System of nonlinear second-order parabolic partial differential equations with interconnected obstacles and oblique derivative boundary conditions on non-smooth time-dependent domains

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Abstract

In this paper, we consider a system of fully non linear second order parabolic partial differential equations with interconnected obstacles and boundary conditions on non smooth time dependent domains. We prove existence and uniqueness of a continuous viscosity solution. This system is the Hamilton-Jacobi-Bellman system of equations associated with a m-switching problem in finite horizon, when the state process is the solution of an obliquely reflected stochastic differential equation in non smooth time-dependent domain. Our approach is based on the study of related system of reflected generalized backward stochastic differential equations with oblique reflection. We show that this system has a unique solution which is the optimal payoff and provides the optimal strategy for the switching problem. Methods of the theory of generalized BSDEs and their connection with PDEs with boundary condition are then used to give a probabilistic representation for the solution of the PDEs system.

Keywords: Viscosity solution, Fully nonlinear partial differential equations, Obliquely reflected diffusion, non-smooth time-dependent domain, Generalized Backward stochastic differential systems, Optimal switching
MSC Classification: 49L25, 60J50, 60H30, 60J60

1 Introduction

Let $d \geq 1$, $T > 0$ and define the time-dependent domain $D = D' \cap \big([0,T] \times \mathbb{R}^d\big)$, given D', a bounded open connected subset of $\mathbb{R}^{1+d}$. For each $t \in [0,T]$, we define the time sections of D as $D_t = \{x : (t,x) \in D\}$. The aim of this paper is to investigate the existence and uniqueness of a solution of the following PDEs system with interconnected obstacles and

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oblique derivative boundary conditions: $\forall T>0$, $\forall i=1,...,m$,

$$\begin{cases} \min\{v^i(t,x)-\max\limits_{j\neq i}(v^j(t,x)-g_{ij}(t,x)); -\partial_t v^i(t,x)-\mathscr{L}v^i(t,x) \\ \qquad -f_i(t,x,(v^j(t,x))_{j=1,...,m},\sigma^\top(t,x)D_x v^j(t,x))\}=0,\ (t,x)\in D^{\mathrm{o}}; \\ \dfrac{\partial v^i}{\partial \gamma}(t,x)+\psi_i(t,x,(v^j(t,x))_{j=1,...,m})=0,\ (t,x)\in \partial D; \\ v^i(T,x)=h_i(x),\ x\in \overline{D}_T. \end{cases} \tag{1.1}$$

The set $\overline{D}_T$ is the adherence of D_T and D^{o}, ∂D are defined by:

$$D^{\mathrm{o}}=D'\cap\big((0,T)\times\mathbb{R}^d\big),$$
$$\partial D=\big(\overline{D}'\setminus D'\big)\cap\big((0,T)\times\mathbb{R}^d\big).$$

The operator $\mathscr{L}$ is given by $\mathscr{L}=\frac{1}{2}Tr\{(\sigma\sigma^\top)D_{xx}(.)\}+b^\top D_x(.)$ and at a boundary point $(t,x)\in\partial D$, the quantity $\frac{\partial}{\partial\gamma}$ is referred to as the oblique derivative along a given direction γ.

The system (1.1) is the Hamilton-Jacobi-Bellman (H-J-B) system of equations associated with a multidimensional switching problem when the diffusion is reflected in the non smooth time-dependent domain D. Switching problems are used in many applied fields like energy storage and finance. This has been investigated in several works (see e.g., [2, 3, 4], etc.). In its general form, when the state process is the solution of a standard SDE which takes its values in $\mathbb{R}^d$, the m-switching problem is given by the solution of a system of reflected BSDEs with oblique reflection. We mention the paper by Hamadene and Zhang [13] that solves a system of reflected BSDEs under a specific monotonicity condition on the driver function $f=(f_i)_{i=1,...,m}$, where each component f_i depends on the whole solution. Based on the latter results, the study of the related H-J-B system of equations was carried on by Hamadene and Morlais [12]. The authors proved that under the monotonicity condition, there exists a unique continuous viscosity solution of the system of PDEs with interconnected obstacles. Later on, Chassagneux, Elie and Kharroubi [5] proved the existence of a unique solution for the system of reflected BSDEs without the monotonicity assumption on the function f. Then, the related PDEs system with interconnected obstacles has been investigated by Hamadene, Mnif and Neffati [11], where a unique continuous viscosity solution has been provided without the monotonicity condition. Afterwards, when the diffusion is required to stay in a bounded time-independent domain, the switching problem has been considered by Boufoussi, Hamadene and Jakani [1]. The existence and uniqueness of an optimal strategy and optimal yield are provided by the unique solution of the associated system of reflected generalized BSDE with oblique reflection. We underline that in this work the results were obtained under monotonicity conditions on both f and the driver function ψ in the generalized integral with respect to an increasing continuous process. This requires that for each $i=1,\ldots,m$, f_i is non-decreasing with respect to the i-th component of $\vec{y}$ when the other components are fixed. The second assumption on ψ_i, which is allowed to depend only on the i-th component of $\vec{y}$, requires that ψ_i is non-increasing. The studies conducted on standard generalized BSDEs given in Pardoux and Zhang [22] and the results on reflected generalized BSDEs established in Ren and Xia [23], the solution of the reflected generalized BSDEs system with interconnected obstacles is obtained as the increasing limit of a sequence of reflected generalized BSDEs. In these two papers, when randomness comes from

a normally reflected diffusion in a bounded time-independent domain, probabilistic representations for solutions of semilinear partial differential equations with Neumann boundary conditions are obtained. This is the key tool to solve the H-J-B system of equations considered in [1]. In the same framework, Lundström and Olofsson [19] deal with similar system of PDEs. The authors prove the existence of a unique viscosity solution of fully nonlinear parabolic partial differential equations system with interconnected obstacles and Neumann boundary conditions using Perron's method.

The main goal of this work is to consider a new class of reflected diffusions in which we allow the domain to vary along time. More precisely, we consider the following type of obliquely reflected SDE in the time-dependent domain D along the oblique direction γ: $\forall s \in [t,T]$,

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r,X_r^{t,x})dr + \int_t^s \sigma(r,X_r^{t,x})dW_r + \int_t^s \gamma(r,X_r^{t,x})d|\Lambda^{t,x}|_r; \\ X_s^{t,x} \in \bar{D}_s,\ |\Lambda^{t,x}|_s = \int_0^s \chi_{\{X_r^{t,x}\in\partial D_r\}} d|\Lambda^{t,x}|_r. \end{cases} \tag{1.2}$$

This type of reflected SDEs has been first introduced by Costantini,Gobet and El Karoui [6] for smooth time-dependent domains when the direction of reflection is normal. Existence and uniqueness of the solution is then established. This result has been generalized by Lundsrtom and Onskog in [20], where the authors have showed the existence and uniqueness of the solution of the reflected SDE (1.2) in non smooth time dependent domain with oblique reflection. In addition, using a pure PDE approach the authors proved existence and uniqueness of a unique viscosity solution of a system of fully non linear PDEs with boundary conditions on time-dependent domain. In the same geometric setting, we consider a more general system that involves interconnected obstacles.

The purpose of this paper is to solve the system of variational inequalities (1.1) and to give a probabilistic representation for its solution using systems of reflected generalized BSDEs with oblique reflection. The approach used in this paper is different from the one used in [1]. The main tool to tackle the PDEs system (1.1) is to establish first the link between standard generalized BSDEs in the case where the state process is a reflected SDE in a time-dependent domain and some PDEs system with boundary conditions as will be stated in section 2. This step requires that the time sections of D should be increasing in time, in other words, $D_t \subset D_{t'}$, whenever $t \leqslant t'$.

Under the above-mentioned geometric setting, let (X,Λ) be the solution of the reflected SDE (1.2). An example of application is the production of electricity in a hydro-power plant that has several working modes. The water is collected in the reservoir, then it is released into turbines, each of them has its mode of production of electricity. This installation raises two concerns:
- The water in the reservoir should be sustained at a certain level.
- The choice of most profitable mode of production.
Let X_s be the level of water in the reservoir at time s. For safety reasons, the level of water should not exceed the capacity of the reservoir that will be denoted $l_{\max}$. On the other hand, the water level should remain above a threshold level $l_{\min}(s)$, which itself is likely to drop off at each time s. Therefore, at each time s, actions should be made in order to satisfy the condition $X_s \in \bar{D}_s := [l_{\min}(s), l_{\max}]$ and to cope with the demand of the power at each time s. As a consequence, the working mode of the station is chosen according to parameters which include the level of water X in the dam which is a reflected stochastic process in the

bounded time dependent domain D. More precisely, let α be a strategy of management of the hydro-power station, then its yield from s to T is given by:

$$\Gamma_s(\alpha) := \mathbb{E}\Big[\int_s^T f_\alpha(r, X_r^{t,x})dr + \int_s^T \psi_\alpha(r, X_r^{t,x})d|\Lambda^{t,x}|_r - G_T^\alpha + h_\alpha(X_T^{t,x})\Big]. \tag{1.3}$$

The quantity $h_\alpha(X_T^{t,x})$ is the terminal profit, $\int_s^T f_\alpha(r, X_r^{t,x})dr$ is the running payoff, whereas $\int_s^T \psi_\alpha(r, X_r^{t,x})d|\Lambda^{t,x}|_r$ stands for the cost of maintaining the level of the dam in the appropriate zone. Finally, the power plan manager aims maximising $\Gamma_s(\alpha)$ by switching between modes at some stopping times. This, induces the total switching cost G_T^α.

The switching problem is related to system of reflected GBSDEs of the following form: $\forall t \leqslant s \leqslant T$, $i = 1, ..., m$

$$\begin{cases} Y_s^i = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^i)dr + \int_s^T \psi_i(r, X_r^{t,x}, \vec{Y}_r^{t,x})d|\Lambda^{t,x}|_r \\ \qquad\qquad + K_T^i - K_s^i - \int_s^T Z_r^i dW_r, \\ Y_s^i \geqslant \max_{j\neq i}(Y_s^j - g_{ij}(s, X_s^{t,x})), \\ \int_0^T \{Y_s^i - \max_{j\neq i}(Y_s^j - g_{ij}(s, X_s^{t,x}))\}dK_s^i = 0. \end{cases} \tag{1.4}$$

This system is different from the one studied in [1]. We allow the generator ψ to depend on the whole solution. Moreover, without assuming any assumption on neither f nor ψ, we establish the existence of unique solution for the generalized reflected BSDEs system which provides the optimal strategy and the optimal yield as well as the solution of the PDEs system (1.1).

This paper is organized as follows. In the next section, we present the assumptions we need on the time-dependent domain D and the direction of reflection γ. We collect also some results on the reflected SDE in the time dependent domain (1.2) and we give some estimates for the solution. Then we make the link between standard generalized BSDEs in the markovian setting i.e., when randomness comes from X solution of (1.2), and systems of fully non linear PDEs and boundary conditions on time-dependent domains. In section 3, we establish existence and uniqueness of the solution for the reflected generalized BSDEs with oblique reflection. We also give a representation of the solution as the value function of our switching problem. Section 4 is devoted to the study of the system of PDEs with interconnected obstacles and boundary conditions on time dependent domain. First, we establish a comparison principle for a particular system of PDEs. Then we make use of the results of sections 2 and 3 to prove the existence of a viscosity solution. Finally, we prove the existence and uniqueness in the general case by a Picard iteration argument.

2 Preliminaries and notations

2.1 Geometry

We consider a non-smooth time-dependent domain as defined in [20]. More precisely, given $d \geq 1$, $T > 0$ and a bounded open connected set $D' \subset \mathbb{R}^{1+d}$. We will refer to

$D = D' \cap \big([0,T] \times \mathbb{R}^d\big)$, as a time-dependent domain. Given D and $t \in [0,T]$, we define the time sections of D as $D_t = \{x : (t,x) \in D\}$. We assume that:

$$D_t \neq \varnothing, \text{ and that } D_t \text{ is open connected for every } t \in [0,T]. \tag{2.1}$$

In order to establish the link between generalized BSDEs and PDEs when the state process is a reflected diffusion in some time-dependent domain, we assume that the following assumption holds throughout this paper:

$$D_t \subset D_{t'}, \text{ whenever } t \leq t',\ t,t' \in [0,T]. \tag{2.2}$$

Let ∂D_t for $t \in [0,T]$ denote the boundary of D_t, the direction of reflection γ is oblique at each boundary point $x \in \partial D_t$ for $t \in [0,T]$ and we assume that γ satisfies:

$$\gamma \in \mathscr{C}_b^{1,2}(\mathbb{R}^{1+d}, B(0,1)). \tag{2.3}$$

and $\gamma(t,x) \in S(0,1)$, $\forall (t,x) \in V$ where V is an open subset such that $D_t^c \subset V$, $\forall t \in [0,T]$. Moreover, there exists a constant $\rho \in (0,1)$ such that the exterior condition holds: $\forall t \in [0,T]$, $x \in \partial D_t$,

$$\bigcup_{0 \leqslant \xi \leqslant \rho} B(x - \xi\gamma(t,x), \xi\rho) \subset D_t^c, \tag{2.4}$$

then γ points into the domain.
Finally, regarding the temporal variation of the domain, we define $d(t,x) := \inf_{y \in D_t} |y - x|$, $\forall t \in [0,T]$, $x \in \mathbb{R}^d$ that is assumed to satisfy for some $p \in (1,\infty)$, $\forall x \in \mathbb{R}^d$

$$d(.,x) \in \mathscr{W}^{1,p}([0,T],[0,\infty]), \tag{2.5}$$

where $\mathscr{W}^{1,p}([0,T],[0,\infty])$ denotes the Sobolev space of functions whose first order weak derivatives belong to $L^p([0,T])$ with Sobolev norm uniformly bounded in space and such that $\partial_t d(t,x)$ the first weak derivative in the sense of distribution is jointly measurable in (t,x). Taking into account (2.4), we can deduce that the interior cone condition is satisfied as well:

$$\bigcup_{0 \leqslant \xi \leqslant \rho} B(x + \xi\gamma(t,x), \xi\rho) \subset \overline{D}_t,\ \forall t \in [0,T],\ x \in \partial D_t. \tag{2.6}$$

Then similarly to Remark 2.1 in [20], we deduce that for all $\theta \in (0,1)$, such that $\theta^2 > 1 - \rho^2$, there exists $\delta > 0$ such that

$$\langle y - x, \gamma(t,x) \rangle \geq -\theta |y - x|,\ \forall x \in \partial D_t,\ y \in \overline{D}_t,\ t \in [0,T], \tag{2.7}$$

for x, y satisfying $|x - y| \leqslant \delta$.
Moreover, by Remark 2.2 in [20], there exists $\hat{\alpha} = 1 - 1/p \in (0,1)$ and $K \in (0,\infty)$ such that for all $s,t \in [0,T]$, $x \in \mathbb{R}^d$,

$$|d(s,x) - d(t,x)| \leq K|s - t|^{\hat{\alpha}}. \tag{2.8}$$

Then, if we consider the function $l(r) = \sup_{\substack{s,t \in [0,T] \\ |s-t| \leqslant r}} \sup_{x \in \bar{D}_s} \inf_{y \in \bar{D}_t} |s - y|$ introduced in [6], then by recalling Remark 2.4 in [20], the condition (2.8) is equivalent to $l(r) \leqslant K r^{\hat{\alpha}}$.

2.2 Reflected SDEs in time-dependent domain

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a fixed probability space on which is defined an n-dimensional Brownian motion $W=(W_t)_{0\leqslant t\leqslant T}$, where $\mathscr{F}=(\mathscr{F}_t)_{0\leq t\leq T}$ is the completed filtration of $(\sigma(W_s,0\leq s\leq t))_{t\leq T}$ with all $\mathbb{P}-$null sets of $\mathscr{F}_0$. Let $b:[0,T]\times\mathbb{R}^d\longrightarrow\mathbb{R}^d$ and $\sigma:[0,T]\times\mathbb{R}^d\longrightarrow\mathbb{R}^d\times\mathbb{R}^n$ be continuous functions. Let us now introduce the notion of reflected SDEs in time-dependent domain that we will adopt throughout our work as it has been considered in [20]. We start by giving a definition which is borrowed from the same work:

Definition 2.1 *A strong solution to the reflected SDEs in $\bar{D}$ driven by W and with coefficients b and σ, direction of reflection along γ and initial condition $(t,x)\in\bar{D}_0$ is a $\mathscr{F}_t$-adapted stochastic process X_t which satisfies $\mathbb{P}$-almost surely, whenever $t\in[0,T]$,*

$$\begin{cases} X_t = x+\int_0^t b(r,X_r)dr+\int_0^t\sigma(r,X_r)dW_r+\Lambda_t;\\ X_t\in\bar{D}_t,\quad |\Lambda|_t=\int_0^t\chi_{\{X_r\in\partial D_r\}}d|\Lambda|_r<\infty.\\ \Lambda_t=\int_0^t\gamma(r,X_r)d|\Lambda|_r. \end{cases} \tag{2.9}$$

We introduce the following assumptions:

(a) The functions b and σ are Lipschitz continuous with respect to x, i.e., there exists a positive constant C such that

$$|b(t,x)-b(t,x')|+|\sigma(t,x)-\sigma(t,x')|\leq C|x-x'|,\ \ \forall(t,x,x')\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d. \tag{2.10}$$

(b) The functions b and σ are of linear growth in (t,x), i.e., there exists a positive constant C such that

$$|b(t,x)|+|\sigma(t,x)|\leq C(1+|x|),\ \ \forall(t,x)\in[0,T]\times\mathbb{R}^d. \tag{2.11}$$

Next, we recall the following result on existence and uniqueness of solution for the reflected SDE (2.9):

Theorem 2.1 *Let D be a time-dependent domain satisfying (2.1),(2.2) and (2.5). If assumption (2.10) holds. Then, the reflected SDE (2.9) has a unique strong solution.*

The next proposition states some crucial properties of the reflected SDE (2.9) that will be needed in the study of our generalized BSDE and PDEs system:

Proposition 2.1 *For each $t\in[0,T]$, there exists a constant C such that for all $x,x'\in\overline{D}_0$,*

$$\begin{aligned} \mathbb{E}[\sup_{0\leqslant s\leqslant T}|X_s^x-X_s^{x'}|^4]&\leq C|x-x'|^4,\\ \mathbb{E}[\sup_{0\leqslant s\leqslant T}||\Lambda^x|_s-|\Lambda^{x'}|_s|^4]&\leq C|x-x'|^4. \end{aligned} \tag{2.12}$$

Moreover, for each $\mu>0$, $s\in[0,T]$, there exists $C(\mu,s)$ such that for all $x\in\overline{D}_0$,

$$\mathbb{E}(e^{\mu|\Lambda^x|_s})\leq C(\mu,s). \tag{2.13}$$

The proof of Proposition 2.1 relies on the following two lemmas. The first one establishes the existence of a class of test functions that interact with γ in a suitable way. The second lemma provides the existence of a non negative test function $\mathscr{C}^{1,2}(\bar{D},\mathbb{R})$ with a gradient aligned with γ at the boundary.

Lemma 2.1 *For any $\theta \in (0,1)$, there exist a family $\{w_\alpha\}_{\alpha>0}$ of functions $w_\alpha \in \mathscr{C}^{1,2}([0,T]\times\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R})$ and positive constants ξ, C such that for all $(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$,*

$$w_\alpha(t,x,y) \geqslant \xi\alpha|x-y|^2, \tag{2.14}$$

$$w_\alpha(t,x,y) \leqslant C\big(\frac{1}{\alpha}+\alpha|x-y|^2\big). \tag{2.15}$$

If $x\in\partial D_t$, we have:

$$\langle D_x w_\alpha(t,x,y),\gamma(t,x)\rangle \leqslant C\alpha|x-y|^2, \text{ whenever } \langle y-x,\gamma(t,x)\rangle \geq -\theta|y-x|, \tag{2.16}$$

and

$$\langle D_y w_\alpha(t,x,y),\gamma(t,x)\rangle \leqslant 0, \text{ whenever } x\in\partial D_t, \langle x-y,\gamma(t,x)\rangle \geq -\theta|y-x|. \tag{2.17}$$

If $y\in\partial D_t$,

$$\langle D_y w_\alpha(t,x,y),\gamma(t,y)\rangle \leqslant C\alpha|x-y|^2, \text{ whenever } \langle x-y,\gamma(t,y)\rangle \geq -\theta|y-x|. \tag{2.18}$$

Moreover,

$$|\partial_t w_\alpha(t,x,y)| \leqslant C\alpha|x-y|^2, \tag{2.19}$$

$$|D_y w_\alpha(t,x,y)| \leqslant C\alpha|x-y|, \tag{2.20}$$

$$|D_x w_\alpha(t,x,y)+D_y w_\alpha(t,x,y)| \leqslant C\alpha|x-y|^2. \tag{2.21}$$

Finally,

$$D_{xx} w_\alpha(t,x,y) \leqslant C\alpha\begin{pmatrix} I & -I\\ -I & I\end{pmatrix} + C\alpha|x-y|^2\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}. \tag{2.22}$$

Lemma 2.2 *There exists a nonnegative function $\alpha\in\mathscr{C}^{1,2}(\bar{D},\mathbb{R})$ such that for $x\in\partial D_t$, $t\in[0,T]$,*

$$\langle D_x\alpha(t,x),\gamma(t,x)\rangle \geqslant 1. \tag{2.23}$$

PROOF.[Proof of Theorem 2.1] The proof of (2.12) is a result of Lemma 5.1 in [20]. To prove (2.13), let $\alpha\in\mathscr{C}^{1,2}(\bar{D},\mathbb{R})$ be the positive function defined in the previous lemma, then by applying Itô formula to $\alpha(s,X_s)$ we get:

$$\begin{aligned}\alpha(t,X_t) = \alpha(0,X_0) &+ \int_0^t \partial_s\alpha(s,X_s)ds + \int_0^t \langle D_x\alpha(s,X_s),b(s,X_s)\rangle ds\\ &+ \int_0^t \langle D_x\alpha(s,X_s),\gamma(s,X_s)\rangle d|\Lambda|_s + \int_0^t \langle D_x\alpha(s,X_s),\sigma(s,X_s)dW_s\rangle\\ &+ \frac{1}{2}\int_0^t D_{xx}\alpha(s,X_s)\sigma(s,X_s)ds.\end{aligned}$$

This implies that,

$$\int_0^t \langle D_x\alpha(s,X_s),\gamma(s,X_s)\rangle d|\Lambda|_s \leqslant \alpha(t,X_t) - \int_0^t \partial_s\alpha(s,X_s)ds \\ - \int_0^t \langle D_x\alpha(s,X_s),b(s,X_s)\rangle ds - \int_0^t \langle D_x\alpha(s,X_s),\sigma(s,X_s)dW_s\rangle \\ - \frac{1}{2}\int_0^t D_{xx}\alpha(s,X_s)\sigma(s,X_s)ds.$$

Note that thanks to Lemma 2.2, we have:

$$\int_0^t \langle D_x\alpha(s,X_s),\gamma(s,X_s)\rangle d|\Lambda|_s \geqslant |\Lambda|_t. \tag{2.24}$$

Since $\alpha \in \mathscr{C}^{1,2}(\bar{D},\mathbb{R})$ and thanks to assumption (2.11) together with the compactness of $\bar{D}$, we have:

$$\alpha(t,X_t) - \int_0^t [\partial_s + \mathscr{L}]\alpha(s,X_s)ds \leqslant c(1+t),$$

where c is a positive constant depending only on α, b, σ and D. It follows that,

$$|\Lambda|_t \leqslant c(1+t) - \int_0^t \langle D_x\alpha(s,X_s),\sigma(s,X_s)dW_s\rangle. \tag{2.25}$$

Then, by denoting $M_t = -\int_0^t \langle D_x\alpha(s,X_s),\sigma(s,X_s)dW_s\rangle$ we have: $\forall \mu > 0$

$$e^{\mu M_t} = 1 - \int_0^t \mu e^{\mu M_s} dM_s - \int_0^t \mu e^{\mu M_s} d\langle M\rangle_s \\ 1 - \mu\int_0^t e^{\mu M_s}\langle D_x\alpha(s,X_s),\sigma(s,X_s)dW_s\rangle + \mu^2 e^{\mu M_s}|D_x\alpha(s,X_s)||\sigma(s,X_s)|ds$$

By the boundedness of σ, α and $D_x\alpha$ on $\bar{D}$, the process $\int_0^{\cdot} e^{\mu M_s}\langle D_x\alpha(s,X_s),\sigma(s,X_s)dW_s\rangle$ is a local martingale and we have:

$$\mathbb{E}(e^{\mu M_t}) \leq 1 + c\mu^2\int_0^t \mathbb{E}(e^{\mu M_s})ds.$$

Using Gronwall's lemma we deduce that

$$\mathbb{E}(e^{\mu M_t}) \leq e^{c\mu^2 t}, \forall t \geqslant 0.$$

Then, back to (2.25) and by taking the exponential on both hand sides, we obtain: $\forall \mu > 0$,

$$\mathbb{E}(e^{\mu|\Lambda|_t}) \leq e^{c\mu(1+t)+c\mu^2 t} \leq c_{\mu,T}, \forall t \geqslant 0.$$

Hence, for $t = T$ and $p \geqslant 1$, we can see that:

$$\mathbb{E}|\Lambda|_T^p \leq \mathbb{E}(e^{p|\Lambda|_T}) \leq c.$$

Next, let $(t,x) \in \overline{D}$ be fixed, we define $(X_s^{t,x},\Lambda_s^{t,x})_{t\leqslant s\leqslant T}$ as the unique solution of:

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r,X_r^{t,x})dr + \int_t^s \sigma(r,X_r^{t,x})dW_r + \int_t^s \gamma(r,X_r^{t,x})d|\Lambda^{t,x}|_r; \\ X_s^{t,x} \in \bar{D}_s,\ |\Lambda^{t,x}|_s = \int_0^s \chi_{\{X_r^{t,x}\in\partial D_r\}} d|\Lambda^{t,x}|_r. \end{cases} \tag{2.26}$$

2.3 Associated generalized BSDE and PDE

Let $(X_s^{t,x}, \Lambda_s^{t,x})_{t\leqslant s\leqslant T}$ be the unique solution of the reflected SDE (2.26). Then, consider the following functions:

$$\begin{aligned} f &: (t,x,y,z) \in [0,T]\times\mathbb{R}^{d+m+k} \longmapsto f(t,x,y,z)\in\mathbb{R}^m,\\ \psi &: (t,x,y) \in [0,T]\times\mathbb{R}^{d+m} \longmapsto \psi(t,x,y)\in\mathbb{R}^m,\\ h &: x\in\mathbb{R}^d \longmapsto h(x)\in\mathbb{R}^m. \end{aligned}$$

We should emphasize that each component f_i of the function f may depend only on the i-th row of Z, then assume that:

(H_0) The function h is continuous.

(H_1) (i) $(t,x)\longmapsto f(t,x,\vec{y},z)$ and $(t,x)\longmapsto \psi(t,x,\vec{y})$ are uniformly continuous with respect to $(\vec{y},z)$ and $\vec{y}$ respectively.

(ii) f and ψ are lipschitz continuous with respect to $(\vec{y},z)$ and $\vec{y}$ respectively.

(iii) $\exists\beta<0$ such that $\langle y-\bar{y}, \psi(t,x,y)-\psi(t,x,\bar{y})\rangle \leq \beta \mid y-\bar{y}\mid^2$.

Then, thanks to [22], the following generalized BSDE: $\forall t\leqslant s\leqslant T$,

$$\begin{aligned} Y_s^{t,x} = h(X_T^{t,x}) &+ \int_s^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr + \int_s^T \psi(r,X_r^{t,x},Y_r^{t,x})d|\Lambda^{t,x}|_r \\ &- \int_s^T Z_r^{t,x}dW_r, \end{aligned} \tag{2.27}$$

has a unique solution. Moreover, taking into account Lemma (2.1), the following estimates hold:

$$\mathbb{E}\left[\sup_{t\leqslant s\leqslant T}\mid Y_s^{t,x}\mid^2 + \int_t^T \mid Y_r^{t,x}\mid^2 d|\Lambda^{t,x}|_r + \int_t^T ||Z_r^{t,x}||^2 d_r\right] < \infty. \tag{2.28}$$

Now, we define:

$$u(t,x) := Y_t^{t,x}. \tag{2.29}$$

This quantity is deterministic since $(Y_s^{t,x})_{t\leqslant s\leqslant T}$ is measurable with respect to the σ-algebra $\sigma(W_r - W_t, t\leqslant r\leqslant s)$. Next, let D be the time-dependent domain defined in paragraph 2.1. and set $\tilde{D} = \overline{D}' \cap \left([0,T]\times\mathbb{R}^d\right)$. It holds that the function u is continuous on $\tilde{D}$. Then it is bounded on $\tilde{D}$.

Remark 2.1 *The continuity of u is proven using similar arguments to the case of standard reflected SDEs. Indeed, under the assumptions that we have made on the domain D and the reflection γ, it turns out that the pair of processes $(X^{t,x},\Lambda^{t,x})$ has the similar properties of the solutions of SDEs with reflection in bounded connected closed time-independent domains (we refer the reader to [21] for further details).*

In order to study the associated PDE system, we introduce some spaces following the notations of [20]:

$$\begin{aligned} D^{\mathrm{o}} &= D' \cap \big((0,T)\times\mathbb{R}^d\big), \\ \partial D &= \big(\overline{D}' \setminus D'\big) \cap \big((0,T)\times\mathbb{R}^d\big). \end{aligned} \tag{2.30}$$

Then, we consider the following system of PDEs: $\forall i = 1,...,m$,

$$\begin{cases} \partial_t u^i(t,x) + \mathscr{L}u^i(t,x) + f_i(t,x,u(t,x),\sigma^\top(t,x)D_x u^i(t,x)) = 0, \ \ (t,x)\in D^{\mathrm{o}}; \\ \dfrac{\partial u^i}{\partial \gamma}(t,x) + \psi_i(t,x,u(t,x)) = 0, \ \ (t,x)\in \partial D; \\ u(T,x) = h(x), \ \ x\in \overline{D}_T, \end{cases} \tag{2.31}$$

where the operator $\mathscr{L}$ is defined by $\mathscr{L} = \frac{1}{2}Tr(\sigma\sigma^\top)D^2_{xx}(.) + b^\top D_x(.)$ and at a point $(t,x)\in \partial D$ we set $\frac{\partial}{\partial\gamma} = \langle \gamma(t,x), D_x(.)\rangle$.

Definition 2.2 (**i**) *A function $u\in USC(\tilde{D})$ is a viscosity subsolution of (2.31) if for any $i\in\{1,...,m\}$ and $\varphi\in\mathscr{C}^{1,2}([0,T]\times\mathbb{R}^d)$, whenever $(t,x)\in\tilde{D}$ is a local maximum of $u^i-\varphi$, then: $\forall i = 1,...,m$*

$$\begin{cases} -\partial_t\varphi(t,x) - \mathscr{L}\varphi(t,x) - f_i(t,x,u(t,x),\sigma^\top(t,x)D_x\varphi(t,x)) \le 0, \ (t,x)\in D^{\mathrm{o}}, \\ \min\{-\partial_t\varphi(t,x) - \mathscr{L}\varphi(t,x) - f_i(t,x,u(t,x),\sigma^\top(t,x)D_x\varphi(t,x)); \\ \qquad\qquad -\langle\gamma(t,x),D_x\varphi(t,x)\rangle - \psi_i(t,x,u(t,x))\} \le 0, \ (t,x)\in\partial D, \\ u_i(T,x)\le h_i(x), x\in\overline{D}_T. \end{cases} \tag{2.32}$$

(**ii**) *A function $u\in LSC(\tilde{D})$ is a viscosity supersolution of (2.31) if for any $i\in\{1,...,m\}$ and for any $\varphi\in\mathscr{C}^{1,2}([0,T]\times\mathbb{R}^d)$ whenever $(t,x)\in\tilde{D}$ is a local minimum of $u^i-\varphi$, then: $\forall i = 1,...,m$*

$$\begin{cases} -\partial_t\varphi(t,x) - \mathscr{L}\varphi(t,x) - f_i(t,x,u(t,x),\sigma^\top(t,x)D_x\varphi(t,x)) \ge 0, \ (t,x)\in D^{\mathrm{o}}, \\ \max\{-\partial_t\varphi(t,x) - \mathscr{L}\varphi(t,x) - f_i(t,x,u(t,x),\sigma^\top(t,x)D_x\varphi(t,x)); \\ \qquad\qquad -\langle\gamma(t,x),D_x\varphi(t,x)\rangle - \psi_i(t,x,u(t,x))\} \ge 0, \ (t,x)\in\partial D, \\ u_i(T,x)\ge h_i(x), x\in\overline{D}_T. \end{cases} \tag{2.33}$$

(**iii**) *Finally, u is a viscosity solution if and only if it is a viscosity supersolution and a subsolution of the system of PDEs (2.31).*

Theorem 2.2 *The function defined by (2.29) is a viscosity solution of the system of PDEs (2.31) in the sense of Definition 2.2.*

PROOF. Step: 1 In the first step, we show that the function u is a viscosity subsolution of (2.31). Let $(t_0,x_0)\in\tilde{D}$ and $\varphi\in\mathscr{C}^{1,2}([0,T]\times\mathbb{R}^d)$ such that $\varphi(t_0,x_0) = u^i(t_0,x_0)$ and $\varphi(t,x)\geqslant u^i(t,x)$ for any $(t,x)\in\tilde{D}$.
1^{st} **case**: We consider the case $(t_0,x_0)\in D^{\mathrm{o}}$ and such that the following holds:

$$-\partial_t\varphi(t_0,x_0) - \mathscr{L}\varphi(t_0,x_0) - f_i(t_0,x_0,u(t_0,x_0),\sigma^\top(t_0,x_0)D_x\varphi(t_0,x_0)) > 0.$$

We then will find a contradiction. Note that by continuity of the data we can find $\varepsilon > 0$ and $\alpha > 0$ such that for each $(t,x) \in \{t : |t-t_0| \leqslant \alpha\} \times \{x : |x-x_0| \leqslant \alpha\} \subset D^{\mathrm{o}}$, we have:

$$-\partial_t \varphi(t,x) - \mathscr{L}\varphi(t,x) - f_i(t,x,u(t,x),\sigma^\top(t,x)D_x\varphi(t,x)) \geqslant \varepsilon. \tag{2.34}$$

Define

$$\tau = \inf\{s \geqslant t_0 : X_s^{t_0,x_0} \notin D_{t_0}\} \wedge \inf\{s \geqslant t_0 : |X_s^{t_0,x_0} - x_0| > \alpha\} \wedge (t_0+\alpha),$$

then thanks to assumption (2.2) we have $D_{t_0} \subset D_s \subset D_{t_0+\alpha}$ which implies that $d|\Lambda^{t_0,x_0}|_s = 0$ over $[t_0,\tau]$. Then, for all $t_0 \leqslant s \leqslant \tau$, we have:

$$Y_s^{t_0,x_0,i} = Y_\tau^{t_0,x_0,i} + \int_s^T f_i(r,X_r^{t_0,x_0},Y_r^{t_0,x_0},Z_r^{t_0,x_0,i})dr - \int_s^T Z_r^{t_0,x_0,i}dW_r.$$

On the other hand, by applying Itô's to $\varphi(s,X_s^{t_0,x_0})$, we get:

$$\begin{aligned}\varphi(\tau,X_\tau^{t_0,x_0}) = \varphi(s,X_s^{t_0,x_0}) &+ \int_s^\tau \partial_r\varphi(r,X_r^{t_0,x_0})dr + \int_s^\tau \mathscr{L}\varphi(r,X_r^{t_0,x_0})dr \\ &- \int_s^\tau \sigma^\top(r,X_r^{t_0,x_0})D_x\varphi(r,X_r^{t_0,x_0})dW_r.\end{aligned}$$

By inequality (2.34), we deduce that,

$$-[\partial_r\varphi + \mathscr{L}_r\varphi](r,X_r^{t_0,x_0}) - f_i(r,X_r^{t_0,x_0},u(r,X_r^{t_0,x_0}),(D_x\varphi\sigma)(r,X_r^{t_0,x_0})) \geqslant \varepsilon.$$

Note that $\varphi(\tau,X_\tau^{t_0,x_0}) \geqslant u^i(\tau,X_\tau^{t_0,x_0}) = Y_\tau^{t_0,x_0,i}$.Then, since $\varphi(t_0,x_0) > \varphi(t_0,X_{t_0}^{t_0,x_0}) - \varepsilon(\tau-t_0)$ and

$$\begin{aligned}\varphi(s,X_s^{t_0,x_0}) - \int_s^\tau \varepsilon dr = \varphi(\tau,X_\tau^{t_0,x_0}) - \int_s^\tau \varepsilon dr &- \int_s^\tau \{[\partial_r\varphi + \mathscr{L}_r\varphi](r,X_r^{t_0,x_0}) + \varepsilon\}dr \\ &- \int_s^\tau \sigma^\top(r,X_r^{t_0,x_0})D_x\varphi(r,X_r^{t_0,x_0})dW_r,\end{aligned}$$

we deduce from comparison theorem 1.4 in [22] that:

$$\varphi(t_0,x_0) - \varepsilon(\tau-t_0) \geqslant u^i(t_0,x_0) = u^i(t_0,x_0).$$

This means that $\varphi(t_0,x_0) > u^i(t_0,x_0)$ which is a contradiction.
2^{sd} **case**: If $(t_0,x_0) \in \partial D$ and if we assume that the following holds,

$$\begin{aligned}\min\{-\partial_t\varphi(t_0,x_0) - \mathscr{L}\varphi(t_0,x_0) - f_i(t_0,x_0,u(t_0,x_0),\sigma^\top(t_0,x_0)D_x\varphi(t_0,x_0)); \\ - \langle\gamma(t_0,x_0),D_x\varphi(t_0,x_0)\rangle - \psi_i(t_0,x_0,u(t_0,x_0))\} > 0.\end{aligned}$$

Then on a neighborhood of (t_0,x_0), there exists $\varepsilon > 0$ and $\alpha > 0$ such that for each $(t,x) \in \{t : |t-t_0| \leqslant \alpha\} \times \{x : |x-x_0| \leqslant \alpha\} \subset \partial D$, we have:

$$\begin{aligned}\min\{-\partial_t\varphi(t,x) - \mathscr{L}\varphi(t,x) - f_i(t,x,u(t,x),\sigma^\top(t,x)D_x\varphi(t,x)); \\ - \langle\gamma(t,x),D_x\varphi(t,x)\rangle - \psi_i(t,x,u(t,x))\} \geqslant \varepsilon.\end{aligned} \tag{2.35}$$

Next, we set:

$$\tau = \inf\{s \geqslant t_0 : |X_s^{t_0,x_0} - x_0| > \alpha\} \wedge (t_0+\alpha).$$

Then, for all $t_0 \leqslant s \leqslant \tau$, we have:

$$Y_s^{t_0,x_0,i} = Y_\tau^{t_0,x_0,i} + \int_s^T f_i(r, X_r^{t_0,x_0}, Y_r^{t_0,x_0}, Z_r^{t_0,x_0,i})dr + \int_s^T \psi_i(r, X_r^{t_0,x_0}, Y_r^{t_0,x_0})d|\Lambda^{t_0,x_0}|_r - \int_s^T Z_r^{t_0,x_0,i} dW_r.$$

Again, by applying Itô's for $\varphi(s, X_s^{t_0,x_0})$ we get:

$$\varphi(s, X_s^{t_0,x_0}) = \varphi(\tau, X_\tau^{t_0,x_0}) - \int_s^\tau \partial_r \varphi(r, X_r^{t_0,x_0})dr - \int_s^\tau \mathscr{L}\varphi(r, X_r^{t_0,x_0})dr + \int_s^\tau \langle \gamma(r, X_r^{t_0,x_0}), D_x\varphi(r, X_r^{t_0,x_0})\rangle d|\Lambda^{t_0,x_0}|_r - \int_s^\tau \sigma^\top(r, X_r^{t_0,x_0}) D_x\varphi(r, X_r^{t_0,x_0}) dW_r.$$

It follows from (2.35) that for any $t_0 \leqslant s \leqslant \tau$,

$$\min\{-[\partial_r\varphi + \mathscr{L}_r\varphi](r, X_r^{t_0,x_0}) - f_i(r, X_r^{t_0,x_0}, u(r, X_r^{t_0,x_0}), (D_x\varphi\sigma)(r, X_r^{t_0,x_0})); - \langle \gamma(r, X_r^{t_0,x_0}), D_x\varphi(r, X_r^{t_0,x_0})\rangle - \psi_i(r, X_r^{t_0,x_0}, Y_r^{t_0,x_0})\} \geqslant \varepsilon.$$

Similarly, as in the first case it suffices to use the comparison for GBSDE [22] to obtain:

$$\varphi(t_0,x_0) > \varphi(t_0,x_0) - \varepsilon(\tau - t_0) \geqslant u^i(t_0, X_{t_0}^{t_0,x_0}) = u^i(t_0,x_0),$$

which is contradictory.
Step: 2 By the same reasoning, we can show that u is a viscosity supersolution of the PDEs system (2.31).

3 System of reflected GBSDEs with oblique reflection

In this section we study a system of reflected GBSDEs with interconnected obstacles. We establish existence and uniqueness of the solution. Our main tool is the relation between reflected generalized BSDEs and optimal m-states switching problem which enables us to write the solution as its value function. A fact that we will often use throughout this paper. Let us introduce the following spaces:

$$\mathscr{H}^2 = \{(\psi_t)_{0\leqslant t\leqslant T}\ \mathscr{F}_t\text{-progressively measurable}/\mathbb{E}[\int_0^T |\psi_t|^2\, dt] < \infty\};$$
$$\mathscr{S}^2 = \{(\psi_t)_{0\leqslant t\leqslant T}\ \mathscr{F}_t\text{-progressively measurable}/\mathbb{E}[\sup_{0\leq t\leq T} |\psi_t|^2] < \infty\};$$
$$\mathscr{A}^2 = \{(K_t)_{0\leqslant t\leqslant T}\ \mathscr{F}_t\text{-adapted continuous increasing}/K_0 = 0, \mathbb{E}[K_T^2] < \infty\}.$$

We will work within a Markovian setting, and we assume that randomness stems from $(X_s^{t,x}, \Lambda_s^{t,x})_{t\leqslant s\leqslant T}$ the solution of the reflected SDE (2.26). Let us introduce the following assumptions: $\forall i \in J := \{1, ..., m\}$,

(H_0) $h_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ is continuous such that

$$h_i(x) \geq \max_{j\neq i}(h_j(x) - g_{ij}(T,x)),\ \forall x \in \mathbb{R}^d. \tag{3.1}$$

(H_1) Let $f_i : [0,T] \times \mathbb{R}^{d\times m\times k} \longrightarrow \mathbb{R}^{d\times m\times k}$ and $\psi_i : [0,T] \times \mathbb{R}^{d\times m} \longrightarrow \mathbb{R}$ be such that

(i) $(t,x) \longrightarrow f_i(t,x,\vec{y},z)$ and $(t,x) \longrightarrow \psi_i(t,x,\vec{y})$ are continuous uniformly w.r.t. $(\vec{y},z)$ and $\vec{y}$ respectively;

(ii) f_i and ψ_i are Lipschitz continuous with respect to $(\vec{y},z)$ and $\vec{y}$ respectively;

(H_2) For any $j \in \{1,...,m\}$, we have:

(i) $g_{ij} : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is continuous, non-negative such that $g_{ii} = 0$;

(ii) g_{ij} satisfies the non-free loop property, i.e.,$\forall (t,x) \in [0,T] \times \mathbb{R}^d$, $\forall i_1,...,i_k$ such that $i_1 \neq i_2$, $i_1 = i_k$ and card$\{i_1,...,i_k\} = k-1$, we have:

$$g_{i_1 i_2}(t,x) + ... + g_{i_{k-1} i_k}(t,x) > 0. \tag{3.2}$$

The system of interest is of the following type: $\forall t \leqslant s \leqslant T$, $i = 1,...,m$

$$\begin{cases} Y^{i,t,x} \in \mathscr{S}^2, K^{i,t,x} \in \mathscr{A}^2, Z^{i,t,x} \in \mathscr{H}^{2,d}; \\ Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T f_i(r,X_r^{t,x},\vec{Y}_r^{t,x},Z_r^{i,t,x})dr + \int_s^T \psi_i(r,X_r^{t,x},\vec{Y}_r^{t,x})d|\Lambda^{t,x}|_r \\ \qquad\qquad + K_T^{i,t,x} - K_s^{i,t,x} - \int_s^T Z_r^{i,t,x} dW_r, \\ Y_s^{i,t,x} \geqslant \max_{j\neq i}(Y_s^{j,t,x} - g_{ij}(s,X_s^{t,x})), \\ \int_0^T \{Y_s^{i,t,x} - \max_{j\neq i}(Y_s^{j,t,x} - g_{ij}(s,X_s^{t,x}))\} dK_s^{i,t,x} = 0. \end{cases} \tag{3.3}$$

Theorem 3.1 *Assume that the data $(h_i)_{i\in J}$, $(f_i)_{i\in J}, (\psi_i)_{i\in J}$ and $(g_{ij})_{(i,j)\in J^2}$ satisfy the assumptions (H_0), (H_1) and (H_2). There exists a unique solution of the system of generalized reflected BSDEs (3.3).*

PROOF. The proof consists in several steps.
Step 1: Let $\mu > 0$ and $\lambda > 0$ be fixed and let $\mathscr{M}^2_{\mu,\lambda}(|\Lambda^{t,x}|)$ denote the set of progressively measurable processes $(\rho_s)_{t\leqslant s\leqslant T}$ such that the following norm is finite:

$$\|\rho\|_{\mathscr{M}^2_{\mu,\lambda}(|\Lambda^{t,x}|)} := \mathbb{E}\left(\int_t^T e^{\mu s+\lambda|\Lambda^{t,x}|_s} \mid \rho_s \mid^2 ds + \int_t^T e^{\mu s+|\Lambda^{t,x}|_s} \mid \rho_s \mid^2 d|\Lambda^{t,x}|_s\right). \tag{3.4}$$

Let $\overrightarrow{u}$ be fixed in $(\mathscr{S}^2)^m$. It follows from Proposition 2.1 that $\vec{u}^i \in \mathscr{M}^2_{\mu,\lambda}(|\Lambda^{t,x}|)$, for every $i = 1,...,m$. Then, we consider the following system: $\forall t \leqslant s \leqslant T$, $\forall i = 1,...,m$,

$$\begin{cases} Y^{u,i} \in \mathscr{S}^2, \quad Z^{u,i} \in \mathscr{H}^{2,d}, \quad K^{u,i} \in \mathscr{A}^2; \\ Y_s^{u,i} = h_i(X_T^{t,x}) + \int_s^T f_i(r,X_r^{t,x},\vec{u}_r,Z_r^{u,i})dr + \int_s^T \psi_i(r,X_r^{t,x},\vec{u}_r)d|\Lambda^{t,x}|_r \\ \qquad\qquad + K_T^{u,i} - K_s^{u,i} - \int_s^T Z_s^{u,i} dW_r, \\ Y_s^{u,i} \geq \max_{j\neq i}(Y_s^{u,j} - g_{ij}(s,X_s^{t,x})), \\ \int_0^T [Y_s^{u,i} - \max_{j\neq i}(Y_s^{u,j} - g_{ij}(s,X_s^{t,x}))] dK_s^{u,i} = 0. \end{cases} \tag{3.5}$$

Note that the data in the above system does no longer depend on the solution. Then under assumptions (H_0), $(H_1)(ii)-(iii)$ and (H_2), and since $\vec{u} \in (\mathscr{M}^2_{\mu,\lambda}(|\Lambda^{t,x}|))^m$ we deduce from Theorem 2.1 in [1] that there exists a triple of processes (Y^u, K^u, Z^u) which is a solution of the system (3.5). Now, let us introduce the following mapping:

$$\Phi(\vec{u}) = (Y^{u,i})_{i=1,\dots,m},$$

Our goal is to show that Φ is a strict contraction on the space $\mathscr{M}^2_{\mu,\lambda}(|\Lambda^{t,x}|)$ equipped with the norm $\|.\|_{\mathscr{M}^2_{\mu,\lambda}(|\Lambda^{t,x}|)}$. The presence of the obstacles does not allow to proceed in a classical way. To handle this, we represent Y^u as the value function of our switching problem. This is related to the solution of a specific standard generalized BSDE which will be given in the next step.

Step 2: Switching problem

First, let us give some notations and definitions related to our optimal switching problem. To begin with, we recall that a switching control α is a pair of subsequences $(\tau_n, \theta_n)_{n\geq 0}$, where τ_n are stopping times such that $\tau_n \leq \tau_{n+1}$ and θ_n is a random variable with values in $\{1,\dots,m\}$. If $\mathbb{P}[\tau_n < T, \forall n \geq 0] = 0$, we say that α is admissible.

Let $i \in \{1,\dots,m\}$ and $s \in [t,T]$, we define the following class of admissible switching controls:

$$\mathscr{D}^i_s = \left\{\alpha = ((\tau_n)_{n\geq 0}, (\theta_n)_{n\geq 0}) \in \mathscr{D},\ \theta_0 = i,\ \tau_0 = 0, \tau_1 \geq s \text{ and } \mathbb{E}(G^\alpha_T)^2 < \infty\right\},$$

where $\mathscr{D}$ denotes the set of admissible switching controls and G^α_s is the cumulative switching costs up to time s, expressed by:

$$G^\alpha_s = \sum_{n\geq 1} g_{\theta_{n-1},\theta_n}(\tau_n)\chi_{[\tau_n \leq s]},\ s < T \text{ and } G^\alpha_T = \lim_{s\to T} G^\alpha_s = \sum_{n\geq 1} g_{\theta_{n-1},\theta_n}(\tau_n)\chi_{[\tau_n < T]}.$$

Let $\alpha = ((\tau_n)_{n\geq 0}, (\theta_n)_{n\geq 0}) \in \mathscr{D}^i_t$. Then, we introduce $h_\alpha(x)$, $f_\alpha(s, X^{t,x}_s)$ and $\psi_\alpha(s, X^{t,x}_s)$ that stand for the reward received at time T, the running reward received on D_s and the additional reward once the boundary ∂D_s is reached under the strategy α.

$$\begin{aligned} h_\alpha(x) &= \sum_{n\geq 0} \xi^{\theta_n}\chi_{[\tau_n \leq T < \tau_{n+1}[}, \\ f_\alpha(s,x,\overrightarrow{y},z) &= \sum_{n\geq 0} f_{\theta_n}(s,x,\overrightarrow{y},z)\chi_{[\tau_n \leq s < \tau_{n+1}[}, \\ \psi_\alpha(s,x,\overrightarrow{y}) &= \sum_{n\geq 0} \psi_{\theta_n}(s,\overrightarrow{y})\chi_{[\tau_n \leq s < \tau_{n+1}[}. \end{aligned} \tag{3.6}$$

Next, let us introduce the following switching equation:

$$\begin{cases} P^\alpha \text{ càdlàg}, \mathbb{E}(\sup\limits_{t\leq s\leq T} |P^\alpha_s|^2) < \infty \text{ and } Q^\alpha \in \mathscr{H}^{2,k}; \\ P^\alpha_s = h_\alpha(X^{t,x}_T) + \int_s^T f_\alpha(r, X^{t,x}_r, \vec{u}_r, Q^\alpha_r)dr + \int_s^T \psi_\alpha(r, X^{t,x}_r, \vec{u}_r)d|\Lambda^{t,x}|_r \\ \qquad\qquad - \int_t^T Q^\alpha_r dW_r - (G^\alpha_T - G^\alpha_s),\ t \leqslant s \leqslant T. \end{cases} \tag{3.7}$$

By setting $\overline{P}^\alpha_t := P^\alpha_t - G^\alpha_t$, we see that the equation (3.7) is a standard Generalized BSDE. In fact, the terminal value satisfies $\mathbb{E}[(h^\alpha(X^{t,x}_T) - G^\alpha_T)^2] < \infty$ and G^α is $\{\mathscr{F}_s\}-$ adapted, then there exists a unique solution (P^α, Q^α) for (3.7) thanks to [22].

Our aim is to get a representation of $Y^{u,i}$ in terms of P^α. To this end, we recall system (3.5) and (3.6) to write the equation of $Y^{u,i}$ between s and τ_1:

$$\begin{aligned}
Y_s^{u,i} =& Y_{\tau_1}^{u,i} + \int_s^{\tau_1} f_i(r, X_r^{t,x}, \vec{u}_r, Z_r^{u,i})ds + \int_s^{\tau_1} \psi_i(r, X_r^{t,x}, \vec{u}_r)d|\Lambda^{t,x}|_r \\
&+ K_{\tau_1}^{u,i} - K_s^{u,i} - \int_s^{\tau_1} Z_r^{u,i} dW_s \\
=& Y_{\tau_1}^{u,i}\chi_{[\tau_1=T]} + Y_{\tau_1}^{u,i}\chi_{[\tau_1<T]} + \int_s^{\tau_1} f_\alpha(r, \vec{u}_r, Z_r^\alpha)dr + \int_s^{\tau_1} \psi_\alpha(r, X_r^{t,x}, \vec{u}_r)d|\Lambda^{t,x}|_r \\
&+ K_{\tau_1}^{u,i} - K_s^{u,i} - \int_s^{\tau_1} Z_r^\alpha dW_s,
\end{aligned}$$

where $Z_r^\alpha = Z_r^{u,\theta_0}\chi_{[\tau_0\le r<\tau_1[} = Z_r^{u,i}$ on $[s,\tau_1]$. Besides, observe that

$$Y_{\tau_1}^{u,i} \ge \max_{j\neq i}(Y_{\tau_1}^{u,j} - g_{ij}(\tau_1, X_{\tau_1}^{t,x})) \ge Y_{\tau_1}^{u,\theta_1} - g_{i\theta_1}(\tau_1, X_{\tau_1}^{t,x}).$$

Thus, $\forall s \in [t,\tau_1]$, $\forall i = 1,...,m$,

$$\begin{aligned}
Y_s^{u,i} =& h^{i=\theta_0}(X_T^{t,x})\chi_{[\tau_1=T]} + \int_s^{\tau_1} f_\alpha(r, X_r^{t,x}, \overrightarrow{u_r}, Z_r^\alpha)dr + \int_s^{\tau_1} \psi_\alpha(r, , X_r^{t,x}, \vec{u}_r)d|\Lambda^{t,x}|_r \\
&+ (Y_{\tau_1}^{u,\theta_1} - g_{i,\theta_1}(\tau_1, X_{\tau_1}^{t,x}))\chi_{[\tau_1<T]} + K_{\tau_1}^{u,i} - K_s^{u,i} - \int_s^{\tau_1} Z_s^\alpha dW_r; \\
\ge& h^{\theta_0}(X_T^{t,x})\chi_{[\tau_1=T]} + Y_{\tau_2}^{u,\theta_1}\chi_{[\tau_1<T]} + \int_s^{\tau_2} f_\alpha(r, X_r^{t,x}, \vec{u}_r, Z_r^\alpha)dr \\
&+ \int_s^{\tau_2} \psi_\alpha(r, , X_r^{t,x}, \vec{u}_r^\alpha)d|\Lambda^{t,x}|_r - g_{i,\theta_1}(\tau_1, X_{\tau_1}^{t,x})\chi_{[\tau_1<T]} + (K_{\tau_2}^{u,\theta_1} - K_{\tau_1}^{u,\theta_1}) \\
&+ (K_{\tau_1}^{u,i} - K_t^{u,i}) - \int_s^{\tau_2} Z_r^\alpha dW_r.
\end{aligned}$$

The last inequality follows from the definition of Y^{u,θ_1} between $[\tau_1,\tau_2]$. By (3.6) and by repeating the same procedure as many times as necessary, we obtain:

$$\begin{aligned}
Y_s^{u,i} \ge& h^\alpha(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, \vec{u}_r, Z_r^\alpha)dr + \int_s^T \psi_\alpha(r, X_r^{t,x}, \vec{u}_r)d|\Lambda^{t,x}|_r \\
&- G_T^\alpha + \tilde{K}_T^\alpha - \int_s^T Z_r^\alpha dW_r.
\end{aligned}$$

By letting $\tilde{K}_T^\alpha := K_{\tau_1}^{u,i} - K_s^{u,i} + \sum_{n\ge1}(K_{\tau_{n+1}}^{u,\theta_n} - K_{\tau_n}^{u,\theta_n})$ which is non negative, we deduce that

$$\begin{aligned}
Y_s^{u,i} \ge& h^\alpha(X_T^{t,x}) + \int_s^T f_\alpha(r, X_r^{t,x}, \vec{u}_r, Z_r^\alpha)dr + \int_s^T \psi_\alpha(r, X_r^{t,x}, \vec{u}_r)d|\Lambda^{t,x}|_r \\
&- G_T^\alpha - \int_s^T Z_r^\alpha dW_r.
\end{aligned}$$

Hence, if (P^α, Q^α) is the solution of the GBSDE (3.7), we have:

$$\begin{aligned}
Y_s^{u,i} - (P_s^\alpha - G_s^\alpha) \ge& \int_s^T \left[f_\alpha(r, X_r^{t,x}, \vec{u}_r, Z_r^\alpha) - f_\alpha(r, X_r^{t,x}, \vec{u}_r, Q_r^\alpha)\right]dr \\
&- \int_s^T (Z_r^\alpha - Q_r^\alpha)dW_r.
\end{aligned}$$

Under assumptions $(H_1)(ii)$ and (iii) and by Girsanov's Theorem there exists a probability measure $\tilde{\mathbb{P}}$ such that $d\tilde{\mathbb{P}} = \varepsilon(M)_T d\mathbb{P}$, where $\varepsilon(M)_s = e^{M_s - \frac{1}{2}\langle M\rangle_s}$ and M_s is defined by: $\forall t \leqslant s \leqslant T$,

$$M_s = \int_0^s \chi_{\{Z_r^\alpha \neq Q_r^\alpha\}}[f_\alpha(r, X_r^{t,x}, \vec{u}_r, Z_r^\alpha) - f_\alpha(r, X_r^{t,x}, \vec{u}_r, Q_r^\alpha)](Z_r^\alpha - Q_r^\alpha)^{-1} dW_r.$$

This implies that the process defined by:$\forall t \leqslant s \leqslant T$,

$$\tilde{W}_s := W_s - \int_0^s \chi_{\{Z_r^\alpha \neq Q_r^\alpha\}}[f_\alpha(r, X_r^{t,x}, \vec{u}_r, Z_r^\alpha) - f_\alpha(r, X_r^{t,x}, \vec{u}_r, Q_r^\alpha)](Z_r^\alpha - Q_r^\alpha)^{-1} dr,$$

is a Brownian motion under $\tilde{\mathbb{P}}$. Therefore, the following inequality holds:

$$Y_s^{u,i} - (P_s^\alpha - G_s^\alpha) \geq \int_s^T (Z_r^\alpha - Q_r^\alpha) d\tilde{W}_r.$$

Then by taking the conditional expectation, we obtain: $\forall t \leqslant s \leqslant T$,

$$Y_s^{u,i} \geq P_s^\alpha - G_s^\alpha, \, \mathbb{P}\text{- a.s. } \forall \alpha \in \mathscr{D}_s^i.$$

Finally, let $\alpha^* = (\tau_n^*, \theta_n^*)_{n\geq 0}$ be the switching strategy defined as follows:

$$\begin{aligned}
&\tau_0^* = 0, \, \theta_0^* = i, \\
&\tau_{n+1}^* = \inf\{s \geq \tau_n^*, Y_s^{u,\theta_n^*} = \max_{j \neq \theta_n^*}(Y_s^{u,j} - g_{\theta_n^*,j}(s, X_s^{t,x}))\} \wedge T, \\
&\theta_{n+1}^* = \operatorname*{argmax}_{j \neq \theta_n^*}(Y_{\tau_{n+1}^*}^{u,j} - g_{\theta_n^*,j}(\tau_{n+1}^*, X_{\tau_{n+1}^*}^{t,x})).
\end{aligned} \tag{3.8}$$

We can show that α^* by showing that $\mathbb{P}[\tau_n^* < T, \forall n \geq 0] = 0$ and $\mathbb{E}(G_T^{\alpha^*})^2 < \infty$. This follows easily from the non-free loop property (consult [1]). Moreover, the following representation holds true

$$Y_s^{u,i} = \operatorname*{esssup}_{\alpha \in \mathscr{D}_s^i}(P_s^\alpha - G_s^\alpha) = P_s^{\alpha^*} - G_s^{\alpha^*}. \tag{3.9}$$

Therefore, the solution of system of reflected GBSDEs (3.5) is unique.
Step 3: Using the last representation, we give estimates for $Y^{u,i}$ in terms of $P_s^\alpha - G_s^\alpha$. In fact, for $\vec{u}$ and $\vec{v}$ fixed in $\mathscr{M}_{\mu,\lambda}^2(|\Lambda^{t,x}|)$ we set:

$$\begin{aligned}
F_i(s, X_s^{t,x}, z) &= f_i(s, X_s^{t,x}, \vec{u}_s, z) \vee f_i(s, X_s^{t,x}, \vec{v}_s, z), \\
\Psi_i(s, X_s^{t,x}) &= \psi_i(s, X_s^{t,x}, u_s^i) \vee \psi_i(s, X_s^{t,x}, v_s^i).
\end{aligned}$$

Next, we introduce the following system of generalized BSDEs with oblique reflection: $\forall t \leqslant s \leqslant T$, $\forall i = 1, ..., m$,

$$\begin{cases}
\tilde{Y}_s^i = h_i(X_T^{t,x}) + \int_s^T F_i(r, X_r^{t,x}, \tilde{Z}_r^i) dr + \int_s^T \Psi_i(r, X_r^{t,x}) d|\Lambda^{t,x}|_r + \tilde{K}_T^i - \tilde{K}_s^i \\
\qquad\qquad - \int_s^T \tilde{Z}_s^i dW_r, \\
\tilde{Y}_s^i \geq \max_{j \neq i}(\tilde{Y}_s^j - g_{ij}(s, X_s^{t,x})), \\
\int_0^T [\tilde{Y}_s^i - \max_{j \neq i}(\tilde{Y}_s^j - g_{ij}(s, X_s^{t,x}))] d\tilde{K}_s^i = 0.
\end{cases} \tag{3.10}$$

Thanks to the previous step, the system (3.10) has unique solution. Then, consider the associated switching equation with the same data, namely:

$$\begin{cases} \tilde{P}^{\alpha} \text{ càdlàg}, \mathbb{E}(\sup\limits_{t\leq s\leq T} \mid \tilde{P}^{\alpha}_s \mid^2) < \infty \text{ and } \tilde{Q}^{\alpha} \in \mathscr{H}^{2,k}; \\ \tilde{P}^{\alpha}_s = h_{\alpha}(X^{t,x}_T) + \int_s^T F_{\alpha}(r, X^{t,x}_r, \tilde{Q}^{\alpha}_r)dr + \int_s^T \Psi_{\alpha}(r, X^{t,x}_r)d|\Lambda^{t,x}|_r - \int_t^T \tilde{Q}^{\alpha}_r dW_r \\ \qquad\qquad -(G^{\alpha}_T - G^{\alpha}_s),\ t \leqslant s \leqslant T. \end{cases} \tag{3.11}$$

By relying on (3.9), we have similar representation for $\tilde{Y}$: $\forall t \leqslant s \leqslant T$,

$$\tilde{Y}^i_s = \operatorname*{esssup}_{\alpha\in\mathscr{D}^i_s}(\tilde{P}^{\alpha}_s - G^{\alpha}_t) = \tilde{P}^{\tilde{\alpha}^*}_t - G^{\tilde{\alpha}^*}_t, \tag{3.12}$$

where $\tilde{\alpha}^*$ is the optimal strategy associated with the switching equation (3.11). We now recall the comparison theorem for system of generalized reflected BSDEs with oblique reflection (see Proposition 2.1 [1]). It should be pointed out that the result is applicable in our case since the generators do not depend on the solution. Therefore, the following inequalities hold:

$$\begin{aligned} Y^{u,i} &\leq \tilde{Y}^i; \\ Y^{v,i} &\leq \tilde{Y}^i. \end{aligned} \tag{3.13}$$

Then by letting $(P'^{\alpha}, Q'^{\alpha})$ be the unique solution of the switching equation (3.7) with data $(h_{\alpha}(X^{t,x}_T), f_{\alpha}(r, X^{t,x}_r, \vec{v}_r, Q^{\alpha}_r), \psi_{\alpha}(r, X^{t,x}_r, \vec{v}_r))$ and taking into account (3.9) and (3.12), we obtain: $\forall i = 1\dots m$, $\forall t \leqslant s \leqslant T$,

$$\begin{aligned} P^{\tilde{\alpha}^*}_s - G^{\tilde{\alpha}^*}_s \leq Y^{u,i}_s \leq \tilde{Y}^i_s = \tilde{P}^{\tilde{\alpha}^*}_s - G^{\tilde{\alpha}^*}_s; \\ {P'}^{\tilde{\alpha}^*}_s - G^{\tilde{\alpha}^*}_s \leq Y^{v,i}_s \leq \tilde{Y}^i_s = \tilde{P}^{\tilde{\alpha}^*}_s - G^{\tilde{\alpha}^*}_s. \end{aligned} \tag{3.14}$$

It yields, $\forall i = 1\dots m$, $\forall t \leqslant s \leqslant T$,

$$\mid Y^{u,i}_s - Y^{v,i}_s \mid \leq \mid \tilde{P}^{\tilde{\alpha}^*}_s - P^{\tilde{\alpha}^*}_s \mid + \mid \tilde{P}^{\tilde{\alpha}^*}_s - {P'}^{\tilde{\alpha}^*}_s \mid . \tag{3.15}$$

Step 4: Φ is a contraction

To start with, note that $(\tilde{P}^{\alpha^*} - P^{\alpha^*}, \tilde{Q}^{\alpha^*} - Q^{\alpha^*})$ is the unique solution for the following generalized BSDE: $\forall t \leqslant s \leqslant T$.

$$\begin{aligned} \tilde{P}^{\tilde{\alpha}^*}_s - P^{\tilde{\alpha}^*}_s &= \int_s^T [F_{\tilde{\alpha}^*}(r, X^{t,x}_r, \tilde{Q}^{\tilde{\alpha}^*}_r) - f_{\tilde{\alpha}^*}(r, X^{t,x}_r, \vec{u}_r, Q^{\tilde{\alpha}^*}_r)]dr \\ &\quad + \int_s^T [\Psi_{\tilde{\alpha}^*}(r, X^{t,x}_r) - \psi_{\tilde{\alpha}^*}(r, X^{t,x}_r, \vec{u}_r)]d|\Lambda^{t,x}|_r - \int_s^T (\tilde{Q}^{\tilde{\alpha}^*}_r - Q^{\tilde{\alpha}^*}_r)dW_r. \end{aligned} \tag{3.16}$$

Then we apply Itô's formula with $e^{\mu s+\lambda|\Lambda^{t,x}|_s} \mid \tilde{P}^{\tilde{\alpha}^*}_s - P^{\tilde{\alpha}^*}_s \mid^2$, we have: $\forall t \leqslant s \leqslant T$,

$$\begin{aligned} &e^{\mu s+\lambda|\Lambda^{t,x}|_s} \mid \tilde{P}^{\tilde{\alpha}^*}_s - P^{\tilde{\alpha}^*}_s \mid^2 + \mu \int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \mid \tilde{P}^{\alpha^*}_r - P^{\alpha^*}_r \mid^2 dr \\ &+ \lambda \int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \mid \tilde{P}^{\alpha^*}_r - P^{\alpha^*}_r \mid^2 d|\Lambda^{t,x}|_r + \int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \|\tilde{Q}^{\tilde{\alpha}^*}_r - Q^{\tilde{\alpha}^*}_r\|^2 dr \\ &= 2\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}(\tilde{P}^{\tilde{\alpha}^*}_r - P^{\tilde{\alpha}^*}_r)[F_{\tilde{\alpha}^*}(r, X^{t,x}_r, \tilde{Q}^{\tilde{\alpha}^*}_r) - f_{\tilde{\alpha}^*}(r, X^{t,x}_r, \vec{u}_r, Q^{\tilde{\alpha}^*}_r)]dr \\ &+ 2\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}(\tilde{P}^{\tilde{\alpha}^*}_r - P^{\tilde{\alpha}^*}_r)[\Psi_{\alpha^*}(r, X^{t,x}_r) - \psi_{\alpha^*}(r, X^{t,x}_r, \vec{u}_r)]d|\Lambda^{t,x}|_r \\ &- 2\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}(\tilde{P}^{\alpha^*}_r - P^{\alpha^*}_r)(\tilde{Q}^{\alpha^*}_r - Q^{\alpha^*}_r)dW_r. \end{aligned}$$

Since $\mid x\vee y-y\mid\leq\mid x-y\mid$ for $x,y\in\mathbb{R}$, we obtain: $\forall t\leqslant s\leqslant T$,

$$
\begin{aligned}
&e^{\mu s+\lambda|\Lambda^{t,x}|_s}\mid\tilde{P}_s^{\tilde{\alpha}^*}-P_s^{\tilde{\alpha}^*}\mid^2+\mu\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\mid\tilde{P}_r^{\alpha^*}-P_r^{\alpha^*}\mid^2 dr\\
&+\lambda\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\mid\tilde{P}_r^{\alpha^*}-P_r^{\alpha^*}\mid^2 d|\Lambda^{t,x}|_r+\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\|\tilde{Q}_r^{\tilde{\alpha}^*}-Q_r^{\tilde{\alpha}^*}\|^2dr\\
&\leq 2\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}|\tilde{P}_r^{\tilde{\alpha}^*}-P_r^{\tilde{\alpha}^*}|\,|f_{\tilde{\alpha}^*}(r,X_r^{t,x},\vec{v}_r,\tilde{Q}_r^{\tilde{\alpha}^*})-f_{\tilde{\alpha}^*}(r,X_r^{t,x},\vec{u}_r,Q_r^{\tilde{\alpha}^*})|dr\\
&+2\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}|\tilde{P}_r^{\tilde{\alpha}^*}-P_r^{\tilde{\alpha}^*}|\,\psi_{\alpha^*}(r,X_r^{t,x},\vec{v}_r)-\psi_{\alpha^*}(r,X_r^{t,x},\vec{u}_r)d|\Lambda^{t,x}|_r\\
&-2\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}(\tilde{P}_r^{\alpha^*}-P_r^{\alpha^*})(\tilde{Q}_r^{\alpha^*}-Q_r^{\alpha^*})dW_r.
\end{aligned}
$$

Recall the Lipschitz continuity of f_α and ψ_α with respect to $(\vec{y},z)$ and $\vec{y}$ respectively. Then as $2\mid ab\mid\leqslant\frac{1}{q}\mid a\mid^2+q\mid b\mid^2$, $q>0$, we get: $\forall t\leqslant s\leqslant T$, $\forall q>4m$,

$$
\begin{aligned}
&e^{\mu s+\lambda|\Lambda^{t,x}|_s}\mid\tilde{P}_s^{\tilde{\alpha}^*}-P_s^{\tilde{\alpha}^*}\mid^2+\mu\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\mid\tilde{P}_r^{\alpha^*}-P_r^{\alpha^*}\mid^2 dr\\
&+\lambda\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\mid\tilde{P}_r^{\alpha^*}-P_r^{\alpha^*}\mid^2 d|\Lambda^{t,x}|_r+\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\|\tilde{Q}_r^{\tilde{\alpha}^*}-Q_r^{\tilde{\alpha}^*}\|^2dr\\
&\leq(qC_f^2+2C_f^2)\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}|\tilde{P}_r^{\tilde{\alpha}^*}-P_r^{\tilde{\alpha}^*}|^2dr+\frac{1}{q}\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\parallel\vec{u}_r-\vec{v}_r\parallel^2 dr\\
&+\frac{1}{2}\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\|\tilde{Q}_r^{\tilde{\alpha}^*}-Q_r^{\tilde{\alpha}^*}\|^2dr+qC_\psi^2\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}|\tilde{P}_r^{\tilde{\alpha}^*}-P_r^{\tilde{\alpha}^*}|^2d|\Lambda^{t,x}|_r\\
&+\frac{1}{q}\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\parallel\vec{u}_r-\vec{v}_r\parallel^2 d|\Lambda^{t,x}|_r-2\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}(\tilde{P}_r^{\alpha^*}-P_r^{\alpha^*})(\tilde{Q}_r^{\alpha^*}-Q_r^{\alpha^*})dW_r,
\end{aligned}
$$

where $C_f:=\max_{i=1,\dots,m}C_{f_i}$ and $C_\psi:=\max_{i=1,\dots,m}C_{\psi_i}$ and for each $i=1,\dots,m$ the positive constants C_{f_i} and C_{ψ_i} stand for the Lipschitz constants of f_i and ψ_i respectively. Then by taking the expectation we have: $\forall t\leqslant s\leqslant T$,

$$
\begin{aligned}
\mathbb{E}\big[e^{\mu s+\lambda|\Lambda^{t,x}|_s}\mid\tilde{P}_s^{\tilde{\alpha}^*}-P_s^{\tilde{\alpha}^*}\mid^2\big]&+(\mu-qC_f^2-2C_f^2)\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\mid\tilde{P}_r^{\alpha^*}-P_r^{\alpha^*}\mid^2 dr\big]\\
&+(\lambda-qC_\psi^2)\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\mid\tilde{P}_r^{\alpha^*}-P_r^{\alpha^*}\mid^2 d|\Lambda^{t,x}|_r\big]\\
&+\frac{1}{2}\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\|\tilde{Q}_r^{\tilde{\alpha}^*}-Q_r^{\tilde{\alpha}^*}\|^2dr\big]\\
\leq\frac{1}{q}\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\parallel\vec{u}_r-\vec{v}_r\parallel^2 dr\big]&\\
&+\frac{1}{q}\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r}\parallel\vec{u}_r-\vec{v}_r\parallel^2 d|\Lambda^{t,x}|_r\big].
\end{aligned}
$$

Then the following estimate holds: $\forall t \leqslant s \leqslant T$,

$$
\begin{aligned}
&(\mu - qC_f^2 - 2C_f^2)\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \mid \tilde{P}_r^{\alpha^*} - P_r^{\alpha^*} \mid^2 dr\big] \\
&\qquad + (\lambda - qC_\psi^2)\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \mid \tilde{P}_r^{\alpha^*} - P_r^{\alpha^*} \mid^2 d|\Lambda^{t,x}|_r\big] \\
&\leq \frac{1}{q}\big(\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \parallel \vec{u}_r - \vec{v}_r \parallel^2 dr\big] \\
&\qquad + \mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \parallel \vec{u}_r - \vec{v}_r \parallel^2 d|\Lambda^{t,x}|_r\big]\big).
\end{aligned}
$$

We can derive an identical estimate for $\mid \tilde{P}_r^{\alpha^*} - {P'}_r^{\alpha^*} \mid$ using the same reasoning. Hence by (3.15), we deduce that: $\forall t \leqslant s \leqslant T$, $\forall i = 1, \ldots, m$,

$$
\begin{aligned}
&(\mu - qC_f^2 - 2C_f^2)\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \mid Y_r^{u,i} - Y_r^{v,i} \mid^2 dr\big] \\
&\qquad + (\lambda - qC_\psi^2)\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \mid Y_r^{u,i} - Y_r^{v,i} \mid^2 d|\Lambda^{t,x}|_r\big] \\
&\leq \frac{4}{q}\big(\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \parallel \vec{u}_r - \vec{v}_r \parallel^2 dr\big] + \\
&\qquad \mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \parallel \vec{u}_r - \vec{v}_r \parallel^2 d|\Lambda^{t,x}|_r\big]\big).
\end{aligned}
$$

Then by taking the sum over $\{1, \ldots, m\}$, we get: $\forall t \leqslant s \leqslant T$,

$$
\begin{aligned}
&(\mu - qC_f^2 - 2C_f^2)\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \sum_{i=1}^m \mid Y_r^{u,i} - Y_r^{v,i} \mid^2 dr\big] \\
&\qquad + (\lambda - qC_\psi^2)\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \sum_{i=1}^m \mid Y_r^{u,i} - Y_r^{v,i} \mid^2 d|\Lambda^{t,x}|_r\big] \\
&\leq m\frac{4}{q}\big(\mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \parallel \vec{u}_r - \vec{v}_r \parallel^2 dr\big] \\
&\qquad + \mathbb{E}\big[\int_s^T e^{\mu r+\lambda|\Lambda^{t,x}|_r} \parallel \vec{u}_r - \vec{v}_r \parallel^2 d|\Lambda^{t,x}|_r\big]\big).
\end{aligned}
$$

Consequently, by choosing $\mu = \mu_0 = 1 + qC_f^2 + 2C_f^2$ and $\lambda = \lambda_0 = 1 + qC_\psi^2$, we have: $\forall t \leqslant T$,

$$
\begin{aligned}
&\mathbb{E}\big[\int_t^T e^{\mu_0 r+\lambda_0|\Lambda^{t,x}|_r} \sum_{i=1}^m \mid \Phi(Y_r^{u,i}) - \Phi(Y_r^{v,i}) \mid^2 dr\big] \\
&\qquad + \mathbb{E}\big[\int_t^T e^{\mu_0 r+\lambda_0|\Lambda^{t,x}|_r} \sum_{i=1}^m \mid \Phi(Y_r^{u,i}) - \Phi(Y_r^{v,i}) \mid^2 d|\Lambda^{t,x}|_r\big] \\
&\leq m\frac{4}{q}\big(\mathbb{E}\big[\int_t^T e^{\mu_0 r+\lambda_0|\Lambda^{t,x}|_r} \parallel \vec{u}_r - \vec{v}_r \parallel^2 dr\big] \\
&\qquad + \mathbb{E}\big[\int_t^T e^{\mu_0 r+\lambda_0|\Lambda^{t,x}|_r} \parallel \vec{u}_r - \vec{v}_r \parallel^2 d|\Lambda^{t,x}|_r\big]\big).
\end{aligned}
$$

Finally, we deduce that Φ is a strict contraction on $\mathscr{M}^2_{\mu_0,\lambda_0}(|\Lambda^{t,x}|)$. Hence, it has a unique fixed point which gives the uniqueness of the solution for the system of generalized reflected BSDEs (3.3).

4 System of variational inequalities with inter-connected obstacles and nonlinear boundary conditions in time-dependent domains

4.1 Definitions and notations

In this section, we turn our attention to a system of fully non linear PDEs with interconnected obstacles. Let D^{o}, $\tilde{D}$ and ∂D be the sets defined by (2.30). Then, consider the PDEs system below: $\forall i = 1,...,m$,

$$\begin{cases} \min\{v^i(t,x) - \max\limits_{j\neq i}(v^j(t,x) - g_{ij}(t,x)); -\partial_t v^i(t,x) - \mathscr{L}v^i(t,x) \\ \qquad -f_i(t,x,(v^j(t,x))_{j=1,...,m}, \sigma^\top(t,x)D_x v^j(t,x))\} = 0,\ (t,x) \in D^{\mathrm{o}}; \\ \dfrac{\partial v^i}{\partial \gamma}(t,x) + \psi_i(t,x,(v^j(t,x))_{j=1,...,m}) = 0,\ (t,x) \in \partial D; \\ v^i(T,x) = h_i(x),\ x \in \overline{D}_T. \end{cases} \tag{4.1}$$

We are concerned with the question of existence and uniqueness of a solution in viscosity sense of the system (4.1). We first evoke some notations and definitions that will be used frequently in the sequel.

Definition 4.1 *For a locally bounded function $u : \tilde{D} \to \mathbb{R}$, we define its lower semicontinuous envelope u_* and its upper semicontinuous envelope u^* as follows:*

$$u_*(t,x) = \underset{\substack{(t',x')\longrightarrow(t,x)\\(t',x')\in\tilde{D}}}{\underline{\lim}} u(t',x') \text{ and } u^*(t,x) = \underset{\substack{(t',x')\longrightarrow(t,x)\\(t',x')\in\tilde{D}}}{\overline{\lim}} u(t',x').$$

The viscosity solution will be studied in terms of parabolic superjet and subjet which in the context of time-dependent domains have been defined in [20]:

Definition 4.2 *Let $E \subset \mathbb{R}^{d+1}$ be arbitrary and $u : E \longmapsto \mathbb{R}$, then the parabolic superjet $\mathscr{P}_E^{2,+}u(s,z)$ contains all triplets $(p,q,M) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$ such that if $(s,z) \in E$, then*

$$u(t,x) \leq u(s,z) + p(t-s) + \langle q, x-z\rangle + \frac{1}{2}\langle x-z, M(x-z)\rangle + o(|\,t-s\,| + |\,x-z\,|^2)),$$
$$\text{as } (t,x) \in E \longrightarrow (s,z).$$

The parabolic subjet is defined as $\mathscr{P}_E^{2,-}u(s,z) = -\mathscr{P}_E^{2,+}u(s,z)$. The closures $\overline{\mathscr{P}}_E^{2,+}(s,z)$ and $\overline{\mathscr{P}}_E^{2,-}u(s,z)$ are defined in analogy with (2.6) and (2.9) in [7].

Definition 4.3 (**i**) *A function $u := (u^1,...,u^m) : \tilde{D} \longrightarrow \mathbb{R}^m$ such that for any $i = 1,\ldots,m$, u^i is usc, is a viscosity subsolution of (4.1) if for any $i = 1,\ldots,m$, $(t,x) \in \tilde{D}$ and $(p,q,M) \in$*

$\overline{\mathscr{P}}^{2,+}_{\tilde{D}} u^i(t,x)$, *we have:*

$$\begin{cases} \min\{u^i(t,x) - \max\limits_{j\neq i}(u^j(t,x) - g_{ij}(t,x)); -p - b(t,x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(t,x)M] \\ \qquad -f_i(t,x,(u^j(t,x))_{j=1,\dots,m}, \sigma^\top(t,x)q)\} \leq 0,\ (t,x) \in D^o, \\ \min\{\min\{u^i(t,x) - \max\limits_{j\neq i}(u^j(t,x) - g_{ij}(t,x)); \\ \qquad -p - b(t,x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(t,x)M] - f_i(t,x,(u^j(t,x))_{j=1,\dots,m}, \sigma^\top(t,x)q)\}; \\ \qquad -\langle \gamma(t,x), q\rangle - \psi_i(t,x,(u^j(t,x))_{j=1,\dots,m})\} \leq 0,\ (t,x) \in \partial D, \\ u^i(T,x) \leq h_i(x), \quad x \in \overline{D}_T. \end{cases} \tag{4.2}$$

(**ii**) *A function* $(u^1,\dots,u^m) : \tilde{D} \longrightarrow \mathbb{R}^m$ *such that for any* $i = 1,\dots,m$, u^i *is lsc, is a viscosity supersolution of* (4.1) *if for any* $i = 1,\dots,m$, $(t,x) \in \tilde{D}$ *and* $(p,q,M) \in \overline{\mathscr{P}}^{2,-}_{\tilde{D}} u^i(t,x)$, *we have:*

$$\begin{cases} \min\{u^i(t,x) - \max\limits_{j\neq i}(u^j(t,x) - g_{ij}(t,x)); -p - b(t,x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(t,x)M] \\ \qquad -f_i(t,x,(u^j(t,x))_{j=1,\dots,m}, \sigma^\top(t,x)q)\} \geq 0,\ (t,x) \in D^o, \\ \max\{\min\{u^i(t,x) - \max\limits_{j\neq i}(u^j(t,x) - g_{ij}(t,x)); \\ \qquad -p - b(t,x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(t,x)M] - f_i(t,x,(u^j(t,x))_{j=1,\dots,m}, \sigma^\top(t,x)q)\}; \\ \qquad -\langle \gamma(t,x), q\rangle - \psi_i(t,x,(u^j(t,x))_{j=1,\dots,m})\} \geq 0,\ (t,x) \in \partial D, \\ u^i(T,x) \geq h_i(x), \quad x \in \overline{D}_T. \end{cases} \tag{4.3}$$

(**ii**) *A locally bounded function* $u : (u^1,\dots,u^m)\overline{D} \longrightarrow \mathbb{R}^m$ *is called a viscosity solution of the PDEs system* (4.1), *if* $(u^{1*},\dots,u^{m*})$ *(resp.* $(u^{1*},\dots,u^{m*})$*) is a viscosity subsolution (resp. supersolution) of the system of PDEs* (4.1).

The solution of the PDEs system (4.1) will be obtained mainly thanks to the connection between the PDEs system (4.1) and the system of reflected generalized BSDEs (3.3). First, we recall the solution of the generalized reflected BSDEs system (3.3). We aim to provide a unique representation of its solution $(Y^{i,t,x})_{i=1,\dots,m}$ in terms of continuous deterministic function $(u^i)_{i=1,\dots,m}$. This will be done in two steps. In the first, we prove the existence of a unique continuous deterministic function which solves the system of PDEs (2.31) in a particular case. Then we extend the result to the general case using a Picard iteration argument.

4.1.1 Results of existence and uniqueness in a particular setting

In this part, we are going to look at a reflected GBSDEs system with interconnected obstacles and its associated PDEs system when the generators take the following form $f_i : [0,T] \times \mathbb{R}^{d\times 1\times k} \longrightarrow \mathbb{R}^{d\times m\times k}$ and $\psi_i : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$, $\forall i = 1,\dots,m$. Then, under the same assumptions of section 3, we introduce the following system of reflected GBSDEs

with interconnected obstacles: $\forall t \leqslant s \leqslant T$, $\forall i = 1, ..., m$,

$$\begin{cases} \tilde{Y}^{i,t,x} \in \mathscr{S}^2, \tilde{K}^{i,t,x} \in \mathscr{A}^2, \tilde{Z}^{i,t,x} \in \mathscr{H}^{2,d}; \\ \tilde{Y}_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, \tilde{Y}_r^{i,t,x}, \tilde{Z}_r^{i,t,x}) dr + \int_s^T \psi_i(r, X_r^{t,x}) d|\Lambda^{t,x}|_r \\ \qquad\qquad + \tilde{K}_T^{i,t,x} - \tilde{K}_s^{i,t,x} - \int_s^T \tilde{Z}_r^{i,t,x} dW_r, \\ \tilde{Y}_s^{i,t,x} \geqslant \max_{j \neq i} (\tilde{Y}_s^{j,t,x} - g_{ij}(s, X_s^{t,x})), \\ \int_0^T \{\tilde{Y}_s^{i,t,x} - \max_{j \neq i} (\tilde{Y}_s^{j,t,x} - g_{ij}(s, X_s^{t,x}))\} d\tilde{K}_s^{i,t,x} = 0. \end{cases} \tag{4.4}$$

Thanks to Theorem 3.1 the above system has a unique solution $(\tilde{Y}^{t,x}, \tilde{K}^{t,x}, \tilde{Z}^{t,x})$.

Now, we consider the PDEs system of the following type: $\forall i = 1, ..., m$

$$\begin{cases} \min\{v^i(t,x) - \max_{j \neq i} (v^j(t,x) - g_{ij}(t,x)); -\partial_t v^i(t,x) - \mathscr{L} v^i(t,x) \\ \qquad - f_i(t, x, v^i(t,x), \sigma^\top(t,x) D_x v^i(t,x))\} = 0, \ (t,x) \in D^o; \\ \dfrac{\partial v^i}{\partial \gamma}(t,x) + \psi_i(t,x) = 0, \ (t,x) \in \partial D; \\ v^i(T,x) = h_i(x), \ x \in \overline{D}_T. \end{cases} \tag{4.5}$$

Theorem 4.1 *There exists a unique family of continuous deterministic functions $(\tilde{u}^i)_{i=1,...,m}$ that solves the PDEs system (4.5). Moreover, $(\tilde{u}^i)_{i=1,...,m}$ is bounded on the compact set $\tilde{D}$ and we have: $\forall i = 1, ..., m$,*

$$\tilde{Y}_s^{i,t,x} = \tilde{u}^i(s, X_s^{t,x}), \forall t \leqslant s \leqslant T. \tag{4.6}$$

The proof of the above theorem is divided into two parts. In the first, we treat the problem of uniqueness that will be stated in the subsequent proposition and its corollary. Then, we make use of some penalized scheme of generalized BSDEs and PDEs to obtain the existence.

Uniqueness In this paragraph, we establish the uniqueness of the solution for system (4.1). A standard tool in the theory of PDEs is to show the comparison principle. We rely essentially on Lemma 6.1 in [20] which is a version of Ishii's Lemma in time-dependent domains.

Proposition 4.1 *Let $(u^i)_{i=1,...,m}$ and $(v^i)_{i=1,...,m}$ be an usc subsolution and a lsc supersolution of system (4.5). Then: $\forall i = 1, \ldots, m$,*

$$u^i \leq v^i, \text{ on } \tilde{D}. \tag{4.7}$$

PROOF. **Step 1:** We first assume that there exists a constant $\nu < -\max_{i=1,...,m} C_f^i$, ($C_f^i$ being the Lipschitz constant of f_i for $i = 1, \ldots, m$) such that $\forall i = 1, \ldots, m$, $\forall t, x, y, \bar{y}, z$, if $y \geq \bar{y}$ then

$$f_i(t,x,y,z) - f_i(t,x,\bar{y},z) \leq \nu(y - \bar{y}). \tag{4.8}$$

Let $(u^i)_{i=1,\dots,m}$ and $(v^i)_{i=1,\dots,m}$ be respectively an usc subsolution and a lsc supersolution of system (4.5). Let $i=1,\dots,m$ be fixed and recall Lemma 2.2, then there exists a positive function $\varphi \in \mathscr{C}^{1,2}(\mathbb{R}^{d+1})$ defined on an open neighborhood of $\tilde{D}$ with $\varphi \geqslant 0$ and such that $\langle \gamma(t,x), D_x\varphi(t,x)\rangle \geq 1$, $\forall x \in \partial D_t$, $\forall t \in [0,T]$. Then put:

$$u^i_\varepsilon(t,x) = u^i(t,x) - \varepsilon\varphi(t,x) - \frac{\varepsilon}{t} - C_\varepsilon \quad \text{and} \quad v^i_\varepsilon(t,x) = v^i(t,x) + \varepsilon\varphi(t,x) + C_\varepsilon,$$

where C_ε is a positive constant that will be chosen later on.
The second step consists of showing that $(u^i_\varepsilon)_{i=1,\dots,m}$ and $(v^i_\varepsilon)_{i=1,\dots,m}$ are respectively subsolution and supersolution of two PDEs systems that will be given explicitly. We start with the family $(u^i_\varepsilon)_{i=1,\dots,m}$. First, note that if $x \in \overline{D}_T$, then $u^i_\varepsilon(T,x) \leq u^i(T,x) \leq h_i(x)$.
Now, let $(t,x) \in \partial D$ and $(p^u_\varepsilon, q^u_\varepsilon, M^u_\varepsilon) \in \overline{\mathscr{P}}^{2,+}_{\tilde{D}} u^i_\varepsilon(t,x)$. Note that $(p^u := p^u_\varepsilon + \partial_t\varphi(t,x) - \frac{\varepsilon}{t^2}, q^u := q^u_\varepsilon + \varepsilon D_x\varphi(t,x), M^u := M^u_\varepsilon + \varepsilon D_{xx}\varphi(t,x)) \in \overline{\mathscr{P}}^{2,+}_{\tilde{D}} u^i(t,x)$ and we have:

$$\begin{aligned}
&\min\{u^i_\varepsilon(t,x) - \max_{j\neq i}(u^j_\varepsilon(t,x) - g_{ij}(t,x));\\
&\quad - p^u_\varepsilon - b(t,x)^\top q^u_\varepsilon - \frac{1}{2}Tr[\sigma\sigma^\top(t,x)M^u_\varepsilon] - f_i(t,x,u^i_\varepsilon(t,x),\sigma^\top(t,x)q^u_\varepsilon)\} \qquad (4.9)\\
&= \min\{u^i(t,x) - \max_{j\neq i}(u^j(t,x) - g_{ij}(t,x));\\
&\quad - p^u - b(t,x)^\top q^u - \frac{1}{2}Tr[\sigma\sigma^\top(t,x)M^u] - f_i(t,x,u^i(t,x),\sigma^\top(t,x)q^u) - \frac{\varepsilon}{t^2}\\
&\quad + \varepsilon b(t,x)^\top D_x\varphi(t,x) + \frac{\varepsilon}{2}Tr[\sigma\sigma^\top(t,x)D_{xx}\varphi(t,x)]\\
&\quad + f_i(t,x,u_i(t,x),\sigma^\top(t,x)q^u) - f_i(t,x,u^i_\varepsilon(t,x),\sigma^\top(t,x)q^u_\varepsilon)\}. \qquad (4.10)
\end{aligned}$$

Then using the continuity of $b, \sigma, D_x\varphi$ and D_{xx} and the compactness of $\tilde{D}$, we deduce that there exists $\eta_1 > 0$ such that:

$$\varepsilon b(t,x)^\top D_x\varphi(t,x) + \frac{\varepsilon}{2}Tr[\sigma\sigma^\top(t,x)D_{xx}\varphi(t,x)] \leqslant \varepsilon\eta_1. \qquad (4.11)$$

Using the the Lipschitz continuity of f_i and assumption (4.8), we get:

$$\begin{aligned}
&f_i(t,x,u_i(t,x),\sigma^\top(t,x)q^u) - f_i(t,x,u^i_\varepsilon(t,x),\sigma^\top(t,x)q^u_\varepsilon)\\
&\qquad\qquad \leq \nu(\frac{\varepsilon}{t} + \varepsilon\varphi(x)) + \nu C_\varepsilon + \varepsilon C^i_f|\sigma^\top D_x\varphi(t,x)|.
\end{aligned}$$

Since φ is positive and $|\sigma^\top D_x\varphi|$ is bounded on the compact set $\tilde{D}$, there exists $\eta_2 > 0$ such that,

$$f_i(t,x,u_i(t,x),\sigma^\top(t,x)q^u) - f_i(t,x,u^i_\varepsilon(t,x),\sigma^\top(t,x)q^u_\varepsilon) \leq -\max_{i=1,\dots,m} C^i_f C_\varepsilon + \varepsilon\eta_2. \qquad (4.12)$$

Then, by combining (4.11) and (4.12) we can find $\eta_3 > 0$ such that:

$$\begin{aligned}
&\varepsilon b(t,x)^\top D_x\varphi(t,x) + \frac{\varepsilon}{2}Tr[\sigma\sigma^\top(t,x)D_{xx}\varphi(t,x)] - \frac{\varepsilon}{t^2}\\
&+ f_i(t,x,u_i(t,x),\sigma^\top(t,x)q) - f_i(t,x,u^i_\varepsilon(t,x),\sigma^\top(t,x)q^\varepsilon) \leqslant -\max_{i=1,\dots,m} C^i_f C_\varepsilon + \varepsilon\eta_3. \qquad (4.13)
\end{aligned}$$

Set $C_\varepsilon = \frac{\varepsilon\eta_3}{\max\limits_{i=1,\dots,m} C_f^i}$, the constant C_ε is positive and tends to zero when $\varepsilon \to 0$. By (4.9) we deduce that:

$$\begin{aligned}
&\min\{u_\varepsilon^i(t,x) - \max_{j\neq i}(u_\varepsilon^j(t,x) - g_{ij}(t,x));\\
&\qquad - p_\varepsilon^u - b(t,x)^\top q_\varepsilon^u - \frac{1}{2}Tr[\sigma\sigma^\top(t,x)M_\varepsilon^u] - f_i(t,x,u_\varepsilon^i(t,x),\sigma^\top(t,x)q_\varepsilon^u)\} \qquad (4.14)\\
&\leqslant \min\{u^i(t,x) - \max_{j\neq i}(u^j(t,x) - g_{ij}(t,x));\\
&\qquad - p^u - b(t,x)^\top q^u - \frac{1}{2}Tr[\sigma\sigma^\top(t,x)M^u] - f_i(t,x,u^i(t,x),\sigma^\top(t,x)q^u)\}. \qquad (4.15)
\end{aligned}$$

For the inequalities on the boundary ∂D, we recall that $\langle\gamma(t,x), D_x\varphi(t,x)\rangle \geqslant 1$ which implies that:

$$\begin{aligned}
-\langle\gamma(t,x), q_\varepsilon^u\rangle - \psi_i(t,x) &= -\langle\gamma(t,x), q^u\rangle - \psi_i(t,x) - \varepsilon\langle\gamma(t,x), D_x\varphi(t,x)\rangle\\
&\leqslant -\langle\gamma(t,x), q\rangle - \psi_i(t,x) - \varepsilon. \qquad (4.16)
\end{aligned}$$

Then, if $(u^i)_{i=1,\dots,m}$ is a subsolution of (4.5), it follows from (4.14) and (4.16) that $(u_\varepsilon^i)_{i=1,\dots,m}$ satisfies the following system of PDEs:

$$\begin{cases}
\min\{u_\varepsilon^i(t,x) - \max\limits_{j\neq i}(u_\varepsilon^j(t,x) - g_{ij}(t,x)); -p_\varepsilon^u - b(t,x)^\top q_\varepsilon^u - \frac{1}{2}Tr[\sigma\sigma^\top(t,x)M_\varepsilon^u]\\
\qquad -f_i(t,x,u_\varepsilon^i(t,x),\sigma^\top(t,x)q_\varepsilon^u)\} \leq 0,\ (t,x)\in D^o,\\
\min\{\min\{u_\varepsilon^i(t,x) - \max\limits_{j\neq i}(u_\varepsilon^j(t,x) - g_{ij}(t,x));\\
\qquad -p_\varepsilon^u - b(t,x)^\top q_\varepsilon^u - \frac{1}{2}Tr[\sigma\sigma^\top(t,x)M_\varepsilon^u] - f_i(t,x,u_\varepsilon^i(t,x),\sigma^\top(t,x)q_\varepsilon^u)\};\\
\qquad -\langle\gamma(t,x), q_\varepsilon^u\rangle - \psi_i(t,x) + \varepsilon\} \leq 0,\ (t,x)\in\partial D,\\
u_\varepsilon^i(T,x) \leq h_i(x), \quad x\in\overline{D}_T.
\end{cases} \qquad (4.17)$$

Similarly, since $(v^i)_{i=1,\dots,m}$ is a supersolution of (4.5), we obtain using the same techniques a system of inequalities for $(v_\varepsilon^i)_{i=1,\dots,m}$. Indeed, for any $i = 1,\dots,m$, $(t,x)\in\tilde{D}$ and $(p_\varepsilon^v, q_\varepsilon^v, M_\varepsilon^v) \in \overline{\mathscr{P}}^{2,-}_{\tilde{D}} v_\varepsilon^i(t,x)$, we have:

$$\begin{cases}
\min\{v_\varepsilon^i(t,x) - \max\limits_{j\neq i}(v_\varepsilon^j(t,x) - g_{ij}(t,x)); -p_\varepsilon^v - b(t,x)^\top q_\varepsilon^v - \frac{1}{2}Tr[\sigma\sigma^\top(t,x)M_\varepsilon^v]\\
\qquad -f_i(t,x,v_\varepsilon^i(t,x),\sigma^\top(t,x)q_\varepsilon^v)\} \geq 0,\ (t,x)\in D^o,\\
\max\{\min\{v_\varepsilon^i(t,x) - \max\limits_{j\neq i}(v_\varepsilon^j(t,x) - g_{ij}(t,x));\\
\qquad -p_\varepsilon^v - b(t,x)^\top q_\varepsilon^v - \frac{1}{2}Tr[\sigma\sigma^\top(t,x)M_\varepsilon^v] - f_i(t,x,v_\varepsilon^i(t,x),\sigma^\top(t,x)q_\varepsilon^v)\};\\
\qquad -\langle\gamma(t,x), q_\varepsilon^v\rangle - \psi_i(t,x)\} \geq 0,\ (t,x)\in\partial D,\\
v_\varepsilon^i(T,x) \geq h_i(x),\ x\in\overline{D}_T.
\end{cases} \qquad (4.18)$$

Consequently, we next show that $u_\varepsilon^i \leqslant v_\varepsilon^i$ on $\tilde{D}$ which helps to avoid the conditions on the boundary ∂D. Then, we can take the limit with respect to ε to get the comparison principle

for u^i and v^i.
Now, note that $u-v$ is usc then it is bounded from above on $\tilde{D}$, then we assume to the contrary that $\max\limits_{\tilde{D}} \max\limits_{i=1,\dots,m} (u^i_\varepsilon - v^i_\varepsilon) > 0$. There exists $(\bar{t},\bar{x}) \in \tilde{D}$ such that

$$\max_{i=1,\dots,m} (u^\varepsilon_i(\bar{t},\bar{x}) - v^i_\varepsilon(\bar{t},\bar{x})) > 0. \tag{4.19}$$

As in [12], thanks to the non-free loop property there exists $k \in \tilde{J} := \{j \in \{1,\dots,m\}, u^j_\varepsilon(\bar{t},\bar{x}) - v^j_\varepsilon(\bar{t},\bar{x}) = \max\limits_{k=1,\dots,m} (u^k_\varepsilon(\bar{t},\bar{x}) - v^k_\varepsilon(\bar{t},\bar{x}))\}$ such that

$$u^k_\varepsilon(\bar{t},\bar{x}) > \max_{j\neq k}(u^j_\varepsilon(\bar{t},\bar{x}) - g_{kj}(\bar{t},\bar{x})). \tag{4.20}$$

Now, in order to avoid the conditions involving the obstacles, we fix $j \in \tilde{J}$ satisfying (4.20), then taking into account the value of $u^i_\varepsilon(T,x)$ and $v^i_\varepsilon(T,x)$ it turns out that the maximum of $\max\limits_{i=1,\dots,m} (u^i_\varepsilon - v^i_\varepsilon)$ over $\tilde{D}$ is achieved in D^o or ∂D.
First, we consider the case where the maximum is achieved on ∂D. Then, taking into account the definition of u^i_ε and the terminal values $u^i_\varepsilon(T,x)$ and $v^i_\varepsilon(T,x)$ there exists $(\bar{t},\bar{x}) \in \partial D$ for $\bar{t} \in (0,T)$. Let $\alpha > 0$ and $w_\alpha \in \mathscr{C}^{1,2}(\mathbb{R}^{d+1})$ be the function defined in Lemma 2.1 and set:

$$\begin{aligned}
&\Phi^\alpha_j(t,x,y) = u^j_\varepsilon(t,x) - v^j_\varepsilon(t,y) - \Psi^\alpha_j(t,x,y),\\
&\Psi^\alpha_j(t,x,y) = w_\alpha(t,x,y) - \psi_j(\bar{t},\bar{x})\langle\gamma(\bar{t},\bar{x}),x-y\rangle + |\,x-\bar{x}\,|^4 + |\,t-\bar{t}\,|^2\,.
\end{aligned}$$

Recall that thanks to Lemma 2.1, there exist $\xi, C > 0$ such that:

$$\xi\,\alpha|x-y|^2 \leqslant w_\alpha(t,x,y) \leqslant C\big(\frac{1}{\alpha} + \alpha|x-y|^2\big). \tag{4.21}$$

Let $(t_\alpha,x_\alpha,y_\alpha)$ be the maximum point of $\Phi^j_\alpha(t,x,y)$ for $t \in [0,T]$ and $(x,y) \in \partial D_t \times \partial D_t$. Then, we can see that,

$$\max_{\partial D}(u^j_\varepsilon - v^j_\varepsilon) - \frac{C}{\alpha} = u^j_\varepsilon(\bar{t},\bar{x}) - v^j_\varepsilon(\bar{t},\bar{x}) - \frac{C}{\alpha}.$$

The inequality (4.21) implies that $w_\alpha(\bar{t},\bar{x},\bar{x}) \leqslant \frac{C}{\alpha}$. Then,

$$\max_{\partial D}(u^j_\varepsilon - v^j_\varepsilon) - \frac{C}{\alpha} \leq \Phi^j_\alpha(\bar{t},\bar{x},\bar{y}) \leq \Phi^j_\alpha(t_\alpha,x_\alpha,y_\alpha). \tag{4.22}$$

Thus,

$$\begin{aligned}
\Phi^j_\alpha(t_\alpha,x_\alpha,y_\alpha) \leq u^j_\varepsilon(t_\alpha,x_\alpha) - v^j_\varepsilon(t_\alpha,y_\alpha) - \xi\,\alpha|x_\alpha-y_\alpha|^2 + \psi_j(\bar{t},\bar{x})\langle\gamma(\bar{t},\bar{x}),x_\alpha-y_\alpha\rangle\\
- |\,x_\alpha-\bar{x}\,|^4 - |\,t_\alpha-\bar{t}\,|^2\,.
\end{aligned}$$

This implies that

$$\begin{cases}
\alpha|x_\alpha-y_\alpha|^2 \underset{\alpha\to\infty}{\longrightarrow} 0,\\
|x_\alpha-y_\alpha|^2 \underset{\alpha\to\infty}{\longrightarrow} 0,\\
(t_\alpha,x_\alpha,y_\alpha) \underset{\alpha\to\infty}{\longrightarrow} (\bar{t},\bar{x},\bar{x}).
\end{cases} \tag{4.23}$$

Moreover, from (4.22), we deduce that,

$$\begin{cases} u_\varepsilon^j(t_\alpha,x_\alpha) \underset{\alpha\to\infty}{\longrightarrow} u_\varepsilon^j(\bar{t},\bar{x}), \\ v_\varepsilon^j(t_\alpha,y_\alpha) \underset{\alpha\to\infty}{\longrightarrow} v_\varepsilon^j(\bar{t},\bar{x}). \end{cases} \tag{4.24}$$

Now, we go back to systems of inequalities (4.17) and (4.18). It turns out that the conditions on the boundary in these system can be reduced to the condition in the interior. In fact, note that if $(t_\alpha,x_\alpha)\in\partial D$, we have:

$$\begin{aligned} \langle\gamma(t_\alpha,x_\alpha),D_x\Psi_j^\alpha(t_\alpha,x_\alpha,y_\alpha)\rangle - \psi_j(t_\alpha,x_\alpha) &= -\langle\gamma(t_\alpha,x_\alpha),D_x w_\alpha(t_\alpha,x_\alpha-y_\alpha)\rangle \\ + \psi_j(\bar{t},\bar{x})\langle\gamma(\bar{t},\bar{x}),\gamma(x_\alpha,y_\alpha)\rangle - 4|x_\alpha-\bar{x}|^2\langle\gamma(x_\alpha,y_\alpha),x_\alpha-\bar{x},\rangle - \psi_j(t_\alpha,x_\alpha). \end{aligned} \tag{4.25}$$

Again, from Lemma 2.1, we have:

$$|D_x w_\alpha(t_\alpha,x_\alpha,y_\alpha)| \leqslant C\alpha(|x_\alpha-y_\alpha|+|x_\alpha-y_\alpha|^2). \tag{4.26}$$

This implies that,

$$\begin{aligned} -\langle\gamma(t_\alpha,x_\alpha),D_x\Psi_j^\alpha(t_\alpha,x_\alpha,y_\alpha)\rangle - \psi_j(t_\alpha,x_\alpha) \\ \geq -C\alpha|\gamma(t_\alpha,x_\alpha)|(|x_\alpha-y_\alpha|+|x_\alpha-y_\alpha|^2) + \psi_j(\bar{t},\bar{x})\langle\gamma(\bar{t},\bar{x}),\gamma(x_\alpha,y_\alpha)\rangle \\ -4|x_\alpha-\bar{x}|^2\langle\gamma(x_\alpha,y_\alpha),x_\alpha-\bar{x},\rangle - \psi_j(t_\alpha,x_\alpha). \end{aligned} \tag{4.27}$$

Therefore, by (4.23) and since γ and ψ_j are continuous, we deduce that the right hand side of (**??**) tends to zero as $\alpha\to\infty$. Then, for α large enough we have: $\forall\varepsilon>0$,

$$-\langle\gamma(t_\alpha,x_\alpha),D_x\Psi_j^\alpha(t_\alpha,x_\alpha,y_\alpha)\rangle - \psi_j(t_\alpha,x_\alpha) \geq \varepsilon. \tag{4.28}$$

If $(t_\alpha,y_\alpha)\in\partial D$, we observe that:

$$\begin{aligned} -\langle\gamma(t_\alpha,y_\alpha),-D_y\Psi_j^\alpha(t_\alpha,x_\alpha,y_\alpha)\rangle - \psi_j(t_\alpha,y_\alpha) &= -\langle\gamma(t_\alpha,y_\alpha),-D_y w_\alpha(t_\alpha,x_\alpha,y_\alpha)\rangle \\ &\quad + \psi_j(\bar{t},\bar{x})\langle\gamma(\bar{t},\bar{x}),\gamma(x_\alpha,y_\alpha)\rangle - \psi_j(t_\alpha,x_\alpha) \\ &\leq |D_x w_\alpha(t_\alpha,x_\alpha,y_\alpha)| + \psi_j(\bar{t},\bar{x})\langle\gamma(\bar{t},\bar{x}),\gamma(x_\alpha,y_\alpha)\rangle - \psi_j(t_\alpha,y_\alpha) \\ &\leq C\alpha|x_\alpha-y_\alpha| + \psi_j(\bar{t},\bar{x})\langle\gamma(\bar{t},\bar{x}),\gamma(x_\alpha,y_\alpha)\rangle - \psi_j(t_\alpha,y_\alpha). \end{aligned} \tag{4.29}$$

The last line is obtained thanks to (4.26) and taking into account (4.23) it converges to zero. This implies that, $\forall\varepsilon>0$

$$-\langle\gamma(t_\alpha,y_\alpha),-D_y\Psi_j^\alpha(t_\alpha,x_\alpha,y_\alpha)\rangle - \psi_j(t_\alpha,y_\alpha) \leq \varepsilon. \tag{4.30}$$

On the other hand, by recalling (4.20) with the use of (4.24), we deduce that for α large enough we have:

$$u_\varepsilon^j(t_\alpha,x_\alpha) > \max_{k\neq j}(u_\varepsilon^k(t_\alpha,x_\alpha) - g_{jk}(t_\alpha,x_\alpha)). \tag{4.31}$$

If the maximum of $\max\limits_{i=1,\dots,m}(u_\varepsilon^i - v_\varepsilon^i)$ over $\tilde{D}$ is achieved in D^{o}, we observe that (t_α,x_α) and (t_α,y_α) are in D^{o}, since $(\bar{t},\bar{x})\in D^{\mathrm{o}}$. In this case, only the inequalities in the interior hold.

Now, we are in a position to apply Lemma 6.1 in [20], which ensures the existence of $(p^u_\varepsilon, q^u_\varepsilon, M^u_\varepsilon) \in \overline{\mathscr{P}}^{2,+}_{\bar{D}} u^i_\varepsilon(t_\alpha, x_\alpha)$ and $(p^v_\varepsilon, q^v_\varepsilon, M^v_\varepsilon) \in \overline{\mathscr{P}}^{2,-}_{\bar{D}} v^i_\varepsilon(t_\alpha, y_\alpha)$ that satisfy:

$$\begin{cases} p^u_\varepsilon - p^v_\varepsilon = \partial_t \Psi^\alpha_j(t_\alpha, x_\alpha, y_\alpha), \\ q^u_\varepsilon = D_x \Psi^\alpha_j(t_\alpha, x_\alpha, y_\alpha), \\ q^v_\varepsilon = (-D_y \Psi^\alpha_j(t_\alpha, x_\alpha, y_\alpha)), \\ \begin{pmatrix} M^u_\varepsilon & 0 \\ 0 & M^v_\varepsilon \end{pmatrix} \leqslant D_{xx} \Psi^\alpha_j(t_\alpha, x_\alpha, y_\alpha) + \frac{1}{\alpha} D_{xx} \Psi^\alpha_j(t_\alpha, x_\alpha, y_\alpha)^2. \end{cases} \tag{4.32}$$

Therefore, by taking into account (4.28), (4.30) and (4.31), we deduce from (4.17) and (4.18) that for α large enough, we have:

$$\begin{aligned} &- \partial_t \Psi^\alpha_j(t_\alpha, x_\alpha, y_\alpha) - b(t_\alpha, x_\alpha)^\top D_x \Psi^\alpha_j(t_\alpha, x_\alpha, y_\alpha) - \frac{1}{2} Tr[\sigma\sigma^\top(t_\alpha, x_\alpha) M^\varepsilon] \\ &\qquad - f_i(t_\alpha, x_\alpha, u^i_\varepsilon(t_\alpha, x_\alpha), \sigma^\top(t_\alpha, x_\alpha) D_x \Psi^\alpha_j(t_\alpha, x_\alpha, y_\alpha)) \leq 0, \\ &- \partial_t \Psi^\alpha_j(t_\alpha, x_\alpha, y_\alpha) - b(t_\alpha, y_\alpha)^\top (-D_y \Psi^\alpha_j(t_\alpha, x_\alpha, y_\alpha)) - \frac{1}{2} Tr[\sigma\sigma^\top(t_\alpha, y_\alpha) M^v_\varepsilon] \\ &\qquad - f_i(t_\alpha, y_\alpha, v^i_\varepsilon(t_\alpha, y_\alpha), \sigma^\top(t_\alpha, y_\alpha)(-D_y \Psi^\alpha_j(t_\alpha, x_\alpha, y_\alpha))) \geq 0. \end{aligned} \tag{4.33}$$

Hence,

$$\begin{aligned} &-(f_j(t_\alpha, x_\alpha, u^j_\varepsilon(t_\alpha, x_\alpha), \sigma^\top(t_\alpha, x_\alpha) q^u_\varepsilon) - f_j(t_\alpha, y_\alpha, v^j_\varepsilon(t_\alpha, y_\alpha), \sigma^\top(t_\alpha, y_\alpha) q^v_\varepsilon)) \\ &\qquad \leq (p^u_\varepsilon - p^v_\varepsilon) + (b(t_\alpha, x_\alpha)^\top q^u_\varepsilon - b(t_\alpha, y_\alpha)^\top q^v_\varepsilon) \\ &\qquad + \frac{1}{2} Tr[\sigma\sigma^\top(t_\alpha, x_\alpha) M^u_\varepsilon - \sigma\sigma^\top(t_\alpha, y_\alpha) M^v_\varepsilon]. \end{aligned}$$

Using the assumptions (4.8) and the Lipschitz continuity of f_j with respect to z, we obtain:

$$\begin{aligned} -\nu(u^j_\varepsilon(t_\alpha, x_\alpha) - v^j_\varepsilon(t_\alpha, y_\alpha) &\leq (p^u_\varepsilon - p^v_\varepsilon) + (b(t_\alpha, x_\alpha)^\top q^u_\varepsilon - b(t_\alpha, y_\alpha)^\top q^v_\varepsilon) \\ &+ \frac{1}{2} Tr[\sigma\sigma^\top(t_\alpha, x_\alpha) M^u_\varepsilon - \sigma\sigma^\top(t_\alpha, y_\alpha) M^v_\varepsilon] + C|\sigma^\top(t_\alpha, x_\alpha) q^u_\varepsilon - \sigma^\top(t_\alpha, y_\alpha) q^v_\varepsilon| \\ &+ (f_j(t_\alpha, x_\alpha, v^j_\varepsilon(t_\alpha, y_\alpha), \sigma^\top(t_\alpha, x_\alpha) q^u_\varepsilon) - f_j(t_\alpha, y_\alpha, v^j_\varepsilon(t_\alpha, y_\alpha), \sigma^\top(t_\alpha, x_\alpha) q^u_\varepsilon)). \end{aligned}$$

Besides, the convergences (4.23) imply that the term on the right hand side converges to zero as $\alpha \to \infty$. Then, passing to the limit it follows from (4.24) that,

$$-\nu(u^j_\varepsilon(\bar{t}, \bar{x}) - v^j_\varepsilon(\bar{t}, \bar{x})) \leq 0.$$

This contradicts assumption (4.19), since ν is assumed to be strictly negative. We conclude that $u^i_\varepsilon \leq v^i_\varepsilon$ for any $i = 1, \ldots, m$. It suffices to take the limit with respect to ε to get $u^i \leq v^i$, $i = 1, \ldots, m$. This ends the proof of the first case since i is arbitrary in $\{1, ..., m\}$.

Step 2: For ν arbitrary in $\mathbb{R}$, let $(\tilde{u}^i)_{i\in\{1,\ldots,m\}}$ and $(v^i)_{i\in\{1,\ldots,m\}}$ be respectively a subsolution and a supersolution of sytem (4.5). Then the families defined by $(\tilde{u}^i(t,x) = e^{\nu t}\tilde{u}^i(t,x))_{i=1,\ldots,m}$ and $(\tilde{v}^i(t,x) = e^{\nu t} v^i(t,x))_{i=1,\ldots,m}$ are respectively subsolution and supersolution of the fol-

lowing system: $\forall i \in \{1,...,m\}$

$$\begin{cases} \min\{\tilde{u}^i(t,x) - \max_{j\neq i}(\tilde{u}^j(t,x) - e^{\nu t} g_{ij}(t,x)); -\partial_t \tilde{u}^i(t,x) - \mathscr{L}\tilde{u}^i(t,x) + \nu \tilde{u}^i(t,x) \\ \quad -e^{\nu t} f_i(t,x,e^{-\nu t}\tilde{u}^i(t,x), e^{-\nu t}\sigma^\top(x) D_x \tilde{u}^i(t,x))\} = 0, \ (t,x) \in D^o; \\ \dfrac{\partial \tilde{u}^i}{\partial \gamma}(t,x) + e^{\nu t}\psi_i(t,x) = 0, \ (t,x) \in \partial D; \\ \tilde{u}^i(T,x) = e^{\nu T} h_i(x), \ x \in \overline{D}_T. \end{cases} \tag{4.34}$$

By choosing ν negative, the functions F_i defined by: $\forall i \in \{1,...,m\}$

$$F_i(t,x,y,z) = -\nu y_i + f_i(t,x,e^{-\nu t}y, e^{-\nu t}z),$$

satisfy assumption (4.8) and the result follows thanks to the first step.

Corollary 4.1 *If the solution of the system of PDEs (4.1) exists, it is unique and continuous, then it is bounded on $\tilde{D}$.*

Existence In this paragraph we establish the existence of a viscosity solution of the PDEs system (4.5). This is obtained mainly thanks to the connection between the latter system and the system of reflected generalized BSDEs (4.4).

Proposition 4.2 *There exists a unique viscosity solution of the PDEs system (2.31) that we denote $(\tilde{u}^i)_{i=1,...,m}$. The function $(\tilde{u}^i)_{i=1,...,m}$ is continuous and bounded on $\tilde{D}$. Moreover, we have: $\forall t \leqslant s \leqslant T$, $\forall i = 1,...,m$,*

$$\tilde{Y}^i_s := \tilde{u}^i(s, X^{t,x}_s). \tag{4.35}$$

PROOF. Let $(\tilde{Y}, \tilde{K}, \tilde{Z})$ be the unique solution of the reflected GBSDEs system (4.4). Next, define the following sequence of reflected GBSDEs, for $i = 1,...,m$ and $n = 0$ set $Y^{i,0} := \underline{Y}$ given by

$$\begin{cases} \underline{Y} \in \mathscr{S}^2, \underline{Z} \in \mathscr{H}^{2,d}; \\ \underline{Y}_s = \min_{i=1,..,m} h_i(X^{t,x}_T) + \int_s^T \min_{i=1,...,m} f_i(r, X^{t,x}_r, \underline{Y}_r, \underline{Z}_r)dr \\ \qquad\qquad + \int_s^T \min_{i=1,..,m} \psi_i(r, X^{t,x}_r) d|\Lambda^{t,x}|_r - \int_s^T \underline{Z}_r dW_r, \ \forall t \leqslant s \leqslant T. \end{cases} \tag{4.36}$$

Then, for $n \geqslant 1$, $\forall t \leqslant s \leqslant T$, $i = 1,...,m$,

$$\begin{cases} \tilde{Y}^{i,n} \in \mathscr{S}^2, \tilde{K}^{i,n} \in \mathscr{A}^2, \tilde{Z}^{i,n} \in \mathscr{H}^{2,d}; \\ \tilde{Y}^{i,n}_s = h_i(X^{t,x}_T) + \int_s^T f_i(r, X^{t,x}_r, \tilde{Y}^{i,n}_r, \tilde{Z}^{i,n}_r)dr + \int_s^T \psi_i(r, X^{t,x}_r) d|\Lambda^{t,x}|_r \\ \qquad\qquad + \tilde{K}^{i,n}_T - \tilde{K}^{i,n}_s - \int_s^T \tilde{Z}^{i,n}_r dW_r, \\ Y^{i,n}_s \geqslant \max_{j\neq i}(\tilde{Y}^{j,n-1}_s - g_{ij}(s, X^{t,x}_s)), \\ \int_0^T \{\tilde{Y}^{i,n}_s - \max_{j\neq i}(\tilde{Y}^{j,n-1}_s - g_{ij}(s, X^{t,x}_s))\} d\tilde{K}^{i,n}_s = 0. \end{cases} \tag{4.37}$$

Finally, consider:

$$\begin{cases} \overline{Y} \in \mathscr{S}^2, \overline{Z} \in \mathscr{H}^{2,d}; \\ \overline{Y}_s = \max\limits_{i=1,..,m} h_i(X_T^{t,x}) + \int_s^T \max\limits_{i=1,..,m} f_i(r, X_r^{t,x}, \overline{Y}_r, \overline{Z}_r)dr \\ \qquad\qquad + \int_s^T \max\limits_{i=1,..,m} \psi_i(r, X_r^{t,x}) d|\Lambda^{t,x}|_r - \int_s^T \overline{Z}_r dW_r, \ \forall t \leqslant s \leqslant T. \end{cases} \tag{4.38}$$

It has been shown in [1], that $\tilde{Y}^{i,n}$ satisfies the following inequality: $\forall i = 1, ..., m$, $\forall n \geqslant 1$,

$$\underline{Y} = \tilde{Y}^{i,0} \leq \tilde{Y}^{i,n} \leq \overline{Y}. \tag{4.39}$$

Moreover, for any $i = 1, \ldots, m$, the sequence $(\tilde{Y}^{i,n})_{n\geqslant 1}$ converges increasingly to $\tilde{Y}^i$ the solution of (3.3). Then by taking the limit with respect to n, we get: $\forall i = 1, \ldots, m$,

$$\underline{Y} = \tilde{Y}^{i,0} \leq \tilde{Y}^{i} \leq \overline{Y}. \tag{4.40}$$

Besides, $\underline{Y}$ and $\overline{Y}$ are solutions of standard GBSDEs then by formula (2.29) there exit $\underline{u}$ and $\overline{u}$ two deterministic continuous functions defined by: $\forall t \leqslant s \leqslant T$,

$$\underline{Y} = \underline{u}(s, X_s^{t,x}) \text{ and } \overline{Y} = \overline{u}(s, X_s^{t,x}). \tag{4.41}$$

Now, we introduce the following sequence of standard GBSDEs, for any $k \geqslant 0$, $\forall t \leqslant s \leqslant T$, $i = 1, ..., m$,

$$\begin{cases} \tilde{Y}^{i,k} \in \mathscr{S}^2, \tilde{Z}^{i,k} \in \mathscr{H}^{2,d}; \\ \tilde{Y}_s^{i,k} = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, \tilde{Y}_r^{i,k}, \tilde{Z}_r^{i,k})dr + \int_s^T \psi_i(r, X_r^{t,x}) d|\Lambda^{t,x}|_r \\ \qquad\qquad + \int_s^T k\big(\tilde{Y}_r^{i,k} - \max\limits_{j\neq i}(\tilde{Y}_r^{j,k} - g_{ij}(s, X_r^{t,x}))\big)^- dr - \int_s^T \tilde{Z}_r^{i,k} dW_r. \end{cases} \tag{4.42}$$

It has been shown in [16] that $(\tilde{Y}^{i,k})_{k\geqslant 1}$ is an increasing sequence that converges to $\tilde{Y}^i$ for each $i = 1, ..., m$. Therefore, it follows from (4.40) that

$$\underline{Y} \leq \tilde{Y}^{i,0} \leq \tilde{Y}^{i,k} \leq \overline{Y}. \tag{4.43}$$

Again, from (2.29) there exists a sequence of deterministic functions $(\tilde{u}^{i,k})_{k\geqslant 1}$ that are continuous and bounded on $\tilde{D}$ such that $\forall t \leqslant s \leqslant T$, $\forall i = 1, ..., m$ we have:

$$\tilde{Y}^{i,k} = \tilde{u}^{i,k}(s, X_s^{t,x}). \tag{4.44}$$

Moreover, for any $k \geqslant 1$, $(\tilde{u}^{i,k})_{i=1,\ldots,m}$ satisfies the following system of PDEs: $\forall i = 1, ..., m$,

$$\begin{cases} \partial_t \tilde{u}^{i,k}(t,x) + \mathscr{L}\tilde{u}^{i,k}(t,x) + f_i(t,x,\tilde{u}^{i,k}(t,x), \sigma^\top(t,x) D_x \tilde{u}^{i,k}(t,x)) \\ \qquad + k\big(\tilde{u}^{i,k}(t,x) - \max\limits_{j\neq i}(\tilde{u}^{j,k}(t,x) - g_{ij}(t,x))\big)^- = 0, \ (t,x) \in D^{\mathrm{o}}; \\ \dfrac{\partial \tilde{u}^{i,k}}{\partial \gamma}(t,x) + \psi_i(t,x) = 0, \ (t,x) \in \partial D; \\ \tilde{u}^{i,k}(T,x) = h_i(x), \ x \in \overline{D}_T. \end{cases} \tag{4.45}$$

Recall (4.40) and (4.41), then for any $i=1,\dots,m$ the sequence $(\tilde{u}^{i,k})_{k\geqslant 1}$ is increasing and satisfies $\underline{u}\leq\tilde{u}^{i,k}\leq\overline{u}$. Hence, there exists a family of deterministic functions $(\tilde{u}^{i})_{i=1,\dots,m}$ such that $\forall i=1,\dots,m$, we have:

$$\begin{cases}\tilde{u}^{i}=\lim\limits_{k\to\infty}\tilde{u}^{i,k},\\ \underline{u}\leq\tilde{u}^{i}\leq\overline{u}.\end{cases}\tag{4.46}$$

Consequently, the family $(\tilde{u}^{i})_{i=1,\dots,m}$ is lsc and bounded on $\tilde{D}$. Note that this result has been proved in more general case (for more details see [16] page 9-10), we shall precise that in this part and for this case we do not assume any monotonicity condition neither on f nor on ψ since the generators do not depend on the other components of the solution.

Next, we rely on the above results in order to show that the family $(u^{i})_{i=1,\dots,m}$ is a viscosity solution of the PDEs system (4.5). To this end, we show that it satisfies the subsolution property and the supersolution property in the sense of Definition 4.3.
Supersolution property: First, recall that $(\tilde{u}^{i,k})_{k\geqslant 1}$ converges increasingly to $\tilde{u}^{i}$, $i=1,...,m$. Then, $(\tilde{u}^{i})_{i=1,\dots,m}$ is lsc on $\tilde{D}$ which means that $\tilde{u}^{i}_{*}=\tilde{u}^{i}$, $i=1,...,m$. Now, fix $i=1,...,m$ then let $(t,x)\in\tilde{D}$ and $(p,q,M)\in\overline{\mathscr{P}}^{2,-}_{\tilde{D}}\tilde{u}^{i}(t,x)$, we need to show that:

$$\begin{cases}\min\{\tilde{u}^{i}(t,x)-\max\limits_{j\neq i}(\tilde{u}^{j}(t,x)-g_{ij}(t,x));-p-b(t,x)^{\top}q-\frac{1}{2}Tr[\sigma\sigma^{\top}(t,x)M]\\ \qquad -f_{i}(t,x,\tilde{u}^{i}(t,x),\sigma^{\top}(t,x)q)\}\geq 0,\ (t,x)\in D^{o},\\ \max\{\min\{\tilde{u}^{i}(t,x)-\max\limits_{j\neq i}(\tilde{u}^{j}(t,x)-g_{ij}(t,x));-p-b(t,x)^{\top}q-\frac{1}{2}Tr[\sigma\sigma^{\top}(t,x)M]\\ \qquad -f_{i}(t,x,\tilde{u}^{i}(t,x),\sigma^{\top}(t,x)q)\};-\langle\gamma(t,x),q\rangle-\psi_{i}(t,x)\}\geq 0,\\ \qquad\qquad (t,x)\in\partial D,\\ \tilde{u}^{i}(T,x)\geq h_{i}(x),\quad x\in\overline{D}_{T}.\end{cases}\tag{4.47}$$

This is obtained using the fact that $(\tilde{u}^{i,k})_{k\geqslant 1}$ is solution of the PDEs system (4.45). Indeed, thanks to Lemma 6.1 in [7] there exist a subsequence $l_k\to\infty$, $(t_l,x_l)_{l\geqslant 1}\in\tilde{D}$ and $(p_l,q_l,M_l)\in\overline{\mathscr{P}}^{2,-}_{\tilde{D}}\tilde{u}^{i,l}(t_l,x_l)$ such that the following convergence holds:

$$(t_l,x_l,\tilde{u}^{i,l}(t_l,x_l),p_l,q_l,M_l)\underset{l\to\infty}{\longrightarrow}(t,x,\tilde{u}^{i}(t,x),p,q,M).\tag{4.48}$$

1st case: If $(t,x)\in D^{o}$, then $(t_l,x_l)_{l\geqslant 1}$ can be chosen in D^{o}, then we obtain from (4.45):

$$\begin{aligned}-p_l-b(t_l,x_l)^{\top}q_l-\frac{1}{2}Tr[\sigma\sigma^{\top}(t_l,x_l)M_l]-f_i(t_l,x_l,\tilde{u}^{i,l}(t_l,x_l),\sigma^{\top}(t_l,x_l)q_l)\\ -k_l\big(\tilde{u}^{i,l}(t_l,x_l)-\max_{j\neq i}(\tilde{u}^{j,l}(t_l,x_l)-g_{ij}(t_l,x_l))\big)^{-}\geq 0.\end{aligned}\tag{4.49}$$

On the one hand, the last inequality implies that:

$$-p_l-b(t_l,x_l)^{\top}q_l-\frac{1}{2}Tr[\sigma\sigma^{\top}(t_l,x_l)M_l]-f_i(t_l,x_l,\tilde{u}^{i,l}(t_l,x_l),\sigma^{\top}(t_l,x_l)q_l)\geq 0.$$

Then by taking the limit as $l\to\infty$, we get:

$$-p-b(t,x)^{\top}q-\frac{1}{2}Tr[\sigma\sigma^{\top}(t,x)M]-f_i(t,x,\tilde{u}^{i}(t,x),\sigma^{\top}(t,x)q)\geq 0.\tag{4.50}$$

On the other hand, recall that $\underline{u}(t_l,x_l) \leq \tilde{u}^{i,l}(t_l,x_l) \leq \overline{u}(t_l,x_l)$. Since $\underline{u}$ and $\overline{u}$ are continuous on the compact $\tilde{D}$, then $(\tilde{u}^{i,l}(t_l,x_l))_{l\geqslant 1}$ is uniformly bounded on $\tilde{D}$ and we can find a subsequence of $(k_l)_{l\geqslant 1}$ for which we have kept the same notation such that $(\tilde{u}^{i,l}(t_l,x_l))_{l\geqslant 1}$ is convergent. Moreover, it follows from (4.49) and (4.50) that we have necessarily:

$$\big(\tilde{u}^{i,l}(t_l,x_l) - \max_{j\neq i}(\tilde{u}^{j,l}(t_l,x_l) - g_{ij}(t_l,x_l))\big)^- \underset{l\to\infty}{\longrightarrow} 0\,. \tag{4.51}$$

Therefore, there exists $l_0 \geqslant 1$ such that $\forall l \geqslant l_0$, we have:

$$\tilde{u}^{i,l}(t_l,x_l) + \big(\tilde{u}^{i,l}(t_l,x_l) - \max_{j\neq i}(\tilde{u}^{j,l}(t_l,x_l) - g_{ij}(t_l,x_l))\big)^- \geqslant \max_{j\neq i}(\tilde{u}^{j,l}(t_l,x_l) - g_{ij}(t_l,x_l)).$$

Using the fact that $(\tilde{u}^{i,l})_{l\geqslant 1}$ is increasing we deduce that:

$$\begin{aligned}\tilde{u}^{i,l}(t_l,x_l) + \big(\tilde{u}^{i,l}(t_l,x_l) - \max_{j\neq i}(\tilde{u}^{j,l}(t_l,x_l) - g_{ij}(t_l,x_l))\big)^- \\ \geqslant \max_{j\neq i}(\tilde{u}^{j,l_0}(t_l,x_l) - g_{ij}(t_l,x_l)).\end{aligned}$$

Hence, by taking the limit as $l\to\infty$ and taking into account the continuity of $(\tilde{u}^{j,l_0})_{\substack{j=1,\dots,m\\ j\neq i}}$, we get:

$$\tilde{u}^{i}(t,x) \geqslant \max_{j\neq i}(\tilde{u}^{j,l_0}(t,x) - g_{ij}(t,x)).$$

Finally, by taking the limit as $l_0\to\infty$, we obtain:

$$\tilde{u}^{i}(t,x) \geqslant \max_{j\neq i}(\tilde{u}^{j}(t,x) - g_{ij}(t,x)). \tag{4.52}$$

It follows from (4.50) and (4.52) that $(\tilde{u}^i)_{i=1,\dots,m}$ is a supersolution of (4.5) on D^{o}.
2^{sd} case: If $(t,x)\in\partial D$, then we can assume that the subsequence $(t_l,x_l)_{l\geqslant 1}$ lies in ∂D. Otherwise, it will be similar to the 1^{st} case. That being said, we get back to (4.45) and we suppose that

$$\begin{aligned}\max\{-p_l - b(t_l,x_l)^\top q_l - \frac{1}{2}Tr[\sigma\sigma^\top(t_l,x_l)M_l] - f_i(t_l,x_l,\tilde{u}^{i,l}(t_l,x_l),\sigma^\top(t_l,x_l)q_l) \\ -k_l\big(\tilde{u}^{i,l}(t_l,x_l) - \max_{j\neq i}(\tilde{u}^{j,l}(t_l,x_l) - g_{ij}(t_l,x_l))\big)^-; -\langle\gamma(t,x),q_l\rangle - \psi_i(t_l,x_l)\} \geq 0.\end{aligned} \tag{4.53}$$

We assume that

$$\begin{aligned}-p_l - b(t_l,x_l)^\top q_l - \frac{1}{2}Tr[\sigma\sigma^\top(t_l,x_l)M_l] - f_i(t_l,x_l,\tilde{u}^{i,l}(t_l,x_l),\sigma^\top(t_l,x_l)q_l) \\ - k_l\big(\tilde{u}^{i,l}(t_l,x_l) - \max_{j\neq i}(\tilde{u}^{j,l}(t_l,x_l) - g_{ij}(t_l,x_l))\big)^- < 0.\end{aligned}$$

Otherwise the supersolution property is obtained by repeating the same calculus as above. Then, if the second term in (4.53) is positive, it suffices to take the limit since the data are continuous. Then, we obtain:

$$-\langle\gamma(t,x),q\rangle - \psi_i(t,x) \geq 0. \tag{4.54}$$

To complete this step we recall that if $x\in\overline{D}_T$, we have $\tilde{u}^i(T,x) = \lim_{k\to\infty}\tilde{u}^{i,k}(T,x)$. Therefore, the function $(\tilde{u}^i)_{i=1,\dots,m}$ is a viscosity supersolution of the PDEs system (4.5).

Subsolution property: Let $(\tilde{u}^{i,*})_{i=1,\dots,m}$ be the upper semicontinuous envelope of $(\tilde{u}^i)_{i=1,\dots,m}$. Then, fix $i=1,\dots,m$ and let $(t,x)\in\tilde{D}$ and $(p,q,M)\in\overline{\mathscr{P}}^{2,+}_{\tilde{D}}\tilde{u}^{i,*}(t,x)$, we need to show that:

$$\begin{cases}\min\{u^{i,*}(t,x)-\max\limits_{j\neq i}(u^{j,*}(t,x)-g_{ij}(t,x));-p-b(t,x)^\top q-\frac{1}{2}Tr[\sigma\sigma^\top(t,x)M]\\ \qquad -f_i(t,x,u^{i,*}(t,x),\sigma^\top(t,x)q)\}\leq 0,\ (t,x)\in D^{\mathrm{o}},\\ \min\{\min\{u^{i,*}(t,x)-\max\limits_{j\neq i}(\tilde{u}^{j,*}(t,x)-g_{ij}(t,x));-p-b(t,x)^\top q-\frac{1}{2}Tr[\sigma\sigma^\top(t,x)M]\\ \qquad -f_i(t,x,u^{i,*}(t,x),\sigma^\top(t,x)q)\};-\langle\gamma(t,x),q\rangle-\psi_i(t,x)\}\leq 0,\\ \qquad\qquad (t,x)\in\partial D,\\ u^{i,*}(T,x)\leq h_i(x),\quad x\in\overline{D}_T.\end{cases} \tag{4.55}$$

By Lemma 6.1 in [7] there exist a subsequence $l_k\to\infty$, $(t_l,x_l)_{l\geqslant 1}\in\tilde{D}$ and $(p_l,q_l,M_l)\in\overline{\mathscr{P}}^{2,+}_{\tilde{D}}\tilde{u}^{i,l}(t_l,x_l)$ satisfying:

$$(t_l,x_l,\tilde{u}^{i,l}(t_l,x_l),p_l,q_l,M_l)\underset{l\to\infty}{\longrightarrow}(t,x,\tilde{u}^{i,*}(t,x),p,q,M). \tag{4.56}$$

1st case: If $(t,x)\in D^{\mathrm{o}}$, we assume $(t_l,x_l)_{l\geqslant 1}\in D^{\mathrm{o}}$. Then using the fact that $(\tilde{u}^{i,l})_{l\geqslant 1}$ is a subsolution of (4.45), we obtain:

$$\begin{aligned}-p_l-b(t_l,x_l)^\top q_l-\frac{1}{2}Tr[\sigma\sigma^\top(t_l,x_l)M_l]-f_i(t_l,x_l,\tilde{u}^{i,l}(t_l,x_l),\sigma^\top(t_l,x_l)q_l)\\ -k_l\big(\tilde{u}^{i,l}(t_l,x_l)-\max_{j\neq i}(\tilde{u}^{j,l}(t_l,x_l)-g_{ij}(t_l,x_l))\big)^-\leq 0.\end{aligned} \tag{4.57}$$

As a first step, we assume that there exists $\varepsilon>0$ such that

$$\tilde{u}^{i,*}(t,x)>\max_{j\neq i}(\tilde{u}^{j,*}(t,x)-g_{ij}(t,x))+\varepsilon,$$

otherwise the subsolution property is satisfied. Moreover, there exists $l_0>0$ such that $\forall l\geqslant l_0$ we have:

$$\tilde{u}^{i,l}(t,x)>\max_{j\neq i}(\tilde{u}^{j,l}(t,x)-g_{ij}(t,x))+\varepsilon,$$

which holds on a neighborhood V_l of (t,x) thanks to the continuity of $(\tilde{u}^{j,k})_{j=1,\dots,m}$ and $(g_{ij})_{i,j=1,\dots,m}$. This implies that,

$$\big(\tilde{u}^{i,l}(t,x)-\max_{j\neq i}(\tilde{u}^{j,l}(t,x)-g_{ij}(t,x))\big)^-=0,\forall(t,x)\in V_l.$$

Note that (t_l,x_l) can be chosen in V_l so that we get:

$$\big(\tilde{u}^{i,l}(t_l,x_l)-\max_{j\neq i}(\tilde{u}^{j,l}(t_l,x_l)-g_{ij}(t_l,x_l))\big)^-=0.$$

Hence, from (4.57) we get:

$$-p_l-b(t_l,x_l)^\top q_l-\frac{1}{2}Tr[\sigma\sigma^\top(t_l,x_l)M_l]-f_i(t_l,x_l,\tilde{u}^{i,l}(t_l,x_l),\sigma^\top(t_l,x_l)q_l)\leq 0.$$

Then the result is obtained by passing to the limit as $l \to \infty$.
2^{sd} case: If $(t,x) \in \partial D$, then we can assume that the subsequence $(t_l,x_l)_{l\geqslant 1}$ lies in ∂D. Then we suppose that

$$\min\{\tilde{u}^{i,*}(t,x) - \max_{j\neq i}(\tilde{u}^{j,*}(t,x) - g_{ij}(t,x));$$
$$-p - b(t,x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(t,x)M] - f_i(t,x,\tilde{u}^{i,*}(t,x),\sigma^\top(t,x)q)\} > 0.$$

In this case, we have necessarily

$$-\langle \gamma(t,x),q\rangle - \psi_i(t,x) \leq 0.$$

Then it suffices to take the limit to obtain the second inequality in (4.55).
3^{rd} case: If $x \in \overline{D}_T$, we need to prove that $\tilde{u}^{i,*}(T,x) \leq h_i(x)$. This will be obtained by showing:

$$\min\{\tilde{u}^{i,*}(T,x) - h_i(x); \tilde{u}^{i,*}(T,x) - \max_{j\neq i}(\tilde{u}^{j,*}(T,x) - g_{ij}(T,x))\} = 0.$$

Let $i \in \{1,...,m\}$ and $x_T \in \overline{D}_T$. Then, we have:

$$\tilde{u}^{i,*}(T,x_T) = \overline{\lim_{\substack{(t',x')\longrightarrow(T,x_T)\\(t',x')\in\tilde{D}}}} u(t',x') \geq \overline{\lim_{\substack{(t',x')\longrightarrow(T,x_T)\\(t',x')\in\tilde{D}}}} u^{i,k}(t',x') = u^{i,k}(T,x_T) \geq h_i(x_T).$$

Moreover $\tilde{u}^i(T,x_T) \geq \max_{j\neq i}(\tilde{u}^j(T,x) - g_{ij}(T,x_T))$. Then by taking the upper semicontinuous envelope we get $\tilde{u}^{i,*}(T,x_T) = \max_{j\neq i}(\tilde{u}^{j,*}(T,x_T) - g_{ij}(T,x_T))$. Hence, $\forall x_T \in \overline{D}_T$,

$$\min\{\tilde{u}^{i,*}(T,x_T) - h_i(x_T); \tilde{u}^{i,*}(T,x_T) - \max_{j\neq i}(\tilde{u}^{j,*}(T,x_T) - g_{ij}(T,x_T))\} \geqslant 0.$$

We assume to the contrary that for $x_T \in \overline{D}_T$, there exists $\varepsilon > 0$ such that:

$$\min\{\tilde{u}^{i,*}(T,x_T) - h_i(x_T); \tilde{u}^{i,*}(T,x_T) - \max_{j\neq i}(\tilde{u}^{j,*}(T,x_T) - g_{ij}(T,x_T))\} = \varepsilon.$$

Let $(t_l,x_l)_{l\geqslant 1}$ be a sequence in $\tilde{D}$ satisfying $(t_l,x_l) \underset{l\to\infty}{\longrightarrow} (T,x_T)$ such that $\tilde{u}^i(t_l,x_l) \underset{l\to\infty}{\longrightarrow} \tilde{u}^{i,*}(T,x_T)$. Now, recall (4.46), by taking the upper semicontinuous envelope and since $\underline{u}$ and $\overline{u}$ are continuous on $\tilde{D}$, we deduce that $u^{i,*}$ is also bounded on $\tilde{D}$. Therefore, using the upper semicontinuity of $\tilde{u}^{i,*}$ we can find $(Q^n)_{n\geqslant 1}$ in $\mathscr{C}^{1,2}(\tilde{D})$ such that $Q^n \underset{n\to\infty}{\longrightarrow} \tilde{u}^{i,*}$. Therefore, by considering $\mathscr{V}_n(T,x_T)$ a neighborhood of (T,x_T), we have:

$$\min\{Q^n(t,x) - h_i(x); Q^n(t,x) - \max_{j\neq i}(\tilde{u}^{j,*}(t,x) - g_{ij}(t,x))\} \geq \varepsilon, \forall (t,x) \in \mathscr{V}_n(T,x_T). \quad (4.58)$$

We can assume that (4.58) holds on:

$$\mathscr{V}_n(t_l,x_l) = [t_l,T] \times \{x \in D_{t_l} / |x - x_l| < \delta^{l,n}\},$$

where $\delta^{l,n} \in (0,1)$ and such that $\mathscr{V}_n(t_l,x_l) \subset \mathscr{V}_n(T,x_T)$. Once again as $u^{i,*}$ is bounded on $\tilde{D}$, there exists $c > 0$ such that $|u^{i,*}| \leq c$ on $\mathscr{V}_n(T,x_T)$. Then we can assume that $Q^n \geq -2c$. Next, define

$$\tilde{Q}^n_l(t,x) = Q^n(t,x) + \frac{4c|x-x_l|^2}{(\delta^{l,n})^2} + \sqrt{T-t}.$$

Note that $\tilde{Q}_l^n \geq Q^n$ and $(u^{i,*} - \tilde{Q}_l^n)(t,x) \leq -c$ for $(t,x) \in [t_l, T] \times \partial B(x_l, \delta^{l,n})$. On the other hand, we have:

$$-\{\partial_t \tilde{Q}_l^n(t,x) + \mathscr{L}\tilde{Q}_l^n(t,x)\} = -\{\partial_t Q^n(t,x) + \partial_t\sqrt{T-t} + D_x b(t,x) \times \{D_x Q^n(t,x) + \frac{8c(x-x_l)}{(\delta^{l,n})^2}\} + \frac{1}{2}\sigma\sigma^\top(t,x)\{D_{xx}Q^n(t,x) + \frac{8c}{(\delta^{l,n})^2}\}\}.$$

Since $\frac{4c|x-x_l|^2}{(\delta^{l,n})^2}$ and Q^n belong to $\mathscr{C}^{1,2}(\tilde{D})$, then the term on the right hand side is bounded, then $\partial_t(\sqrt{T-t}) \underset{t\to T}{\longrightarrow} -\infty$, it suffices to choose l large enough in order to get:

$$-\{\partial_t \tilde{Q}_l^n(t,x) + \mathscr{L}\tilde{Q}_l^n(t,x)\} \geq 0, \forall (t,x) \in \mathscr{V}_n(t_l, x_l).$$

Next, recall the reflected SDE (2.26) and set:

$$\tau_n^l = \inf\{s \geqslant t_l;\ (s, X_s^{t_l,x_l}) \in \mathscr{V}_n(t_l,x_l)^c\} \wedge T,$$
$$\tau_l = \inf\{s \geqslant t_l;\ u^i(s, X_s^{t_l,x_l}) = \max_{j\neq i}(u^j(s, X_s^{t_l,x_l}) - g_{ij}(s, X_s^{t_l,x_l}))\} \wedge T.$$

Then we can apply Ito's formula to $\tilde{Q}(., X^{t_l,x_l})$ stopped at time $\tau_n^l \wedge \tau_l$ and taking into account the fact that $d|\Lambda^{t_l,x_l}| = 0$ on $[t_l, \tau_n^l \wedge \tau_l]$. The remaining of the proof is similar to the proof of Theorem 3.2 in [1], we omit any further details. To sum up, we have showed that the family of functions $(u^i)_{i=1,\dots,m}$ is a viscosity solution of the PDEs system (4.5). Then, thanks to the comparison principle stated in Proposition 4.7, we know that $(u^i)_{i=1,\dots,m}$ is continuous and bounded on $\tilde{D}$.

4.1.2 General case

Theorem 4.2 *Assume that assumptions (H_0), (H_1) and (H_1) are satisfied. Then the system of PDEs (4.1) has a unique viscosity solution $(u^i)_{i=1,\dots,m}$. Moreover, the function $(u^i)_{i=1,\dots,m}$ is continuous and bounded on $\tilde{D}$ and it is given by $\forall t \leqslant s \leqslant T$, $\forall i = 1,\dots,m$,*

$$Y_s^i := u^i(s, X_s^{t,x}). \tag{4.59}$$

PROOF. Let $i = 1,\dots,m$ be fixed and define the following sequence, $(Y^{i,0}, K^{i,0}, Z^{i,0}) = (0,0,0)$ and for $\ell \geqslant 1$ we set: $\forall t \leqslant s \leqslant T$,

$$\begin{cases} Y_s^{i,\ell} = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, (u^{i,\ell-1}(r, X_r^{t,x}))_{i=1,\dots,m}, Z_r^{i,\ell})dr \\ \quad + \int_s^T \psi_i(r, X_r^{t,x}, (u^{i,\ell-1}(r, X_r^{t,x}))_{i=1,\dots,m})d|\Lambda^{t,x}|_r + K_T^{i,\ell} - K_s^{i,\ell} - \int_s^T Z_r^{i,\ell} dW_r, \\ Y_s^{i,\ell} \geqslant \max_{j\neq i}(Y_s^{j,\ell} - g_{ij}(s, X_s^{t,x})), \\ \int_0^T \{Y_s^{i,\ell} - \max_{j\neq i}(Y_s^{j,\ell} - g_{ij}(s, X_s^{t,x}))\} dK_s^{i,\ell} = 0. \end{cases} \tag{4.60}$$

Note that this system has a unique solution thanks Theorem 3.1. Moreover, by Proposition 4.2 and provided $(u^{i,\ell-1}(r, X_r^{t,x}))_{i=1,\dots,m}$, there exists a deterministic continuous function $(u^{i,\ell})_{i=1,\dots,m}$ which is bounded on $\tilde{D}$ and defined by: $\forall t \leqslant s \leqslant T$, $\forall i = 1,\dots,m$,

$$Y_s^{i,\ell} := u^{i,\ell}(s, X_s^{t,x}). \tag{4.61}$$

Next we show the uniform convergence of $(u^{i,\ell})_{\ell\geqslant 1}$, $\forall i=1,\dots,m$. This will be done by means of the switching representation of the solution $Y^{i,\ell}$ given by the switching representation (3.9). Then, consider the following switching equation: $\forall t\leqslant s\leqslant T$

$$\begin{cases} P^{\alpha,\ell} \text{ càdlàg, } \mathbb{E}(\sup\limits_{t\leq s\leq T} \mid P_s^{\alpha,\ell}\mid^2)<\infty \text{ and } Q^{\alpha,\ell}\in\mathscr{H}^{2,k};\\ P_s^{\alpha,\ell}=h_\alpha(X_T^{t,x})+\int_s^T f_\alpha(r,X_r^{t,x},(u^{i,\ell-1}(r,X_r^{t,x}))_{i=1,\dots,m},Q_r^{\alpha,\ell})dr\\ \quad+\int_s^T \psi_\alpha(r,X_r^{t,x},(u^{i,\ell-1}(r,X_r^{t,x}))_{i=1,\dots,m})d|\Lambda^{t,x}|_r-\int_t^T Q_r^{\alpha,\ell}dW_r-(G_T^\alpha-G_s^\alpha), \end{cases} \tag{4.62}$$

and we have: $\forall t\leqslant s\leqslant T$, $\forall i=1,\dots,m$,

$$Y_s^{i,\ell}=\operatorname*{esssup}_{\alpha\in\mathscr{D}_s^i}(P_s^{\alpha,\ell}-G_s^\alpha). \tag{4.63}$$

Now, for fixed $i=1,\dots,m$, and $\ell,\kappa\geqslant 1$, we set:

$$\begin{aligned} F_i^{\ell,\kappa}(s,X_s^{t,x},z)&=f_i(s,X_s^{t,x},(u^{i,\ell-1}(s,X_s^{t,x}))_{i=1,\dots,m},z)\vee f_i(s,X_s^{t,x},(u^{i,\kappa-1}(s,X_s^{t,x}))_{i=1,\dots,m},z),\\ \Psi_i^{\ell,\kappa}(s,X_s^{t,x})&=\psi_i(s,X_s^{t,x},(u^{i,\ell-1}(s,X_s^{t,x}))_{i=1,\dots,m}\vee\psi_i(s,X_s^{t,x},(u^{i,\kappa-1}(s,X_s^{t,x}))_{i=1,\dots,m}). \end{aligned}$$

Then, we consider the associated system of generalized BSDEs with oblique reflection: $\forall i=1,\dots,m$,

$$\begin{cases} Y_s^{i,\ell,\kappa}=h_i(X_T^{t,x})+\int_s^T F_i^{\ell,\kappa}(r,X_r^{t,x},Z_r^{i,\ell,\kappa})dr+\int_s^T\Psi_i^{\ell,\kappa}(r,X_r^{t,x})d|\Lambda^{t,x}|_r\\ \qquad\qquad +K_T^{i,\ell,\kappa}-K_s^{i,\ell,\kappa}-\int_s^T Z_r^{i,\ell,\kappa}dW_r,\\ Y_s^{i,\ell,\kappa}\geqslant\max\limits_{j\neq i}(Y_s^{j,\ell,\kappa}-g_{ij}(s,X_s^{t,x})),\\ \int_0^T\{Y_s^{i,\ell,\kappa}-\max\limits_{j\neq i}(Y_s^{j,\ell,\kappa}-g_{ij}(s,X_s^{t,x}))\}dK_s^{i,\ell,\kappa}=0. \end{cases} \tag{4.64}$$

Again, by Theorem 3.1 the solution of the above system exists and is unique. Moreover, the following representation holds:

$$Y_s^{i,\ell,\kappa}=\operatorname*{esssup}_{\alpha\in\mathscr{D}_s^i}(P_s^{\alpha,\ell,\kappa}-G_s^\alpha)=P_s^{\alpha^*,\ell,\kappa}-G_s^{\alpha^*}. \tag{4.65}$$

The pair $(P^{\alpha,\ell,\kappa},Q^{\alpha,\ell,\kappa})$ denotes the unique solution of the switching equation (4.62) when the data are replaced with $\big(h_\alpha(X_T^{t,x}),F_\alpha^{\ell,\kappa}(s,X_s^{t,x},z),\Psi_\alpha^{\ell,\kappa}(s,X_s^{t,x})\big)$. Besides, since the generators $F_i^{\ell,\kappa},\Psi_i^{\ell,\kappa}$ do not depend on $\vec{y}$, we can use the comparison result for the solutions of reflected GBSDEs system given by Proposition 2.1 in [1] and we have:

$$\begin{aligned} Y^{i,\ell}&\leq Y^{i,\ell,\kappa}=P_s^{\alpha^*,\ell,\kappa}-G_s^{\alpha^*};\\ Y^{i,\kappa}&\leq Y^{i,\ell,\kappa}=P_s^{\alpha^*,\ell,\kappa}-G_s^{\alpha^*}, \end{aligned} \tag{4.66}$$

where α^* is the optimal strategy provided by $Y^{i,\ell,\kappa}$.
Therefore, by repeating the same calculus as in the proof of Theorem 3.1 (step 4). It follows that for $q>0$ and by setting $\mu_0=1+qC_f^2+2C_f^2$ and $\lambda_0=1+qC_\psi^2$, we have: $\forall t\leqslant s\leqslant T$,

$$\begin{aligned} &\mathbb{E}\big[\mid u^{i,\ell}(s,X_s^{t,x})-u^{i,\kappa}(s,X_s^{t,x})\mid^2\big]\leq\frac{4}{q}\\ &\quad\times\Big(\mathbb{E}\Big[\int_t^T e^{\mu_0 r+\lambda_0|\Lambda^{t,x}|_r}\parallel(u^{i,\ell-1}(r,X_r^{t,x}))_{i=1,\dots,m}-(u^{i,\kappa-1}(r,X_r^{t,x}))_{i=1,\dots,m}\parallel^2 dr\Big]\\ &+\mathbb{E}\Big[\int_t^T e^{\mu_0 r+\lambda_0|\Lambda^{t,x}|_r}\parallel(u^{i,\ell-1}(r,X_r^{t,x}))_{i=1,\dots,m}-(u^{i,\kappa-1}(r,X_r^{t,x}))_{i=1,\dots,m}\parallel^2 d|\Lambda^{t,x}|_r\Big]\Big). \end{aligned}$$

This implies that for $s=t$, we get:

$$|\, u^{i,\ell}(t,x)-u^{i,\kappa}(t,x)\,|^2\leq \frac{4}{q}\mathbb{E}\Big[\int_t^T e^{\mu_0 T+\lambda_0|\Lambda^{t,x}|_T}$$
$$\times\,\|\,(u^{i,\ell-1}(r,X_r^{t,x}))_{i=1,\dots,m})-(u^{i,\kappa-1}(r,X_r^{t,x}))_{i=1,\dots,m})\,\|^2\,(dr+d|\Lambda^{t,x}|_r)\Big].$$

Recall that $X^{t,x}$ is the solution of the reflected SDE (2.26), this implies that $(s,X^{t,x})$ belongs to the bounded domain $\overline{D}$ which is a subset of $\tilde{D}$ for any $t\leqslant s\leqslant T$. Then, thanks to the boundedness of $(u^{i,\ell-1})_{i=1,\dots,m}$, we obtain:

$$|\, u^{i,\ell}(t,x)-u^{i,\kappa}(t,x)\,|^2\leq \frac{4}{q}\mathbb{E}\Big[\int_t^T e^{\mu_0 T+\lambda_0|\Lambda^{t,x}|_T}$$
$$\times\Big(\sum_{i=1}^m\sup_{(t,x)\in\tilde{D}}|\, u^{i,\ell-1}(t,x)-u^{i,\kappa-1}(t,x)\,|^2\Big)(dr+d|\Lambda^{t,x}|_r)\Big].$$

It follows that:

$$|\, u^{i,\ell}(t,x)-u^{i,\kappa}(t,x)\,|^2\leq \frac{4}{q}e^{\mu_0 T}\mathbb{E}\big[e^{\lambda_0|\Lambda^{t,x}|_T}(T+|\Lambda^{t,x}|_T)\big]$$
$$\times\sum_{i=1}^m\sup_{(t,x)\in\tilde{D}}|\, u^{i,\ell-1}(t,x)-u^{i,\kappa-1}(t,x)\,|^2.$$

Next, we take the supremum over $\tilde{D}$ and the sum over i, then we have:

$$\sum_{i=1}^m\sup_{(t,x)\in\tilde{D}}|\, u^{i,\ell}(t,x)-u^{i,\kappa}(t,x)\,|^2\leq m\frac{4}{q}e^{\mu_0 T}\mathbb{E}\big[e^{\lambda_0|\Lambda^{t,x}|_T}(T+|\Lambda^{t,x}|_T)\big]$$
$$\times\sum_{i=1}^m\sup_{(t,x)\in\tilde{D}}|\, u^{i,\ell-1}(t,x)-u^{i,\kappa-1}(t,x)\,|^2.$$

Finally, we choose $q>4me^{\mu_0 T}\mathbb{E}\big[e^{\lambda_0|\Lambda^{t,x}|_T}(T+|\Lambda^{t,x}|_T)\big]$ which yields the following estimate: $\ell,\kappa\geqslant 1$,

$$\sum_{i=1}^m\sup_{(t,x)\in\tilde{D}}|\, u^{i,\ell}(t,x)-u^{i,\kappa}(t,x)\,|^2\leq\varpi\sum_{i=1}^m\sup_{(t,x)\in\tilde{D}}|\, u^{i,\ell-1}(t,x)-u^{i,\kappa-1}(t,x)\,|^2,$$

where $\varpi<1$. Thus, the following convergence holds:

$$\lim_{\ell,\kappa\to\infty}\sum_{i=1}^m\sup_{(t,x)\in\tilde{D}}|\, u^{i,\ell}(t,x)-u^{i,\kappa}(t,x)\,|^2=0.$$

Therefore, there exists $(u^i)_{i=1,\dots,m}$ a family of functions such that $(u^{i,\ell})_{\ell\geqslant 1}$ converges uniformly on $\tilde{D}$ to u^i for any $i=1,\dots,m$. Moreover, $(u^i)_{i=1,\dots,m}$ is continuous and bounded on $\tilde{D}$. Then, we have necessarily $Y_s^i:=u^i(s,X_s^{t,x})$. To conclude, it remains to consider the system of PDEs below:$\forall i=1,\dots,m$

$$\begin{cases}\min\{v^i(t,x)-\max\limits_{j\neq i}(v^j(t,x)-g_{ij}(t,x));-\partial_t v^i(t,x)-\mathscr{L}v^i(t,x)\\ \qquad -f_i(t,x,(u^i(t,x))_{i=1,\dots,m},\sigma^\top(t,x)D_x v^j(t,x))\}=0,\ (t,x)\in D^{\rm o};\\ \dfrac{\partial v^i}{\partial\gamma}(t,x)+\psi_i(t,x,(u^i(t,x))_{i=1,\dots,m})=0,\ (t,x)\in\partial D;\\ v^i(T,x)=h_i(x),\ x\in\overline{D}_T.\end{cases}\tag{4.67}$$

Then, consider the associated system of reflected GBSDEs:$\forall t \leqslant s \leqslant T$, $\forall i = 1, ..., m$,

$$\begin{cases} Y_s^i = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, (u^i(r, X_r^{t,x}))_{i=1,...,m}, Z_r^i)dr \\ \qquad + \int_s^T \psi_i(r, X_r^{t,x}, (u^i(r, X_r^{t,x}))_{i=1,...,m})d|\Lambda^{t,x}|_r + K_T^i - K_s^i - \int_s^T Z_r^i dW_r, \\ Y_s^i \geqslant \max_{j \neq i}(Y_s^j - g_{ij}(s, X_s^{t,x})), \\ \int_0^T \{Y_s^i - \max_{j \neq i}(Y_s^j - g_{ij}(s, X_s^{t,x}))\}dK_s^i = 0. \end{cases} \tag{4.68}$$

It follows from Theorem 4.1 that $(u^i)_{i=1,...,m}$ is solution of the PDEs system (4.67).

It remains to show that $(u^i)_{i=1,...,m}$ is the unique solution of the PDEs system (4.67). For this, we assume that there exists $(\tilde{u}^i)_{i=1,...,m}$ a continuous function on $\tilde{D}$ which is solution of the system of PDEs (4.67). Then, let $i = 1, \ldots, m$ and $\tilde{Y}^i \in \mathscr{S}^2$ such that $\tilde{Y}_s^i = \tilde{u}^i(s, X_s^{t,x})$, $\forall t \leqslant s \leqslant T$. Then recall the contraction Φ defined in section 3 and set:

$$(\overline{Y}^i)_{i=1,...,m} = \Phi((\tilde{Y}^i)_{i=1,...,m}).$$

This means that $(\overline{Y}, \overline{K}, \overline{Z})$ is the unique solution of the following system of reflected GBSDEs with interconnected obstacles: $\forall t \leqslant s \leqslant T$, $\forall i = 1, ..., m$,

$$\begin{cases} \overline{Y}^i \in \mathscr{S}^2, \overline{K}^i \in \mathscr{A}^2, \overline{Z}^i \in \mathscr{H}^{2,d}; \\ \overline{Y}_s^i = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, (\tilde{u}^j(r, X_r^{t,x}))_{j=1,...,m}, \overline{Z}_r^i)dr \\ \qquad + \int_s^T \psi_i(r, X_r^{t,x}, (\tilde{u}^j(r, X_r^{t,x}))_{j=1,...,m})d|\Lambda^{t,x}|_r + \overline{K}_T^i - \overline{K}_s^i - \int_s^T \overline{Z}_r^i dW_r, \\ \overline{Y}_s^i \geqslant \max_{j \neq i}(\overline{Y}_s^j - g_{ij}(s, X_s^{t,x})), \\ \int_0^T \{\overline{Y}_s^i - \max_{j \neq i}(\overline{Y}_s^j - g_{ij}(s, X_s^{t,x}))\}d\overline{K}_s^i = 0. \end{cases} \tag{4.69}$$

Now, we rely on Theorem 4.1 which provides the existence of $(\overline{u}^i)_{i=1,...,m}$ a continuous function on $\tilde{D}$ such that $\overline{Y}_s^i = \overline{u}^i(s, X_s^{t,x})$, $\forall t \leqslant s \leqslant T$. Moreover, $(\overline{u}^i)_{i=1,...,m}$ is the unique viscosity solution of the following system of PDEs: $\forall i = 1, ..., m$

$$\begin{cases} \min\{\overline{u}^i(t,x) - \max_{j \neq i}(\overline{u}^j(t,x) - g_{ij}(t,x)); -\partial_t \overline{u}^i(t,x) - \mathscr{L}\overline{u}^i(t,x) \\ \qquad - f_i(t, x, (\tilde{u}^j(t,x))_{j=1,...,m}, \sigma^\top(t,x) D_x \overline{u}^j(t,x))\} = 0, \ (t,x) \in D^o; \\ \dfrac{\partial \overline{u}^i}{\partial \gamma}(t,x) + \psi_i(t, x, (\tilde{u}^j(t,x))_{j=1,...,m}) = 0, \ (t,x) \in \partial D; \\ \overline{u}^i(T,x) = h_i(x), \ x \in \overline{D}_T. \end{cases} \tag{4.70}$$

But $(\tilde{u}^i)_{i=1,...,m}$ is also solution of system (4.70). Therefore, for any $i = 1, \ldots, m$ we have $\overline{u}^i = \tilde{u}^i$ hence $\overline{Y}^i = \tilde{Y}^i$,i.e., $(\overline{Y}^i)_{i=1,...,m} = \Phi\big((\overline{Y}^i)_{i=1,...,m}\big)$. On the other hand $(Y^i)_{i=1,...,m}$ is the unique fixed point of Φ. This implies that we have necessarily $(\tilde{Y}^i)_{i=1,...,m} = (Y^i)_{i=1,...,m}$. Thus, for any $i = 1, \ldots, m$ we have $\overline{u}^i = \tilde{u}^i$, which proves our claim.

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