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# A note on Fourier eigenfunctions in four dimensions

Daniel Lautzenheiser

**Abstract.** In this note, we exhibit a weakly holomorphic modular form for use in constructing a Fourier eigenfunction in four dimensions. Such auxiliary functions may be of use to the D4 checkerboard lattice and the four dimensional sphere packing problem.

**Keywords.** sphere packing, Fourier analysis, modular forms.

**2010 Mathematics Subject Classification.** 52C26,52C17

## 1. Introduction

The sphere packing density is the proportion of  $\mathbb{R}^d$  occupied by non-overlapping unit balls. In recent years, the sphere packing problem of finding the most dense arrangement of spheres in  $\mathbb{R}^d$  has regained interest. This problem has been recently solved in 8 and 24 dimensions [Via17, CKM17] by means of constructing a specialized radial Schwartz function using Fourier and complex analytic methods. The sphere packing problem has otherwise only been solved in dimensions 1,2, and 3 [Tot43, Hal05]. A recent work [CLS] conjectures conditions for which the sphere packing problem in dimension 4 may be solved. In this paper, we present a function for possible use toward this conjecture.

If  $\Lambda$  is a lattice in  $\mathbb{R}^d$  with minimal nonzero vector length  $\rho$ , then a sphere packing associated to  $\Lambda$  may be defined by placing spheres of radius  $\rho/2$  at each lattice point. In this case, there is one sphere for each copy of the lattice fundamental cell  $\mathbb{R}^d/\Lambda$  and the sphere packing density is the ratio

$$\frac{\text{Vol}(B_{\rho/2}^d)}{\text{Vol}(\mathbb{R}^d/\Lambda)}.$$

Let  $f : \mathbb{R}^d \rightarrow \mathbb{C}$  be integrable with Fourier transform

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} dx, \quad \xi \in \mathbb{R}^d.$$

If  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is a Schwartz function, Poisson summation enforces

$$\sum_{x \in \Lambda} f(x) = \frac{1}{\text{Vol}(\mathbb{R}^d/\Lambda)} \sum_{y \in \Lambda^*} \widehat{f}(y)$$

where  $\Lambda^*$  is the dual lattice. We can modify Poisson summation to form an inequality

$$f(0) \geq \sum_{x \in \Lambda} f(x) = \frac{1}{\text{Vol}(\mathbb{R}^d/\Lambda)} \sum_{y \in \Lambda^*} \widehat{f}(y) \geq \frac{\widehat{f}(0)}{\text{Vol}(\mathbb{R}^d/\Lambda)}$$

if  $f$  is positive definite ( $\widehat{f} \geq 0$ ) and  $f(x) \leq 0$  for all  $\|x\| \geq \rho$ . This yields  $\text{Vol}(\mathbb{R}^d/\Lambda) \geq 1$  if  $\widehat{f}(0) = f(0)$ . In light of the lattice sphere packing density, the existence of an auxiliary function  $f$  turning the above inequalities into equalities shows that the  $\Lambda$  lattice sphere packing density is  $\leq \text{Vol}(B_{\rho/2}^d)$ . This argument, first put forth in [CoEl03], converts the sphere packing problem into an

analysis problem. To construct a *magic function*  $f$  in which the above inequalities are tight ( $f$  and  $\widehat{f}$  also vanish on nonzero lattice points), it is typical to write  $f = f_+ + f_-$  where  $\widehat{f_+} = f_+$  and  $\widehat{f_-} = -f_-$  and  $f_+, f_-$  have sign radius

$$r_f = \inf\{M \geq 0 : f(\{|x| \geq M\}) \subseteq [0, \infty)\}$$

equal to  $\rho$ . Minimizing the quantity  $(r_f r_{\widehat{f}})^{1/2}$  is a type of uncertainty principle which is currently only solved in dimension  $d = 12$  when  $\widehat{f} = f$  and dimensions  $d \in \{1, 8, 24\}$  when  $\widehat{f} = -f$ , with the cases  $d \in \{8, 24\}$  corresponding to the  $-1$  eigenfunctions constructed in the solving of the sphere packing problems in dimensions 8 and 24 [Lo83, BCK10, GSS17, CoGo19, GoRa20].

We construct a radial Schwartz function  $f_+ : \mathbb{R}^4 \rightarrow \mathbb{R}$  such that  $\widehat{f_+} = f_+$  with sign radius  $r_{f_+} = \sqrt{2}$ , which is also the minimal vector length in the  $D_4$  checkerboard lattice [CoSl13].  $f_+$  is not sharp in the sense of minimizing the sign radius since numerical results in [CoGo19] demonstrate auxiliary  $f$  with  $r_f < 0.97$ . It is possible, in light of the slackness conditions imposed in conjecture 6.1 of [CLS] that  $f_+$  may be of use toward the sphere packing problem in dimension 4.

To make this note more self-contained, we will include some of the details and arguments from [Via17, CKM17, CoGo19, RoWa20, RaVi19].

## 2. A $+1$ eigenfunction in four dimensions

We let  $q = e^{2\pi iz}$  with  $\text{Im}(z) > 0$ . Define the Jacobi theta series

$$\begin{aligned}\Theta_2(z) &= \sum_{n \in \mathbb{Z} + \frac{1}{2}} q^{\frac{1}{2}n^2}, \\ \Theta_3(z) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2}, \\ \Theta_4(z) &= \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2}.\end{aligned}$$

Under the action of  $z \mapsto -1/z$ , these theta functions satisfy

$$\begin{aligned}\Theta_2(-1/z) &= (-iz)^{1/2} \Theta_4(z), \\ \Theta_3(-1/z) &= (-iz)^{1/2} \Theta_3(z), \\ \Theta_4(-1/z) &= (-iz)^{1/2} \Theta_2(z).\end{aligned}$$

Under the action of  $z \mapsto z + 1$  they satisfy

$$\begin{aligned}\Theta_2(z + 1) &= e^{i\pi/4} \Theta_2(z), \\ \Theta_3(z + 1) &= \Theta_4(z), \\ \Theta_4(z + 1) &= \Theta_3(z).\end{aligned}$$

The fourth powers  $\Theta_2^4, \Theta_3^4, \Theta_4^4$  have the following leading terms at the cusp  $i\infty$ :

$$\begin{aligned}\Theta_2^4 &= 16q^{1/2} + 64q^{3/2} + O(q^{5/2}), \\ \Theta_3^4 &= 1 + 8q^{1/2} + 24q + 32q^{3/2} + 24q^2 + O(q^{5/2}), \\ \Theta_4^4 &= 1 - 8q^{1/2} + 24q - 32q^{3/2} + 24q^2 + O(q^{5/2}).\end{aligned}$$

From these  $q$ -expansions one can observe the *Jacobi identity*:  $\Theta_3^4 = \Theta_2^4 + \Theta_4^4$ . Additionally,  $\Theta_2^4$ ,  $\Theta_3^4$ , and  $\Theta_4^4$  generate the ring of holomorphic modular forms of weight  $k = 2$  for the congruence subgroup  $\Gamma(2)$  [Bru08]. We normalize the *modular discriminant*  $\Delta$ , which can be written in several ways:

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} = \eta(z)^{24} = \sum_{n \geq 1} \tau(n) q^n = \frac{E_4^3 - E_6^2}{1728}.$$

The function  $\Delta$  is a weight 12 cusp form for  $SL_2(\mathbb{Z})$  [MDG]. A salient point for our purposes is that  $\Delta(it) > 0$  for  $t > 0$ . This can be seen upon setting  $z = it$  in the above product expansion.

For  $x \in \mathbb{R}^d$ , the Fourier transform of a Gaussian function is

$$e^{\pi i \|x\|^2 z} \rightarrow (-iz)^{-d/2} e^{\pi i \|x\|^2 (-1/z)}.$$

Thus, a natural way to construct Fourier eigenfunctions is to work with linear combinations of Gaussians - the continuous version being the Laplace transform. For  $r > \sqrt{2}$ , let

$$a(r) = -4i \sin(\pi r^2/2)^2 \int_0^\infty \varphi(it) e^{-\pi r^2 t} dt$$

which may be interpreted as the Laplace transform of a weighting function  $g(t) = \varphi(it)$  evaluated at  $\pi r^2$ , multiplied by the root forcing function  $-4i \sin(\pi r^2/2)^2$ . We derive conditions on  $\varphi$  that make  $\widehat{a} = a$ . We may interpret  $a(r) = a(x)$  for  $x \in \mathbb{R}^4$  with  $\|x\| = r$  since we are interested in radial functions [BCK10, CoEl03]. Letting  $z = it$  rotates the integration to the positive imaginary axis giving

$$\begin{aligned} a(r) &= \int_0^{i\infty} \varphi(z) e^{\pi i r^2 (z+1)} dz + \int_0^{i\infty} \varphi(z) e^{\pi i r^2 (z-1)} dz - 2 \int_0^{i\infty} \varphi(z) e^{\pi i r^2 z} dz \\ &= \int_1^{1+i\infty} \varphi(z-1) e^{\pi i r^2 z} dz + \int_{-1}^{-1+i\infty} \varphi(z+1) e^{\pi i r^2 z} dz - 2 \int_0^{i\infty} \varphi(z) e^{\pi i r^2 z} dz. \end{aligned}$$

Addressing the convergence of the integral at  $i\infty$  and near the real line, if  $|\varphi(it)| = O(e^{2\pi t})$  as  $t \rightarrow \infty$  and  $|\varphi(it)| = O(e^{-\pi/t})$  as  $t \rightarrow 0^+$ , we may apply Cauchy's theorem and write  $\int_1^{1+i\infty} + \int_{i\infty}^i + \int_i^1 = 0$  and  $\int_{-1}^{-1+i\infty} + \int_{i\infty}^i + \int_i^{-1} = 0$ . The path of integration is chosen to be perpendicular to the real line at the endpoints 1 and  $-1$ . If also  $\varphi(z+2) = \varphi(z)$ , then

$$\begin{aligned} a(r) &= 2 \int_i^{i\infty} (\varphi(z+1) - \varphi(z)) e^{\pi i r^2 z} dz + \int_1^i \varphi(z-1) e^{\pi i r^2 z} dz \\ &\quad + \int_{-1}^i \varphi(z+1) e^{\pi i r^2 z} dz - 2 \int_0^i \varphi(z) e^{\pi i r^2 z} dz. \end{aligned}$$

Taking the Fourier transform and maintaining the variable  $r$  (since the Fourier transform of a radial function is radial), substitute  $\omega = \frac{-1}{z}$ :

$$\begin{aligned} \widehat{a}(r) &= 2 \int_i^0 (\varphi(\frac{-1}{\omega} + 1) - \varphi(\frac{-1}{\omega})) i^{-d/2} \omega^{d/2-2} e^{\pi i r^2 \omega} d\omega \\ &\quad + \int_{-1}^i \varphi(\frac{-1}{\omega} - 1) i^{-d/2} \omega^{d/2-2} e^{\pi i r^2 \omega} d\omega \\ &\quad + \int_1^i \varphi(\frac{-1}{\omega} + 1) i^{-d/2} \omega^{d/2-2} e^{\pi i r^2 \omega} d\omega \\ &\quad - 2 \int_{i\infty}^i \varphi(\frac{-1}{\omega}) i^{-d/2} \omega^{d/2-2} e^{\pi i r^2 \omega} d\omega. \end{aligned}$$

Note that we are exchanging the Fourier transform with the contour integrals. The equality  $\widehat{a} = a$  can be achieved if equality happens at the level of the integrand functions. Comparing the terms in the integral  $\int_i^{i\infty}$  results in

$$\varphi(z+1) - \varphi(z) = \varphi\left(\frac{-1}{z}\right) i^{-d/2} z^{d/2-2}. \quad (2.1)$$

Comparing terms in  $\int_0^i$  results in  $\varphi(z) = (\varphi\left(\frac{-1}{z}\right) + 1) - \varphi\left(\frac{-1}{z}\right) i^{-d/2} z^{d/2-2}$ , which is the same as (2.1) after inverting back  $z \mapsto \frac{-1}{z}$ . Comparing terms in  $\int_{-1}^i$ , we get

$$\varphi(z+1) = \varphi\left(\frac{-1}{z}\right) + 1 i^{-d/2} z^{d/2-2}. \quad (2.2)$$

From (2.1),

$$\begin{aligned} \varphi(z+1) &= \varphi(z) + \varphi\left(\frac{-1}{z}\right) i^{-d/2} z^{d/2-2} \\ &= (\varphi\left(\frac{-1}{z}\right) + 1) - \varphi\left(\frac{-1}{z}\right) i^{-d/2} z^{d/2-2} + \varphi\left(\frac{-1}{z}\right) i^{-d/2} z^{d/2-2} \\ &= \varphi\left(\frac{-1}{z}\right) + 1 i^{-d/2} z^{d/2-2}. \end{aligned}$$

Thus (2.2) follows from (2.1). The comparison in  $\int_{-1}^i$  is similar.

**Proposition 2.1.** *Let  $d = 4$  and let*

$$\varphi = \frac{\Theta_4^{12}(\Theta_3^{12} + \Theta_2^{12})}{\Delta}. \quad (2.3)$$

*Then,  $f_+(r) = ia(r)$  is a radial Schwartz function invariant under the Fourier transform and  $f_+$  has sign radius  $r_{f_+} = \sqrt{2}$ .*

*Proof.* Based on the functional equations for  $\Theta_2, \Theta_3, \Theta_4$  and  $\Delta$ ,

$$\varphi(z+1) = \frac{\Theta_3^{12}(\Theta_4^{12} - \Theta_2^{12})}{\Delta}$$

and

$$\varphi(-1/z) = \frac{-z^6 \Theta_2(z)^{12} (-z^6 \Theta_3(z)^{12} - z^6 \Theta_4(z)^{12})}{z^{12} \Delta(z)} = \frac{\Theta_2^{12}(\Theta_3^{12} + \Theta_4^{12})}{\Delta}.$$

Thus,  $\varphi$  satisfies (2.1) for  $d = 4$ . To show that  $\varphi$  is a weight 0 weakly holomorphic modular form for the congruence subgroup  $\Gamma(2)$ , written  $\varphi \in M_0^1(\Gamma(2))$ , it suffices to check that  $\varphi(\gamma z) = \varphi(z)$  for a set of generators of  $\Gamma(2)$ .  $\Gamma(2)$  is generated by  $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ , and  $-I$ . Observe that  $\varphi\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} z\right) = \varphi(z+2) = \varphi(z)$  directly from the definitions of  $\Delta, \Theta_2, \Theta_3$ , and  $\Theta_4$ . Using (2.1),

$$\begin{aligned} \varphi\left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} z\right) &= \varphi\left(\frac{z}{2z+1}\right) = \varphi\left(\frac{-1}{-1/z-2}\right) \\ &= \varphi\left(\frac{-1}{z}-2\right) - \varphi\left(\frac{-1}{z}-2+1\right) \\ &= \varphi\left(\frac{-1}{z}\right) - \left(\varphi\left(\frac{-1}{z}\right) - \varphi\left(\frac{-1}{-1/z}\right)\right) = \varphi(z). \end{aligned}$$

The first few Fourier coefficients of  $\varphi$  are

$$\begin{aligned} \varphi(z) &= q^{-1} - 24 + 4096q^{1/2} - 98028q + O(q^{3/2}), \\ \varphi(-1/z) &= 8192q^{1/2} + O(q^{3/2}). \end{aligned} \quad (2.4)$$

The asymptotics and the fact that  $f_+$  is a Schwartz function follow directly from the growth rate of these Fourier coefficients since  $\varphi$  is a weakly holomorphic modular form [Bru02].  $f_+$  has sign radius  $\sqrt{2}$  since we may observe that for  $t > 0$ ,  $\Delta(it) > 0$  and each  $\Theta_j(it) \in \mathbb{R}$ . Thus,  $f_+(r) \geq 0$  for  $r > \sqrt{2}$ .

Based on the  $q$ -series expansion (2.4), it is possible to write

$$\begin{aligned} a(r) &= -4i \sin(\pi r^2/2)^2 \int_0^\infty (e^{2\pi t} - 24 + O(e^{-\pi t})) e^{-\pi r^2 t} dt \\ &= -4i \sin(\pi r^2/2)^2 \left( \frac{1}{\pi(r^2 - 2)} - \frac{24}{\pi r^2} + \int_0^\infty (\varphi(it) - e^{2\pi t} + 24) e^{-\pi r^2 t} dt \right) \end{aligned}$$

which converges for all  $r \geq 0$ . In particular,  $f_+(0) = 0$ , so  $f_+$  is not positive definite.

### 3. A -1 eigenfunction and concluding remarks

To construct a  $-1$  eigenfunction, it is possible to begin with a similar function shape

$$b(r) = -4 \sin(\pi r^2/2)^2 \int_0^{i\infty} \phi(-1/z) z^{d/2-2} e^{\pi i r^2 z} dz.$$

Comparing integrand terms as before, we get

$$\phi\left(\frac{-1}{z-1}\right) (z-1)^{d/2-2} + \phi\left(\frac{-1}{z+1}\right) (z+1)^{d/2-2} - 2\phi\left(\frac{-1}{z}\right) z^{d/2-2} = 2\phi(z). \quad (3.5)$$

So, the form given for  $b(r)$  enforces (3.5) when  $\hat{b} = -b$ . Using the Eisenstein series  $E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$  (which is usually introduced as a basic example of a quasi-modular form) and the transformation property

$$E_2\left(\frac{-1}{z}\right) = z^2 E_2(z) + \frac{6z}{\pi i},$$

a straightforward calculation shows that  $E_2$  satisfies (3.5) for  $d = 4$ . Thus, setting  $\phi = E_2$  above gives a  $-1$  eigenfunction also with sign radius  $\sqrt{2}$ . So,  $b(r)$  provides a simple construction of an eigenfunction with the basic depth 1 Eisenstein series  $E_2$  within the integral. As before,  $b(r)$  is not sharp since the numerical bounds from table 4.1 in [CoGo19] give  $A_-(4) \leq 1.204 < \sqrt{2}$ . Furthermore, using the checkerboard lattice  $D_4$  from [CoSl13] with minimal radius  $\sqrt{2}$ , the dual lattice  $D_4^*$  is homothetic to  $D_4$  but will have minimal radius 1. We can scale  $D_4$  so that  $D_4$  and  $D_4^*$  both have minimal radius  $2^{1/4}$ , however constructing eigenfunctions  $f : \mathbb{R}^4 \rightarrow \mathbb{R}$  of the above shape with sign radius  $2^{1/4}$  appears to remain a difficult problem.

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