



# Long Time Gyrokinetic Equations

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# LONG TIME GYROKINETIC EQUATIONS

CHRISTOPHE CHEVERRY AND SHAHNAZ FARHAT

**ABSTRACT.** The aim of this text is to elucidate the oscillating patterns [9] which are generated in a toroidal plasma by a strong external magnetic field and a nonzero electric field. It is also to justify and then study new modulation equations which are valid for longer times than before. Oscillating coherent structures are induced by the collective motions of charged particles which satisfy a system of ODEs implying a large parameter, the gyrofrequency  $\varepsilon^{-1} \gg 1$ . By exploiting the properties of underlying integrable systems, we can complement the KAM picture [1, 3] and go beyond the classical results about gyrokinetics [1, 2, 3, 5]. The purely magnetic situation was addressed in [7, 8]. We are concerned here with the numerous additional difficulties due to the influence of a nonzero electric field.

**KEYWORDS.** Nonlinear differential equations; Nearly integrable systems; WKB analysis; Toroidal magnetized plasmas; Coherent structures; Gyrokinetic equations.

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## 1. INTRODUCTION

In Subsection 1.1, we present the general ideas. In Subsection 1.2, we define the two key concepts of *average flow* and *mean flow*. In Subsection 1.3, we clarify the assumptions made on the external electromagnetic field. In Subsection 1.4, we specify the coherent structures; we highlight the applications to toroidal plasmas; and we detail our plan.

**1.1. Motivations.** We study the motions of charged particles in the presence of a fixed external electromagnetic field  $(E, B)$  with  $E \neq 0$  and  $|B| \gg 1$ . The case where  $E \equiv 0$  was recently published in [7, 8]. As is well-known, the electric and magnetic forces affect the trajectories of charged particles through the Lorentz force. They can be exploited in order to confine plasmas through for instance toroidal devices (like tokamaks). The purpose is to ensure that the plasma stays away from the walls.

Understanding the transport processes in axisymmetric configurations is an old challenging goal. The subject has a long history with a huge number of contributions. There are many approaches, including the KAM method [1, 3] and **gyrokinetics** [2, 5]. The difference of our viewpoint with KAM theory is - away from a set of zero measure in the phase space (corresponding to the presence of separatrices) - a global quantitative space-time description [9] of plasma oscillations, in terms of explicit phases and profiles. Its main novelty compared to gyrokinetics is the deterministic rigorous investigation of longer times. And, in connection with [7, 9, 8], the originality of the present paper is the addition of a non trivial electric field  $E \neq 0$ .

The main tool to confine plasmas is a large fixed external magnetic field  $B$ . But it is also essential to explore what happens under the simultaneous presence of a smaller electric field  $E \neq 0$ . Indeed, the influence of such  $E \neq 0$  cannot be ignored. First, due to the coupling between the Vlasov and Maxwell equations, a nonzero electric field is spontaneously generated. Secondly, in experiments, such non trivial  $E$  is a known cause of disruptions. When  $E \neq 0$ , the kinetic energy is no more conserved. The particles can be accelerated or decelerated, leading to the development of instabilities. Such instabilities often proceed from the production of whistler waves [7, 9], which have their origins in the collective organization of charged particles [10]. It is therefore important to detect the oscillating coherent structures which are induced by the *inhomogeneous* features of the electromagnetic field  $(E, B)$ .

In the case  $E \equiv 0$ , this program has been achieved in [7, 8] with difficulties coming from the variations of  $B$ . When  $E \neq 0$ , the discussion is even more complicated (with new geometrical structures appearing at the level of Paragraph 3.2.1). Still, it falls within the scope of the WKB analysis introduced in [11]. This is not obvious, far from it. Thus, our first task is to make sure that all prerequisites of [11] are satisfied when dealing with concrete data emanating from the modeling of tokamaks. Once done, the results of [11] furnish long time modulation equations (Theorem 1.9) where it is possible to measure the impact of  $E$ . Then, our second motivation is to investigate the stability issues at the level of these reduced equations. In particular, we want to examine the potential effects of the condition  $E \neq 0$ .

A relativistic charged particle starting at time  $t = 0$  from the position  $(x_0, v_0) \in \mathbb{R}^3 \times \mathbb{R}^3$  is moving in the phase space  $\mathbb{R}^3 \times \mathbb{R}^3$  according to the dimensionless ordinary differential equation

$$(1.1) \quad \begin{cases} \frac{dx}{dt} &= v, \\ \frac{dv}{dt} &= (1 - |v|^2)^{1/2} \left\{ -\frac{1}{\varepsilon} v \wedge B(x) - E(x) + (v \cdot E(x)) v \right\}. \end{cases}$$

The integral curves associated to (1.1) give rise to the characteristics of the Vlasov equation. The origin of (1.1) is recalled in Subsection 2.1. In physics, the parameter  $\varepsilon$  comes from the inverse of the electron gyrofrequency. In practice, it is small, such that  $\varepsilon \simeq 10^{-4} \ll 1$ . This is why it is often viewed as going to zero. The regime of intermediate times (related to gyrokinetics) is when  $t \sim 1$ . We are concerned here with the study of (1.1) for longer times, when  $\tau := \varepsilon t \sim 1$  (or  $t \sim \varepsilon^{-1}$ ). Thus, with  ${}^t(\tilde{x}, \tilde{v})(x_0, v_0; \tau) := {}^t(x, v)(x_0, v_0; \tau/\varepsilon)$  we have to consider:

$$(1.2) \quad \begin{cases} \frac{d\tilde{x}}{d\tau} &= \frac{1}{\varepsilon} \tilde{v}, \\ \frac{d\tilde{v}}{d\tau} &= \frac{1}{\varepsilon} (1 - |\tilde{v}|^2)^{1/2} \left\{ -\frac{1}{\varepsilon} \tilde{v} \wedge B(\tilde{x}) - E(\tilde{x}) + (\tilde{v} \cdot E(\tilde{x})) \tilde{v} \right\}. \end{cases}$$

Knowing that  $\varepsilon \ll 1$ , the right hand side of (1.2) is very large. We can recognize a leading term of size  $\varepsilon^{-2}$  yielding the (well-known for  $t \sim 1$ ) repercussions of the strong external magnetic field (the gyromotions). But there are also terms of size  $\varepsilon^{-1}$  which take into account the impact of the electric field. This leads to a number of extra effects that we aimed to better understand. Here, for some  $\varepsilon_0 \in ]0, 1[$ , we assume that  $\varepsilon \in ]0, \varepsilon_0]$ . To clearly exhibit the hierarchy of effects in terms of powers of  $\varepsilon$ , the parameter  $\varepsilon$  is intended to go to zero.

In contrast to preceding works [7, 8], in the present situation, the kinetic energy is in general not conserved. Still, we can describe the leading behavior of the solution  $(\tilde{x}, \tilde{v})$  to (1.2). On this basis, in Section 4, we can exhibit a spatial confinement property (related to  $\tilde{x}$ ) for a whole range of electric fields  $E$  adjusted as in Assumption 1.7. In addition, when the electric field is radial, that is when the scalar electric potential  $\Phi$  depends only on  $r$ , we can also prove a long time ( $\tau \sim 1$ ) stability property concerning the momentum component  $\tilde{v}$ .

**1.2. Notions of average and mean flows.** For realistic choices of  $(E, B)$ , we would like to deduce from (1.2) simplified models which are valid for  $\tau \sim 1$ . To fit with the toroidal geometry, in Subsection 2.2, we change the phase space  $\mathbb{R}^3 \times \mathbb{R}^3$  into  $(\mathbb{R} \times \mathbb{T} \times \mathbb{T})^2$ , where  $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$  is the torus. We also extract from (1.1) a special class of a nonlinear differential equations having the following form

$$(1.3) \quad \partial_t \begin{pmatrix} \mathbf{z} \\ v \end{pmatrix} = \begin{pmatrix} \mathcal{A} \\ \mathcal{V} \end{pmatrix} \left( \varepsilon; \mathbf{z}; \frac{v}{\varepsilon} \right), \quad \begin{pmatrix} \mathbf{z} \\ v \end{pmatrix} (0) = \begin{pmatrix} \mathbf{z}_0 \\ \mathbf{v}_0 \end{pmatrix},$$

where the new dependent variables are  $\mathbf{z} = {}^t(r, \theta, \mathbf{v}, \zeta) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}$  and  $v \in \mathbb{T}$ , and where  $\mathbf{z}_0 \in \mathbb{R}^4$  and  $\mathbf{v}_0 \in \mathbb{T}$  stand for initial data (which may depend smoothly on  $\varepsilon \in [0, \varepsilon_0]$ ). The vector valued function  $\mathcal{A}$  does not involve  $\varepsilon$ , whereas the scalar component  $\mathcal{V}$  does. More precisely, the source term  ${}^t(\mathcal{A}, \mathcal{V})$  looks like

$$(1.4) \quad C^\infty(\mathbb{R}_+^* \times \mathbb{T} \times \mathbb{R} \times \mathbb{T} \times \mathbb{T}; \mathbb{R}^4) \ni \mathcal{A}(\varepsilon; \mathbf{z}; \nu) \equiv \mathcal{A}(\mathbf{z}; \nu) = {}^t(\mathcal{A}_r, \mathcal{A}_\theta, \mathcal{A}_v, \mathcal{A}_\zeta)(\mathbf{z}; \nu),$$

$$(1.5) \quad C^\infty([0, \varepsilon_0] \times \mathbb{R}_+^* \times \mathbb{T} \times \mathbb{R} \times \mathbb{T} \times \mathbb{T}; \mathbb{R}) \ni \mathcal{V}(\varepsilon; \mathbf{z}; \nu) = \mathcal{V}_0(\mathbf{z}) + \varepsilon \mathcal{V}_1(\mathbf{z}; \nu).$$

Most importantly, the source terms  $\mathcal{A}$  and  $\mathcal{V}$  are periodic in the last variable  $\nu$  of period  $2\pi$ . This means that, when passing from (1.1) to (1.3), the singular weight  $\varepsilon^{-1}$  has been converted into fast oscillations carried by  $\nu$ . The profile  $\mathcal{A}(\mathbf{z}; \nu)$  can be decomposed according to

$$(1.6) \quad \mathcal{A}(\mathbf{z}; \nu) = \overline{\mathcal{A}}(\mathbf{z}) + \mathcal{A}^*(\mathbf{z}; \nu), \quad \overline{\mathcal{A}}(\mathbf{z}) := \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A}(\mathbf{z}; \nu) d\nu,$$

where  $\overline{\mathcal{A}} = {}^t(\overline{\mathcal{A}}_r, \overline{\mathcal{A}}_\theta, \overline{\mathcal{A}}_v, \overline{\mathcal{A}}_\zeta)$  is the mean part and  $\mathcal{A}^* = {}^t(\mathcal{A}_r^*, \mathcal{A}_\theta^*, \mathcal{A}_v^*, \mathcal{A}_\zeta^*)$  is the oscillating part. The same applies to  $\mathcal{V} = \overline{\mathcal{V}} + \mathcal{V}^*$ . The function  $\mathbf{z}(\cdot)$  is the sum of the *average flow*  $Z(\mathbf{z}_0; t)$  defined below plus deviations.

**Definition 1.1** (The *average flow* associated with the intermediate time evolution in  $t$  of  $\mathbf{z}$ ). The *average flow*  $Z(\mathbf{z}_0; t)$  on  $\mathbb{R}^4$  that is associated with (1.3) is the (locally or globally defined) mapping  $Z : \mathbb{R}^4 \times \mathbb{R} \rightarrow \mathbb{R}^4$  obtained by solving

$$(1.7) \quad \partial_t Z = \overline{\mathcal{A}}(Z), \quad Z(\mathbf{z}_0; 0) = \mathbf{z}_0, \quad Z = {}^t(Z_r, Z_\theta, Z_v, Z_\zeta).$$

The average flow issued from (1.1) is studied in Section 3. It reflects the main behavior of the dynamics during intermediate times. It is described in Subsection 3.1. It plays a key role in what happens next, when  $\tau \sim 1$ . Indeed, to have access to [11], one of the difficulties is to prove that, for all fixed  $\mathbf{z}_0$ , the average flow  $Z(\mathbf{z}_0; \cdot)$  is globally defined and periodic in  $t$  (with a period which may depend on  $\mathbf{z}_0$ ). Subsection 3.2 is designated to this purpose.

Once this is done, we can normalize the period of the average flow. To this end, in Paragraph 4.1.1, we change adequately (in a way depending on  $\mathbf{z}_0$ ) the time variable  $t$  into  $s \sim t$  (with  $s \neq t$  but  $s \sim t$  still representing intermediate times). Then, we can interpret the long time behavior in terms of  $\tau := \varepsilon s$  (where  $\tau \neq \tau$  but  $\tau \sim \tau$ ). Instead of  ${}^t(\mathbf{z}, v)$ , we work with  ${}^t(\mathbf{z}, \mathbf{v})$  as in (4.4). By this way, in Section 4, we can enter into the framework of [11]. As a matter of fact, the system (1.2) is recast as follows

$$(1.8) \quad \partial_\tau \begin{pmatrix} \mathbf{z} \\ \mathbf{v} \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} \left( \varepsilon, \mathbf{z}_0; \mathbf{z}; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon} \right), \quad \begin{pmatrix} \mathbf{z} \\ \mathbf{v} \end{pmatrix} (0) = \begin{pmatrix} \mathbf{z}_0 \\ \mathbf{v}_0 \end{pmatrix},$$

where the functions  $\mathbf{A}$  and  $\mathbf{V}$  satisfy

$$(1.9) \quad C^\infty(\mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{T} \times \mathbb{T}; \mathbb{R}^4) \ni \mathbf{A}(z_0; z; s, \nu) = {}^t(\mathbf{A}_r, \mathbf{A}_\theta, \mathbf{A}_v, \mathbf{A}_\zeta)(z_0; z; s, \nu),$$

$$(1.10) \quad C^\infty([0, \varepsilon_0] \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{T} \times \mathbb{T}; \mathbb{R}) \ni \mathbf{V}(\varepsilon, z_0; z; s, \nu) = (\mathbf{V}_0 + \varepsilon \mathbf{V}_1)(z_0; z; s, \nu).$$

Compare (1.2) with (1.8). The singular weight  $\varepsilon^{-2}$  has disappeared but there remains a large factor  $\varepsilon^{-1}$  multiplied by a double periodic function. Indeed, the source term  ${}^t(\mathbf{A}, \mathbf{V})$  is periodic of period  $2\pi$  in both variables  $s$  and  $\nu$ .

**Definition 1.2** (The *mean flow* associated with the intermediate time evolution in  $s$  of  $z$ ). Let  $\bar{\mathbf{A}}(z_0; z; s)$  be defined from  $\mathbf{A}$  as in (1.6). Given a position  $z_0 \in \mathbb{R}^4$ , the mean flow  $\Xi_{\text{mf}}(z_0; \mathfrak{z}; s)$  on  $\mathbb{R}^4$  associated with (1.8) is the (locally or globally defined) mapping  $\Xi_{\text{mf}}(z_0; \cdot) : \mathbb{R}^4 \times \mathbb{R} \rightarrow \mathbb{R}^4$  which is obtained by solving

$$(1.11) \quad \partial_s \Xi_{\text{mf}}(z_0; \mathfrak{z}; s) = \bar{\mathbf{A}}(z_0; \Xi_{\text{mf}}(z_0; \mathfrak{z}; s); s), \quad \Xi_{\text{mf}}(z_0; \mathfrak{z}; 0) = \mathfrak{z}.$$

*Remark 1.3.* As already explained, the average flow is the gate to the construction of the mean flow. It should not be confused with the mean flow. The two equations (1.7) and (1.11) are distinct. We have  $\bar{\mathbf{A}} \neq \bar{\mathbf{A}}$ , not least because  $\bar{\mathbf{A}}$  and  $\bar{\mathbf{A}}$  depend on distinct variables. The above notion of mean flow is consistent with the one of [11, Definition 1].  $\circ$

In accordance with the above remark, we give below an overview about the distinct time variables which are used in this introduction. We have mentioned the following time variables:  $t$ ,  $\tau = \varepsilon t$ ,  $s$  and  $\tau = \varepsilon s$ . Each time variable owns its special class of notations.

◁ For the time variable  $t$ , we implement:

- $(x, v)$  which solves the characteristics equations (1.1) of the Vlasov system;
- $(z, v)$  which solves the system (1.3). This system is extracted from the system (1.1) on  $(x, v)$  in Subsection 2.2. The latter is achieved by reformulating the equations for  $x$  after writing it in toroidal coordinates and reformulating the equations for  $v$  after decomposing it in the spherical coordinates. It is obvious then that  $v \neq v = \varepsilon v$ ;
- the average flow  $Z$  which solves (1.7). Its equation is extracted from the equation on  $z$  in (1.3) after applying the averaging operator (1.6) with respect to  $v$ .

◁ For the time variable  $\tau$ , we use  $(\tilde{x}, \tilde{v})$  which solves the characteristic equations (1.2). The latter system is just the system (1.1) interpreted in the long time variable  $\tau = \varepsilon t$ .

◁ For the time variable  $s$  and  $\tau = \varepsilon s$ , we use:

- $(z, v)$  which solves the system (1.8). The latter system is the system (1.3) on  $(z, v)$  interpreted in the time variable  $\tau$ . This is clarified in Subsection 4.1;
- the mean flow  $\Xi_{\text{mf}}$  which solves (1.11). We will show in Subsection 4.1 that the mean flow  $\Xi_{\text{mf}}$  is related to the average flow  $Z$ .

The oscillating coherent structures can be detected on the WKB approximation in  $(\tau, z_0, v_0)$  of the solutions  ${}^t(z_\varepsilon, v_\varepsilon)(\tau, z_0, v_0)$  to (1.8). As shown in [11], such expansions become available provided that the two following criterions are verified.

**Condition 1.4** (Complete integrability of the *mean flow*). For all  $(z_0, \mathfrak{z}) \in \mathbb{R}^4 \times \mathbb{R}^4$ , the mean flow  $\Xi_{\text{mf}}(z_0; \mathfrak{z}; \cdot)$  is globally defined and periodic of period  $2\pi$ .

**Condition 1.5** (Constraints on  $V_0$ ). The function  $V_0$  is positive and does not depend on  $\nu \in \mathbb{T}$ .

The validity of Conditions 1.4 and 1.5 in the case of functions  $\mathbf{A}$  and  $\mathbf{V}$  issued from (1.1) is verified in Subsection 4.1. This means that the results of [11] can be applied concerning (1.1). Related applications are developed in Section 4.

**1.3. Assumptions on the external field.** In Paragraph 1.3.1, we implement the toroidal coordinates. In Paragraph 1.3.2, we describe the structure of the external magnetic field  $B$ . In Paragraph 1.3.3, we detail the content of the electric field  $E$ . By doing so, we take care to adjust  $B$  and  $E$  in a manner which is consistent with what could be expected in tokamaks.

1.3.1. *Toroidal description of tokamaks.* To simplify the discussion, we introduce geometrical coordinates. It is easy to see that every point  $x \in \mathbb{R}^3$  in the cartesian coordinates can also be identified by a radial coordinate  $r \in \mathbb{R}_+$  (the distance from the magnetic axis) and by the two rotational invariances  $\theta \in \mathbb{T}$  (the poloidal angle) and  $\phi \in \mathbb{T}$  (the toroidal angle). These new coordinates are associated with the following decomposition

$$(1.12) \quad x = X(r, \theta, \phi) = (R(r, \theta) \cos \phi, R(r, \theta) \sin \phi, r \sin \theta), \quad R(r, \theta) = R_0 + r \cos \theta.$$

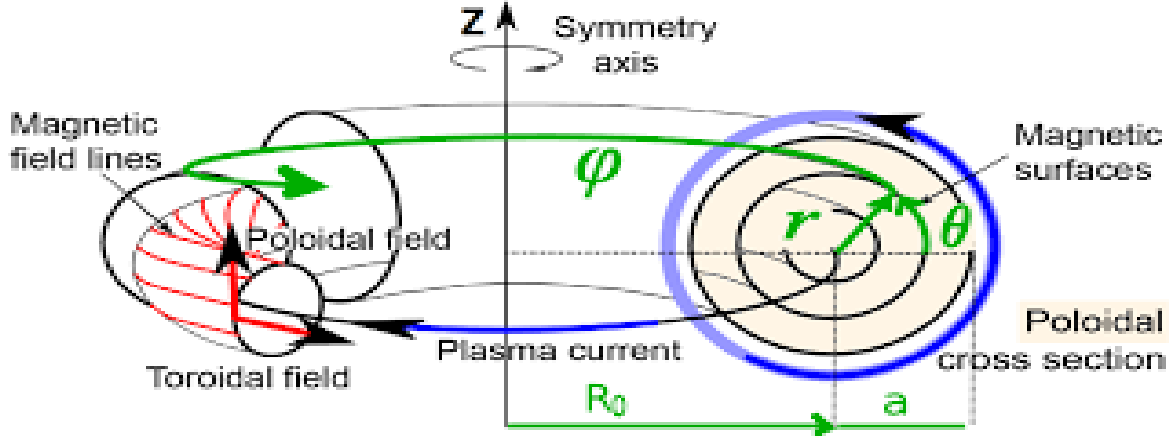


FIGURE 1. Schematic diagram of the tokamak-plasma geometry.

Retain that  $0 \leq r \leq a < R_0$  where  $a$  and  $R_0$  are respectively the minor radius and major radius of the tokamak, see Figure 1. We denote by  $e_r$ ,  $e_\theta$  and  $e_\phi$  the orthonormal basis corresponding to the selection of  $(r, \theta, \phi)$ , which is given by

$$(1.13) \quad \begin{aligned} e_r(\theta, \phi) &= (\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta), \\ e_\theta(\theta, \phi) &= (-\sin \theta \cos \phi, -\sin \theta \sin \phi, \cos \theta), \\ e_\phi(\phi) &= (-\sin \phi, \cos \phi, 0). \end{aligned}$$

Remark that  $e_r \wedge e_\theta = -e_\phi$  and recall that  $\nabla = e_r \partial_r + e_\theta \frac{1}{r} \partial_\theta + e_\phi \frac{1}{R_0} \partial_\phi$ .

1.3.2. *Adjusting the magnetic field.* The function  $B(\cdot)$  inside (1.1) has  $\varepsilon^{-1}$  in factor (with  $\varepsilon \ll 1$ ) for reasons clarified in Subsection 2.1. It represents the dimensionless version of the external magnetic field, and therefore it is of amplitude one. We impose common restrictions on  $B$ , compatible with all the contributions [6, 12, 13, 15, 16, 19, 20]. In particular, the reader can refer to the [Notes on Tokamak equilibrium](#) [15].

**Assumption 1.6** (Toroidal external magnetic field). The magnetic field  $B$  is issued from a scalar magnetic potential  $A(\cdot)$  through the relation  $B = \nabla \wedge A$  where, given some auxiliary function  $I \in C^1([0, a]; \mathbb{R}_+^*)$  which must be associated with a poloidal current, we have

$$A(r, \theta) := -r \nabla \phi - g(r, \theta) \nabla \theta, \quad g(r, \theta) := \int_0^r \frac{\tilde{r} I(\tilde{r})}{R_0 + \tilde{r} \cos \theta} d\tilde{r}.$$

The above magnetic field  $B(\cdot)$  can always be decomposed (see Figure 2) according to

$$B = \underbrace{B_t}_{\text{Toroidal magnetic field}} + \underbrace{B_p}_{\text{Poloidal magnetic field}}, \quad B_t \parallel e_\phi, \quad B_p \parallel e_\theta, \quad B_t \perp B_p.$$

In practice, the poloidal magnetic field is produced by the plasma current flowing in the toroidal direction. It helps to confine the plasma and to get some equilibrium. As seen in Figure 2, the magnetic field lines follow helical paths around the torus.

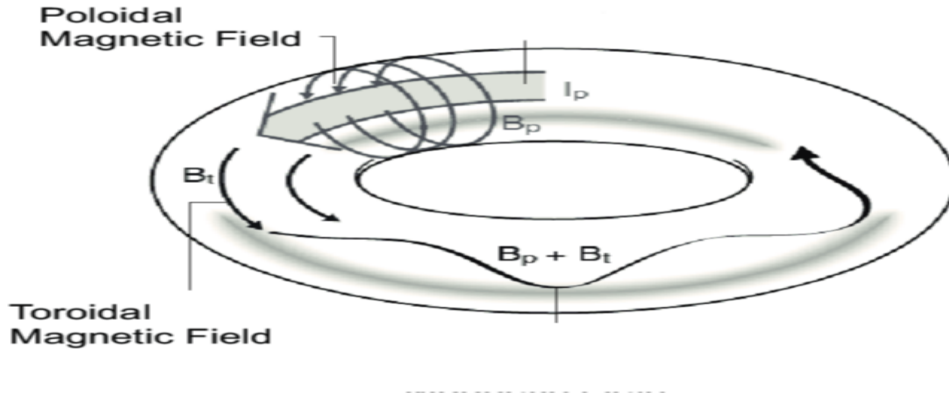


FIGURE 2. The decomposition of the magnetic field.

1.3.3. *Adjusting the electric field  $E$ .* In the journal [12], the authors take measurements that fit (after nondimensionalization) with the condition  $|E| \simeq 1$ . Thus, from now on, the electric field  $E$  inside (1.1) is supposed to be of amplitude one. Due to the cylindrical symmetry, it is reasonable to assume some invariance with respect to  $\phi$ .

**Assumption 1.7** (Axisymmetric external electric field). The electric field  $E$  is issued from a scalar electric potential  $\Phi$  which does not depend on  $\phi$ . More precisely, we can find a smooth (non constant) function  $\Phi : [0, a] \times \mathbb{R} \rightarrow \mathbb{R}$  such that

$$(1.14) \quad E(r, \theta, \phi) = \nabla \Phi = \partial_r \Phi(r, \theta) e_r(\theta, \phi) + r^{-1} \partial_\theta \Phi(r, \theta) e_\theta(\theta, \phi).$$

Even though  $\partial_\phi \Phi \equiv 0$ , the above field  $E(\cdot)$  does depend on  $\phi$  through  $e_r(\cdot)$  and  $e_\theta(\cdot)$ . When moreover  $\Phi(r, \theta) \equiv \Phi(r)$ , the electric field  $E \equiv E_r(r, \theta, \phi) \equiv \partial_r \Phi e_r$  is said to be *radial*. There is no definitive consensus concerning the choice of  $E$ . But the formula (1.14) is sufficiently general to take into account most of the phenomena.

The model (1.14) is selected in accordance with the authoritative publications [19, Section 2.18] and [20] where, as in the next Paragraph 2.2.1, there are three nontrivial components  $E_r$ ,  $E_\parallel$  and  $E_\perp$  emanating from  $\nabla \Phi$ . In [20], the authors consider the relativistic guiding center motion of charged particles in magnetic fields consistent with toroidal Tokamak equilibrium. They take into account that runaway electrons increase their kinetic energy due to the accelerating force of the toroidal electric field. They try to know what is the orbit of a runaway electron which is produced on a given magnetic surface after it has performed a very large number of toroidal revolutions. They search for whether the observed phenomena can be explained by the change of the runaway electron orbits due to a flattening of the current and pressure profiles. In this process, the acceleration due to the toroidal electric field is essential. We share here the same concerns but from a mathematical viewpoint.

From knowledge of  $E$ , we can define the *separatrix region*  $\mathcal{S}$  according to a procedure that is described in Subsection 3.2. The (closed) set  $\mathcal{S}$  is used in Theorem 1.8 below. It is properly defined at the level of (3.29).

**1.4. Results, outcomes and plan.** Paragraph 1.4.1 is devoted to a first application about coherent structures. Paragraph 1.4.2 focusses on the properties of the long time gyrokinetic equations. Paragraph 1.4.3 details the plan of the article.

1.4.1. *Coherent structures.* As long as  $t \sim 1$ , the trajectory of charged particles is a helix that winds rapidly around the magnetic field lines. The *gyrokinetic theory* describes perfectly the (relatively) slow motion of the guiding center. But what happens next? For  $t \gg 1$ , that is for  $t \sim \varepsilon^{-1}$  or  $\tau \sim 1$ , the motions become much more complicated. As explained below, their collective behaviors are organized around multiscale structures which can be completely described through WKB expansions.



**Theorem 1.8** (Long time global picture of the flow: emergence of coherent structures). *Suppose that Assumptions 1.6 and 1.7 hold. Select in the phase space any bounded connected open set  $\Omega$  whose closure  $\bar{\Omega}$  does not intersect the separatrix region  $\mathcal{S}$ . Then, for all  $\varepsilon \in ]0, \varepsilon_0]$  with  $\varepsilon_0$  small enough, there exists some  $\tilde{T} \in \mathbb{R}_+^*$  such that for all initial data  $(x_0, v_0) \in \Omega$ , the system (1.2) has a solution on  $[0, \tilde{T}]$ . In addition, we can find smooth profiles*

$$\begin{aligned} \tilde{X}_j(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}) &\in C^\infty(\Omega \times [0, \tilde{T}] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}^3), & j \in \mathbb{N}, \\ \tilde{V}_j(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}) &\in C^\infty(\Omega \times [0, \tilde{T}] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}^3), & j \in \mathbb{N}, \end{aligned}$$

*adjusted in such a way that the long time behavior of the solution  $(\tilde{x}_\varepsilon, \tilde{v}_\varepsilon)$  of (1.2) can be described in the sup-norm as follows*

$$(1.15) \quad \begin{pmatrix} \tilde{x}_\varepsilon \\ \tilde{v}_\varepsilon \end{pmatrix}(x_0, v_0; \tau) \underset{\varepsilon \rightarrow 0}{\sim} \sum_{j=0}^{+\infty} \varepsilon^j \begin{pmatrix} \tilde{X}_j \\ \tilde{V}_j \end{pmatrix} \left( x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2} \right).$$

*The profiles  $\tilde{X}_j$  and  $\tilde{V}_j$  are identified by modulation equations (which can be made explicit). Given  $P(\cdot)$  as in Lemma 3.7, the expression  $\Psi^1$  is given by*

$$(1.16) \quad \Psi^1(x_0, v_0; \tau) := \frac{2\pi\tau}{P(z_0(x_0, v_0))}.$$

*The function  $\Psi_\varepsilon^2$  plays the role of a phase; it is oscillating; and it is defined by*

$$(1.17) \quad \begin{aligned} \Psi_\varepsilon^2(x_0, v_0; \tau) &:= \langle \overline{\mathcal{V}}_{-1} \rangle(z_0(x_0, v_0), \Psi^1(x_0, v_0; \tau)) \\ &+ \varepsilon \left\{ [\partial_\tau \langle \overline{\mathcal{V}}_{-1} \rangle](z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau)) s^*(z_0(x_0, v_0); \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}) \right. \\ &\left. + \overline{\mathcal{V}}_0(z_0(x_0, v_0), 0; \Psi^1(x_0, v_0; \tau), \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon} + s^*(z_0(x_0, v_0); \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon})) \right\}, \end{aligned}$$

*where  $s^*$  is introduced in Lemma 3.8, whereas  $\langle \overline{\mathcal{V}}_{-1} \rangle$  and  $\overline{\mathcal{V}}_0$  are computed in Proposition 4.2.*

Gyrokinetics deals with  $t \sim 1$  (that is with  $x$  and  $v$ ) and fast gyromotions around the field lines, which occur at the frequency  $\varepsilon^{-1}$  in terms of the time scale  $t$ . On the other hand, Theorem 1.8 addresses longer times  $\tau \sim 1$  (which imply  $\tilde{x}$  and  $\tilde{v}$ ) and extra fast oscillating behaviors, which arise at the frequency  $\varepsilon^{-1}$  in terms of the time scale  $\tau$ . Depending on the chosen set  $\Omega$ , we are faced with librations (in the case of trapped particles in banana orbits) or rotations (in the case of passing particles), see Figure 3. The expansion (1.15) makes visible the complex macroscopic juxtaposition of all these oscillations.

**1.4.2. Outcomes.** By analogy with the usual gyrokinetic equations, which are valid for  $t \sim 1$ , the modulation equations leading to the main profiles  $\tilde{X}_0$  and  $\tilde{V}_0$  are referred to as the *long time gyrokinetic equations*. They are of practical prime interest because they reveal the leading behavior of the solutions (for  $\tau \sim 1$ ). This is why they are studied below in detail.

**Theorem 1.9** (Long time gyrokinetic equations). *The profile  $\tilde{X}_0(\cdot)$  does not depend on  $\tilde{\nu}$ , while the profile  $\tilde{V}_0(\cdot)$  does. These functions can be expressed in terms of the average flow  $Z$ , which is the solution to (1.7), as follows*

$$(1.18) \quad \tilde{X}_0(x_0, v_0; \tau, \tilde{s}) := X(\langle \overline{\mathcal{Z}}_{0r} \rangle; Z_\theta(\langle \overline{\mathcal{Z}}_0 \rangle; P \tilde{s}/2\pi), \overline{\Phi}_0(\tau, \tilde{s})),$$

$$(1.19) \quad \begin{aligned} \tilde{V}_0(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}) &:= V(\langle \overline{\mathcal{Z}}_{0r} \rangle; Z_\theta(\langle \overline{\mathcal{Z}}_0 \rangle; P \tilde{s}/2\pi), \overline{\Phi}_0(\tau, \tilde{s}), \\ &Z_v(\langle \overline{\mathcal{Z}}_0 \rangle; P \tilde{s}/2\pi), Z_\zeta(\langle \overline{\mathcal{Z}}_0 \rangle; P \tilde{s}/2\pi), \tilde{\nu} + \overline{\mathcal{V}}_1(\tau, \tilde{s})), \end{aligned}$$

*where  $X$  and  $V$  are explicit functions given by (2.4) and (2.8),  $P$  is as in Lemma 3.7,  $\overline{\Phi}_0$  is issued from (4.51) and (4.53), and the scalar function  $\overline{\mathcal{V}}_1$  is as in (4.23). In the right hand*



side of (1.18) and (1.19),  $\langle \bar{\mathfrak{Z}}_0 \rangle$  is for the function  $\langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau) = {}^t(\langle \bar{\mathfrak{Z}}_{0r} \rangle, \langle \bar{\mathfrak{Z}}_{0\theta} \rangle, \langle \bar{\mathfrak{Z}}_{0v} \rangle, \langle \bar{\mathfrak{Z}}_{0\zeta} \rangle)(z_0; \tau)$  which is determined by solving the nonlinear Cauchy problem

$$(1.20) \quad \partial_\tau \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau) = \langle \bar{\mathfrak{A}}_1 \rangle(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau)), \quad \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; 0) = z_0,$$

where  $z_0 = {}^t(r_0, \theta_0, v_0, \zeta_0)$  may be computed from  $(x_0, v_0)$  through (2.4) and (2.8), and where the expression of  $\langle \bar{\mathfrak{A}}_1 \rangle = {}^t(\langle \bar{\mathfrak{A}}_{1r} \rangle, \langle \bar{\mathfrak{A}}_{1\theta} \rangle, \langle \bar{\mathfrak{A}}_{1v} \rangle, \langle \bar{\mathfrak{A}}_{1\zeta} \rangle)$  is given by (4.68). Moreover, as claimed in Lemma 4.11, we find that  $\langle \bar{\mathfrak{A}}_{1r} \rangle(z_0; \mathfrak{z}) = 0$  for all  $(z_0, \mathfrak{z}) \in \mathbb{R}^d \times \mathbb{R}^d$ , so that  $\langle \bar{\mathfrak{Z}}_{0r} \rangle(z_0; \tau) = r_0$ .

The above theorem has multiple implications:

- A model for the leading behavior of the flow  $(\tilde{x}, \tilde{v})$  is provided by:

$$(1.21) \quad \begin{aligned} \tilde{x}(x_0, v_0; \tau) &= \tilde{X}_0\left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}\right) + \mathcal{O}(\varepsilon), \\ \tilde{v}(x_0, v_0; \tau) &= \tilde{V}_0\left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2}\right) + \mathcal{O}(\varepsilon). \end{aligned}$$

The behavior of  $\tilde{x}$  and  $\tilde{v}$  are mainly governed respectively by the profiles  $\tilde{X}_0$  and  $\tilde{V}_0$ ;

- Looking at (1.18), we find  $\partial_{\tilde{s}} \tilde{X}_0 \neq 0$  and  $\partial_{\tilde{v}} \tilde{X}_0 \equiv 0$ . This implies that the propagation of the oscillating singularities at the leading order of the spatial part  $\tilde{x}$  is completely achieved by  $\Psi^1(x_0, v_0; \tau)$ . On the other hand, we have  $\partial_{\tilde{v}} \tilde{X}_1 \neq 0$ . This implies that the spatial part  $\tilde{x}$  involves small amplitude oscillations related to  $\Psi_\varepsilon^2(x_0, v_0; \tau)$  associated with higher frequency  $\varepsilon^{-2}$ . The rapid variation with respect to  $\tilde{v}$  is activated at the level of  $\tilde{X}_j$  as soon as  $j \geq 1$ ;
- Looking at (1.19), we find  $\partial_{\tilde{s}} \tilde{V}_0 \neq 0$  and  $\partial_{\tilde{v}} \tilde{V}_0 \neq 0$ . The large amplitude oscillations related to  $\Psi^1(x_0, v_0; \tau)$  and  $\Psi_\varepsilon^2(x_0, v_0; \tau)$  emerge in the velocity part  $\tilde{v}$  at the level of  $\tilde{V}_j$  as soon as  $j \geq 0$ ;
- The expressions  $\tilde{X}_0$  and  $\tilde{V}_0$  consist of distinct parts. There are first the two diffeomorphic maps  $X$  as in (2.4) and  $V$  as in (2.8) composed with the average flow  $Z$ , the scalar function  $\bar{\mathcal{V}}_1$ , and the toroidal angle  $\bar{\Phi}_0$ . Secondly, there is the contribution  $\langle \bar{\mathfrak{Z}}_0 \rangle$ . Now, the access to  $\langle \bar{\mathfrak{Z}}_0 \rangle$  is complicated. It requires the determination of  $\langle \bar{\mathfrak{A}}_1 \rangle$ . Computing or even finding the simplified expression of  $\langle \bar{\mathfrak{A}}_1 \rangle$  requires long computations which are achieved in Paragraph 4.3.1;
- There is a cancellation effect induced by the action of the operator  $\langle \cdot \rangle$  on  $\bar{\mathfrak{A}}_1(\cdot)$ . Indeed, in view of Theorem 1.9, we have  $\langle \bar{\mathfrak{A}}_{1r} \rangle \equiv 0$ . This implies that the spatial component  $\langle \bar{\mathfrak{Z}}_{0r} \rangle$  is not activated when solving (1.20). And thus, it suffices to look at the remaining three components  $\langle \bar{\mathfrak{Z}}_{0\theta} \rangle$ ,  $\langle \bar{\mathfrak{Z}}_{0v} \rangle$  and  $\langle \bar{\mathfrak{Z}}_{0\zeta} \rangle$ . We find also that  $\langle \bar{\mathfrak{A}}_{1v} \rangle \equiv 0$  when the electric field is radial, that is when  $\Phi = \Phi(r)$ , leading in this simplified case to a stability property concerning the component  $\langle \bar{\mathfrak{Z}}_{0v} \rangle$ .

1.4.3. *Plan of the text.* The present text is organized as follows.

Section 2 furnishes useful tools to deal with the axisymmetric configuration.

- In Subsection 2.1, we clarify the origin of (1.1).
- In Subsection 2.2, the space coordinate  $x$  is changed into the toroidal coordinates, while the velocity  $v$  is expressed in spherical coordinates coming from a basis  $(e_r, e_\perp, e_\parallel)$  where the vector  $e_\parallel$  represents the (unitary) direction of the magnetic field. The system (1.1) is then interpreted in terms of these new coordinates.

Section 3 describes the properties of the average flow.

- In Subsection 3.1, we specify the time evolution equations for the average flow.
- In Subsection 3.2, we study the periodic properties of the average flow.

Section 4 is devoted to applications.

- In Subsection 4.1, we show that our analysis is consistent with the one presented in [11].
- In Subsection 4.2, we prove Theorem 1.8.
- In Subsection 4.3, we exhibit and study the long time gyrokinetic equations.

## 2. MODELING OF THE DYNAMICS

In the present chapter, we are concerned with the equations satisfied by a charged particle in the presence of a prescribed electromagnetic field (which is not affected by the repartition of the charged particles). We consider the motion of an electron with mass  $m$  and electric charge  $-e$  (of negative sign) in a given electric field  $\mathbf{E}$  and magnetic field  $\mathbf{B}$ . In Subsection 2.1, we explain the origin of (1.1). In Subsection 2.2, we adapt the phase space and the equations in accordance with the geometries induced by the shape of the toroidal device and by the variations of  $\mathbf{B}$ .

**2.1. Nondimensionalization.** Under the influence of a Lorentz force  $\mathbf{F}$ , a relativistic charged particle is moving according to

$$(2.1) \quad \begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{v}, \\ \frac{d\mathbf{p}}{dt} = \mathbf{F} := -e [\mathbf{E}(\mathbf{x}) + \mathbf{v} \wedge \mathbf{B}(\mathbf{x})], \end{cases}$$

with the relations

$$\mathbf{v} = \frac{\mathbf{p}}{\gamma m}, \quad \gamma = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2} = \left(1 + \frac{\mathbf{p}^2}{m^2 c^2}\right)^{1/2}, \quad |\mathbf{v}| < c,$$

where  $c \simeq 3 \times 10^8 \text{ ms}^{-1}$  is the speed of light,  $e \simeq 1.6 \times 10^{-19} \text{ C}$  is the elementary charge, and  $m \simeq 9.1 \times 10^{-31} \text{ kg}$  is the electron mass.

**Lemma 2.1** (Equivalent system). *The system (2.1) amounts to the same thing as*

$$(2.2) \quad \begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{v}, \\ \frac{d\mathbf{v}}{dt} = -\frac{e}{m} \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{1/2} [\mathbf{E}(\mathbf{x}) + \mathbf{v} \wedge \mathbf{B}(\mathbf{x}) - c^{-2} (\mathbf{v} \cdot \mathbf{E}(\mathbf{x})) \mathbf{v}]. \end{cases}$$

*Proof.* The chain rule yields

$$\frac{\gamma^3}{c^2} m \left( \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \right) \mathbf{v} + \gamma m \frac{d\mathbf{v}}{dt} = -e [\mathbf{E} + \mathbf{v} \wedge \mathbf{B}].$$

We can decompose the acceleration  $a := d\mathbf{v}/dt$  into the sum of two parts  $a_{\parallel}$  and  $a_{\perp}$  with  $a_{\parallel}$  parallel to the velocity  $\mathbf{v}$  and  $a_{\perp}$  perpendicular to it. Thus, we have  $\mathbf{v} \cdot a_{\perp} = 0$  and  $\mathbf{v} \cdot a_{\parallel} = \mathbf{v} \cdot a$ . This furnishes  $(\mathbf{v} \cdot a_{\parallel})\mathbf{v} = \mathbf{v}^2 a_{\parallel}$  and therefore

$$\begin{aligned} \mathbf{F} &= \frac{\gamma^3 m \mathbf{v}^2}{c^2} a_{\parallel} + \gamma m (a_{\parallel} + a_{\perp}) = \gamma^3 m \left( \frac{\mathbf{v}^2}{c^2} + \frac{1}{\gamma^2} \right) a_{\parallel} + \gamma m a_{\perp} \\ &= \gamma^3 m \left( \frac{\mathbf{v}^2}{c^2} + 1 - \frac{\mathbf{v}^2}{c^2} \right) a_{\parallel} + \gamma m a_{\perp} = \gamma^3 m a_{\parallel} + \gamma m a_{\perp}. \end{aligned}$$

This can be used to calculate acceleration from force, yielding

$$(2.3) \quad a = \frac{1}{m \gamma} \left( \mathbf{F} - \frac{(\mathbf{v} \cdot \mathbf{F})\mathbf{v}}{c^2} \right).$$

Since  $\mathbf{v} \cdot \mathbf{F} = -e \mathbf{v} \cdot \mathbf{E}$ , after substitution inside (2.3), we find (2.2).  $\square$

Let us assume that  $\mathbf{x} \in \Omega \subset \mathbb{R}^3$  where  $\Omega$  is a bounded open domain of size  $\mathbf{L}$ . We then consider the nondimensional variable  $x := \mathbf{x}/\mathbf{L}$  with  $|x| < 1$ . We also introduce the normalized velocity  $v := \mathbf{v}/c$  with  $|v| < 1$ . Taking into account these selections, it is more convenient to work with the time variable  $t := c \mathbf{t}/\mathbf{L}$ . It remains to interpret the electric and magnetic fields in terms of

these variables. So, we select  $E(x) := \underline{\mathbf{E}}^{-1} \mathbf{E}(Lx)$  and  $B(x) := \underline{\mathbf{B}}^{-1} \mathbf{B}(Lx)$  where  $\underline{\mathbf{E}}$  and  $\underline{\mathbf{B}}$  are typical amplitudes of respectively  $\mathbf{E}$  and  $\mathbf{B}$ . The system (2.2) becomes

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = (1 - v^2)^{1/2} \left\{ -\frac{e \mathbf{L} \underline{\mathbf{B}}}{m c} v \wedge B(x) - \frac{e \mathbf{L} \underline{\mathbf{E}}}{m c^2} [E(x) - (v \cdot E(x)) v] \right\}. \end{cases}$$

The inverse of the electron gyrofrequency is the dimensionless parameter  $\varepsilon := (mc)/(e \mathbf{L} \underline{\mathbf{B}})$ . Common tokamaks are built with  $L \simeq 5m$  and  $\underline{\mathbf{B}} \simeq 5T$  so that  $\varepsilon \simeq 10^{-4}$ . On the other hand, in line with [14, 17, 20], we can consider that

$$\frac{e \mathbf{L} \underline{\mathbf{E}}}{m c^2} = 1.$$

This is consistent with the realistic selections of  $\underline{\mathbf{E}} \simeq 5 \times 10^5 V m^{-1}$ . This furnishes (1.1) as a starting point for our analysis.

**2.2. Geometric reformulation of the equations.** The cartesian coordinates are not suitable to differentiate the various types of motion. In Paragraph 2.2.1, we present a toroidal description of  $x$ , and we decompose  $v$  in spherical coordinates emanating from an orthonormal basis following the variations of  $B$ . In Paragraph 2.2.2, we reformulate the equations for  $x$  accordingly. Then, in Paragraph 2.2.3, we do the same for  $v$ .

2.2.1. *Change of coordinates for  $x$  and  $v$ .* Recall (1.13). Remark that

$$(2.4) \quad x = X(r, \theta, \phi) = R_0 (\cos \phi, \sin \phi, 0) + r e_r, \quad \dot{x} = \dot{r} e_r + r \dot{\theta} e_\theta + R(r, \theta) \dot{\phi} e_\phi$$

as well as

$$\nabla_x = e_r \partial_r + e_\theta \frac{1}{r} \partial_\theta + e_\phi \frac{1}{R} \partial_\phi$$

Picture 1 represents  $r, \theta, \phi$  (and the shape of the confined plasma). From now on, we will work with  $(r, \theta, \phi) \in [0, a] \times \mathbb{T}^2$  instead of  $x$ . By assumption 1.6, the function  $B(\cdot)$  takes the form

$$B = I(r) \nabla \phi + \nabla \phi \wedge \nabla r, \quad I(r) \in \mathbb{R}_+^*,$$

which leads to

$$B = I(r) R^{-1} e_\phi + R^{-1} e_\phi \wedge e_r = \frac{I(r)}{R(r, \theta)} e_\phi - \frac{1}{R(r, \theta)} e_\theta.$$

Retain that

$$B(r, \theta) = B_\theta(r, \theta) e_\theta + B_\phi(r, \theta) e_\phi, \quad B_\theta(r, \theta) := -\frac{1}{R(r, \theta)}, \quad B_\phi(r, \theta) := \frac{I(r)}{R(r, \theta)}.$$

We aim to straighten out the field lines in an orthogonal way. For that, we define

$$e_\parallel(r, \theta, \phi) := -\frac{1}{\sqrt{1 + I(r)^2}} e_\theta(\theta, \phi) + \frac{I(r)}{\sqrt{1 + I(r)^2}} e_\phi(\phi).$$

The magnetic field then can be written

$$B(X(r, \theta, \phi)) = b(r, \theta) e_\parallel(r, \theta, \phi), \quad b(r, \theta) := \frac{1}{R(r, \theta)} \sqrt{1 + I(r)^2},$$

where  $b > 0$  is the norm of the magnetic field  $B$ . Introduce the angle  $w(r) \in ]-\pi/2, 0[$  between the two directions  $e_\phi$  and  $e_\parallel$ . We have

$$e_\parallel(r, \theta, \phi) = \sin \omega(r) e_\theta(\theta, \phi) + \cos \omega(r) e_\phi(\phi).$$

In the basis  $(e_\phi, e_\theta)$ , this gives rise to

$$(\cos \omega(r), \sin \omega(r)) = \left( \frac{I(r)}{\sqrt{1 + I(r)^2}}, -\frac{1}{\sqrt{1 + I(r)^2}} \right).$$

The unitary direction  $e_{\parallel}$  can be completed in order to obtain a positively oriented orthonormal basis  $(e_r, e_{\perp}, e_{\parallel})$  with  $e_r \wedge e_{\perp} = -e_{\parallel}$ . We find that

$$e_{\perp}(r, \theta, \phi) := \cos \omega(r) e_{\theta}(\theta, \phi) - \sin \omega(r) e_{\phi}(\phi).$$

We can also pass from  $(e_{\perp}, e_{\parallel})$  to  $(e_{\theta}, e_{\phi})$  through

$$(2.5) \quad e_{\theta} = +\cos \omega e_{\perp} + \sin \omega e_{\parallel}, \quad e_{\phi} = -\sin \omega e_{\perp} + \cos \omega e_{\parallel}.$$

From (1.14) and (2.5), we have

$$E(x) = E_r(r, \theta) e_r + E_{\perp}(r, \theta) e_{\perp} + E_{\parallel}(r, \theta) e_{\parallel},$$

with

$$(2.6) \quad E_r(r, \theta) = \partial_r \Phi(r, \theta), \quad E_{\perp}(r, \theta) = r^{-1} \cos \omega(r) \partial_{\theta} \Phi(r, \theta),$$

and

$$(2.7) \quad E_{\parallel}(r, \theta) = r^{-1} \sin \omega(r) \partial_{\theta} \Phi(r, \theta).$$

On the other hand,  $v(t)$  can be decomposed according to the spherical coordinates associated with the basis  $(e_r, e_{\perp}, e_{\parallel})$  as follows

$$(2.8) \quad \begin{aligned} v &= V(r, \theta, \phi, v, \zeta, \nu) \\ &= v \left[ \cos \zeta \cos \nu e_r(\theta, \phi) + \cos \zeta \sin \nu e_{\perp}(r, \theta, \phi) + \sin \zeta e_{\parallel}(r, \theta, \phi) \right]. \end{aligned}$$

with radial distance  $v := |v| \in \mathbb{R}_+$ , azimuth angle  $\zeta \in [-\pi/2, \pi/2]$ , and polar angle  $\nu \in \mathbb{T}$ . The two maps  $V(r, \theta, \phi, 0, \cdot, \cdot)$  and  $V(r, \theta, \phi, v, \pm\pi/2, \cdot)$  are not one-to-one. Unless otherwise stated, we will work away from  $v = 0$  (with  $0 < v < 1$ ) and away from parallel and antiparallel situations (with  $\zeta \notin \{-\pi/2, +\pi/2\}$ ). The number  $\nu$  allows to describe rotations in the plane perpendicular to  $e_{\parallel}$ . From the perspective of gyrokinetics (as long as  $t \simeq 1$ ), the coordinates  $r, \theta, \phi, v$  and  $\zeta$  undergo slow variations, whereas  $\nu$  is subjected to fast fluctuations.

**2.2.2. Equations in toroidal coordinates for  $x$ .** The purpose of this paragraph is to interpret (1.1) in terms of  $r, \theta, \phi, v, \zeta$  and  $\nu$ . To this end, we follow [8] with some adaptations due to the presence of the electric field. The spatial part  $(r, \theta, \phi)$  can be handled as follows.

**Lemma 2.2.** *[Spatial toroidal equations] The first equation inside (1.1) furnishes*

$$(2.9) \quad \begin{pmatrix} \frac{dr}{dt} \\ \frac{d\theta}{dt} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_r \\ \mathcal{A}_{\theta} \end{pmatrix} := v \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r} \cos \omega(r) & \frac{1}{r} \sin \omega(r) \end{pmatrix} \begin{pmatrix} \cos \zeta \cos \nu \\ \cos \zeta \sin \nu \\ \sin \zeta \end{pmatrix},$$

together with

$$(2.10) \quad \frac{d\phi}{dt} = \mathcal{A}_{\phi} := \frac{v}{R(r, \theta)} (-\sin \omega(r) \cos \zeta \sin \nu + \cos \omega(r) \sin \zeta).$$

At time  $t = 0$ , we start with  $(r, \theta, \phi)(0) = (r_0, \theta_0, \phi_0)$ .

*Proof.* We can express the spherical coordinates (2.8) of  $v$  in the basis  $(e_r, e_{\theta}, e_{\phi})$  through

$$(2.11) \quad \begin{aligned} v &= v \cos \zeta \cos \nu e_r(\theta, \phi) + v [\cos \zeta \sin \nu \cos \omega(r) + \sin \zeta \sin \omega(r)] e_{\theta}(\theta, \phi) \\ &\quad + v [\sin \zeta \cos \omega(r) - \cos \zeta \sin \nu \sin \omega(r)] e_{\phi}(\phi). \end{aligned}$$

Just compare the second part of (2.4) to the equation (2.11) to find (2.9) and (2.10).  $\square$

2.2.3. *Equations in spherical coordinates for  $v$ .* Below, we interpret the second equation of (1.1) in terms of  $r, \theta, \phi, v, \zeta$  and  $\nu$ .

**Lemma 2.3** (Velocity equations in spherical coordinates). *We have*

$$(2.12) \quad \begin{cases} \dot{v} = \mathcal{A}_v := -(1-v^2)^{3/2} [\cos \zeta \cos \nu E_r + \cos \zeta \sin \nu E_\perp + \sin \zeta E_\parallel], \\ \dot{\zeta} = \mathcal{A}_\zeta := v \mathcal{F}_\zeta^g + \frac{(1-v^2)^{1/2}}{v} [\sin \zeta \cos \nu E_r + \sin \zeta \sin \nu E_\perp - \cos \zeta E_\parallel], \\ \dot{\nu} = \frac{v \mathcal{F}_\nu^g}{\cos \zeta} + \frac{(1-v^2)^{1/2}}{v \cos \zeta} [\sin \nu E_r - \cos \nu E_\perp - \frac{1}{\varepsilon} v b \cos \zeta], \end{cases}$$

where  $\mathcal{F}_\zeta^g$  and  $\mathcal{F}_\nu^g$  are geometrical coefficients given by

$$\begin{aligned} \mathcal{F}_\zeta^g(r, \zeta, \theta, \nu) &:= \omega'(r) \cos \zeta \cos \nu \sin \nu \\ &\quad - \frac{1}{r} \sin \omega \cos \nu (\cos \omega \cos \zeta \sin \nu + \sin \omega \sin \zeta) \\ &\quad - \frac{1}{R} (\cos \theta \cos \omega \cos \nu - \sin \theta \sin \nu) (-\sin \omega \cos \zeta \sin \nu + \cos \omega \sin \zeta). \end{aligned}$$

$$\begin{aligned} \mathcal{F}_\nu^g(r, \theta, \zeta, \nu) &:= -\omega'(r) \cos \zeta (\cos \nu)^2 \sin \zeta \\ &\quad - \frac{1}{r} (\cos \omega \sin \nu \cos \zeta + \sin \omega \sin \zeta) (\sin \omega \sin \nu \sin \zeta + \cos \omega \cos \zeta) \\ &\quad - \frac{1}{R} (-\sin \omega \cos \zeta \sin \nu + \cos \omega \sin \zeta) (\cos \omega \cos \theta \sin \nu + \sin \theta \cos \nu) \sin \zeta \\ &\quad + \frac{1}{R} (-\sin \omega \cos \zeta \sin \nu + \cos \omega \sin \zeta) \cos \zeta \sin \omega \cos \theta. \end{aligned}$$

Note that the right hand sides of (2.12) are not defined when  $v = 0$  or when  $\zeta \in \{-\pi/2, +\pi/2\}$ . For this reason, at time  $t = 0$ , we start with  ${}^t(v, \zeta, \nu)(0) = {}^t(v_0, \zeta_0, \nu_0)$  satisfying

$$(2.13) \quad 0 < v_0 < 1, \quad \zeta_0 \notin \{-\pi/2, +\pi/2\}.$$

The system (2.12) involves all components  $E_r$ ,  $E_\perp$  and  $E_\parallel$  of the electric field  $E$ . The effects on the motion of these components are interpreted in [19, Sections 2.5, 2.6, 3.9]. In particular:

- The perpendicular component  $E_\perp$  takes part in the so called  $E \times B$  drift which may induce small displacements in perpendicular directions to both fields  $E$  and  $B$ ;
- The parallel component  $E_\parallel$  lies in the magnetic surface. It can be responsible for the acceleration or deceleration of particles along the magnetic field.

*Proof.* We introduce first the auxilliary orthonormal basis  $(e_v^1, e_v^2, e_v^3)$  such that  $e_v^1 \wedge e_v^2 = e_v^3$  and built with

$$\begin{aligned} e_v^1 &:= +\cos \zeta \cos \nu e_r + \cos \zeta \sin \nu e_\perp + \sin \zeta e_\parallel, \\ e_v^2 &:= -\sin \zeta \cos \nu e_r - \sin \zeta \sin \nu e_\perp + \cos \zeta e_\parallel, \\ e_v^3 &:= -\sin \nu e_r + \cos \nu e_\perp. \end{aligned}$$

We can also express  $(e_r, e_\perp, e_\parallel)$  in terms of this new basis:

$$\begin{aligned} e_r &:= \cos \zeta \cos \nu e_v^1 - \sin \zeta \cos \nu e_v^2 - \sin \nu e_v^3, \\ e_\perp &:= \cos \zeta \sin \nu e_v^1 - \sin \zeta \sin \nu e_v^2 + \cos \nu e_v^3, \\ e_\parallel &:= \sin \zeta e_v^1 + \cos \zeta e_v^2. \end{aligned}$$

By construction, we have  $v = v e_v^1$  which can be used to compute the second equation of (1.1) in terms of  $r, \theta, \phi, v, \zeta$  and  $\nu$ . To this end, we can exploit the three following intermediate formulas:

(1) The first one is

$$\begin{aligned} \dot{v} = & \dot{v} e_v^1 + v \dot{\zeta} e_v^2 + v \cos \zeta \dot{\nu} e_v^3 + v \cos \zeta (\dot{\theta} \cos \omega - \dot{\phi} \sin \omega \cos \theta) e_v^3 \\ & + v (\dot{\theta} \sin \omega + \dot{\phi} \cos \omega \cos \theta) (-\sin \zeta e_r + \cos \zeta \cos \nu e_{\parallel}) \\ & + v (\dot{r} \omega'(r) + \dot{\phi} \sin \theta) (\sin \zeta e_{\perp} - \cos \zeta \sin \nu e_{\parallel}). \end{aligned}$$

(2) The second is

$$\begin{pmatrix} \cos \nu & -\sin \nu \\ \sin \nu & +\cos \nu \end{pmatrix} \begin{pmatrix} -\sin \zeta e_r + \cos \zeta \cos \nu e_{\parallel} \\ +\sin \zeta e_{\perp} - \cos \zeta \sin \nu e_{\parallel} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin \zeta \end{pmatrix} \begin{pmatrix} e_v^2 \\ e_v^3 \end{pmatrix}.$$

(3) The third is

$$B(X) \wedge V = -v b(r, \theta) \cos \zeta e_v^3.$$

By combining (1) and (2), we get

$$\begin{aligned} \dot{v} = & \dot{v} e_v^1 + v \dot{\zeta} e_v^2 + v \cos \zeta \dot{\nu} e_v^3 + v \cos \zeta (\dot{\theta} \cos \omega - \dot{\phi} \sin \omega \cos \theta) e_v^3 \\ & + v (\dot{\theta} \sin \omega + \dot{\phi} \cos \omega \cos \theta) \sin \nu \sin \zeta e_v^3 + v (\dot{r} \omega'(r) + \dot{\phi} \sin \theta) \cos \nu \sin \zeta e_v^3 \\ & + v (\dot{\theta} \sin \omega + \dot{\phi} \cos \omega \cos \theta) \cos \nu e_v^2 - v (\dot{r} \omega'(r) + \dot{\phi} \sin \theta) \sin \nu e_v^2. \end{aligned}$$

This allow us to write the second equation of (1.1) in the basis  $(e_v^1, e_v^2, e_v^3)$  according to

$$\begin{aligned} 0 = & (1 - v^2)^{-1/2} \dot{v} + E - (v \cdot E) v + \frac{1}{\varepsilon} v \wedge B \\ = & [(1 - v^2)^{-1/2} \dot{v} + (1 - v^2) (\cos \zeta \cos \nu E_r + \cos \zeta \sin \nu E_{\perp} + \sin \zeta E_{\parallel})] e_v^1 \\ & + [(1 - v^2)^{-1/2} v \dot{\zeta} + (1 - v^2)^{-1/2} v (\dot{\theta} \sin \omega + \dot{\phi} \cos \omega \cos \theta) \cos \nu \\ & - (1 - v^2)^{-1/2} v (\dot{r} \omega'(r) + \dot{\phi} \sin \theta) \sin \nu - \sin \zeta \cos \nu E_r - \sin \zeta \sin \nu E_{\perp} + \cos \zeta E_{\parallel}] e_v^2 \\ & + [(1 - v^2)^{-1/2} v \cos \zeta \dot{\nu} + (1 - v^2)^{-1/2} v (\dot{\theta} \sin \omega + \dot{\phi} \cos \omega \cos \theta) \sin \nu \sin \zeta \\ & + (1 - v^2)^{-1/2} v (\dot{r} \omega'(r) + \dot{\phi} \sin \theta) \cos \nu \sin \zeta \\ & + (1 - v^2)^{-1/2} v \cos \zeta (\dot{\theta} \cos \omega - \dot{\phi} \sin \omega \cos \theta) - \sin \nu E_r + \cos \nu E_{\perp} + \frac{1}{\varepsilon} v b \cos \zeta] e_v^3. \end{aligned}$$

As  $(e_v^1, e_v^2, e_v^3)$  is a basis, all components must be zero. The one in factor of  $e_v^1$  leads to the first equation of (2.12). Those in factor of  $e_v^2$  and  $e_v^3$ , after replacing the time derivatives of  $\dot{r}, \dot{\theta}, \dot{\phi}$  as in (2.9) and (2.10), give rise respectively to the second and third equation of (2.12).  $\square$

To absorb the large factor  $\varepsilon^{-1}$  in the last equation of (2.12), we perform now a sort of filtering method. Recall that we have decomposed the velocity in spherical coordinates where  $v$  as well as  $\zeta$  are intended to be slow scales whereas  $\nu \in \mathbb{T}$  is a fast one. This implies that it is natural to seek  $\nu$  in the form  $\nu = \varepsilon^{-1} v$  with, at time  $t = 0$ , the condition  $v(0) = \varepsilon \nu_0$ . Then, comparing this to the third equation in (2.12), we get

$$(2.14) \quad \dot{v} = \mathcal{V}_0(r, \theta, v) + \varepsilon \mathcal{V}_1(r, \theta, v, \zeta, \nu), \quad v(0) = \varepsilon \nu_0,$$

where the two source terms  $\mathcal{V}_0$  and  $\mathcal{V}_1$  are given by

$$(2.15) \quad \mathcal{V}_0(r, \theta, v) := -(1 - v^2)^{1/2} b(r, \theta) < 0,$$

$$(2.16) \quad \mathcal{V}_1(r, \theta, v, \zeta, \nu) := v (\cos \zeta)^{-1} \mathcal{F}_\nu^g + v^{-1} (\cos \zeta)^{-1} (1 - v^2)^{1/2} [\sin \nu E_r - \cos \nu E_{\perp}].$$

The right hand sides of (2.9) and (2.12) do not depend on  $\phi$ . Thus, we can put  $\phi$  aside. The strategy to solve (1.1) is to focus on (2.9)-(2.12), and then (ultimately) to consider (2.10) to recover  $\phi$ . By construction, the system (2.9)-(2.12) amounts to the same thing as (1.3) with

$$(2.17) \quad z := {}^t(r, \theta, v, \zeta), \quad \mathcal{A} := {}^t(\mathcal{A}_r, \mathcal{A}_\theta, \mathcal{A}_v, \mathcal{A}_\zeta), \quad v := \varepsilon \nu, \quad \mathcal{V} := \mathcal{V}_0 + \varepsilon \mathcal{V}_1, \quad \nu_0 = \varepsilon \nu_0.$$

At time  $t = 0$ , we start with  $z(0) = z_0 := {}^t(r_0, \theta_0, v_0, \zeta_0)$  where

$$(2.18) \quad 0 < r_0 < a < R_0, \quad 0 \leq \theta_0 < 2\pi, \quad 0 < v_0 < 1, \quad -\frac{\pi}{2} < \zeta_0 < \frac{\pi}{2}.$$

The nonlinear source term  ${}^t(\mathcal{A}, \mathcal{V})(\varepsilon; z; \nu)$  satisfies (1.4) and (1.5). In particular, it is periodic of period  $2\pi$  with respect to the last variable  $\nu$ . The Cauchy-Lipschitz theorem guarantees the existence of unique local solution to (1.3). Since both  $\mathcal{A}$  and  $\mathcal{V}$  are bounded for  $z$  in a compact set and  $\nu \in \mathbb{T}$ , the lifespan  $T_\varepsilon$  is such that  $T_\varepsilon \geq T$  for some  $T > 0$ .

### 3. WORK OF PREPARATION

We consider here the average flow (in the sense of Definition 1.1) which is associated with the system (1.3) on  $r, \theta, v$  and  $\zeta$  derived from (1.1). Thus, with  $Z = {}^t(Z_r, Z_\theta, Z_v, Z_\zeta)$  and  $\bar{\mathcal{A}} = {}^t(\bar{\mathcal{A}}_r, \bar{\mathcal{A}}_\theta, \bar{\mathcal{A}}_v, \bar{\mathcal{A}}_\zeta)$ , we look at the flow generated by (1.7). In Subsection 3.1, its structure is made explicit. In Subsection 3.2, its periodic properties (which are essential to eliminate problems of secular growth) are investigated.

**3.1. Structure of the average flow.** In Subsection 3.1.1, we detail the content of (1.7) for the realistic data issued from (2.9) and (2.12). In Subsection 3.1.2, we exhibit two invariant quantities. In Subsection 3.1.3, this information is exploited to show that the lifespan associated with (1.7) is infinite. Finally, in Subsection 3.1.4, we come back to the parallel case.

**3.1.1. Equations for the average flow.** The first step is to clarify the structure of (1.7).

**Lemma 3.1** (The system of odes in  $]0, a[ \times \mathbb{T} \times ]0, 1[ \times ]-\pi/2, \pi/2[$  satisfied by the average flow).  
We find that

$$(3.1) \quad \frac{dZ_r}{dt} = 0, \quad Z_r(0) = r_0.$$

For the remaining components, we obtain the following system of coupled equations

$$(3.2) \quad \begin{cases} \frac{dZ_\theta}{dt} = +r_0^{-1} \sin \omega(r_0) Z_v \sin Z_\zeta, & Z_\theta(z_0; 0) = \theta_0, \\ \frac{dZ_v}{dt} = -(1 - Z_v^2)^{3/2} \sin Z_\zeta E_{\parallel}(r_0, Z_\theta), & Z_v(z_0; 0) = v_0, \\ \frac{dZ_\zeta}{dt} = -Z_v (2R(r_0, Z_\theta))^{-1} \sin \omega(r_0) \sin Z_\theta \cos Z_\zeta \\ \quad - Z_v^{-1} (1 - Z_v^2)^{1/2} \cos Z_\zeta E_{\parallel}(r_0, Z_\theta), & Z_\zeta(z_0; 0) = \zeta_0. \end{cases}$$

From (3.1), we know that  $Z_r(t) = r_0$  for all  $t \in \mathbb{R}$ . In comparison with (2.12), observe that both components  $E_r$  and  $E_\perp$  have disappeared from (3.2).

*Proof.* We have  $\bar{\mathcal{A}}_r = 0$  so that (3.1) is obvious. By formula (1.6), we can see that

◁ The mean value of  $\mathcal{A}_\theta$  is

$$\begin{aligned} \bar{\mathcal{A}}_\theta(z) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{v}{r_0} [\cos \zeta \sin \nu \cos \omega(r_0) + \sin \zeta \sin \omega(r_0)] d\nu \\ &= r_0^{-1} v \sin \zeta \sin \omega(r_0). \end{aligned}$$

◁ The mean value of  $\mathcal{A}_v$  is

$$\begin{aligned} \bar{\mathcal{A}}_v(z) &= \frac{1}{2\pi} \int_0^{2\pi} -(1 - v^2)^{3/2} [\cos \zeta \cos \nu E_r + \cos \zeta \sin \nu E_\perp + \sin \zeta E_{\parallel}] d\nu \\ &= -(1 - v^2)^{3/2} \sin \zeta E_{\parallel}(r_0, \theta). \end{aligned}$$



◁ The mean value of  $\mathcal{A}_\zeta$  is

$$\begin{aligned}\bar{\mathcal{A}}_\zeta(z) &= \frac{1}{2\pi} \int_0^{2\pi} [\nu \mathcal{F}_\zeta^g + \nu^{-1} (1 - \nu^2)^{1/2} (\sin \zeta \cos \nu E_r + \sin \zeta \sin \nu E_\perp - \cos \zeta E_\parallel)] d\nu \\ &= -\nu^{-1} (1 - \nu^2)^{1/2} \cos \zeta E_\parallel - \frac{1}{2\pi} \frac{\nu}{R} \sin \theta \cos \zeta \sin \omega \int_0^{2\pi} \frac{1 + \cos 2\nu}{2} d\nu \\ &= -\nu^{-1} (1 - \nu^2)^{1/2} \cos \zeta E_\parallel(r_0, \theta) - \frac{\nu}{2R(r_0, \theta)} \sin \theta \cos \zeta \sin \omega(r_0).\end{aligned}$$

It is now clear that (1.7) implies (3.2).  $\square$

3.1.2. *Two invariant quantities.* With  $\Phi$  as in (2.7), define

$$(3.3) \quad H_1(r_0; \theta, \nu, \zeta) := \nu (1 - \nu^2)^{-1/2} (R_0 + r_0 \cos \theta)^{1/2} \cos \zeta,$$

$$(3.4) \quad H_2(r_0; \theta, \nu, \zeta) \equiv \mathcal{H}_2(r_0; \theta, \nu) := \Phi(r_0, \theta) + (1 - \nu^2)^{-1/2}.$$

**Lemma 3.2** (Constants of motion). *For all  $r_0 \in \mathbb{R}_+^*$ , the two functions  $H_1(r_0; \cdot)$  and  $H_2(r_0; \cdot)$  given by (3.3) and (3.4) are constant along the trajectories induced by (3.2), in the sense that*

$$(3.5) \quad \frac{d}{dt} [H_i(r_0; Z_\theta, Z_\nu, Z_\zeta)] = 0, \quad \forall i \in \{1, 2\}.$$

*Proof.* Compute

$$\begin{aligned}(3.6) \quad \partial_\theta H_1(r_0; \theta, \nu, \zeta) &= -\frac{1}{2} r_0 \nu (1 - \nu^2)^{-1/2} R(r_0, \theta)^{-1/2} \sin \theta \cos \zeta, \\ \partial_\nu H_1(r_0; \theta, \nu, \zeta) &= \sqrt{R(r_0, \theta)} \cos \zeta [(1 - \nu^2)^{-1/2} + \nu^2 (1 - \nu^2)^{-3/2}], \\ \partial_\zeta H_1(r_0; \theta, \nu, \zeta) &= -\nu (1 - \nu^2)^{-1/2} \sqrt{R(r_0, \theta)} \sin \zeta, \\ \partial_\theta H_2(r_0; \theta, \nu) &= r_0 (\sin \omega(r_0))^{-1} E_\parallel(r_0, \theta) = \partial_\theta \Phi(r_0, \theta), \\ \partial_\nu H_2(r_0; \theta, \nu) &= \nu (1 - \nu^2)^{-3/2}.\end{aligned}$$

We derive now the two function  $H_1$  and  $H_2$  with respect to time to get

$$\begin{aligned}(3.7) \quad \frac{d}{dt} H_1(r_0; Z_\theta, Z_\nu, Z_\zeta) &= \partial_\theta H_1(r_0; Z_\theta, Z_\nu, Z_\zeta) \frac{dZ_\theta}{dt} + \partial_\nu H_1(r_0; Z_\theta, Z_\nu, Z_\zeta) \frac{dZ_\nu}{dt} \\ &\quad + \partial_\zeta H_1(r_0; Z_\theta, Z_\nu, Z_\zeta) \frac{dZ_\zeta}{dt}, \\ \frac{d}{dt} H_2(r_0; Z_\theta, Z_\nu) &= \partial_\theta H_2(r_0; Z_\theta, Z_\nu) \frac{dZ_\theta}{dt} + \partial_\nu H_2(r_0; Z_\theta, Z_\nu) \frac{dZ_\nu}{dt}.\end{aligned}$$

Substitute in (3.7) the terms  $dZ_\theta/dt$ ,  $dZ_\nu/dt$  and  $dZ_\zeta/dt$  as indicated in (3.2) as well as the partial derivatives of  $H_i$  with  $i \in \{1, 2\}$  as indicated in (3.6) to get that (3.5) holds true.  $\square$

In general (at least when  $\zeta \neq 0$ ), the level surfaces of  $H_1$  and  $H_2$  intersect transversally in the three dimensional space  $\mathbb{T} \times ]0, 1[ \times \mathbb{T}$  to define curves which correspond to integral curves of (3.2).

3.1.3. *Study of the lifespan.* The conservation of  $H_1$  and  $H_2$  allows to extract the following interesting information.

**Lemma 3.3** (Infinite lifespan). *Fix any  $r_0 \in ]0, a]$ . Then, for all  $(\theta_0, \nu_0, \zeta_0)$  satisfying (2.18), the solution to the nonlinear system (3.2) exists for all times  $t \in \mathbb{R}$ .*

*Proof.* We denote by  $S(z_0) \in \mathbb{R}_+^* \cup \{+\infty\}$  the lifespan associated with (3.2). From (3.5) with  $i = 2$ , we can deduce that

$$(3.8) \quad (1 - Z_\nu(t)^2)^{-1/2} = -\frac{r_0}{\sin \omega(r_0)} \int_{\theta_0}^{Z_\theta(t)} E_\parallel(r_0, \tilde{\theta}) d\tilde{\theta} + (1 - \nu_0^2)^{-1/2}.$$

The function  $E_{\parallel}(r_0, \cdot)$  is a periodic function with mean value equal to zero. Define

$$-\infty < m(r_0) := \min_{Z_{\theta} \in \mathbb{R}} \int_0^{Z_{\theta}} E_{\parallel}(r_0, \tilde{\theta}) d\tilde{\theta} \leq 0, \quad 0 \leq M(r_0) := \max_{Z_{\theta} \in \mathbb{R}} \int_0^{Z_{\theta}} E_{\parallel}(r_0, \tilde{\theta}) d\tilde{\theta} < +\infty.$$

Recall that  $\omega(r_0) \in ]-\pi/2, 0[$  so that  $\sin \omega(r_0) < 0$ . We see that

$$\forall t \in [0, S(z_0)[, \quad (1 - Z_v(t)^2)^{-1/2} \leq -\frac{r_0}{\sin \omega(r_0)} M(r_0) + (1 - v_0^2)^{-1/2}.$$

This implies that

$$(3.9) \quad \exists \eta_1 \in ]0, 1[; \quad \forall t \in [0, S(z_0)[, \quad Z_v(t) \leq 1 - \eta_1.$$

From (3.5) with  $i = 1$ , we can infer that

$$\begin{aligned} Z_v &= v_0 \frac{(1 - Z_v^2)^{1/2}}{(1 - v_0^2)^{1/2}} \frac{(R_0 + r_0 \cos \theta_0)^{1/2}}{(R_0 + r_0 \cos Z_{\theta})^{1/2}} \frac{\cos \zeta_0}{\cos Z_{\zeta}} \\ &\geq v_0 \frac{\sqrt{\eta_1}}{(1 - v_0^2)^{1/2}} \frac{(R_0 + r_0 \cos \theta_0)^{1/2}}{(R_0 + r_0)^{1/2}} \cos \zeta_0 > 0. \end{aligned}$$

By this way, we get

$$(3.10) \quad \exists \eta_2 \in ]0, 1[; \quad \forall t \in [0, S(z_0)[, \quad \eta_2 \leq Z_v(t).$$

But we can also assert that

$$\begin{aligned} \cos Z_{\zeta} &= v_0 \frac{(1 - Z_v^2)^{1/2}}{(1 - v_0^2)^{1/2}} \frac{(R_0 + r_0 \cos \theta_0)^{1/2}}{(R_0 + r_0 \cos Z_{\theta})^{1/2}} \frac{\cos \zeta_0}{Z_v} \\ &\geq v_0 \frac{\sqrt{\eta_1}}{(1 - v_0^2)^{1/2}} \frac{(R_0 + r_0 \cos \theta_0)^{1/2}}{(R_0 + r_0)^{1/2}} \cos \zeta_0 > 0, \end{aligned}$$

which means that

$$(3.11) \quad \exists \eta_3 \in ]0, 1[; \quad \forall t \in [0, S(z_0)[, \quad -\frac{\pi}{2} + \eta_3 \leq Z_{\zeta}(t) \leq \frac{\pi}{2} - \eta_3.$$

Thus, the flow remains in a compact set of  $\mathbb{T} \times ]0, 1[ \times ]-\pi/2, \pi/2[$ . By extension theorem, this implies that  $S(z_0) = +\infty$ .  $\square$

In view of (2.8), we have

$$v \cdot e_{\parallel} = v \sin \zeta = v \cos(\pi/2 - \zeta).$$

Thus, the real number  $\pi/2 - \zeta \in ]0, \pi[$  represents the angle between the direction  $v$  and the magnetic field  $B$ . The endpoints  $\zeta = -\pi/2$  and  $\zeta = \pi/2$  correspond respectively to antiparallel and parallel situations.

**3.1.4. Solving the average flow in the parallel case.** As already observed, the system (2.12) is not meaningful when  $\cos \zeta = 0$ . By contrast, the right hand sides of (3.2) are defined and smooth with respect to  $Z_{\zeta}$  without any restriction on  $Z_{\zeta}$ . By continuity, any (local or global if it does exist) solution to (3.2) issued from  $\zeta_0 = -\pi/2$  or  $\zeta_0 = \pi/2$  tells us what happens (approximately) when  $\cos \zeta_0 > 0$  is small. We work with  $\zeta_0 = \pi/2$ , the other case  $\zeta_0 = -\pi/2$  being very similar. In the limiting case  $\zeta_0 = \pi/2$ , we find that  $Z_{\zeta}(t) = \pi/2$  for all time  $t$ , and we have just to solve

$$(3.12) \quad \begin{cases} \frac{dZ_{\theta}}{dt} = +r_0^{-1} Z_v \sin \omega(r_0), & Z_{\theta}(z_0; 0) = \theta_0, \\ \frac{dZ_v}{dt} = -(1 - Z_v^2)^{3/2} E_{\parallel}(r_0, Z_{\theta}), & Z_v(z_0; 0) = v_0. \end{cases}$$

The functional  $H_1$  (which is zero) does not help. On the other hand, the conservation of  $H_2$  yields (3.9) as before. To simplify, let us assume that

$$-\frac{r_0 m(r_0)}{\sin \omega(r_0)} + (1 - v_0^2)^{-1/2} > 1.$$

Using (3.8), this ensures that  $Z_v > 0$  remains away from zero. As a consequence, we have

$$\exists \eta_4 > 0; \quad \forall t \in \mathbb{R}_+, \quad \frac{dZ_\theta}{dt} \leq -\eta_4 < 0.$$

Define  $T := \inf \{ t \in \mathbb{R}_+^* ; Z_\theta(t) = \theta_0 - 2\pi \}$ . From (3.8), we see that  $Z_v(T) = v_0$ . Thus, at the time  $T$ , we return to the same initial data as in (3.12). The solution of (3.12) viewed in  $\mathbb{T} \times ]0, 1[$  is therefore periodic of period  $T \in \mathbb{R}_+^*$ . This brief discussion suggests that (3.2) could generate similar periodic features. The next subsection checks this point.

**3.2. Periodic properties of the average flow.** We start with initial data as in (2.18). The next step is to show that (3.2) is made of a family of Hamiltonian subsystems sharing the same simplified form, namely (3.23). This is achieved in Paragraph 3.2.1 after a series of change of variables. In the resulting Hamiltonian subsystems, as observed in Paragraph 3.2.2, we can decompose the phase space into the union of distinct subsets leading to different types of motion: libration, rotation or convergence to a fixed point (separatrices). In the two first cases, we can construct action-angle variables. In Paragraph 3.2.3, we select a domain  $\Omega$  that avoids the separatrix region. We observe that the flow issued from positions inside  $\Omega$  is periodic. In Paragraph 3.2.3, we exhibit extra almost periodic features.

**3.2.1. Foliation by Hamiltonian subsystems.** Define

$$\begin{aligned} \mathcal{U}_v : ]0, 1[ &\longrightarrow ]1, +\infty[ & \mathcal{U}_\zeta : ]-\pi/2, \pi/2[ &\longrightarrow \mathbb{R} \\ Z_v &\longmapsto (1 - Z_v^2)^{-1/2}, & Z_\zeta &\longmapsto \sqrt{2} \operatorname{sgn} Z_\zeta (-\ln \cos Z_\zeta)^{1/2}. \end{aligned}$$

It is clear that  $\mathcal{U}_v : ]0, 1[ \rightarrow ]1, +\infty[$  is one-to-one. The case of  $\mathcal{U}_\zeta$  is examined below.

**Lemma 3.4** (About  $\mathcal{U}_\zeta$ ). *The function  $\mathcal{U}_\zeta : ]-\pi/2, \pi/2[ \rightarrow \mathbb{R}$  is a smooth diffeomorphic map.*

*Proof.* Due to the factor  $\operatorname{sgn} Z_\zeta$ , remark that  $\mathcal{U}_\zeta(Z_\zeta) \sim Z_\zeta$  near  $Z_\zeta = 0$ . This indicates that  $\mathcal{U}_\zeta$  should be a local diffeomorphism near  $Z_\zeta = 0$ . For a global argument, consider the function

$$\begin{aligned} G : ]-\pi/2, \pi/2[ &\longrightarrow \mathbb{R}_+ \\ Z_\zeta &\longmapsto -\ln \cos Z_\zeta. \end{aligned}$$

Observe that  $G(0) = G'(0) = 0$ . On the other hand  $G''(Z_\zeta) = (\cos Z_\zeta)^{-2}$ . The integral version of Taylor's theorem yields

$$2 G(Z_\zeta) = -2 \ln \cos Z_\zeta = Z_\zeta^2 g(Z_\zeta), \quad g(Z_\zeta) := 2 \int_0^1 (1-s) (\cos(Z_\zeta s))^{-2} ds.$$

Remark that  $g(\cdot)$  is a smooth positive even function on the interval  $]-\pi/2, \pi/2[$ , which is strictly increasing on  $[0, \pi/2[$  and such that  $g(0) = 1$ . This implies that  $Z_\zeta \mapsto Z_\zeta \sqrt{g(Z_\zeta)} \equiv \mathcal{U}_\zeta(Z_\zeta)$  is well-defined, smooth and odd. It has a positive derivative, and it tends to  $\pm\infty$  when  $Z_\zeta$  goes to  $\pm\pi/2$ . Therefore, it gives rise to a diffeomorphism from  $]-\pi/2, \pi/2[$  onto  $\mathbb{R}$ .  $\square$

To elucidate the internal structure of (3.2), the first step is to replace the state variable  $Z$  by  $\mathcal{Z} = {}^t(Z_r, Z_\theta, Z_v, Z_\zeta) = \mathcal{U}(Z)$  where  $\mathcal{U}$  is the diffeomorphic map

$$(3.13) \quad \begin{aligned} \mathcal{U} : ]0, a] \times \mathbb{T} \times ]0, 1[ \times ]-\pi/2, \pi/2[ &\longrightarrow ]0, a] \times \mathbb{T} \times ]1, +\infty[ \times \mathbb{R} \\ (Z_r, Z_\theta, Z_v, Z_\zeta) &\longmapsto (Z_r, Z_\theta, \mathcal{U}_v(Z_v), \mathcal{U}_\zeta(Z_\zeta)). \end{aligned}$$

By convention, we can set  $\mathcal{U}_r(Z_r) := Z_r$  and  $\mathcal{U}_\theta(Z_\theta) := Z_\theta$ . The second step is to change the time variable  $t$ . To this end, introduce

$$\begin{aligned} f : ]-\pi/2, \pi/2[ &\longrightarrow \mathbb{R}_+^* \\ Z_\zeta &\longmapsto \mathcal{U}'_\zeta(Z_\zeta) \cos Z_\zeta. \end{aligned}$$

By differentiating the identity  $\mathcal{U}_\zeta(Z_\zeta)^2 = -2 \ln \cos Z_\zeta$ , we can deduce the extra relation

$$(3.14) \quad \forall Z_\zeta \in ]-\pi/2, \pi/2[, \quad \sin Z_\zeta = f(Z_\zeta) \mathcal{U}_\zeta(Z_\zeta).$$

We now follow the evolution of  $Z(z_0; \cdot)$  in terms of the new state variable  $\mathcal{Z}$  and also according to some adapted internal clock associated with the new time variable

$$(3.15) \quad \mathfrak{t}(z_0; t) := \frac{\sin \omega(r_0)}{r_0} \int_0^t Z_v(z_0; \tilde{t}) f \circ Z_\zeta(z_0; \tilde{t}) d\tilde{t}.$$

Recall that  $\sin \omega(r_0) < 0$ . Due to (3.9), (3.10) and (3.11), we have

$$(3.16) \quad \exists (\eta_5, \eta_6) \in \mathbb{R}^2; \quad 0 < \eta_5 \leq -\frac{dt}{d\mathfrak{t}} \leq \eta_6.$$

Thus, we can change the time variable  $t \in \mathbb{R}$  into  $\mathfrak{t} \in \mathbb{R}$  (with a one-to-one global correspondence). Let  $t(z_0; \cdot)$  be the inverse function of  $\mathfrak{t}(z_0; \cdot)$ . We can define

$$(3.17) \quad \mathcal{Z}(z_0; \mathfrak{t}) := \mathcal{U} \circ Z(z_0; t(z_0; \mathfrak{t})).$$

**Lemma 3.5** (The system of odes in  $]0, a[\times \mathbb{T} \times ]1, +\infty[\times \mathbb{R}$  satisfied by  $\mathcal{Z}(z_0; \cdot)$ ). *Starting from  $(r_0, \theta_0, v_0, \zeta_0)$  as in (2.18), define*

$$\mathcal{Z}_0 \equiv \mathcal{Z}_0(z_0) = (\mathcal{Z}_{0r}, \mathcal{Z}_{0\theta}, \mathcal{Z}_{0v}, \mathcal{Z}_{0\zeta})(z_0) := (r_0, \theta_0, \mathcal{U}_v(v_0), \mathcal{U}_\zeta(\zeta_0)).$$

We find that

$$(3.18) \quad \frac{d\mathcal{Z}_r}{d\mathfrak{t}} = 0, \quad \mathcal{Z}_r(0) = \mathcal{Z}_{0r}.$$

On the other hand, the system (3.2) is equivalent to

$$(3.19) \quad \begin{cases} \frac{d\mathcal{Z}_\theta}{d\mathfrak{t}} = +\mathcal{Z}_\zeta, & \mathcal{Z}_\theta(0) = \mathcal{Z}_{0\theta}, \\ \frac{d\mathcal{Z}_v}{d\mathfrak{t}} = -\partial_\theta \Phi(r_0, \mathcal{Z}_\theta) \mathcal{Z}_\zeta, & \mathcal{Z}_v(0) = \mathcal{Z}_{0v}, \\ \frac{d\mathcal{Z}_\zeta}{d\mathfrak{t}} = -\frac{r_0 \sin \mathcal{Z}_\theta}{2R(r_0, \mathcal{Z}_\theta)} - \frac{\mathcal{Z}_v}{\mathcal{Z}_v^2 - 1} \partial_\theta \Phi(r_0, \mathcal{Z}_\theta), & \mathcal{Z}_\zeta(0) = \mathcal{Z}_{0\zeta}. \end{cases}$$

From (3.18), we know that  $\mathcal{Z}_r(\mathfrak{t}) = r_0$  for all  $\mathfrak{t} \in \mathbb{R}$ .

*Proof.* The line (3.18) is straightforward. In view of (3.2) and (3.15), the first equation of (3.19) is satisfied if and only if

$$\frac{d\mathcal{Z}_\theta}{d\mathfrak{t}} = \frac{dt}{d\mathfrak{t}} \frac{d\mathcal{Z}_\theta}{dt} = \frac{\sin \mathcal{Z}_\zeta}{f \circ Z_\zeta} = \mathcal{Z}_\zeta = \mathcal{U}_\zeta(\mathcal{Z}_\zeta)$$

which is guaranteed by (3.14). The second equation of (3.19) comes from

$$\frac{d\mathcal{Z}_v}{d\mathfrak{t}} = \frac{dt}{d\mathfrak{t}} \mathcal{U}'_v(\mathcal{Z}_v) \frac{d\mathcal{Z}_v}{dt} = -\frac{r_0}{\sin \omega(r_0)} \frac{\sin \mathcal{Z}_\zeta}{f \circ Z_\zeta} E_\parallel(r_0, \mathcal{Z}_\theta)$$

together with (2.7) and (3.14). The last equation of (3.19) can be obtained by following the same lines of computations.  $\square$

From (3.9), (3.10) and (3.11), we know that  $\mathcal{Z}$  remains in a compact set of  $\mathbb{R} \times \mathbb{T} \times ]1, +\infty[\times \mathbb{R}$ . On the other hand, the invariant functionals  $H_1(r_0; \cdot)$  and  $H_2(r_0; \cdot)$  can be expressed in terms of  $\mathcal{Z}$ . They simply become

$$\mathcal{H}_1(\mathcal{Z}) := (\mathcal{Z}_v^2 - 1)^{1/2} (R_0 + \mathcal{Z}_r \cos \mathcal{Z}_\theta)^{1/2} e^{-\mathcal{Z}_\zeta^2/2}, \quad \mathcal{H}_2(\mathcal{Z}) := \Phi(\mathcal{Z}_r, \mathcal{Z}_\theta) + \mathcal{Z}_v.$$

Of course, the relation (3.5) becomes

$$(3.20) \quad \frac{d}{d\mathfrak{t}} [\mathcal{H}_i(\mathcal{Z})] = 0, \quad \forall i \in \{1, 2\}.$$

The information (3.20) for  $i = 2$  is particularly interesting. Starting from the initial data  $\mathcal{Z}_0$  having (second) energy  $\mathcal{H}_{20} := \Phi(\mathcal{Z}_{0r}, \mathcal{Z}_{0\theta}) + \mathcal{Z}_{0v}$ , we can compute the component  $\mathcal{Z}_v(\mathfrak{t})$  as a function of  $\mathcal{Z}_\theta(\mathfrak{t})$  according to

$$(3.21) \quad \mathcal{Z}_v(\mathfrak{t}) = \mathcal{F}_{r_0, \mathcal{H}_{20}}(\mathcal{Z}_\theta(\mathfrak{t})), \quad \mathcal{F}_{r_0, \mathcal{H}_{20}}(\mathcal{Z}_\theta) := \mathcal{H}_{20} - \Phi(r_0, \mathcal{Z}_\theta).$$

Consider the two-dimensional manifold

$$\mathcal{E}_{r_0, \mathcal{H}_{20}} := \{ \mathcal{Z} \in ]0, a] \times \mathbb{T} \times ]1, +\infty[ \times \mathbb{R} ; \mathcal{Z}_r = r_0, \Phi(r_0, \mathcal{Z}_\theta) + \mathcal{Z}_v = \mathcal{H}_{20} \}.$$

This surface may be parametrized by  $q \equiv \mathcal{Z}_\theta$  and  $p \equiv \mathcal{Z}_\zeta$  in the following way

$$\mathcal{E}_{r_0, \mathcal{H}_{20}} := \{ \mathcal{P}\mathcal{F}_{r_0, \mathcal{H}_{20}}(q, p) ; q \in \mathbb{T}, p \in \mathbb{R} \}, \quad \mathcal{P}\mathcal{F}_{r_0, \mathcal{H}_{20}}(q, p) := (r_0, q, \mathcal{F}_{r_0, \mathcal{H}_{20}}(q), p).$$

Now, starting from any position  $\mathcal{P}\mathcal{F}_{r_0, \mathcal{H}_{20}}(q_0, p_0)$  inside  $\mathcal{E}_{r_0, \mathcal{H}_{20}}$ , which is associated with the choice of some  $(q_0, p_0)$ , we can use (3.21) instead of the equation on  $\mathcal{Z}_v$  to compute  $\mathcal{Z}_v(t)$ . As a consequence, limited to  $\mathcal{E}_{r_0, \mathcal{H}_{20}}$ , the third equation of (3.19) implies a right hand side that can be viewed as minus the derivative with respect to  $\mathcal{Z}_\theta$  of the function

$$(3.22) \quad \mathcal{K}_{r_0, \mathcal{H}_{20}}(\mathcal{Z}_\theta) := -\frac{1}{2} \ln(R_0 + r_0 \cos \mathcal{Z}_\theta) - \frac{1}{2} \ln(\mathcal{F}_{r_0, \mathcal{H}_{20}}(\mathcal{Z}_\theta)^2 - 1).$$

By this way, we have extracted from (3.19) a two-dimensional systems in  $\mathbb{T} \times \mathbb{R}$  looking like

$$(3.23) \quad \begin{cases} \frac{dq}{dt} = +p, & q(0) = q_0, \\ \frac{dp}{dt} = -\partial_q \mathcal{K}_{r_0, \mathcal{H}_{20}}(q), & p(0) = p_0. \end{cases}$$

In so doing, we have highlighted a foliation of the dynamics generated by (3.18)-(3.19) by self-contained subsystems (which are indexed by  $r_0$  and  $\mathcal{H}_{20}$ ) on  $(q, p) := (\mathcal{Z}_\theta, \mathcal{Z}_\zeta)$ , where  $p$  is the conjugate variable of  $q$ . For the sake of simplicity, we will sometimes skip the two indices  $r_0$  and  $\mathcal{H}_{20}$ , and simply use  $\mathcal{K} \equiv \mathcal{K}_{r_0, \mathcal{H}_{20}}$ .

The reduced equations (3.23) describe the motion of a body of mass 1 subject to a conservative potential energy  $\mathcal{K}$ . The Hamiltonian is given by  $\mathcal{H}(q, p) := (p^2/2) + \mathcal{K}(q)$ . Since  $\mathcal{K}(\cdot)$  is a periodic function of period  $2\pi$ , the system (3.23) is of pendulum type.

**3.2.2. Reminders about the pendulum.** The system (3.23) on  $\mathbb{T} \times \mathbb{R}$  is well known in classical dynamics. When  $\mathcal{K}(q) = \cos q$ , this is just the simple pendulum which is addressed in detail in [4]. The more general case, when  $\mathcal{K}$  is for instance as in (3.22), is discussed in the [lecture notes](#) of M.V.N. Murthy or in [18]. For general potentials  $\mathcal{K}$ , we cannot solve (3.23) explicitly. But still we can explore its properties. In this paragraph, we highlight particulars that are needed for the subsequent discussion.

**Definition 3.6** (Critical value). A critical value (sometimes called critical energy) is a number  $E_{crit}$  such that  $E_{crit} = \mathcal{K}(q)$  for some  $q$  satisfying  $\partial_q \mathcal{K}(q) = 0$ .

The set of critical values is denoted by

$$\mathcal{C} := \{ E_{crit} \in \mathbb{R} ; \exists q \in \mathbb{T} \text{ such that } \partial_q \mathcal{K}(q) = 0 \text{ and } \mathcal{K}(q) = E_{crit} \}.$$

The set  $\mathcal{C} \subset \mathbb{T}$  is compact with  $\#\mathcal{C} \geq 1$ . By Sard's lemma, it is of measure zero. For the sake of simplicity, let us first recall what happens when  $\mathcal{C}$  is finite of cardinal  $m$ , that is when

$$\mathcal{C} = \{ E_{crit}^1, \dots, E_{crit}^m \}, \quad E_{crit}^1 < \dots < E_{crit}^m.$$

Then, there is a partition of the phase space into  $\mathbb{T} \times \mathbb{R} = \mathcal{L} \cup \mathcal{R} \cup \mathcal{S}$  corresponding to the distinct types of motion indicated below:

- **The region of libration motion:** .

$$\mathcal{L} := \cup_{j=0}^{m-1} \mathcal{L}^j, \quad \mathcal{L}^j := \{ (q, p) \in \mathbb{T} \times \mathbb{R} ; E_{crit}^j < \mathcal{H}(q, p) < E_{crit}^{j+1} \}.$$

The particles issued from some  $(q_0, p_0) \in \mathcal{L}$  are *trapped*. Define  $\mathcal{H}_0 := \mathcal{H}(p_0, q_0)$ . Identify the two turning points  $q_1$  and  $q_2$  associated to the energy level  $\mathcal{H}_0$ , which are the largest

$q_1$  and the smallest  $q_2$  satisfying  $q_1 \leq q \leq q_2$  together with  $\mathcal{K}(q_1) = \mathcal{K}(q_2) = \mathcal{H}_0$ . Then, the solution  $(q, p)(\cdot)$  is periodic of period

$$(3.24) \quad 0 < \mathcal{P} \equiv \mathcal{P}(\mathcal{H}_0) \equiv \mathcal{P}(z_0) := \int_{q_1}^{q_2} \frac{\sqrt{2}}{\sqrt{\mathcal{H}_0 - \mathcal{K}(q)}} dq.$$

Physically, the particles are bouncing back and forth between two poloidal angles  $\theta_1 \equiv q_1$  and  $\theta_2 \equiv q_2$ . They explore only a part of the torus (in the  $\theta$ -variable).

- **The region of rotation motion:**

$$\mathcal{R} \equiv \mathcal{L}^m := \{(q, p) \in \mathbb{T} \times \mathbb{R}; \mathcal{H}(q, p) > E_{crit}^m\}.$$

The particles issued from some  $(q_0, p_0) \in \mathcal{R}$  are *passing*. They are in transit in  $q$  with momentum  $p$  not changing sign. This corresponds to high levels of energy  $\mathcal{H}$ , with  $\partial_t q$  of fixed sign. The position  $q(t)$  is going to  $\pm\infty$  when  $t \rightarrow \pm\infty$  according to the sign  $\pm$  of  $p_0$ . Select some  $(q_0, p_0) \in \mathcal{R}$ . Then, viewed on  $\mathbb{T} \times \mathbb{R}$ , the motion is periodic of period

$$(3.25) \quad 0 < \mathcal{P} \equiv \mathcal{P}(\mathcal{H}_0) \equiv \mathcal{P}(z_0) := \frac{1}{\sqrt{2}} \int_0^{2\pi} \frac{1}{\sqrt{\mathcal{H}_0 - \mathcal{K}(q)}} dq.$$

- **The separatrix region:**

$$\mathcal{S} := \cup_{j=1}^m \mathcal{S}^j, \quad \mathcal{S}^j := \{(q, p) \in \mathbb{T} \times \mathbb{R}; \mathcal{H}(q, p) = E_{crit}^j\}.$$

The particles issued from some  $(q_0, p_0) \in \mathcal{S}$  are converging to a fixed point. The level sets  $\mathcal{S}^j$  of  $\mathcal{H}$  may be viewed as boundaries which separate the preceding different types of trajectories (having distinct homotopy classes).

When the cardinal of  $\mathcal{C}$  is infinite, by extension, we can still define the (closed) separatrix region  $\mathcal{S} := \mathcal{H}^{-1}(\mathcal{C}) \subset \mathbb{T} \times \mathbb{R}$  and the (open) rotation region  $\mathcal{R} := \{(q, p) \in \mathbb{T} \times \mathbb{R}; \mathcal{H}(q, p) > \sup \mathcal{C}\}$ . The complement set of  $\mathcal{S} \cup \mathcal{R}$  is the (open) libration region  $\mathcal{L}$ .

The phase portrait of the simple pendulum is represented below. We find that  $\mathcal{C} = \{-1, 1\}$  so that  $m = 2$ . The region of libration motion contains the closed trajectories (homeomorphic to circles with nonzero radius) which are drawn inside the two red curves. The region of rotation motion contains the trajectories which are outside the red curves. The separatrix region is represented by the two red curves plus a (non visible) equilibrium position placed at  $(\pi, 0)$ .

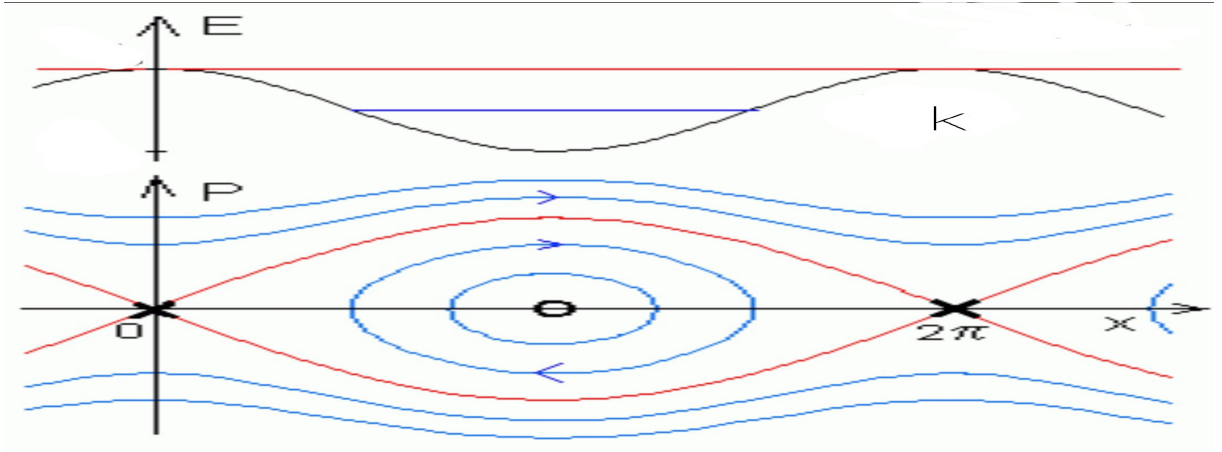


FIGURE 3. The phase portrait when  $\mathcal{K}(q) = \cos q$

Inside  $\mathcal{L}$  or  $\mathcal{R}$ , we can use a canonical transformation to action-angle variables  $(I, \Theta)$ . The new Hamiltonian is denoted by  $W$ . It is independent of  $\Theta$ . Thus, it suffices to solve

$$(3.26) \quad \begin{cases} \partial_t I = 0, & I(0) = I_0, \\ \partial_t \Theta = \partial_I W(I), & \Theta(0) = \Theta_0. \end{cases}$$

The solution to this Cauchy problem is globally defined. It is simply

$$(3.27) \quad (\mathbf{I}, \Theta)(\mathbf{I}_0, \Theta_0; \mathbf{t}) = (\mathbf{I}_0, \Theta_0 + \partial_{\mathbf{t}} W(\mathbf{I}_0) \mathbf{t}), \quad 0 < \partial_{\mathbf{t}} W(\mathbf{I}_0) = 2\pi/\mathcal{P}, \quad \mathbf{t} \in \mathbb{R}.$$

To normalize the period of the solution, we can change the time variable  $t$  (or  $\mathbf{t}$ ) into  $s$  through

$$(3.28) \quad s \equiv s(z_0; t) := \partial_{\mathbf{t}} W(\mathbf{I}_0) \mathbf{t}(z_0; t).$$

Indeed, expressed in terms of  $s$ , the flow associated to (3.26) just becomes

$$(\mathbf{I}, \Theta)(\mathbf{I}_0, \Theta_0; s) = (\mathbf{I}_0, \Theta_0 + s), \quad s \in \mathbb{R},$$

which is a uniform translation.

**3.2.3. Periodic properties away from separatrices.** In the present situation, the potential  $\mathcal{K}$  does depend on  $(r_0, \mathcal{H}_{20})$ , or more generally on  $z_0$ . The same applies to the corresponding separatrix region  $\mathcal{S}_{r_0, \mathcal{H}_{20}}$ , the period  $\mathcal{P} \equiv \mathcal{P}(z_0)$ , the construction of  $W \equiv W(z_0)$ , and so on  $\dots$ , including the passage from  $\mathbf{t}$  to  $s$ . Look at

$$(3.29) \quad \mathcal{S} := \{ \mathcal{P}_{r_0, \mathcal{H}_{20}}(\mathcal{S}_{r_0, \mathcal{H}_{20}}); r_0 \in ]0, a], \mathcal{H}_{20} \geq \inf \Phi(r_0, \cdot) + 1 \} \\ \subset ]0, a] \times \mathbb{T} \times ]1, +\infty[ \times \mathbb{R}.$$

We want to avoid the positions which emerge from  $\mathcal{S}$  because they involve qualitatively different behaviors. What happens in the proximity of  $\mathcal{S}$  is complicated. On the other hand, we can select in the phase space any bounded connected open domain  $\mathcal{D}$  (made of libration or rotation) such that  $\bar{\mathcal{D}} \cap \mathcal{S} = \emptyset$ . The open set  $\Omega$  of Theorem 1.8 is just the pullback of such domain  $\mathcal{D}$  in the original phase space variables  $(x, v)$ . In other words

$$(3.30) \quad \Omega := \{ (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3; \exists (r, \theta, \phi, v, \zeta, \nu) \in ]0, a[ \times \mathbb{T}^2 \times ]0, 1[ \times ] -\frac{\pi}{2}, \frac{\pi}{2}[ \times \mathbb{T}; \\ \mathcal{U}(r, \theta, v, \zeta) \in \mathcal{D}, (x, v) = (X(r, \theta, \phi), V(r, \theta, \phi, v, \zeta, \nu)) \}.$$

**Lemma 3.7** (Periodic behaviors of the average flow). *Select  $z_0 \in \mathcal{U}^{-1}(\mathcal{D})$ . The average flow issued from such  $z_0$ , when it is interpreted in the time variable  $s$ , is periodic in  $s$  with period  $2\pi$ . On the other hand*

$$(3.31) \quad \exists ! \mathcal{P}(z_0) \in \mathbb{R}_+^*; \quad \mathbf{t}(z_0; -\mathcal{P}(z_0)) = -\mathcal{P}(z_0),$$

and the flow  $Z(z_0, \cdot)$  is periodic in  $t$  of period  $\mathcal{P}(z_0)$ .

*Proof.* Taking into account (3.15) and (3.27), looking at  $Z(z_0, \cdot)$  in terms of the time variable  $s$  means to consider the function

$$(3.32) \quad s \mapsto Z(z_0; \mathbf{t}(z_0; \mathcal{P}(z_0) s/2\pi)) = \mathcal{U}^{-1} \circ \mathcal{Z}(z_0; \mathcal{P}(z_0) s/2\pi).$$

The periodicity for the component  $Z_r(z_0, \cdot)$  is obvious because this function is simply constant. On the other hand, our study of the reduced system (3.23) has revealed that  $(\mathcal{Z}_\theta, \mathcal{Z}_\zeta)(z_0, \cdot)$  is periodic in  $\mathbf{t}$  with a period  $\mathcal{P}(z_0)$  given by (3.24) or (3.25). Exploiting (3.19), this implies for the remaining component  $\mathcal{Z}_v$  that

$$\mathcal{Z}_v(\mathbf{t} + \mathcal{P}) = \mathcal{Z}_v(\mathbf{t}) - \int_{\mathbf{t}}^{\mathbf{t} + \mathcal{P}} \partial_\theta \Phi(r_0, \mathcal{Z}_\theta(s)) \mathcal{Z}_\zeta(s) ds = \mathcal{Z}_v(\mathbf{t}) - \int_0^{\mathcal{P}} \partial_\theta \Phi(r_0, \mathcal{Z}_\theta(s)) \frac{d\mathcal{Z}_\theta}{ds}(s) ds \\ = \mathcal{Z}_v(\mathbf{t}) - \int_{\theta_0}^{\mathcal{Z}_\theta(\mathcal{P})} \partial_\theta \Phi(r_0, \theta) d\theta = \mathcal{Z}_v(\mathbf{t}) + \Phi(r_0, \theta_0) - \Phi(r_0, \mathcal{Z}_\theta(\mathcal{P})) = \mathcal{Z}_v(\mathbf{t}).$$

This clearly means that  $\mathcal{Z}_v(z_0, \cdot)$ , and therefore the field  $Z(z_0, \cdot)$  as a whole, is periodic in  $\mathbf{t}$  of period  $\mathcal{P} \equiv \mathcal{P}(z_0)$ . This period for  $Z(z_0, \cdot)$  becomes  $2\pi$  at the level of the map (3.32).

It is now a matter of transferring this periodic information on  $\mathcal{Z}$  to  $Z$ . By construction, we have  $\mathbf{t}(z_0; 0) = 0$ , and the condition (3.16) ensures that (3.31) is satisfied by a unique  $\mathcal{P}(z_0) \in \mathbb{R}_+^*$ . Then, using (3.17), we can assert that

$$Z(z_0; -\mathcal{P}(z_0)) = \mathcal{U}^{-1} \circ \mathcal{Z}(z_0; -\mathcal{P}(z_0)) = \mathcal{U}^{-1} \circ \mathcal{Z}(z_0; 0) = Z(z_0; 0).$$



This means that  $Z(z_0; \cdot)$  is coming back to the initial condition  $Z(z_0; 0)$  at the time  $t = -P(z_0)$ . By Cauchy-Lipschitz theorem, the flow  $Z(z_0; \cdot)$  is periodic of period  $P(z_0)$ .  $\square$

**3.2.4. Linear growths plus periodic oscillations.** Given a periodic function  $F$  of period  $P$ , we can decompose  $F$  into a mean part  $\langle F \rangle_P$  plus a contribution  $F^*$  with zero mean according to

$$(3.33) \quad F(t) = \langle F \rangle_P + F^*(t), \quad \langle F \rangle_P := \frac{1}{P} \int_0^P F(t) dt, \quad \int_0^P F^*(t) dt = 0.$$

With this convention, we have  $\langle F \rangle_{2\pi} \equiv \bar{F}$ , where the operator  $\bar{F}$  is as in (1.6). Moreover

$$(3.34) \quad \int_0^t F(s) ds = \langle F \rangle_P t + IF^*(t), \quad IF^*(t) := \int_0^t F^*(s) ds = IF^*(t + P),$$

where the exponent  $*$  serves to highlight the periodicity. The above integration of  $F$  reveals a linear part (which is non-trivial as soon as  $\langle F \rangle_P \neq 0$ ) plus a periodic contribution. This principle applies to certain quantities that have been exhibited before. For instance, it allows to further investigate the change of time variable  $s(z_0; \cdot)$  introduced at the level of (3.28).

**Lemma 3.8.** *[Link between  $t$  and  $s$ ] For all time  $t \in \mathbb{R}$ , with  $s(z_0; \cdot)$  as in (3.28), we have*

$$(3.35) \quad s \equiv s(z_0; t) = \frac{2\pi t}{P(z_0)} + s^*\left(z_0; \frac{2\pi t}{P(z_0)}\right), \quad s^*(z_0; t + 2\pi) = s^*(z_0; t).$$

*Proof.* We can apply (3.34) to the product  $(Z_v f \circ Z_\zeta)(z_0; t)$  which, in view of Lemma 3.7, is periodic of period  $P(z_0)$ . This furnishes

$$t(z_0; t) := \frac{\sin \omega(r_0)}{r_0} \left[ \langle (Z_v f \circ Z_\zeta)(z_0; \cdot) \rangle_{P(z_0)} t + I(Z_v f \circ Z_\zeta)(z_0; \cdot)^*(t) \right],$$

with from Lemma 3.7

$$\begin{aligned} \langle (Z_v f \circ Z_\zeta)(z_0; \cdot) \rangle_{P(z_0)} &= \frac{1}{P(z_0)} \int_0^{P(z_0)} (Z_v f \circ Z_\zeta)(z_0; t) dt \\ &= \frac{1}{P(z_0)} \int_{-P(z_0)}^0 (Z_v f \circ Z_\zeta)(z_0; t) dt \\ &= -\frac{1}{P(z_0)} \frac{r_0}{\sin \omega(r_0)} t(z_0; -P(z_0)) = \frac{P(z_0) r_0}{P(z_0) \sin \omega(r_0)}. \end{aligned}$$

Coming back to (3.28), we find (3.35) with

$$s^*(z_0; t) = \frac{2\pi}{P(z_0)} \frac{\sin \omega(r_0)}{r_0} I(Z_v f \circ Z_\zeta)(z_0; \cdot)^*\left(\frac{P(z_0) t}{2\pi}\right),$$

which is indeed periodic of period  $2\pi$ .  $\square$

The two physical quantities  $\phi$  (toroidal angle) and  $v$  (slow gyromotion of the polar angle  $\nu$  associated with the velocity  $v$ ) have been put aside respectively at the level of Lemma 2.2 and inside the system (1.3). For the sake of completeness, we indicate below their approximate behavior. Looking at (2.10) and (2.14), neglecting the impact of  $\sin \nu$  (due to the cancellations induced by the rapid oscillations) and the  $O(\varepsilon)$ -term, it is expected that  $\phi \sim \bar{\phi}$  and  $v \sim \bar{v}$  with  $\bar{\phi}$  and  $\bar{v}$  subject to

$$\begin{aligned} \frac{d\bar{\phi}}{dt} &= \bar{\mathcal{A}}_\phi(Z) = R(Z_r, Z_\theta)^{-1} Z_v \cos \omega(Z_r) \sin Z_\zeta, & \bar{\phi}(0) &= \phi_0, \\ \frac{d\bar{v}}{dt} &= \mathcal{V}_0(Z) = -\sqrt{1 - Z_v^2} b(Z_r, Z_\theta), & \bar{v}(0) &= 0. \end{aligned}$$

It is worth noting that neither  $\bar{\phi}$  nor  $\bar{v}$  are periodic. Instead, applying again (3.34), we find

$$\begin{aligned} \bar{\phi}(t) &= \langle \bar{\mathcal{A}}_\phi \circ Z(z_0; \cdot) \rangle_{P(z_0)} t + I(\bar{\mathcal{A}}_\phi \circ Z(z_0; \cdot))^*(t), \\ \bar{v}(t) &= \langle \mathcal{V}_0 \circ Z(z_0; \cdot) \rangle_{P(z_0)} t + I(\mathcal{V}_0 \circ Z(z_0; \cdot))^*(t), \end{aligned}$$

where the coefficients in factor of  $t$  are (in general) non zero. These expressions are composed with  $\cos$  or  $\sin$  at the level of (2.4) and (2.8). As a consequence, even at a rough level of description, the complete flow associated with (1.1) inherits almost periodic (and not only purely periodic) features.

#### 4. APPLICATIONS

Subsection 4.1 shows that the description of (1.2) falls within the scope of [11] in the case of the realistic data that we have introduced. Subsection 4.2 describes the coherent structures underlying the flow issued from (1.2) with, at the end (Paragraph 4.2.4) the proof of Theorem 1.8. Subsection 4.3 exhibits and studies the long time gyrokinetic equations in order to draw out some implications.

**4.1. A manageable framework.** We want to apply the results of the article [11]. To this end, in Paragraph 4.1.1, we explain how to get a system of nonlinear differential equations which is consistent with the one presented in [11]. Then, in Paragraph 4.1.2, we check that Conditions 1.4 and 1.5 (required in [11] to apply Theorems 2, 3 and 4 there) are satisfied.

**4.1.1. A singular system of odes with multiscale periodic features.** Like  $t(z_0; \cdot)$ , the change of time variable  $s(z_0; \cdot) : \mathbb{R} \rightarrow \mathbb{R}$  introduced at the level of (3.28) is a global diffeomorphism. The associated inverse is denoted by  $t(z_0; s)$ .

**Lemma 4.1.** *The function  $\partial_t s(z_0; t(z_0; \cdot))$  is negative, and it is periodic of period  $2\pi$ .*

*Proof.* From (3.27) and (3.28), the derivative of  $s$  is found to be

$$(4.1) \quad \partial_t s(z_0; t) = \underbrace{\partial_t W(I_0)}_{>0} \frac{dt(z_0; t)}{dt}.$$

The negative sign of  $\partial_t s$  follows then from (3.16) where  $dt/dt < 0$ . Due to (3.35), we have

$$s + 2\pi = s(z_0; t(z_0; s)) + 2\pi = \frac{2\pi}{P(z_0)} [t(z_0; s) + P(z_0)] + s^*\left(z_0; \frac{2\pi}{P(z_0)} t(z_0; s)\right).$$

By definition, the left hand side is the same as  $s(z_0; t(z_0; s + 2\pi))$  where  $t(z_0; s)$  is the inverse of  $s(z_0; \cdot)$ . On the other hand, due again to (3.35) and since  $s^*(z_0; \cdot)$  is periodic of period  $2\pi$ , the right hand side coincides with  $s(z_0; t(z_0; s) + P(z_0))$ . This is possible if and only if

$$(4.2) \quad t(z_0; s + 2\pi) = t(z_0; s) + P(z_0).$$

Use again (3.15) and (3.27), the expression (4.1) becomes

$$(4.3) \quad \partial_t s(z_0; t) = \frac{2\pi}{P(z_0)} \frac{\sin \omega(r_0)}{r_0} (Z_v f \circ Z_\zeta)(z_0; t) < 0.$$

Now, knowing from Lemma 3.7 that  $\partial_t s(z_0; t)$  is periodic in  $t$  of period  $P(z_0)$ , we can easily deduce the expected result.  $\square$

The system (1.3) can be expressed in the time variable  $s$  and then in terms of  $\tau = \varepsilon s$ . To this end, it suffices to consider

$$(4.4) \quad \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} (\varepsilon, z_0, \mathbf{v}_0; \tau) := \begin{pmatrix} \tilde{z} \\ \tilde{\mathbf{v}} \end{pmatrix} (\varepsilon, z_0, \mathbf{v}_0; t(z_0; \tau/\varepsilon)),$$

and then, with  $\mathbf{v}_0 = \varepsilon \nu_0$ , to replace (1.3) by (1.8) where

$$(4.5) \quad \begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} (\varepsilon, z_0; z; s, \nu) := \partial_t s(z_0; t(z_0; s))^{-1} \begin{pmatrix} \mathcal{A} \\ \mathcal{V} \end{pmatrix} (\varepsilon, z; \nu).$$

The above expression  $\mathbf{A}$  (like  $\mathcal{A}$ ) does not depend on  $\varepsilon$ , whereas  $\mathbf{V}$  (like  $\mathcal{V}$ ) does. The source term  ${}^t(\mathbf{A}, \mathbf{V})$  is a smooth function. From Lemma 4.1, it is periodic in  $s$  with period  $2\pi$ . In view of the preceding discussion, we can assert that  ${}^t(\mathbf{A}, \mathbf{V})$  satisfies (1.9) and (1.10).

One may wonder why it is necessary to pass from  $t$  to  $s$ . The reason is that a *uniform* period (not depending on  $z_0$ ) is crucial with a view to applying [11]. Otherwise, the WKB construction (the determination of the phases and the profiles) does not work due to instabilities with respect to variations in the initial data.

We now deal with the system of ODE (1.8). It is of the same type as the one addressed in [11] where in the latter reference we have that

- the solution  $(z, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}$ . Then since in our case, we have defined  $z = {}^t(r, \theta, \mathbf{v}, \zeta)$ , this means we work with  $n = 4$ ;
- the source terms  $\mathbf{A}$  and  $\mathbf{V}$  are periodic in the last two variables of periods respectively  $2\pi$  and  $T_r(z_0)$ . This means  $T_r(z_0) = 2\pi$  in our case.

In Subsection 4.2, we want to apply Theorems 2 and 3 from [11]. In Subsection 4.3, we plan to implement Theorem 4 from [11]. But to this end, first and foremost, we need to check that all requirements of [11], which have been reformulated into Conditions 1.4 and 1.5 in our introduction, are satisfied. This what is done below.

4.1.2. *Verification of the prerequisites.* Given a function  $Z(s, \nu)$  which is periodic of period  $2\pi$  in both variables  $s$  and  $\nu$ , define

$$(4.6) \quad \overline{Z}(s) := \frac{1}{2\pi} \int_0^{2\pi} Z(s, \nu) d\nu, \quad Z^*(s, \nu) := Z(s, \nu) - \overline{Z}(s),$$

$$(4.7) \quad \langle \overline{Z} \rangle := \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} Z(s, \nu) ds d\nu, \quad \overline{Z}^*(s) := \overline{Z}(s) - \langle \overline{Z} \rangle.$$

Retain that this induces a decomposition of  $Z$  in three components according to

$$(4.8) \quad Z(s, \nu) = \langle \overline{Z} \rangle + \overline{Z}^*(s) + Z^*(s, \nu).$$

- *Validity of Condition 1.4.* This condition is about the mean flow  $\Xi_{\text{mf}}$  which, as indicated in (1.11), is the solution to

$$\partial_s \Xi_{\text{mf}}(z_0; \mathfrak{z}; s) = \overline{\mathbf{A}}(z_0; \Xi_{\text{mf}}(z_0; \mathfrak{z}; s); s), \quad \Xi_{\text{mf}}(z_0; \mathfrak{z}; 0) = \mathfrak{z},$$

where  $\mathbf{A}$  is defined in (4.5) and  $\overline{\mathbf{A}}$  is as indicated in (4.6). Thus, the first thing to do is to make the link between  $\Xi_{\text{mf}}$  and the average flow  $Z$  of Lemma 3.1. To this end, we claim that

$$(4.9) \quad \Xi_{\text{mf}}(z_0; \mathfrak{z}; s) = Z(\mathfrak{z}; t(z_0, s)),$$

where  $t(z_0; \cdot)$  is the inverse of the map  $s(z_0; \cdot)$ . Then, to prove the claim (4.9), we compute

$$(4.10) \quad \partial_s [Z(\mathfrak{z}; t(z_0; s))] = \partial_s t(z_0; s) \partial_t Z(\mathfrak{z}; t(z_0; s)).$$

To proceed, we consider the following useful identities.

- ◁ From (1.7), we have  $\partial_t Z(\mathfrak{z}; t(z_0; s)) = \overline{\mathbf{A}}(Z(\mathfrak{z}; t(z_0; s)))$  where  $\mathcal{A} = {}^t(\mathcal{A}_r, \mathcal{A}_\theta, \mathcal{A}_v, \mathcal{A}_\zeta)$  is defined explicitly as indicated in (2.9) and (2.12);
- ◁ Since  $t(z_0; \cdot)$  is the inverse of  $s(z_0; \cdot)$ , we have  $\partial_s t(z_0; s) = \partial_t s(z_0; t(z_0; s))^{-1}$ ;
- ◁ We apply (4.6) to the first equation of (4.5) to see that

$$\overline{\mathbf{A}}(z_0; Z(\mathfrak{z}; t(z_0; s)); s) = \partial_t s(z_0; t(z_0; s))^{-1} \overline{\mathbf{A}}(Z(\mathfrak{z}; t(z_0; s)));$$

- ◁ From the proof of Lemma 4.9, we have  $t(z_0; 0) = 0$ . This implies that

$$Z(\mathfrak{z}; t(z_0; 0)) = Z(\mathfrak{z}; 0) = \mathfrak{z}.$$

Exploiting the above identities, the equation (4.10) becomes

$$(4.11) \quad \partial_s [Z(\mathfrak{z}; t(z_0; s))] = \overline{\mathbf{A}}(z_0; Z(\mathfrak{z}; t(z_0; s)); s), \quad Z(\mathfrak{z}; t(z_0; 0)) = \mathfrak{z}.$$

It bears noting that (4.11) coincides with the above equation of the mean flow  $\Xi_{\text{mf}}$ . Thus, by Cauchy-Lipschitz theorem, we must have (4.9).

Since  $t(z_0, \cdot)$  and  $Z(\mathfrak{z}; \cdot)$  are globally defined (Lemma 3.3), the same applies to  $\Xi_{\text{mf}}(z_0; \mathfrak{z}; \cdot)$ . Moreover, with Lemma 3.7 and (4.2), we can see on (4.9) that, for all  $(z_0, \mathfrak{z})$ , the map  $\Xi_{\text{mf}}(z_0; \mathfrak{z}; \cdot)$  is as required periodic of period  $2\pi$ .

By this way, we have proved that the first requirement to apply Theorems 2 and 3 in [11] is satisfied. Below, we consider the second one which is condition 1.5.

• *Validity of Condition 1.5.* The second equation in the system (1.8) reads

$$\partial_\tau \mathbf{v} = \frac{1}{\varepsilon} \mathbf{V}_0\left(z_0; z; \frac{\tau}{\varepsilon}\right) + \mathbf{V}_1\left(z_0; z; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon}\right)$$

where, in view of (2.15) and (4.5), we have

$$\mathbf{V}_0(z_0; z; s) = -\partial_t s(z_0; t(z_0; s))^{-1} \sqrt{1 - v^2} b(r, \theta) > 0,$$

which is positive and does not depend on  $\nu$  (whereas  $\mathbf{V}_1$  does). This is Condition 1.5.

In conclusion, all the prerequisites to apply Theorems 2, 3 and 4 in [11] are in our hands.

**4.2. Coherent structures.** In Paragraph 4.2.1, we apply [11] to get the oscillating description of the solutions  $(z, \mathbf{v})$  to (1.8). Then, to show Theorem 1.8, we have to rebuild  $(\tilde{x}, \tilde{v})(\varepsilon, x_0, v_0; \tau)$  from  $(z, \mathbf{v})(\varepsilon, z_0, \mathbf{v}_0; \tau)$ . To this end, in Paragraph 4.2.2, we replace  $(z, \mathbf{v})$ ,  $(z_0, \mathbf{v}_0)$  and  $\tau$  in terms of respectively  $(x, v)$ ,  $(x_0, v_0)$  and  $\tau$ . This is not sufficient since, in view of (2.4), (2.8) and (2.17), the access to  $(x, v)$  or  $(\tilde{x}, \tilde{v})$  from these data needs also the knowledge of  $\phi$ . Therefore, in Paragraph 4.2.3, we exhibit separately the WKB expansion of the toroidal angle  $\phi$ . Finally, in Paragraph 4.2.4, we can conclude with the proof of Theorem 1.8.

**4.2.1. Coherent structures associated with (1.8).** Recall that the map  $\mathcal{U}$  has been defined at the level of Paragraph 3.2.1, see (3.13). On the other hand, as explained in Paragraph 3.2.3, a bounded connected set  $\mathcal{D} \subset B(0, R]$  with  $R \in \mathbb{R}_+^*$  has been selected away from the separatrix region  $\mathcal{S}$ . Now, given  $\mathcal{Z}_0 \in \mathcal{D}$  as in Lemma 3.5, we have  $z_0 = \mathcal{U}^{-1}(\mathcal{Z}_0) \in \mathcal{U}^{-1}(\mathcal{D}) \subset \mathcal{U}^{-1}(B(0, R])$  where the latter is a compact set of  $]0, a] \times \mathbb{T} \times ]0, 1[ \times ]-\pi/2, \pi/2[$ . As a direct consequence of [11, Theorems 2 and 3], we have the following result.

**Proposition 4.2.** *[The WKB expansion of the flow induced by the system (1.8) from [11]] Suppose that Assumptions 1.6 and 1.7 hold. Let  $z_0 \in \mathcal{U}^{-1}(\mathcal{D})$  and  $\mathbf{v}_0 \in \mathbb{T}$ . The lifespan  $\mathcal{T}(\varepsilon, z_0, \mathbf{v}_0)$  that is associated with the system (1.8) is uniformly bounded below by some  $\mathcal{T}(z_0, \mathbf{v}_0) \in \mathbb{R}_+^*$ . Moreover, there exist smooth profiles  $Z_j = {}^t(Z_{jr}, Z_{j\theta}, Z_{jv}, Z_{j\zeta})$  and  $\mathcal{Y}_j$  as follows*

$$\begin{aligned} Z_j(z_0, \mathbf{v}_0; \tau, s, \nu) &\in C^\infty(\mathcal{U}^{-1}(\mathcal{D}) \times \mathbb{T} \times [0, \mathcal{T}(z_0, \mathbf{v}_0)] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}^4), \\ \mathcal{Y}_j(z_0, \mathbf{v}_0; \tau, s, \nu) &\in C^\infty(\mathcal{U}^{-1}(\mathcal{D}) \times \mathbb{T} \times [0, \mathcal{T}(z_0, \mathbf{v}_0)] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}), \end{aligned}$$

which, taking into account (4.6) and (4.7), are such that

$$(4.12) \quad \mathcal{V}_{-1} \equiv \langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \tau), \quad \mathcal{V}_0 \equiv \overline{\mathcal{V}}_0(z_0, \mathbf{v}_0; \tau, s), \quad Z_0 \equiv \overline{Z}_0(z_0; \tau, s),$$

and which are adjusted in such a way that, in terms of the sup norm, we have

$$(4.13) \quad \begin{aligned} z(\varepsilon, z_0, \mathbf{v}_0; \tau) &= \sum_{j=0}^{+\infty} \varepsilon^j Z_j\left(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \tau)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon})}{\varepsilon}\right), \\ \mathbf{v}(\varepsilon, z_0, \mathbf{v}_0; \tau) &= \sum_{j=-1}^{+\infty} \varepsilon^j \mathcal{Y}_j\left(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \tau)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon})}{\varepsilon}\right). \end{aligned}$$

From [11, Theorem 4], with  $\langle \overline{\mathcal{V}}_{-1} \rangle$  determined as indicated in this Theorem 4, and using (4.9), we have also

$$(4.14) \quad \overline{Z}_0(z_0; \tau, s) = \Xi_{\text{mf}}(z_0; \langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \tau); s) = Z(\langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \tau); t(z_0, s)).$$

As explained in the next paragraphs, Theorem 1.8 is a corollary of Proposition 4.2. To see why, we have now to reverse the procedure going from  $(x, v)$  to  $(z, \mathbf{v})$ . The first step is to replace

inside (4.13) the time variable  $\tau$  by  $\tau$  and the state variable  $(z, \mathbf{v})$  by  $(\mathbf{z}, v)$  - in fact by  $(\tilde{z}, \tilde{v})$  with  $(\tilde{z}, \tilde{v})$  as in (4.17).

4.2.2. *Coherent structures associated with the long time study of (1.3).* By construction, we have  $\tau = \varepsilon t$  and  $\tau = \varepsilon s$  as well as  $s = s(z_0; t)$ . Taking into account Lemma 3.8, we can directly pass from  $\tau$  to  $\tau$  through

$$(4.15) \quad \tau = \varepsilon s\left(z_0; \frac{\tau}{\varepsilon}\right) = \frac{2\pi\tau}{P(z_0)} + \varepsilon s^*\left(z_0; \frac{2\pi\tau}{\varepsilon P(z_0)}\right).$$

On the other hand, the change (4.4) can be reformulated as

$$(4.16) \quad \begin{pmatrix} \mathbf{z} \\ v \end{pmatrix} \left(\varepsilon, z_0, \mathbf{v}_0; \frac{\tau}{\varepsilon}\right) = \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} \left(\varepsilon, z_0, \mathbf{v}_0; \varepsilon s\left(z_0; \frac{\tau}{\varepsilon}\right)\right).$$

Amongst other things, the formula (1.15) allows to capture the dependence of the flow  $(\tilde{x}, \tilde{v})(\cdot)$  on the initial data  $(x_0, v_0)$ . To take this into account, we have to look inside (4.4) at  $(z_0, \mathbf{v}_0)$  as a function of  $(x_0, v_0)$ . According to (2.4) and (2.8), we have

$$(x_0, v_0) = (X(r_0, \theta_0, \phi_0), V(r_0, \theta_0, \phi_0, v_0, \zeta_0, \nu_0)), \quad z_0 = {}^t(r_0, \theta_0, v_0, \zeta_0).$$

The map  $(X, V)(\cdot)$  is smooth and one-to-one. This implies that both  $z_0$  and  $\nu_0$  are smoothly determined by  $(x_0, v_0)$ . In other words, they can be viewed as functions  $z_0(x_0, v_0)$  and  $\nu_0(x_0, v_0)$  of  $(x_0, v_0)$  where  $(x_0, v_0)$  is selected in the bounded connected open set  $\Omega \subset \mathbb{R}^3 \times \mathbb{R}^3$  which is introduced in Theorem 1.8, see also (3.30). Following this perspective, we define  $\Psi^1$  as in (1.16) and, since  $\mathbf{v}_0 = \varepsilon \nu_0$ , we introduce

$$(4.17) \quad \begin{pmatrix} \tilde{z} \\ \tilde{v} \end{pmatrix} (\varepsilon, x_0, v_0; \tau) := \begin{pmatrix} \mathbf{z} \\ v \end{pmatrix} \left(\varepsilon, z_0(x_0, v_0), \varepsilon \nu_0(x_0, v_0); \frac{\tau}{\varepsilon}\right).$$

We denote by  $\tilde{r}, \tilde{\theta}, \tilde{v}$  and  $\tilde{\zeta}$  the components of  $\tilde{z}$ . Recall (3.16), (3.27) and (3.28) which furnish

$$\tau = \varepsilon \frac{2\pi}{\mathcal{P}(z_0(x_0, v_0))} \mathbf{t}(z_0(x_0, v_0); \frac{\tau}{\varepsilon}), \quad 0 < \tilde{\eta}_5 \leq \left|\frac{d\tau}{d\tau}\right| \leq \tilde{\eta}_6.$$

By construction, the expression  $(\tilde{z}, \tilde{v})$  is defined on the interval  $[0, \tilde{\mathcal{T}}(\varepsilon, x_0, v_0)[$  where  $\tilde{\mathcal{T}}$  can be deduced from  $\mathcal{T}(\varepsilon, z_0, \mathbf{v}_0)$  by inverting (uniformly in  $\varepsilon$ ) the above relation from  $\tau$  to  $\tau$ . In so doing, we find that

$$(4.18) \quad \exists \tilde{\mathcal{T}}(x_0, v_0) \in \mathbb{R}_+^*; \quad \forall \varepsilon \in ]0, \varepsilon_0], \quad \tilde{\mathcal{T}}(x_0, v_0) \leq \tilde{\mathcal{T}}(\varepsilon, x_0, v_0).$$

**Lemma 4.3** (The WKB expansion interpreted at the level of  $(\tilde{z}, \tilde{v})$ ). *Suppose that Assumptions 1.6 and 1.7 hold true. Take  $\Omega$  and  $\tilde{\mathcal{T}}$  as indicated above. Then, for all  $\varepsilon \in ]0, \varepsilon_0]$  with  $\varepsilon_0$  small enough, the field  ${}^t(\tilde{z}, \tilde{v})(\varepsilon, x_0, v_0; \cdot)$  is defined on  $[0, \tilde{\mathcal{T}}(x_0, v_0)]$  and there exist smooth profiles  $\tilde{Z}_j = {}^t(\tilde{Z}_{jr}, \tilde{Z}_{j\theta}, \tilde{Z}_{jv}, \tilde{Z}_{j\zeta})$  and  $\tilde{\mathcal{V}}_j$  as follows*

$$\begin{aligned} \tilde{Z}_j(x_0, v_0; \tau, \tilde{s}, \tilde{v}) &\in C^\infty(\Omega \times [0, \tilde{\mathcal{T}}(x_0, v_0)] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}^4), \\ \tilde{\mathcal{V}}_j(x_0, v_0; \tau, \tilde{s}, \tilde{v}) &\in C^\infty(\Omega \times [0, \tilde{\mathcal{T}}(x_0, v_0)] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}), \end{aligned}$$

and such that

$$(4.19) \quad \begin{aligned} \tilde{z}(\varepsilon, x_0, v_0; \tau) &= \sum_{j=0}^{+\infty} \varepsilon^j \tilde{Z}_j\left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2}\right), \\ \tilde{v}(\varepsilon, x_0, v_0; \tau) &= \sum_{j=-1}^{+\infty} \varepsilon^j \tilde{\mathcal{V}}_j\left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2}\right) \\ &= \frac{1}{\varepsilon} \Psi_\varepsilon^2(x_0, v_0; \tau) + \mathcal{O}(\varepsilon), \end{aligned}$$

where  $\Psi^1(x_0, v_0; \tau)$  and  $\Psi_\varepsilon^2(x_0, v_0; \tau)$  are respectively as in (1.16) and (1.17).

Before proceeding to the proof of the above Lemma 4.3, note that we have used many notations to explain the complicated oscillating structure of the solutions. Let us make distinction between different notations that were and will be used in this article.

- ◁ The function  $\Psi^1$  plays the role of a phase which is linear in  $\tau$ ;
- ◁ The function  $\Psi_\varepsilon^2$  is also a phase which oscillates *weakly* with respect to  $\varepsilon$ . Indeed, at the level of (1.17), the weight  $\varepsilon$  is in factor of profiles involving the frequency  $\varepsilon^{-1}$ ;
- ◁ Up to now, we have introduced three distinct couple of variables which solve three distinct systems:
  - $(z, v)$  is related to the time  $t$ . It describes the characteristic equations of the Vlasov equation (1.1);
  - $(z, \mathbf{v})$  is related to the time  $\tau$ . Its expansions are described by the profiles  $(Z_j, \mathcal{V}_j)$  in Proposition 4.2;
  - $(\tilde{z}, \tilde{v})$  is related to the time  $\tau$ . It describes the characteristic equations of the Vlasov system (1.2). Its expansions are described by the profiles  $(\tilde{Z}_j, \tilde{\mathcal{V}}_j)$  in Lemma 4.3;
- ◁ We have introduced the profiles  $(\tilde{X}_j, \tilde{V}_j)$  in Theorem 1.8. It describes the asymptotic expansion related to  $(\tilde{x}, \tilde{v})$ . This is extracted from the profiles  $(\tilde{Z}_j, \tilde{\mathcal{V}}_j)$  as well as the two diffeomorphic maps  $X$  as in (2.4) and  $V$  as in (2.8). And thus, we can assert that every notation with the symbol tilde  $\tilde{\cdot}$  is designated to describe the system (1.2) at time  $\tau$ ;
- ◁ From the above discussions, it is obvious that  $X \neq \tilde{X}_j$  and  $V \neq \tilde{\mathcal{V}}_j \neq \tilde{V}_j$  as well as  $\tilde{v} \neq \tilde{v}$ ;
- ◁ The following terms  $\mathcal{V}_0, \mathcal{V}_1$  in (1.5),  $\mathbf{V}_0, \mathbf{V}_1$  in (1.10),  $\mathbf{V}_0$  in (4.38),  $\tilde{V}_j$  in (1.15) and  $V$  in (2.8) are also distinct. Indeed, we have that
  - $\mathcal{V}_0, \mathcal{V}_1$  and  $\mathbf{V}_0, \mathbf{V}_1$  are the source terms associated respectively to  $v$  and  $\mathbf{v}$ ;
  - $\mathbf{V}_0$  is defined in terms of the source term  $\mathbf{V}_0$  as indicated in (4.38);
  - $\tilde{V}_j$  represents the profiles which are used to describe the oscillatory structure of  $\tilde{v}$ ;
  - $V$  is a diffeomorphic map;
- ◁ The distinction between  $s, \nu$  (Proposition 4.2) and  $\tilde{s}, \tilde{\nu}$  (Theorem 1.8 and Lemma 4.3) is to highlight that different phases come to replace different variables;
- ◁ The following terms  $Z$  in (1.7),  $Z_j$  in (4.13) and  $\tilde{Z}_j$  in (4.19) are also distinct:
  - $Z$  is the average flow;
  - $Z_j$  and  $\tilde{Z}_j$  represent the profiles associated to  $z$  and  $\tilde{z}$  respectively.

*Proof.* Combining (4.15), (4.16), (4.17) and (1.16), we find that

$$\begin{pmatrix} \tilde{z} \\ \tilde{v} \end{pmatrix}(\varepsilon, x_0, v_0; \tau) = \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} \left( \varepsilon, z_0(x_0, v_0), \varepsilon \nu_0(x_0, v_0); \right. \\ \left. \Psi^1(x_0, v_0; \tau) + \varepsilon s^*(z_0(x_0, v_0); \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}) \right).$$

To go further, we exploit (4.13) where  $\tau$  is replaced as indicated above by  $\Psi^1 + \varepsilon s^*$ . We argue with  $\tilde{z}$ , the case of  $\tilde{v}$  being completely similar. By this way, we find that

$$\begin{aligned} \tilde{z}(\varepsilon, x_0, v_0; \tau) = & \sum_{j=0}^{+\infty} \varepsilon^j Z_j \left( z_0(x_0, v_0), \varepsilon \nu_0(x_0, v_0); \Psi^1 + \varepsilon s^*, \frac{\Psi^1}{\varepsilon} + s^*, \right. \\ & \left. \frac{\langle \tilde{\mathcal{V}}_{-1} \rangle(z_0(x_0, v_0); \Psi^1 + \varepsilon s^*)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0(z_0(x_0, v_0), \varepsilon \nu_0(x_0, v_0); \Psi^1 + \varepsilon s^*, \frac{\Psi^1}{\varepsilon} + s^*)}{\varepsilon} \right). \end{aligned}$$

The first line involves only oscillations with respect to  $\Psi^1$  (including those induced by  $s^*$ ) which can be incorporated inside profiles  $\tilde{Z}_j$  as indicated in (4.19). By expanding the second line in

powers of  $\varepsilon$ , we find

$$\begin{aligned} \frac{\langle \overline{\mathcal{V}}_{-1} \rangle(z_0(x_0, v_0); \Psi^1 + \varepsilon s^*)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0(z_0(x_0, v_0), \varepsilon \nu_0(x_0, v_0); \Psi^1 + \varepsilon s^*, \frac{\Psi^1}{\varepsilon} + s^*)}{\varepsilon} \\ = \frac{1}{\varepsilon^2} \Psi_\varepsilon^2(x_0, v_0; \tau) + \mathcal{O}(1), \end{aligned}$$

with  $\Psi_\varepsilon^2$  as in (1.17) and where again the  $\mathcal{O}(1)$  may imply oscillations at the frequency  $\varepsilon^{-1}$  with respect to  $\Psi^1$  which can contribute to the  $\tilde{Z}_j$ . This explains the origin of (4.19).  $\square$

Retain that the preceding construction yields

$$(4.20) \quad \tilde{Z}_0 \equiv \overline{\tilde{Z}}_0(x_0, v_0; \tau, \tilde{s}) = \overline{Z}_0\left(z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau), \tilde{s} + s^*(z_0(x_0, v_0); \tilde{s})\right),$$

$$(4.21) \quad \tilde{\mathcal{V}}_{-1} \equiv \langle \overline{\tilde{\mathcal{V}}}_{-1} \rangle(x_0, v_0; \tau) = \langle \overline{\mathcal{V}}_{-1} \rangle(z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau)),$$

$$(4.22) \quad \tilde{\mathcal{V}}_0 \equiv \overline{\tilde{\mathcal{V}}}_0(x_0, v_0; \tau, \tilde{s}) = \partial_\tau \langle \overline{\mathcal{V}}_{-1} \rangle(z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau)) s^*(z_0(x_0, v_0); \tilde{s}) \\ + \overline{\mathcal{V}}_0(z_0(x_0, v_0), 0; \Psi^1(x_0, v_0; \tau), \tilde{s} + s^*(z_0(x_0, v_0), \tilde{s})),$$

$$(4.23) \quad \tilde{\mathcal{V}}_1 \equiv \overline{\tilde{\mathcal{V}}}_1(x_0, v_0; \tau, \tilde{s}) = \frac{1}{2} \partial_\tau^2 \langle \overline{\mathcal{V}}_{-1} \rangle(z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau)) s^*(z_0(x_0, v_0); \tilde{s})^2 \\ + \partial_\tau \overline{\mathcal{V}}_0(z_0(x_0, v_0), 0; \Psi^1(x_0, v_0; \tau), \tilde{s} + s^*(z_0(x_0, v_0), \tilde{s})) s^* \\ + \partial_{v_0} \overline{\mathcal{V}}_0(z_0(x_0, v_0), 0; \Psi^1(x_0, v_0; \tau), \tilde{s} + s^*(z_0(x_0, v_0), \tilde{s})) \nu_0 \\ + \overline{\mathcal{V}}_1(z_0(x_0, v_0), 0; \Psi^1(x_0, v_0; \tau), \tilde{s} + s^*(z_0(x_0, v_0), \tilde{s})).$$

At the level of (4.19), we can already recognize the oscillating structure of (1.15). But, there are still complications since, as explained in Subsection 2.2 - see especially (2.4), (2.8) and (2.17), to deduce  $(\tilde{x}, \tilde{v})$  from  $(\tilde{z}, \tilde{v})$ , we also need access to  $\phi$ .

4.2.3. *Asymptotic behavior of the toroidal angle.* In line with (4.17), introduce

$$(4.24) \quad \tilde{\phi}(\varepsilon, x_0, v_0; \tau) := \phi(\varepsilon, z_0(x_0, v_0), \varepsilon \nu_0(x_0, v_0), \phi_0(x_0, v_0); \frac{\tau}{\varepsilon}).$$

From (2.10), with  $\mathcal{A}_\phi$  as in (2.10), we know that

$$(4.25) \quad \frac{d\tilde{\phi}}{d\tau} = \frac{1}{\varepsilon} \mathcal{A}_\phi\left(\tilde{z}, \frac{\tilde{v}}{\varepsilon}\right) = \frac{1}{\varepsilon} \frac{\tilde{v}}{R(\tilde{r}, \tilde{\theta})} \left( -\sin \omega(\tilde{r}) \cos \tilde{\zeta} \sin\left(\frac{\tilde{v}}{\varepsilon}\right) + \cos \omega(\tilde{r}) \sin \tilde{\zeta} \right).$$

On the other hand, from the last line of (4.19), we can infer that

$$(4.26) \quad \frac{\tilde{v}}{\varepsilon} = \frac{1}{\varepsilon^2} \Psi_\varepsilon^2(x_0, v_0; \tau) + \sum_{j=1}^{+\infty} \varepsilon^{j-1} \tilde{\mathcal{V}}_j\left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2}\right).$$

Applying Lemma 4.3 and (4.26) to replace  $\tilde{v}/\varepsilon$  as well as rules of composition, it follows that the source term of (4.25) can be expanded (for adequate profiles  $\mathcal{A}_{\phi j}$ ) according to

$$(4.27) \quad \frac{1}{\varepsilon} \mathcal{A}_\phi\left(\tilde{z}, \frac{\tilde{v}}{\varepsilon}\right) = \sum_{j=-1}^{+\infty} \varepsilon^j \mathcal{A}_{\phi j}\left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2}\right).$$

In view of (4.20) and (4.23), knowing that  $\tilde{Z}_0 \equiv \overline{\tilde{Z}}_0(\tau, \tilde{s})$  and  $\tilde{\mathcal{V}}_1 \equiv \overline{\tilde{\mathcal{V}}}_1(\tau, \tilde{s})$ , the term with  $\varepsilon^{-1}$  in factor inside (4.27) yields

$$\begin{aligned} \mathcal{A}_{\phi-1}(x_0, v_0; \tau, \tilde{s}, \tilde{v}) &= \mathcal{A}_\phi(\overline{\tilde{Z}}_0(x_0, v_0; \tau, \tilde{s}), \tilde{v} + \overline{\tilde{\mathcal{V}}}_1(x_0, v_0; \tau, \tilde{s})) \\ &= \left[ \frac{\overline{\tilde{Z}}_{0v}}{R(\overline{\tilde{Z}}_{0r}, \overline{\tilde{Z}}_{0\theta})} \left( -\sin \omega(\overline{\tilde{Z}}_{0r}) \cos \overline{\tilde{Z}}_{0\zeta} \sin(\tilde{v} + \overline{\tilde{\mathcal{V}}}_1) + \cos \omega(\overline{\tilde{Z}}_{0r}) \sin \overline{\tilde{Z}}_{0\zeta} \right) \right] (x_0, v_0; \tau, \tilde{s}). \end{aligned}$$



**Lemma 4.4** (The double mean of  $\mathcal{A}_{\phi-1}$ ). *With  $\mathcal{A}_{\phi-1}$  as above, we have*

$$(4.28) \quad \langle \bar{\mathcal{A}}_{\phi-1} \rangle(x_0, v_0; \tau) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \mathcal{A}_{\phi-1}(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}) \, d\tilde{\nu} \, d\tilde{s} = 0.$$

*Proof.* Consider (4.20) together with (4.14) to extract

$$(4.29) \quad \bar{Z}_0(x_0, v_0; \tau, \tilde{s}) = Z\left(\langle \bar{\mathcal{Z}}_0 \rangle(z_0(x_0, v_0); \Psi^1(\tau)); t(z_0(x_0, v_0); \tilde{s} + s^*(z_0(x_0, v_0); \tilde{s}))\right),$$

where  $Z$  is the solution to (3.2) and  $t(z_0; \cdot)$  is the inverse of  $s(z_0; \cdot)$ . In particular, due to (3.1), we find that

$$(4.30) \quad \bar{Z}_{0r} \equiv \langle \bar{Z}_{0r} \rangle(x_0, v_0; \tau) = \langle \bar{\mathcal{Z}}_{0r} \rangle(z_0(x_0, v_0); \Psi^1(\tau)).$$

With the aid of (4.29) and (4.30), the mean of  $\mathcal{A}_{\phi-1}$  with respect to  $\tilde{\nu} \in \mathbb{T}$  is

$$\begin{aligned} \bar{\mathcal{A}}_{\phi-1}(\tau, \tilde{s}) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{Z_v}{R(\langle \bar{\mathcal{Z}}_{0r} \rangle, Z_\theta)} \left[ -\sin \omega(\langle \bar{\mathcal{Z}}_{0r} \rangle) \cos Z_\zeta \sin(\tilde{\nu} + \bar{\mathcal{V}}_1) + \cos \omega(\langle \bar{\mathcal{Z}}_{0r} \rangle) \sin Z_\zeta \right] d\tilde{\nu} \\ &= \left[ \frac{Z_v}{R(\langle \bar{\mathcal{Z}}_{0r} \rangle, Z_\theta)} \cos \omega(\langle \bar{\mathcal{Z}}_{0r} \rangle) \sin Z_\zeta \right] \left( \langle \bar{\mathcal{Z}}_0 \rangle(z_0; \Psi^1(\tau)); t(z_0; \tilde{s} + s^*(z_0; \tilde{s})) \right). \end{aligned}$$

Now, we can compute then the mean of  $\bar{\mathcal{A}}_{\phi-1}$  with respect to  $\tilde{s} \in \mathbb{T}$  to find that

$$\begin{aligned} \langle \bar{\mathcal{A}}_{\phi-1} \rangle(x_0, v_0; \tau) &= \frac{1}{2\pi} \cos \omega(\langle \bar{\mathcal{Z}}_{0r} \rangle) \int_0^{2\pi} \left[ \frac{Z_v}{R(\langle \bar{\mathcal{Z}}_{0r} \rangle, Z_\theta)} \sin Z_\zeta \right] \left( \langle \bar{\mathcal{Z}}_0 \rangle; t(z_0; \tilde{s} + s^*(z_0; \tilde{s})) \right) d\tilde{s} \\ &= \frac{1}{P(z_0)} \cos \omega(\langle \bar{\mathcal{Z}}_{0r} \rangle) \int_0^{P(z_0)} \left[ \frac{Z_v}{R(\langle \bar{\mathcal{Z}}_{0r} \rangle, Z_\theta)} \sin Z_\zeta \right] \left( \langle \bar{\mathcal{Z}}_0 \rangle; t(z_0; \underbrace{\frac{2\pi\tilde{t}}{P(z_0)} + s^*(z_0; \frac{2\pi\tilde{t}}{P(z_0)})}_{=s(z_0; \tilde{t})}) \right) d\tilde{t} \\ &= \frac{1}{P(z_0)} \cos \omega(\langle \bar{\mathcal{Z}}_{0r} \rangle) \int_0^{P(z_0)} \left[ \frac{Z_v}{R(\langle \bar{\mathcal{Z}}_{0r} \rangle, Z_\theta)} \sin Z_\zeta \right] (\langle \bar{\mathcal{Z}}_0 \rangle; \tilde{t}) \, d\tilde{t}, \end{aligned}$$

where, to obtain the third line, we took the change of variable  $\tilde{s} = 2\pi\tilde{t}/P(z_0)$  and we exploited (3.35). Now, looking at the first equation of (3.2), we have

$$(4.31) \quad [Z_v \sin Z_\zeta] (\langle \bar{\mathcal{Z}}_0 \rangle, \tilde{t}) = \frac{dZ_\theta}{d\tilde{t}} \frac{\langle \bar{\mathcal{Z}}_{0r} \rangle}{\sin \omega(\langle \bar{\mathcal{Z}}_{0r} \rangle)}.$$

This furnishes

$$\langle \bar{\mathcal{A}}_{\phi-1} \rangle(x_0, v_0; \tau) = \frac{1}{P(z_0)} \frac{\langle \bar{\mathcal{Z}}_{0r} \rangle \cos \omega(\langle \bar{\mathcal{Z}}_{0r} \rangle)}{\sin \omega(\langle \bar{\mathcal{Z}}_{0r} \rangle)} \int_0^{P(z_0)} \frac{1}{R_0 + \langle \bar{\mathcal{Z}}_{0r} \rangle \cos Z_\theta(\langle \bar{\mathcal{Z}}_0 \rangle; \tilde{t})} \frac{dZ_\theta(\langle \bar{\mathcal{Z}}_0 \rangle, \tilde{t})}{d\tilde{t}} d\tilde{t}.$$

This must be zero since the average flow  $Z(\langle \bar{\mathcal{Z}}_0 \rangle; \cdot)$  is periodic of period  $P(z_0)$ .  $\square$

We seek a WKB solution  $\tilde{\phi}_a^\varepsilon$  to (4.25) in a similar form as (4.27), that is

$$(4.32) \quad \tilde{\phi}_a^\varepsilon(x_0, v_0; \tau) = \sum_{j=-1}^{+\infty} \varepsilon^j \tilde{\Phi}_j \left( x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2} \right).$$

The idea is to perform a formal analysis at the level of (4.25) using (4.32) to find constraints on the profiles  $\tilde{\Phi}_j$ . To this end, we need first to compute the derivative of the oscillating phase  $\Psi_\varepsilon^2$  with respect to time variable  $\tau$ . From (1.17), it is found to be

$$\begin{aligned} (4.33) \quad \partial_\tau \Psi_\varepsilon^2(\tau) &= \frac{2\pi}{P(z_0)} \left( 1 + \partial_t s^*(z_0; \frac{\Psi^1}{\varepsilon}) \right) (\partial_s \bar{\mathcal{V}}_0 + \partial_\tau \langle \bar{\mathcal{V}}_{-1} \rangle)(z_0, 0; \Psi^1, \frac{\Psi^1}{\varepsilon} + s^*(z_0; \frac{\Psi^1}{\varepsilon})) \\ &\quad + \varepsilon \frac{2\pi}{P(z_0)} (\partial_\tau \bar{\mathcal{V}}_0 + \partial_\tau^2 \langle \bar{\mathcal{V}}_{-1} \rangle s^*(z_0; \frac{\Psi^1}{\varepsilon}))(z_0, 0; \Psi^1, \frac{\Psi^1}{\varepsilon} + s^*(z_0; \frac{\Psi^1}{\varepsilon})). \end{aligned}$$

To proceed, we need to simplify (4.33). To this end, we exploit two helpful information. The first is issued from (3.35) which gives rise to

$$(4.34) \quad \partial_t s(z_0; \frac{\tau}{\varepsilon}) = \frac{2\pi}{P(z_0)} \left( 1 + \partial_t s^*(z_0; \frac{\Psi^1}{\varepsilon}) \right).$$

The second tool is given in the following lemma.

**Lemma 4.5** (Simplification). *With  $\mathcal{V}_0$  as in (2.15), we have*

$$(4.35) \quad \begin{aligned} & (\partial_s \overline{\mathcal{V}}_0 + \partial_\tau \langle \overline{\mathcal{V}}_{-1} \rangle)(z_0, 0; \Psi^1, \frac{\Psi^1}{\varepsilon} + s^*(z_0; \frac{\Psi^1}{\varepsilon})) \\ &= \partial_t s(z_0; \frac{\tau}{\varepsilon})^{-1} \mathcal{V}_0 \left( \Xi_{\text{mf}}(z_0; \langle \overline{\mathcal{V}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})) \right). \end{aligned}$$

*Proof.* First from (3.35), we have

$$(4.36) \quad s(z_0; \frac{\tau}{\varepsilon}) = \frac{\Psi^1(\tau)}{\varepsilon} + s^*(z_0; \frac{\Psi^1(\tau)}{\varepsilon}).$$

Exploit (4.36) together with [11, Section 3, equations (3.41) and (3.42)] to get

$$\begin{aligned} \partial_\tau \langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \Psi^1) &= \langle \overline{\mathcal{V}}_0 \rangle(z_0; \langle \overline{\mathcal{V}}_0 \rangle(z_0; \Psi^1)), \\ \partial_s \overline{\mathcal{V}}_0(z_0, 0; \Psi^1, \underbrace{\frac{\Psi^1}{\varepsilon} + s^*(z_0; \frac{\Psi^1}{\varepsilon})}_{=s(z_0; \frac{\tau}{\varepsilon})}) &= \overline{\mathcal{V}}_0^*(z_0; \langle \overline{\mathcal{V}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})). \end{aligned}$$

Then exploiting the above two expressions leads to

$$(4.37) \quad \begin{aligned} & \partial_s \overline{\mathcal{V}}_0(z_0, 0; \Psi^1, \frac{\Psi^1}{\varepsilon} + s^*(z_0; \frac{\Psi^1}{\varepsilon})) + \partial_\tau \langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \Psi^1) \\ &= \overline{\mathcal{V}}_0(z_0; \langle \overline{\mathcal{V}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})) \equiv V_0(z_0; \langle \overline{\mathcal{V}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})), \end{aligned}$$

where we have introduced the term  $V_0$  as indicated in [11, Section 2, equation (2.22)]. We have

$$(4.38) \quad V_0(z_0; \langle \overline{\mathcal{V}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})) := V_0\left(z_0; \Xi_{\text{mf}}(z_0; \langle \overline{\mathcal{V}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})); s(z_0; \frac{\tau}{\varepsilon})\right).$$

Then, using (4.38), (4.37) becomes

$$(4.39) \quad \begin{aligned} & \partial_s \overline{\mathcal{V}}_0(z_0, 0; \Psi^1, \frac{\Psi^1}{\varepsilon} + s^*(z_0; \frac{\Psi^1}{\varepsilon})) + \partial_\tau \langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \Psi^1) \\ &= V_0\left(z_0; \Xi_{\text{mf}}(z_0; \langle \overline{\mathcal{V}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})); s(z_0; \frac{\tau}{\varepsilon})\right) \\ &= \partial_t s(z_0; \underbrace{t(z_0; s(z_0; \frac{\tau}{\varepsilon}))}_{= \frac{\tau}{\varepsilon} \text{ since } t \text{ is inverse of } s})^{-1} \mathcal{V}_0\left(\Xi_{\text{mf}}(z_0; \langle \overline{\mathcal{V}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon}))\right) \\ &= \partial_t s(z_0; \frac{\tau}{\varepsilon})^{-1} \mathcal{V}_0\left(\Xi_{\text{mf}}(z_0; \langle \overline{\mathcal{V}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon}))\right), \end{aligned}$$

where in the line before the last, we have used (4.5). □

With the aid of the two identities (4.34) and (4.35), the line (4.33) reduces to

$$(4.40) \quad \partial_\tau \Psi_\varepsilon^2(\tau) = \mathcal{V}_0 + \varepsilon \frac{2\pi}{P(z_0)} (\partial_\tau \overline{\mathcal{V}}_0 + \partial_\tau^2 \langle \overline{\mathcal{V}}_{-1} \rangle s^*).$$

In view of (4.40), the equation (4.25) is formally equivalent to

$$(4.41) \quad \sum_{j=-1}^{+\infty} \varepsilon^j \partial_\tau \tilde{\Phi}_j + \sum_{j=-1}^{+\infty} \varepsilon^{j-1} \frac{2\pi}{P(z_0)} \left\{ \partial_s \tilde{\Phi}_j + (\partial_\tau \overline{\mathcal{V}}_0 + \partial_\tau^2 \langle \overline{\mathcal{V}} - 1 \rangle s^*) \partial_{\tilde{\nu}} \tilde{\Phi}_j \right\} \\ + \sum_{j=-1}^{+\infty} \varepsilon^{j-2} \mathcal{V}_0 \partial_{\tilde{\nu}} \tilde{\Phi}_j = \sum_{j=-1}^{+\infty} \varepsilon^j \mathcal{A}_{\phi j}(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}).$$

**Lemma 4.6** (The WKB expansion of the toroidal coordinate  $\tilde{\phi}$ ). *Suppose that Assumptions 1.6 and 1.7 hold. Let  $\Omega$  as in Theorem 1.8 and  $\tilde{T}$  as in (4.18). For all  $\varepsilon \in ]0, \varepsilon_0]$  with  $\varepsilon_0$  small enough, the function  $\tilde{\phi}(\varepsilon, x_0, v_0; \cdot)$  is defined on  $[0, \tilde{T}(x_0, v_0)]$ . Moreover, there exist smooth profiles*

$$\tilde{\Phi}_j(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}) \in C^\infty(\Omega \times [0, \tilde{T}] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}), \quad j \in \mathbb{N},$$

such that the expression  $\tilde{\phi}_a^\varepsilon$  which is built as indicated in (4.32) with  $\tilde{\Phi}_{-1} \equiv 0$  is a solution to the cascade of equations that is issued from (4.41). This furnishes through (4.32) an approximated solution  $\tilde{\phi}_a^\varepsilon$  which is close in the sup norm to the solution  $\tilde{\phi}$  of (4.25) with, for all  $N \in \mathbb{N}$ , a precision of order  $\mathcal{O}(\varepsilon^N)$ .

In what follows, we need to invert the derivative  $\partial_{\tilde{s}}$  and  $\partial_{\tilde{\nu}}$ . To this end, introduce the set  $L_*^1(\mathbb{T})$  made of periodic functions with zero mean, namely

$$L_*^1(\mathbb{T}) := \left\{ \mathcal{Z} \in L^1(\mathbb{T}); \int_0^{2\pi} \mathcal{Z}(\theta) d\theta = 0 \right\}.$$

We can define an operator  $\partial_{\tilde{s}}^{-1}$  or  $\partial_{\tilde{\nu}}^{-1} : L_*^1(\mathbb{T}) \longrightarrow L_*^1(\mathbb{T})$  according to

$$(4.42) \quad \partial_{\tilde{s}}^{-1} \mathcal{Z}(\tilde{s}) := \int_0^{\tilde{s}} \mathcal{Z}(r) dr - \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^s \mathcal{Z}(r) dr \right) ds, \\ \partial_{\tilde{\nu}}^{-1} \mathcal{Z}(\tilde{\nu}) := \int_0^{\tilde{\nu}} \mathcal{Z}(r) dr - \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^s \mathcal{Z}(r) dr \right) ds.$$

*Proof.* The matter here is to determine the profiles  $\tilde{\Phi}_j$  using the constraint (4.41). Define

$$(4.43) \quad \Gamma_{-3}(\tilde{\Phi}_{-1}) := \mathcal{V}_0 \partial_{\tilde{\nu}} \tilde{\Phi}_{-1},$$

$$(4.44) \quad \Gamma_{-2}(\tilde{\Phi}_{-1}, \tilde{\Phi}_0) := \frac{2\pi}{P(z_0)} \left\{ \partial_s \tilde{\Phi}_{-1} + \partial_{\tilde{\nu}} \tilde{\Phi}_{-1} (\partial_\tau \overline{\mathcal{V}}_0 + \partial_\tau^2 \langle \overline{\mathcal{V}} - 1 \rangle s^*) \right\} + \mathcal{V}_0 \partial_{\tilde{\nu}} \tilde{\Phi}_0.$$

The above two expressions (4.43) and (4.44) are respectively the  $\varepsilon^{-3}$  and  $\varepsilon^{-2}$  terms collected from (4.41). In a similar fashion, we pick the  $\varepsilon^j$  terms from (4.41) and we gather them in the expression  $\Gamma_j$  which, for  $j \geq -1$ , is defined by

$$(4.45) \quad \Gamma_j(\tilde{\Phi}_j, \tilde{\Phi}_{j+1}, \tilde{\Phi}_{j+2}) := \partial_\tau \tilde{\Phi}_j + \frac{2\pi}{P(z_0)} \left\{ \partial_s \tilde{\Phi}_{j+1} + \partial_{\tilde{\nu}} \tilde{\Phi}_{j+1} (\partial_\tau \overline{\mathcal{V}}_0 + \partial_\tau^2 \langle \overline{\mathcal{V}} - 1 \rangle s^*) \right\} \\ + \mathcal{V}_0 \partial_{\tilde{\nu}} \tilde{\Phi}_{j+2} - \mathcal{A}_{\phi j}(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}).$$

Introduce the following hypothesis

$$\mathcal{H}_{\phi j} : \left\{ \text{The profiles } \tilde{\Phi}_{-1}, \dots, \tilde{\Phi}_j, \overline{\tilde{\Phi}}_{j+1}^*, \tilde{\Phi}_{j+1}^*, \tilde{\Phi}_{j+2}^* \text{ are known on the domain } [0, \tilde{T}] \times \mathbb{T}^2 \right\}$$

which reflects the outcome of solving successively  $\Gamma_k = 0$  from  $k = -3$  up to  $k = j$ . To prove Lemma 4.6, it suffices to show that  $\mathcal{H}_{\phi j}$  is verified for all  $j \geq -1$ . This is achieved in two steps:

- A) First, we prove that the preliminary condition  $\mathcal{H}_{\phi -1}$  is verified. This amounts to the determination of the profiles  $\tilde{\Phi}_{-1}$ ,  $\overline{\tilde{\Phi}}_0^*$ ,  $\tilde{\Phi}_0^*$  and  $\tilde{\Phi}_1^*$ ;
- B) Secondly, we prove the validity of the hypothesis  $\mathcal{H}_{\phi j}$  by using an induction argument.

In what follows, we first check A). Then, we consider B).

A) To obtain the starting point  $\mathcal{H}_{\phi-1}$ , we can limit ourselves at looking at the constraints  $\Gamma_k = 0$  for  $k = -3, -2, -1$ . The constraint  $\Gamma_{-3} = 0$  with  $\Gamma_{-3}$  emanating from (4.43) implies that

$$(4.46) \quad \underbrace{\mathcal{V}_0}_{\neq 0} \partial_{\tilde{\nu}} \tilde{\Phi}_{-1} = 0 \quad \Rightarrow \quad \tilde{\Phi}_{-1} \equiv \bar{\tilde{\Phi}}_{-1}(x_0, v_0; \tau, \tilde{s}).$$

With the simplification (4.46), the constraint  $\Gamma_{-2} = 0$  becomes

$$(4.47) \quad \frac{2\pi}{P(z_0)} \partial_{\tilde{s}} \bar{\tilde{\Phi}}_{-1} + \mathcal{V}_0 \partial_{\tilde{\nu}} \tilde{\Phi}_0^* = 0.$$

Since  $\mathcal{V}_0$  does not depend on  $\tilde{\nu}$ , the equation (4.47) amounts to the same thing as

$$(4.48) \quad \bar{\tilde{\Phi}}_{-1} \equiv \langle \bar{\tilde{\Phi}}_{-1} \rangle(x_0, v_0; \tau), \quad \tilde{\Phi}_0 \equiv \bar{\tilde{\Phi}}_0(x_0, v_0; \tau, \tilde{s}).$$

With the aid of (4.48), the equation  $\Gamma_{-1} = 0$  reduces to

$$(4.49) \quad \partial_{\tau} \langle \bar{\tilde{\Phi}}_{-1} \rangle + \frac{2\pi}{P(z_0)} \partial_{\tilde{s}} \bar{\tilde{\Phi}}_0^* + \mathcal{V}_0 \partial_{\tilde{\nu}} \tilde{\Phi}_1^* = \mathcal{A}_{\phi-1}(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}).$$

Take in (4.49) the average in both variables  $(\tilde{s}, \tilde{\nu}) \in \mathbb{T}^2$  to get

$$(4.50) \quad \partial_{\tau} \langle \bar{\tilde{\Phi}}_{-1} \rangle = \langle \bar{\mathcal{A}}_{\phi-1} \rangle(x_0, v_0; \tau).$$

Due to (4.28) and since  $\langle \bar{\tilde{\Phi}}_{-1} \rangle(x_0, v_0; 0) = 0$ , we find that  $\bar{\tilde{\Phi}}_{-1}(x_0, v_0; \tau, \tilde{s}) \equiv \langle \bar{\tilde{\Phi}}_{-1} \rangle(x_0, v_0; \tau) \equiv 0$ . The double mean value  $\langle \bar{\tilde{\Phi}}_{-1} \rangle$  is not activated so that  $\partial_{\tau} \langle \bar{\tilde{\Phi}}_{-1} \rangle(x_0, v_0; \tau) = 0$ . Then, the equation (4.49) averaged with respect to  $\tilde{\nu} \in \mathbb{T}$  furnishes

$$(4.51) \quad \frac{2\pi}{P(z_0)} \partial_{\tilde{s}} \bar{\tilde{\Phi}}_0^* = \bar{\mathcal{A}}_{\phi-1}(x_0, v_0; \tau, \tilde{s}) \equiv \bar{\mathcal{A}}_{\phi-1}^*(x_0, v_0; \tau, \tilde{s}).$$

Since the right hand side of (4.51) belongs to  $L_*^1(\mathbb{T})$  with respect to  $\tilde{s}$ , we can invert  $\partial_{\tilde{s}}$  as indicated in (4.42) to recover  $\bar{\tilde{\Phi}}_0^*$ . Then, substitute  $\partial_{\tilde{s}} \bar{\tilde{\Phi}}_0^*$  as in (4.51) into (4.49) to obtain

$$(4.52) \quad \partial_{\tilde{\nu}} \tilde{\Phi}_1^* = \frac{1}{\mathcal{V}_0} \mathcal{A}_{\phi-1}^*(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}).$$

Once more, since the right hand side of (4.52) belongs to  $L_*^1(\mathbb{T})$  with respect to  $\tilde{\nu}$ , we can invert  $\partial_{\tilde{\nu}}$  as indicated in (4.42) to get  $\tilde{\Phi}_1^*$ . We have determined  $\bar{\tilde{\Phi}}_{-1} \equiv 0$ ,  $\bar{\tilde{\Phi}}_0^*$ ,  $\tilde{\Phi}_0^* \equiv 0$  and  $\tilde{\Phi}_1^*$  which means that the starting point  $\mathcal{H}_{\phi-1}$  is verified.

B) We assume that  $\mathcal{H}_{\phi k}$  is verified for  $k = -1$  up to  $k = j - 1$ . This means that  $\tilde{\Phi}_{-1}, \dots, \tilde{\Phi}_{j-1}, \bar{\tilde{\Phi}}_j^*, \tilde{\Phi}_j^*, \tilde{\Phi}_{j+1}^*$  have been identified on the domain  $[0, \tilde{\mathcal{T}}] \times \mathbb{T}^2$ . The aim is to prove that  $\mathcal{H}_{\phi j}$  holds true. Recall that  $\tilde{\Phi}_j$  can be decomposed according to (4.8). To obtain  $\mathcal{H}_{\phi j}$ , we need to determine  $\langle \bar{\tilde{\Phi}}_j \rangle, \bar{\tilde{\Phi}}_{j+1}^*, \tilde{\Phi}_{j+2}^*$ . Look at the equation  $\langle \bar{\Gamma}_j \rangle = 0$  with  $\Gamma_j$  as in (4.45), that is

$$(4.53) \quad \partial_{\tau} \langle \bar{\tilde{\Phi}}_j \rangle = \langle \bar{\mathcal{A}}_{\phi j} \rangle(x_0, v_0; \tau),$$

which must be completed with the initial data coming from  $\tilde{\phi}(x_0, v_0; 0) = \phi_0$ . The linear Cauchy differential equation (4.53) has a solution on the whole interval  $[0, \tilde{\mathcal{T}}]$ , leading to  $\langle \bar{\tilde{\Phi}}_j \rangle$ . Next, we consider the equation  $\bar{\Gamma}_j = 0$  where  $\bar{\tilde{\Phi}}_j$  is written in the form  $\bar{\tilde{\Phi}}_j = \bar{\tilde{\Phi}}_j^* + \langle \bar{\tilde{\Phi}}_j \rangle$  and where  $\partial_{\tau} \langle \bar{\tilde{\Phi}}_j \rangle$  is replaced as indicated in (4.53). This furnishes

$$(4.54) \quad \bar{\tilde{\Phi}}_{j+1}^* = \frac{P(z_0)}{2\pi} \partial_{\tilde{s}}^{-1} [\bar{\mathcal{A}}_{\phi j}^*(x_0, v_0; \tau, \tilde{s}) - \partial_{\tau} \bar{\tilde{\Phi}}_j^*].$$

There remains to extract  $\tilde{\Phi}_{j+2}^*$  from the equation  $\Gamma_j = 0$  through

$$\tilde{\Phi}_{j+2}^* = \frac{1}{\mathcal{V}_0} \partial_{\tilde{\nu}}^{-1} \left[ \mathcal{A}_{\phi j}^* - \partial_{\tau} \tilde{\Phi}_j^* - \frac{2\pi}{\mathbb{P}(z_0)} \left\{ \partial_{\tilde{s}} \tilde{\Phi}_{j+1}^* + \partial_{\tilde{\nu}} \tilde{\Phi}_{j+1}^* (\partial_{\tau} \overline{\mathcal{V}}_0 + \partial_{\tau}^2 \langle \overline{\mathcal{V}} - 1 \rangle s^*) \right\} \right].$$

In summary, we have  $\mathcal{H}_{\phi j}$ . Thus, for all  $N \in \mathbb{N}$ , we know that  $\partial_{\tau}(\tilde{\phi} - \tilde{\phi}_a^{\varepsilon}) = \mathcal{O}(\varepsilon^N)$  and therefore (just by integration) that  $\tilde{\phi} - \tilde{\phi}_a^{\varepsilon} = \mathcal{O}(\varepsilon^N)$ .  $\square$

4.2.4. *Proof of Theorem 1.8.* According to (2.4), (2.8) and (2.17), the solution  ${}^t(x, v)$  to the system (1.1) looks like

$$(4.55) \quad \begin{pmatrix} x \\ v \end{pmatrix}(\varepsilon, x_0, v_0; t) = \begin{pmatrix} X(r, \theta, \phi) \\ V(r, \theta, \phi, v, \zeta, v/\varepsilon) \end{pmatrix}(z_0(x_0, v_0), \varepsilon \nu_0(x_0, v_0), \phi_0(x_0, v_0); t).$$

This means, using (4.17) as well as  $(\tilde{x}, \tilde{v})(x_0, v_0; \tau) = (x, v)(x_0, v_0; \frac{\tau}{\varepsilon})$ , that

$$(4.56) \quad \begin{pmatrix} \tilde{x} \\ \tilde{v} \end{pmatrix}(\varepsilon, x_0, v_0; \tau) = \begin{pmatrix} X(\tilde{r}, \tilde{\theta}, \tilde{\phi}) \\ V(\tilde{r}, \tilde{\theta}, \tilde{\phi}, \tilde{v}, \tilde{\zeta}, \tilde{v}/\varepsilon) \end{pmatrix}(\varepsilon, x_0, v_0; \tau).$$

The existence part of Theorem 1.8 is guaranteed by Lemmas 4.3 and 4.6. On the other hand, the expansion of  $\tilde{x}$  and  $\tilde{v}$  can be recovered by applying Taylor expansion in (4.56) to the two functions  $X(\cdot)$  and  $V(\cdot)$  after substituting  $\tilde{z} = (\tilde{r}, \tilde{\theta}, \tilde{v}, \tilde{\zeta})$ ,  $\tilde{\phi}$  and  $\tilde{v}/\varepsilon$  as indicated respectively in Lemma 4.3, Lemma 4.6 and (4.26). By this way, we can recognize (1.15).

4.3. **The long time gyrokinetic equations.** The goal of this section is to understand the long time leading behavior of the solution to (1.2) under Assumptions 1.6 and 1.7. In view of (4.19), (4.20), (4.21), (4.22), (4.23), (4.26) and (4.48), the identity (4.56) gives rise to

$$(4.57) \quad \tilde{x}(x_0, v_0; \tau) = X\left(\overline{\tilde{Z}}_{0r}, \overline{\tilde{Z}}_{0\theta}, \overline{\tilde{\Phi}}_0\right)(x_0, v_0; \tau, \frac{\Psi^1}{\varepsilon}) + \mathcal{O}(\varepsilon),$$

$$(4.58) \quad \tilde{v}(x_0, v_0; \tau) = V\left(\overline{\tilde{Z}}_{0r}, \overline{\tilde{Z}}_{0\theta}, \overline{\tilde{\Phi}}_0, \overline{\tilde{Z}}_{0v}, \overline{\tilde{Z}}_{0\zeta}, \frac{\Psi_{\varepsilon}^2}{\varepsilon^2} + \overline{\mathcal{V}}_1\right)(x_0, v_0; \tau, \frac{\Psi^1}{\varepsilon}) + \mathcal{O}(\varepsilon).$$

Substitute  $s(z_0; \frac{\tau}{\varepsilon})$  as indicated in (4.36) inside (4.29) to see that

$$(4.59) \quad \overline{\tilde{Z}}_0(x_0, v_0; \tau, \frac{\Psi^1(\tau)}{\varepsilon}) = Z\left(\langle \overline{\tilde{\mathcal{Z}}}_0 \rangle(z_0; \Psi^1(\tau)); t(z_0; s(z_0; \frac{\tau}{\varepsilon}))\right) = Z\left(\langle \overline{\tilde{\mathcal{Z}}}_0 \rangle(z_0; \Psi^1(\tau)); \frac{\tau}{\varepsilon}\right).$$

With the aid of (4.59) and since we have the simplification (4.30), in line with (1.15), the equations (4.57) and (4.58) become (1.21), where, with  $\langle \overline{\tilde{\mathcal{Z}}}_0 \rangle \equiv \langle \overline{\tilde{\mathcal{Z}}}_0 \rangle(z_0(x_0, v_0); \Psi^1(\tau))$  and  $\mathbb{P} \equiv \mathbb{P}(z_0(x_0, v_0))$ , we have introduced  $\tilde{X}_0$  and  $\tilde{V}_0$  as indicated in (1.18) and (1.19).

In contrast with the classical gyrokinetic equations, the *long time* gyrokinetic ansatz  $\tilde{X}_0$  implies large amplitude oscillations (carried by  $\tilde{s}$ ). In the formulas (1.18) and (1.19) for  $\tilde{X}_0$  and  $\tilde{V}_0$ , the following components can be directly determined from the data:  $X$  and  $V$  are given by (2.4) and (2.8); the average flow  $Z$  can be deduced from (3.1) and (3.2); the scalar function  $\overline{\mathcal{V}}_1$  comes from (4.23) together with the WKB construction of Proposition 4.2; and  $\overline{\tilde{\Phi}}_0$  is issued from (4.51) and (4.53) which are based on  $\overline{\mathcal{A}}_{\phi-1}^*$  and  $\overline{\mathcal{A}}_{\phi 0}$ . Once done, the access to the main profiles  $\tilde{X}_0$  and  $\tilde{V}_0$  is entirely driven by  $\langle \overline{\tilde{\mathcal{Z}}}_0 \rangle(z_0; \tau)$ .

Now, we recall how to determine  $\langle \overline{\tilde{\mathcal{Z}}}_0 \rangle(z_0; \tau)$  precisely from [11]. The result below is proved in [11, Theorem 4] where the source terms  $\mathbf{A}$  and  $\mathbf{V}$  in the latter reference are supposed to be expanded in powers of  $\varepsilon$  up to order  $N \in \mathbb{N}$  as follows

$$(4.60) \quad \mathbf{A} = \sum_{j=0}^N \varepsilon^j \mathbf{A}_j + \mathcal{O}(\varepsilon^{N+1}), \quad \mathbf{V} = \sum_{j=0}^N \varepsilon^j \mathbf{V}_j + \mathcal{O}(\varepsilon^{N+1}).$$

**Proposition 4.7** (Theorem 4 in [11]: Reduced equations). *The function  $\langle \bar{\mathfrak{Z}}_0 \rangle$  in the right hand side of (4.14) can be determined by solving*

$$(4.61) \quad \partial_\tau \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau) = \langle \bar{\mathbf{A}}_1 \rangle(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau)), \quad \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; 0) = z_0,$$

where, with  $\partial_\nu^{-1}$  as in (4.42), we have introduced

$$(4.62) \quad \begin{aligned} \mathbf{A}_1(z_0; \mathfrak{z}; s, \nu) &:= D_{\mathfrak{z}} \Xi_{\text{mf}}(z_0; \mathfrak{z}; s)^{-1} \\ &\left\{ [\mathbf{A}_1 + \mathbf{V}_0^{-1} (\partial_\nu^{-1} \mathbf{A}_0^* \cdot \nabla_z) \mathbf{A}_0 - \partial_s (\mathbf{V}_0^{-1} \partial_\nu^{-1} \mathbf{A}_0^*) - \mathbf{V}_1 \mathbf{V}_0^{-1} \mathbf{A}_0^* \right. \\ &\quad \left. + (\partial_\nu^{-1} \mathbf{A}_0^* \cdot \nabla_z) (\mathbf{V}_0^{-1}) \mathbf{A}_0^*] (z_0; \Xi_{\text{mf}}(z_0; \mathfrak{z}; s); s, \nu) \right\} \end{aligned}$$

and where the access to the double mean value  $\langle \bar{\mathbf{A}}_1 \rangle$  is furnished by (4.7).

Then to apply the above proposition, the first thing to do is to adapt the arguments to our context. In our case, due to (1.9) and (1.10), the source terms  $\mathbf{A}$  and  $\mathbf{V}$  in (4.60) are as follows

$$(4.63) \quad \mathbf{A} = \mathbf{A}_0, \quad \mathbf{A}_j = 0, \quad \forall j \geq 1,$$

$$(4.64) \quad \mathbf{V} = \mathbf{V}_0 + \varepsilon \mathbf{V}_1, \quad \mathbf{V}_j = 0, \quad \forall j \geq 2.$$

Then, Proposition 4.7 with the choice of  $\mathbf{A}$  and  $\mathbf{V}$  in (4.63) and (4.64) implies that the function  $\langle \bar{\mathfrak{Z}}_0 \rangle$  can be determined by solving (4.61) where the expression  $\mathbf{A}_1 = {}^t(\mathbf{A}_{1r}, \mathbf{A}_{1\theta}, \mathbf{A}_{1\nu}, \mathbf{A}_{1\zeta})$  in (4.62) is reduced to

$$(4.65) \quad \begin{aligned} \mathbf{A}_1(z_0; \mathfrak{z}; s, \nu) &:= D_{\mathfrak{z}} \Xi_{\text{mf}}(z_0; \mathfrak{z}; s)^{-1} \\ &\left\{ [\mathbf{V}_0^{-1} (\partial_\nu^{-1} \mathbf{A}^* \cdot \nabla_z) \mathbf{A} - \partial_s (\mathbf{V}_0^{-1} \partial_\nu^{-1} \mathbf{A}^*) - \mathbf{V}_1 \mathbf{V}_0^{-1} \mathbf{A}^* \right. \\ &\quad \left. + (\partial_\nu^{-1} \mathbf{A}^* \cdot \nabla_z) (\mathbf{V}_0^{-1}) \mathbf{A}^*] (z_0; \Xi_{\text{mf}}(z_0; \mathfrak{z}; s); s, \nu) \right\}. \end{aligned}$$

In Paragraph 4.3.1, we explore the effect on  $\mathbf{A}_1$  of the mean operator along  $(s, \nu) \in \mathbb{T}^2$ . Then, in Paragraph 4.3.2, we exhibit a spatial confinement property. In Paragraph 4.3.3, we also exhibit a conservation of the kinetic energy in the special case when the electric field is radial. Finally, in Paragraph 4.3.4, we present some computations explicitly as an appendix.

4.3.1. *Computation of the source term  $\langle \bar{\mathbf{A}}_1 \rangle$ .* Let us examine the averaging operation on  $\mathbf{A}_1$  along  $\nu \in \mathbb{T}$ . For simplicity and to be consistent with [11], we write  $\Xi_{\text{mf}} \equiv \Xi_0 = {}^t(\Xi_{0r}, \Xi_{0\theta}, \Xi_{0\nu}, \Xi_{0\zeta})$ .

**Lemma 4.8** (Finding  $\bar{\mathbf{A}}_1$ ). *We have*

$$(4.66) \quad \bar{\mathbf{A}}_1(z_0; \mathfrak{z}; s) = D_{\mathfrak{z}} \Xi_0(z_0; \mathfrak{z}; s)^{-1} \partial_t s(z_0; t(z_0, s))^{-1} \mathcal{M}(\Xi_0(z_0; \mathfrak{z}; s)),$$

where  $\mathcal{M} = {}^t(\mathcal{M}_r, \mathcal{M}_\theta, \mathcal{M}_\nu, \mathcal{M}_\zeta)$  is as follows

$$\begin{aligned} \mathcal{M}_r(r, \theta, \nu, \zeta) &:= -\frac{I(r)}{1 + I(r)^2} \left[ \frac{R(r, \theta)}{r} \partial_\theta \Phi(r, \theta) + \nu^2 (1 - \nu^2)^{-1/2} \sin \theta \left( \frac{\cos^2 \zeta}{2} + \sin^2 \zeta \right) \right], \\ \mathcal{M}_\theta(r, \theta, \nu, \zeta) &:= \frac{1}{r} \cos \omega b^{-1} E_r - \frac{1}{2rR} \nu^2 (1 - \nu^2)^{-1/2} b^{-1} \sin^2 \omega \cos \omega \cos^2 \zeta \cos \theta \\ &\quad - \frac{1}{2r} \nu^2 (1 - \nu^2)^{-1/2} \partial_r (b^{-1}) \cos \omega \cos^2 \zeta \\ &\quad + \nu^2 (1 - \nu^2)^{-1/2} b^{-1} \left[ -\frac{1}{rR} \cos^3 \omega \sin^2 \zeta \cos \theta - \frac{1}{r^2} \cos \omega \sin^2 \omega \sin^2 \zeta \right. \\ &\quad \left. + \frac{1}{2r} \omega' \sin \omega \cos^2 \zeta + \frac{1}{2r^2} (\cos \omega \cos^2 \zeta - \cos^3 \omega \cos^2 \zeta) \right], \end{aligned}$$

$$\begin{aligned}
\mathcal{M}_v(r, \theta, v, \zeta) &:= \frac{v}{2} (1 - v^2) \cos^2 \zeta \left[ \partial_r(b^{-1}) E_\perp - \partial_\theta(b^{-1}) E_r \frac{\cos \omega}{r} \right] \\
&+ \frac{1}{2} v (1 - v^2) b^{-1} \left[ \partial_r E_\perp \cos^2 \zeta - \partial_\theta E_r \cos^2 \zeta \frac{\cos \omega}{r} + \frac{2}{R} E_r \sin^2 \zeta \cos \omega \sin \theta \right. \\
&+ \left. E_\perp \left( \frac{2 \sin^2 \omega \sin^2 \zeta + \cos^2 \omega \cos^2 \zeta}{r} + \frac{2 \sin^2 \zeta \cos^2 \omega + \sin^2 \omega \cos^2 \zeta}{R} \cos \theta \right) \right], \\
\mathcal{M}_\zeta(r, \theta, v, \zeta) &:= \cos \zeta \sin \zeta \left\{ \frac{b^{-1}}{2} \left( \frac{\cos \omega}{r} \partial_\theta E_r - \partial_r E_\perp \right) \right. \\
&+ \frac{1}{2} \left( \frac{\cos \omega}{r} \partial_\theta(b^{-1}) E_r - \partial_r(b^{-1}) E_\perp \right) + \frac{1}{R} E_r b^{-1} \sin \theta \cos \omega \\
&+ \frac{v^2}{2} (1 - v^2)^{-1/2} \left[ \frac{1}{R} \sin \theta (\omega' \sin \omega b^{-1} - \partial_r(b^{-1}) \cos \omega) \right. \\
&\quad \left. - \frac{1}{r^2} \sin^2 \omega \cos \omega \partial_\theta(b^{-1}) - \frac{1}{rR} \cos^3 \omega \cos \theta \partial_\theta(b^{-1}) \right] \\
&+ \left. E_\perp b^{-1} \left( \frac{\sin^2 \omega}{r} + \cos \theta \frac{\cos^2 \omega}{R} - \frac{\cos^2 \omega}{2r} - \frac{\sin^2 \omega \cos \theta}{2R} \right) \right\}.
\end{aligned}$$

*Proof.* We are interested in finding  $\bar{A}_1(z_0, \mathfrak{z}; s)$ . For this reason, we do not have to take into account the contribution  $\partial_s(V_0^{-1} \partial_\nu^{-1} A_0^*)$  inside (4.65). Indeed, its mean with respect to  $\nu \in \mathbb{T}$  is just equal to zero since the term  $V_0$  is independent of  $\nu \in \mathbb{T}$ . Now, in view of (4.5) as well as since  $D_3 \Xi_{\text{mf}}^{-1}$  and  $\partial_t s^{-1}$  are independent of  $\nu \in \mathbb{T}$ , we can average (4.65) to get

$$\bar{A}_1 \equiv \begin{pmatrix} \bar{A}_{1r} \\ \bar{A}_{1\theta} \\ \bar{A}_{1v} \\ \bar{A}_{1\zeta} \end{pmatrix} (z_0; \mathfrak{z}; s) = D_3 \Xi_0(z_0; \mathfrak{z}; s)^{-1} \partial_t s(z_0; t(z_0, s))^{-1} \begin{pmatrix} \mathcal{M}_r \\ \mathcal{M}_\theta \\ \mathcal{M}_v \\ \mathcal{M}_\zeta \end{pmatrix} (\Xi_0(z_0; \mathfrak{z}; s)),$$

where we have introduced the terms  $\mathcal{M}_\star$  with  $\star \in \{r, \theta, v, \zeta\}$  as follows

$$(4.67) \quad \mathcal{M}_\star = \mathcal{V}_0^{-1} \overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z)} \mathcal{A}_\star - \mathcal{V}_0^{-1} \bar{\mathcal{V}}_1 \mathcal{A}_\star^* - \mathcal{V}_0^{-2} \overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z)} (\mathcal{V}_0) \mathcal{A}_\star^*.$$

The passage from the above line to the explicit expressions for the  $\mathcal{M}_\star$  enumerated in Lemma 4.8 is detailed in Paragraph 4.3.4.  $\square$

Let us now examine the effect of the mean operator along  $s \in \mathbb{T}$  to the function  $\bar{A}_1(z_0, \mathfrak{z}, s)$  through the following lemma.

**Lemma 4.9** (Finding  $\langle \bar{A}_1 \rangle$ ). *We have*

$$(4.68) \quad \langle \bar{A}_1 \rangle(z_0; \mathfrak{z}) = \frac{1}{2\pi} \int_0^{\mathbb{P}(z_0)} D_{z_0} Z(\mathfrak{z}; \tilde{t})^{-1} \mathcal{M}(Z(\mathfrak{z}; \tilde{t})) d\tilde{t},$$

where  $\mathcal{M}$  is identified in Lemma 4.8 and  $\mathbb{P}(z_0)$  is the period provided by (3.31).

*Proof.* In view of (4.9), we can see that

$$(4.69) \quad D_3 \Xi_0(z_0; \mathfrak{z}; s) = D_3 Z(\mathfrak{z}; t(z_0; s)).$$

With the aid of (4.9), (4.69) and Lemma 4.8, we get

$$(4.70) \quad \langle \bar{A}_1 \rangle(z_0; \mathfrak{z}) = \frac{1}{2\pi} \int_0^{2\pi} D_3 Z(\mathfrak{z}; t(z_0; s))^{-1} \partial_t s(z_0; t(z_0, s))^{-1} \mathcal{M}(Z(\mathfrak{z}; t(z_0; s))) ds.$$

Now, inside (4.70), we perform the change of variable  $\tilde{t} = t(z_0; s)$ , and since  $t(z_0; \cdot)$  is the inverse of  $s(z_0; \cdot)$ , we get

$$d\tilde{t} = \partial_s t(z_0; s) ds = (\partial_t s(z_0; t(z_0, s)))^{-1} ds.$$



From (3.15) and (3.28), it is obvious that  $s(z_0; 0) = 0$  and thus  $t(z_0; 0) = 0$ , we exploit then (4.2) with  $s = 0$  to see that  $t(z_0; 2\pi) = P(z_0)$ . We get at the end

$$\begin{aligned} \langle \bar{A}_1 \rangle(z_0; \mathfrak{z}) &= \frac{1}{2\pi} \int_0^{2\pi} D_{z_0} Z(\mathfrak{z}; \underbrace{t(z_0; s)}_{=\tilde{t}})^{-1} \mathcal{M}(Z(\mathfrak{z}; \underbrace{t(z_0; s)}_{=\tilde{t}})) \underbrace{(\partial_t s(z_0; t(z_0, s)))^{-1}}_{=d\tilde{t}} ds \\ &= \frac{1}{2\pi} \int_0^{P(z_0)} D_{z_0} Z(\mathfrak{z}; \tilde{t})^{-1} \mathcal{M}(Z(\mathfrak{z}; \tilde{t})) d\tilde{t}. \end{aligned}$$

□

**4.3.2. Confinement properties.** Of particular interest is the radius  $\tilde{r}$  which appears as the first component of  $\tilde{z} = {}^t(\tilde{r}, \tilde{\theta}, \tilde{v}, \tilde{\zeta})$ . It owns an asymptotic expansion as indicated in (4.19) with the main term  $\tilde{Z}_{0r}$  as in (4.30). In other words, we have

$$(4.71) \quad \tilde{r}(x_0, v_0; \tau) = \langle \bar{\mathfrak{Z}}_{0r} \rangle(z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau)) + \mathcal{O}(\varepsilon),$$

where  $\langle \bar{\mathfrak{Z}}_{0r} \rangle$  solves

$$(4.72) \quad \partial_\tau \langle \bar{\mathfrak{Z}}_{0r} \rangle = \langle \bar{A}_{1r} \rangle(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau)), \quad \langle \bar{\mathfrak{Z}}_{0r} \rangle(z_0; 0) = r_0.$$

It is interesting to see that we have a stability property related to  $\langle \bar{\mathfrak{Z}}_{0r} \rangle$ . This is illustrated in the two following lemmas.

**Lemma 4.10** (Conservative form). *We have*

$$(4.73) \quad \mathcal{M}_r(Z(\mathfrak{z}; t)) = \frac{d}{dt} \left[ F_r(Z(\mathfrak{z}; t)) \right],$$

where  $Z(\mathfrak{z}; t) = {}^t(\mathfrak{z}_r, Z_\theta, Z_v, Z_\zeta)(\mathfrak{z}, t)$  is the average flow solving the system (3.2) with initial data  $\mathfrak{z}$  (instead of  $z_0$ ) and where we have introduced  $F_r(\cdot)$  as follows

$$F_r(r, \theta, v, \zeta) := \frac{I(r)}{1 + I(r)^2} \frac{1}{\sin \omega(r)} v (1 - v^2)^{-1/2} R(r, \theta) \sin \zeta.$$

*Proof.* Note first that since  $Z_r(\mathfrak{z}; t) \equiv \mathfrak{z}_r$ , we have

$$(4.74) \quad \frac{d}{dt} \left[ F_r(Z(\mathfrak{z}; t)) \right] = \partial_\theta F_r(Z(\mathfrak{z}; t)) \frac{dZ_\theta}{dt} + \partial_v F_r(Z(\mathfrak{z}; t)) \frac{dZ_v}{dt} + \partial_\zeta F_r(Z(\mathfrak{z}; t)) \frac{dZ_\zeta}{dt}.$$

The partial derivatives of  $F_r$  are given by

$$\begin{aligned} \partial_\theta F_r(r, \theta, v, \zeta) &= -\frac{I(r)}{1 + I(r)^2} \frac{r}{\sin \omega(r)} v (1 - v^2)^{-1/2} \sin \theta \sin \zeta, \\ \partial_v F_r(r, \theta, v, \zeta) &= +\frac{I(r)}{1 + I(r)^2} \frac{1}{\sin \omega(r)} (1 - v^2)^{-3/2} R(r, \theta) \sin \zeta, \\ \partial_\zeta F_r(r, \theta, v, \zeta) &= +\frac{I(r)}{1 + I(r)^2} \frac{1}{\sin \omega(r)} v (1 - v^2)^{-1/2} R(r, \theta) \cos \zeta. \end{aligned} \quad (4.75)$$

Exploit (4.75) and (3.2) inside (4.74) to obtain that

$$\begin{aligned} \frac{d}{dt} \left[ F_r(Z(\mathfrak{z}; t)) \right] &= -\frac{I(\mathfrak{z}_r)}{1 + I(\mathfrak{z}_r)^2} \left[ Z_v^2 (1 - Z_v^2)^{-1/2} \sin Z_\theta (\sin Z_\zeta)^2 \right. \\ &\quad + \frac{1}{\sin \omega(\mathfrak{z}_r)} R(\mathfrak{z}_r, Z_\theta) E_\parallel(\mathfrak{z}_r, Z_\theta) (\sin Z_\zeta)^2 \\ &\quad + \frac{Z_v^2}{2} (1 - Z_v^2)^{-1/2} \sin Z_\theta (\cos Z_\zeta)^2 \\ &\quad \left. + \frac{1}{\sin \omega(\mathfrak{z}_r)} R(\mathfrak{z}_r, Z_\theta) E_\parallel(\mathfrak{z}_r, Z_\theta) (\cos Z_\zeta)^2 \right]. \end{aligned} \quad (4.76)$$

From (2.7), we have  $E_\parallel(\mathfrak{z}_r, Z_\theta) = 1/\mathfrak{z}_r \sin \omega(\mathfrak{z}_r) \partial_\theta \Phi(\mathfrak{z}_r, Z_\theta)$ . Insert the latter into (4.76) to display the term  $\mathcal{M}_r$  of Lemma 4.8. □

**Lemma 4.11** (Cancellation property). *We have  $\langle \bar{A}_{1r} \rangle(z_0; \mathfrak{z}) = 0$  implying that  $\langle \bar{\mathfrak{Z}}_{0r} \rangle(z_0; \tau) = r_0$  for all  $\tau$ .*

*Proof.* Note that since  $Z(\mathfrak{z}; t) = \mathfrak{z}_r$ , the first row of the matrix  $(D_{z_0} Z(\mathfrak{z}; t))^{-1}$  is  $(1 \ 0 \ 0 \ 0)$ . Exploit (4.73) inside (4.68). This gives

$$\langle \bar{A}_{1r} \rangle(z_0; \mathfrak{z}) = \frac{1}{2\pi} \int_0^{P(z_0)} \mathcal{M}_r(Z(\mathfrak{z}; \tilde{t})) \, d\tilde{t} = \frac{1}{2\pi} \int_0^{P(z_0)} \frac{d}{d\tilde{t}} \left[ F_r(Z(\mathfrak{z}; \tilde{t})) \right] d\tilde{t},$$

which must be zero since the average flow  $Z(\mathfrak{z}; \cdot)$  is periodic of period  $P(z_0)$  by Lemma 3.7.  $\square$

From (4.72), it follows that  $\langle \bar{\mathfrak{Z}}_{0r} \rangle(z_0; \tau) = r_0$  for all  $\tau$ . Coming back to (4.59) together with (3.1), and then using (2.4), we can see that

$$|\tilde{x}(x_0, v_0; \tau) - R_0| = r_0 + \mathcal{O}(\varepsilon), \quad \forall \tau \in [0, \tilde{T}(x_0, v_0)].$$

This implies that the portion of the plasma that is issued from  $z_0 \in \mathcal{U}^{-1}(\mathcal{D})$  remains trapped during long times inside a toroidal chamber (in particular, it does not collide with the walls). Thus, for such positions  $z_0$ , the conclusion is that the introduction of an electric field satisfying Assumption 1.7 is not a factor of deconfinement.

Still, looking at (4.59) together with (4.57)-(4.58), the leading long time behavior of  $\tilde{x}$  and  $\tilde{v}$  involves large amplitude oscillations at frequencies  $\varepsilon^{-1}$  and  $\varepsilon^{-2}$ . Let us now look more closely at what happens concerning  $\tilde{v}$ .

**4.3.3. The case of a radial electric field.** We suppose in this paragraph that the electric field is radial. In other words, we work with  $\Phi \equiv \Phi(r)$  in Assumption 1.7. This implies that the two components of the electric field:  $E_\perp$  in (2.6) and  $E_\parallel$  in (2.7) are such that  $E_\parallel = E_\perp = 0$ . In this case, the system for the average flow (3.2) with initial data  $\mathfrak{z}$  instead of  $z_0$  is reduced to

$$(4.77) \quad Z_r(\mathfrak{z}; t) = \mathfrak{z}_r,$$

$$(4.78) \quad dZ_\theta/dt = +\mathfrak{z}_v \mathfrak{z}_r^{-1} \sin \omega(\mathfrak{z}_r) \sin Z_\zeta, \quad Z_\theta(\mathfrak{z}; 0) = \mathfrak{z}_\theta,$$

$$(4.79) \quad Z_v(\mathfrak{z}; t) = \mathfrak{z}_v,$$

$$(4.80) \quad dZ_\zeta/dt = -\mathfrak{z}_v (2R(\mathfrak{z}_r, Z_\theta))^{-1} \sin \omega(\mathfrak{z}_r) \sin Z_\theta \cos Z_\zeta, \quad Z_\zeta(\mathfrak{z}; 0) = \mathfrak{z}_\zeta.$$

On the other hand, the expression of the term  $\mathcal{M}_v$  in Lemma 4.8 is also simplified to

$$(4.81) \quad \mathcal{M}_v(r, \theta, v, \zeta) = v (1 - v^2) \frac{I(r)}{1 + I(r)^2} \partial_r \Phi(r) \sin \theta \left( \frac{\cos^2 \zeta}{2} + \sin^2 \zeta \right).$$

Then, in view of (4.19), (4.29) and (4.79), we can assert that the momentum  $\tilde{v}$  which appears as the third component of  $\tilde{\mathfrak{z}} = {}^t(\tilde{r}, \tilde{\theta}, \tilde{v}, \tilde{\zeta})$  owns the following expansion

$$(4.82) \quad \tilde{v}(x_0, v_0; \tau) = \langle \bar{\mathfrak{Z}}_{0v} \rangle(z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau)) + \mathcal{O}(\varepsilon),$$

where  $\langle \bar{\mathfrak{Z}}_{0v} \rangle$  solves

$$(4.83) \quad \partial_\tau \langle \bar{\mathfrak{Z}}_{0v} \rangle = \langle \bar{A}_{1v} \rangle(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau)), \quad \langle \bar{\mathfrak{Z}}_{0v} \rangle(z_0; 0) = v_0.$$

Since the average flow  $Z(\mathfrak{z}; t) = {}^t(\mathfrak{z}_r, Z_\theta(\mathfrak{z}; t), \mathfrak{z}_v, Z_\zeta(\mathfrak{z}; t))$  is as in (4.77), (4.78), (4.79) and (4.80), we get from (4.68) that

$$(4.84) \quad \langle \bar{A}_{1v} \rangle(z_0; \mathfrak{z}) = \frac{1}{2\pi} \int_0^{P(z_0)} \mathcal{M}_v(Z(\mathfrak{z}; \tilde{t})) \, d\tilde{t}.$$

Below, we show that we have a stability property related to  $\langle \bar{\mathfrak{Z}}_{0v} \rangle$ .

**Lemma 4.12** (Conservative form for the momentum term). *For all  $\mathfrak{z}$  in a compact set, we have that  $\mathcal{M}_v$  is conservative in the sense*

$$(4.85) \quad \mathcal{M}_v(Z(\mathfrak{z}; t)) = \frac{d}{dt} \left[ F_v(Z(\mathfrak{z}; t)) \right],$$

where  $Z(\mathfrak{z}; t) = {}^t(\mathfrak{z}_r, Z_\theta, \mathfrak{z}_v, Z_\zeta)(\mathfrak{z}, t)$  is the average flow as in (4.77), (4.78), (4.79) and (4.80) and where we have introduced  $F_v(\cdot)$  as follows

$$F_v(r, \theta, v, \zeta) := -(1 - v^2) \frac{I(r)}{1 + I(r)^2} \frac{1}{\sin \omega(r)} \partial_r \Phi(r) R(r, \theta) \sin \zeta.$$

*Proof.* Note first that since  $Z_r(\mathfrak{z}; t) \equiv \mathfrak{z}_r$  and  $Z_v(\mathfrak{z}; t) \equiv \mathfrak{z}_v$ , we have

$$(4.86) \quad \frac{d}{dt} [F_v(Z(\mathfrak{z}; t))] = \partial_\theta F_v(Z(\mathfrak{z}; t)) \frac{dZ_\theta}{dt} + \partial_\zeta F_v(Z(\mathfrak{z}; t)) \frac{dZ_\zeta}{dt}.$$

With  $R(\cdot)$  as in (1.12), the partial derivatives of  $F_v$  are computed as follows

$$(4.87) \quad \begin{aligned} \partial_\theta F_v(r, \theta, v, \zeta) &= \frac{rI(r)}{1 + I(r)^2} \frac{1}{\sin \omega(r)} \partial_r \Phi(r) \sin \theta (1 - v^2) \sin \zeta, \\ \partial_\zeta F_v(r, \theta, v, \zeta) &= -\frac{I(r)}{1 + I(r)^2} \frac{1}{\sin \omega(r)} \partial_r \Phi(r) R(r, \theta) (1 - v^2) \cos \zeta. \end{aligned}$$

Exploit (4.87), and then substitute  $dZ_\theta/dt$ ,  $dZ_\zeta/dt$  as indicated in (4.78) and (4.80) inside (4.86) to find that

$$\begin{aligned} \frac{d}{dt} [F_v(Z(\mathfrak{z}; t))] &= \frac{I(\mathfrak{z}_r)}{1 + I(\mathfrak{z}_r)^2} \frac{\mathfrak{z}_r}{\sin \omega} \partial_r \Phi(\mathfrak{z}_r) \sin Z_\theta (1 - \mathfrak{z}_v^2) \sin Z_\zeta \frac{\mathfrak{z}_v}{\mathfrak{z}_r} \sin \omega \sin Z_\zeta \\ &+ \frac{I(\mathfrak{z}_r)}{1 + I(\mathfrak{z}_r)^2} \frac{1}{\sin \omega} \partial_r \Phi(\mathfrak{z}_r) R(\mathfrak{z}_r, Z_\theta) (1 - \mathfrak{z}_v^2) \cos Z_\zeta \mathfrak{z}_v (2R(\mathfrak{z}_r, Z_\theta))^{-1} \sin \omega \sin Z_\theta \cos Z_\zeta \\ &= \mathcal{M}_v(Z(\mathfrak{z}; t)). \end{aligned}$$

□

Now, we exploit (4.85) inside (4.84), we find that  $\langle \bar{A}_{1v} \rangle(z_0; \mathfrak{z}) = 0$  since the average flow  $Z(\mathfrak{z}; \cdot)$  is periodic of period  $P(z_0)$  by Lemma 3.7. This implies that  $\langle \bar{\mathfrak{J}}_{0v} \rangle(z_0; \tau) = v_0$  for all  $\tau$ . And thus, the equation (4.82) becomes

$$(4.88) \quad \tilde{v}(x_0, v_0; \tau) = v_0(x_0, v_0) + \mathcal{O}(\varepsilon).$$

This means that the kinetic energy is, at least at the leading order, a conserved quantity.

4.3.4. *Appendix.* To proceed, we need to consider the Fourier series expansions with respect to the variable  $\nu \in \mathbb{T}$  of the functions  $\mathcal{A}(z, \cdot)$  and  $\mathcal{V}_j(z, \cdot)$  for  $j \in \{0, 1\}$ , which can be written as

$$\begin{aligned} \mathcal{A}_\star(z, \nu) &= \bar{\mathcal{A}}_\star(z) + \sum_{n=1}^2 \frac{1}{n} \mathcal{C} \mathcal{A}_{\star n}(z) \cos(n\nu) + \sum_{n=1}^2 \frac{1}{n} \mathcal{S} \mathcal{A}_{\star n}(z) \sin(n\nu), \quad \star \in \{r, \theta, v, \zeta\}, \\ \mathcal{V}_j(z, \nu) &= \bar{\mathcal{V}}_j(z) + \sum_{n=1}^2 \frac{1}{n} \mathcal{C} \mathcal{V}_{jn}(z) \cos(n\nu) + \sum_{n=1}^2 \frac{1}{n} \mathcal{S} \mathcal{V}_{jn}(z) \sin(n\nu), \quad j \in \{0, 1\}. \end{aligned}$$

From (2.9), we know that

$$\begin{aligned} \mathcal{C} \mathcal{A}_{r1} &= v \cos \zeta, & \mathcal{S} \mathcal{A}_{r1} &= 0, & \mathcal{C} \mathcal{A}_{rn} &= \mathcal{S} \mathcal{A}_{rn} = 0, & \forall n \geq 2, \\ \mathcal{C} \mathcal{A}_{\theta 1} &= 0, & \mathcal{S} \mathcal{A}_{\theta 1} &= \frac{v}{r} \cos \omega \cos \zeta, & \mathcal{C} \mathcal{A}_{\theta n} &= \mathcal{S} \mathcal{A}_{\theta n} = 0, & \forall n \geq 2. \end{aligned}$$

From (2.12), we can deduce in particular that

$$\begin{aligned}
\mathcal{CA}_{v1} &= -(1-v^2)^{3/2} \cos \zeta E_r, & \mathcal{SA}_{v1} &= -(1-v^2)^{3/2} \cos \zeta E_\perp, \\
\mathcal{SA}_{\zeta 1} &= v^{-1} (1-v^2)^{1/2} \sin \zeta E_\perp + \frac{v}{R} \sin \theta \cos \omega \sin \zeta, \\
\mathcal{CA}_{\zeta 1} &= v^{-1} (1-v^2)^{1/2} \sin \zeta E_r - \frac{v}{r} \sin^2 \omega \sin \zeta - \frac{v}{R} \cos \theta \cos^2 \omega \sin \zeta, \\
\mathcal{CA}_{\zeta 2} &= \frac{v}{R} \sin \theta \sin \omega \cos \zeta, \\
\mathcal{SA}_{\zeta 2} &= v \omega' \cos \zeta - \frac{v}{r} \sin \omega \cos \omega \cos \zeta + \frac{v}{R} \cos \theta \cos \omega \sin \omega \cos \zeta,
\end{aligned}$$

as well as  $\mathcal{CA}_{vn} = \mathcal{SA}_{vn} = \mathcal{CA}_{\zeta n} = \mathcal{SA}_{\zeta n} = 0$  for all  $n \geq 2$ . On the other hand, from (2.15), it is obvious that  $\mathcal{CV}_{0n} = 0$  and  $\mathcal{SV}_{0n} = 0$  for all  $n \in \mathbb{N}$ . Looking at (2.16), we can assert that

$$\begin{aligned}
\mathcal{CV}_{11} &= -\frac{v}{R} \sin^2 \zeta \cos \omega \cos^{-1} \zeta \sin \theta - v^{-1} \cos^{-1} \zeta (1-v^2)^{1/2} E_\perp, \\
\mathcal{SV}_{11} &= -\frac{v}{r} \cos^2 \omega \cos \zeta - \frac{v}{r} \sin^2 \omega \sin^2 \zeta \cos^{-1} \zeta - \frac{v}{R} \sin^2 \zeta \cos^{-1} \zeta \cos^2 \omega \cos \theta \\
&\quad - \frac{v}{R} \sin^2 \omega \cos \zeta \cos \theta + v^{-1} \cos^{-1} \zeta (1-v^2)^{1/2} E_r, \\
\mathcal{CV}_{12} &= -v \omega' \sin \zeta + \frac{v}{r} \cos \omega \sin \omega \sin \zeta - \frac{v}{R} \sin \zeta \sin \omega \cos \omega \cos \theta, \\
\mathcal{SV}_{12} &= \frac{v}{R} \sin \zeta \sin \omega \sin \theta.
\end{aligned}$$

We now come back to (4.67) in order to compute the different terms occurring in  $\mathcal{S}_*$ . We have

$$\begin{aligned}
(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) &= (\partial_\nu^{-1} \mathcal{A}_r^* \partial_r) + (\partial_\nu^{-1} \mathcal{A}_\theta^* \partial_\theta) + (\partial_\nu^{-1} \mathcal{A}_v^* \partial_v) + (\partial_\nu^{-1} \mathcal{A}_\zeta^* \partial_\zeta) \\
&= \mathcal{CA}_{r1} \sin \nu \partial_r - \mathcal{SA}_{\theta 1} \cos \nu \partial_\theta + [\mathcal{CA}_{v1} \sin \nu - \mathcal{SA}_{v1} \cos \nu] \partial_v \\
&\quad + \left[ \mathcal{CA}_{\zeta 1} \sin \nu - \mathcal{SA}_{\zeta 1} \cos \nu + \frac{1}{4} \mathcal{CA}_{\zeta 2} \sin 2\nu - \frac{1}{4} \mathcal{SA}_{\zeta 2} \cos 2\nu \right] \partial_\zeta.
\end{aligned}$$

It follows that

$$\begin{aligned}
\overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) \mathcal{A}_r} &= -\frac{1}{2} [\mathcal{SA}_{v1} \partial_v (\mathcal{CA}_{r1}) + \mathcal{SA}_{\zeta 1} \partial_\zeta (\mathcal{CA}_{r1})] \\
&= \frac{(1-v^2)^{3/2}}{2} \cos^2 \zeta E_\perp + \frac{v^2}{2R} \sin^2 \zeta \cos \omega \sin \theta + \frac{(1-v^2)^{1/2}}{2} \sin^2 \zeta E_\perp. \\
\overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) \mathcal{A}_\theta} &= \frac{1}{2} [\mathcal{CA}_{r1} \partial_r (\mathcal{SA}_{\theta 1}) + \mathcal{CA}_{v1} \partial_v (\mathcal{SA}_{\theta 1}) + \mathcal{CA}_{\zeta 1} \partial_\zeta (\mathcal{SA}_{\theta 1})] \\
&= \frac{1}{2} v^2 \cos^2 \zeta \frac{\partial}{\partial r} \left( \frac{\cos \omega}{r} \right) - \frac{(1-v^2)^{3/2}}{2r} \cos^2 \zeta \cos \omega E_r \\
&\quad + \frac{v^2}{2r} \sin^2 \zeta \cos \omega \left( \frac{\sin^2 \omega}{r} + \frac{\cos \theta \cos^2 \omega}{R} \right) - \frac{(1-v^2)^{1/2}}{2r} \sin^2 \zeta \cos \omega E_r. \\
\overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) \mathcal{A}_v} &= \frac{1}{2} \left[ -\mathcal{SA}_{\theta 1} \partial_\theta (\mathcal{CA}_{v1}) - \mathcal{SA}_{v1} \partial_v (\mathcal{CA}_{v1}) - \mathcal{SA}_{\zeta 1} \partial_\zeta (\mathcal{CA}_{v1}) \right. \\
&\quad \left. + \mathcal{CA}_{r1} \partial_r (\mathcal{SA}_{v1}) + \mathcal{CA}_{v1} \partial_v (\mathcal{SA}_{v1}) + \mathcal{CA}_{\zeta 1} \partial_\zeta (\mathcal{SA}_{v1}) \right] \\
&= -\frac{v}{2} (1-v^2)^{3/2} \cos^2 \zeta \partial_r E_\perp + \frac{v}{2r} (1-v^2)^{3/2} \cos^2 \zeta \cos \omega \partial_\theta E_r \\
&\quad - \frac{v}{2R} (1-v^2)^{3/2} \sin^2 \zeta \cos \omega \sin \theta E_r - \frac{v}{2} (1-v^2)^{3/2} \sin^2 \zeta E_\perp \left( \frac{\sin^2 \omega}{r} + \frac{\cos \theta \cos^2 \omega}{R} \right).
\end{aligned}$$

$$\begin{aligned}
\overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) \mathcal{A}_\zeta} &= \frac{1}{2} \left[ -\mathcal{S} \mathcal{A}_{\theta 1} \partial_\theta (\mathcal{C} \mathcal{A}_{\zeta 1}) - \mathcal{S} \mathcal{A}_{v 1} \partial_v (\mathcal{C} \mathcal{A}_{\zeta 1}) - \mathcal{S} \mathcal{A}_{\zeta 1} \partial_\zeta (\mathcal{C} \mathcal{A}_{\zeta 1}) \right. \\
&\quad \left. + \mathcal{C} \mathcal{A}_{r 1} \partial_r (\mathcal{S} \mathcal{A}_{\zeta 1}) + \mathcal{C} \mathcal{A}_{v 1} \partial_v (\mathcal{S} \mathcal{A}_{\zeta 1}) + \mathcal{C} \mathcal{A}_{\zeta 1} \partial_\zeta (\mathcal{S} \mathcal{A}_{\zeta 1}) \right] \\
&\quad + \frac{1}{16} \left[ \mathcal{C} \mathcal{A}_{\zeta 2} \partial_\zeta (\mathcal{S} \mathcal{A}_{\zeta 2}) - \mathcal{S} \mathcal{A}_{\zeta 2} \partial_\zeta (\mathcal{C} \mathcal{A}_{\zeta 2}) \right] \\
&= \frac{v^2}{2R^2} \cos \zeta \sin \zeta \sin \theta \cos \theta \cos \omega (\cos^2 \omega - 1) + \frac{(1-v^2)^{1/2}}{2} \sin \zeta \cos \zeta \left( \partial_r E_\perp - \frac{\cos \omega}{r} \partial_\theta E_r \right) \\
&\quad - \frac{v^2}{2R} \omega' \cos \zeta \sin \zeta \sin \theta \sin \omega - \frac{v^2}{2rR} \cos \zeta \sin \zeta \sin \theta \cos^3 \omega \\
&\quad - \frac{(1-v^2)^{3/2}}{2} \sin \zeta \cos \zeta \left( \frac{\sin^2 \omega}{r} E_\perp + \sin \theta \frac{\cos \omega}{R} E_r + \cos \theta \frac{\cos^2 \omega}{R} E_\perp \right).
\end{aligned}$$

We find also

$$\overline{\mathcal{V}_1 \mathcal{A}_r^*} = \frac{1}{2} \mathcal{C} \mathcal{A}_{r 1} \mathcal{C} \mathcal{V}_{11} = -\frac{v^2}{2R} \sin^2 \zeta \cos \omega \sin \theta - \frac{(1-v^2)^{1/2}}{2} E_\perp,$$

as well as

$$\begin{aligned}
\overline{\mathcal{V}_1 \mathcal{A}_\theta^*} &= \frac{1}{2} \mathcal{S} \mathcal{A}_{\theta 1} \mathcal{S} \mathcal{V}_{11} = -\frac{v^2}{2r^2} \cos^3 \omega \cos^2 \zeta - \frac{v^2}{2r^2} \cos \omega \sin^2 \omega \sin^2 \zeta \\
&\quad + \frac{(1-v^2)^{1/2}}{2r} \cos \omega E_r - \frac{v^2}{2rR} \sin^2 \zeta \cos^3 \omega \cos \theta - \frac{v^2}{2rR} \sin^2 \omega \cos \omega \cos^2 \zeta \cos \theta.
\end{aligned}$$

$$\begin{aligned}
\overline{\mathcal{V}_1 \mathcal{A}_v^*} &= \frac{1}{2} [\mathcal{C} \mathcal{A}_{v 1} \mathcal{C} \mathcal{V}_{11} + \mathcal{S} \mathcal{V}_{11} \mathcal{S} \mathcal{A}_{v 1}] = \frac{v}{2r} (1-v^2)^{3/2} E_\perp \left[ \cos^2 \omega \cos^2 \zeta + \sin^2 \omega \sin^2 \zeta \right] \\
&\quad + \frac{v(1-v^2)^{3/2}}{2R} E_r \sin^2 \zeta \cos \omega \sin \theta + \frac{v}{2R} (1-v^2)^{3/2} E_\perp \cos \theta \left[ \sin^2 \zeta \cos^2 \omega + \sin^2 \omega \cos^2 \zeta \right].
\end{aligned}$$

$$\begin{aligned}
\overline{\mathcal{V}_1 \mathcal{A}_\zeta^*} &= \frac{1}{2} [\mathcal{C} \mathcal{A}_{\zeta 1} \mathcal{C} \mathcal{V}_{11} + \mathcal{S} \mathcal{A}_{\zeta 1} \mathcal{S} \mathcal{V}_{11}] + \frac{1}{8} [\mathcal{C} \mathcal{A}_{\zeta 2} \mathcal{C} \mathcal{V}_{12} + \mathcal{S} \mathcal{A}_{\zeta 1} \mathcal{S} \mathcal{V}_{12}] \\
&= -\frac{v^2}{2rR} \sin \theta \cos^3 \omega \sin \zeta \cos \zeta - \frac{v^2}{2R^2} \sin \zeta \cos \zeta \sin \theta \cos \theta \sin^2 \omega \cos \omega \\
&\quad + \frac{(1-v^2)^{1/2}}{2R} \sin \theta \cos \omega \sin \zeta \cos \zeta E_r \\
&\quad + \frac{(1-v^2)^{1/2}}{2} \sin \zeta \cos \zeta E_\perp (\sin^2 \omega - \cos^2 \omega) \left( \frac{1}{r} - \frac{\cos \theta}{R} \right).
\end{aligned}$$

There remains to compute

$$\begin{aligned}
\overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) (\mathcal{V}_0) \mathcal{A}_r^*} &= -\frac{1}{2} \mathcal{C} \mathcal{A}_{r 1} [\mathcal{S} \mathcal{A}_{\theta 1} \partial_\theta \mathcal{V}_0 + \mathcal{S} \mathcal{A}_{v 1} \partial_v \mathcal{V}_0] \\
&= \frac{v^2(1-v^2)^{1/2}}{2r} \partial_\theta b \cos \omega \cos^2 \zeta + \frac{v^2(1-v^2)}{2} b E_\perp \cos^2 \zeta,
\end{aligned}$$

$$\begin{aligned}
\overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) (\mathcal{V}_0) \mathcal{A}_\theta^*} &= \frac{1}{2} \mathcal{S} \mathcal{A}_{\theta 1} [\mathcal{C} \mathcal{A}_{r 1} \partial_r \mathcal{V}_0 + \mathcal{C} \mathcal{A}_{v 1} \partial_v \mathcal{V}_0] \\
&= \frac{v}{2r} \cos \omega \cos \zeta \left[ -v(1-v^2)^{1/2} \partial_r b \cos \zeta - v(1-v^2) b \cos \zeta E_r \right], \\
\overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) (\mathcal{V}_0) \mathcal{A}_v^*} &= \frac{1}{2} [\mathcal{C} \mathcal{A}_{r 1} \partial_r \mathcal{V}_0 \mathcal{S} \mathcal{A}_{v 1} - \mathcal{S} \mathcal{A}_{\theta 1} \partial_\theta \mathcal{V}_0 \mathcal{C} \mathcal{A}_{v 1}] \\
&= \frac{v}{2} (1-v^2)^2 \cos^2 \zeta \left( \partial_r b E_\perp - \frac{\cos \omega}{r} \partial_\theta b E_r \right),
\end{aligned}$$

as well as

$$\begin{aligned}
& \overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) (\mathcal{V}_0) \mathcal{A}_\zeta^*} = \frac{1}{2} \left[ \mathcal{C} \mathcal{A}_{r1} \mathcal{S} \mathcal{A}_{\zeta 1} \partial_r \mathcal{V}_0 - \mathcal{S} \mathcal{A}_{\theta 1} \mathcal{C} \mathcal{A}_{\zeta 1} \partial_\theta \mathcal{V}_0 \right. \\
& \quad \left. + (\mathcal{C} \mathcal{A}_{v1} \mathcal{S} \mathcal{A}_{\zeta 1} - \mathcal{S} \mathcal{A}_{v1} \mathcal{C} \mathcal{A}_{\zeta 1}) \partial_v \mathcal{V}_0 \right] \\
& = -\frac{v^2}{2r} (1-v^2)^{1/2} \partial_\theta b \cos \omega \cos \zeta \sin \zeta \left( \frac{\sin^2 \omega}{r} + \frac{\cos \theta \cos^2 \omega}{R} \right) \\
& \quad - \frac{1}{2} (1-v^2) E_\perp \partial_r b \sin \zeta \cos \zeta + \frac{1}{2r} (1-v^2) \partial_\theta b \cos \omega \cos \zeta \sin \zeta E_r \\
& \quad - \frac{v^2}{2} (1-v^2) \cos \zeta \sin \zeta E_\perp b \left( \frac{\sin^2 \omega}{r} + \frac{\cos \theta \cos^2 \omega}{R} \right) \\
& \quad - \frac{v^2}{2R} (1-v^2)^{1/2} \partial_r b \cos \zeta \cos \omega \sin \theta \sin \zeta - \frac{v^2}{2R} (1-v^2) \sin \theta \cos \omega \sin \zeta \cos \zeta b E_r.
\end{aligned}$$

We can compile the above results at the level of (4.67) to recover the content of the  $\mathcal{M}_\star$  inside Lemma 4.8.

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