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Limit law for Zagreb and Wiener indices of random exponential recursive trees

Ali Q. M. Al-Saedi* Ramin Imany Nabiyyi† Mehri Javanian‡

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Abstract

The Wiener index is the sum of distances of all pairs of nodes in a graph; and the Zagreb index is defined as the sum of squares of the out-degrees of nodes in a rooted tree. In this note, we calculate the first two moments of the Wiener and Zagreb indices of random exponential recursive trees (random ERTs) from two systems of recurrence relations. Then, by an application of the contraction method, we characterize the limit law for a scaled Zagreb index of ERTs. Via martingale convergence theorem, we also show the almost sure convergence and quadratic mean convergence of an appropriate scaled Wiener index that is indicative the distance of two randomly chosen nodes.

AMS 2000 Subject Classification : 05C05, 05C09, 60F25.

Key words: random exponential recursive trees, Wiener index, Zagreb index, limit law, L_2 -norm, Wasserstein metric, contraction method, martingale convergence theorem.

1 Introduction and setting

A *random recursive tree* of size n is starting with a root, i.e. node 1, and node 2 connected to the root node. Afterwards, at step i ($i = 3, \dots, n$) node i is connected uniformly at random to one of the prior nodes $1, \dots, i - 1$ (see Figure 1). Random recursive trees have been used as a model for various phenomena such as the stochastic growth of networks [3]; genealogy of ancient and medieval texts [13]; and the spread of epidemics [12]. For only a few of variations of recursive trees, see [6], [7], [8], [9], [10] and [18]; for fundamental properties of recursive trees, see [17].

Since only one node is added at each step in the growth of a recursive tree, then these slow-growing models cannot be suitable for fast-growing phenomena such as the growing the Twitter and the spreading of infectious diseases. In [11], it has been proposed a fast-growing recursive tree

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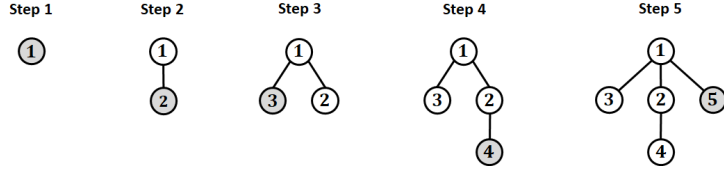


Figure 1: The steps of evolution of a recursive tree of size 5.

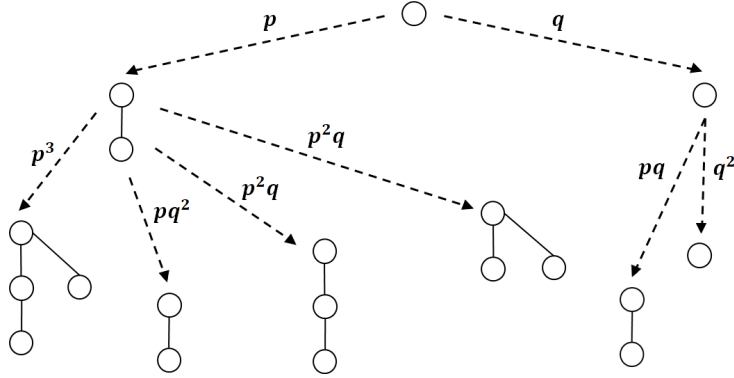


Figure 2: The first two steps of evolution of the exponential recursive tree of index p . The probability of getting each tree from the previous one appears above the arrow leading to it.

model, i.e. the *exponential recursive tree* (ERT), where every node independently attracts a new node with probability p , or fails to do at each step in the growth of the tree. The parameter p is called the *index* of the ERT. The *age* of an ERT is the number of steps it takes to grow the tree.

A *random ERT of index p* grown as follows: Initially (at age 0), there is only one node. For $n \geq 1$, at the n th step (at age n), every node independently attracts a new node with probability $p \in (0, 1)$, or not to attract with probability $q := 1 - p$. Figure 2 shows all ERTs of ages 0, 1 and 2. The *size* (the number of nodes) and the *total path length* (the sum of the distances between all nodes and the root) have been studied for ERTs by [1].

The *out-degree* of a node is the number of its children in a rooted tree. Javanian and Vahidi-Asl [5] and Javanian [8] have given the mean, variance and the normal limiting distribution of the out-degree of a node in recursive trees and scaled attachment recursive trees, respectively.

By *Zagreb index*, we mean the sum of out-degrees of all nodes in a rooted tree. The mean, variance and limiting distribution of Zagreb index for recursive trees have been investigated in [19].

The *Wiener index*, a distance-based topological index, is defined as the sum of the distances between all nodes in a graph. For recursive trees, Neininger [15] has obtained the expectations, the variances and a bivariate limit law for the Wiener index and the total path length. It also has been obtained a strong limit theorem for the Wiener index of recursive trees in [4].

Our interest here is to study the Zagreb and Wiener indices of ERTs. We give the first two moments of the Zagreb and Wiener indices of random ERTs. By the contraction method, we also characterize the limiting distribution of a scaled Zagreb and Wiener indices for random ERTs.

We use the recursive structure of the random ERTs under consideration in order to setup some recurrences for the Zagreb and Wiener indices. We first need to state the argument made in the proof of Lemma 1 in [1] as follows: At the first step (at age $n = 1$) of the growth of a random ERT of index p , the root node either succeeds to attract a node (event \mathcal{R}) or not. We denote the indicator function of the event \mathcal{R} by \mathbb{I} . If the event \mathcal{R} does not occur, then after $n - 1$ steps, the root of the tree fathers an exponential recursive tree T'_{n-1} . Alternatively, if \mathcal{R} occurs, then after $n - 1$ steps, the root of the tree and its only child that appeared in step 1, produce two independent exponential recursive trees T''_{n-1} and T'''_{n-1} , respectively. Now, we hook T''_{n-1} to T'''_{n-1} to construct a random ERT of index p at age n . Let (S'_{n-1}, X'_{n-1}) , (S''_{n-1}, X''_{n-1}) and (S'''_{n-1}, X'''_{n-1}) be the pairs of the sizes and the total path lengths in T'_{n-1} , T''_{n-1} and T'''_{n-1} , respectively. Then for a random ERT of index p at age n with the size S_n and the total path length X_n (see [1] and [11]),

$$S_n \stackrel{d}{=} (1 - \mathbb{I})S'_{n-1} + \mathbb{I}(S''_{n-1} + S'''_{n-1}), \quad (1)$$

$$X_n \stackrel{d}{=} (1 - \mathbb{I})X'_{n-1} + \mathbb{I}(X''_{n-1} + X'''_{n-1} + S'''_{n-1}), \quad (2)$$

where the symbol $\stackrel{d}{=}$ denotes the equality in distribution; the pairs (S'_{n-1}, X'_{n-1}) , (S''_{n-1}, X''_{n-1}) and (S'''_{n-1}, X'''_{n-1}) are independent copies of (S_{n-1}, X_{n-1}) and independent of \mathbb{I} . Moreover, the almost sure convergence of the scaled size and total path length are shown in [11] and [1], as $n \rightarrow \infty$,

$$\frac{S_n}{(p+1)^n} \xrightarrow{\text{a.s.}} S_*, \quad \left(\mathbb{E}[S_*] = 1, \mathbb{E}[S_*^2] = \frac{2}{p+1} \right), \quad (3)$$

for a limiting variable S_* characterized by inductively-constructed moments.

To setup the recurrences for the Zagreb and Wiener indices, let (Z'_{n-1}, W'_{n-1}) , (Z''_{n-1}, W''_{n-1}) and (Z'''_{n-1}, W'''_{n-1}) be the pairs of the Zagreb indices and the Wiener indices in T'_{n-1} , T''_{n-1} and T'''_{n-1} , respectively. Let B''_{n-1} be the out-degree of the root of T''_{n-1} , then $B''_{n-1} \stackrel{d}{=} \text{Binomial}(n-1, p)$. Thus for a random ERT of index p at age n with the Zagreb index Z_n and the Wiener index W_n , by direct enumeration we obtain the recurrences

$$Z_n \stackrel{d}{=} (1 - \mathbb{I})Z'_{n-1} + \mathbb{I}(Z''_{n-1} + Z'''_{n-1} + 2B''_{n-1} + 1), \quad (4)$$

where $Z'_n \stackrel{d}{=} Z''_n \stackrel{d}{=} Z'''_n \stackrel{d}{=} Z_n$; \mathbb{I} , Z'_n , (Z''_n, B''_n) and Z'''_n are independent; and

$$W_n \stackrel{d}{=} (1 - \mathbb{I})W'_{n-1} + \mathbb{I}(W''_{n-1} + W'''_{n-1} + X''_{n-1}S'''_{n-1} + X'''_{n-1}S''_{n-1} + S''_{n-1}S'''_{n-1}), \quad (5)$$

where the triples (S''_n, X''_n, W''_n) and (S'''_n, X'''_n, W'''_n) are independent copies of (S_n, X_n, W_n) and are independent of W'_n and \mathbb{I} .

In Section 2, we calculate the first two moments of Z_n and W_n . In Section 3 and Section 4, using the contraction method and martingale convergence theorem, respectively, we show the convergence in distribution, in L^2 and almost sure convergence for the scaled Zagreb index and Wiener index, that is, as $n \rightarrow \infty$,

$$\frac{Z_n}{(p+1)^n} \xrightarrow{d} \hat{Z}, \quad \frac{W_n}{n(p+1)^{2n}} \xrightarrow{L^2 \text{ \& a.s.}} \frac{p}{p+1} S_*^2,$$

where the random variable \hat{Z} is the unique distributional fixed-point of a distributional recurrence.

2 The first two moments

In this section, we calculate the first two moments of Z_n and W_n by the equations (4) and (5). The authors of [11] and [1] (Theorem 2.1 and Section 5, respectively) have found

$$\mathbb{E}[S_n] = (p+1)^n, \quad \mathbb{E}[S_n^2] = 2(p+1)^{2n-1} - (1-p)(p+1)^{n-1}, \quad \mathbb{E}[X_n] = np(p+1)^{n-1}.$$

Theorem 1. *Let Z_n be the Zagreb index in a random ERT of age n and index p . We have*

$$\begin{aligned} \mathbb{E}[Z_n] &= 3(p+1)^n - 2np - 3, & \text{for } n \geq 1, \\ \mathbb{E}[Z_n^2] &= 18(p+1)^{2n-1} - 12pn(p+1)^n + (4p^2 - 9p - 31)(p+1)^{n-1} \\ &\quad + 4n^2p^3 - 4p(p^2 - 2p - 2)n - 4p + 13, & \text{for } n \geq 1. \end{aligned}$$

Proof. Taking the expectation of the equation (4), one gets

$$\mathbb{E}[Z_n] = (p+1)\mathbb{E}[Z_{n-1}] + 2p^2(n-1) + p,$$

with initial condition $\mathbb{E}[Z_0] = 0$. Iterating this recurrence we find

$$\mathbb{E}[Z_n] = \sum_{i=1}^n (p+1)^{n-i} (2p^2(i-1) + p).$$

This sum yields the explicit solution as stated in the theorem.

Raise both sides of the distributional equation (4) to the second power and take expectations:

$$\begin{aligned} \mathbb{E}[Z_n^2] &= \mathbb{E} \left[(1 - \mathbb{I})(Z'_{n-1})^2 + \mathbb{I} \left((Z''_{n-1})^2 + (Z'''_{n-1})^2 + (2B''_{n-1} + 1)^2 \right. \right. \\ &\quad \left. \left. + 2(Z''_{n-1})(Z'''_{n-1}) + 2(Z''_{n-1})(2B''_{n-1} + 1) + 2(Z'''_{n-1})(2B''_{n-1} + 1) \right) \right]. \end{aligned}$$

Since $Z'_n \stackrel{d}{=} Z''_n \stackrel{d}{=} Z'''_n \stackrel{d}{=} Z_n$; $B''_n \stackrel{d}{=} \text{Binomial}(n, p)$; \mathbb{I} , Z'_n , (Z''_n, B''_n) and Z'''_n are independent, then

$$\begin{aligned} \mathbb{E}[Z_n^2] &= (p+1)\mathbb{E}[Z_{n-1}^2] + 2p(\mathbb{E}[Z_{n-1}])^2 + (4p^2(n-1) + 4p)\mathbb{E}[Z_{n-1}] \\ &\quad + 4p\mathbb{E}[Z_{n-1}B_{n-1}] + 4p^3n^2 + (8p^2 - 12p^3)n + 8p^3 - 8p^2 + p, \end{aligned}$$

where $B_n \stackrel{d}{=} B''_n$. In this recurrence, we need the expectation of $\mathbb{E}[Z_{n-1}B_{n-1}]$. This can be obtained from a recurrence for $\mathbb{E}[Z_{n-1}B_{n-1}]$ itself.

Let B'_{n-1} be the out-degree of the root of T'_{n-1} . Then for a random ERT of age n with the root of out-degree B_n , we have

$$B_n \stackrel{d}{=} (1 - \mathbb{I})B'_{n-1} + \mathbb{I}(B''_{n-1} + 1). \quad (6)$$

Multiply (4) and (6), and take expectations to get

$$\mathbb{E}[Z_n B_n] = \mathbb{E}[Z_{n-1} B_{n-1}] + (p^2(n-1) + 2p)\mathbb{E}[Z_{n-1}] + 2p^3n^2 + (5p^2 - 6p^3)n + 4p^3 - 5p^2 + p.$$

with the initial value $\mathbb{E}[Z_0 B_0] = 0$. This gives the solution

$$\mathbb{E}[Z_n B_n] = 3pn(p+1)^n + 3(1-p)(p+1)^n - (p^3 + p^2)n^2 + (p^3 + p^2 - 5p)n + 3p - 3.$$

Consequently

$$\begin{aligned} \mathbb{E}[Z_n^2] &= p \sum_{i=1}^n (p+1)^{n-i} \left(2(\mathbb{E}[Z_{i-1}])^2 + (4p(i-1) + 4)\mathbb{E}[Z_{i-1}] \right. \\ &\quad \left. + 4\mathbb{E}[Z_{i-1} B_{i-1}] + 4p^2 i^2 + (8p - 12p^2)i + 8p^2 - 8p + 1 \right) \\ &= 18(p+1)^{2n-1} - 12pn(p+1)^n + (4p^2 - 9p - 31)(p+1)^{n-1} \\ &\quad + 4n^2 p^3 - 4p(p^2 - 2p - 2)n - 4p + 13, \end{aligned}$$

where the calculation has been done by MAPLE software. □

Theorem 2. *Let W_n be the wiener index in a random ERT of age n and index p . We have*

$$\begin{aligned} \mathbb{E}[W_n] &= 2pn(p+1)^{2n-2} - (p+1)^{2n-1} + (p+1)^{n-1}, \quad \text{for } n \geq 1, \\ \mathbb{E}[W_n^2] &= \frac{24(p+6)p^2}{(p+2)(p^2+3p+3)} n^2 (p+1)^{4n-5} + \mathcal{O}(n(p+1)^{4n-5}), \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Proof. Take expectations both sides of the equation (5). By the identical distribution of W'_n, W''_n, W'''_n and W_n ; and the independence of $\mathbb{I}, (S''_n, X''_n)$ and (S'''_n, X'''_n) , we get

$$\begin{aligned} \mathbb{E}[W_n] &= (p+1)\mathbb{E}[W_{n-1}] + 2p\mathbb{E}[X_{n-1}]\mathbb{E}[S_{n-1}] + p(\mathbb{E}[S_{n-1}])^2 \\ &= (p+1)\mathbb{E}[W_{n-1}] + 2p^2(n-1)(p+1)^{2n-3} + p(p+1)^{2n-2}, \end{aligned}$$

with $\mathbb{E}[W_0] = 0$. By iterating, we have

$$\mathbb{E}[W_n] = \sum_{i=1}^n (p+1)^{n-i} \left(2p^2(i-1)(p+1)^{2i-3} + p(p+1)^{2i-2} \right),$$

that is obtained as claimed in the theorem.

For obtaining $\mathbb{E}[W_n^2]$, raise both sides of (5) to the second power and take expectations. The second moment recurrence takes the form,

$$\begin{aligned} \mathbb{E}[W_n^2] &= (p+1)\mathbb{E}[W_{n-1}^2] + 2p\mathbb{E}[S_{n-1}^2]\mathbb{E}[X_{n-1}^2] + p(\mathbb{E}[S_{n-1}^2])^2 + 2p(\mathbb{E}[W_{n-1}])^2 \\ &\quad + 4p\mathbb{E}[W_{n-1}X_{n-1}]\mathbb{E}[S_{n-1}] + 4p\mathbb{E}[W_{n-1}S_{n-1}]\mathbb{E}[X_{n-1}] + 4p\mathbb{E}[W_{n-1}S_{n-1}]\mathbb{E}[S_{n-1}] \\ &\quad + 2p(\mathbb{E}[X_{n-1}S_{n-1}])^2 + 4p\mathbb{E}[X_{n-1}S_{n-1}]\mathbb{E}[S_{n-1}^2]. \end{aligned} \tag{7}$$

In the recurrence (7), we need the expectations $\mathbb{E}[X_n^2]$, $\mathbb{E}[W_n X_n]$, $\mathbb{E}[W_n S_n]$ and $\mathbb{E}[X_n S_n]$.

From (1), (2) and (5), we have the following recurrences, for $n \geq 1$,

$$\begin{aligned}
\mathbb{E}[X_n^2] &= (p+1)\mathbb{E}[X_{n-1}^2] + p\mathbb{E}[S_{n-1}^2] + 2p(\mathbb{E}[X_{n-1}])^2 + 2p\mathbb{E}[X_{n-1}S_{n-1}] + 2p\mathbb{E}[X_{n-1}]\mathbb{E}[S_{n-1}] \\
\mathbb{E}[X_n S_n] &= (p+1)\mathbb{E}[X_{n-1}S_{n-1}] + 2p\mathbb{E}[X_{n-1}]\mathbb{E}[S_{n-1}] + p\mathbb{E}[S_{n-1}^2] + p(\mathbb{E}[S_{n-1}])^2 \\
\mathbb{E}[W_n S_n] &= (p+1)\mathbb{E}[W_{n-1}S_{n-1}] + 2p\mathbb{E}[S_{n-1}]\mathbb{E}[X_{n-1}S_{n-1}] + 2p\mathbb{E}[X_{n-1}]\mathbb{E}[S_{n-1}^2] \\
&\quad + 2p\mathbb{E}[W_{n-1}]\mathbb{E}[S_{n-1}] + 2p\mathbb{E}[S_{n-1}]\mathbb{E}[S_{n-1}^2] \\
\mathbb{E}[W_n X_n] &= (p+1)\mathbb{E}[W_{n-1}X_{n-1}] + 2p\mathbb{E}[X_{n-1}^2]\mathbb{E}[S_{n-1}] + 2p\mathbb{E}[X_{n-1}]\mathbb{E}[X_{n-1}S_{n-1}] \\
&\quad + 3p\mathbb{E}[S_{n-1}]\mathbb{E}[X_{n-1}S_{n-1}] + 2p\mathbb{E}[X_{n-1}]\mathbb{E}[W_{n-1}] + p\mathbb{E}[W_{n-1}S_{n-1}] \\
&\quad + p\mathbb{E}[W_{n-1}]\mathbb{E}[S_{n-1}] + p\mathbb{E}[X_{n-1}]\mathbb{E}[S_{n-1}^2] + p\mathbb{E}[S_{n-1}]\mathbb{E}[S_{n-1}^2],
\end{aligned}$$

with initial conditions $\mathbb{E}[X_0^2] = 0$, $\mathbb{E}[W_0 X_0] = 0$, $\mathbb{E}[W_0 S_0] = 0$ and $\mathbb{E}[X_0 S_0] = 0$.

The above recurrence equations are of the form $y_n = (p+1)y_{n-1} + h_n$, with solution

$$y_n = \sum_{i=1}^n (p+1)^{n-i} h_i. \quad (8)$$

Therefore, after some calculations, their solutions are obtained as follows

$$\begin{aligned}
\mathbb{E}[X_n^2] &= 2p^2 n^2 (p+1)^{2n-3} + 2p(1-p)n(p+1)^{2n-3} + 2(1-p)(p+1)^{2n-3} \\
&\quad - (1-p)p^2 n^2 (p+1)^{n-3} - 3p(1-p)n(p+1)^{n-3} - 2(1-p)(p+1)^{n-3}, \\
\mathbb{E}[X_n S_n] &= 2pn(p+1)^{2n-2} + (1-p)(p+1)^{2n-2} - p(1-p)n(p+1)^{n-2} - (1-p)(p+1)^{n-2}, \\
\mathbb{E}[W_n S_n] &= \frac{12p}{p+2} n(p+1)^{3n-3} - \frac{4(3p^2+5p+1)}{(p+2)^2} (p+1)^{3n-3} - 4p(1-p)n(p+1)^{2n-3} \\
&\quad + (2+4p-2p^2)(p+1)^{2n-3} - \frac{2(1-p)(p^2+2p+2)}{(p+2)^2} (p+1)^{n-2}, \\
\mathbb{E}[W_n X_n] &= \frac{12p^2}{p+2} n^2 (p+1)^{3n-4} - \frac{2p(11p^2+15p-8)}{(p+2)^2} n(p+1)^{3n-4} \\
&\quad - \frac{2(1-p)(2p+3)(2p-1)}{(p+2)^2} (p+1)^{3n-4} - 4(1-p)p^2 n^2 (p+1)^{2n-4} \\
&\quad + 2p(2p-1)(3-p)n(p+1)^{2n-4} - (1-p)(p^2-6p-1)(p+1)^{2n-4} \\
&\quad - \frac{2p(1-p)(p^2+2p+2)}{(p+2)^2} n(p+1)^{n-3} + \frac{(1-p^2)(p^3-3p^2-10p-10)}{(p+2)^2} (p+1)^{n-4}.
\end{aligned}$$

Again, according to (8), the solution to the recurrence (7) is

$$\begin{aligned}
\mathbb{E}[W_n^2] &= \frac{24(p+6)p^2}{(p+2)(p^2+3p+3)} n^2 (p+1)^{4n-5} - \frac{8p(7p^5+81p^4+272p^3+369p^2+171p-18)}{(p+2)^2(p^2+3p+3)^2} \\
&\quad \times n(p+1)^{4n-5} + \frac{8(4p^8+58p^7+286p^6+687p^5+883p^4+614p^3+276p^2+180p+99)}{(p+2)^2(p^2+3p+3)^3} (p+1)^{4n-5} \\
&\quad + \mathcal{O}(n^2(p+1)^{3n}), \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Hence we obtain the assertion. \square

3 Quadratic mean convergence for the Zagreb index

In this section we apply the contraction method in [16] or the multivariate contraction method in [14] to state the limiting distribution of $\hat{Z}_n := \frac{Z_n}{(1+p)^n}$.

We first need to set up some notation as follows: For a random variable X , we write $X \sim \lambda$ if the distribution of X is λ , i.e. the law $\mathcal{L}(X)$ of X is λ . The symbol $\|X\|_2 := (\mathbb{E}[|X|^2])^{1/2}$ denotes the usual L_2 -norm of X . The Wasserstein-metric ℓ_2 is defined on the space of probability distributions with finite second moments by

$$\ell_2(X, Y) := \ell_2(\mathcal{L}(X), \mathcal{L}(Y)) := \ell_2(\lambda, \nu) := \inf\{\|X - Y\|_2 : X \sim \lambda, Y \sim \nu\}.$$

By \mathcal{M}_2 the space of all probability distributions λ with mean 3 (as in Theorem 1) and finite second moment is denoted. The metric space (\mathcal{M}_2, ℓ_2) is complete and convergence in ℓ_2 is equivalent to convergence in distribution plus convergence of the second moments.

By equation (4) we have

$$\begin{aligned} \frac{Z_n}{(1+p)^n} &\stackrel{d}{=} \frac{\mathbb{I}}{1+p} \left(\frac{Z''_{n-1}}{(1+p)^{n-1}} + \frac{Z'''_{n-1}}{(1+p)^{n-1}} \right) \\ &\quad + \frac{1-\mathbb{I}}{1+p} \cdot \frac{Z'_{n-1}}{(1+p)^{n-1}} + \mathbb{I} \cdot \frac{2B''_{n-1} + 1}{(1+p)^n}. \end{aligned} \quad (9)$$

Theorem 3. *Let Z_n be the Zagreb index of a random ERT of age n and index p . The normalized Zagreb index $\hat{Z}_n := \frac{Z_n}{(1+p)^n}$ satisfies the distributional recursion*

$$\hat{Z}_n \stackrel{d}{=} \frac{\mathbb{I}}{1+p} \cdot (\hat{Z}''_{n-1} + \hat{Z}'''_{n-1}) + \frac{1-\mathbb{I}}{1+p} \cdot \hat{Z}'_{n-1} + \mathbb{I} \cdot \frac{2B_{n-1} + 1}{(1+p)^n}, \quad (10)$$

where $B_n \stackrel{d}{=} \text{Binomial}(n, p)$, \hat{Z}'_n , (\hat{Z}''_n, B_n) , \hat{Z}'''_n and \mathbb{I} are independent, $\hat{Z}'_n \stackrel{d}{=} \hat{Z}''_n \stackrel{d}{=} \hat{Z}'''_n \stackrel{d}{=} \hat{Z}_n$, and \mathbb{I} is a Bernoulli random variable with success probability p , then $\hat{Z}_n \xrightarrow{d} \hat{Z}$, as $n \rightarrow \infty$, where the random variable \hat{Z} is the unique distributional fixed-point of

$$\hat{Z} \stackrel{d}{=} \frac{\mathbb{I}}{1+p} (\hat{Z}'' + \hat{Z}''') + \frac{1-\mathbb{I}}{1+p} \hat{Z}', \quad (11)$$

with \hat{Z}' , \hat{Z}'' and \hat{Z}''' are independent copies of \hat{Z} and independent of \mathbb{I} .

Proof. The equation (10) is an immediate consequence of the equation (9), where $\hat{Z}'_n := \frac{Z'_n}{(1+p)^n}$, $\hat{Z}''_n := \frac{Z''_n}{(1+p)^n}$ and $\hat{Z}'''_n := \frac{Z'''_n}{(1+p)^n}$. Using Chebyshev's inequality and $\mathbb{E}[B_{n-1}] = (n-1)p$, we have the convergence in probability $\frac{2B_{n-1} + 1}{(1+p)^n} \xrightarrow{p} 0$. That is,

$$\hat{Z}_n \stackrel{d}{=} \frac{\mathbb{I}}{1+p} \cdot (\hat{Z}''_{n-1} + \hat{Z}'''_{n-1}) + \frac{1-\mathbb{I}}{1+p} \cdot \hat{Z}'_{n-1} + o_p(1), \quad (12)$$

where $o_p(1)$ denotes a quantity tending to zero in probability. For purposes of convergence we can, and hence will, ignore the $o_p(1)$ term.

Consider the well-defined transformation $T : \mathcal{M}_2 \rightarrow \mathcal{M}_2$,

$$T(\lambda) := \mathcal{L}\left(\frac{\mathbb{I}}{1+p}(\hat{Z}'_\lambda + \hat{Z}''_\lambda) + \frac{1-\mathbb{I}}{1+p}\hat{Z}'_\lambda\right), \quad (13)$$

where $\hat{Z}'_\lambda \stackrel{d}{=} \hat{Z}'$, $\hat{Z}''_\lambda \stackrel{d}{=} \hat{Z}''$ and $\hat{Z}'''_\lambda \stackrel{d}{=} \hat{Z}'''$ have λ as distribution and are independent of \mathbb{I} .

At a first step, we have to prove that the transformation T has a unique fixed point with respect to the ℓ_2 -metric. Let $\lambda, \nu \in \mathcal{M}_2$ be given. By (13), we have

$$\begin{aligned} T(\lambda) &= \mathcal{L}\left(\frac{\mathbb{I}}{1+p}(\hat{Z}''_\lambda + \hat{Z}'''_\lambda) + \frac{1-\mathbb{I}}{1+p}\hat{Z}'_\lambda\right), \\ T(\nu) &= \mathcal{L}\left(\frac{\mathbb{I}}{1+p}(\hat{Z}''_\nu + \hat{Z}'''_\nu) + \frac{1-\mathbb{I}}{1+p}\hat{Z}'_\nu\right), \end{aligned}$$

and

$$\begin{aligned} \ell_2^2(T(\lambda), T(\nu)) &\leq 2 \frac{\mathbb{E}[\mathbb{I}^2]}{(1+p)^2} \mathbb{E}[|\hat{Z}'_\lambda - \hat{Z}'_\nu|^2] + \frac{\mathbb{E}[(1-\mathbb{I})^2]}{(1+p)^2} \mathbb{E}[|\hat{Z}'_\lambda - \hat{Z}'_\nu|^2] \\ &= \frac{1}{1+p} \mathbb{E}[|\hat{Z}'_\lambda - \hat{Z}'_\nu|^2]. \end{aligned}$$

Therefore, we have

$$\ell_2^2(T(\lambda), T(\nu)) \leq \frac{1}{1+p} \ell_2^2(\lambda, \nu).$$

Since $\frac{1}{1+p} < 1$, we deduce that T is a contraction with respect to the ℓ_2 -metric. Thus, Banach's fixed point theorem provides existence and uniqueness of solutions of the fixed point equation $T(\lambda) = \lambda$. By (12), if \hat{Z}_n converges in distribution to some random variable \hat{Z} , then \hat{Z} satisfies (11). Consequently, the distribution of \hat{Z} will be λ_0 , the unique fixed point of T , i.e. $\mathcal{L}(\hat{Z}) = T(\lambda_0) = \lambda_0$.

Therefore, we have to prove $\lim_{n \rightarrow \infty} \ell_2(\hat{Z}_n, \hat{Z}) = 0$ to conclude $\hat{Z}_n \xrightarrow{d} \hat{Z}$.

Since $\hat{Z}'_n \stackrel{d}{=} \hat{Z}''_n \stackrel{d}{=} \hat{Z}'''_n \stackrel{d}{=} \hat{Z}_n$ and $\hat{Z}' \stackrel{d}{=} \hat{Z}'' \stackrel{d}{=} \hat{Z}''' \stackrel{d}{=} \hat{Z}$, we deduce

$$\begin{aligned} \lim_{n \rightarrow \infty} \ell_2^2(\hat{Z}_n, \hat{Z}) &\leq \frac{p}{(1+p)^2} \lim_{n \rightarrow \infty} \left(\left\| \hat{Z}''_{n-1} - \hat{Z}'' \right\|_2^2 + \left\| \hat{Z}'''_{n-1} - \hat{Z}''' \right\|_2^2 \right) \\ &\quad + \frac{1-p}{(1+p)^2} \lim_{n \rightarrow \infty} \left\| \hat{Z}'_{n-1} - \hat{Z}' \right\|_2^2 \\ &\leq \frac{1}{1+p} \lim_{n \rightarrow \infty} \left\| \hat{Z}_n - \hat{Z} \right\|_2. \end{aligned}$$

Therefore, we have

$$\lim_{n \rightarrow \infty} \ell_2^2(\hat{Z}_n, \hat{Z}) \leq \frac{1}{1+p} \lim_{n \rightarrow \infty} \ell_2^2(\hat{Z}_n, \hat{Z}).$$

This is true only if $\lim_{n \rightarrow \infty} \ell_2(\hat{Z}_n, \hat{Z}) = 0$. □

4 Almost sure convergence for the Wiener index

In T_n , a random ERT of age n and index p , assume we numbered the nodes of the tree arbitrarily with the numbers $1, 2, \dots, S_n$. Let $D_{i,j}$ be the distance between nodes i and j (the number of edges on the path joins the nodes i and j), for $i, j \in \{1, 2, \dots, S_n\}$. The Wiener index of T_n is

$$W_n = \sum_{i=1}^{S_n} \sum_{\substack{j=1 \\ i < j}}^{S_n} D_{i,j}.$$

Theorem 4. *Let W_n be the Wiener index in a random ERT of age n and index p . As $n \rightarrow \infty$, we have almost sure convergence and in L^2*

$$\frac{W_n}{n(p+1)^{2n}} \xrightarrow{L^2 \text{ \& a.s.}} \frac{p}{p+1} S_*^2.$$

Proof. Let \mathbb{I}_k be a sequence of independent Bernoulli(p) random variables, defined on the same probability space as the trees. Consider $\{\mathbb{I}_k\}_{k=1}^\infty$ to be independent of the structure of the tree at any age, too. If each of two nodes i and j or both succeed in attracting a new node, then the contribution of the nodes i and j of T_{n-1} into W_n , is $D_{i,j}\mathbb{I}_{i < j} + (D_{i,j} + 1)\mathbb{I}_j + (D_{i,j} + 2)\mathbb{I}_i\mathbb{I}_j\mathbb{I}_{i < j}$, for $i, j \in \{1, 2, \dots, S_{n-1}\}$, and we write

$$\begin{aligned} W_n &= W_{n-1} + \sum_{i=1}^{S_{n-1}} \sum_{j=1}^{S_{n-1}} \mathbb{I}_j(D_{i,j} + 1) + \sum_{i=1}^{S_{n-1}} \sum_{\substack{j=1 \\ i < j}}^{S_{n-1}} \mathbb{I}_i\mathbb{I}_j(D_{i,j} + 2) \\ &= W_{n-1} + \sum_{i=1}^{S_{n-1}} \mathbb{I}_i + \sum_{\substack{i=1 \\ i \neq j}}^{S_{n-1}} \sum_{j=1}^{S_{n-1}} \mathbb{I}_j(D_{i,j} + 1) + \sum_{i=1}^{S_{n-1}} \sum_{\substack{j=1 \\ i < j}}^{S_{n-1}} \mathbb{I}_i\mathbb{I}_j(D_{i,j} + 2). \end{aligned}$$

Let \mathcal{F}_n be the sigma field generated by the first n steps of the evolution of T_n . Then

$$\begin{aligned} \mathbb{E}[W_n | \mathcal{F}_{n-1}] &= W_{n-1} + 2pW_{n-1} + pS_{n-1}^2 + p^2W_{n-1} + p^2S_{n-1}^2 - p^2S_{n-1} \\ &= (p+1)^2W_{n-1} + p(p+1)S_{n-1}^2 - p^2S_{n-1}. \end{aligned} \tag{14}$$

Let $M_n := \alpha_n W_n + \beta_n S_n^2 + \gamma_n S_n$, for specified factors α_n, β_n and γ_n that transformed M_n a martingale. So, by (8), (14) and a martingalization procedure, we have

$$\begin{aligned} \mathbb{E}[M_n | \mathcal{F}_{n-1}] &= \alpha_n(p+1)^2W_{n-1} + \alpha_n p(p+1)S_{n-1}^2 - \alpha_n p^2S_{n-1} \\ &\quad + \beta_n \mathbb{E}[S_n^2 | \mathcal{F}_{n-1}] + \gamma_n \mathbb{E}[S_n | \mathcal{F}_{n-1}] \\ &= \alpha_n(p+1)^2W_{n-1} + \alpha_n p(p+1)S_{n-1}^2 - \alpha_n p^2S_{n-1} \\ &\quad + \beta_n(p+1)^2S_{n-1}^2 + \beta_n p(1-p)S_{n-1} + \gamma_n(p+1)S_{n-1} \\ &= \alpha_n(p+1)^2W_{n-1} + (\alpha_n p(p+1) + \beta_n(p+1)^2)S_{n-1}^2 \\ &\quad + (\gamma_n(p+1) + \beta_n p(1-p) - \alpha_n p^2)S_{n-1} \\ &= M_{n-1} = \alpha_{n-1}W_{n-1} + \beta_{n-1}S_{n-1}^2 + \gamma_{n-1}S_{n-1}. \end{aligned}$$

This is possible, if we choose

$$\begin{aligned}\alpha_{n-1} &= \alpha_n(p+1)^2, \\ \beta_{n-1} &= \alpha_n p(p+1) + \beta_n(p+1)^2 \\ \gamma_{n-1} &= \gamma_n(p+1) + \beta_n p(1-p) - \alpha_n p^2.\end{aligned}$$

From these solvable recurrences, we obtain

$$\begin{aligned}\alpha_n &= \frac{\alpha_0}{(p+1)^{2n}}, & \beta_n &= \frac{\beta_0}{(p+1)^{2n}} - \frac{\alpha_0 p n}{(p+1)^{2n+1}} \\ \gamma_n &= \frac{\gamma_0}{(p+1)^n} - \frac{\alpha_0 p(1-p)n + \alpha_0 p + \beta_0 p^2 - \beta_0 + \alpha_0}{(p+1)^{2n+2}} + \frac{\beta_0 p^2 + \alpha_0 p + \alpha_0 - \beta_0}{(p+1)^{n+2}}.\end{aligned}$$

For simplicity, we take $\alpha_0 = 1 = \gamma_0$ and $\beta_0 = 0$. Thus, by (10), (11), (12) and (13),

$$M_n = \frac{1}{(p+1)^{2n}} W_n - \frac{pn}{(p+1)^{2n+1}} S_n^2 + \left(\frac{p+2}{(p+1)^{n+1}} - \frac{p(1-p)n+p+1}{(p+1)^{2n+2}} \right) S_n,$$

is a martingale with bounded moments

$$\begin{aligned}\mathbb{E}[M_n] &= \frac{1}{(p+1)^{2n}} \mathbb{E}[W_n] - \frac{pn}{(p+1)^{2n+1}} \mathbb{E}[S_n^2] + \left(\frac{p+2}{(p+1)^{n+1}} - \frac{p(1-p)n+p+1}{(p+1)^{2n+2}} \right) \mathbb{E}[S_n] \\ &= \frac{2pn(p+1)^{2n-2} - (p+1)^{2n-1} + (p+1)^{n-1}}{(p+1)^{2n}} - \frac{2pn(p+1)^{2n-1} - p(1-p)n(p+1)^{n-1}}{(p+1)^{2n+1}} \\ &\quad - \frac{p(1-p)n+p+1}{(p+1)^{n+2}} + \frac{p+2}{p+1} = 1 < \infty,\end{aligned}$$

$$\begin{aligned}\mathbb{E}[M_n^2] &= \frac{1}{(p+1)^{4n}} \mathbb{E}[W_n^2] + \frac{p^2 n^2}{(p+1)^{4n+2}} \mathbb{E}[S_n^4] + \left(\frac{p+2}{(p+1)^{n+1}} - \frac{p(1-p)n+p+1}{(p+1)^{2n+2}} \right)^2 \mathbb{E}[S_n^2] \\ &\quad + 2 \left(\frac{p+2}{(p+1)^{3n+1}} - \frac{p(1-p)n+p+1}{(p+1)^{4n+2}} \right) \left(\mathbb{E}[W_n S_n] - \frac{pn}{p+1} \mathbb{E}[S_n^3] \right) - \frac{2pn}{(p+1)^{4n+1}} \mathbb{E}[W_n S_n^2] \\ &= \frac{1}{(p+1)^{4n}} \mathbb{E}[W_n^2] + \frac{p^2 n^2}{(p+1)^{4n+2}} \mathbb{E}[S_n^4] - \frac{2pn}{(p+1)^{4n+1}} \mathbb{E}[W_n S_n^2] + \mathcal{O}(1) \\ &= \frac{1}{(p+1)^{4n}} \left(\frac{24(p+6)p^2}{(p+2)(p^2+3p+3)} n^2 (p+1)^{4n-5} \right. \\ &\quad \left. - \frac{8p(7p^5+81p^4+272p^3+369p^2+171p-18)}{(p+2)^2(p^2+3p+3)^2} n(p+1)^{4n-5} \right) \\ &\quad + \frac{p^2 n^2}{(p+1)^{4n+2}} \left(\frac{24(p+6)(p+1)^{4n-3}}{(p+2)(p^2+3p+3)} \right) - \frac{2pn}{(p+1)^{4n+1}} \left(\frac{24p(p+6)n(p+1)^{4n-4}}{(p+2)(p^2+3p+3)} \right. \\ &\quad \left. - \frac{4(7p^5+81p^4+272p^3+369p^2+171p-18)}{(p+2)^2(p^2+3p+3)^2} (p+1)^{4n-4} \right) + \mathcal{O}(1) \\ &= 0 + \mathcal{O}(1) < \infty, \quad \text{as } n \rightarrow \infty,\end{aligned}$$

where

$$\begin{aligned}\mathbb{E}[S_n^3] &= \frac{12(p+1)^{3n-2}}{p+2} - 6(1-p)(p+1)^{2n-2} + \frac{(p^2-3p+2)(p+1)^{n-1}}{p+2}, \\ \mathbb{E}[S_n^4] &= \frac{24(p+6)(p+1)^{4n-3}}{(p+2)(p^2+3p+3)} - \frac{72(1-p)(p+1)^{3n-3}}{p+2} \\ &\quad - \frac{2(1-p)(7p^2-p-14)(p+1)^{2n-3}}{p+2} - \frac{(1-p)(p^4-6p^3+2p^2+3p+6)(p+1)^{n-2}}{(p+2)(p^2+3p+3)}, \\ \mathbb{E}[W_n S_n^2] &= \frac{24p(p+6)n(p+1)^{4n-4}}{(p+2)(p^2+3p+3)} - \frac{4(7p^5+81p^4+272p^3+369p^2+171p-18)}{(p+2)^2(p^2+3p+3)^2} (p+1)^{4n-4} \\ &\quad - \frac{60p(1-p)}{p+2} n(p+1)^{3n-4} - \frac{4(15p^3+p^2-35p+1)(p+1)^{3n-4}}{(p+1)^2} + \mathcal{O}(n(p+1)^{2n}), \quad \text{as } n \rightarrow \infty.\end{aligned}$$

So, by the martingale convergence theorem (Theorem 6.6.9 in [2]), M_n converges to some M_* almost surely and in L^2 . Consequently,

$$\begin{aligned}\frac{M_n}{n} &= \frac{W_n}{n(p+1)^{2n}} - \frac{p}{p+1} \cdot \left(\frac{S_n}{(p+1)^n} \right)^2 \\ &\quad + \left(\frac{p+2}{(p+1)^{n+1}} - \frac{p(1-p)n+p+1}{(p+1)^{2n+2}} \right) \cdot \frac{S_n}{n(p+1)^n} \xrightarrow{L^2 \text{ \& a.s.}} 0.\end{aligned}$$

Hence, by (3), the result follows. \square

Corollary 5. *Let D_n^* be the distance between two randomly chosen nodes in a random ERT of age n and index p . Then we have*

$$\frac{\mathbb{E}[D_n^*]}{n} \longrightarrow \frac{2p}{p+1}.$$

Proof. In T_n , a random ERT of age n and index p , we have

$$\mathbb{E}[D_n^*|T_n] = \frac{2 \sum_{\substack{i=1 \\ i < j}}^{S_n} \sum_{j=1}^{S_n} D_{i,j}}{S_n(S_n-1)} = \frac{2W_n}{S_n(S_n-1)}.$$

Therefore, by (3) and Theorem 4, we have

$$\frac{\mathbb{E}[D_n^*|T_n]}{n} = \frac{2W_n/(n(p+1)^{2n})}{S_n(S_n-1)/((p+1)^{2n})} \xrightarrow{\text{a.s.}} \frac{2p}{p+1}.$$

Since $W_n \leq \frac{1}{2}S_n(S_n-1)n$, then $\mathbb{E}[D_n^*|T_n]/n \leq 1$. Consequently, by dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[D_n^*]}{n} = \lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{\mathbb{E}[D_n^*|T_n]}{n} \right] = \mathbb{E} \left[\lim_{n \rightarrow \infty} \frac{\mathbb{E}[D_n^*|T_n]}{n} \right].$$

This yields the result. \square

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