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A Generalization of Iseki's Formula and The Jacobi Theta Function

Maher Me'meh and Ali Saraeb

Abstract. In this paper we give a generalization of Iseki's formula and use it to prove the transformation law of $\theta_1(z, \tau)$.

Keywords. Jacobi Theta Function, Dedekind Eta function, Iseki's Formula, Dedekind Sums.

2010 Mathematics Subject Classification. 11F20, 11F50, 11F55.

1. Introduction

The Dedekind Eta function, defined as

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}),$$

plays an important role in the study of Modular and Jacobi forms. Its transformation over a matrix $A \in \text{SL}_2(\mathbb{Z})$, where with no loss of generality $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c > 0$, is given by

$$\eta(A\tau) = \varepsilon(A)(-i(c\tau + d))^{1/2}\eta(\tau).$$

One obtains the Dedekind eta-character " $\varepsilon(A)$ " which is defined as, (check, [Rad12], III)

$$\varepsilon(A) = \begin{cases} \left(\frac{d}{c}\right) i^{(1-c)/2} e^{(\pi i / 12)(bd(1-c^2) + c(a+d))} & \text{if } c \text{ odd} \\ \left(\frac{c}{d}\right) e^{\pi d i / 4} e^{(\pi i / 12)(ac(1-d^2) + d(b-c))} & \text{if } d \text{ odd,} \end{cases}$$

where $(d, c) = 1$, and $\left(\frac{c}{d}\right)$ is the Legendre-Jacobi symbol given by

$$\left(\frac{c}{d}\right) = \prod_{i=1}^n \left(\frac{c}{p_i}\right)^{\alpha_i},$$

if $d = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots p_n^{\alpha_n}$, where each p_i is a prime and each $\left(\frac{c}{p_i}\right)$ is the regular Legendre symbol defined as

$$\left(\frac{c}{p_i}\right) = \begin{cases} 1 & \text{if } c \text{ is a quadratic residue mod } p_i \\ -1 & \text{if } c \text{ is a quadratic non-residue mod } p_i \\ 0 & \text{if } c \text{ is a multiple of } p_i \end{cases}$$

However, it turns out that computations done using this definition of the eta-character can get really messy. On the other hand, Sho Iseki proved in 1952 [Apo89, p.53, Theorem 3.5] the transformation law of the eta function using a functional equation which will be addressed below. Using his proof he was able to write the eta-character using Dedekind sums which proved to be much easier in terms of computations. The eta-character turns out to be

$$\varepsilon(A) = \exp \left(\pi i \left(\frac{a+d}{12c} + s(-d, c) \right) \right),$$

where

$$s(d, c) = \sum_{r=1}^{c-1} \frac{r}{c} \left(\frac{dr}{c} - \left[\frac{dr}{c} \right] - \frac{1}{2} \right)$$

is the Dedekind sum and $(d, c) = 1$.

Since the eta-character appears significantly in the transformation laws of Jacobi theta functions and Jacobi forms ([Rad12],X), we generalize Iseki's proof of the eta function and apply the generalization to Jacobi theta function $\vartheta_1(z, \tau)$, which is defined as

$$\vartheta_1(z, \tau) = -i\zeta^{1/2}q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)(1 - \zeta q^n)(1 - \zeta^{-1}q^{n-1}), \tag{1.1}$$

where $\zeta = e^{2\pi iz}$ and $q = e^{2\pi i\tau}$, $z \in \mathbb{C}$ and $\tau \in \mathcal{H}$, where \mathcal{H} is the upper half plane. We first generalize Iseki's functional equation of his Λ function defined by

$$\Lambda(\alpha, \beta, z) = \sum_{n=0}^{\infty} \left(-\log(1 - e^{-2\pi((n+\alpha)z-i\beta)}) - \log(1 - e^{-2\pi((n+1-\alpha)z+i\beta)}) \right)$$

and

$$\Lambda(\alpha, \beta, z) = \Lambda(1 - \beta, \alpha, z^{-1}) + g_0(\alpha, \beta, z)$$

to four variables using methods from Fourier analysis then we employ this tool to prove the transformation law of ϑ_1 .

2. Generalization of Iseki's formula

Theorem 2.1. *If $Re(w) > 0$, $0 < \alpha < 1$, θ is real, and $0 < \beta + \theta < 1$, then*

$$\Lambda(\alpha, \beta, w, \theta) = \Lambda(1 - \beta, \alpha, w^{-1}, -i\theta/w) + g_0(\alpha, \beta, w, \theta), \tag{2.2}$$

where

$$\Lambda(\alpha, \beta, w, \theta) = - \sum_{n=0}^{\infty} \log(1 - e^{2\pi i\theta} e^{-2\pi((n+\alpha)w-i\beta)}) + \log(1 - e^{-2\pi i\theta} e^{-2\pi((n+1-\alpha)w+i\beta)}), \tag{2.3}$$

with

$$g_0(\alpha, \beta, w, \theta) = \frac{\pi}{w} B_2(\beta + \theta) - \pi w B_2(\alpha) + 2\pi i B_1(\alpha) B_1(\beta + \theta), \tag{2.4}$$

and B_n is the n^{th} Bernoulli polynomial coming from the Taylor expansion of

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=1}^{\infty} B_n(x) \frac{t^n}{n!},$$

with an explicit formula

$$B_n(x) = \sum_{m=0}^n \frac{1}{m+1} \sum_{k=0}^m (-1)^k \binom{m}{k} (x+k)^n.$$

Proof: The proof utilizes the following well-known identities from Fourier analysis (check [Bro59, 5.2, p. 370])

$$\frac{e^{2\pi m\alpha w}}{1 - e^{2\pi mw}} + \frac{1}{2\pi w m} = \frac{1}{2\pi i} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i\alpha n}}{w m i + n}. \quad (2.5)$$

Replacing w by w^{-1} and m and $-m$, we get

$$\frac{e^{-2\pi m\alpha w^{-1}}}{1 - e^{-2\pi m w^{-1}}} - \frac{1}{2\pi w^{-1} m} = -\frac{1}{2\pi i} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i\alpha n}}{w^{-1} m i - n}. \quad (2.6)$$

We also have

$$\frac{1}{m(w m i - n)} = -\frac{1}{m n} + \frac{w}{n(n i + w m)}. \quad (2.7)$$

We first observe that (2.3) can be rewritten as follows

$$\begin{aligned} \Lambda(\alpha, \beta, w, \theta) &= \sum_{m=1}^{\infty} \frac{e^{2\pi i m \beta}}{m} \frac{e^{-2\pi m \alpha w}}{1 - e^{-2\pi m w}} e^{2\pi i m \theta} - \sum_{m=1}^{\infty} \frac{e^{-2\pi i m \beta}}{m} \frac{e^{2\pi m \alpha w}}{1 - e^{2\pi m w}} e^{-2\pi i m \theta} \\ &= - \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e^{-2\pi i m \beta}}{m} \frac{e^{2\pi m \alpha w}}{1 - e^{2\pi m w}} e^{-2\pi i m \theta}. \end{aligned} \quad (2.8)$$

Multiplying both sides of equation (2.5) by $\frac{-1}{2\pi m i} e^{-2\pi i m \beta} e^{-2\pi i m \theta}$ and then summing from $m = -\infty$ to $+\infty$, we rewrite (2.8) as follows

$$\begin{aligned} \Lambda(\alpha, \beta, w, \theta) &= -\frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{-2\pi i m \beta}}{m} \frac{e^{2\pi i n \alpha}}{w m i + n} e^{-2\pi i m \theta} + \frac{1}{2\pi w} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e^{-2\pi i m(\beta+\theta)}}{m^2} \\ &= -\frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{-2\pi i m \beta}}{m} \frac{e^{-2\pi i n \alpha}}{w m i - n} e^{-2\pi i m \theta} + \frac{1}{2\pi w} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e^{2\pi i m(\beta+\theta)}}{m^2} \\ &= -\frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} (A_m(\alpha, \beta, w, \theta)) + \frac{1}{2\pi w} F_2(\beta + \theta), \end{aligned} \quad (2.9)$$

where

$$A_m(\alpha, \beta, w, \theta) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{-2\pi i m \beta}}{m} \frac{e^{-2\pi i n \alpha}}{w m i - n} e^{-2\pi i m \theta}, \quad (2.10)$$

and

$$F_n(x) = \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e^{2\pi i m x}}{m^n} = \begin{cases} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e^{-2\pi i m x}}{m^n} & \text{if } n \text{ is even.} \\ - \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e^{-2\pi i m x}}{m^n} & \text{if } n \text{ is odd.} \end{cases} \quad (2.11)$$

Using (2.7), (2.10) becomes as follows

$$A_m(\alpha, \beta, w, \theta) = -\frac{e^{-2\pi im(\beta+\theta)}}{m} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{-2\pi in\alpha}}{n} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e^{-2\pi im\beta} \frac{e^{-2\pi in\alpha}}{n(\frac{ni}{w} + m)} e^{-2\pi im\theta}.$$

By using $F_1(x) = -2\pi i B_1(x)$, $F_2(x) = \frac{-(2\pi i)^2}{2!} B_2(x)$ [Apo89, theorem 3.5, p:56], (2.7), (2.10), and (2.11) and by carefully manipulating the signs of m and n in the summands, we observe that

$$\begin{aligned} & -\frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} A_{m,n}(\alpha, \beta, w, \theta) \\ &= \frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e^{-2\pi im(\beta+\theta)}}{m} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{-2\pi in\alpha}}{n} - \frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e^{-2\pi im(\beta+\theta)} \frac{e^{-2\pi in\alpha}}{n(\frac{ni}{w} + m)} \\ &= \frac{1}{2\pi i} F_1(\beta + \theta) F_1(\alpha) - \frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e^{-2\pi in(\beta+\theta)} \frac{e^{-2\pi im\alpha}}{m(\frac{mi}{z} + n)} \\ &= 2\pi i B_1(\beta + \theta) B_1(\alpha) - \frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e^{2\pi in(\beta+\theta)} \frac{e^{-2\pi im\alpha}}{m(\frac{mi}{w} - n)} \\ &= 2\pi i B_1(\beta + \theta) B_1(\alpha) + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e^{-2\pi m(1-\beta-\theta)w^{-1}}}{1 - e^{-2\pi mw^{-1}}} \frac{e^{-2\pi im\alpha}}{m} - \frac{w}{2\pi} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e^{-2\pi im\alpha}}{m^2} \\ &= 2\pi i B_1(\beta + \theta) B_1(\alpha) - \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e^{-2\pi im\alpha}}{m} \frac{e^{-2\pi m(1-\beta)w^{-1}}}{1 - e^{2\pi mw^{-1}}} e^{2\pi im(-i\theta w^{-1})} - \frac{w}{2\pi} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e^{-2\pi im\alpha}}{m^2} \\ &= 2\pi i B_1(\beta + \theta) B_1(\alpha) + \Lambda(1 - \beta, \alpha, w^{-1}, -i\theta/w) - \pi w B_2(\alpha). \tag{2.12} \end{aligned}$$

Plugging (2.12) in (2.9), we get that

$$\Lambda(\alpha, \beta, w, \theta) = \Lambda(1 - \beta, \alpha, w^{-1}, -i\theta/w) + \frac{\pi}{w} B_2(\beta + \theta) - \pi w B_2(\alpha) + 2\pi i B_1(\beta + \theta) B_1(\alpha).$$

This completes the proof of Theorem 2.1.

We now use Theorem 2.1. to prove the transformation law for ϑ_1 under the elements of the full modular group $SL_2(\mathbb{Z})$.

Theorem 2.2. For $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $c > 0$, we have

$$\vartheta_1\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = \varepsilon_1(A) (-i(c\tau + d))^{1/2} e^{\frac{\pi icz^2}{c\tau + d}} \vartheta_1(z, \tau). \tag{2.13}$$

Here ε appears in the transformation law of the Dedekind eta function as mentioned in the introduction, where

$$\varepsilon_1(A) = -i\varepsilon^3 = -i \cdot \exp\left(3\pi i \left(\frac{a+d}{12c} + s(-d, c)\right)\right).$$

and

$$s(h, k) = \sum_{r=1}^{k-1} \frac{r}{k} \left(\frac{hr}{k} - \left[\frac{hr}{k} \right] - \frac{1}{2} \right)$$

is the Dedekind sum for $k > 0$ and $(k, h) = 1$.

Proof. We will start by deriving equivalent functional equations to (2.13). Firstly note that (2.13) is equivalent to

$$\log \left(\vartheta_1 \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) \right) = \log(\varepsilon_1(A)) + \frac{1}{2} \log(-i(c\tau + d)) + \frac{\pi icz^2}{c\tau + d} + \log(\vartheta_1(z, \tau)) \quad (2.14)$$

where the logarithm is taken over the principle branch.

Note that using the definition of Dedekind eta function, we have

$$\prod_{n=1}^{\infty} (1 - q^n) = \eta(\tau) \cdot e^{-\frac{\pi i \tau}{12}}.$$

Hence (1.1) becomes

$$\vartheta_1(z, \tau) = -i\zeta^{1/2} q^{1/8} \left(\eta(\tau) e^{-\frac{\pi i \tau}{12}} \right) \prod_{n=1}^{\infty} (1 - \zeta q^n) (1 - \zeta^{-1} q^{n-1}), \quad (2.15)$$

where again $\zeta = e^{2\pi iz}$ and $q = e^{2\pi i \tau}$.

As a result of (2.15), (2.14) is equivalent to

$$\begin{aligned} & \log \left(-i e^{\frac{\pi iz}{c\tau + d}} e^{\frac{\pi i}{4} \left(\frac{a\tau + b}{c\tau + d} \right)} \right) + \log \left(\eta \left(\frac{a\tau + b}{c\tau + d} \right) \right) - \frac{\pi i}{12} \left(\frac{a\tau + b}{c\tau + d} \right) \\ & + \sum_{n=1}^{\infty} \log \left(1 - e^{\frac{2\pi iz}{c\tau + d}} e^{2n\pi i \left(\frac{a\tau + b}{c\tau + d} \right)} \right) + \sum_{n=1}^{\infty} \log \left(1 - e^{\frac{-2\pi iz}{c\tau + d}} e^{2(n-1)\pi i \left(\frac{a\tau + b}{c\tau + d} \right)} \right) \\ & = \log \left(-i \cdot \exp \left(3\pi i \left(\frac{a+d}{12c} + s(-d, c) \right) \right) \right) + \frac{1}{2} \log(-i(c\tau + d)) + \frac{\pi icz^2}{c\tau + d} \\ & + \log \left(-i e^{\pi iz} e^{\frac{\pi i \tau}{4}} \right) + \log(\eta(\tau)) - \frac{\pi i \tau}{12} + \sum_{n=1}^{\infty} \log \left(1 - e^{2\pi iz} e^{2n\pi i \tau} \right) + \sum_{n=1}^{\infty} \log \left(1 - e^{-2\pi iz} e^{2(n-1)\pi i \tau} \right). \end{aligned}$$

Using the fact that, (check [Apo89, 3.4, p.52])

$$\log \left(\eta \left(\frac{a\tau + b}{c\tau + d} \right) \right) = \pi i \left(\frac{a+d}{12c} \right) + \pi i s(-d, c) + \frac{1}{2} \log(-i(c\tau + d)) + \log(\eta(\tau)),$$

we obtain

$$\begin{aligned} & \frac{\pi iz}{c\tau + d} + \frac{\pi i}{6} \left(\frac{a\tau + b}{c\tau + d} \right) + \sum_{n=1}^{\infty} \log \left(1 - e^{\frac{2\pi iz}{c\tau + d}} e^{2n\pi i \left(\frac{a\tau + b}{c\tau + d} \right)} \right) + \sum_{n=1}^{\infty} \log \left(1 - e^{\frac{-2\pi iz}{c\tau + d}} e^{2(n-1)\pi i \left(\frac{a\tau + b}{c\tau + d} \right)} \right) \\ & = \log(-i) + 2\pi i \left(\frac{a+d}{12c} \right) + 2\pi i s(-d, c) + \frac{\pi icz^2}{c\tau + d} + \pi iz + \frac{\pi i \tau}{6} + \sum_{n=1}^{\infty} \log \left(1 - e^{2\pi iz} e^{2n\pi i \tau} \right) \\ & + \sum_{n=1}^{\infty} \log \left(1 - e^{-2\pi iz} e^{2(n-1)\pi i \tau} \right). \end{aligned}$$

Relocating the terms we get

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \log(1 - e^{\frac{2\pi iz}{c\tau+d}} e^{2n\pi i(\frac{a\tau+b}{c\tau+d})}) + \sum_{n=1}^{\infty} \log(1 - e^{-\frac{2\pi iz}{c\tau+d}} e^{2(n-1)\pi i(\frac{a\tau+b}{c\tau+d})}) \\
 &= \sum_{n=1}^{\infty} \log(1 - e^{2\pi iz} e^{2n\pi i\tau}) + \sum_{n=1}^{\infty} \log(1 - e^{-2\pi iz} e^{2(n-1)\pi i\tau}) + \log(-i) + 2\pi i(\frac{a+d}{12c}) + 2\pi is(-d, c) \\
 &+ \frac{\pi icz^2}{c\tau+d} - \frac{\pi iz}{c\tau+d} + \pi iz + \frac{\pi i}{6} \left(\tau - \frac{a\tau+b}{c\tau+d} \right). \tag{2.16}
 \end{aligned}$$

Now we prove an equivalence of (2.16) by introducing a classical change of variable, we set

$$-i(c\tau + d) = v \quad a = H, \quad c = k \text{ and } h = -d,$$

where it is clear that $Hh \equiv -1 \pmod{k}$.

Under this change of variable, we have

$$\tau = \frac{iv + h}{k} \quad \text{and} \quad \frac{a\tau + b}{c\tau + d} = \frac{1}{k} \left(H + \frac{i}{v} \right).$$

Moreover,

$$\frac{\pi i}{6} \left(\tau - \frac{a\tau + b}{c\tau + d} \right) = -2\pi i \left(\frac{a+d}{12c} \right) - \frac{\pi}{6k} \left(v - \frac{1}{v} \right).$$

Plugging in (2.16), where $\log(-i) = -\frac{\pi i}{2}$, we obtain that (2.16) and hence our functional equation (2.13) is equivalent to

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \log(1 - e^{\frac{2\pi z}{v}} e^{\frac{2n\pi i}{k}(H+\frac{i}{v})}) + \log(1 - e^{-\frac{2\pi z}{v}} e^{\frac{2(n-1)\pi i}{k}(H+\frac{i}{v})}) = \sum_{n=1}^{\infty} \log(1 - e^{2\pi iz} e^{\frac{2n\pi i}{k}(h+iv)}) \\
 &+ \log(1 - e^{-2\pi iz} e^{\frac{2(n-1)\pi i}{k}(h+iv)}) + 2\pi is(h, k) - \frac{\pi}{6k} \left(v - \frac{1}{v} \right) - \frac{\pi i}{2} + \frac{\pi kz^2}{v} + \pi iz - \frac{\pi z}{v}, \tag{2.17}
 \end{aligned}$$

which is what we desire to prove.

We now follow Iseki's proof closely and we let

$$\beta = \frac{\phi}{k} \text{ where } 0 \leq \phi \leq k - 1.$$

In order to use Theorem 2.1, we need $0 < \beta + \theta < 1$ for which one can easily prove that it is equivalent to having $0 < \theta < \frac{1}{k}$.

We will use Theorem 2.2 to prove the functional equation (2.17) for θ real and in the open interval $(0, \frac{1}{k})$. Also, we will divide the proof into two cases, for $k = 1$ and $k > 1$, and then use analytic continuation to extend the result to the whole plane.

$$\Lambda(\alpha, \beta, w, \theta) = \Lambda(1 - \beta, \alpha, 1/w, -i\theta/w) + g_0(\alpha, \beta, w, \theta). \tag{2.18}$$

For $k = 1$ we obtain from (2.17)

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \log(1 - e^{\frac{2\pi z}{v}} e^{2n\pi i(H+\frac{i}{v})}) + \log(1 - e^{-\frac{2\pi z}{v}} e^{2(n-1)\pi i(H+\frac{i}{v})}) \\
 &= \sum_{n=1}^{\infty} \log(1 - e^{2\pi iz} e^{2n\pi i(h+iv)}) + \log(1 - e^{-2\pi iz} e^{2(n-1)\pi i(h+iv)}) - \frac{\pi}{6} \left(v - \frac{1}{v} \right) - \frac{\pi i}{2} + \frac{\pi z^2}{v} - \frac{\pi z}{v} + \pi iz.
 \end{aligned}$$

Note that $e^{2n\pi i H} = 1$, so we end up with

$$\begin{aligned} & \sum_{n=1}^{\infty} \log(1 - e^{\frac{2\pi z}{v}} e^{\frac{-2n\pi}{v}}) + \log(1 - e^{-\frac{2\pi z}{v}} e^{\frac{-2(n-1)\pi}{v}}) \\ &= \sum_{n=1}^{\infty} \log(1 - e^{2\pi iz} e^{-2n\pi v}) + \sum_{n=1}^{\infty} \log(1 - e^{-2\pi iz} e^{-2(n-1)\pi v}) - \frac{\pi}{6} \left(v - \frac{1}{v} \right) - \frac{\pi i}{2} + \frac{\pi z^2}{v} + \pi iz - \frac{\pi z}{v}. \end{aligned} \quad (2.19)$$

Setting $\beta = 0$ and $\alpha \rightarrow 1$, from (2.18), we see that

$$\begin{aligned} & - \sum_{n=0}^{\infty} \log(1 - e^{-2\pi i\theta} e^{-2\pi(n)w}) + \log(1 - e^{2\pi i\theta} e^{-2\pi(n+1)w}) \\ &= - \sum_{n=0}^{\infty} \log(1 - e^{-2\pi \frac{\theta}{w}} e^{\frac{-2\pi(n)}{w}}) + \log(1 - e^{2\pi \frac{\theta}{w}} e^{\frac{-2\pi(n+1)}{w}}) - \frac{\pi w}{6} + \frac{\pi}{w} (\theta^2 - \theta + \frac{1}{6}) - \frac{\pi i}{2} + \pi i\theta \end{aligned}$$

Relocating the terms,

$$\begin{aligned} & \sum_{n=1}^{\infty} \log(1 - e^{\frac{2\pi\theta}{w}} e^{\frac{-2\pi n}{w}}) + \sum_{n=1}^{\infty} \log(1 - e^{-\frac{2\pi\theta}{w}} e^{\frac{-2\pi(n-1)}{w}}) \\ &= \sum_{n=1}^{\infty} \log(1 - e^{2\pi i\theta} e^{-2\pi n w}) + \sum_{n=1}^{\infty} \log(1 - e^{-2\pi i\theta} e^{-2\pi(n-1)w}) \\ &= -\frac{\pi}{6} \left(w - \frac{1}{w} \right) + \frac{\pi\theta^2}{w} - \frac{\pi\theta}{w} - \frac{\pi i}{2} + \pi i\theta. \end{aligned} \quad (2.20)$$

This is exactly (2.19) if we let $\theta = z$ and $w = v$. This proves the transformation law when $k = 1$.

For $k > 1$ we let

$$\alpha = \frac{\mu}{k} \quad \text{where } 1 \leq \mu \leq k-1,$$

and writing $h\mu = qk + \phi$ we choose

$$\beta = \frac{\phi}{k} \quad \text{where } 1 \leq \phi \leq k-1.$$

Note that $\phi \equiv h\mu \pmod{k}$ so $-H\phi \equiv -Hh\mu \equiv \mu \pmod{k}$, and hence $-H\phi/k \equiv \mu/k \pmod{1}$. Therefore

$$\alpha = \mu/k \equiv -H\phi/k \pmod{1}$$

$$\beta = \phi/k \equiv h\mu/k \pmod{1}.$$

Plugging in (2.18) where again $w = v$ and $\theta = z$ we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \log(1 - e^{\frac{-2\pi z}{v}} e^{-2\pi((n+\beta)v^{-1}+i\alpha)}) + \sum_{n=0}^{\infty} \log(1 - e^{\frac{2\pi z}{v}} e^{-2\pi((n+1-\beta)v^{-1}-i\alpha)}) \\ &= \sum_{n=0}^{\infty} \log(1 - e^{2\pi iz} e^{-2\pi((n+\alpha)v-i\beta)}) + \sum_{n=0}^{\infty} \log(1 - e^{-2\pi iz} e^{-2\pi((n+1-\alpha)v+i\beta)}) \\ & - \pi v \left(\alpha^2 - \alpha + \frac{1}{6} \right) + \frac{\pi}{v} \left((\beta+z)^2 - (\beta+z) + \frac{1}{6} \right) + 2\pi i \left(\alpha - \frac{1}{2} \right) \left(\beta - \frac{1}{2} \right) + 2\pi zi \left(\alpha - \frac{1}{2} \right). \end{aligned}$$

Using $\alpha \equiv -H\phi/k \pmod{1}$ and $\beta \equiv h\mu/k \pmod{1}$, we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \log(1 - e^{\frac{-2\pi z}{v}} e^{-2\pi((n+\phi/k)v^{-1}-i\frac{H\phi}{k})}) + \sum_{n=0}^{\infty} \log(1 - e^{\frac{2\pi z}{v}} e^{-2\pi((n+1-\phi/k)v^{-1}+i\frac{H\phi}{k})}) \\ &= \sum_{n=0}^{\infty} \log(1 - e^{2\pi iz} e^{-2\pi((n+\mu/k)v-i\frac{h\mu}{k})}) + \sum_{n=0}^{\infty} \log(1 - e^{-2\pi iz} e^{-2\pi((n+1-\mu/k)v+i\frac{h\mu}{k})}) \end{aligned}$$

$$-\pi v \left(\left(\frac{\mu}{k} \right)^2 - \frac{\mu}{k} + \frac{1}{6} \right) + \frac{\pi}{v} \left(\left(\frac{\phi}{k} + z \right)^2 - \left(\frac{\phi}{k} + z \right) + \frac{1}{6} \right) + 2\pi i \left(\frac{\mu}{k} - \frac{1}{2} \right) \left(\frac{\phi}{k} - \frac{1}{2} \right) + 2\pi z i \left(\frac{\mu}{k} - \frac{1}{2} \right). \tag{2.21}$$

Note that $\log(1 - e^{-2\pi(x+mi)}) = \log(1 - e^{-2\pi x})$, i.e it's periodic of period i so the above can be written

$$\begin{aligned} & \sum_{n=0}^{\infty} \log(1 - e^{-\frac{2\pi z}{v}} e^{-2\pi(\frac{(nk+\phi)(\frac{1}{v}-iH)}{k})}) + \sum_{n=0}^{\infty} \log(1 - e^{-\frac{2\pi z}{v}} e^{-2\pi(\frac{(nk+k+\phi)(\frac{1}{v}-iH)}{k})}) \\ &= \sum_{n=0}^{\infty} \log(1 - e^{2\pi iz} e^{-2\pi(\frac{(nk+\mu)(v-ih)}{k})}) + \sum_{n=0}^{\infty} \log(1 - e^{-2\pi iz} e^{-2\pi(\frac{(nk+k+\mu)(v-ih)}{k})}) \\ &-\pi v \left(\left(\frac{\mu}{k} \right)^2 - \frac{\mu}{k} + \frac{1}{6} \right) + \frac{\pi}{v} \left(\left(\frac{\phi}{k} \right)^2 + 2z\left(\frac{\phi}{k}\right) + z^2 - \left(\frac{\phi}{k} + z \right) + \frac{1}{6} \right) + 2\pi i \left(\frac{\mu}{k} - \frac{1}{2} \right) \left(\frac{\phi}{k} - \frac{1}{2} \right) + 2\pi z i \left(\frac{\mu}{k} - \frac{1}{2} \right) \end{aligned}$$

Now sum both sides on μ from $\mu = 1, 2, \dots, k - 1$ and also notice that

$$\{nk + \mu, n = 0, 1, 2, \dots; \mu = 1, 2, \dots, k - 1\} = \{r : r \not\equiv 0 \pmod{k}\},$$

and the same goes for the set of number $nk + k - \mu$, and since $\phi \equiv h\mu \pmod{k}$ as μ runs over the number $1, 2, \dots, k - 1$ so does ϕ but in some other order. Hence we get

$$\begin{aligned} & \sum_{\substack{r=1 \\ r \not\equiv 0 \pmod{k}}}^{\infty} \log(1 - e^{-\frac{2\pi z}{v}} e^{-2\pi r(\frac{(\frac{1}{v}-iH)}{k})}) + \sum_{\substack{r=1 \\ r \not\equiv 0 \pmod{k}}}^{\infty} \log(1 - e^{-\frac{2\pi z}{v}} e^{-2\pi r(\frac{(\frac{1}{v}-iH)}{k})}) \\ &= \sum_{\substack{r=1 \\ r \not\equiv 0 \pmod{k}}}^{\infty} \log(1 - e^{2\pi iz} e^{-2\pi r(\frac{(v-ih)}{k})}) + \sum_{\substack{r=1 \\ r \not\equiv 0 \pmod{k}}}^{\infty} \log(1 - e^{-2\pi iz} e^{-2\pi r(\frac{(v-ih)}{k})}) \\ &-\pi v \sum_{\mu=1}^{k-1} \left(\left(\frac{\mu}{k} \right)^2 - \frac{\mu}{k} + \frac{1}{6} \right) + \frac{\pi}{v} \sum_{\mu=1}^{k-1} \left(\left(\frac{\phi}{k} \right)^2 + 2z\left(\frac{\phi}{k}\right) + z^2 - \left(\frac{\phi}{k} + z \right) + \frac{1}{6} \right) \\ &+ 2\pi i \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \frac{1}{2} \right) \left(\frac{\phi}{k} - \frac{1}{2} \right) + 2\pi z i \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \frac{1}{2} \right). \end{aligned} \tag{2.22}$$

Checking [Apo89, 3.7, p: 61], one can see that

$$\sum_{\mu=1}^{\infty} \left(\frac{\mu}{k} - \frac{1}{2} \right) \left(\frac{\phi}{k} - \frac{1}{2} \right) = \sum_{\mu=1}^{\infty} \left(\frac{\mu}{k} \right) \left(\frac{\phi}{k} - \frac{1}{2} \right) = s(h, k),$$

so (22) transforms into

$$\begin{aligned} & \sum_{\substack{r=1 \\ r \not\equiv 0 \pmod{k}}}^{\infty} \log(1 - e^{-\frac{2\pi z}{v}} e^{2\pi ir(\frac{(\frac{i}{v}+H)}{k})}) + \sum_{\substack{r=1 \\ r \not\equiv 0 \pmod{k}}}^{\infty} \log(1 - e^{-\frac{2\pi z}{v}} e^{2\pi ir(\frac{(\frac{i}{v}+H)}{k})}) \\ &= \sum_{\substack{r=1 \\ r \not\equiv 0 \pmod{k}}}^{\infty} \log(1 - e^{2\pi iz} e^{2\pi ir(\frac{(vi+h)}{k})}) + \sum_{\substack{r=1 \\ r \not\equiv 0 \pmod{k}}}^{\infty} \log(1 - e^{-2\pi iz} e^{2\pi ir(\frac{(vi+h)}{k})}) \\ &-\pi v \sum_{\mu=1}^{k-1} \left(\left(\frac{\mu}{k} \right)^2 - \frac{\mu}{k} + \frac{1}{6} \right) + \frac{\pi}{v} \sum_{\mu=1}^{k-1} \left(\left(\frac{\phi}{k} \right)^2 + 2z\left(\frac{\phi}{k}\right) + z^2 - \left(\frac{\phi}{k} + z \right) + \frac{1}{6} \right) \\ &+ 2\pi i s(h, k) + 2\pi z i \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \frac{1}{2} \right). \end{aligned} \tag{2.23}$$

Notice that the four sums resemble the desired form, so we intend to look at the residues g_0 , where

$$\begin{aligned}
& -\pi v \sum_{\mu=1}^{k-1} \left(\left(\frac{\mu}{k} \right)^2 - \frac{\mu}{k} + \frac{1}{6} \right) + \frac{\pi}{v} \sum_{\mu=1}^{k-1} \left(\left(\frac{\phi}{k} \right)^2 + 2z \left(\frac{\phi}{k} \right) + z^2 - \left(\frac{\phi}{k} + z \right) + \frac{1}{6} \right) \\
& + 2\pi i s(h, k) + 2\pi z i \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \frac{1}{2} \right) \\
& = -\pi v \left(\frac{1}{k^2} \cdot \frac{k(k-1)(2k-1)}{6} \right) + \frac{\pi}{v} \left(\frac{1}{k} \cdot \frac{k(k-1)}{2} \right) - \pi v \left(\frac{k-1}{6} \right) \\
& + 2\pi i s(h, k) + \frac{\pi}{v} \left(\frac{1}{k^2} \cdot \frac{k(k-1)(2k-1)}{6} + (2z-1) \left(\frac{k-1}{2} \right) + (z^2 - z + \frac{1}{6})(k-1) \right). \tag{2.24}
\end{aligned}$$

Note that

$$\sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \frac{1}{2} \right) = \frac{1}{k} \left(\frac{k(k-1)}{2} \right) - \frac{(k-1)}{2} = 0,$$

so the last term in (2.23) cancels. After some simplification of the terms, (2.24) becomes

$$\begin{aligned}
& -\frac{\pi}{6k}(k-1)(2k-1) \left(v - \frac{1}{v} \right) + \frac{\pi}{3}(k-1) \left(v - \frac{1}{v} \right) + \frac{k\pi z^2}{v} - \frac{\pi z^2}{v} \\
& = \frac{\pi}{6} \left(v - \frac{1}{v} \right) - \frac{\pi}{6k} \left(v - \frac{1}{v} \right) + \frac{k\pi z^2}{v} - \frac{\pi z^2}{v}.
\end{aligned}$$

Hence (2.23) is equivalent to

$$\begin{aligned}
& \sum_{\substack{r=1 \\ r \neq 0 \pmod{k}}}^{\infty} \log(1 - e^{-\frac{2\pi z}{v}} e^{2\pi i r (\frac{i}{k} + H)}) + \sum_{\substack{r=1 \\ r \neq 0 \pmod{k}}}^{\infty} \log(1 - e^{-\frac{2\pi z}{v}} e^{2\pi i r (\frac{i}{k} + H)}) \\
& = \sum_{\substack{r=1 \\ r \neq 0 \pmod{k}}}^{\infty} \log(1 - e^{2\pi i z} e^{2\pi i r (\frac{vi+h}{k})}) + \sum_{\substack{r=1 \\ r \neq 0 \pmod{k}}}^{\infty} \log(1 - e^{-2\pi i z} e^{2\pi i r (\frac{vi+h}{k})}) \\
& + 2\pi i s(h, k) + \frac{\pi}{6} \left(v - \frac{1}{v} \right) - \frac{\pi}{6k} \left(v - \frac{1}{v} \right) + \frac{k\pi z^2}{v} - \frac{\pi z^2}{v}. \tag{2.25}
\end{aligned}$$

Adding equation (2.25) to equation (2.20), which corresponds to the case when $k = 1$:

$$\begin{aligned}
& \sum_{n=1}^{\infty} \log(1 - e^{-\frac{2\pi z}{v}} e^{-\frac{2\pi n}{v}}) + \sum_{n=1}^{\infty} \log(1 - e^{-\frac{2\pi z}{v}} e^{-\frac{2\pi(n-1)}{v}}) \\
& = \sum_{n=1}^{\infty} \log(1 - e^{-2\pi i z} e^{-2\pi(n-1)v}) + \sum_{n=1}^{\infty} \log(1 - e^{2\pi i z} e^{-2\pi n v}) - \frac{\pi}{6} \left(v - \frac{1}{v} \right) + \frac{\pi z^2}{v} - \frac{\pi z}{v} - \frac{\pi i}{2} + \pi i z.
\end{aligned}$$

Adding equation (2.20) accounts for the missing r where $r \equiv 0 \pmod{k}$ if we write $r = mk$, then the functional equation becomes

$$\begin{aligned}
& \sum_{r=1}^{\infty} \log(1 - e^{-\frac{2\pi z}{v}} e^{2\pi i r (\frac{H+i}{k})}) + \sum_{r=1}^{\infty} \log(1 - e^{-\frac{2\pi z}{v}} e^{2\pi i (r-1) (\frac{H+i}{k})}) \\
& = \sum_{r=1}^{\infty} \log(1 - e^{2\pi i z} e^{2\pi i r (\frac{h+iv}{k})}) + \sum_{r=1}^{\infty} \log(1 - e^{-2\pi i z} e^{2\pi i (r-1) (\frac{h+iv}{k})}) \\
& + 2\pi i s(h, k) - \frac{\pi}{6k} \left(v - \frac{1}{v} \right) + \frac{\pi k z^2}{v} - \frac{\pi z}{v} + \pi i z - \frac{\pi i}{2}.
\end{aligned}$$

This is exactly (2.17), and this completes the proof of theorem 2 for $0 < \theta < \frac{1}{k}$ where $\theta = z$. However by considering equation (2.13), both sides are entire in z , and the set $E = (0, \frac{1}{k})$ clearly has a limit point. Hence by proving that the functional equation holds for $\theta \in E$ then it holds for all $z \in \mathbb{C}$, and hence our proof is done.

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