2 \cdot \frac{p+1}{2} - 1\right)^2} \binom{2 \cdot \frac{p+1}{2}}{\frac{p+1}{2}}^2 \left(\frac{x-1}{2}\right)^{\frac{p+1}{2}} \\ &= \frac{2p(p+1) + (p+1)^2 - 1}{16^{\frac{p+1}{2}} p^2} \binom{p+1}{\frac{p+1}{2}}^2 \left(\frac{1-x}{2}\right)^{\frac{p+1}{2}} \end{aligned}$$

$$\begin{aligned}
&\equiv \frac{4p}{16^{\frac{p-1}{2}} \cdot 16p^2} \cdot \left(\frac{(p+1)p}{\frac{p+1}{2} \cdot \frac{p+1}{2}} \right)^2 \left(\frac{p-1}{2} \right)^2 \left(\frac{1-x}{2} \right)^{\frac{p+1}{2}} \\
&\equiv \frac{4p}{16^{\frac{p-1}{2}}} \left(\frac{p-1}{2} \right)^2 \left(\frac{1-x}{2} \right)^{\frac{p-1}{2}} \left(\frac{1-x}{2} \right) \\
&\equiv \frac{2p(1-x)}{16^{\frac{p-1}{2}}} \left(\frac{p-1}{2} \right)^2 \left(\frac{1-x}{2} \right)^{\frac{p-1}{2}} \pmod{p^2}
\end{aligned}$$

so that we can write Eq. (4) as

$$\begin{aligned}
P_{\frac{p-3}{2}}(x) &\equiv (2p+1) \left[- \sum_{k=0}^{\frac{p-3}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{x-1}{2} \right)^k \right. \\
&\quad \left. + \frac{p}{16^{\frac{p-1}{2}}} \left(\frac{p-1}{2} \right)^2 \left(\frac{1-x}{2} \right)^{\frac{p-1}{2}} - \frac{2p(1-x)}{16^{\frac{p-1}{2}}} \left(\frac{p-1}{2} \right)^2 \left(\frac{1-x}{2} \right)^{\frac{p-1}{2}} \right] \\
&\quad - 2px \left[\sum_{k=0}^{\frac{p-3}{2}} \frac{1}{16^k} \binom{2k}{k}^2 \left(\frac{1-x}{2} \right)^k + \frac{1}{16^{\frac{p-1}{2}}} \left(\frac{p-1}{2} \right)^2 \left(\frac{1-x}{2} \right)^{\frac{p-1}{2}} \right] \\
&\equiv - \sum_{k=0}^{\frac{p-3}{2}} \frac{(2p+1)(4pk + 4k^2 - 1)}{16^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{1-x}{2} \right)^k \\
&\quad + \frac{p}{16^{\frac{p-1}{2}}} \left(\frac{p-1}{2} \right)^2 \left(\frac{1-x}{2} \right)^{\frac{p-1}{2}} - \frac{2p(1-x)}{16^{\frac{p-1}{2}}} \left(\frac{p-1}{2} \right)^2 \left(\frac{1-x}{2} \right)^{\frac{p-1}{2}} \\
&\quad - \sum_{k=0}^{\frac{p-3}{2}} \frac{2px}{16^k} \binom{2k}{k}^2 \left(\frac{1-x}{2} \right)^k - \frac{2px}{16^{\frac{p-1}{2}}} \left(\frac{p-1}{2} \right)^2 \left(\frac{1-x}{2} \right)^{\frac{p-1}{2}} \\
&= - \sum_{k=0}^{\frac{p-3}{2}} \frac{1}{16^k} \left[\frac{(2p+1)(4pk + 4k^2 - 1)}{(2k-1)^2} + 2px \right] \binom{2k}{k}^2 \left(\frac{1-x}{2} \right)^k \\
&\quad - \frac{p}{16^{\frac{p-1}{2}}} \left(\frac{p-1}{2} \right)^2 \left(\frac{1-x}{2} \right)^{\frac{p-1}{2}} \\
&\equiv - \sum_{k=0}^{\frac{p-3}{2}} \frac{4(2px + 2p + 1)k^2 - 4p(2x-1)k + 2px - 2p - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{1-x}{2} \right)^k \\
&\quad - p \left(\frac{p-1}{2} \right)^2 \left(\frac{1-x}{2} \right)^{\frac{p-1}{2}} \pmod{p^2},
\end{aligned}$$

where we utilize $16^{\frac{p-1}{2}} \equiv \left(\frac{16}{p} \right) = 1 \pmod{p}$ by Legendre's original definition. \square

Theorem 2.6. *Let p be an odd prime and let x be a variable. Then*

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left(x^k - (-1)^{\frac{p+1}{2}} (1-x)^k \right) \equiv 0 \pmod{p^2}.$$

Proof. By (3) and Lemma 2.3 we note that

$$\begin{aligned} & \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{1-x}{2} \right)^k \\ &= \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{x-1}{2} \right)^k \\ &\equiv -P_{\frac{p+1}{2}}(x) \\ &= -(-1)^{\frac{p+1}{2}} P_{\frac{p+1}{2}}(-x) \\ &\equiv -(-1)^{\frac{p+1}{2}} \left[- \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{-x-1}{2} \right)^k \right] \\ &= (-1)^{\frac{p+1}{2}} \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left(\frac{x+1}{2} \right)^k \pmod{p^2} \end{aligned}$$

and so

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left[\left(\frac{1-x}{2} \right)^k - (-1)^{\frac{p+1}{2}} \left(\frac{x+1}{2} \right)^k \right] \equiv 0 \pmod{p^2}.$$

Now putting $x := 1 - 2y$ in the above identity, we conclude that

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left(y^k - (-1)^{\frac{p+1}{2}} (1-y)^k \right) \equiv 0 \pmod{p^2}. \quad \square$$

Theorem 2.7. *Let p be an odd prime. Then*

$$\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{(-1)^{\frac{p-3}{4}}}{2^{\frac{p+1}{2}}} \binom{\frac{p+1}{2}}{\frac{p+1}{4}} \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. When $p \equiv 1 \pmod{4}$, taking $x = \frac{1}{2}$ in Theorem 2.6 we obtain

$$0 \equiv \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{16^k (2k-1)^2} \binom{2k}{k}^2 \left(\left(\frac{1}{2} \right)^k + \left(\frac{1}{2} \right)^k \right)$$

$$= 2 \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 \pmod{p^2}$$

then since $2 \nmid p^2$, we have $\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 \equiv 0 \pmod{p^2}$. Next when $p \equiv 3 \pmod{4}$, setting $x = 0$ in Lemma 2.3 and applying Eq. (3) we get

$$\begin{aligned} \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 &= \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-16)^k (2k-1)^2} \binom{2k}{k}^2 \left(-\frac{1}{2}\right)^k \\ &\equiv -P_{\frac{p+1}{2}}(0) \\ &= -\frac{(-1)^{\frac{p+1}{4}}}{2^{\frac{p+1}{2}}} \binom{\frac{p+1}{2}}{\frac{p+1}{4}} \\ &= \frac{(-1)^{\frac{p-3}{4}}}{2^{\frac{p+1}{2}}} \binom{\frac{p+1}{2}}{\frac{p+1}{4}} \pmod{p^2}. \end{aligned} \quad \square$$

Theorem 2.8. *Let p be an odd prime and $r \in \{1, 2, \dots, \frac{p+1}{2}\}$. Then*

$$\begin{aligned} &\sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 k(k-1) \cdots (k-r+1) \\ &\equiv \begin{cases} 0 \pmod{p^2} & \text{if } 4 \mid (p+3-2r), \\ (-1)^{\frac{p-3+2r}{4}} 2^{-\frac{p+1}{2}} \frac{(\frac{p+1}{2} + r)!}{\frac{p+1-2r}{4}! \frac{p+1+2r}{4}!} \pmod{p^2} & \text{if } 4 \mid (p+1-2r). \end{cases} \end{aligned}$$

Proof. First, according to Lemma 2.3 we can see that

$$\begin{aligned} &\left. \frac{d^r}{dx^r} P_{\frac{p+1}{2}}(x) \right|_{x=0} \\ &\equiv - \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-32)^k (2k-1)^2} \binom{2k}{k}^2 \cdot \left. \frac{d^r}{dx^r} (x-1)^k \right|_{x=0} \\ &= - \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-32)^k (2k-1)^2} \binom{2k}{k}^2 \cdot k(k-1) \cdots (k-r+1) (x-1)^{k-r} \Big|_{x=0} \quad (5) \\ &= - \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{(-32)^k (2k-1)^2} \binom{2k}{k}^2 k(k-1) \cdots (k-r+1) (-1)^{k-r} \\ &= (-1)^{-r+1} \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 k(k-1) \cdots (k-r+1) \pmod{p^2}. \end{aligned}$$

Second by (1), we have

$$\begin{aligned}
 & \left. \frac{d^r}{dx^r} P_{\frac{p+1}{2}}(x) \right|_{x=0} \\
 &= \frac{1}{2^{\frac{p+1}{2}}} \sum_{k=0}^{\lfloor \frac{p+1}{4} \rfloor} \frac{(-1)^k (p+1-2k)!}{k! \left(\frac{p+1}{2}-k\right)! \left(\frac{p+1}{2}-2k\right)!} \cdot \left. \frac{d^r}{dx^r} x^{\frac{p+1}{2}-2k} \right|_{x=0} \\
 &= \frac{1}{2^{\frac{p+1}{2}}} \sum_{k=0}^{\lfloor \frac{p+1}{4} \rfloor} \frac{(-1)^k (p+1-2k)!}{k! \left(\frac{p+1}{2}-k\right)! \left(\frac{p+1}{2}-2k\right)!} \cdot \left(\frac{p+1}{2}-2k\right) \left(\frac{p+1}{2}-2k-1\right) \\
 & \quad \times \dots \left(\frac{p+1}{2}-2k-r+1\right) \cdot \begin{cases} 0 & \text{if } \frac{p+1}{2}-2k-r \neq 0, \\ \delta_{\frac{p+1}{2}-2k-r} & \text{if } \frac{p+1}{2}-2k-r = 0 \end{cases} \quad (6) \\
 &= \frac{1}{2^{\frac{p+1}{2}}} \sum_{k=0}^{\lfloor \frac{p+1}{4} \rfloor} \frac{(-1)^k (p+1-2k)!}{k! \left(\frac{p+1}{2}-k\right)! \left(\frac{p+1}{2}-2k-r\right)!} \\
 & \quad \times \begin{cases} 0 & \text{if } r \not\equiv \frac{p+1}{2} \pmod{2}, \\ \delta_{\frac{p+1}{2}-2k-r} & \text{if } r = \frac{p+1}{2} - 2k \end{cases} \\
 &= \begin{cases} 0 & \text{if } r \not\equiv \frac{p+1}{2} \pmod{2}, \\ \frac{1}{2^{\frac{p+1}{2}}} \cdot \frac{(-1)^k (p+1-2k)!}{k! \left(\frac{p+1}{2}-k\right)!} & \text{if } r = \frac{p+1}{2} - 2k. \end{cases}
 \end{aligned}$$

Therefore equating (5) with (6), we deduce that

$$\begin{aligned}
 & \sum_{k=0}^{\frac{p+1}{2}} \frac{4pk + 4k^2 - 1}{32^k (2k-1)^2} \binom{2k}{k}^2 k(k-1) \dots (k-r+1) \\
 & \equiv \begin{cases} 0 \pmod{p^2} & \text{if } r \not\equiv \frac{p+1}{2} \pmod{2}, \\ \frac{(-1)^{r-1+k} (p+1-2k)!}{2^{\frac{p+1}{2}} k! \left(\frac{p+1}{2}-k\right)!} \pmod{p^2} & \text{if } r = \frac{p+1}{2} - 2k \end{cases} \\
 & = \begin{cases} 0 \pmod{p^2} & \text{if } 4 \mid (p+3-2r), \\ (-1)^{\frac{p-3+2r}{4}} 2^{-\frac{p+1}{2}} \frac{\left(\frac{p+1}{2}+r\right)!}{\frac{p+1-2r}{4}! \frac{p+1+2r}{4}!} \pmod{p^2} & \text{if } 4 \mid (p+1-2r). \end{cases}
 \end{aligned}$$

□

Corollary 2.9. *Let p be an odd prime. Then*

(a)

$$\begin{aligned} & \sum_{k=0}^{\frac{p+1}{2}} \frac{k(4pk + 4k^2 - 1) \binom{2k}{k}^2}{32^k (2k-1)^2} \\ & \equiv \begin{cases} (-1)^{\frac{p-1}{4}} 2^{-\frac{p+1}{2}} \frac{\frac{p+3}{2}!}{\frac{p-1}{4}! \frac{p+3}{4}!} \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{k=0}^{\frac{p+1}{2}} \frac{k^2(4pk + 4k^2 - 1) \binom{2k}{k}^2}{32^k (2k-1)^2} \\ & \equiv \begin{cases} (-1)^{\frac{p-1}{4}} 2^{-\frac{p+1}{2}} \frac{\frac{p+3}{2}!}{\frac{p-1}{4}! \frac{p+3}{4}!} \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{\frac{p+1}{4}} 2^{-\frac{p+1}{2}} \frac{\frac{p+5}{2}!}{\frac{p-3}{4}! \frac{p+5}{4}!} \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Proof. The proof of part (a) is obvious by letting $r = 1$ in Theorem 2.8. In a similar manner, $r = 2$ in Theorem 2.8 shows that

$$\begin{aligned} & \sum_{k=0}^{\frac{p+1}{2}} \frac{(4pk + 4k^2 - 1) \binom{2k}{k}^2}{32^k (2k-1)^2} k(k-1) \\ & \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{\frac{p+1}{4}} 2^{-\frac{p+1}{2}} \frac{\frac{p+5}{2}!}{\frac{p-3}{4}! \frac{p+5}{4}!} \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Using this fact, the result of part (a), and $k^2 = k(k-1) + k$ we can deduce part (b). \square

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