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# Godement–Jacquet $L$ -function, some conjectures and some consequences

Amrinder Kaur and Ayyadurai Sankaranarayanan

**Abstract.** In this paper, we investigate the mean square estimate for the logarithmic derivative of the Godement–Jacquet  $L$ -function  $L_f(s)$  assuming the Riemann hypothesis for  $L_f(s)$  and Rudnick–Sarnak conjecture.

**Keywords.** Godement–Jacquet  $L$ -function, Rudnick–Sarnak conjecture, Hecke–Maass form, Riemann Hypothesis.

**2010 Mathematics Subject Classification.** 11F30, 11N75

## 1. Introduction

Let  $n \geq 2$ , and let  $v = (v_1, v_2, \dots, v_{n-1}) \in \mathbb{C}^{n-1}$ . A Maass form [Gol06] for  $SL(n, \mathbb{Z})$  of type  $v$  is a smooth function  $f \in \mathcal{L}^2(SL(n, \mathbb{Z}) \backslash \mathcal{H}^n)$  which satisfies

1.  $f(\gamma z) = f(z)$ , for all  $\gamma \in SL(n, \mathbb{Z})$ ,  $z \in \mathcal{H}^n$ ,
2.  $Df(z) = \lambda_D f(z)$ , for all  $D \in \mathfrak{D}^n$  where  $\mathfrak{D}^n$  is the center of the universal enveloping algebra of  $\mathfrak{gl}(n, \mathbb{R})$  and  $\mathfrak{gl}(n, \mathbb{R})$  is the Lie algebra of  $GL(n, \mathbb{R})$ ,
3.  $\int_{(SL(n, \mathbb{Z}) \cap U) \backslash U} f(uz) du = 0$ ,

for all upper triangular groups  $U$  of the form

$$U = \left\{ \begin{pmatrix} I_{r_1} & & & \\ & I_{r_2} & & * \\ & & \ddots & \\ & & & I_{r_b} \end{pmatrix} \right\},$$

with  $r_1 + r_2 + \dots + r_b = n$ . Here,  $I_r$  denotes the  $r \times r$  identity matrix, and  $*$  denotes arbitrary real entries.

A Hecke–Maass form is a Maass form which is an eigenvector for the Hecke operators algebra. Let  $f(z)$  be a Hecke–Maass form of type  $v = (v_1, v_2, \dots, v_{n-1}) \in \mathbb{C}^{n-1}$  for  $SL(n, \mathbb{Z})$ . Then it has the Fourier expansion

$$f(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash SL(n-1, \mathbb{Z})} \sum_{m_1=1}^{\infty} \dots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A(m_1, \dots, m_{n-1})}{\prod_{j=1}^{n-1} |m_j|^{\frac{j(n-j)}{2}}} \times W_J \left( M \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, v, \psi_{1, \dots, 1, \frac{m_{n-1}}{|m_{n-1}|}} \right),$$

where

$$M = \begin{pmatrix} m_1 \dots m_{n-2} \cdot |m_{n-1}| & & & & \\ & \ddots & & & \\ & & m_1 m_2 & & \\ & & & m_1 & \\ & & & & 1 \end{pmatrix},$$

$$A(m_1, \dots, m_{n-1}) \in \mathbb{C}, \quad A(1, \dots, 1) = 1,$$

$$\psi_{1, \dots, 1, \epsilon} \left( \begin{pmatrix} 1 & u_{n-1} & & & \\ & 1 & u_{n-2} & & * \\ & & \ddots & \ddots & \\ & & & 1 & u_1 \\ & & & & 1 \end{pmatrix} \right) = e^{2\pi i(u_1 + \dots + u_{n-2} + \epsilon u_{n-1})},$$

$U_{n-1}(\mathbb{Z})$  denotes the group of  $(n-1) \times (n-1)$  upper triangular matrices with 1s on the diagonal and an integer entry above the diagonal and  $W_J$  is the Jacquet Whittaker function.

If  $f(z)$  is a Maass form of type  $(v_1, \dots, v_{n-1}) \in \mathbb{C}^{n-1}$ , then

$$\tilde{f}(z) := f(w \cdot (z^{-1})^T \cdot w),$$

$$w = \begin{pmatrix} & & & (-1)^{\lfloor \frac{n}{2} \rfloor} \\ & & 1 & \\ & \ddots & & \\ 1 & & & \end{pmatrix}$$

is a Maass form of type  $(v_{n-1}, \dots, v_1)$  for  $SL(n, \mathbb{Z})$  called the *dual Maass form*. If  $A(m_1, \dots, m_{n-1})$  is the  $(m_1, \dots, m_{n-1})$ -Fourier coefficient of  $f$ , then  $A(m_{n-1}, \dots, m_1)$  is the corresponding Fourier coefficient of  $\tilde{f}$ .

We note that the Fourier coefficients  $A(m_1, \dots, m_{n-1})$  satisfy the multiplicative relations

$$A(m_1 m'_1, \dots, m_{n-1} m'_{n-1}) = A(m_1, \dots, m_{n-1}) \cdot A(m'_1, \dots, m'_{n-1}),$$

if

$$(m_1 \dots m_{n-1}, m'_1 \dots m'_{n-1}) = 1,$$

$$A(m, 1, \dots, 1)A(m_1, \dots, m_{n-1}) = \sum_{\substack{\prod_{l=1}^n c_l = m \\ c_1 | m_1, c_2 | m_2, \dots, c_{n-1} | m_{n-1}}} A\left(\frac{m_1 c_n}{c_1}, \frac{m_2 c_1}{c_2}, \dots, \frac{m_{n-1} c_{n-2}}{c_{n-1}}\right),$$

and

$$A(m_{n-1}, \dots, m_1) = \overline{A(m_1, \dots, m_{n-1})}.$$

**Definition 1.1.** [Gol06] The Godement–Jacquet  $L$ -function  $L_f(s)$  attached to  $f$  is defined for  $\Re(s) > 1$  by

$$L_f(s) = \sum_{m=1}^{\infty} \frac{A(m, 1, \dots, 1)}{m^s} = \prod_p \prod_{i=1}^n (1 - \alpha_{p,i} p^{-s})^{-1},$$

where  $\{\alpha_{p,i}\}, 1 \leq i \leq n$  are the complex roots of the monic polynomial

$$X^n + \sum_{j=1}^{n-1} (-1)^j A(\overbrace{1, \dots, 1}^{j-1 \text{ terms}}, p, 1, \dots, 1) X^{n-j} + (-1)^n \in \mathbb{C}[X], \quad \text{and}$$

$$A(\overbrace{1, \dots, 1}^{j-1}, p, 1, \dots, 1) = \sum_{1 \leq i_1 < \dots < i_j \leq n} \alpha_{p,i_1} \dots \alpha_{p,i_j}, \quad \text{for } 1 \leq j \leq n-1.$$

$L_f(s)$  satisfies the functional equation:

$$\begin{aligned} \Lambda_f(s) &:= \prod_{i=1}^n \pi^{\frac{-s + \lambda_i(v_f)}{2}} \Gamma\left(\frac{s - \lambda_i(v_f)}{2}\right) L_f(s) \\ &= \Lambda_{\tilde{f}}(1-s), \end{aligned}$$

where  $\tilde{f}$  is the Dual Maass form.

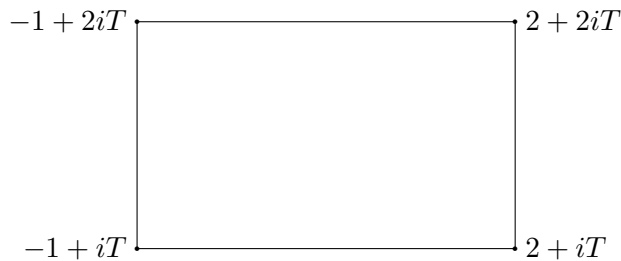
In the case of Godement–Jacquet  $L$ -function, Yujiao Jiang and Guangshi Lü [JiLu17] have studied cancellation on the exponential sum  $\sum_{m \leq N} \mu(m) A(m, 1) e^{2\pi i m \theta}$  related to  $SL(3, \mathbb{Z})$  where  $\theta \in \mathbb{R}$ .

Throughout the paper, we assume that  $f$  is self dual i.e.,  $\tilde{f} = f$ .  
 $\epsilon, \epsilon_1$  and  $\eta$  always denote any small positive constants.

If  $N_f(T)$  denotes the number of zeros of  $L_f(s)$  in the rectangle mentioned below, then from the functional equation and the argument principle of complex function theory we have,

$$N_f(T) \sim c(n)T \log T,$$

where  $c(n)$  is a non zero constant depending only on the degree  $n$  of  $L_f(s)$ .



**(i) The generalized Ramanujan conjecture:**

It asserts that

$$|A(m, 1, \dots, 1)| \leq d_n(m)$$

where  $d_n(m)$  is the number of representations of  $m$  as the product of  $n$  natural numbers. The current best estimates are due to Kim and Sarnak [Kim03] for  $2 \leq n \leq 4$  and Luo, Rudnick and Sarnak for  $n \geq 5$

$$\begin{aligned} |A(m)| &\leq m^{\frac{7}{64}} d(m), \\ |A(m, 1)| &\leq m^{\frac{5}{14}} d_3(m), \\ |A(m, 1, 1)| &\leq m^{\frac{9}{22}} d_4(m), \\ |A(m, 1, \dots, 1)| &\leq m^{\frac{1}{2} - \frac{1}{n^2+1}} d_n(m). \end{aligned}$$

We note that the generalized Ramanujan conjecture is equivalent to

$$|\alpha_{p,i}| = 1 \quad \forall \text{ primes } p \text{ and } i = 1, 2, \dots, n.$$

Other estimates are equivalent to

$$|\alpha_{p,i}| \leq p^{\theta_n} \quad \forall \text{ primes } p \text{ and } i = 1, 2, \dots, n \text{ where}$$

$$\theta_2 := \frac{7}{64}, \quad \theta_3 := \frac{5}{14}, \quad \theta_4 := \frac{9}{22}, \quad \theta_n := \frac{1}{2} - \frac{1}{n^2+1} (n \geq 5).$$

**(ii) Ramanujan’s generalized weak conjecture:**

We formulate this conjecture as:

For  $n \geq 2$ , the inequality

$$|\alpha_{p,i}| \leq p^{\frac{1}{4} - \epsilon_1}$$

holds for some small  $\epsilon_1 > 0$ , for every prime  $p$  and for  $i = 1, 2, \dots, n$ . Of course, this weak conjecture holds good for  $n = 2$ . For  $n \geq 3$ , this conjecture is still open.

Taking the logarithmic derivative of  $L_f(s)$ , we have

$$-\frac{L'_f(s)}{L_f(s)} := \sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^s} = \sum_{m=1}^{\infty} \frac{\Lambda(m) a_f(m)}{m^s}$$

where  $a_f(m)$  is multiplicative and

$$a_f(p^r) = \sum_{i=1}^n \alpha_{p,i}^r$$

for any integer  $r \geq 1$ .

In particular,

$$a_f(p) = \sum_{i=1}^n \alpha_{p,i} = A(p, 1, \dots, 1).$$

**(iii) Rudnick–Sarnak conjecture:**

For any fixed integer  $r \geq 2$ ,

$$\sum_p \frac{|a_f(p^r)|^2 (\log p)^2}{p^r} < \infty.$$

We know that this conjecture is true for  $n \leq 4$ . (See [Ki06, RuSa96].)

**(iv) Riemann hypothesis for  $L_f(s)$ :**

It asserts that  $L_f(s) \neq 0$  in  $\Re(s) > \frac{1}{2}$ .

The aim of this paper is to establish:

**Theorem 1.1.** *Ramanujan's weak conjecture implies Rudnick–Sarnak conjecture.*

**Remark 1.2.** Theorem 1.1 is indicated in [Ki06].

**Theorem 1.3.** *Assume  $n \geq 5$  be any arbitrary but fixed integer. Let  $\epsilon$  be any small positive constant and  $T \geq T_0$  where  $T_0$  is sufficiently large. Assume the Rudnick–Sarnak conjecture and Riemann hypothesis for  $L_f(s)$ . Then the estimate:*

$$\int_T^{2T} \left| \frac{L'_f}{L_f}(\sigma_0 + it) \right|^2 dt \ll_{f,n,\epsilon,\eta} T(\log T)^{2\eta}$$

holds for  $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$  with  $\eta$  being some constant satisfying  $0 < \eta < \frac{1}{2}$ .

**Remark 1.4.** Since Rudnick–Sarnak conjecture is true for  $2 \leq n \leq 4$ , Theorem 1.3 holds just with the assumption of Riemann hypothesis for  $L_f(s)$  whenever  $2 \leq n \leq 4$ .

**Remark 1.5.** It is not difficult to see from our arguments that only assuming Riemann Hypothesis for  $L_f(s)$ , Theorem 1.3 can be upheld for any  $\sigma_0$  satisfying  $1 - \frac{1}{n^2+1} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$  by using the bound  $\theta_n = \frac{1}{2} - \frac{1}{n^2+1}$  of Luo, Rudnick and Sarnak.

It is also not difficult to see from our arguments that the generalized Ramanujan conjecture and the Riemann hypothesis for  $L_f(s)$  together imply the bound

$$\int_T^{2T} \left| \frac{L'_f}{L_f}(\sigma_0 + it) \right|^2 dt \ll_{f,n,\epsilon} T \tag{1.1}$$

to hold for any  $\sigma_0$  satisfying  $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$ .

Though we expect the bound stated in Equation 1.1 to hold unconditionally for  $\sigma_0$  in the said range, this seems to be very hard.

## 2. Some Lemmas

**Lemma 2.1.** *If  $f(s)$  is regular and*

$$\left| \frac{f(s)}{f(s_0)} \right| < e^M \quad (M > 1)$$

in  $|s - s_0| \leq r_1$ , then for any constant  $b$  with  $0 < b < \frac{1}{2}$ ,

$$\left| \frac{f'}{f}(s) - \sum_{\rho} \frac{1}{s - \rho} \right| \ll_b \frac{M}{r_1}$$

in  $|s - s_0| \leq \left(\frac{1}{2} - b\right) r_1$ , where  $\rho$  runs over all zeros of  $f(s)$  such that  $|\rho - s_0| \leq \frac{r_1}{2}$ .

*Proof.* See Lemma  $\alpha$  in Section 3.9 of [TiHe86] or see [RaSa91].

**Lemma 2.2.** Let  $N_f^*(T)$  denote the number of zeros of  $L_f(s)$  in the region  $0 \leq \sigma \leq 1$ ,  $0 \leq t \leq T$ . Then,

$$N_f^*(T+1) - N_f^*(T) \ll_n \log T.$$

*Proof.* Let  $n(r_1)$  denote the number of zeros of  $L_f(s)$  in the circle with centre  $2 + iT$  and radius  $r_1$ . By Jensen's theorem,

$$\int_0^3 \frac{n(r_1)}{r_1} dr_1 = \frac{1}{2\pi} \int_0^{2\pi} \log \left| L_f \left( 2 + iT + 3e^{i\theta} \right) \right| d\theta - \log |L_f(2 + iT)|.$$

From the functional equation, we observe that

$$|L_f(s)| \ll_f t^A \quad \text{for } -1 \leq \sigma \leq 5 \text{ where } A \text{ is some fixed positive constant,}$$

and hence we have,

$$\log \left| L \left( 2 + iT + 3e^{i\theta} \right) \right| \ll A \log T.$$

Note that

$$\begin{aligned} \left| 1 - \frac{\alpha_{p,i}}{p^{2+it}} \right| &\geq 1 - \frac{|\alpha_{p,i}|}{p^2} \\ &\geq 1 - \frac{p^{\frac{1}{2}}}{p^2} \\ &= 1 - \frac{1}{p^{\frac{3}{2}}}. \end{aligned}$$

Thus we have,

$$\begin{aligned} |L_f(2 + it)| &= \prod_p \prod_{i=1}^n \left| \left( 1 - \frac{\alpha_{p,i}}{p^{2+it}} \right) \right|^{-1} \\ &\leq \prod_p \prod_{i=1}^n \left( 1 - \frac{1}{p^{\frac{3}{2}}} \right)^{-1} \\ &\leq \left( \zeta \left( \frac{3}{2} \right) \right)^n \\ &\ll_n 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^3 \frac{n(r_1)}{r_1} dr_1 &\ll A \log T + A \ll \log T, \\ \int_0^3 \frac{n(r_1)}{r_1} dr_1 &\geq \int_{\sqrt{5}}^3 \frac{n(r_1)}{r_1} dr_1 \geq n(\sqrt{5}) \int_{\sqrt{5}}^3 \frac{dr_1}{r_1} \geq c.n(\sqrt{5}). \end{aligned}$$

Hence,

$$N_f^*(T+1) - N_f^*(T) \ll_n \log T.$$

**Lemma 2.3.** *Let  $a_m (m=1,2,\dots,N)$  be any set of complex numbers. Then*

$$\int_T^{2T} \left| \sum_{m=1}^N a_m m^{-it} \right|^2 dt = \sum_{m=1}^N |a_m|^2 (T + O(m)).$$

**Lemma 2.4.** *Let  $b_m$  be any set of complex numbers such that  $\sum m (|b_m|)^2$  is convergent. Then*

$$\int_T^{2T} \left| \sum_{m=1}^{\infty} b_m m^{-it} \right|^2 dt = \sum_{m=1}^{\infty} |b_m|^2 (T + O(m)).$$

*Proof.* See [MoVa74] or [Ram79] for Montgomery and Vaughan theorem.

Hereafter,  $Y \geq 10$  is an arbitrary parameter depending on  $T$  which will be chosen suitably later. Also,  $\sigma_0$  satisfies the inequality  $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$  for any small positive constant  $\epsilon$ .

**Lemma 2.5.** *For  $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$ , we have*

$$\sum_{m > \frac{Y}{2} (\log Y)^2} \frac{m |\Lambda_f(m)|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} \ll 1.$$

*Proof.* We have,

$$\begin{aligned} \sum_{m > \frac{Y}{2} (\log Y)^2} \frac{m |\Lambda_f(m)|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} &\ll \sum_{m > \frac{Y}{2} (\log Y)^2} \frac{m |\Lambda_f(m)|^2 e^{-\frac{m}{Y}} \frac{Y^2}{m^2}}{m^{2\sigma_0}} \\ &\ll Y^2 \sum_{m > \frac{Y}{2} (\log Y)^2} \frac{|\Lambda_f(m)|^2 e^{-\frac{m}{Y}}}{m^{1+2\sigma_0}}. \end{aligned}$$



Since  $\frac{m}{Y} \geq \frac{1}{2}(\log Y)^2$  for  $m \geq \frac{Y}{2}(\log Y)^2$ , we have  $e^{\frac{m}{Y}} \gg Y^B$  for any large positive constant  $B$ . Therefore,

$$\begin{aligned} \sum_{m > \frac{Y}{2}(\log Y)^2} \frac{m |\Lambda_f(m)|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} &\ll \frac{Y^2}{Y^B} \sum_{m > \frac{Y}{2}(\log Y)^2} \frac{|\Lambda_f(m)|^2}{m^{1+2\sigma_0}} \\ &\ll 1. \end{aligned}$$

**Lemma 2.6.** *Assuming Rudnick–Sarnak conjecture and taking  $Y$  sufficiently large, we have*

$$\sum_{m \leq \frac{Y}{2}(\log Y)^2} \frac{|\Lambda_f(m)|^2}{m^{2\sigma_0}} e^{-\frac{2m}{Y}} \ll (\log Y)^2.$$

*Proof.* Note that

$$\sum_{m \leq \frac{Y}{2}(\log Y)^2} \frac{|\Lambda_f(m)|^2}{m^{2\sigma_0}} e^{-\frac{2m}{Y}} \leq \sum_{p \leq \frac{Y}{2}(\log Y)^2} \frac{(\log p)^2 |a_f(p)|^2}{p^{2\sigma_0}} + \sum_{r=2}^{\left[\frac{\log \frac{Y}{2}}{\log 2}\right]+1} \sum_p \frac{(\log p)^2 |a_f(p^r)|^2}{(p^r)^{2\sigma_0}},$$

and

$$|a_f(p)| = \left| \sum_{i=1}^n \alpha_{p,i} \right| = |A(p, 1, \dots, 1)|.$$

We have,

$$\begin{aligned} \sum_{m \leq Y} \frac{c_m}{m^l} &= \int_1^Y \frac{d\left(\sum_{m \leq u} c_m\right)}{u^l} \\ &= \frac{\sum_{m \leq u} c_m}{u^l} \Big|_1^Y - \int_1^Y (-l) \frac{\sum_{m \leq u} c_m}{u^{l+1}} du. \end{aligned}$$

From Remark 12.1.8 of [Gol06], we have

$$\sum_{m_1^{n-1} m_2^{n-2} \dots m_{n-1} \leq Y} |A(m_1, m_2, \dots, m_{n-1})|^2 \ll_f Y.$$

Therefore,

$$\sum_{m \leq Y} |A(m, 1, \dots, 1)|^2 \leq \sum_{m_1^{n-1} m_2^{n-2} \dots m_{n-1} \leq Y} |A(m_1, m_2, \dots, m_{n-1})|^2 \ll_f Y.$$

Taking  $l = 2\sigma_0$  and  $c_m = |A(m, 1, \dots, 1)|^2$ ,

$$\sum_{m \leq \frac{Y}{2} (\log Y)^2} \frac{|A(m, 1, \dots, 1)|^2}{m^{2\sigma_0}} \ll 1.$$

Hence,

$$\sum_{p \leq \frac{Y}{2} (\log Y)^2} \frac{(\log p)^2 |a_f(p)|^2}{p^{2\sigma_0}} \ll (\log Y)^2 \sum_{m \leq \frac{Y}{2} (\log Y)^2} \frac{|A(m, 1, \dots, 1)|^2}{m^{2\sigma_0}} \ll (\log Y)^2.$$

By Rudnick–Sarnak conjecture and the bound  $|\alpha_{p,i}| \leq p^{\theta_n}$  with  $\theta_n = \frac{1}{2} - \frac{1}{n^2+1}$ ,

$$\sum_{r \geq 2} \sum_p \frac{(\log p)^2 |a_f(p^r)|^2}{p^r}$$

converges (as in proof of Theorem 1.1) and in particular,

$$\sum_{r=2}^{\left\lfloor \frac{\log \frac{Y}{2}}{\log 2} \right\rfloor + 1} \sum_p \frac{(\log p)^2 |a_f(p^r)|^2}{p^r} \ll 1.$$

Therefore,

$$\sum_{m \leq \frac{Y}{2} (\log Y)^2} \frac{|\Lambda_f(m)|^2}{m^{2\sigma_0}} \ll (\log Y)^2.$$

### 3. Proof of Theorem 1.1

Assuming  $|\alpha_{p,i}| \leq p^{\theta_n}$  with  $\theta_n \leq \frac{1}{4} - \epsilon_1$ , we need to prove that for every integer  $n \geq 5$  and for every integer  $r \geq 2$ ,

$$\sum_p \frac{(\log p)^2 |a_f(p^r)|^2}{p^r} < \infty.$$

It is enough to show that

$$\sum_{r=2}^{\infty} \sum_p \frac{(\log p)^2 |a_f(p^r)|^2}{p^r} < \infty.$$

Using

$$a_f(p^r) := \sum_{i=1}^n \alpha_{p,i}^r \quad \text{and} \quad |\alpha_{p,i}| \leq p^{\theta_n}$$

we get,

$$\begin{aligned}
\sum_{r=2}^{\infty} \sum_p \frac{(\log p)^2 |a_f(p^r)|^2}{p^r} &\leq \sum_{r=2}^{\infty} \sum_p \frac{(\log p)^2 \left( \sum_{i=1}^n p^{r\theta_n} \right)^2}{p^r} \\
&= \sum_{r=2}^{\infty} \sum_p \frac{(\log p)^2 n^2 p^{2r\theta_n}}{p^r} \\
&\leq n^2 \sum_p (\log p)^2 \sum_{r=2}^{\infty} \frac{p^{2r(\frac{1}{4}-\epsilon_1)}}{p^r} \\
&= n^2 \sum_p (\log p)^2 \sum_{r=2}^{\infty} \frac{1}{p^{\frac{r}{2}+2r\epsilon_1}} \\
&= n^2 \sum_p (\log p)^2 \frac{p^{-(1+4\epsilon_1)}}{1-p^{-(\frac{1}{2}+2\epsilon_1)}} \\
&= n^2 \sum_p (\log p)^2 \frac{1}{p^{\frac{1}{2}+2\epsilon_1} (p^{\frac{1}{2}+2\epsilon_1} - 1)} \\
&\ll_{n,\epsilon_1} 1.
\end{aligned}$$

This proves Theorem 1.1.

#### 4. Proof of Theorem 1.3

First, we wish to approximate  $\frac{L'_f}{L_f}(s)$  uniformly for  $\frac{1}{2} < \sigma_0 \leq \sigma \leq \sigma_1 < 1$  when  $T \leq t \leq 2T$ . We assume throughout below the Riemann hypothesis for  $L_f(s)$ .

From the work of Godement–Jacquet [GoJa06], it is known that the function  $L_f(s)$  is of finite order in any bounded vertical strip. Hence, we can very well assume that

$$L_f(s) \ll T^A = e^{A \log T}$$

for  $-1 \leq \sigma \leq 2$ ,  $T \leq t \leq 2T$  and  $A$  some fixed positive constant.

Taking  $s_0 = 2 + it$  with  $t \in \mathbb{R}$ , we have

$$L_f(2 + it) = \prod_p \prod_{i=1}^n \left( 1 - \frac{\alpha_{p,i}}{p^{2+it}} \right)^{-1}.$$

Observe that

$$\begin{aligned} \left| 1 - \frac{\alpha_{p,i}}{p^{2+it}} \right| &\leq 1 + \frac{|\alpha_{p,i}|}{p^2} \\ &\leq 1 + \frac{p^{\theta_n}}{p^2} \\ &= 1 + \frac{1}{p^{2-\theta_n}} \\ &\leq 1 + \frac{1}{p^{\frac{3}{2}}} \end{aligned}$$

because  $\theta_n \leq \frac{1}{2}$  for  $n \geq 2$ .

Therefore,

$$\begin{aligned} |L_f(2+it)| &\geq \prod_p \prod_{i=1}^n \left( 1 + \frac{1}{p^{\frac{3}{2}}} \right)^{-1} \\ &= \prod_p \left( 1 + \frac{1}{p^{\frac{3}{2}}} \right)^{-n} \\ &= \prod_p \left( \frac{1 - \frac{1}{p^{\frac{3}{2}}}}{1 - \frac{1}{p^3}} \right)^n \\ &= \left( \frac{\zeta(3)}{\zeta\left(\frac{3}{2}\right)} \right)^n \end{aligned}$$

which is a constant depending only on  $n$ . Therefore,  $L_f(2+it) \neq 0 \forall t \in \mathbb{R}$ .

Hence from Lemma 2.1, with  $r = 12$ ,  $s_0 = 2 + iT$ ,  $f(s) = L_f(s)$ ,  $M = A \log T$ , we obtain

$$-\frac{L'_f}{L_f}(s) = \sum_{|s-s_0| \leq 6} \frac{1}{s-\rho} + O(\log T).$$

For  $|s-s_0| \leq 3$  and so in particular for  $-1 \leq \sigma \leq 2, t = T$ , replacing  $T$  by  $t$  in the particular case, we obtain

$$-\frac{L'_f}{L_f}(s) = \sum_{|\rho-s_0| \leq 6} \frac{1}{s-\rho} + O(\log t).$$

Any term occurring in  $\sum_{|t-\gamma| \leq 1} \frac{1}{s-\rho}$  but not in  $\sum_{|s-s_0| \leq 6} \frac{1}{s-\rho}$  is bounded and the number of such terms does not exceed

$$N_f^*(t+6) - N_f^*(t-6) \ll \log t,$$

where  $N_f^*(t)$  is the number of zeros of  $L_f(s)$  in the region  $0 \leq \sigma \leq 1$  and  $0 \leq t \leq T$ . Thus, we get

$$-\frac{L'_f}{L_f}(s) = \sum_{|t-\gamma| \leq 1} \frac{1}{s-\rho} + O(\log t).$$

Assume  $\frac{1}{2} < \sigma < 1$  and  $T \leq t \leq 2T$ , then

$$\sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^s} e^{-\frac{m}{Y}} = -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{L'_f}{L_f}(s+w)\Gamma(w)Y^w dw.$$

Note also that from the above reasoning

$$\frac{L'_f}{L_f}(s) \ll \log t \quad \text{on any line } \sigma \neq \frac{1}{2}.$$

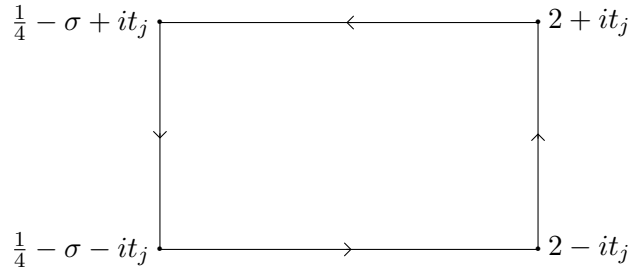
Also,

$$\frac{L'_f}{L_f}(s) \ll \frac{\log t}{\min(|t-\gamma|)} + \log t \quad \text{uniformly for } -1 \leq \sigma \leq 2.$$

From Lemma 2.2, we observe that each interval  $(j, j+1)$  contains values of  $t$  whose distance from the ordinate of any zero exceeds  $\frac{A}{\log j}$ , there is a  $t_j$  in any such interval for which

$$\frac{L'_f}{L_f}(s) \ll (\log t)^2 \quad \text{where } -1 \leq \sigma \leq 2 \text{ and } t = t_j.$$

Applying Cauchy’s residue theorem to the rectangle, we get



$$\begin{aligned} & \frac{1}{2\pi i} \left( \int_{2-it_j}^{2+it_j} + \int_{2+it_j}^{\frac{1}{4}-\sigma+it_j} + \int_{\frac{1}{4}-\sigma+it_j}^{\frac{1}{4}-\sigma-it_j} + \int_{\frac{1}{4}-\sigma-it_j}^{2-it_j} \right) \frac{L'_f}{L_f}(s+w)\Gamma(w)Y^w dw \\ &= \frac{L'_f}{L_f}(s) + \sum_{-t_j < \gamma < t_j} \Gamma(\rho-s)Y^{\rho-s}. \end{aligned}$$

In the sum appearing on the right hand side above, zeros  $\rho$  are counted with its multiplicity if there are any multiple zeros. The integrals along the horizontal lines tend to zero as  $j \rightarrow \infty$  since gamma function decays exponentially and  $Y$  is going to be at most a power of  $T$  only, so that

$$\sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^s} e^{-\frac{m}{Y}} = \frac{1}{2\pi i} \int_{\frac{1}{4}-\sigma-i\infty}^{\frac{1}{4}-\sigma+i\infty} \frac{L'_f}{L_f}(s+w)\Gamma(w)Y^w dw - \frac{L'_f}{L_f}(s) - \sum_{\rho} \Gamma(\rho-s)Y^{\rho-s}.$$

Note that  $\Gamma(w) \ll e^{-A|v|}$  so that the integral on  $\Re(w) = \frac{1}{4} - \sigma$  is

$$\begin{aligned} &\ll \int_{-\infty}^{\infty} e^{-A|v|} \log(|t+v|+2) Y^{\frac{1}{4}-\sigma} dv \\ &\ll \int_0^{2t} e^{-A|v|} \log(10|t|+2) Y^{\frac{1}{4}-\sigma} dv + \left( \int_{-\infty}^0 + \int_{2t}^{\infty} \right) e^{-A|v|} \log(|v|+10) Y^{\frac{1}{4}-\sigma} dv \\ &\ll Y^{\frac{1}{4}-\sigma} \log T + Y^{\frac{1}{4}-\sigma} \\ &\ll Y^{\frac{1}{4}-\sigma} \log T. \end{aligned}$$

Note that for  $\frac{1}{2} < \sigma_0 \leq \sigma \leq \sigma_1 < 1$ ,

$$|\Gamma(\rho - s)| < A_1 e^{-A_2|\gamma-t|}$$

uniformly for  $\sigma$  in the said range. Therefore,

$$\sum_{\rho} |\Gamma(\rho - s)| < A_1 \sum_{\rho} e^{-A_2|\gamma-t|} = A_1 \sum_{m=1}^{\infty} \sum_{m-1 \leq \gamma \leq m} e^{-A_2|t-\gamma|}.$$

The number of terms in the inner sum is

$$\ll \log(|t| + m) \ll \log |t| + \log(m+1)$$

and hence

$$\begin{aligned} \sum_{\rho} |\Gamma(\rho - s)| &\ll \sum_{m=1}^{\infty} e^{-A_2 m} (\log |t| + \log(m+1)) \ll \log T, \\ \left| \sum_{\rho} \Gamma(\rho - s) Y^{\rho-s} \right| &\ll Y^{\frac{1}{2}-\sigma} \log T. \end{aligned}$$

Thus for  $\frac{1}{2} < \sigma_0 \leq \sigma \leq \sigma_1 < 1$ , we have

$$-\frac{L'_f}{L_f}(s) = \sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^s} e^{-\frac{m}{Y}} + O_f(Y^{\frac{1}{2}-\sigma} \log T).$$

Thus for  $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$  and  $T \leq t \leq 2T$ , we obtain

$$\left| \frac{L'_f}{L_f}(\sigma_0 + it) \right|^2 \ll \left| \sum_{m=1}^{\infty} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0+it}} \right|^2 + \left( Y^{\frac{1}{2}-\sigma_0} \log T \right)^2.$$

Thus,

$$\int_T^{2T} \left| \frac{L'_f}{L_f}(\sigma_0 + it) \right|^2 dt \ll_f \int_T^{2T} \left| \sum_{m=1}^{\infty} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0+it}} \right|^2 dt + Y^{1-2\sigma_0} T (\log T)^2.$$

We note that

$$\left| \sum_{m=1}^{\infty} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0+it}} \right|^2 \ll \left| \sum_{m \leq \frac{Y}{2} (\log Y)^2} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0+it}} \right|^2 + \left| \sum_{m > \frac{Y}{2} (\log Y)^2} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0+it}} \right|^2,$$

and hence

$$\int_T^{2T} \left| \frac{L'_f}{L_f}(\sigma_0 + it) \right|^2 dt \ll_f \int_T^{2T} \left| \sum_{m \leq \frac{Y}{2} (\log Y)^2} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0+it}} \right|^2 + \int_T^{2T} \left| \sum_{m > \frac{Y}{2} (\log Y)^2} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0+it}} \right|^2 + Y^{1-2\sigma_0} T (\log T)^2.$$

By Montgomery–Vaughan theorem (Lemmas 2.3 and 2.4) and Lemma 2.5, we get

$$\begin{aligned} \int_T^{2T} \left| \frac{L'_f}{L_f}(\sigma_0 + it) \right|^2 dt &\ll_f \sum_{m \leq \frac{Y}{2} (\log Y)^2} \frac{|\Lambda_f(m)|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} (T + O(m)) \\ &+ \sum_{m > \frac{Y}{2} (\log Y)^2} \frac{|\Lambda_f(m)|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} (T + O(m)) + Y^{1-2\sigma_0} T (\log T)^2 \\ &\ll_f T \sum_{m \leq \frac{Y}{2} (\log Y)^2} \frac{|\Lambda_f(m)|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} + \sum_{m \leq \frac{Y}{2} (\log Y)^2} m \frac{|\Lambda_f(m)|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} \\ &+ T \sum_{m > \frac{Y}{2} (\log Y)^2} \frac{|\Lambda_f(m)|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} + \sum_{m > \frac{Y}{2} (\log Y)^2} m \frac{|\Lambda_f(m)|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} \\ &+ Y^{1-2\sigma_0} T (\log T)^2. \end{aligned}$$

By Lemmas 2.5 and 2.6, we obtain

$$\int_T^{2T} \left| \frac{L'_f}{L_f} \left( \frac{1}{2} + \epsilon + it \right) \right|^2 dt \ll_{f,n,\epsilon} T (\log Y)^2 + Y (\log Y)^4 + Y^{1-2\sigma_0} T (\log T)^2.$$

We choose  $Y = \exp\{(\log T)^\eta\}$  with any  $\eta$  satisfying  $0 < \eta < \frac{1}{2}$  so that we obtain

$$\int_T^{2T} \left| \frac{L'_f}{L_f}(\sigma_0 + it) \right|^2 dt \ll_{f,n,\epsilon,\eta} T (\log T)^{2\eta}.$$

This proves Theorem 1.3.

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### Amrinder Kaur

School of Mathematics and Statistics  
University of Hyderabad  
Hyderabad - 500046, Telangana, India.  
e-mail: amrinder1kaur@gmail.com

### Ayyadurai Sankaranarayanan

School of Mathematics and Statistics  
University of Hyderabad  
Hyderabad - 500046, Telangana, India.  
e-mail: sank@uohyd.ac.in