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# NULL-CONTROLLABILITY FOR THE WEAKLY DISSIPATIVE FRACTIONAL HEAT EQUATIONS

PAUL ALPHONSE AND ARMAND KOENIG

ABSTRACT. We prove that the fractional heat equations posed on the whole Euclidean space  $\mathbb{R}^n$  and associated with the operators  $(-\Delta)^{s/2}$  are exactly null-controllable from control supports which are sufficiently “exponentially thick”, when  $0 < s \leq 1$ . This is a first positive null-controllability result in this weak dissipation regime. When  $0 < s < 1$ , we also give a necessary condition of the same form. This notion of exponential thickness is stronger than the geometric notion of thickness known to be a necessary and sufficient condition to obtain positive null-controllability results in the strong dissipation regime  $s > 1$ . Inspired by the construction of the Smith-Volterra-Cantor sets, we also provide examples of non-trivial exponentially thick control supports.

## 1. INTRODUCTION

This paper is part of the study of the null-controllability of the fractional heat equations

$$(E_s) \quad \begin{cases} (\partial_t + (-\Delta)^{s/2})f(t, x) = \mathbb{1}_\omega u(t, x), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ f(0, \cdot) = f_0 \in L^2(\mathbb{R}^n), \end{cases}$$

where  $s > 0$  is a positive real number and  $\omega \subset \mathbb{R}^n$  is a measurable set with positive measure whose geometry is to be understood.

Although the null-controllability properties of parabolic equations posed on bounded domains of  $\mathbb{R}^n$  are known for years [11], the same study for parabolic equations considered on the whole Euclidean space  $\mathbb{R}^n$ , as the equations  $(E_s)$ , is quite recent. It follows from the works [1, 2, 3, 5, 6, 7, 13, 16, 17] that the null-controllability properties of such models, and also their approximate null-controllability or the stabilization properties, are associated with the geometric notion of thickness, defined as follows

**Definition 1.** Given  $\gamma \in (0, 1)$  and  $L > 0$ , the set  $\omega \subset \mathbb{R}^n$  is said to be  $\gamma$ -*thick* at scale  $L$  when it is measurable and satisfies

$$\forall x \in \mathbb{R}^n, \quad \text{Leb}(\omega \cap B(x, L)) \geq \gamma \text{Leb}(B(x, L)),$$

where  $\text{Leb}$  denotes the Lebesgue measure in  $\mathbb{R}^n$ .

It was in particular proven in the works [1, 6, 17] that in the strong dissipation regime  $s > 1$ , the thickness is a necessary and sufficient geometric condition that allows to obtain positive null-controllability results for the equation  $(E_s)$ . The present paper addresses the question of the null-controllability of the equation  $(E_s)$  in the low-dissipation regime  $0 < s \leq 1$ , for which very few results have been obtained so far. It is for example stated in the works [9, 12] that in dimension  $n = 1$ , the equation  $(E_s)$  is not null-controllable from a control support  $\omega$  which is a strict open subset of  $\mathbb{R}$ . We refine this result by proving that the null-controllability properties of the equation  $(E_s)$  is associated

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with the following geometric condition for the control support  $\omega$ , which we will call “exponential thickness”

**Definition 2.** Given  $\alpha > 0$ , the set  $\omega \subset \mathbb{R}^n$  is said to be  $\alpha$ -*exponentially thick* when it is measurable and there exist  $C > 0$ ,  $c > 0$  and  $L_0 > 0$  such that for every  $0 < L < L_0$  and  $x \in \mathbb{R}^n$ ,

$$\text{Leb}(\omega \cap B(x, L)) \geq ce^{-CL^{-\alpha}} \text{Leb}(B(x, L)).$$

This definition can be rephrased as “ $\omega$  is  $ce^{-CL^{-\alpha}}$ -thick at every scale  $0 < L < L_0$ ”. The notion of exponential thickness is therefore far stronger than the notion of thickness. Our first result states that in the weak dissipation regime  $0 < s < 1$ , the fractional heat equation  $(E_s)$  is not null-controllable from a control support  $\omega \subset \mathbb{R}^n$  which is not  $2s/(1-s)$ -exponentially thick. We also prove that in the case  $0 < s \leq 1$ , the equation  $(E_s)$  is exactly null-controllable from any  $\alpha$ -exponentially thick set  $\omega \subset \mathbb{R}^n$ , with  $0 < \alpha < s$ , and in any positive time  $T > 0$ .

An example of  $\alpha$ -exponential thick is of course the whole space  $\mathbb{R}^n$ , but it might be difficult to visualize non trivial examples of sets satisfying this property. In dimension  $n = 1$ , inspired by the construction of the Cantor-Smith-Volterra sets, we give examples of subsets of  $\mathbb{R}$  which are exactly  $\alpha$ -exponentially thick, in the sense that they are not  $\alpha'$ -exponentially thick with  $\alpha' > \alpha$ .

*Outline of the work.* In Section 2, we present in details the main results contained in this work. Section 3 is then devoted to prove that the fractional heat equations are null-controllable from exponentially thick control supports, the necessity of this geometric condition being investigated in Section 4. In Section 5, we construct examples of exponentially thick sets.

## 2. MAIN RESULTS

This section is devoted to present in details the main results contained in this work.

**2.1. Null-controllability.** Let us first recall the definition of null-controllability.

**Definition 3.** Let  $T > 0$  and  $\omega \subset \mathbb{R}^n$  be a measurable set with positive measure. The equation  $(E_s)$  is said to be *null-controllable* from the control support  $\omega$  in time  $T > 0$  when for all  $f_0 \in L^2(\mathbb{R}^n)$ , there exists a control  $u \in L^2((0, T) \times \omega)$  such that the mild solution of  $(E_s)$  satisfies  $f(T, \cdot) = 0$ .

By the Hilbert Uniqueness Method (see, e.g., [4, Theorem 2.44]), the null-controllability of the fractional heat equation  $(E_s)$  is equivalent to the observability of the fractional heat semigroup, whose we recall in the following definition.

**Definition 4.** Let  $T > 0$ , and let  $\omega \subset \mathbb{R}^n$  be measurable. The fractional heat semigroup  $(e^{-t(-\Delta)^{s/2}})_{t \geq 0}$  is said to be exactly observable from the set  $\omega$  at time  $T$  if there exists a positive constant  $C_{\omega, T} > 0$  such that for all  $g \in L^2(\mathbb{R}^n)$ ,

$$\|e^{-T(-\Delta)^{s/2}} g\|_{L^2(\mathbb{R}^n)}^2 \leq C_{\omega, T} \int_0^T \|e^{-t(-\Delta)^{s/2}} g\|_{L^2(\omega)}^2 dt.$$

We first prove that the exponential thickness is a necessary geometric condition to prove exact null-controllability results for the equation  $(E_s)$  in the weak dissipation regime  $0 < s < 1$ .

**Theorem 5.** *Let  $s \in (0, 1)$ ,  $T > 0$  and  $\omega \subset \mathbb{R}^n$  be measurable. If the fractional heat equation  $(E_s)$  is null-controllable from  $\omega$  at time  $T$ , then  $\omega$  is  $2s/(1-s)$ -exponentially thick.*

*Remark 6.* We do not consider the critical case  $s = 1$ , whose understanding remains an open problem.

We then prove that the exponential thickness is not only a necessary condition for the null-controllability of the equation  $(E_s)$ , but is also a sufficient condition.

**Theorem 7.** For all  $s \in (0, 1]$ , the fractional heat equation  $(E_s)$  is null-controllable from any  $\alpha$ -exponentially thick set  $\omega \subset \mathbb{R}^n$ , with  $0 < \alpha < s$ , and in any positive time  $T > 0$ . Moreover, there exists a positive constant  $C > 0$  such that for all  $T > 0$  and  $f \in L^2(\mathbb{R}^n)$ ,

$$\|e^{-T(-\Delta)^{s/2}} f\|_{L^2(\mathbb{R}^n)}^2 \leq C \exp\left(\frac{C}{T^{\alpha/(s-\alpha)}}\right) \frac{1}{T} \int_0^T \|e^{-t(-\Delta)^{s/2}} f\|_{L^2(\omega)}^2 dt.$$

*Remark 8.* This provides a positive null-controllability result for the fractional heat equation  $(E_s)$  in the regime  $0 < s \leq 1$  (notice that the critical case  $s = 1$  is also allowed). Recall from [15, Theorem 3.8] or [1, Remark 1.13] that in the regime  $s > 1$ , the following exact observability estimates have been obtained for all thick set  $\omega \subset \mathbb{R}^n$ ,  $T > 0$  and  $f \in L^2(\mathbb{R}^n)$

$$\|e^{-T(-\Delta)^{s/2}} f\|_{L^2(\mathbb{R}^n)}^2 \leq C \exp\left(\frac{C}{T^{1/(s-1)}}\right) \frac{1}{T} \int_0^T \|e^{-t(-\Delta)^{s/2}} f\|_{L^2(\omega)}^2 dt.$$

The above observability estimate holds in particular for any  $\alpha$ -exponential thick set  $\omega \subset \mathbb{R}^n$ .

**2.2. Examples.** To end this section, we present examples of exponentially thick control supports. Their construction is based on the one of the Smith-Volterra-Cantor sets, defined as follows

**Definition 9** (Smith-Volterra-Cantor sets). Let  $(r_n)_{n \in \mathbb{N}}$  be a sequence of real numbers such that  $0 < r_n < 1$ . For  $n \in \mathbb{N}$ , let  $K_n$  be the closed subset of  $[0, 1]$ , finite union of closed disjoint intervals, defined inductively by the following procedure.

- $K_0 := [0, 1]$ .
- If  $K_n = \bigcup_k I_{nk}$ , where the  $(I_{nk})_k$  are disjoint closed intervals, remove from  $I_{nk}$  the middle part of size  $r_n \text{Leb}(I_{nk})$  and call the resulting sets<sup>1</sup>  $I'_{nk}$ . Then set  $K_{n+1} := \bigcup_k I'_{nk}$ .

Let  $K := \bigcap_{n \in \mathbb{N}} K_n$ . The set  $K$  is the Smith-Volterra-Cantor set associated to the sequence  $(r_n)_n$ .

**Theorem 10.** Let  $\alpha > 0$ ,  $c \in (0, 1)$  and  $C > 0$ . Set  $r_n := c \exp(-C2^{n\alpha})$ . Let  $K$  be the associated Smith-Volterra-Cantor set and  $\omega := \mathbb{R} \setminus K$ . There exist  $c', C', L_0 > 0$  such that for every  $0 < L < L_0$ ,

$$c' \exp(-C'L^{-\alpha}) \leq \inf_{x \in \mathbb{R}} \frac{\text{Leb}(\omega \cap B(x, L))}{\text{Leb}(B(x, L))} \leq C' \exp(-c'L^{-\alpha}).$$

In other words,  $\omega$  is  $\alpha$ -exponentially thick, and not better.

The set  $\omega$  defined in the statement of theorem 10 is  $\alpha$ -exponentially thick, but does not have full measure. It is in fact the *simplest* of such sets that we could think of.

The  $\omega$  defined in theorem 10 is a subset of  $\mathbb{R}$ . To get non-trivial  $\alpha$ -exponentially thick sets in higher dimension, we can just take  $\omega^n$ .

### 3. SUFFICIENT CONDITION

This section is devoted to the proof of Theorem 7, which states that in the weak dissipation regime  $s \in (0, 1]$ , the fractional heat equation  $(E_s)$  is exactly null-controllable from any  $\alpha$ -exponentially thick set  $\omega \subset \mathbb{R}^n$ , with  $0 < \alpha < s$ , and in any positive time  $T > 0$ . This done in two steps: first proving a spectral estimate reminiscent of Jerison and Lebeau's spectral inequality [8, Theorem 14.6] or Logvinenko-Sereda-Kovrijkine estimate [10], and second using Lebeau and Robbiano's method, as stated in the following theorem that was proved by Miller [14, Theorem 2.2] (see also [15, Theorem 2.8]).

<sup>1</sup>I.e., if  $I_{nk} = [a_{nk}, b_{nk}]$ , set  $b'_{nk} := (a_{nk}(1 + r_n) + b_{nk}(1 - r_n))/2$  and  $a'_{nk} = (a_{nk}(1 - r_n) + b_{nk}(1 + r_n))/2$ , and finally  $I'_{nk} := [a_{nk}, b'_{nk}] \cup [a'_{nk}, b_{nk}]$ .

**Theorem 11.** *Let  $A$  be a non-negative selfadjoint operator on  $L^2(\mathbb{R}^n)$ , and let  $\omega \subset \mathbb{R}^d$  be measurable. Suppose that there are  $d_0 > 0$ ,  $d_1 \geq 0$ , and  $\zeta \in (0, 1)$  such that for all  $\lambda \geq 0$  and  $f \in \text{Ran } \mathbb{1}_{(-\infty, \lambda]}(A)$ ,*

$$\|f\|_{L^2(\mathbb{R}^n)} \leq d_0 e^{d_1 \lambda^\zeta} \|f\|_{L^2(\omega)}.$$

*Then, there exist positive constants  $c_1, c_2, c_3 > 0$ , only depending on  $\zeta$ , such that for all  $T > 0$  and  $g \in L^2(\mathbb{R}^n)$  we have the observability estimate*

$$\|e^{-TA}g\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{C_{obs}}{T} \int_0^T \|e^{-tA}g\|_{L^2(\omega)}^2 dt,$$

where the positive constant  $C_{obs} > 0$  is given by

$$C_{obs} = c_1 d_0 (2d_0 + 1)^{c_2} \exp\left(c_3 \left(\frac{d_1}{T^\zeta}\right)^{\frac{1}{1-\zeta}}\right).$$

In the rest of this section, in order to alleviate the text, we will denote the spectral subspaces associated with the Laplacian as follows

$$\mathcal{E}_\lambda = \text{Ran } \mathbb{1}_{(-\infty, \lambda]}(-\Delta), \quad \lambda \geq 0.$$

Let us now prove the following spectral estimates.

**Proposition 12.** *Let  $\alpha \in (0, 1)$  and let  $\omega \subset \mathbb{R}^n$  be a measurable set that is  $\alpha$ -exponentially thick. Then, there exists a positive constant  $c > 0$  such that*

$$\forall \lambda > 0, \forall f \in \mathcal{E}_\lambda, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq c e^{c\lambda^{\alpha/2}} \|f\|_{L^2(\omega)}.$$

*Remark 13.* Recall from Kovrikin's estimate [10, Theorem 3] that there exists a universal positive constant  $K > 0$  depending only on the dimension  $n$  such that for all  $(\gamma, L)$ -thick set  $\omega \subset \mathbb{R}^n$ , with  $\gamma \in (0, 1]$  and  $L > 0$ , we have

$$(1) \quad \forall \lambda \geq 0, \forall f \in \mathcal{E}_\lambda, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{K}{\gamma}\right)^{K(1+L\sqrt{\lambda})} \|f\|_{L^2(\omega)}.$$

Our spectral estimates therefore improve the ones known for the thick sets.

*Proof of proposition 12.* Let us consider some  $\lambda > 0$  and  $f \in \mathcal{E}_\lambda$  be fixed. Since an  $\alpha$ -exponentially thick set is  $c e^{-CL^{-\alpha}}$ -thick at every scale  $0 < L < L_0$ , for some  $c > 0$ ,  $C > 0$  and  $L_0 > 0$ , we deduce from (1) that

$$\forall L \in (0, L_0), \quad \|f\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{K}{c e^{-CL^{-\alpha}}}\right)^{K(1+L\sqrt{\lambda})} \|f\|_{L^2(\omega)}.$$

Assume for a moment that  $\lambda \geq 1$ . Then, by choosing  $L = L_0/\sqrt{\lambda}$  in the above estimate, we obtain that

$$\|f\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{K}{c}\right)^{K(1+L_0)} e^{CK(1+L_0)L_0^{-\alpha}\lambda^{\alpha/2}} \|f\|_{L^2(\omega)}.$$

This is the expected estimate when  $\lambda \geq 1$ . For the case  $0 < \lambda < 1$ , we use again Kovrikin's estimate to find a  $C_1$  such that for every  $f \in \mathcal{E}_\lambda \subset \mathcal{E}_1$ ,  $\|f\|_{L^2(\mathbb{R}^n)} \leq C_1 \|f\|_{L^2(\omega)}$ . Therefore, for any  $c_0 > 0$ , and in particular for  $c_0 = CL(1 + L_0)L_0^{-\alpha}$

$$\|f\|_{L^2(\mathbb{R}^n)} \leq C_1 e^{c_0 \lambda^{\alpha/2}} \|f\|_{L^2(\omega)}.$$

This ends the proof of proposition 12. □

*Proof of theorem 7.* Given a positive time  $T > 0$  and a  $\alpha$ -exponentially thick set  $\omega \subset \mathbb{R}^n$ , with  $0 < \alpha < s$ , we are now in position to prove an exact observability estimate for the fractional heat semigroup from  $\omega$  at time  $T$ . First notice that

$$\mathbb{1}_{(-\infty, \lambda]}((-\Delta)^{s/2}) = \mathbb{1}_{(-\infty, \lambda^{2/s}]}(-\Delta), \quad \lambda \geq 0.$$

We therefore deduce from Proposition 12 that there exists a positive constant  $c > 0$  such that

$$\forall \lambda > 0, \forall f \in \text{Ran } \mathbb{1}_{(-\infty, \lambda]}((-\Delta)^{s/2}), \quad \|f\|_{L^2(\mathbb{R}^n)} \leq ce^{c\lambda^{\alpha/s}} \|f\|_{L^2(\omega)}.$$

Theorem 11 then implies that there exists a positive constant  $C > 0$  such that for all  $T > 0$  and  $g \in L^2(\mathbb{R}^n)$ ,

$$\|e^{-T(-\Delta)^{s/2}} g\|_{L^2(\mathbb{R}^n)}^2 \leq C \exp\left(\frac{C}{T^{\alpha/(s-\alpha)}}\right) \frac{1}{T} \int_0^T \|e^{-t(-\Delta)^{s/2}} g\|_{L^2(\omega)}^2 dt.$$

This is the expected estimate.  $\square$

#### 4. NECESSARY CONDITION

The aim of this section is to prove Theorem 5. This necessary condition is proved by testing the observability inequality on coherent states, as in [9].

*Proof of theorem 5.* We detail the proof in dimension 1. The proof in higher dimension is very similar. Indeed, all the tools and theorems we use can be adapted in higher dimension (see [9, §4.3]).

*Step 1: Observability inequality.* — As in the proof of theorem 7 (and see [4, Theorem 2.44]), the exact null-controllability of the fractional heat equation  $(E_s)$  on  $\omega$  in time  $T$  is equivalent to the following observability inequality: for every  $g_0 \in L^2(\mathbb{R})$ , the solution  $g$  of  $(\partial_t + (-\Delta)^{s/2})g(t, x) = 0$ ,  $g(0, \cdot) = g_0$  satisfies

$$(2) \quad \|g(T, \cdot)\|_{L^2(\mathbb{R})} \leq C \|g_0\|_{L^2((0, T) \times \omega)}.$$

Throughout this proof,  $c$  and  $C$  denote constants that can change from line to line.

*Step 2: Construction of test functions.* — We want to find a lower bound on  $\text{Leb}(\omega \cap B(x, L))$ . Since the fractional heat equation is invariant by translation, we may assume that  $x = 0$ .

Let  $\xi_0 > 0$  and  $\chi \in C_c^\infty(-\xi_0, \xi_0)$  such that  $\chi \equiv 1$  on a neighborhood of 0. For  $h > 0$ , set

$$(3) \quad \begin{aligned} g_h(t, x) &:= h \int_{\mathbb{R}} \chi(h\xi - \xi_0) e^{-(h\xi - \xi_0)^2/2h + ix\xi - t|\xi|^s} d\xi \\ &:= \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{-(\xi - \xi_0)^2/2h + ix\xi/h - t|\xi|^s/h^{-s}} d\xi. \end{aligned}$$

Then  $g_h$  is a solution to the fraction heat equation  $(E_s)$ .

*Step 3: Estimates  $g_h$ .* — With the notations of [9, Section 3.2], we have  $g_h(t, x) = I_{t, h, 1}(x)$  with  $\rho_t(\xi) = -t\xi^s$ . According to [9, Proposition 3.5] (with  $X = [0, T]$ ), there exists  $\eta > 0$  such that uniformly in  $0 \leq t \leq T$  and  $|x| < \eta$ ,

$$(4) \quad g_h(t, x) = \sqrt{2\pi h} e^{ix\xi_0/h - x^2/2h - t(\xi_0 + ix)^s h^{-s} + O(h^{1-2s})} (1 + O(h^{1-s})).$$

Hence,

$$(5) \quad \|g_h(T, \cdot)\|_{L^2(\mathbb{R})} \geq \|g_h(T, \cdot)\|_{L^2(|x| < \eta)} \geq ce^{-Ch^{-s}}.$$

Moreover, according to [9, Proposition 3.7] (with  $X = [0, T]$ ), uniformly in  $0 \leq t \leq T$  and  $|x| > \eta$

$$(6) \quad g_h(t, x) = O(|x|^{-2} e^{-c/h}).$$

Hence, for every  $h > 0$  small enough

$$\begin{aligned} \|g_h\|_{L^2((0,T)\times\omega)}^2 &= \|g_h\|_{L^2((0,T)\times\{\eta<|x|\}\cap\omega)}^2 + \|g_h\|_{L^2((0,T)\times\{\eta<|x|\}\cap\omega)}^2 \\ &\leq Ce^{-c/h} + \|g_h\|_{L^2((0,T)\times\{|x|<\eta\}\cap\omega)}^2. \end{aligned}$$

Now, set  $\beta := (1 - s)/2$ . According to the asymptotics (4), there exist  $C, c > 0$  such that for every  $r > 0$  and every  $h > 0$  small enough so that  $rh^\beta < \eta$ ,

$$\begin{aligned} \|g_h\|_{L^2((0,T)\times\omega)}^2 &\leq Ce^{-c/h} + \|g_h\|_{L^2(0,T)\times\{rh^\beta<|x|<\eta\}}^2 + \|g_h\|_{L^2(0,T)\times\{|x|<rh^\beta\}}^2 \\ &\leq Ce^{-c/h} + Che^{-cr^2h^{2\beta-1}+O(h^{1-2s})} + Ch \text{Leb}(\omega \cap B(0, rh^\beta)). \end{aligned}$$

Since  $2\beta - 1 = -s$  and  $1 - 2s > -s$ , there exist  $C, c > 0$  such that for every  $r > 0$  and every  $h > 0$  small enough (depending on  $r$ ),

$$(7) \quad \|g_h\|_{L^2((0,T)\times\omega)}^2 \leq Ce^{-cr^2h^{-s}} + Ch \text{Leb}(\omega \cap B(0, rh^\beta)).$$

*Step 4: Conclusion.* — If the observability inequality (2) holds, according to eqs. (5) and (7), there exist  $c, C > 0$  such that for any  $r > 0$  and every  $h > 0$  small enough (depending on  $r$ ):

$$ce^{-Ch^{-s}} \leq Ce^{-cr^2h^{-s}} + Ch \text{Leb}(\omega \cap B(0, rh^\beta)).$$

If we choose  $r$  large enough, we can absorb the  $e^{-cr^2h^{-s}}$  in the left-hand side. Hence, for every  $h > 0$  small enough,

$$e^{-Ch^{-s}} \leq C \text{Leb}(\omega \cap B(0, rh^\beta)).$$

Setting  $L = rh^\beta$ , i.e.,  $h = cL^{1/\beta}$ ,

$$\text{Leb}(\omega \cap B(0, L)) \geq ce^{-CL^{-s/\beta}} = ce^{-CL^{-2s/(1-s)}}. \quad \square$$

## 5. EXAMPLES OF EXPONENTIALLY-THICK SETS

Let us first recall some basic facts about Smith-Volterra-Cantor sets. At each step in the construction of a Smith-Volterra-Cantor set, we remove a subset of measure  $(1 - r_n) \text{Leb}(K_n)$  from  $K_n$  (see definition 9), hence:

**Proposition 14.** *With the notations of definition 9,  $\text{Leb}(K_n) = \prod_{k=0}^{n-1} (1 - r_k)$  and  $\text{Leb}(K) = \prod_{n=0}^{+\infty} (1 - r_n)$ . In particular,  $\text{Leb}(K) > 0$  if and only if  $\sum_n r_n < +\infty$ .*

Our first result is a some upper and lower bounds on the thickness of the complement of Smith-Volterra-Cantor sets.

**Proposition 15.** *Let  $(r_n)_n \in (0, 1)^\mathbb{N}$ . Assume that  $\sum_n r_n < +\infty$ . Let  $K$  be the associated Smith-Volterra-Cantor set. Set  $c_0 := \text{Leb}(K)$  and  $\omega := \mathbb{R} \setminus K$ . There exist  $c > 0, C > 0$  and  $L_0 > 0$  such that for every  $0 < L < L_0$ ,*

$$c \sum_{k \geq \log_2(3c_0/L)} r_k \leq \inf_{x \in \mathbb{R}} \frac{\text{Leb}(\omega \cap B(x, L))}{\text{Leb}(B(x, L))} \leq C \sum_{k \geq \log_2(c_0/4L)} r_k,$$

where  $\log_2$  is the base 2 logarithm  $\log_2(x) = \ln(x)/\ln(2)$ .

With a more careful analysis in the proof below, it seems we could improve this inequality by replacing the  $\log_2(3c_0/L)$  by  $\log_2(\kappa c_0/L)$  with some  $\kappa < 3$ . We don't know what the optimal  $\kappa$  is. We don't pursue this because we don't need such a sharp estimate. In fact proposition 15 is already sharper than we need for our purposes.

*Proof.* Remark that we only need to estimate  $\text{Leb}(\omega \cap B(x, L))$  for  $x \in [0, 1]$ . Indeed, if for instance  $x > 1$ ,  $\omega \cap B(x, L)$  contains at least  $[x, x + L]$ , hence  $\text{Leb}(\omega \cap B(x, L)) / \text{Leb}(B(x, L)) \in (1/2, 1)$ .

*Step 1: Notations and preliminary computations.* — In this proof, we denote by  $I_{nk}$  the intervals that appears in the construction of  $K$ , as defined in definition 9. We denote the length of  $I_{nk}$  (which does not depend on  $k$ ) by  $\ell_n$ . We have

$$(8) \quad \ell_n = \frac{1 - r_{n-1}}{2} \ell_{n-1}.$$

Notice that

$$\text{Leb}(I_{nk} \cap \omega) = \text{Leb}(I_{nk}) - \text{Leb}(I_{nk} \cap K) = \ell_n \left(1 - \prod_{k \geq n} (1 - r_k)\right).$$

In addition, we can estimate the right-hand side in the following way

$$\begin{aligned} 1 - \prod_{k \geq n} (1 - r_k) &= 1 - \exp\left(\sum_{k \geq n} \ln(1 - r_k)\right) \\ &= 1 - \exp\left(\sum_{k \geq n} -r_k(1 + o_k(1))\right) \\ &= 1 - \exp\left(- (1 + o_n(1)) \sum_{k \geq n} r_k\right) \\ &= 1 - \left(1 - (1 + o_n(1)) \sum_{k \geq n} r_k\right) \\ &= (1 + o_n(1)) \sum_{k \geq n} r_k. \end{aligned}$$

In the third and fourth relations, we used that  $\sum_k r_k < +\infty$ . Thus,

$$(9) \quad \text{Leb}(I_{nk} \cap \omega) = (1 + o_n(1)) \ell_n \sum_{k \geq n} r_k.$$

*Step 2: Lower bound when  $L$  is comparable to  $\ell_n$ .* — Let  $L > 0$  and  $n \in \mathbb{N}$  be such that  $2\ell_n \leq L \leq 6\ell_n$ . Let  $x \in [0, 1]$ .

If  $\omega \cap B(x, L)$  contains an interval of length  $\geq \ell_n/2$ ,  $\text{Leb}(\omega \cap B(x, L)) \geq \ell_n/2$ .

If that is not the case, then,  $\text{distance}(x, K_n) < \ell_n/4$ . Since  $L \geq 2\ell_n$ , this implies that  $B(x, L)$  contains some  $I_{nk}$ . Hence, according to eq. (9),

$$\text{Leb}(\omega \cap B(x, L)) \geq \text{Leb}(\omega \cap I_{nk}) = (1 + o_n(1)) \ell_n \sum_{k \geq n} r_k.$$

Putting the two cases together:

$$\inf_{x \in \mathbb{R}} \frac{\text{Leb}(\omega \cap B(x, L))}{\text{Leb}(B(x, L))} \geq \frac{\ell_n}{2L} \min\left(c \sum_{k \geq n} r_k, \frac{1}{2}\right) \geq \frac{c\ell_n}{2L} \sum_{k \geq n} r_k.$$

Since  $L \leq 6\ell_n$ ,

$$(10) \quad \inf_{x \in \mathbb{R}} \frac{\text{Leb}(\omega \cap B(x, L))}{\text{Leb}(B(x, L))} \geq c \sum_{k \geq n} r_k,$$

this inequality being valid whenever  $2\ell_n \leq L \leq 6\ell_n$ .

*Step 3: Upper bound when  $L$  is comparable to  $\ell_n$ .* — Let  $L > 0$  and  $n \in \mathbb{N}$  be such that  $\ell_n/3 \leq 2L \leq \ell_n$ . Let  $x \in [0, 1]$  in the middle of a  $I_{nk}$ , so that  $B(x, \ell_n/2) = I_{nk}$ . Then  $B(x, L) \subset I_{nk}$ , and according to eq. (9),

$$\text{Leb}(\omega \cap B(x, L)) \leq \text{Leb}(I_{nk} \cap \omega) = (1 + o_n(1)) \ell_n \sum_{k \geq n} r_k.$$



Since,  $\ell_n/3 \leq 2L$ ,

$$(11) \quad \inf_{x \in \mathbb{R}} \frac{\text{Leb}(\omega \cap B(x, L))}{\text{Leb}(B(x, L))} \leq C \sum_{k \geq n} r_k,$$

this inequality being valid whenever  $\ell_n/3 \leq 2L \leq \ell_n$ .

*Step 4: Solving the inequality  $a\ell_n \leq L \leq b\ell_n$ .* — Let  $L > 0$  and  $0 < \kappa < 1$ . Set  $n(L) := \lceil \log_2(c_0/L) \rceil$ , where  $\lceil \cdot \rceil$  is the ceiling function. We aim to prove that for  $L$  small enough,  $\kappa\ell_{n(L)} \leq L \leq 2\ell_{n(L)}$ .

According to the definition of  $I_{nk}$ ,  $\ell_n = 2^{-n} \prod_{k=0}^{n-1} (1 - r_k)$ . Recall that  $c_0 = \prod_{k=0}^{+\infty} (1 - r_k) > 0$ . Define  $\epsilon_n$  by  $1 + \epsilon_n = \left( \prod_{k \geq n} (1 - r_k) \right)^{-1}$ . Then  $\epsilon_n > 0$  and  $\epsilon_n \rightarrow 0$  as  $n \rightarrow +\infty$ . With this notation, we have the equivalences

$$\begin{aligned} \kappa\ell_n \leq L \leq 2\ell_n &\Leftrightarrow 2^{-n}\kappa c_0(1 + \epsilon_n) \leq L \leq 2^{1-n}c_0(1 + \epsilon_n) \\ &\Leftrightarrow -n + \log_2(\kappa c_0(1 + \epsilon_n)) \leq \log_2(L) \leq 1 - n + \log_2(c_0(1 + \epsilon_n)) \\ &\Leftrightarrow \log_2\left(\frac{c_0}{L}\right) + \log_2(\kappa) + \log_2(1 + \epsilon_n) \leq n \leq \log_2\left(\frac{c_0}{L}\right) + 1 + \log_2(1 + \epsilon_n). \end{aligned} \quad (12)$$

According to the definition of the ceiling function,

$$\log_2\left(\frac{c_0}{L}\right) \leq n(L) < \log_2\left(\frac{c_0}{L}\right) + 1.$$

Since  $\log_2(\kappa) < 0$ ,  $\log_2(1 + \epsilon_n) > 0$  and  $\epsilon_n \rightarrow 0$ , the inequalities (12) are satisfied for  $n = n(L)$  and small enough  $L > 0$ .

*Step 5: Conclusion.* — Applying the previous step with  $\kappa = 2/3$ , and replacing  $L$  by  $L/3$ , we see that lower bound (10) holds for  $n = \lceil \log_2\left(\frac{3c_0}{L}\right) \rceil$  when  $L$  is small enough. Hence, the lower-bound stated in proposition 15 holds.

Applying Step 4 with  $\kappa = 2/3$  and  $L$  replaced by  $4L$ , we get that the upper bound (11) holds with  $n = \lceil \log_2\left(\frac{c_0}{4L}\right) \rceil$  and  $L$  small enough, which gives the stated upper bound.  $\square$

Applying this general bound, we now prove that with the proper sequence  $(r_n)_n$ , the complement of the associated Smith-Volterra-Cantor set is  $\alpha$ -exponentially thick.

*Proof of Theorem 10.* Since  $r_n = c \exp(-C2^{n\alpha})$  decays faster than exponentially,  $\sum_n r_n < +\infty$ . Moreover, as  $n \rightarrow +\infty$

$$\sum_{k \geq n} r_k = r_n(1 + o(1)).$$

Plugging this into the bounds of proposition 15 proves the claimed bounds.  $\square$

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