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# On fixed divisors of the values of the minimal polynomials over $\mathbb{Z}$ of algebraic numbers

M. Ayad, A. Bayad and O. Kihel

**Abstract.** Let  $K$  be a number field of degree  $n$ ,  $A$  be its ring of integers, and  $A_n$  ( resp.  $K_n$ ) be the set of elements of  $A$  ( resp.  $K$ ) which are primitive over  $\mathbb{Q}$ . For any  $\gamma \in K_n$ , let  $F_\gamma(x)$  be the unique irreducible polynomial in  $\mathbb{Z}[x]$ , such that its leading coefficient is positive and  $F_\gamma(\gamma) = 0$ . Let  $i(\gamma) = \gcd_{x \in \mathbb{Z}} F_\gamma(x)$ ,  $i(K) = \text{lcm}_{\theta \in A_n} i(\theta)$  and  $\hat{i}(K) = \text{lcm}_{\gamma \in K_n} i(\gamma)$ . For any  $\gamma \in K_n$ , there exists a unique pair  $(\theta, d)$ , where  $\theta \in A_n$  and  $d$  is a positive integer such that  $\gamma = \theta/d$  and  $\theta \not\equiv 0 \pmod{p}$  for any prime divisor  $p$  of  $d$ . In this paper, we study the possible values of  $\nu_p(d)$  when  $p|i(\gamma)$ . We introduce and study a new invariant of  $K$  defined using  $\nu_p(d)$ , when  $\gamma$  describes  $K_n$ . In the last theorem of this paper, we establish a generalisation of a theorem of MacCluer.

**Keywords.** Values of polynomials, Denominators of algebraic numbers, Splitting of prime numbers.

**2010 Mathematics Subject Classification.** 11R04, 12Y05.

## 1. Introduction

Let  $K$  be a number field of degree  $n \geq 2$  and  $A$  be its ring of integers. Denote by  $A_n$  ( resp.  $K_n$ ) be the set of elements of  $A$  ( resp.  $K$ ) which are primitive over  $\mathbb{Q}$ , that is those elements which generate  $K$  over  $\mathbb{Q}$ . For any primitive polynomial  $g(x) \in \mathbb{Z}[x]$ , we define the integer  $i(g)$  by

$$i(g) = \gcd_{x \in \mathbb{Z}} g(x).$$

Let  $\gamma$  be an algebraic number. When we refer to the minimal polynomial of  $\gamma$  over  $\mathbb{Z}$ , we mean the unique polynomial  $F_\gamma(x) = a_n x^n + \dots + a_1 x + a_0 \in \mathbb{Z}[x]$ , irreducible such that  $a_n > 0$  and  $F_\gamma(\gamma) = 0$ . The leading coefficient  $a_n$  will be denoted by  $c(\gamma)$ . We set  $i(\gamma) = i(F_\gamma)$ . In [GuMc70], Gunji and McQuillan defined the integer  $i(K)$  by

$$i(K) = \text{lcm}_{\theta \in A_n} i(\theta).$$

MacCluer [Mac71] proved that a given prime  $p$  divides  $i(K)$  if and only if the number of prime ideals of  $A$  lying over  $p$  is at least equal to  $p$ . In [GuMc70] or [AyKi11], it is proved that there exists  $\theta \in A_n$  such that

$$i(K) = i(\theta).$$

The smallest positive integer  $d$  such that  $d\gamma$  is an algebraic integer is called the denominator of  $\gamma$  and will be denoted by  $d(\gamma)$ .

Arno et al proved in [ARW96] that the density of the set of the algebraic numbers  $\gamma$  such that  $c(\gamma) = d(\gamma)$  is equal to  $1/\zeta(3) = 0.8319\dots$ . In [ABK15], for a fixed number field  $K$ , the set

$$T_p(k) = \{t \geq 1, \text{ there exists } \gamma \in K_n, \nu_p(d(\gamma)) = k \text{ and } \nu_p(c(\gamma)) = t\}$$

is connected to the splitting of the prime  $p$  in  $K$ .

In this paper, among other results, we study the possible values of  $\nu_p(d)$  when  $p \mid i(\gamma)$ . After recalling some lemmas in section 2, it is proved in Theorem 3.1 that if  $\theta \not\equiv 0 \pmod{pA}$  and  $p \mid i(\theta/p^k)$ , for some positive integer  $k$ , then the  $p$ -adic valuation of the leading coefficient of the minimal polynomial of  $\theta/p^k$  belongs to the set  $\{k, k+1, \dots, (n-p)k\}$ . Furthermore any element of this set may occur. Section 4 shows that given  $\theta \in A_n$  such that  $\theta \not\equiv 0 \pmod{pA}$ , it is possible to find explicitly the values of  $k \in \mathbb{N}$ , if any, such that  $p \mid i(\theta/p^k)$ . Fix  $\theta \in A_n$  such that  $\theta \not\equiv 0 \pmod{pA}$  and define the set  $V_p(\theta) = \{k \in \mathbb{N} ; p \mid i(\theta/p^k)\}$ . On the one hand it is shown that  $|V_p(\theta)|$  for  $\theta \in A_n$  is bounded by some constant depending on  $n$  and  $p$ . On the other hand the values of  $k$  are bounded by a constant depending on  $p$  and  $\nu_p(N_{K/\mathbb{Q}}(\theta))$ , where  $\nu_p(a)$  denotes the  $p$ -adic valuation of  $a$ . Section 5 deals with this last bound. It is shown that, even if we fix the field  $K$ , the values of  $k$  may be greater than any given positive constant. In this section, we give examples of Galois number fields  $K$  of degree 4 (resp. 3) for which the set of the values of  $k$ , when  $\theta$  runs in  $A_n$  is  $\{0\}$  (resp.  $\mathbb{N}$ ). Throughout this paper we denote by  $\mathbb{N}$  the set of nonnegative integers. The paper ends with remarks and open questions.

## 2. Indices and denominators of algebraic numbers

Let  $K$  be a number field of degree  $n$ ,  $A$  its ring of integers,  $\gamma \in K_n$  and let  $g(x) = a_n x^n + \dots + a_1 x + a_0 \in \mathbb{Z}[x]$  be the unique primitive polynomial of degree  $n$  such that  $a_n > 0$  and  $g(\gamma) = 0$ . We denote this polynomial  $F_\gamma(x)$ . This polynomial will be called the minimal polynomial of  $\gamma$  over  $\mathbb{Z}$ . The leading coefficient  $a_n$  will be denoted  $c(\gamma)$ .

Let

$$\mathfrak{J}(\gamma) = \{m \in \mathbb{Z}, m\gamma \in A\},$$

then  $\mathfrak{J}(\gamma)$  is a nonzero ideal of  $\mathbb{Z}$ , hence a principal ideal generated by some positive integer denoted by  $d(\gamma)$ . The integer  $d(\gamma)$  is called the denominator of  $\gamma$ . Since  $c(\gamma) \in \mathfrak{J}(\gamma)$ , then  $d(\gamma) \mid c(\gamma)$ . Write  $\gamma = \frac{\theta}{d(\gamma)}$ , where  $\theta \in A_n$ , then  $\theta$  is unique and we call it the numerator of  $\gamma$ . Let  $f(x)$  be the minimal polynomial of  $\theta$  over  $\mathbb{Q}$ , then  $g(x) = f(d(\gamma)x)/\text{cont}(f(d(\gamma)x)$ , where the abbreviation  $\text{cont}(h(x))$  denotes the content of the polynomial  $h(x)$ . From [ARW96] we have the following result:

**Lemma 2.1.** *For any prime  $p$ , we have*

$$\nu_p(d(\gamma)) = \max \left( 0, \max_{j=0}^{n-1} \left\lceil \frac{\nu_p(a_n) - \nu_p(a_j)}{n-j} \right\rceil \right). \quad (2.1)$$

From Lemma 2.1 we see that any prime factor  $p$  of  $c(\gamma)$  divides  $d(\gamma)$ .

Summarizing the relations between  $c(\gamma)$  and  $d(\gamma)$ , we have:

**Remark 2.1.** *Let  $K$  be a number field of degree  $n$  and  $\gamma \in K_n$ , then  $d(\gamma)$  and  $c(\gamma)$  have the same prime factors and for any prime  $p$ , we have  $\nu_p(d(\gamma)) \leq \nu_p(c(\gamma)) \leq n\nu_p(d(\gamma))$ .*

For any  $\theta \in A_n$ , let  $F_\theta(x)$  its minimal polynomial over  $\mathbb{Q}$ . Following [GuMc70], we define the integers

$$i(\theta) = \gcd_{x \in \mathbb{Z}} F_\theta(x) \text{ and } i(K) = \text{lcm}_{\theta \in A_n} i(\theta). \quad (2.2)$$

For the integers  $i(\theta)$  and  $i(K)$  we have the following results:

- i) C. R. MacCluer proved in [Mac71] that for a given prime number  $p$ ,  $p$  divides  $i(K)$  if and only if the number of prime ideals of  $A$  lying over  $p$  is at least equal to  $p$ .
- ii) In [GuMc70] and [AyKi11], it is proved that there exists  $\theta \in A_n$  such that  $i(K) = i(\theta)$ , and that  $i(K) = \text{lcm}_{\theta \in A} i(\theta)$ .

We extend the definition of  $i(\theta)$  and  $i(K)$  to algebraic numbers as follows. Given any  $\gamma \in K_n$ , we define

$$i(\gamma) := \text{gcd}_{x \in \mathbb{Z}} F_\gamma(x) \text{ and } \hat{i}(K) = \text{lcm}_{\gamma \in K_n} i(\gamma).$$

We quote from [AyKi11] the following result.

**Lemma 2.2.** *Let  $g(x) \in \mathbb{Z}[x]$ . Write  $g(x)$  in the form*

$$g(x) = b_n(x)_n + \cdots + b_1(x)_1 + b_0,$$

where  $b_0, \dots, b_n \in \mathbb{Z}$ ,

$$(x)_0 = 1 \text{ and } (x)_j := x(x-1) \cdots (x-(j-1)), \text{ for } j \geq 1.$$

Then, we have the identity

$$i(g) = \text{gcd}_{j=0}^n (j!b_j).$$

**Corollary 2.1.** *The integers  $i(\gamma)$ ,  $i(K)$  and  $\hat{i}(K)$  divide  $n!$ .*

**Remark 2.2.** *Clearly, from the definitions of  $i(K)$  and  $\hat{i}(K)$ , we see that  $i(K)$  divides  $\hat{i}(K)$ .*

### 3. Study of the denominators of some algebraic numbers

We prove the following lemma, which is useful for the rest of this paper:

**Lemma 3.1.** *Let  $p$  be a prime number,  $\gamma$  be an algebraic number and  $d(\gamma)$  be its denominator. Write  $d(\gamma)$  in the form  $d(\gamma) = p^k \cdot q$ , where  $\text{gcd}(p, q) = 1$  and let  $\mu = q\gamma$ . Then we have*

$$d(\mu) = p^k, \nu_p(c(\mu)) = \nu_p(c(\gamma)) \text{ and } \nu_p(i(\gamma)) = \nu_p(i(\mu)).$$

*Proof:* Let  $\gamma = \theta/pk_q$  where  $\theta$  is an algebraic integer such that  $\theta \not\equiv 0 \pmod{p}$  and  $\theta \not\equiv 0 \pmod{l}$  for any prime factor  $l$  of  $q$ . Then  $\mu = q\gamma = \theta/pk$ , hence  $d(\mu) = pk$ . Let  $n$  be the degree of  $\gamma$  over  $\mathbb{Q}$  and let  $g(x) = a_nx_n + \cdots + a_1x + a_0$  be its minimal polynomial over  $\mathbb{Z}$ . Since  $a_n = c(\gamma)$  and  $d(\gamma)|c(\gamma)$ , then  $q|a_n$ . This implies that the polynomial  $h(x) = q^{n-1}g(x/q) = (a_n/q)x^n + a_{n-1}x^{n-1} + \cdots + a_1q^{n-2}x + a_0q^{n-1}$  has integral coefficients and vanishes for  $\mu$ . Therefore, the minimal polynomial of  $\mu$  over  $\mathbb{Z}$  is given by  $f(x) = h(x)/\text{cont}(h)$ . Write  $\text{cont}(h)$  in the form  $\text{cont}(h) = \text{gcd}(q, \text{cont}(h)) \cdot \lambda$ , where  $\lambda$  is a positive integer. We show that  $\lambda = 1$ . Suppose that there exists a prime number  $l|\lambda$ , then  $l|\text{cont}(h)$  and  $l \nmid q$ . Since  $l|a_0q^{n-1}, a_1q^{n-2}, \dots, a_{n-1}, a_n|q$ , then  $l|a_0, \dots, a_n$  which is a contradiction, hence  $\lambda = 1$ . Therefore  $\text{cont}(h)|q$ . Since

$$c(\mu) = (a_n/q)/\text{cont}(h) = a_n/q\text{cont}(h) = c(\gamma)/q\text{cont}(h),$$

then  $V_p(c(\mu)) = V_p(c(\gamma))$ . We now prove the last statement. Suppose that  $p^u | i(\gamma)$  for some positive integer  $u$  and  $x_0 \in \mathbb{Z}$ . Since  $\gcd(p, q) = 1$ , there exists  $y_0 \in \mathbb{Z}$  such that the congruence  $qy_0 \equiv x_0 \pmod{p^u}$  holds. In particular  $g(y_0) \equiv 0 \pmod{p^u}$ , so that we have,  $f(x_0) = h(x_0)/\text{cont}(h) = q^{n-1}g(y_0)/\text{cont}(h) \equiv 0 \pmod{p^u}$ . Since  $x_0$  was arbitrary, then  $p^u | i(\mu)$ . Conversely, suppose that  $p^u | i(\mu)$  and let  $x_0$  and  $y_0$  as above. Then  $f(x_0) = 0 \pmod{p^u}$ , hence  $q^{n-1}g(y_0)/\text{cont}(h) \equiv 0 \pmod{p^u}$ , thus  $g(y_0) \equiv 0 \pmod{p^u}$ . Therefore  $p^u | i(\gamma)$ .

We state the main result of this section.

**Theorem 3.1.** *Let  $K$  be a number field of degree  $n$  over  $\mathbb{Q}$  and  $p$  be a prime number, Let  $\gamma \in K_n$  such that  $p | i(\gamma)$ ,  $c = c(\gamma)$ ,  $d = d(\gamma)$  and  $k = v_p(d) \geq 1$ . Then  $p < n$  and  $k \leq v_p(c) \leq (n - p)k$ .*

*Proof:* Set  $d = p^k q$  with  $\gcd(p, q) = 1$ . Let  $\mu = q\gamma$ , then  $d(\mu) = p^k$  and by Lemma 3.1, we have  $v_p(c(\mu)) = v_p(c(\gamma))$  and  $v_p i(\mu) = v_p i(\gamma) \geq 1$ . Therefore,  $p | i(\mu)$ . The minimal polynomial of  $\mu$  over  $\mathbb{Z}$  has the form:

$$g(x) = p^t(x)_n + b_{n-1}(x)_{n-1} + \cdots + b_1(x)_1 + b_0, \text{ with } k \leq t \leq nk.$$

By corollary 2.1,  $p | n!$ , hence  $p \leq n$ . If  $p = n$ , since  $p | j! b_j$  for all  $j = 0, \dots, n - 1$ , then we conclude that  $p | b_0, b_1, \dots, b_{n-1}$ . Therefore  $g(x)$  is reducible in  $\mathbb{Z}[x]$ , which is a contradiction, hence  $p < n$ . In this case, since  $p | b_0, b_1, \dots, b_{p-1}$ , then we may write  $g(x)$  in the form

$$g(x) = x(x-1) \cdots (x-(p-1)) \left( p^t x^{n-p} + \tilde{a}_{n-p-1} x^{n-p-1} + \cdots + \tilde{a}_1 x + \tilde{a}_0 \right) \\ + p \left( c_{p-1} x^{p-1} + \cdots + c_1 x + c_0 \right)$$

where all the coefficients  $\tilde{a}_i, c_j$  are integral. Since  $g(x)$  is irreducible in  $\mathbb{Z}[x]$ , there exists  $j \in \{0, \dots, n - p - 1\}$  such that  $p \nmid \tilde{a}_j$ . Denote by  $j_o$  the greatest of these integers. Let  $\theta \in A_n$  be the unique element such that  $\gamma = \frac{\theta}{p^k}$ . Then we have

$$\theta(\theta - p^k) \cdots (\theta - (p-1)p^k) \left( p^t \theta^{n-p} + p^k \tilde{a}_{n-p-1} \theta^{n-p-1} \right. \\ \left. + \cdots + p^{k(n-p-j_o)} \tilde{a}_{j_o} \theta^{j_o} + \cdots + p^{k(n-p-1)} \tilde{a}_1 \theta + p^{k(n-p)} \tilde{a}_0 \right) \\ + p \cdot p^{k(n-(p-1))} (c_{p-1} \theta^{p-1} + c_{p-2} p^k \theta^{p-2} + \cdots + c_0 p^{k(p-1)}) = 0.$$

Write this equation in the form:

$$p^t \theta^n + u_{n-1} \theta^{n-1} + \cdots + u_p \theta^p + \cdots + u_1 \theta + u_0 = 0.$$

Since  $\theta$  is integral, it follows in particular that  $p^t | u_j$  for  $j = p, \dots, n - 1$ . We can set

$$\theta(\theta - p^k) \cdots (\theta - (p-1)p^k) = \theta^p + \sigma_1 \theta^{p-1} + \cdots + \sigma_{p-1} \theta.$$

Then

$$\left\{ \begin{array}{l} \sigma_1 = -(p^k + \cdots + p^k(p-1)) = p^k \frac{p(p-1)}{2} \\ \sigma_2 = p^{2k} \sum_{\substack{i \neq j \\ i, j \in \{1, \dots, p-1\}}} ij \\ \dots \\ \sigma_{p-1} = (-1)^{p-1} p^{k(p-1)} (p-1)! \end{array} \right.$$

We have

$$(x^p + \sigma_1 x^{p-1} + \cdots + \sigma_{p-1} x) \left( p^t x^{n-p} + \cdots + p^{k(n-p-j_0)} \tilde{a}_{j_0} x^{j_0} + \cdots + \tilde{a}_0 p^{k(n-p)} \right) + p p^{k(n-(p-1))} \left( c_{p-1} x^{p-1} + \cdots + c_1 x \right) = p^t x^n + u_{n-1} x^{n-1} + \cdots + u_p x^p + \cdots + u_1 x + u_0,$$

hence

$$\begin{cases} u_{n-1} &= p^k \tilde{a}_{n-p-1} + p^t \sigma_1 \\ u_{n-2} &= p^{2k} \tilde{a}_{n-p-2} + p^k \tilde{a}_{n-p-1} \sigma_1 + p^t \sigma_2 \\ &\dots \\ u_{j_0+p} &= \tilde{a}_{j_0} p^{k(n-p-j_0)} + \tilde{a}_{j_0+1} p^{k(n-p-(j_0+1))} \sigma_1 + \cdots + \tilde{a}_{j_0+m} p^{k(n-p-(j_0+m))} \sigma_m \\ &+ \cdots + p^t \sigma_{n-(j_0+p)} \end{cases}$$

The first equation implies that  $p^t | p^k \tilde{a}_{n-p-1}$ . Then the second implies that  $p^t | p^{2k} \tilde{a}_{n-p-2}$ . Iterating the process, the last equation gives  $p^t | \tilde{a}_{j_0} p^{k(n-p-j_0)}$ . Since  $p \nmid \tilde{a}_{j_0}$ , then

$$t \leq k(n-p-j_0) \leq k(n-p).$$

**Theorem 3.2.** *Let  $p$  be a prime number,  $n$  and  $k$  be positive integers,  $p < n$ . Then for any integer  $t$ , such that  $k \leq t \leq (n-p)k$ , there exist infinitely many algebraic numbers  $\gamma \in \mathbb{C}$  of degree  $n$  such that  $p | i(\gamma)$ ,  $\nu_p(c(\gamma)) = t$  and  $\nu_p(d(\gamma)) = k$ .*

*Proof:* Dividing  $t$  by  $k$ , we have two possibilities:

$$t = (n-i)k + \alpha \text{ with } 0 < \alpha < k \text{ and } p < i \leq n-1 \quad (3.3)$$

$$t = (n-i)k \text{ with } p \leq i \leq n-1 \quad (3.4)$$

- First case:  $t = (n-i)k + \alpha$  with  $0 < \alpha < k$  and  $p < i \leq n-1$ . Then, we have

$$t > (n-i)\alpha + \alpha = \alpha(n-i+1), \text{ hence } \alpha < \frac{t}{n-(i+1)}.$$

On the other hand, choose integers  $a_0, \dots, a_n$  such that

$$\begin{cases} \gcd(a_0, \dots, a_n) = 1, \\ \nu_p(a_j) = t, \text{ for } j > i \\ \nu_p(a_i) = \alpha, \text{ (note that } \alpha \neq 0) \\ \nu_p(a_{i-1}) = 0 \\ \nu_p(a_j) = 1, \text{ for } j < i-1. \end{cases} \quad (3.5)$$

Consider the polynomials

$$f(x) = \sum_{j=0}^n a_j(x)_j = \sum_{j=0}^n \tilde{a}_j x^j$$

and

$$g(x) = \tilde{a}_n x^n + q^{e_{n-1}} \tilde{a}_{n-1} x^{n-1} + \cdots + q^{e_1} \tilde{a}_1 x + q \tilde{a}_0 := \sum_{j=0}^n b_j x^j,$$

where  $q$  is a prime number such that  $q \equiv 1 \pmod{p}$ ,  $q \nmid \tilde{a}_0$  and the exponents  $e_j$  are arbitrary fixed positive integers. Clearly  $g(x)$  is irreducible in  $\mathbb{Z}[x]$  by Eisenstein's Theorem. Let  $\gamma$  be a root of  $g(x)$ . Since  $p \leq i-1$ , then by Lemma 2.2, we conclude that  $p \mid i(\gamma)$  and since  $g(x) \equiv f(x) \pmod{p}$  for any  $x \in \mathbb{Z}$ , then  $p \mid i(\gamma)$ . We look at the  $p$ -adic valuations of the  $\tilde{a}_j$ . Recall that  $\tilde{a}_n = c(\gamma)$  and  $\tilde{a}_0 = a_0$ , hence  $\nu_p(\tilde{a}_n) = t$  and  $\nu_p(\tilde{a}_0) = 1$ . We claim that

$$\begin{cases} \nu_p(\tilde{a}_j) \geq t, & \text{for } j > i \\ \nu_p(\tilde{a}_i) = \alpha, \\ \nu_p(\tilde{a}_{i-1}) = 0. \end{cases} \quad (3.6)$$

For any  $j \geq 1$  we have  $\tilde{a}_j = a_j + \sum_{l=j+1}^n a_l c_l$ , where  $c_l \in \mathbb{Z}$  for any  $l$ .

If  $j > i$  then  $\nu_p(a_j) = \nu_p(a_{j+1}) = \dots = \nu_p(a_l) = t$ , hence  $\nu_p(\tilde{a}_j) \geq t$ .

For  $j = i$  we have  $\nu_p(a_j) = \alpha < t$  and  $\nu_p(a_l) = t$  for  $l = i+1, \dots, n$ , hence  $\nu_p(\tilde{a}_i) = \alpha$ .

For  $j = i-1$  we have  $\nu_p(a_{i-1}) = 0$  and  $\nu_p(a_l) \geq \alpha$  for  $l = i, \dots, n$ , hence  $\nu_p(\tilde{a}_{i-1}) = 0$ . Thus we obtain the desired claim.

To compute the  $p$ -adic valuation of the denominator of  $\gamma$ , we use Lemma 2.1. For  $j > i$ , we have

$$\frac{\nu_p(b_n) - \nu_p(b_j)}{n-j} = \frac{\nu_p(\tilde{a}_n) - \nu_p(\tilde{a}_j)}{n-j} = \frac{t - t_j}{n-j} \leq 0, \text{ because } t_j \geq t.$$

For  $j = i$ , we have

$$\frac{\nu_p(b_n) - \nu_p(b_i)}{n-i} = \frac{\nu_p(\tilde{a}_n) - \nu_p(\tilde{a}_i)}{n-i} = \frac{t - \alpha}{n-i} = k.$$

For  $j < i$  we have

$$\frac{\nu_p(b_n) - \nu_p(b_j)}{n-j} = \frac{\nu_p(\tilde{a}_n) - \nu_p(\tilde{a}_j)}{n-j} = \frac{t - t_j}{n-j} \leq \frac{t}{n-(i-1)} \leq \frac{(n-i)k + \alpha}{n-(i-1)} < \frac{(n-i)k + k}{n-(i-1)} = k,$$

hence  $\nu_p(d(\gamma)) = k$ .

- Second case:  $t = (n-i)k$  with  $p \leq i \leq n-1$ . Choose integers  $a_0, \dots, a_n$  such that  $\gcd(a_0, \dots, a_n) = 1$  and

$$\begin{cases} \nu_p(a_j) \geq t, & \text{for } j > i, \\ \nu_p(a_i) = 0, \\ \nu_p(a_j) = 1, & \text{for } j < i, \\ \nu_p(a_n) = t. \end{cases} \quad (3.7)$$

Consider the polynomials

$$f(x) = \sum_{j=0}^n a_j(x)_j = \sum_{j=0}^n \tilde{a}_j x^j$$

and

$$g(x) = \tilde{a}_n x^n + q^{e_{n-1}} \tilde{a}_{n-1} x^{n-1} + \dots + q^{e_1} \tilde{a}_1 x + q \tilde{a}_0 := \sum_{j=0}^n b_j x^j,$$

where  $q$  and the  $e_j$  have the same meaning as in the preceding case. Clearly  $g(x)$  is irreducible in  $\mathbb{Z}[x]$  by Eisenstein's Theorem. Let  $\gamma$  be a root of  $g(x)$ . Since  $p \leq i$ , then by Lemma 2.2, we conclude that  $p|i(f)$  and since  $g(x) \equiv f(x) \pmod{p}$  for any  $x \in \mathbb{Z}$ , then  $p|i(\gamma)$ . We look at the  $p$ -adic valuations of the  $\tilde{a}_j$ . We have  $\tilde{a}_n = a_n$  hence  $\nu_p(\tilde{a}_n) = t$ . For  $j > i$ , we have

$$\tilde{a}_j = a_j + \sum_{l=j+1}^n a_l c_l \text{ where } c_l \in \mathbb{Z} \text{ and we have } \nu_p(a_j) \geq t \text{ and } \nu_p(a_l) \geq t \text{ for } l \geq j + 1, \text{ hence } \nu_p(\tilde{a}_j) \geq t.$$

For  $j = i$ , we have  $\nu_p(a_i) = 0$  and  $\nu_p(a_l) = t, l \geq j + 1$ , hence  $\nu_p(\tilde{a}_i) = 0$ .

For  $j < i$ , we have  $\nu_p(\tilde{a}_j) \geq 0$ .

For  $j < i$ , we have  $\nu_p(\tilde{a}_j) \geq 0$ .

We compute the  $p$ -adic valuation of the denominator of  $\gamma$  by using Lemma 2.1.

For  $j > i$ , we have

$$\frac{\nu_p(b_n) - \nu_p(b_j)}{n - j} = \frac{\nu_p(\tilde{a}_n) - \nu_p(\tilde{a}_j)}{n - j} = \frac{t - t_j}{n - j} \leq 0, \text{ because } t_j \geq t.$$

For  $j = i$ , we have

$$\frac{\nu_p(b_n) - \nu_p(b_i)}{n - i} = \frac{\nu_p(\tilde{a}_n) - \nu_p(\tilde{a}_i)}{n - i} = \frac{t}{n - i} = k.$$

For  $j < i$  we have

$$\frac{\nu_p(b_n) - \nu_p(b_j)}{n - j} = \frac{\nu_p(\tilde{a}_n) - \nu_p(\tilde{a}_j)}{n - j} < \frac{t - t_j}{n - i} \leq \frac{t}{n - i} = k,$$

hence  $\nu_p(d(\gamma)) = k$ .

Since we can choose  $q$  and the  $e_j$  in an infinite number of ways, then the number of  $\gamma$ 's is infinite.  $\square$

#### 4. Upper bounds for the enumeration of the denominators of some algebraic numbers

**Proposition 4.1.** *Let  $\theta \in A_n$ , and  $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$  its minimal polynomial over  $\mathbb{Q}$ . Suppose that  $\theta \not\equiv 0 \pmod{pA}$ . Construct the Newton polygon of  $f(x)$  by plotting in the  $(x, y)$  plan the points  $A_i$  whose coordinates are  $(i, \nu_p(a_i))$  for all  $i \in \{0, \dots, n\}$  such that  $a_i \neq 0$ . Suppose that there exists  $k \geq 1$  such that  $p|i(\theta/p^k)$ . Then there exists two integers  $m, M$  such that  $1 \leq m < M \leq n - 1$  and the line joining the points  $A_m$  and  $A_M$  has the following equation*

$$y + kx - u = 0, \text{ where } kM \leq u < \nu_p(a_0) \text{ and } u = \nu_p(\text{cont}(f(p^k x))).$$

Moreover all the points  $A_i$  such that  $i < m$  or  $i > M$  belong to the domain of all points  $(x, y)$  such that  $y + kx - u > 0$ . If  $m < i < M$ , then we have  $\nu_p(a_i) + ki - u \geq 0$ .

*Proof:* The minimal polynomial over  $\mathbb{Z}$  of  $\theta/p^k$  is given by

$$f(p^k x)/p^u = p^{nk-u}x^n + p^{(n-1)k-u}a_{n-1}x^{n-1} + \dots + p^{k-u}a_1x + p^{-u}a_0 := g(x),$$



where  $u = \nu_p(\text{cont}(f(p^k x)))$ . Let

$$I = \{i : 1 \leq i \leq n-1, a_i \neq 0 \text{ and } ik + \nu_p(a_i) - u = 0\}.$$

Since  $\theta/p^k$  is not integral then  $nk - u > 0$ . Since  $g(0) \equiv 0 \pmod{p}$ , then  $\nu_p(a_0) - u > 0$ . Adding these two facts to the property that  $g(x)$  is primitive implies that  $I \neq \emptyset$ . Furthermore  $g(1) \equiv 0 \pmod{p}$ , hence  $|I| \geq 2$ .

Let  $m = \inf(I)$  and  $M = \max(I)$ . Clearly the equation of the line joining the points  $A_m$  and  $A_M$  is given by:  $y + kx - u = 0$ . Moreover a point  $(i, \nu_p(a_i))$  of the Newton polygon belongs to this line if and only if  $i \in I$ . The definition of  $m$  and  $M$  implies the properties of the points  $A_i$  and of  $u$ .  $\square$

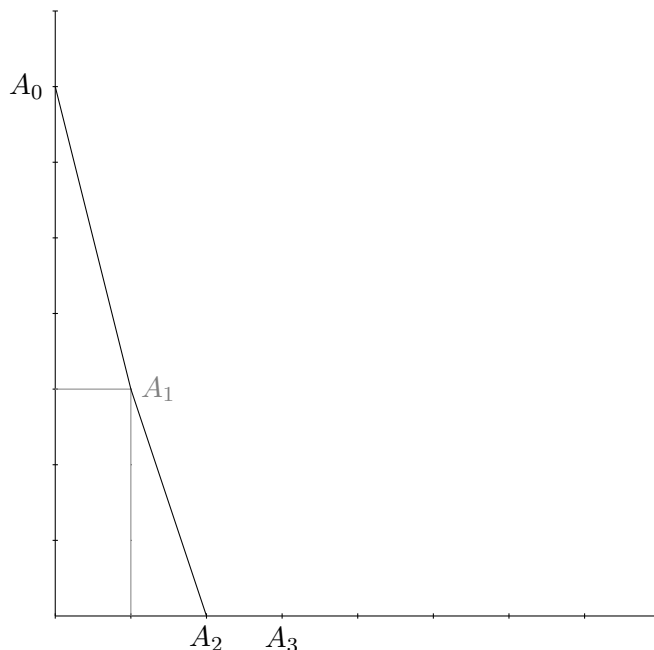
**Remark 4.1.** Proposition 4.1 shows that  $-k$  is the slope of some line joining two points  $A_m$  and  $A_M$ . Moreover all the others points belong to the same side of the line (or on the line). Therefore, if we fix a prime  $p$  and an algebraic integer  $\theta$  such that  $\theta \not\equiv 0 \pmod{pA}$ , it is possible to find explicitly all the values of  $k$  such that  $p \mid i(\theta/p^k)$ . This proposition shows also that the set of such nonnegative integers  $k$  is finite (may be empty).

**Example 4.1.** Let  $t \geq 2$  be an integer,  $f(x) = x^3 + x^2 + 2^t x + 2^{t+1}$  and  $\theta_t$  be a root of  $f(x)$ . It is seen that  $f(x)$  is irreducible over  $\mathbb{Q}$ : if not, it has a root  $a/b$  in  $\mathbb{Q}$  with  $a, b \in \mathbb{Z}$  and  $\gcd(a, b) = 1$ . Substitution then yields

$$a^3 + a^2b + 2^t ab^2 + 2^{t+1}b^3 = 0,$$

implying  $b \mid a^3$ . Thus,  $b = \pm 1$ , and we then obtain  $a \mid 2^{t+1}$ . Letting  $a = 2^i$ , we obtain  $2^{3i} + 2^{t+i} = 2^{2i} + 2^{t+1}$ , implying that  $t+i \leq t+1$  so  $i = 1$  which is impossible.

For any nonnegative integer, let  $\gamma_{t,k} = \theta_t/2^k$ . We show that  $V_2(\theta_t) = \{0, t\}$ . Clearly,  $2 \mid i(\theta_t)$ , hence  $0 \in V_2(\theta_t)$ . The Newton diagram for  $p = 2$  has the following shape:



The possible edges of the convex hull which may give rise to values of  $k \in V_2(\theta_t)$ ,  $k \geq 1$ , are  $[A_0A_1]$  and  $[A_1A_2]$ . Their slopes are equal to  $-t - 1$  and  $t$  respectively. Thus,  $k = t + 1$  or  $k = t$ .

If  $k = t + 1$ , the minimal polynomial of  $\gamma_{t,k}$  over  $\mathbb{Z}$  is given by  $g(x) = 2^{t+2}x^3 + 2x^2 + x + 1$ . This shows that  $2 \nmid i(g(x))$ , hence  $t + 1 \notin V_2(\theta_t)$ .

If  $k = t$ , the minimal polynomial of  $\gamma_{t,k}$  over  $\mathbb{Z}$  is given by  $h(x) = 2x^3 + x^2 + x + 2$ . This shows that  $2 \mid i(h(x))$ , hence  $t \in V_2(\theta_t)$ . Thus,  $V_2(\theta_t) = \{0, t\}$ .

We state now our main result on the upper bounds for the enumeration of the denominators of algebraic numbers  $\gamma$  such that  $p \mid i(\gamma)$ .

**Theorem 4.1.** *Let  $\theta$  be a root of  $f(x) \in \mathbb{Z}[x]$ , monic irreducible,  $p$  a prime number such that  $\theta \not\equiv 0 \pmod{p}$  and let  $a_0 = f(0)$ . We set*

$$V_p(\theta) = \left\{ k \geq 0; p \mid i(\theta/p^k) \right\}.$$

Suppose that  $V_p(\theta) \neq \emptyset$  then we have

$$|V_p(\theta)| \leq \frac{n-1}{p-1}, \tag{4.8}$$

$$\sum_{k \in V_p(\theta)} k < \frac{\nu_p(a_0)}{p}. \tag{4.9}$$

For the proof of this theorem, we need the following lemma.

**Lemma 4.1.** *Let  $p$  be a prime number and  $g(x) = a_Mx^M + \dots + a_mx^m$ ,  $M > m > 0$  such that  $p \nmid a_M$ . If  $p \mid i(g)$ , then  $M - m \geq p - 1$ .*

*Proof:* Suppose that  $p \mid i(g)$ , then clearly  $p \mid i(xg_1)$ , where  $g_1(x) = a_Mx^{M-m} + \dots + a_m$ . Write  $xg_1$  in the form

$$xg_1(x) = a_M(x)_{M-m+1} + \sum_{j < M-m+1} b_j(x)_j,$$

then by Lemma 2.2  $p \mid (M - m + 1)!a_M$ , hence  $p \mid (M - m + 1)!$ . Therefore  $p \leq M - m + 1$ .  $\square$

*Proof of Theorem 4.1.* Suppose that the complete list of elements of  $V_p(\theta)$  is given by  $k_1 < k_2 < \dots < k_z$ . We have  $k_1 = 0$  if and only if  $p \mid i(\theta)$ . Set  $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ . For each  $j = 1, \dots, z$ , let  $g_j(x)$  be the minimal polynomial of  $\theta/p^{k_j}$  over  $\mathbb{Z}$ .

Set  $g_j(x) = p^{t_j}x^n + b_{n-1}^{(j)}x^{n-1} + \dots + b_1^{(j)}x + b_0^{(j)}$ , we have

$$g_1(x) = \begin{cases} f(x) & \text{if } k_1 = 0 \\ f(p^{k_1}x)p^{-u_1} & \text{if } k_1 \geq 1 \end{cases} \tag{4.10}$$

and  $g_{j+1}(x) = g_j(p^{k_{j+1}-k_j}x)p^{-u_{j+1}}$  for  $j = 1, \dots, z - 1$  and  $u_1, \dots, u_z$  are positive integers.

For any  $j = 1, \dots, z$ , let  $I_j = \{i \in \{1, \dots, n - 1\}, \nu_p(b_i^{(j)}) = 0\}$ . Since  $t_j > 0$ ,  $\nu_p(b_0^{(j)}) > 0$  and  $g_j(x)$  is irreducible in  $\mathbb{Z}[x]$ , then it follows that  $I_j \neq \emptyset$ . Let  $m_j = \inf(I_j)$  and  $M_j = \sup(I_j)$ . Since  $g_j(1) \equiv 0 \pmod{p}$  then  $|I_j| \geq 2$  and  $m_j < M_j$ . Clearly  $m_j \geq 1$  and  $M_j \leq n - 1$ . We claim that:

- $M_j - m_j \geq p - 1$  for  $j = 1, \dots, z$ .

- $u_1 \geq k_1 M_1$  and  $u_j \geq (k_j - k_{j-1})M_j$  for  $j = 2, \dots, z$ .
- $n \geq M_1 > m_1 \geq M_2 > m_2 \geq \dots \geq M_z > m_z \geq 1$ .

The first claim follows from Lemma 4.1. If  $k_1 = 0$ , then from (4.10) we have  $u_1 = 0 = k_1 M_1$ . From the definition of  $M_1$  and the definition of the interval  $I_1$ , it follows that

$$\nu_p(b_{M_1}^{(1)}) = 0.$$

Therefore, if  $k_1 \geq 1$ , equation (10) and the above equality imply that

$$0 = \nu_p(b_{M_1}^{(1)}) = \nu_p(a_{M_1}) + k_1 M_1 - u_1,$$

hence  $u_1 \geq k_1 M_1$ . Similarly for  $j = 2, \dots, z$ , we have

$$0 = \nu_p(b_{M_j}^{(j)}) = \nu_p(b_{M_j}^{(j-1)}) + (k_j - k_{j-1})M_j - u_j,$$

hence  $u_j \geq (k_j - k_{j-1})M_j$ , which proves the second part of the claim. For the last part of the claim it is sufficient to prove that  $m_j \geq M_{j+1}$  for  $j = 1, \dots, z-1$ . For, suppose that  $m_j < M_{j+1}$  for some  $j \in \{1, \dots, z-1\}$ . We have  $0 = \nu_p(b_{m_j}^{(j)})$ , hence

$$\nu_p(b_{m_j}^{(j+1)}) = \nu_p(b_{m_j}^{(j)} \cdot p^{(k_{j+1}-k_j)m_j} \cdot p^{-u_{j+1}}) = (k_{j+1} - k_j)m_j - u_{j+1}.$$

We deduce that  $(k_{j+1} - k_j)m_j \geq u_{j+1}$  and then  $(k_{j+1} - k_j)M_{j+1} > u_{j+1}$ . It follows that

$$\nu_p(b_{M_{j+1}}^{(j+1)}) = \nu_p(b_{M_{j+1}}^{(j)}) + (k_{j+1} - k_j)M_{j+1} - u_{j+1} > 0,$$

which contradicts the definition of  $M_{j+1}$  and completes the proof of the claim.

We now come back to the proof of Theorem 4.1.

*Completion of proof of Theorem 4.1:* We use the first and the third points of the claim. We have

$$n \geq M_1 > M_2 > \dots > M_{z-1} > M_z \geq p > 1.$$

Using the claim, we obtain

$$n - p \geq M_1 - M_z = (M_1 - M_2) + \dots + (M_{z-1} - M_z) \geq (p-1)(z-1)$$

hence

$$z \leq \frac{n-p}{p-1} + 1 = \frac{n-1}{p-1}.$$

Therefore (4.8) is proved.

We prove the inequality (4.9) of Theorem 4.1. We have  $b_0^{(1)} = a_0 p^{-u_1}$ ,  $b_0^{(j+1)} = b_0^{(j)} p^{-u_{j+1}}$  for  $j = 1, \dots, z-1$  and since  $g_z(0) \equiv 0 \pmod{p}$ , then  $\nu_p(b_0^{(z)}) > 0$ . Hence  $u_1 + u_2 + \dots + u_z < \nu_p(a_0)$ . On the other hand, using the first and the second parts of the claim, we obtain

$$\begin{aligned} u_1 + u_2 + \dots + u_z &\geq k_1 M_1 + (k_2 - k_1)M_2 + \dots + (k_z - k_{z-1})M_z \\ &\geq k_1 z p + (k_2 - k_1)(z-1)p + \dots + (k_z - k_{z-1})p \\ &= p(k_1 z + k_2 z - k_1 z - k_2 + k_1 + k_3 z - k_2 z - 2k_3 + 2k_2 + \dots + k_z - k_{z-1}) \\ &= p((k_1 + k_2 + \dots + k_z)). \end{aligned}$$

Therefore, we have

$$\nu_p(a_0) > \sum_{j=1}^z u_j \geq p \sum_{j=1}^z k_j,$$

hence

$$\sum_{j=1}^z k_j < \nu_p(a_0)/p.$$

**Remark 4.2.** *Theorem 4.1, shows that if  $\nu_p(a_0) \leq p$ , then  $V_p(\theta) = \{0\}$  or  $V_p(\theta) = \emptyset$ .*

The following result shows that the bound (4.8) in Theorem 4.1 is the best possible. More precisely, we have

**Proposition 4.2.** *Let  $p$  and  $q$  be distinct prime numbers such that  $q \equiv 1 \pmod{p}$ ,  $\theta$  be a root of*

$$f(x) = x^n + \sum_{i=1}^N a_i x^{n-i(p-1)} + qp^\lambda,$$

where  $N = \lfloor (n-1)/(p-1) \rfloor$ ,  $a_i = (-1)^i qp^{(p-1)i(i-1)/2}$ , for  $i = 1, \dots, N$  and

$$\lambda > \frac{2n(N-1) - (p-1)((N-1)^2 + N-1)}{2}.$$

Then

$$|V_p(\theta)| = \left\lfloor \frac{n-1}{p-1} \right\rfloor.$$

*Proof:* Clearly, by Eisenstein's criterion,  $f(x)$  is irreducible over  $\mathbb{Q}$ . The coefficient of  $x^{n-(p-1)}$  is coprime to  $p$ , hence  $\theta \not\equiv 0 \pmod{pA}$ . By Theorem 4.1, we have

$$|V_p(\theta)| \leq \left\lfloor \frac{n-1}{p-1} \right\rfloor.$$

We show that the integers  $0, 1, \dots, \left\lfloor \frac{n-1}{p-1} \right\rfloor - 1$  belong to  $V_p(\theta)$  and this will complete the proof of Proposition 4.2. Since  $f(x) \equiv x^n - qx^{n-(p-1)} \pmod{p\mathbb{Z}[x]}$  and  $q \equiv 1 \pmod{p}$ , then  $f(x) \equiv x^{n-(p-1)}(x^p - 1) \pmod{p\mathbb{Z}[x]}$ . Thus,  $p \mid i(f)$  and  $0 \in V_p(\theta)$ . Set  $a_0 = 1$  and fix  $k \in \left\{1, \dots, \left\lfloor \frac{n-1}{p-1} \right\rfloor - 1\right\}$ . We have

$$f(p^k x) = p^{nk} x^n + \sum_{i=1}^N a_i p^{nk-i(p-1)k} x^{n-i(p-1)} + qp^\lambda.$$

We claim, omitting the proofs that

$$\nu_p(a_k p^{nk-k(p-1)k}) = \nu_p(a_{k+1} p^{nk-k(p-1)(k+1)}) = \frac{2nk - (p-1)(k^2 + k)}{2}$$

and

$$\nu_p(a_i p^{nk-i(p-1)k}) > \frac{2nk - (p-1)(k^2 + k)}{2} \text{ if } i \neq k, k+1.$$

Moreover, since the function  $x \mapsto \psi(x) = 2nx - (p-1)(x^2 + x)$  is increasing in  $[0, N-1]$  and since  $\lambda > \frac{2n(N-1) - (p-1)(N-1)^2 + N-1}{2}$ , then  $\lambda > \frac{2nk - (p-1)(k^2 + k)}{2}$ . It follows that  $\text{cont}(f(p^k x)) = \frac{2nk - (p-1)(k^2 + k)}{2}$  and the minimal polynomial over  $\mathbb{Z}$  of  $\gamma_k = \frac{\theta}{p^k}$  is given by

$$g_k(x) = f(p^k x) p^{-(2nk - (p-1)k^2 + k)/2}.$$

From the above it is seen that

$$g_k(x) \equiv (-1)^k x^{n-k(p-1)} + (-1)^{k+1} x^{n-(k+1)(p-1)} \pmod{p} \equiv (-1)^k x^{n-(k+1)(p-1)} (x^{p-1} - 1) \pmod{p},$$

hence  $p \mid i(\gamma_k)$ , thus  $k \in V_p(\theta)$ .  $\square$

**Corollary 4.1.** *Let  $p$  be a prime number, and  $\theta$  be a root of*

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x],$$

*irreducible over  $\mathbb{Q}$ . Suppose that there exists  $i \in \{0, \dots, \min(p, n) - 1\}$  such that  $\nu_p(a_i) = 0$ . Then*

$$\theta \not\equiv 0 \pmod{pA} \text{ and } V_p(\theta) = \emptyset \text{ or } V_p(\theta) = \{0\}.$$

*Moreover if  $p > n$ , then  $V_p(\theta) = \emptyset$ .*

*Proof:* Let  $\alpha$  be an algebraic integer and  $g(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$  be its minimal polynomial over  $\mathbb{Q}$ . It is easy to prove that  $\alpha \equiv 0 \pmod{p}$  if and only if  $\nu_p(b_i) \geq n - i$  for  $i = 0, \dots, n - 1$ . Our assumption then implies that  $\theta \not\equiv 0 \pmod{pA}$ . Suppose that  $V_p(\theta) \neq \emptyset$  and let  $k \in V_p(\theta)$ . Assume that  $k \geq 1$  and let  $\gamma = \theta/p^k$  and  $u = \text{cont}(f(p^k x))$ . Then the minimal polynomial of  $\gamma$  over  $\mathbb{Z}$  is given by

$$g(x) = f(p^k x) p^{-u} = p^{nk-u} x^n + p^{(n-1)k-u} a_{n-1} x^{n-1} + \cdots + p^{-u} a_0.$$

As in Theorem 4.1, let

$$I = \{j \in \{1, \dots, n-1\}; \nu_p(a_j) + kj - u = 0\}, m = \inf(I), M = \sup(I).$$

Suppose first that  $m \leq i$ . We have  $ik - u = \nu_p(a_i) + ik - u \geq 0$ . Since  $M - m \geq p - 1$ , then  $M > i$  which implies  $\nu_p(a_M) + Mk - u \geq Mk - u > ik - u \geq 0$ , a contradiction. We deduce that  $m > i$  and then  $\nu_p(a_m) + km - u \geq km - u > ki - u = \nu_p(a_i) + ki - u \geq 0$ , a contradiction again. Therefore  $V_p(\theta) = \{0\}$ .  $\square$

## 5. A new invariant of number fields and a generalisation of MacCluer's Theorem

Let  $p$  be a fixed prime integer. We have shown that for any algebraic integer  $\theta$  such that  $\theta \not\equiv 0 \pmod{pA}$ ,  $p \mid i(\theta/p^k)$  for some  $k \geq 1$ , then  $k < \nu_p(N_{\mathbb{Q}(\theta)/\mathbb{Q}}(\theta))/p$ . Does there exist some constant  $c > 0$  such that if  $\theta \in \overline{\mathbb{Q}}$ ,  $\theta \not\equiv 0 \pmod{pA}$  and  $p \mid i(\theta/p^k)$  then  $k < c$ ?

Even if we fix the degree  $n$  of  $\theta$  and suppose that the constant  $c$  depends on  $n$ , the answer is negative as it is shown by the following result.

**Proposition 5.1.** *Let  $n, N$  be positive integers and  $p$  be a prime number such that  $p < n$ . Then there exists an integer  $k > N$  and an algebraic integer  $\theta$  of degree  $n$  such that*

$$\theta \not\equiv 0 \pmod{pA} \text{ and } p \mid i(\theta/p^k).$$

*Proof:* Let  $F$  be a number field of degree  $n - 1$  such that  $p \mid i(F)$ . In particular, we can take  $F$  such that  $p$  completely splits in  $F$ , so that  $p \mid i(F)$  by MacCluer's Theorem. Such a field  $F$  exists by Tchebotarev's theorem [Neu99]. Let  $\alpha$  be a primitive element of  $F/\mathbb{Q}$ . Suppose that  $\alpha$  is integral and  $p \mid i(\alpha)$ . Let  $F_\alpha(x)$  be the minimal polynomial of  $\alpha$  over  $\mathbb{Q}$ . Let  $q$  be a prime number such that

$$q \neq p \text{ and } q \nmid N_{F/\mathbb{Q}}(\alpha).$$

Let  $t$  be an integer such that  $t > nN$  and let  $g(x) = p^t x^n + qF_\alpha(x)$ . Then Eisenstein's criterion shows that  $g(x)$  is irreducible over  $\mathbb{Q}$ . Obviously  $g(x)$  is primitive, hence it is irreducible in  $\mathbb{Z}[x]$ . Let  $\gamma$  be a root of  $g(x)$ , then clearly  $p \mid i(\gamma)$  and  $d(\gamma) = p^k$  for some positive integer  $k$  such that  $k \leq t \leq nk$ , hence  $k \geq t/n > N$ . The algebraic integer  $\theta = p^k \gamma$  satisfies all the conditions of the proposition and the proof is complete.  $\square$

Let  $K$  be a number field of degree  $n$  over  $\mathbb{Q}$  and  $A$  be its ring of integers. We define the integer  $v_p(K)$  as follows.

**Definition 5.1.** *Let*

$$V_p(K) = \left\{ k \geq 0, \text{ there exists } \theta \in A_n, \theta \not\equiv 0 \pmod{pA}, \text{ and } p \mid i(\theta/p^k) \right\},$$

and we define

$$v_p(K) = \begin{cases} -\infty & \text{if } V_p(K) = \emptyset, \\ \infty & \text{if } V_p(K) \text{ is infinite,} \\ \max(V_p(K)) & \text{if } V_p(K) \text{ is finite.} \end{cases}$$

**Remark 5.1.** *By Theorem 3.1, we have  $v_p(K) = -\infty$  if and only if  $p \nmid i(K)$ . So there is no need to give examples illustrating this fact. Theorem 3.1 again shows that if the degree of the number field  $K$  is a prime  $p$  then  $v_p(K) = 0$  if  $p \mid i(K)$  and  $v_p(K) = -\infty$  if  $p \nmid i(K)$ .*

In the following we compute explicitly  $v_2(K)$  for some number fields of degree 3 or 4 over  $\mathbb{Q}$ .

**Proposition 5.2. (Galois field of degree 4)** *Let  $K/\mathbb{Q}$  be a Galois number field of degree 4 in which the prime 2 splits into a product of two prime ideals having their residual degree equal to 2. Then we have  $v_2(K) = 0$ .*

*Proof:* By MacCluer's theorem,  $2 \mid i(K)$ , hence  $0 \in V_p(K)$ . Let  $\mathfrak{p}$  and  $\mathfrak{p}'$  be the conjugate prime ideals of  $A$  lying over 2 and having their residual degree equal to 2. Suppose that  $2 \mid i(\theta/2^k)$  for some  $k \geq 1$  and  $\theta \in A_n$  such that  $\theta \not\equiv 0 \pmod{2A}$ . Since  $N_{K/\mathbb{Q}}(\theta) \equiv 0 \pmod{2}$ , then we may suppose that  $\mathfrak{p}^e \parallel \theta$  and  $\mathfrak{p}' \nmid \theta$  for some  $e \geq 1$ . We suppose that the conjugates  $\theta_1 = \theta, \theta_2, \theta_3, \theta_4$  of  $\theta$  satisfy the following conditions:

$$\mathfrak{p}^e \parallel \theta_1, \mathfrak{p}^e \parallel \theta_3, \mathfrak{p}' \nmid \theta_1 \theta_3, \mathfrak{p}'^e \parallel \theta_2, \mathfrak{p}'^e \parallel \theta_4, \mathfrak{p} \nmid \theta_2 \theta_4.$$

Let  $f(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in \mathbb{Z}[x]$  be the minimal polynomial of  $\theta$  over  $\mathbb{Q}$ . Let  $g(x) \in \mathbb{Z}[x]$  be the minimal polynomial of  $\gamma = \theta/2^k$  over  $\mathbb{Z}$ , then

$$g(x) = f(2^kx) \cdot 2^{-u} = 2^{4k-u}x^4 + 2^{3k-u}a_3x^3 + 2^{2k-u}a_2x^2 + 2^{k-u}a_1x + 2^{-u}a_0,$$

where  $u$  is the content of  $f(2^kx)$ . Using the elementary symmetric functions of the  $\theta_j$  and our assumption on their  $\mathfrak{p}$ -adic and  $\mathfrak{p}'$ -adic valuations, we get

$$\nu_2(a_0) = 2e, \nu_2(a_1) \geq e \text{ and } \nu_2(a_2) = 0.$$

If  $k \geq e$ , then  $\nu_2(2^{4k-u}) \geq 4e - u$ ,  $\nu_2(2^{3k-u}a_3) \geq 3e - u$ ,  $\nu_2(2^{2k-u}a_2) = 2k - u \geq 2e - u$ ,  $\nu_2(2^{k-u}a_1) \geq 2e - u$ ,  $\nu_2(2^{-u}a_0) = 2e - u$ . Since these five 2-adic valuations must be nonnegative, then  $u \leq 2e$ . Furthermore one (at least) of these valuations must be 0, hence  $u = 2e$ . In this case,  $g(0) \not\equiv 0 \pmod{2}$  which is a contradiction to  $2|i(\gamma)$ . If  $k < e$ , then  $\nu_2(2^{4k-u}) = 4k - u$ ,  $\nu_2(2^{3k-u}a_3) \geq 3k - u$ ,  $\nu_2(2^{2k-u}a_2) = 2k - u$ ,  $\nu_2(2^{k-u}a_1) > 2k - u$ ,  $\nu_2(2^{-u}a_0) > 2k - u$ . Using similar arguments as in the preceding case, we obtain  $u = 2k$ . We conclude that all the coefficients of  $g(x)$  have their 2-adic valuations positive except the coefficient of  $x^2$  which has a 2-adic valuation equal to 0. In this case also we reach a contradiction since  $g(1) \not\equiv 0 \pmod{2}$ . It follows that  $V_2(K) = \{0\}$  and  $\nu_2(K) = 0$ .  $\square$

For the proof of the next proposition, we will need the following lemma.

**Lemma 5.1. (Engstrom)** *Let  $K$  be a number field,  $A$  be its ring of integers and  $p$  be a prime integer. Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_s$  be distinct prime ideals of  $A$  lying over  $p$  and let  $\Phi_1(x), \dots, \Phi_s(x)$  be monic irreducible polynomials over  $\mathbb{F}_p$  not necessarily distincts of degree  $d_1, \dots, d_s$  respectively, where  $d_i$  divides the residual degree of  $\mathfrak{p}_i$ . Let  $h_1, \dots, h_s$  be positive integers. Then there exists a primitive element  $\theta \in A$  such that  $\mathfrak{p}_i^{h_i} \parallel \Phi_i(\theta)$  for  $i = 1, \dots, s$*

*Proof:* see [Eng30].  $\square$

**Proposition 5.3. (Cubic Galois)** *Let  $K/\mathbb{Q}$  be a Galois number field of degree 3 in which the prime 2 splits completely. Then  $V_2(K) = \mathbb{N}$ .*

*Proof:* Let  $k$  and  $e$  be positive integers such that  $e > k$ . Let  $\mathfrak{p}_1, \mathfrak{p}_2$  and  $\mathfrak{p}_3$  be the prime ideals of  $A$  lying over 2. By Lemma 5.1 there exists  $\theta \in A_n$  such that

$$\mathfrak{p}_1^e \parallel \theta, \mathfrak{p}_2^k \parallel \theta \text{ and } \mathfrak{p}_3 \nmid \theta.$$

Assume that the conjugates of  $\theta$ ,  $\theta_1 = \theta, \theta_2, \theta_3$  are labelled in order to satisfy the following conditions:

$$\begin{aligned} \mathfrak{p}_2^e \parallel \theta_2, \mathfrak{p}_3^k \parallel \theta_2, \mathfrak{p}_1 \nmid \theta_2, \\ \mathfrak{p}_3^e \parallel \theta_3, \mathfrak{p}_1^k \parallel \theta_3, \mathfrak{p}_2 \nmid \theta_3. \end{aligned}$$

Let  $f(x) = x^3 + a_2x^2 + a_1x + a_0 \in \mathbb{Z}[x]$  be the minimal polynomial of  $\theta$  over  $\mathbb{Q}$ . Expressing  $a_0, a_1$  and  $a_2$  in terms of  $\theta_1, \theta_2, \theta_3$ , we get

$$\nu_2(a_0) = e + k, \nu_2(a_1) = k \text{ and } \nu_2(a_2) = 0.$$

We have

$$f(2^k x) = 2^{3k} x^3 + 2^{2k} a_2 x^2 + 2^k a_1 x + a_0.$$

Set

$$b_3 = 2^{3k}, b_2 = 2^{2k} a_2, b_1 = 2^k a_1, b_0 = a_0.$$

Using the 2-adic valuation of  $a_0, a_1, a_2$  we obtain

$$\nu_2(b_1) = \nu_2(b_2) = 2k, \nu_2(b_0) = e + k > 2k, \nu_2(b_3) = 3k > 2k.$$

Therefore  $\text{cont}(f(2^{2k}x)) = 2^{2k}$  and the minimal polynomial of  $\theta/2^k$  is given by

$$g(x) = f(2^k x) \cdot 2^{-2k}.$$

Clearly we have  $g(0) \equiv g(1) \equiv 0 \pmod{2}$  hence  $2|i(\theta/p^k)$ . Since the prime 2 splits completely in  $K$ , then  $0 \in V_2(K)$ . Therefore  $V_2(K) = \mathbb{N}$  and  $v_2(K) = \infty$ .  $\square$

**Remark 5.2.** *Our result in the sequel can be viewed as a generalization of MacCluer's theorem which establishes a relation between the number of prime ideals of  $A$  lying over  $p$  and the property of  $p$  to be a divisor of  $i(K)$ .*

Fix a prime number  $p$  and define, for any primitive element  $\theta \in A$  of  $K$ , the integer  $j_p(\theta)$  as follows.

**Definition 5.2.** *Let  $F_\theta(x)$  be the minimal polynomial of  $\theta$  over  $\mathbb{Q}$ . Let  $j_p(\theta)$  be the largest integer  $y$ , if it exists,  $1 \leq y \leq p$  such that  $F_\theta(1) \equiv F_\theta(2) \equiv \dots \equiv F_\theta(y) \equiv 0 \pmod{p}$ . If not set  $j_p(\theta) = 0$ . We define also  $j_p(K) = \max_{\theta \in A_n} j_p(\theta)$ .*

**Theorem 5.1.** *Let  $r$  be the number of prime ideals of  $A$  lying over  $p$ . Then*

$$j_p(K) = \inf(r, p).$$

Moreover

$$p|i(K) \iff j_p(K) = p.$$

*Proof:* Suppose first that  $r \leq p$  and let  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$  be the distinct prime ideals of  $A$  lying over  $p$ . By Lemma 5.1 there exists  $\theta \in A_n$  such that  $\theta \equiv i \pmod{p}$  for  $i = 1, \dots, r$ . It follows that the minimal polynomial  $F_\theta(x)$  of  $\theta$  satisfies the condition

$$F_\theta(x) \equiv (x-1)(x-2) \cdots (x-r)g(x) \pmod{p}$$

hence  $j_p(\theta) \geq r$  which implies that  $j_p(K) \geq r$ . On the other hand, let  $\theta \in A_n$  such that

$$j_p(K) = j_p(\theta) := t,$$

then

$$F_\theta(x) \equiv (x-1)(x-2) \cdots (x-t)g(x) \pmod{p}$$

hence, by Hensel's Lemma, we deduce that  $F_\theta(x)$  has at least  $t$  irreducible factors over  $\mathbb{Z}_p$ , the ring of  $p$ -adic integers. Again by Theorem 5.1 of chap. 2 of [Jan96], we have  $t \leq r$ . We conclude that



$j_p(K) = r = \inf(p, r)$ .

Suppose now that  $r > p$ . By Lemma 5.1, let  $\theta \in A_n$  such that  $\theta \equiv i \pmod{p}$  for  $i = 1, \dots, p$ . Then

$$F_\theta(x) \equiv (x-1)(x-2)\cdots(x-p)g(x) \pmod{p}.$$

therefore we have  $j_p(\theta) \geq p$  which implies  $j_p(K) \geq p$ . From the definition we have  $j_p(K) \leq p$ , hence  $j_p(K) = p = \inf(r, p)$ .

We now prove the last statement of the proposition. We have

$$p|i(K) \iff r \geq p \text{ (by MacCluer's theorem)} \iff \inf(r, p) = p \iff j_p(K) = p. \quad \square$$

## 6. Concluding remarks

**Questions** Let  $K$  be a number field of degree  $n$ . If  $[K : \mathbb{Q}] = 2$ , then by Corollary 2.1,  $i(K)$  and  $\hat{i}(K)$  are equal to 1 or 2. Theorem 3.1 shows that  $2 \nmid i(\gamma)$  if  $\gamma \notin A_n$ , hence  $i(K) = \hat{i}(K) \in \{1, 2\}$ .

If  $[K : \mathbb{Q}] = 3$ , then  $i(K)$  and  $\hat{i}(K) \in \{1, 2, 3, 6\}$ . Moreover, Theorem 3.1 shows that  $3 \mid \hat{i}(K)$  if and only if  $3 \mid i(K)$ .

Suppose that there exists  $\gamma = \theta/2^k$  with  $k \geq 1$ ,  $k \leq t \leq 3k$  and  $\theta \not\equiv 0 \pmod{p}$  such that  $2 \mid i(\gamma)$ . Let  $g(x) = 2^t x^3 + b_2 x^2 + b_1 x + b_0$  be the minimal polynomial of  $\gamma$  over  $\mathbb{Z}$ . Since  $g(0) \equiv 0 \pmod{2}$ , then  $b_0 \equiv 0 \pmod{2}$ . Since  $g(1) \equiv 0 \pmod{2}$ , then  $b_1 + b_2 \equiv 0 \pmod{2}$ , thus  $b_1 \equiv b_2 \pmod{2}$ . Moreover, since  $g(x)$  is primitive, then  $b_1 \equiv b_2 \equiv 1 \pmod{2}$ . By Theorem 3.1,  $t \leq k$ . Since  $k \leq t$ , then  $k = t$ . The minimal polynomial of  $\theta$  is then given by

$$f(x) = x^3 + b_2 x^2 + b_1 2^t x + b_0 2^{2t}.$$

This shows that  $2 \mid i(f)$  and then  $2 \mid i(\theta)$ , thus  $2 \mid i(K)$ . We conclude that  $i(K) = \hat{i}(K)$ .

Let  $K$  be a number field of degree  $n$  and let  $\gamma \in K_n \setminus A_n$ . Set  $\gamma = \theta/d$ , where  $d$  is an integer at least equal to 2 such that  $\theta \not\equiv 0 \pmod{p}$  for any prime divisor  $p$  of  $d$ . It is proved in Lemma 3.1 that if  $d = p^k q$  with  $\gcd(p, q) = 1$  and  $k \geq 1$ , then  $p \mid i(\gamma)$  if and only if  $p \mid i(\theta/p^k)$ . We ask that following: Is it true that if  $p \mid i(\theta/p^k)$  with  $k \geq 1$  and  $\theta \not\equiv 0 \pmod{p}$ , then  $p \mid i(K)$ ? Do we have  $\hat{i}(K) = i(K)$ ?

Recall that  $\nu_p(K)$  is the greatest element of the set  $V_p(K)$ , when this set is finite. Do we have  $\{0, 1, \dots, \nu_p(K)\} = V_p(K)$ ? The example given in section 4 shows that  $V_p(\theta_t) = \{0, t\} \neq \{0, 1, \dots, t\}$ . We may ask a similar question when  $V_p(K)$  is infinite. Do we have  $V_p(K) = \mathbb{N}$ ?

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