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SPECTRAL DECONVOLUTION OF MATRIX MODELS: THE ADDITIVE CASE

PIERRE TARRAGO

ABSTRACT. We implement a complex analytic method to build an estimator of the spectrum of a matrix perturbed by the addition of a random matrix noise in the free probabilistic regime. This method, which has been previously introduced by Arizmendi, Tarrago and Vargas, is done in two steps: the first step consists in a fixed point method to compute the Stieltjes transform of the desired distribution in a certain domain, and the second step is a classical deconvolution by a Cauchy distribution, whose parameter depends on the intensity of the noise. We thus reduce the spectral distribution problem to a classical one. We also provide explicit bounds for the mean squared error of the first step under the assumption that the distribution of the noise is unitary invariant. In the case where the unknown measure is sparse, we prove that the resulting estimator converges to the desired measure at speed $O(1/\sqrt{N})$ in the 1-Wasserstein distance, where N is the dimension of the matrix.

1. INTRODUCTION

Recovery of data from noisy signal is a recurrent problem in many areas of mathematics (geology, wireless communication, finance, electroencephalography...). From a statistical point of view, this can be seen as the recovery of a probability distribution from a sample of the distribution perturbed by a noise. In the simplest case, the perturbation is the addition of a random noise independent from the signal, and the process of recovering the original probability distribution from a noisy one is called deconvolution. In [Fan91, Fan92], Fan presented a first general approach to the deconvolution of probability distributions, which allowed to both recover the original data and to get a bound on the accuracy of the recovery. Since this seminal paper, several progresses have been made towards a better understanding of the classical deconvolution of probability measures, see for example [Lac06] in the density case or [DP17] in the atomic setting.

We are interested in the broader problem of the recovery of data in a non-commutative setting. Generally speaking, we are given a matrix $g(A, B)$, which is an algebraic combination of a possibly random matrix B representing the data we want to recover and a random matrix A representing the noise, and the goal is to recover the matrix B . Taking A and B diagonals and independent with entries of each matrix iid and considering the case $g(A, B) = A + B$ is equivalent to the classical deconvolution problem. This non commutative generalization has already seen many applications in the simplest cases of g being the addition or multiplication of matrices, [BBP17, LW04, BABP16]. Yet, the recovery of B is a complicated process already in those situations and we propose to address the additive case in the present manuscript: we provide a method to recover the spectral distribution of B , and we give precise bounds on the accuracy of the method in the case where this distribution is sparse.

Let us first discuss some important theoretical aspects of the non-commutative setting. A first difference with the classical case is the notion of independence. In the classical case, independence is a fundamental hypothesis in the success of the deconvolution, which allows to translate sum of random variables into convolution of distributions. In the non-commutative setting, one can generally consider two main hypotheses of independence: either the entries of A and B are assumed to be independent and the entries of A are assumed iid (up to a symmetry if A is self-adjoint), or the distribution of the noise matrix A is assumed to be invariant by unitary conjugation. Both notions generally yield similar results but require different tools. In this paper, we focus on the second hypothesis of a unitary invariant noise, which has already been studied in [BABP16, BGEM19, LP11]. Note that in the case of Gaussian matrices with

independent entries, the hypothesis of unitary invariance of the distribution is also satisfied, and both notions of independence coincide. The results of the present paper extend of course to the case of orthogonal invariant noises, up to numerical constants.

The second question is the scope of the deconvolution process: assuming B is self-adjoint, a perfect recovery of B would mean the recovery of both its eigenvalues and its eigenbasis. However, when the noise is unitary invariant, the recovery of the eigenbasis is almost impossible due to the delocalization phenomenon [Kar15, BES17]. Indeed, in general all eigenvectors of the received matrix $A + B$ have all coordinates having a same order of magnitude in the basis of eigenvectors of A , which prevents from guessing the latter. On the contrary, we will show that it is always possible to recover, to some extent, the eigenvalues of B , with an accuracy improving when the size of the matrices grows. In some cases, obtaining the spectrum of B is a first step towards a better recovery of B . This is the main approach of [LP11] in the case of a multiplicative noise to estimate large covariance matrices, which has led to the successful shrinkage method of [LW04, LW15]. This method has been generalized in [BABP16, BGEM19] to provide a general method to build estimators of the matrix B in the additive and multiplicative case when the distribution of the noise matrix A is assumed unitary invariant: once again, this approach uses the knowledge of the spectral distribution of B as an oracle, and the missing step of the latter method is precisely a general way of estimating the spectral distribution of B .

In the classical deconvolution, the known fact that the Fourier transform of the convolution of two probability measures is the product of the Fourier transform of both original measures has been the starting point of the pioneering work of Fan [Fan91]. Indeed, apart from definition issues, one can see the classical deconvolution as the division of the Fourier transform of the received signal by the Fourier transform of the noise. In the non-commutative setting, there is no close formula describing the spectrum of algebraic combination of finite size matrices, which prevents any hope of concrete formulas in the finite case. However, as the size goes to infinity, the spectral properties of algebraic combinations of independent random matrices is described by the theory of free probability introduced by Voiculescu [Voi91]. In particular, the spectral distribution of the sum of independent unitary invariant random matrices is closed to the so-called free additive convolution of the spectral distributions of each original matrix. Based on this theory and complex analysis, the subordination method (see [Bia98, Bel05, BB07, Voi00, BMS17]) provides us tools to compute very good approximations of the spectrum of sums of independent random matrices in the same flavor as the multiplication of the Fourier transforms in the classical case.

In [ATV17], Arizmendi, Vargas and the author developed an approach to the spectral deconvolution by inverting the subordination method. This approach showed promising results on simulations, and the goal of this paper is to show theoretically that it successfully achieves the spectral deconvolution of random matrix models in the additive case. We also provide first concentration bounds on the result of the deconvolution, in the vein of Fan's results on the classical deconvolution [Fan91]. A companion paper [Tar] proves similar result in the multiplicative case. In his first two papers dealing with deconvolution, Fan already noted that the accuracy of the deconvolution greatly worsens as the noise gets smoother, and improves with the smoothness of the unknown distribution. This can be seen at the level of the Fourier transform approach. Indeed, the Fourier transform of a smooth noise is rapidly decreasing to zero at infinity and thus the convolution with a smooth noise sets the Fourier transform of the original distribution exponentially close to zero for higher modes, acting as a low-pass filter. When the original distribution has non-trivial higher modes, it is thus extremely difficult to recover those higher frequencies in the deconvolution, which translates into a poor concentration bound on the accuracy of the process. When the original distribution is also very smooth, those higher modes do not contribute to the distribution and thus the recovery is still accurate. In the supersmooth case where the Fourier transform of the noise is decreasing exponentially to zero at infinity, the accuracy is logarithmically decreasing with the size of the sample, except when the original distribution is also supersmooth [Lac06].

In [BB04], Belinschi and Bercovici proved that the free additive and multiplicative convolutions of probability measures are always analytic, except at some exceptional points. As the

spectral deconvolution is close to reversing a free convolution, we should expect the behavior of the spectral convolution to be close to the ultrasmooth case of Fan. This phenomenon appears in the method proposed in [ATV17], which first builds an estimator $\widehat{\mathcal{C}}_B$ of the convolution \mathcal{C}_B of the desired distribution with a certain Cauchy distribution, and then achieve the classical deconvolution of $\widehat{\mathcal{C}}_B$ by this Cauchy distribution, which is a supersmooth. Therefore, the accuracy of the spectral deconvolution method should be approximately the one of a deconvolution by a Cauchy transform. We propose then to measure the accuracy of the method by two main quantities: the parameter of the Cauchy transform involved in the first step of the deconvolution, and the size of the matrices. We show that the parameter of the Cauchy transform, which gives the range of Fourier modes we can recover, depends mainly on the intensity of the noise, while the precision of the recovery of \mathcal{C}_B depends on the size N of the matrices. This is similar to the situation in the classical case [Fan91], where the size of the matrices is replaced by the size of the samples. The concentration bounds we get for the estimator of \mathcal{C}_B in the additive case depend on the first six moments of the spectral distribution of A and B . Parallel to our work, Maïda et al. [MNN⁺20] have successfully used the method from [ATV17] to study the backward free Fokker-Planck equation. In the course of their study, they also managed to improve the method of [ATV17] in the case of a semi-circular noise and to measure the accuracy of the method in the case of a backward Dyson Brownian motion.

Let us describe the organization of the manuscript. In Section 2, we explain precisely the models, recall the deconvolution procedure implemented in [ATV17] and some methods to achieve a classical deconvolution and state convergence in expectation of our estimator. This section is self-contained for a reader only interested in an overview of the deconvolution and its practical implementation and accuracy, and in particular the free probabilistic background is postponed to next section. We also provide simulations to illustrate the deconvolution procedure and to show how the concentration bounds compare to simulated errors. We aimed at providing explicit constants for every probabilistic bounds. Expressions of these constants are given in Appendix C. In Section 3, we introduce all necessary background to prove the concentration bounds, and we prove the main theorem of the manuscript assuming several results whose proofs are postponed to next sections. In Section 4, we prove the concentration of the estimator assuming the spectral measure of $A + B$ is close to a free additive convolution in a precise sense (see Definition 3.4 for a formal definition). In Section 5, we prove concentration results regarding the classical deconvolution to recover a sparse measure. In Section 6, we prove that in the case of a unitary invariant noise, the almost free probabilistic behavior defined in Definition 3.4 is achieved with some explicit bounds only depending on the first six moments of A and B : to this end, we introduce matricial subordination functions of Pastur and Vasilchuk [PV00], which is the main tool of the proof. The latter also heavily relies on integration formulas and concentration bounds on the unitary groups, which are respectively described in Appendix A and B.

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2. DESCRIPTION OF THE MODEL AND OF THE RESULTS

2.1. Notations. In the sequel, N is a positive number denoting the dimension of the matrices, \mathbb{C} denotes the field of complex numbers, and \mathbb{C}^+ denotes the half-space of complex numbers with positive imaginary part. For $K > 0$, we denote by \mathbb{C}_K the half-space of complex numbers with imaginary part larger than K .

We write $\mathcal{H}_N(\mathbb{C})$ for the space of N -dimensional self-adjoint matrices. When $X \in \mathcal{H}_N(\mathbb{C})$, we denote by $X = X^+ + X^-$ the unique decomposition of X such that $X^+ \geq 0$ and $X^- \leq 0$. The matrix X^+ is called the positive part of X and X^- its negative part. We recall that the normalized trace $\text{tr}(X)$ of X is equal to $\frac{1}{N} \sum_{i=1}^N X_{ii}$. The resolvent G_X of G is defined on \mathbb{C}^+ by

$$G_X(z) = (X - z)^{-1}.$$

When $X \in \mathcal{H}_N(\mathbb{C})$, we denote by $\lambda_1^X, \dots, \lambda_N^X$ its eigenvalues and by

$$\mu_X = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i^X}$$

its spectral distribution. We use the convention to use capital letters to denote matrices, and corresponding small letter with index $i \in \mathbb{N}$ to denote the i -th moment of the corresponding spectral distribution. For example, if X is Hermitian and $i \in \mathbb{N}$, then

$$x_i = \text{tr}(X^i) = \frac{1}{N} \sum_{i=1}^N \lambda_i^X.$$

We also write x_i^0 for the i -th centered moment of X , namely

$$x_i^0 = \text{tr}((X - \text{tr}(X))^i).$$

In particular, $x_1^0 = 0$ and $x_2^0 = \text{Var}(\mu_X)$, the variance of μ_X . Finally, we write $\sigma_X = \sqrt{\text{Var}(\mu_X)}$ for the standard deviation of μ_X , $\theta_X = \frac{x_4^0}{\sigma_X^4}$ for the kurtosis of X and x_∞ for the infinity norm of X .

When μ is a probability distribution on \mathbb{R} and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function, we set $\mu(f) = \int_{\mathbb{R}} f(t) d\mu(t)$ and we write $\mu(k)$ for the k -th moment of μ , when it is well defined. When μ admits moments of order 2, we denote by $\text{Var}(\mu) = \mu(2) - \mu(1)^2$ the variance of μ . The Stieltjes transform of a probability measure μ is the analytic function defined on \mathbb{C}^+ by

$$m_\mu(z) = \int_{\mathbb{R}} \frac{1}{t - z} d\mu(t).$$

In the special case where $\mu = \mu_X$ for some Hermitian matrix X , we simply write m_X instead of m_{μ_X} .

2.2. Unitary invariant model and reduction of the problem. The main topic of this paper is the estimation of the spectral density of a matrix which is modified by an additive matrix noise. Hence, we fix a Hermitian matrix $B = B^* \in \mathcal{M}_N(\mathbb{C})$, which is called the *signal matrix*. We denote by $\lambda_1, \dots, \lambda_N$ its eigenvalues and by $\mu_B = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}$ its spectral distribution. Additionally, we consider a random Hermitian matrix $A \in \mathcal{M}_N(\mathbb{C})$, called the *noise matrix*, whose spectral distribution μ_A is therefore random. We suppose that the random distribution μ_A satisfies the following properties.

Condition 2.1. *There exists a known probability measure μ_1 with moments of order 6 and a constant $C_A > 0$ such that:*

- (1) $\mu_1(1) = 0$,
- (2) *there exists a constant $\kappa > 0$ such that*

$$|a_i| \leq \left(1 + \frac{\kappa}{\sqrt{N}}\right)^i |\mu_1(i)|,$$

for $1 \leq i \leq 6$, where we recall that $a_i = \mu_A(i) = \text{tr}(A^i)$, and

- (3) *there exists $C_{\text{noise}} > 0$ such that for any C^1 function $f : \mathbb{R} \rightarrow \mathbb{C}$,*

$$\mathbb{E}(|\mu_A(f) - \mu_1(f)|^2) \leq \frac{C_{\text{noise}}^2 \mathbb{E} \|\nabla f\|_2^2}{N},$$

where f is considered as a function from $\mathcal{H}_N(\mathbb{C}) \rightarrow \mathbb{C}$ with $f(A) = \frac{1}{N} \sum_{i=1}^N f(\lambda_i^A)$, and \mathbb{E} denotes the expectation with respect to the random matrix A .

The first assumption of Condition 2.1 is a simple scaling to simplify the formulas of the manuscript. The second assumption is mostly technical, and can be relaxed at the cost of coarsening the concentration bounds. Indeed, we use several constants involving moments of the unknown distribution μ_A , and the bounding assumption of Condition 2.1 allows us to use the moments of μ_1 instead. This bound generally holds with probability $1 - \exp(-c'N)$ for some c' depending on the moment and on the class of matrix model. Finally, the last condition is

usually also satisfied in most known cases. See [GZ00] for concentration inequalities in the case where A is either Wigner or Wishart (see also [AGZ10, Section 4.4.1]). Then, we consider the following additive recovery problem

Problem 2.2 (Additive deconvolution). *Given $H = B + UAU^*$ with U Haar unitary, $\mu_B(1) = 0$ and μ_A satisfying Condition 2.1, reconstruct μ_B ,*

The assumption on the first moment of B is merely technical and can be relaxed without any consequence on the results.

2.3. Deconvolution procedure. We now explain the deconvolution procedure leading to an estimator $\widehat{\mu_B}$ of μ_B . This deconvolution is done in two steps. The first step is to build an estimator $\widehat{\mathcal{C}_B[\eta]}$ of the classical convolution $\mathcal{C}_B[\eta] := \mu_B * \text{Cauchy}[\eta]$ of μ_B with a Cauchy distribution $\text{Cauchy}[\eta]$ of parameter η . Let us recall that

$$d\text{Cauchy}[\eta](t) = \frac{1}{\pi} \frac{\eta}{t^2 + \eta^2},$$

for $t \in \mathbb{R}$. The estimator only exists for η larger than some threshold depending on the moments of the noise. Then, the second step is to build an estimator $\widehat{\mu_B}$ of μ_B from $\widehat{\mathcal{C}_B[\eta]}$ by simply doing the classical deconvolution of $\widehat{\mathcal{C}_B[\eta]}$ by the noise $\text{Cauchy}[\eta]$.

2.3.1. Obtaining the estimator $\widehat{\mathcal{C}_B}$. The first step is quite new [ATV17] and requires complex analytic tools. Recall the Stieltjes inversion formula, saying that for $t \in \mathbb{R}$ we have

$$d\mathcal{C}_B[\eta](t) = \frac{1}{\pi} \Im m_B(t + i\eta),$$

where m_B is the Stieltjes transform of μ_B introduced in Section 2.1. Using this formula, we build $\widehat{\mathcal{C}_B[\eta]}$ by first constructing an estimator of m_B which exists on the upper half-plane \mathbb{C}_η . In our case, we can simply take $\eta = 2\sqrt{2}\sigma_1$, where $\sigma_1 = \sqrt{\text{Var}(\mu_1)}$. We then have the following convergence result from [ATV17].

Theorem 2.3. [ATV17] *There exist two analytic functions $\omega_1, \omega_3 : \mathbb{C}_{2\sqrt{2}\sigma_1} \rightarrow \mathbb{C}^+$ such that for all $z \in \mathbb{C}_{2\sqrt{2}\sigma_1}$,*

- $\Im \omega_1(z) \geq \frac{\Im z}{2}, \Im \omega_3(z) \geq \frac{3\Im z}{4},$
- $\omega_1(z) + z = \omega_3(z) - \frac{1}{m_{\mu_1}(\omega_1(z))} = \omega_3(z) - \frac{1}{m_H(\omega_3(z))}.$

Moreover, setting $h_{\mu_1}(w) = -w - \frac{1}{m_{\mu_1}(w)}$, $\omega_3(z)$ is the unique fixed point of the function $K_z(w) = z - h_{\mu_1}(w - \frac{1}{m_H(w)} - z)$ in $\mathbb{C}_{3\Im(z)/4}$ and we have

$$\omega_3(z) = \lim K_z^{\text{on}}(w),$$

for all $w \in \mathbb{C}_{3/4\Im(z)}$.

The functions ω_1, ω_3 are called subordination functions for the free deconvolution. The last part of the latter theorem is important, since it yields a concrete method to build the function ω_3 by iteration of the map K_z . This iteration converges quickly thanks to its contraction properties with respect to the Schwartz distance. The constant $2\sqrt{2}$ has been improved to 2 in [MNN⁺20] in the case where μ_1 is a semi-circular distribution. The above theorem leads then to the construction of $\widehat{\mathcal{C}_B[\eta]}$.

Definition 2.4. *The additive Cauchy estimator of μ_B for $\eta \geq 2\sqrt{2}\sigma_1$ at $t \in \mathbb{R}$ is*

$$\widehat{\mathcal{C}_B[\eta]}(t) = \frac{1}{\pi} \Im \left[m_H(\omega_3(t + 2\sqrt{2}\sigma_1 i)) \right],$$

where ω_3 is defined in Theorem 2.3.

Let us explain the free probabilistic intuition behind this definition (see Section 3.2 for more background on free probability). In the ideal situation where $\mu_H = \mu_1 \boxplus \mu_B$, then $m_{\mu_B}(z) = m_H(\omega_3(z)) = m_{\mu_1}(\omega_1(z))$ for all $z \in \mathbb{C}_{2\sqrt{2}\sigma_1}$. In general we never have the exact relation $\mu_H = \mu_1 \boxplus \mu_B$, but by Theorem 3.3 $\mu_H \simeq \mu_A \boxplus \mu_B$ and by Condition 2.1, $\mu_A \simeq \mu_1$; hence

we have the approximate free convolution $\mu_H \simeq \mu_1 \boxplus \mu_B$, and thus $m_{\mu_B}(z) \simeq m_H(\omega_3(z))$ on $\mathbb{C}_{2\sqrt{2}\sigma_1}$. Then, taking the imaginary part gives the approximated value of \mathcal{C}_B by Stieltjes inversion formula.

2.3.2. Estimating the distribution μ_B . The last step is to recover μ_B from $\widehat{\mathcal{C}_B[\eta]}$, which is a classical deconvolution of $\widehat{\mathcal{C}_B[\eta]}$ by the Cauchy distribution $\text{Cauchy}[\eta]$ with $\eta \geq 2\sqrt{2}\sigma_1$ in our setting. Deconvolving a positive measure is a classical problem in statistic which has been deeply studied since the first results of Fan [Fan91]. The main feature of our situation is the supersmooth aspect of the Cauchy distribution. In particular, the convergence of the deconvolution may be very slow depending on the smoothness of the original measure. There are two main situations, which are solved differently:

- the original μ_B is close to a probability distribution with a density in $L^2(\mathbb{R})$: in this case, it is better to take a Fourier approach. This density case, which is tackled in a future paper in collaboration with Emilien Joly [JT], requires an estimation of the L^2 -distance between $\widehat{\mathcal{C}_B[\eta]}$ and $\mathcal{C}_B[\eta]$. Such estimation is given in Theorem 2.6.
- the original measure μ_B is *sparse*, meaning that it consists of few atoms. We mainly focus on the off-the-grid deconvolution, although an on-the-grid approach is also possible. In this case, we reduce the problem to a L^1 -minimization procedure with respect to some Fourier measurement. Namely, denote by \mathcal{F} the Fourier transform on \mathbb{R} , and fix some $t_0 > 0$, $M \in \mathbb{N}$ (see Section 5 for an appropriate choice of t_0, M according to the error the estimation of $\mathcal{C}_B[\eta]$). Then, set $\widehat{\mu}_B = \hat{\mu}/\hat{\mu}(\mathbb{R})$, with

$$(1) \quad \hat{\mu} = \arg \min_{\nu \in \mathcal{M}^+([-t_0/2, t_0/2])} \|\mathcal{F}(\nu) - \exp(\eta \cdot) \mathcal{F}(F)\|_2^{t_0, M} + \lambda \|\nu\|_{TV},$$

$$\|\nu\|_{TV} \leq 1$$

where $F(t) = \widehat{\mathcal{C}_B[\eta]}|_{[-R_0, R_0]}(t)$, $\mathcal{M}^+([-t_0/2, t_0/2])$ denotes the space of positive measures on $[-t_0/2, t_0/2]$, $\|f\|_2^{t_0, M} = \sqrt{\sum_{k=-M}^M f(2\pi k/t_0)^2}$ and λ, R_0 are parameters to tune depending on the expected error between $\mathcal{C}_B[\eta]$ and $\widehat{\mathcal{C}_B[\eta]}$. This minimization problem can be solved by a constrained quadratic programming method (see [BV04]). The constraints of the domain on which the minimization is achieved actually enforces the sparsity of the solution. When the atoms of μ are well-separated, this deconvolution successfully recovers μ . The accuracy of this method is proven in the present paper.

We focus here on the second situation of a sparse signal. As it is often required in the deconvolution of low-pass filter, see [CFG14], we impose a minimal separation condition on the support of the signal. Namely, there exists a constant $d > 0$ such that

$$(2) \quad \Delta(\mu) := \min_{\substack{t_i, t_j \in \text{Supp}(\mu) \\ t_i \neq t_j}} |t_i - t_j| \geq 5/2 \cdot d\eta,$$

where η is the parameter of the Cauchy distribution involved in the deconvolution process. The numerical constant $5/2$ reflects the difficulty of recovering close spikes from a low-pass filter, and several other numerical constants depend on this choice. The lower this constant is, the more accurate is the deconvolution process. The choice $5/2$ allowed us compute the corresponding constants used in Section 5. In [CFG14], it is claimed that it can be lowered to 1.87.

Then, choosing an adequate sampling rate M and range t_0 , see (34), we have the following convergence result with respect to the Wasserstein distance W_1 .

Proposition 2.5. *Let $p \in [2, \infty]$, $\lambda_0 > 0$ and let μ_0 be a probability distribution on \mathbb{R} having a finite p -moment bounded by $M_p > 0$ and satisfying the condition (2), and suppose that $F = \mu_0 * \text{Cauchy}[\eta] + n$ with n being a noise such that $\|n\|_{L^1(\mathbb{R})} \leq \epsilon$. Then, there exists $t_0 = O(\lambda_0^{1/p})$ such that the solution $\hat{\mu}$ of (1) with $\lambda = \frac{e\sqrt{t_0}}{\sqrt{2\pi\eta}} \exp(2\pi/d)\lambda_0$ and $M = t_0/(d\eta)$ satisfies the inequality*

$$W_1(\mu, \hat{\mu}/\hat{\mu}(\mathbb{R})) \leq a(t_0^{-1/2})t_0^{3/2} \exp(2\pi/d)(\epsilon + \lambda_0) + bd\eta^{3/4}t_0^{1/4} \exp(\pi/d)\sqrt{\epsilon + \lambda_0},$$

with $a(t)$ is an affine function given in (36) and b is a numerical constant given in (37). Moreover for $p \geq 3$, we have

$$W_1(\mu, \hat{\mu}/\hat{\mu}(\mathbb{R})) = O((\epsilon + \lambda_0)^{1/2-1/(4p)})$$

as ϵ goes to zero.

The expression of t_0 in the above Proposition is given in (34). Remark in particular that for $p = \infty$ and $\lambda_0 = \epsilon$, we get

$$W_1(\mu, \hat{\mu}/\hat{\mu}(\mathbb{R})) = O(\sqrt{\epsilon}).$$

In the case of a low-pass filter, more general results are available in case the unknown distribution has atoms closer than the minimal separation distance, see [Ben17, DDP17, DP17]. The recovery is still possible, but the error term grows exponentially with the number of atoms which are closer than the threshold $d\eta$, and the known constants are quite large.

2.4. Mean-square bounds on the recovery of $\mathcal{C}_B[\eta]$. Recall that $\mathcal{C}_B[\eta] = \mu_B * \text{Cauchy}(\eta)$. We now state the concentration bounds for the estimators we constructed before, which involve moments of A and B up to order 6. We chose to avoid any simplification which would hinder the accuracy of the constants or restrict their domain of validity, since any numerical computing environment can easily compute the expressions obtained. Despite some increased complexity, the simulations in the next section show some promising result on the precision in known cases. The reader should refer to Appendix C to get a full picture of the constants involved.

Theorem 2.6 (Mean L^1 and L^2 distance). *Let $\eta \geq 2\sqrt{2}\sigma_1$. There exists $C_{\text{threshold}}[\eta] > 0$ such that for $N \geq C_{\text{threshold}}[\eta]$,*

$$\sqrt{\mathbb{E} \left(\left\| \widehat{\mathcal{C}}_B[\eta] - \mathcal{C}_B[\eta] \right\|_{L^2}^2 \right)} \leq \frac{C_{MSE}(\eta, N)}{N}$$

and

$$\mathbb{E} \left[\left\| \widehat{\mathcal{C}}_B[\eta]^{R_0} - \mathcal{C}_B[\eta] \right\|_{L^1} \right] \leq \frac{C_{L^1}(\eta, N) \log N}{N},$$

where $\widehat{\mathcal{C}}_B^{R_0} = \mathbf{1}_{[-R_0, R_0]} \widehat{\mathcal{C}}_B$ for some suitable R_0 and $C_{MSE}(\eta, N)$, $C_{L^1}(\eta, N)$ are decreasing in N .

The constants $C_{\text{threshold}}[\eta]$, $C_{MSE}(\eta, N)$, $C_{L^1}(\eta, N)$ and R_0 are respectively given in (7), (17), (21) and (23).

2.5. Accuracy of the spectral deconvolution in the sparse case. We are only dealing with the case of a sparse distribution in this paper, the case of distribution with a density being tackled in the forthcoming paper [JT]. Combining Theorem 2.6 with Proposition 2.5, we get the following Markov-type bound on the accuracy of the deconvolution.

Theorem 2.7. *Suppose that $N^2 \geq C_{\text{threshold}}[2\sqrt{2}\sigma_1]$ and $p \in [6, +\infty]$, $M_p > 0$ such that the p -th moment of μ_B is smaller than M_p . Assume moreover that μ_B satisfies to (2) for some $d > 0$. Then, the solution $\widehat{\mu}_B$ of (1) with $\lambda = \frac{C_\lambda}{N}$ where C_λ is given in (8) satisfies the mean 1-Wasserstein error*

$$\mathbb{E}(W_1(\widehat{\mu}_B, \mu_B)) \leq C_{\text{sparse}}(N) \left(\frac{\log N}{N} \right)^{1/2-1/(4p)},$$

where W_1 denotes the 1-Wasserstein distance and C_{sparse} is decreasing in N and given in (10).

Remark that C_{sparse} only depends on d , the p -th moment of μ_B and the six first moments of μ_1 and μ_B . In particular, if we assume μ_B as a support bounded by some constant B_∞ and satisfies (2) for some fixed $d > 0$, we have

$$\mathbb{E}(W_1(\widehat{\mu}_B, \mu_B)) = O \left(\sqrt{\frac{\log N}{N}} \right).$$

2.6. Simulations. We provide here some simulations to show the accuracy and limits of the concentration bounds we found on the mean squared error in Section 2.4. We perform the first step of the deconvolution as explained in Section 2.3 and compute the error with $\mathcal{C}_B(\eta)$, and then compare this error with the constant we computed according to the formulas in Theorem 2.6. We consider a data matrix B which is diagonal with iid entries following a real standard Gaussian distribution, and a noise matrix A which follows a GUE distribution (namely, $A = (X + X^*)/\sqrt{2}$, with the entries of X iid following a complex centered distribution with variance $1/N$). Hence, μ_A is close to a standard semi-circular distribution μ_1 in the sense of Condition 2.1. Then, we consider the additive model $H = B + UAU^*$ (even if the presence of U is redundant, since the distribution of A is already unitary invariant). We performed the iteration procedure explained in Theorem 2.3 at $\eta = 2\sqrt{2}\sigma_1 = 2\sqrt{2}$. In Figure 1, we show an example of the spectral distribution of H , the result of the first step of the deconvolution, and then the result of the deconvolution after the classical deconvolution by a Cauchy distribution (we used here a constrained Tychonov method see [Neu88]), and a comparison with μ_B .

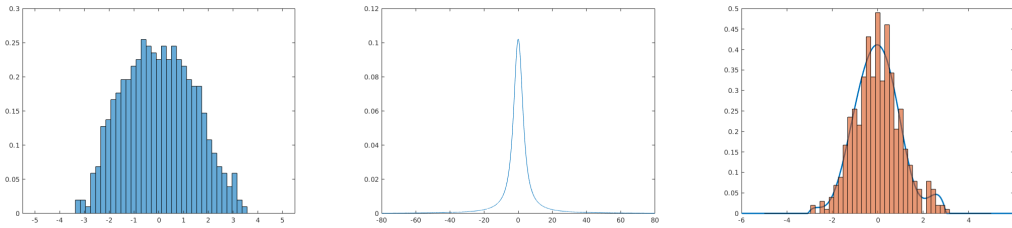


FIGURE 1. Histogram of the eigenvalues of H , result of the first step of the deconvolution, result of the second step of the deconvolution and comparison with the histogram of μ_B ($N = 500$).

The result is very accurate, which is not surprising due to the analyticity property of the Gaussian distribution (see the discussion in Section 2.5). Then, we simulate the standard error \sqrt{MSE} with a sampling of deconvolutions with the size N going from 50 to 2000. The lower bound on N for the validity of Theorem 2.6 is 4, which is directly satisfied. We can then compare the simulated standard deviation to the square root of the bound given in Theorem 2.6. The results are displayed in Figure 2. The first diagram is a graph of the estimated square root of MSE and the second one is the graph of the theoretical constant we computed according to N . The third graph is a ratio of both quantities according to N .

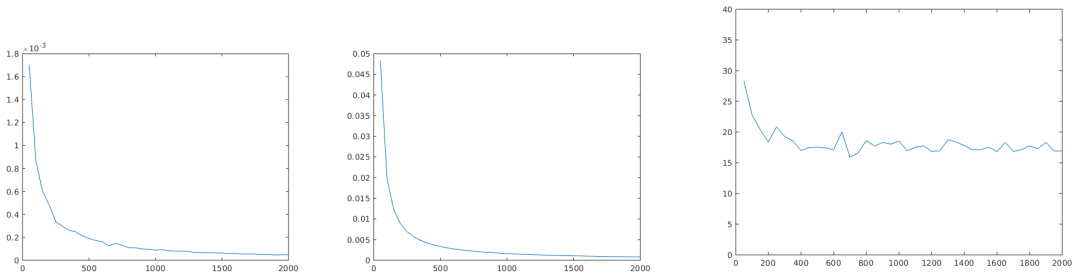


FIGURE 2. Simulation of \sqrt{MSE} in the additive case for N from 50 to 2000 (with a sampling of size 100 for each size), theoretical bound on \sqrt{MSE} provided in Theorem 2.6, and ratio of the theoretical bound on the simulated error.

We see that the error on the bound is better when N is larger. When N is small, the term C_1N^{-1} is non negligible, and approximations in the concentration results of the subordination function in Section 6 contribute to this higher ratio. When N gets larger, the term C_1N^{-1} vanishes and the ratio between the theoretical constant and the estimated error gets better. There is certainly room for improvement: for example, we know that the variance of the

Cauchy transform of H converges to some explicit constant depending on A and B , see [PS11]. It would be interesting to provide a bound on this constant which would only depend on the first moments of B .

3. APPROXIMATE SUBORDINATION AND PROOF OF THE MAIN RESULTS

We introduce here the notion of approximate subordination and prove Theorem 2.6 and 2.7. These proofs rely on several results which are proven in the different following sections.

3.1. Probability measures, cumulants and analytic transforms. Let μ be a probability measure on \mathbb{R} . Recall that $\mu(k)$ denotes the k -th moment of μ , when it is defined.

3.1.1. Free cumulants. Throughout this manuscript, free probability theory will be present without being really mentioned. In particular, several quantities involve free cumulants of probability measures and mixed moments of free random variables, which have been introduced by Speicher in [Spe94]. Since we will only use moments of low orders, we won't develop the general theory of free cumulants and the interested reader should refer to [NS06] for more information on the subject, in particular to learn about the non-crossing partitions picture explaining the formulas below.

The free cumulant of order r of μ is denoted by $k_r(\mu)$. In this paper, we use only the first three free cumulants, which are the following:

$$k_1(\mu) = \mu(1), \quad k_2(\mu) = \text{Var}(\mu) = \mu(2) - \mu(1)^2, \quad k_3(\mu) = \mu(3) - 3\mu(2)\mu(1) + 2\mu(1)^3.$$

If μ, μ' are two probability measures on \mathbb{R} and \vec{k}, \vec{k}' are words of integers of length r with $r > 0$ we denote by $m_{\mu, \mu'}(\vec{k}, \vec{k}')$ the mixed moments of μ, μ' when they are assumed in free position (see [NS06] for more background on free random variables). Once again, we only need the formulas of $m_{\mu, \mu'}(\vec{k}, \vec{k}')$ for few values of \vec{k}, \vec{k}' , which are as follow:

$$m_{\mu, \mu'}(k, k') = \mu(k)\mu'(k'),$$

$$m_{\mu, \mu'}(k_1 \cdot k_2, k'_1 \cdot k'_2) = \mu(k_1 + k_2)\mu'(k'_1)\mu'(k'_2) + \mu(k_1)\mu(k_2)\mu'(k'_1 + k'_2) - \mu(k_1)\mu(k_2)\mu'(k'_1)\mu'(k'_2),$$

and, writing 1^3 for the word $1 \cdot 1 \cdot 1$,

$$\begin{aligned} m_{\mu, \mu'}(k_1 \cdot k_2 \cdot k_3, 1^3) &= \mu'(1)^3 \mu(k_1 + k_2 + k_3) \\ &\quad + \mu'(1) \text{Var}(\mu') \left(\mu(k_1 + k_2)\mu(k_3) + \mu(k_2 + k_3)\mu(k_1) + \mu(k_3 + k_1)\mu(k_2) \right) \\ &\quad + k_3(\mu') \mu(k_1)\mu(k_2)\mu(k_3). \end{aligned}$$

By abuse of notation, we simply write $k_r(X)$ for $k_r(\mu_X)$ and $m_{X, X'}(\vec{k}, \vec{k}')$ for $m_{\mu_X, \mu_{X'}}(\vec{k}, \vec{k}')$, when X, X' are self-adjoint matrices.

3.1.2. Analytic transforms of probability distributions. The Stieltjes transform of a probability distribution μ is the analytic function $m_\mu : \mathbb{C}^+ \rightarrow \mathbb{C}$ defined by the formula

$$m_\mu(z) = \int_{\mathbb{R}} \frac{1}{t - z} d\mu(t), \quad z \in \mathbb{C}^+.$$

We can recover a distribution from its Stieltjes transform through the Stieltjes Inversion formula, which gives μ in terms of m_μ as

$$d\mu(t) = \frac{1}{\pi} \lim_{y \rightarrow 0} \Im m_\mu(t + iy)$$

in a weak sense. We will mostly explore spectral distributions through their Stieltjes transforms, since the latter have very good analytical properties. The first important property is that $m_\mu(\mathbb{C}^+) \subset \mathbb{C}^+$. Actually, Nevanlinna's theory provides a reciprocal result.

Theorem 3.1. [MS17, Theorem 3.10] *Suppose that $m : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ is such that*

$$-iym(iy) \xrightarrow[n \rightarrow \infty]{} 1,$$

then there exists a probability measure ρ such that $m = m_\rho$.

We will use the following transforms of m_μ , whose given properties are direct consequences of Nevanlinna's theorem and the expansion at infinity $m_\mu(z) = -\sum_{k=0}^r \frac{\mu(r)}{z^{r+1}} + o(z^{-(r+2)})$, when μ admits moments of order up to $r > 0$.

- the reciprocal Cauchy transform of μ , $F_\mu : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ with $F_\mu(z) = \frac{-1}{m_\mu(z)}$. If μ admits moments of order two, we have the following important formula [MS17, Lemma 3.20], which will be used throughout the paper,

$$(3) \quad F_\mu(z) = z - \mu(1) + \text{Var}(\mu)m_\rho(z),$$

for some probability measure ρ . In particular,

$$(4) \quad \Im[F_\mu(z)] \geq \Im z.$$

When μ admits a moment of order three, then ρ has a moment of order one which is given by the formula

$$(5) \quad \rho(1) = \frac{\mu(3) - 2\mu(1)\mu(2) + \mu(1)^3}{\text{Var}(\mu)}.$$

- the h -transform of μ , $h_\mu = F_\mu(z) - z$. By (4), $h_\mu : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ and $h_\mu(z) = \text{Var}(\mu)m_\rho(z) - \mu(1)$ for $z \in \mathbb{C}^+$.

We write F_X and h_X instead of F_{μ_X} and h_{μ_X} for $X \in \mathcal{M}_N(\mathbb{C})$ self-adjoint.

3.2. Free convolution of measures. From the seminal work of Voiculescu [Voi91], it is known that for N large, the spectral distribution of $H = UAU^* + B$ (resp. $M = A^{1/2}UBU^*A^{1/2}$) with U Haar unitary is close in probability to a deterministic measure called the free additive (resp. multiplicative) convolution of μ_A and μ_B and denoted by $\mu_A \boxplus \mu_B$ (resp. $\mu_A \boxtimes \mu_B$), see below for a more precise statement. For more background on free convolutions and their relation with random matrices, see [MS17]. In this manuscript, we will only use the following characterization of the free additive convolutions, called the *subordination phenomenon*. This characterization has been fully developed by [BB07, Bel05], after having been introduced by [Bia98] and [Voi00]. For readers not familiar with free probabilistic concepts, the following can be understood as a definition of the free additive convolution.

Theorem 3.2. [BB07] *Suppose that $\mu_1 \boxplus \mu_2 = \mu_3$. Then, for $z \in \mathbb{C}^+$, we have $m_{\mu_3}(z) = m_{\mu_2}(\omega_2(z)) = m_{\mu_1}(\omega_1(z))$, where $\omega_2(z)$ is the unique fixed point of the function $K_z : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ given by*

$$K_z(w) = h_{\mu_1}(h_{\mu_2}(w) + z) + z,$$

and ω_1 and ω_2 satisfy the relation

$$(6) \quad \omega_1(z) + \omega_2(z) = z - \frac{1}{m_{\mu_3}(z)}.$$

Moreover, ω_1, ω_2 are analytic functions on \mathbb{C}^+ and we have

$$\omega_2(z) = \lim_{n \rightarrow \infty} K_z^{\circ n}(w)$$

for all $w \in \mathbb{C}^+$. The functions ω_1 and ω_2 are called the *subordination functions* for the free additive convolution.

These two iterative procedures should be understood as the main implementation scheme for concrete applications, whereas the fixed point equations give the precise definition of both convolutions. The fundamental result relating free probability to random matrices is the convergence of the spectral distribution of sum of random matrices conjugated by Haar unitaries towards the free additive convolution.

Theorem 3.3. [Voi91, Spe93, PV00, Vas01] *Suppose that $(A_N, B_N)_{N \geq 0}$ are two sequences of matrices, with $A_N, B_N \in \mathcal{M}_N(\mathbb{C})$ self-adjoint, and let U_N be a random unitary matrix distributed according to the Haar measure. Then, if $\mu_{A_N} \xrightarrow[\text{weakly}]{a.s.} \mu_1$ and $\mu_{B_N} \xrightarrow[\text{weakly}]{a.s.} \mu_2$ with $\sup_N(\max(\mu_{A_N}(2), \mu_{B_N}(2))) < +\infty$, then*

$$\mu_{A_N + UB_N U^*} \xrightarrow[\text{weakly}]{a.s.} \mu_1 \boxplus \mu_2.$$

Since those first results, several progresses have been made towards a better comprehension of the above convergences. In particular, concentration inequalities for the convergence of the spectral distribution are given in [BES17, Kar15, MM13] in the additive case, leading to the so-called local laws of the spectral distribution up to an optimal scale (see also [EKN20] for concentration inequalities for arbitrary polynomials of matrices). Let us mention also the recent results of [BGH20], which establish a large deviation principle for the convergence of the spectral distribution in the additive case.

3.3. Local laws and approximate free probabilities. As mentioned in the previous section, as the dimension N goes to infinity the behavior of the spectral distribution of $A + UBU^*$ is close to the free additive convolution of μ_A and μ_B . One way to quantify such convergence is to show that the subordination phenomenon depicted in Theorem 3.2 holds approximately for the Stieltjes transforms of μ_A and μ_B at some point $z \in \mathbb{C}^+$. The closer to the real axis one can prove such behavior, the closer is the distribution μ_{A+UBU^*} to $\mu_A \boxplus \mu_B$. We thus introduce the following notion of approximate subordination.

Definition 3.4. *Let $A, B \in \mathcal{M}_N(\mathbb{C})$ be two (possibly random) self-adjoint matrices, for which we assume for simplicity that $\mathbb{E} \operatorname{tr}(A) = \mathbb{E} \operatorname{tr}(B) = 0$, and let $\eta, C_A, C_B, c > 0$. We say that the pair (A, B) satisfies an (c, C_A, C_B) -approximate additive subordination property (or simply AAS property) at η with bound if for all $z \in \mathbb{C}$ with $\Im z \geq \eta$, there exist $\omega_A(z), \omega_B(z) \in \mathbb{C}$ such that almost-surely*

- $\Im \omega_A(z), \Im \omega_B(z) \geq (1 - \frac{c}{N^2}) \Im z$
- $|\omega_A(z) - z| \leq \frac{\sigma_B}{\Im z} + \frac{c}{N^2} \Im z$ and $|\omega_B(z) - z| \leq \frac{\sigma_A}{\Im z} + \frac{c}{N^2} \Im z$.
- $\omega_A(z) + \omega_B(z) = z - \frac{1}{\mathbb{E}_U m_{A+B}(z)}$,
- $|\mathbb{E}_U m_{A+B}(z) - m_A(\omega_A(z))| \leq \frac{C_A}{|z|N^2}$,
- $|\mathbb{E}_U m_{A+B}(z) - m_B(\omega_B(z))| \leq \frac{C_B}{|z|N^2}$.

Note that $\omega_A(z)$ and $\omega_B(z)$ can be random if A or B are random. If (A, B) would satisfy a $(0, 0, 0)$ -average approximate subordination at η , by Theorem 3.2 one would have $\mathbb{E} \mu_{A+B} = \mu_A \boxplus \mu_B$. Hence, this notions quantifies how far the system is from an ideal system for which the spectral distribution of the sum of A and B is actually the free additive definition. The first rigorous result in this direction has been obtained by Kargin [Kar15], who proved that for U Haar unitary, the pair (A, UBU^*) satisfies a (c, C_A, C_B) -AAS property with $C_A, C_B \asymp C' \eta^{-6}$ and $c \asymp c' \eta^{-7}$ and with C', c' depending on the operator norm of A and B . It has been a challenging task to improve this bound, and the best (and optimal) result has been obtained by Bao, Erdos and Schnelli [BES17] who proved that (A, UBU^*) satisfies a (c, C_A, C_B) -AAS property with $C_A, C_B \asymp C' \eta^{-(2+\epsilon)}$ and $c \asymp c' \eta^{-(2+\epsilon)}$ with ϵ as small as wanted, and with C', c' depending in a non-trivial way on the analytic properties of μ_A and μ_B . In our case, improving the negative powers of η in the constants c, C_A, C_B is not so important, since we will use this property away from the real axis. However, since the goal is to recover the unknown spectral distribution μ_A , one needs to get constants depending on few properties of A and B . From those perspectives, the optimal results from [BES17] are not suitable for our framework. Actually, improving the method of Kargin with matrix Hölder inequalities allows us to get constants only depending on the six first moments of A and B . This results is summarized in the following proposition.

Proposition 3.5. *Let A and B be two self-adjoint matrices, U a random Haar unitary element and $\eta = \kappa \sigma_1$ with $\kappa > 0$. There exist constant c, C_A, C_B only depending on η and the first*

six moments of A and B such that for $N > \sqrt{3c}$, the pair (UAU^*, B) satisfies a (c, C_A, C_B) -approximate additive subordination property.

The proof of this proposition and explicit descriptions of c, C_A, C_B are postponed to Section 6. See also directly Appendix C for a direct expression of the constants.

3.4. Proof of the main theorem. The proof of Theorem 2.6 is a straightforward deduction of Proposition 3.5 and the results of the next sections.

Proof of Theorem 2.6. Let A be a self-adjoint noise matrix satisfying Condition 2.1 and B a self-adjoint matrix. Then, by Proposition 3.5, for each realization of A the pair (UAU^*, B) satisfies the AAS-property with $\widetilde{C}_A, \tilde{c}$ respectively given in (47) (46) and \widetilde{C}_B obtained from \widetilde{C}_A by switching the role of A and B . Since these three constants are increasing in the moments of A , by Condition 2.1, (UAU^*, B) satisfies the AAS-property with constants C_A, C_B, c obtained from $\widetilde{C}_A, \widetilde{C}_B, \tilde{c}$ by replacing $|a_i|$ by $\left(1 + \frac{\kappa}{\sqrt{N}}\right)^i |\mu_1(i)|$ and σ_A by $\left(1 + \frac{\kappa}{\sqrt{N}}\right) \sigma_1$. Hence, by Proposition 4.5 and Proposition 4.6, for $N \geq C_{threshold}$ with

$$(7) \quad C_{threshold} = \sqrt{3c}.$$

we have

$$\sqrt{\mathbb{E} \left(\left\| \widehat{\mathcal{C}}_B[\eta] - \mathcal{C}_B[\eta] \right\|_{L^2}^2 \right)} \leq \frac{C_{MSE}(\eta, N)}{N}$$

and

$$\mathbb{E} \left[\left\| \widehat{\mathcal{C}}_B[\eta]^{R_0} - \mathcal{C}_B[\eta] \right\|_{L^1} \right] \leq \frac{C_{L^1}(\eta, N)}{N},$$

with the constants given as in the statement of the theorem for C_A, C_B, c given above. \square

Using the L^1 -estimate of the latter theorem, we can now prove our main result.

Proof of Theorem 2.7. Suppose that $\bar{\mu}_B$ satisfies (2) for some $d > 0$ and that the p -th moment of μ_B is smaller than M_p for some $M_p > 0$. Set $\eta_0 = 2\sqrt{2}\sigma_1$, $\lambda_0 = \frac{C_{L^1}(\eta_0, N) \log N}{N}$, and

$$t_0 = \max \left(2\pi\eta_0, 256d\eta_0/5, 2 \exp(-2\pi/(pd)) \left[6M_p \sqrt{1 + 4\pi/d/\lambda_0} \right]^{1/p} \right).$$

Hence, by Proposition 5.2, for $F \in L^1(\mathbb{R})$, the solution $\hat{\mu}$ of (1) with $\lambda = \frac{C_\lambda \log N}{N}$ where

$$(8) \quad C_\lambda = \frac{e\sqrt{t_0}}{\sqrt{2\pi\eta}} \exp(2\pi/d)(N\lambda_0/\log N) = \frac{e\sqrt{t_0}}{2^{5/4}\sqrt{\pi\sigma_1}} \exp(2\pi/d)C_{L^1}(\eta_0, N)$$

satisfies

$$W_1(\mu_B, \hat{\mu}/\hat{\mu}(\mathbb{R})) \leq a(t_0^{-1/2})t_0^{3/2} \exp(2\pi/d)(\epsilon + \lambda_0) + bd\eta_0^{3/4}t_0^{1/4} \exp(\pi/d)\sqrt{\epsilon + \lambda_0},$$

where $\epsilon = \|F - \mathcal{C}_B[\eta_0]\|_{L^1}$. Hence, if F is a random function, taking the expectation and using the concavity of the square root yield that

$$(9) \quad \mathbb{E}W_1(\mu_B, \hat{\mu}/\hat{\mu}(\mathbb{R})) \leq a(t_0^{-1/2})t_0^{3/2} \exp(2\pi/d)(\mathbb{E}\epsilon + \lambda_0) + bd\eta_0^{3/4}t_0^{1/4} \exp(\pi/d)\sqrt{\mathbb{E}\epsilon + \lambda_0},$$

where ϵ is the random error term $\|F - \mathcal{C}_B[\eta_0]\|_{L^1}$. Applying (9) to $F = \widehat{\mathcal{C}}_B^{R_0}$ for R_0 given in (23) for η_0 gives then by Theorem 2.6 for $N \geq C_{threshold}[\eta_0]$

$$\begin{aligned} \mathbb{E}W_1(\mu_B, \hat{\mu}/\hat{\mu}(\mathbb{R})) &\leq a(t_0^{-1/2})t_0^{3/2} \exp(2\pi/d) \left(\frac{C_{L^1}(\eta_0, N) \log N}{N} + \lambda_0 \right) \\ &\quad + bd\eta_0^{3/4}t_0^{1/4} \exp(\pi/d) \sqrt{\frac{C_{L^1}(\eta_0, N) \log N}{N} + \lambda_0}. \end{aligned}$$

By the choice of λ_0 and by setting $\widehat{\mu}_B = \widehat{\mu}/\widehat{\mu}(\mathbb{R})$, we finally have

$$\begin{aligned} \mathbb{E}W_1(\mu_B, \widehat{\mu}_B) &\leq 2a(t_0^{-1/2})t_0^{3/2} \exp(2\pi/d) \frac{C_{L^1}(\eta_0, N) \log N}{N} \\ &\quad + 2bd\eta_0^{3/4}t_0^{1/4} \exp(\pi/d) \sqrt{\frac{C_{L^1}(\eta_0, N) \log N}{N}}. \end{aligned}$$

Remark that

$$\begin{aligned} &t_0(\log N/N)^{1/p} \\ &= \max \left(\max(2\pi, 256d/5)\eta_0 \left(\frac{\log N}{N}\right)^{1/p}, 2 \exp(-2\pi/(pd)) \left[6M_p \sqrt{1 + 4\pi/d}/C_{L^1}(\eta_0, N)\right]^{1/p} \right) \\ &\leq \max \left(\max(2\pi, 256d/5)\eta_0 \left(\frac{\log N}{N}\right)^{1/p}, 2 \exp(-2\pi/(pd)) \left[6M_p \sqrt{1 + 4\pi/d}/C_{L^1}(\infty, \eta_0)\right]^{1/p} \right) \\ &:= C_t(N), \end{aligned}$$

where $C_{L^1}(\infty, \eta_0) = \lim_{n \rightarrow \infty} C_{L^1}(N, \eta_0) = \frac{2K}{\pi}$ with K given in (22) for $\eta = \eta_0$. Hence, $C_t(N)$ is decreasing in N and

$$\begin{aligned} \mathbb{E}W_1(\mu_B, \widehat{\mu}_B) &\leq 2a(t_0^{-1/2})C_t(N)^{3/2} \exp(2\pi/d) C_{L^1}(N, \eta_0) \left(\frac{\log N}{N}\right)^{1-3/(2p)} \\ &\quad + 2bd\eta_0^{3/4} C_t^{1/4} \exp(\pi/d) \sqrt{C_{L^1}(N, \eta_0)} \left(\frac{\log N}{N}\right)^{1/2-1/(4p)} \\ &\leq C_{\text{sparse}}(N) \left(\frac{\log N}{N}\right)^{1/2-1/(4p)} \end{aligned}$$

with

$$(10) \quad C_{\text{sparse}}(N) = 2bd\eta_0^{3/4} C_t(N)^{1/4} \exp(\pi/d) \sqrt{C_{L^1}(N, \eta_0)} + 2a(t_0^{-1/2})C_t(N)^{3/2} \exp(2\pi/d) C_{L^1}(N, \eta_0) \left(\frac{\log N}{N}\right)^{1/2-5/(4p)},$$

where b is the numerical constant given in (37) and $a(t)$ is defined in (36) for η_0 . For $p \geq 3$, we indeed have that $C_{\text{sparse}}(N)$ is decreasing in N . \square

4. STABILITY RESULTS FOR THE DECONVOLUTION

In this section, we show that the AAS-property introduced in the Section 3.3 yields an estimation of the error of the estimator $\widehat{C}_B[\eta]$. Throughout this section, $\eta > 2\sqrt{2}\sigma_1$ is fixed and we assume that $A, B \in \mathcal{M}_N(\mathbb{C})$ satisfy the AAS-property at $3\eta/4$ with constant (c, C_A, C_B) and that μ_A satisfies Condition 2.1, and we set $H = A + B$. We need to take into account the error term from the fluctuations of m_H around their average and fluctuations from μ_A around μ_1 (recall the definition of μ_1 from Condition 2.1).

Stability results are obtained using the coercive property of the reciprocal Cauchy transform, which is summarized in the next lemma.

Lemma 4.1. *Let μ be a probability measure with variance σ^2 . For all $z, z' \in \mathbb{C}^+$,*

$$F_\mu(z) - F_\mu(z') = (z - z')(1 + \tau_\mu(z, z')),$$

with $|\tau_\mu(z, z')| \leq \frac{\sigma^2}{\Im z \Im z'}$.

Proof. By (3),

$$F_\mu(z) = z - \mu(1) + \sigma^2 m_\rho(z),$$

with ρ a probability measure on \mathbb{R} . Then, for $z, z' \in \mathbb{C}_\sigma$,

$$F_\mu(z) - F_\mu(z') = z - z' + \sigma^2(m_\rho(z) - m_\rho(z')).$$

Moreover,

$$m_\rho(z) - m_\rho(z') = \int_{\mathbb{R}} \frac{1}{t-z} d\rho(t) - \int_{\mathbb{R}} \frac{1}{t-z'} d\rho(t) = (z-z') \int_{\mathbb{R}} \frac{1}{(t-z)(t-z')} d\rho(t),$$

which implies the first statement of the lemma. The second statement is given by the inequality $\left| \int_{\mathbb{R}} \frac{1}{(t-z)(t-z')} d\rho(t) \right| \leq \frac{1}{\Im z \Im z'}$. \square

Following a similar pattern as for previous notations, we simply write τ_X instead of τ_{μ_X} for X self-adjoint in $\mathcal{M}_N(\mathbb{C})$.

4.1. Pointwise estimate. Fix $z \in \mathbb{C}_{2\sqrt{2}\sigma_1}$ and let $(\omega_1, \omega_3) \in \mathbb{C}^+ \times \mathbb{C}^+$ be the solution of the system

$$(11) \quad \begin{cases} \omega_1 + z = \omega_3 + F_H(\omega_3) \\ \omega_1 + z = \omega_3 + F_{\mu_1}(\omega_1) \end{cases},$$

which, by Theorem 2.3, exists and satisfies

$$\Im \omega_3 \geq \frac{3\Im(z)}{4}, \quad \Im \omega_1 \geq \frac{\eta}{2},$$

with $\eta = \Im z$. By the AAS-property on (A, B) for $3\eta/4$ and the fact that $\Im \omega_3 \geq 3\eta/4$, there exist $\omega_A(z), \omega_B(z) \in \mathbb{C}$ satisfying the inequalities of Definition 3.4 at $\omega_3(z)$. Introduce moreover the (random) error terms

$$\delta_H(z) = m_H(\omega_3(z)) - \mathbb{E}m_H(\omega_3(z)), \quad \delta_A(z) = m_A(\omega_A \circ \omega_3(z)) - m_{\mu_1}(\omega_A \circ \omega_3(z)).$$

The dependence of the latter functions in z will often be dropped in the sequel.

Lemma 4.2. For $N > \sqrt{3c}$, $\omega_A, \omega_B \in \mathbb{C}_{\eta/2}$ and

$$(m_B(z) - m_H(\omega_3)) = \frac{Lm_B(z)}{m_{\mu_1}(\omega_A)} (m_A(\omega_A) - \mathbb{E}m_H(\omega_3)) + m_B(\omega_B) - \mathbb{E}m_H(\omega_3) \\ - \frac{Lm_B(z)}{m_{\mu_1}(\omega_A)} \delta_A + \left(\frac{m_B(z)}{m_H(\omega_3)} L\tau_{\mu_1}(\omega_1, \omega_A) - 1 \right) \delta_H,$$

with

$$\mathcal{L} = \left(1 + \frac{m_B(\omega_B) - \mathbb{E}m_H(\omega_3)}{\mathbb{E}m_H(\omega_3)} \right) \frac{1 + \tau_B(\omega_B, z)}{1 + \tau_{\mu_1}(\omega_1, \omega_A)}.$$

Proof. By the AAS-property of (A, B) at $3\eta/4$ and the fact that $\Im \omega_3 \geq 3\eta/4$, we have for $N > \sqrt{3c}$

$$\Im \omega_A \geq \left(1 - \frac{c}{3c} \right) \Im \omega_3 > \eta/2,$$

and the same holds for ω_B . Then, note that

$$m_B(z) - m_H(\omega_3) = m_B(z) - m_B(\omega_B) + m_B(\omega_B) - \mathbb{E}m_H(\omega_3) + \mathbb{E}m_H(\omega_3) - m_H(\omega_3).$$

First,

$$(12) \quad m_B(z) - m_B(\omega_B) = -\frac{1}{F_B(z)} + \frac{1}{F_B(\omega_B)} = (F_B(z) - F_B(\omega_B)) m_B(z) m_B(\omega_B) \\ = (z - \omega_B)(1 + \tau_B(\omega_B, z)) m_B(z) m_B(\omega_B),$$

where we used Lemma 4.1 in the last inequality. Then, using the relation satisfied by ω_B and z yields

$$\omega_B - z = \omega_3 + F_{\bar{H}}(\omega_3) - \omega_A - \omega_3 - F_H(\omega_3) + \omega_1 \\ = \omega_1 - \omega_A + F_{\bar{H}}(\omega_3) - F_H(\omega_3),$$

where $F_{\bar{H}} = \frac{-1}{\mathbb{E}m_H}$. Then, by Lemma 4.1 and the relation $F_{\mu_1}(\omega_1) = F_H(\omega_3)$, with $\tau_1 = \tau_{\mu_1}(\omega_1, \omega_A)$,

$$\begin{aligned} \omega_B - z &= \frac{F_{\mu_1}(\omega_1) - F_{\mu_1}(\omega_A)}{1 + \tau_{\mu_1}(\omega_1, \omega_A)} + F_{\bar{H}}(\omega_3) - F_H(\omega_3) \\ &= \frac{F_H(\omega_3) - F_{\bar{H}}(\omega_3) + F_{\bar{H}}(\omega_3) - F_{\mu_1}(\omega_A)}{1 + \tau_1} + F_{\bar{H}}(\omega_3) - F_H(\omega_3) \\ &= (F_{\bar{H}}(\omega_3) - F_H(\omega_3)) \frac{\tau_1}{1 + \tau_1} + \frac{F_{\bar{H}}(\omega_3) - F_{\mu_1}(\omega_A)}{1 + \tau_1} \\ &= \frac{F_{\bar{H}}(\omega_3)F_H(\omega_3)\tau_1}{1 + \tau_1} (\mathbb{E}m_H(\omega_3) - m_H(\omega_3)) + \frac{F_{\mu_1}(\omega_A)F_{\bar{H}}(\omega_3)}{1 + \tau_1} (\mathbb{E}m_H(\omega_3) - m_{\mu_1}(\omega_A)). \end{aligned}$$

Write temporarily $\epsilon_B = \frac{m_B(\omega_B) - \mathbb{E}m_H(\omega_3)}{\mathbb{E}m_H(\omega_3)}$, $\epsilon_A = \frac{m_{\mu_1}(\omega_A) - \mathbb{E}m_H(\omega_3)}{m_{\mu_1}(\omega_A)}$. Hence, putting the latter relation in (12) yields

$$m_B(z) - m_B(\omega_B) = m_B(z) \left(L\tau_1 \frac{\delta_H(\omega_3)}{m_H(\omega_3)} + L\epsilon_A \right),$$

with

$$(13) \quad \mathcal{L} = \frac{m_B(\omega_B)}{\mathbb{E}m_H(\omega_3)} \frac{1 + \tau_2}{1 + \tau_1} = (1 + \epsilon_B) \frac{1 + \tau_2}{1 + \tau_1},$$

where $\tau_2 = \tau_B(\omega_B, z)$. Hence, using the first relation of the proof gives then

$$\begin{aligned} (m_B(z) - m_H(\omega_3)) &= \frac{\mathcal{L}m_B(z)}{m_{\mu_1}(\omega_A)} (m_{\mu_A}(\omega_A) - \mathbb{E}m_H(\omega_3)) + m_B(\omega_B) - \mathbb{E}m_H(\omega_3) \\ &\quad - \frac{\mathcal{L}m_B(z)}{m_{\mu_1}(\omega_A)} \delta_A + \left(\frac{m_B(z)}{m_H(\omega_3)} \mathcal{L}\tau_1 - 1 \right) \delta_H. \end{aligned}$$

□

From the latter lemma we express the distance between $m_B(z)$ and $m_{\mu_H}(\omega_3)$ in terms of the fluctuations δ_H and δ_A .

Proposition 4.3. For $N > \sqrt{3c}$,

$$|m_B(z) - m_H(\omega_3)| \leq \frac{C_1(\eta)}{|z|N^2} + \frac{C_2(\eta)}{|z|} |\omega_A \delta_A| + \frac{C_3(\eta)}{|z|} |\omega_3 \delta_H|,$$

where $C_1(\eta), C_2(\eta)$ and $C_3(\eta)$ are respectively given in (14), (15) and (16).

Proof. By the AAS-property of (A, B) at $3\eta/4$ and the fact that $\Im\omega_3 \geq 3\eta/4$,

$$|\mathbb{E}m_H(\omega_3) - m_{\mu_1}(\omega_A(z))| \leq \frac{C_A}{|\omega_3|N^2},$$

and

$$|\mathbb{E}m_H(\omega_3) - m_B(\omega_B(z))| \leq \frac{C_B}{|\omega_3|N^2},$$

with $C_A, C_B > 0$ independent of $\omega_3 \in \mathbb{C}_{3\eta/4}$. Hence, in particular, by the definition of L from (13), we get

$$|\mathcal{L}| \leq \left(1 + \frac{C_B}{\mathbb{E}m_H(\omega_3)|\omega_3|(\Im\omega_3)^2 N^2} \right) \left| \frac{1 + \tau_B(\omega_B, z)}{1 + \tau_{\mu_1}(\omega_1, \omega_A)} \right|.$$

By Lemma 4.2, we have $\Im\omega_A, \Im\omega_B \geq \eta/2$, and by Theorem 2.3 $\Im\omega_1 \geq \eta/2$, thus

$$|\tau_{\mu_1}(\omega_1, \omega_A)| = \left| \int_{\mathbb{R}} \frac{\sigma_1^2 d\rho(t)}{(\omega_1 - t)(\omega_A - t)} \right| \leq \frac{4\sigma_1^2}{\eta^2}, \quad |\tau_B(\omega_B, z)| = \left| \int_{\mathbb{R}} \frac{\sigma_B^2 d\rho'(t)}{(\omega_B - t)(z - t)} \right| \leq \frac{2\sigma_B^2}{\eta^2}.$$

Hence, since by (43) we have $\left| \frac{1}{\omega_3 \mathbb{E}m_H(\omega_3)} \right| \leq 1 + \frac{a_2+b_2}{\Im \omega_3^2}$,

$$|\mathcal{L}| \leq \left(1 + \frac{C_B \left(1 + \frac{a_2+b_2}{(\Im \omega_3)^2} \right)}{N^2} \right) \frac{1 + 2\sigma_B^2/\eta^2}{1 - 4\sigma_1^2/\eta^2} \leq \left(1 + \frac{C_B \left(1 + \frac{16(a_2+b_2)}{9\eta^2} \right)}{N^2} \right) \frac{1 + 2\sigma_B^2/\eta^2}{1 - 4\sigma_1^2/\eta^2}.$$

Therefore, by Lemma 4.2

$$|m_B(z) - m_H(\omega_3)| \leq \frac{C_1(\eta)}{|z|N^2} + \frac{C_2(\eta)}{|z|} |\omega_A \delta_A| + \frac{C_3(\eta)}{|z|} |\omega_3 \delta_H|,$$

with, recalling that $\Im \omega_A \geq \eta/2$, using that $|\omega_A - \omega_3| \leq \frac{\sigma_B^2}{\Im \omega_3} + \Im \omega_3/3$ by the AAS-property of (A, B) and the hypothesis $N^2 > \sqrt{3}c$, and $|z - \omega_3| \leq 1 + \frac{2\sigma_1^2}{\eta}$ by Theorem 2.3,

$$\begin{aligned} C_1(\eta) &= |\mathcal{L}| \cdot \frac{|F_{\mu_1}(\omega_A)|}{|\omega_A|} \cdot \frac{|\omega_A|}{|\omega_3|} \cdot |\omega_3| \cdot |m_A(\omega_A) - \mathbb{E}m_H(\omega_3)| \cdot |m_B(z)z| \\ &\quad + \left| \frac{z}{\omega_3} \right| \cdot |\omega_3| \cdot |m_B(\omega_B) - \mathbb{E}m_H(\omega_3)| \\ &\leq \left(1 + \frac{2\sigma_1^2}{\eta^2} \right) C_B + \left(1 + \frac{C_B \left(1 + \frac{16(a_2+b_2)}{9\eta^2} \right)}{N^2} \right) \cdot \frac{1 + 2\sigma_B^2/\eta^2}{1 - 4\sigma_1^2/\eta^2} \cdot \left(1 + \frac{4\sigma_1^2}{\eta^2} \right) \\ (14) \quad &\quad \cdot \left(\frac{4}{3} + \frac{16\sigma_B^2}{9\eta^2} \right) C_A \left(1 + \frac{\sigma_B}{\eta} \right), \end{aligned}$$

$$\begin{aligned} C_2(\eta) &= |\mathcal{L}| \frac{|F_{\mu_1}(\omega_A)|}{|\omega_A|} |zm_B(z)| \\ (15) \quad &\leq \left(1 + \frac{C_B \left(1 + \frac{16(a_2+b_2)}{9\eta^2} \right)}{N^2} \right) \cdot \frac{1 + 2\sigma_B^2/\eta^2}{1 - 4\sigma_1^2/\eta^2} \cdot \left(1 + \frac{4\sigma_1^2}{\eta^2} \right) \cdot \left(1 + \frac{\sigma_B}{\eta} \right), \end{aligned}$$

and

$$C_3(\eta) = \left| \tau_{\mu_1}(\omega_1, \omega_A) \frac{zm_B(z)}{\omega_3 m_H(\omega_3)} \mathcal{L} - \frac{z}{\omega_3} \right|.$$

Using $z - \omega_3 = h_{\mu_1}(\omega_1)$ to expand the right hand side of the latter equation gives then

$$\tau_{\mu_1}(\omega_1, \omega_A) \frac{zm_B(z)}{\omega_3 m_H(\omega_3)} \mathcal{L} - \frac{z}{\omega_3} = -1 - \frac{h_{\mu_1}(\omega_1)}{\omega_3} + \mathcal{L} \tau_{\mu_1}(\omega_1, \omega_A) (1 + \tilde{m}_B(z)) \left(1 + \frac{\sigma_H^2}{\omega_3} m_\rho(\omega_3) \right),$$

and finally

$$\begin{aligned} C_3(\eta) &\leq 1 + \frac{8\sigma_1^2}{3\eta^2} \\ (16) \quad &+ \left(1 + \frac{C_B \left(1 + \frac{16(a_2+b_2)}{9\eta^2} \right)}{N^2} \right) \cdot \frac{1 + 2\sigma_B^2/\eta^2}{1 - 4\sigma_1^2/\eta^2} \cdot \frac{4\sigma_1^2}{\eta^2} \cdot \left(1 + \frac{\sigma_B}{\eta} \right) \cdot \left(1 + \frac{16\sigma_H^2}{9\eta^2} \right). \end{aligned}$$

□

4.2. L^2 -estimates. Building on the previous stability results, we deduce an estimate of the L^2 -distance between $\widehat{\mathcal{C}_B[\eta]}$ and $\mathcal{C}_B[\eta]$. In this section, we fix a parameter $\eta > 0$ which denotes the imaginary part of the line on which the first part of the deconvolution process is achieved (see Section 2.3 for an explanation of the method). Then, for each $t \in \mathbb{R}$, the deconvolution process associates to each sample of H an estimator $\widehat{\mathcal{C}_B[\eta]}(t)$ of $\mathcal{C}_B[\eta](t)$ given by $\widehat{\mathcal{C}_B[\eta]}(t) = \frac{1}{\pi} m_H(\omega_3(t + i\eta))$, with ω_3 the subordination function respectively given by Theorem 2.3. Let us first bound the fluctuation term coming from the noise in Proposition 4.3.

Lemma 4.4. *Suppose that $\eta \geq 2\sqrt{2}\sigma_1$ and assume that for (A, B) has the AAS-property at $3\eta/4$ with constants $C_A, C_B, c > 0$. Then for $N \geq \sqrt{3c}$*

$$\sqrt{\mathbb{E}|\omega_A \delta_A|^2} \leq \frac{2C_{noise} \left(1 + \frac{2(1+c/N)\sqrt{\mu_1(2)}}{\eta}\right)}{\eta N},$$

where we recall that $\omega_A = \omega_A(\omega_3(z))$ and $\delta_A = m_A(\omega_A \circ \omega_3(z)) - m_{\mu_1}(\omega_A \circ \omega_3(z))$.

Proof. Note first that the function $f_z : t \rightarrow \frac{z}{t-z}$ is C^1 for $z \in \mathbb{C}^+$, and, viewed as a function on $\mathcal{H}_N(\mathbb{C})$, we have for $A \in \mathcal{H}_N(\mathbb{C})$

$$\nabla f_z(A)(X) = \text{tr} \left(z \frac{1}{A-z} X \frac{1}{A-z} \right) = \text{tr} \left(\frac{z}{(A-z)^2} X \right).$$

Hence, $\|\nabla f_z(A)\|_2 = \frac{1}{N} \left\| \frac{z}{(A-z)^2} \right\|_2 \leq \frac{1}{N} \left(\left\| \frac{1}{A-z} \right\|_2 + \left\| \frac{A}{(A-z)^2} \right\|_2 \right)$ and thus, with the second hypothesis of Condition 2.1, $\|\nabla f_z(A)\|_2^2 \leq \frac{1}{N} \left(\frac{1}{\eta} + \frac{(1+c/N)\sqrt{\mu_1(2)}}{\eta^2} \right)^2$, where $\eta = \Im z$. This implies by the third hypothesis of Condition 2.1

$$\sqrt{\mathbb{E}|\omega_A \delta_A|^2} \leq \frac{C_{noise} \left(1 + \frac{(1+c/N)\sqrt{\mu_1(2)}}{\Im \omega_A}\right)}{\Im \omega_A N}.$$

By Lemma 4.2 we have $\Im \omega_A \geq \eta/2$ for $N > \sqrt{3c}$, which gives the result. \square

Using the latter inequality, we deduce the following estimate in the additive case.

Proposition 4.5. *Suppose that $\eta \geq 2\sqrt{2}\sigma_1$ and assume that for (A, B) has the AAS-property at $3\eta/4$ with constants $C_A, C_B, c > 0$. Then for $N \geq \sqrt{3c}$,*

$$\mathbb{E} \left(\left\| \widehat{\mathcal{C}_B[\eta]} - \mathcal{C}_B[\eta] \right\|_{L^2}^2 \right) \leq \frac{C_{MSE}(N, \eta)^2}{N^2}$$

with

$$(17) \quad C_{MSE}(\eta, N) = \frac{\sqrt{\pi}}{\sqrt{\eta}} \left(\frac{2C_2(\eta)C_{noise} [1 + 2(1+c/N)(\sigma_1/\eta)]}{\eta} + \frac{8\sqrt{2}C_3(\eta)}{3\eta} \sqrt{\sigma_A^2 + 4^2 \frac{\sigma_A^2 \sigma_B^2 + a_4}{3^2 \eta^2}} + \frac{C_1(\eta)}{N} \right),$$

where the functions C_1, C_2, C_3 are respectively given in (14), (15) and (16).

Proof. By the AAS-property of (A, B) and Proposition 4.3, for $z = t + i\eta$ with $\eta > 2\sqrt{2}\sigma_1$ and $N > \sqrt{3c}$,

$$|m_B(z) - m_H(\omega_3(z))| \leq \frac{C_1(\eta)}{|z|N^2} + \frac{C_2(\eta)}{|z|} |\omega_A \delta_A| + \frac{C_3(\eta)}{|z|} |\omega_3(z) \delta_H|,$$

with $C_1(\eta), C_2(\eta), C_3(\eta)$ given in Proposition 4.3 and where $\omega_A := \omega_A(\omega_3(z))$. Hence,

$$(18) \quad \mathbb{E}(|m_B(z) - m_H(\omega_3(z))|^2) \leq \frac{1}{|z|^2} \left[\frac{C_1(\eta)}{N^2} + C_2(\eta) \sqrt{\mathbb{E}(|\omega_A \delta_A|^2)} + C_3(\eta) \sqrt{\mathbb{E}(|\omega_3 \delta_H|^2)} \right]^2.$$

First, by Lemma 4.4, we have

$$\sqrt{\mathbb{E}(|\omega_A \delta_A(\omega_A)|^2)} \leq \frac{2C_A \left(1 + \frac{2(1+c/N)\sqrt{\mu_1(2)}}{\eta}\right)}{\eta N}.$$

Then, by Lemma B.4, the hypotheses $\text{tr}(A) = 0$ and $\text{tr}(B) = 0$ and the fact that $\Im\omega_3(z) \geq 3\eta/4$ by Theorem 2.3,

$$(19) \quad \mathbb{E}(|\omega_3\delta_H|^2) \leq \frac{8}{N^2(\Im\omega_3)^2} \left(\sigma_A^2 + \frac{\sigma_A^2\sigma_B^2 + a_4}{(\Im\omega_3)^2} \right) \leq \frac{2^7}{3^2N^2\eta^2} \left(\sigma_A^2 + 4^2 \frac{\sigma_A^2\sigma_B^2 + a_4}{3^2\eta^2} \right).$$

Hence,

$$(20) \quad \begin{aligned} & \left[\frac{C_1(\eta)}{N^2} + C_2(\eta)\sqrt{\mathbb{E}(|\omega_A\delta_A|^2)} + C_3(\eta)\sqrt{\mathbb{E}(|\omega_3\delta_H|^2)} \right]^2 \\ & \leq \left(\frac{C_1(\eta)}{N^2} + \frac{2C_2(\eta)C_{noise} \left(1 + \frac{2(1+c/N)\sqrt{\mu_1(2)}}{\eta} \right)}{\eta N} + \frac{8\sqrt{2}C_3(\eta)}{3\eta N} \sqrt{\sigma_A^2 + 4^2 \frac{\sigma_A^2\sigma_B^2 + a_4}{3^2\eta^2}} \right)^2 \\ & \leq \frac{1}{N^2} \left(\frac{2C_2(\eta)C_{noise} \left(1 + \frac{2(1+c/N)\sqrt{\mu_1(2)}}{\eta} \right)}{\eta} + \frac{8\sqrt{2}C_3(\eta)}{3\eta} \sqrt{\sigma_A^2 + 4^2 \frac{\sigma_A^2\sigma_B^2 + a_4}{3^2\eta^2}} + \frac{C_1(\eta)}{N} \right)^2. \end{aligned}$$

Since, $\int_{\mathbb{R}} \frac{dt}{|t+i\eta|^2} = \frac{\pi}{\eta}$, the latter inequality together with (18) and the Stieltjes inversion formula yields

$$\mathbb{E} \left(\left\| \widehat{\mathcal{C}}_B[\eta] - \mathcal{C}_B[\eta] \right\|_{L^2}^2 \right) \leq \frac{C_{MSE}(N, \eta)^2}{N^2} \leq \frac{C_{MSE}(\eta, N)^2}{N}$$

with,

$$C_{MSE}(\eta, N) = \frac{\sqrt{\pi}}{\sqrt{\eta}} \left(\frac{2C_2(\eta)C_{noise} [1 + 2(1+c/N)(\sigma_1/\eta)]}{\eta} + \frac{8\sqrt{2}C_3(\eta)}{3\eta} \sqrt{\sigma_A^2 + 4^2 \frac{\sigma_A^2\sigma_B^2 + a_4}{3^2\eta^2}} + \frac{C_1(\eta)}{N} \right).$$

□

4.3. L^1 -estimates. In view of applying Proposition 2.5, let us prove an L^1 -estimates for on grid evaluation.

Proposition 4.6. *Suppose that $\eta \geq 2\sqrt{2}\sigma_1$ and assume that for (A, B) has the AAS-property at $3\eta/4$ with constants $C_A, C_B, c > 0$. Then, for $N > 3\sqrt{c}$, we have*

$$\mathbb{E} \left\| \widehat{\mathcal{C}}_B[\eta]^{R_0} - \mathcal{C}_B \right\|_{L^1} \leq \frac{C_{L^1}(\eta, N) \log N}{N}$$

with

$$(21) \quad C_{L^1}(\eta, N) = \frac{2K}{\pi} \left[1 + \frac{1}{\log N} \left(\log \left(\frac{\eta + \sigma_B^2/\eta}{K} \right) + 1 \right) \right],$$

where K is given in (22).

Proof. Let $R > 0$ to be defined later, we are then interested in the error term

$$\Theta_R = \int_{-R}^R \left| \widehat{\mathcal{C}}_B(t) - \mathcal{C}_B(t) \right| dt + \int_{|t|>R} \mathcal{C}_B(t) dt.$$

and remark that for $\eta \geq 2\sqrt{2}\sigma_1$,

$$\begin{aligned} \Theta_R & \leq \int_{-R}^R \frac{1}{\pi} |m_H(\omega_3(t+i\eta)) - m_B(t+i\eta)| + \int_{|t|>R} \mathcal{C}_B(t) dt \\ & := \Theta_R^{(1)} + \Theta_R^{(2)}. \end{aligned}$$

By the AAS-property of (A, B) , Proposition 4.3 for $N > \sqrt{3c}$, Cauchy-Schwartz inequality and Lemma 4.4 and (19) we get

$$\mathbb{E} |m_H(\omega_3(t + i\eta) - m_B(t + i\eta))| \leq \frac{K_1}{N|t + i\eta|} \leq \frac{K}{N} \min\left(\frac{1}{t}, \eta^{-1}\right)$$

with

(22)

$$K[\eta] = \frac{C_1(\eta)}{N} + \left(\frac{2C_2(\eta)C_{noise} \left(1 + \frac{2(1+c/N)\sqrt{\mu_1(2)}}{\eta}\right)}{\eta} + \frac{2^{7/2}C_3(\eta)}{3\eta} \sqrt{\sigma_A^2 + 4^2 \frac{\sigma_A^2 \sigma_B^2 + a_4}{3^2 \eta^2}} \right),$$

where the functions C_1, C_2, C_3 are respectively given in (14), (15) and (16). Hence, integrating on $|t| \leq R$ yields a the first bound

$$\mathbb{E}\Theta_R^{(1)} \leq \frac{2K}{\pi N} \int_0^R \min(\eta^{-1}, t^{-1}) \leq \frac{2K}{\pi N} (1 + \log R),$$

which is valid for $R > \eta$. On the other hand, recall that

$$\mathcal{C}_B(t) = -\frac{1}{\pi} \Im m_B(t + i\eta),$$

For $z \in \mathbb{C}^+$ and μ a probability measure with second moment, we have

$$m_\mu(z) = -\frac{1}{z} + \frac{1}{z^2} \left(-\mu(1) + \int_{\mathbb{R}} \frac{t^2}{t - z} d\mu(t) \right)$$

and thus

$$\Im m_\mu(z) \leq \Im(z^{-1}) + \frac{\mu(1) + \mu(2)/(\Im z)}{|z|^2} \leq \frac{1}{|z|^2} \left(\Im z + \mu(1) + \frac{\mu(2)}{\Im z} \right).$$

Hence, using that $\mu_B(1) = 0$ and $\mu_B(2) = \sigma_B^2$,

$$\Theta_R^{(2)} \leq \int_{|t| > R} \mathcal{C}_B(t) dt \leq \frac{2(\eta + \sigma_B^2/\eta)}{\pi} \int_{t \geq R} \frac{1}{t^2} dt \leq \frac{2(\eta + \sigma_B^2/\eta)}{\pi R}.$$

Therefore,

$$\mathbb{E}\Theta_R \leq \frac{2K}{\pi N} (1 + \log R) + \frac{2(\eta + \sigma_B^2/\eta)}{\pi R}.$$

Since for $A, B > 0$, the function $t \mapsto A \log(t) + \frac{B}{t}$ reaches its minimum value $A \log(B/A) + A$ at $t = B/A$, then for $A = \frac{2K}{\pi N}$, $B = \frac{2(\eta + \sigma_B^2/\eta)}{\pi}$ and

$$(23) \quad R_0 = \frac{B}{A} = \frac{N(\eta + \sigma_B^2/\eta)}{K},$$

we get

$$\mathbb{E} \left\| \widehat{\mathcal{C}_B[\eta]}^{R_0} - \mathcal{C}_B \right\|_{L^1} \leq \frac{2 \cdot K \log N}{\pi N} \left(1 + \left(\log \left(\frac{(\eta + \sigma_B^2/\eta)}{K} \right) + 1 \right) / \log N \right).$$

□

5. SUPERRESOLUTION TO THE CAUCHY DECONVOLUTION

We prove here Proposition 2.5, which is an adaptation of the method of [CFG14] for an ideal low-pass filter to the deconvolution by a Cauchy distribution. Originally, superresolution has been applied to the deconvolution problem on a circle of a signal convoluted with a sine kernel. Recently, there have been some generalization to the real line with other deconvolution problems [DP17], for example in the Gaussian setting. For the sake of brevity, we will simply reduce here our problem to a deconvolution of low-pass filter. In this section, suppose that $\eta > 0$, μ is a discrete distribution on the real line with p moment smaller than M_p , and that $\mathcal{C} : \mathbb{R} \rightarrow \mathbb{R}$ is such that

$$(24) \quad \|\mathcal{C}(t) - \mu * \text{Cauchy}[\eta](t)\|_{L^1} \leq \epsilon.$$

We can translate this information into the Fourier transform of μ . Set $F(t) = \exp(\eta|t|)\mathcal{F}[\mathcal{C}](t)$, and for $c > 0, M \in \mathbb{N}^*$ and $f : \mathbb{R} \rightarrow \mathbb{R}$, write $\|f\|_2^{M,c} = \sqrt{\sum_{k=-M}^M f(2\pi k/c)^2}$.

Lemma 5.1. *For $t_0 > 0$ such that $\mu|_{[-t_0/2, t_0/2]} \geq 1/2$, set $\tilde{\mu} = \frac{1}{\mu([-t_0/2, t_0/2])}\mu|_{[-t_0/2, t_0/2]}$. Then,*

$$\|F - \tilde{F}[\tilde{\mu}]\|_2^{t_0, M} \leq \frac{\sqrt{t_0}}{\sqrt{2\pi\eta}} \exp(2\pi\eta(M+1)/t_0)\epsilon + \sqrt{2M+1} \frac{3 \cdot 2^p M_p}{t_0^p},$$

and

$$W_1(\tilde{\mu}, \mu) \leq \frac{3 \cdot 2^{p-1} M_p}{t_0^{p-1}},$$

where W_1 denotes the 1-Wasserstein distance.

Proof. Let $c > 0$, and $M \in \mathbb{N}^*$. For $t \in \mathbb{R}$, (24) yields

$$|\mathcal{F}[\mathcal{C}](t) - \mathcal{F}[\mu * \text{Cauchy}[\eta]](t)| \leq \epsilon.$$

Since $\mathcal{F}[\mu * \text{Cauchy}[\eta]](t) = \exp(-\eta|t|)\mathcal{F}[\mu](t)$, we deduce that

$$(25) \quad |F(t) - \mathcal{F}[\mu](t)|^2 = |\exp(2\eta|t|)\mathcal{F}[\mathcal{C}](t) - \mathcal{F}[\mu](t)|^2 \leq \exp(2\eta|t|)\epsilon^2.$$

Hence, summing the latter inequality for $t = k/c, -M \leq k \leq M$ yields

$$\begin{aligned} \sum_{k=-M}^M |F(t) - \mathcal{F}[\mu](t)|^2 &\leq 2\epsilon^2 \sum_{k=0}^M \exp(4\pi\eta k/c) \leq 2\epsilon^2 \frac{\exp(4\pi(M+1)\eta/c) - 1}{\exp(4\pi\eta/c) - 1} \\ &\leq \frac{\epsilon^2 c}{2\pi\eta} \exp(4\pi(M+1)\eta/c), \end{aligned}$$

which gives

$$\|F - \mathcal{F}[\mu]\|_2^{c, M} \leq \frac{\sqrt{c}}{\sqrt{2\pi\eta}} \exp(2\pi\eta(M+1)/c)\epsilon.$$

Let us then reduce the problem to a finite interval. Note first that by the Markov inequality and the finiteness of the p -moment of μ , we have for $t > 0$

$$\mu(-\infty, t] \cup [t, +\infty) \leq \frac{M_p}{t^p}.$$

For $t_0 > 0$ and $\tilde{\mu} = \frac{1}{\mu([-t_0/2, t_0/2])}\mu|_{[-t_0/2, t_0/2]}$, we then have

$$|\mathcal{F}[\tilde{\mu}](t) - \mathcal{F}[\mu](t)| \leq \left(1 - \frac{1}{\mu([-t_0/2, t_0/2])}\right) + \mu(\mathbb{R} \setminus [-t_0/2, t_0/2]) \leq \frac{3 \cdot 2^p M_p}{t_0^p},$$

and thus for $M \geq 1$ we have

$$\|\mathcal{F}[\tilde{\mu}] - \mathcal{F}[\mu]\|_2^{t_0, M} \leq \sqrt{2M+1} \frac{3 \cdot 2^p M_p}{t_0^p}.$$

Finally,

$$(26) \quad \|F - \tilde{F}[\tilde{\mu}]\|_2^{t_0, M} \leq \frac{\sqrt{t_0}}{\sqrt{2\pi\eta}} \exp(2\pi\eta(M+1)/t_0)\epsilon + \sqrt{2M+1} \frac{3 \cdot 2^p M_p}{t_0^p}.$$

Finally, for $f : \mathbb{R} \rightarrow \mathbb{R}$ 1-Lipschitz with $f(0) = 0$, we have

$$\begin{aligned} \left\| \int_{\mathbb{R}} f d\mu - d\tilde{\mu} \right\| &\leq \left\| \int_{-t_0/2}^{t_0/2} f \left(1 - \frac{1}{\mu([-t_0/2, t_0/2])}\right) d\mu \right\| + \left\| \int_{\mathbb{R} \setminus [-t_0/2, t_0/2]} f d\mu \right\| \\ &\leq t_0/2 \frac{1 - \mu([-t_0/2, t_0/2])}{\mu([-t_0/2, t_0/2])} + \int_{\mathbb{R} \setminus [-t_0/2, t_0/2]} |t| d\mu, \end{aligned}$$

where we used that $|f(t)| \leq t$ on the last inequality. By the Markov inequality, we thus have

$$W_1(\tilde{\mu}, \mu) \leq t_0 \frac{2^p M_p}{t_0^p} + \frac{2^{p-1} M_p}{t_0^{p-1}} \leq \frac{3 \cdot 2^{p-1} M_p}{t_0^{p-1}}.$$

□

For $\lambda > 0$ and $F : \mathbb{R} \rightarrow \mathbb{R}$, let $\hat{\mu}$ the minimizer of the minimization problem

$$(27) \quad \arg \min_{\substack{\nu \in \mathcal{M}^+([-t_0/2, t_0/2]) \\ \|\nu\|_{TV} \leq 1}} \|\mathcal{F}[\nu] - F\|_2^{t_0, M} + 2\lambda \|\nu\|_{TV}.$$

It can be shown by considering the subgradient of the constrained problem at the minimizer that $\hat{\mu}$ is a purely atomic measure with a finite support. One can thus write $\hat{\mu} = \sum_{t \in \hat{T}} \hat{a}_t \delta_t$ for some subset \hat{T} of $[-t_0/2, t_0/2]$.

Proposition 5.2. *Suppose that μ is a purely atomic probability measure with a support $T \subset [-t_0/2, t_0/2]$ satisfying a minimum separation distance given by*

$$\inf_{t, t' \in T, t \neq t'} |t - t'| \geq \frac{5t_0}{2M},$$

with $M \geq 128$. Then, if $\hat{\mu}$ is a solution to (27) with F such that $\|F - \mathcal{F}[\mu]\|_2^{t_0, M} \leq \tau$, we have

$$W_1(\mu, \hat{\mu}/\hat{\mu}(\mathbb{R})) \leq 3(K_1 M/5 + t_0/K_2)(\tau + \lambda) + t_0/M \sqrt{2/K_3} \sqrt{\tau + \lambda}$$

for some numerical constants $K_1, K_2, K_3 > 0$.

The numerical constants K_1, K_2, K_3 given in the latter Proposition can be explicitly deduced from [CFG14, FG13]: namely, one can choose, $K_1 = 101.3$, $K_2 = 0,0157$ and $K_3 = 0,3353$.

Proof. Set $\mu = \sum_{t \in T} a_t \delta_t$, and introduce the probability measures $\mu' = \sum_{t \in T} a_t \delta_{t/t_0}$, $\hat{\mu}' = \sum_{t \in \hat{T}} \hat{a}_t \delta_{t/t_0}$. Note first that for $t \in \mathbb{R}$, $\mathcal{F}[\mu'](t) = \mathcal{F}[\mu](t * t_0)$ and $\mathcal{F}[\hat{\mu}'](t) = \mathcal{F}[\hat{\mu}](t * t_0)$, so that with $\tilde{F}(t) = F(t/t_0)$ we have by Lemma 5.1

$$\|\tilde{F} - \mathcal{F}[\hat{\mu}']\|_2^{1, M} \leq \tau.$$

By a rescaling of (27) by t_0 , we then have

$$(28) \quad \hat{\mu}' = \arg \min_{\substack{\nu \in \mathcal{M}^+([-1/2, 1/2]) \\ \|\nu\|_{TV} \leq 1}} \|\mathcal{F}[\nu] - \tilde{F}\|_2^{1, M} + 2\lambda \|\nu\|_{TV}.$$

Moreover, since $\hat{\mu}'$ is the minimizer of (28), we have

$$(29) \quad \|\hat{\mu}'\|_{TV} \leq 1 = \|\mu'\|_{TV}, \quad \|\mathcal{F}[\hat{\mu}'] - \tilde{F}\|_2^{1, M} \leq \|\mathcal{F}[\mu'] - \tilde{F}\|_2^{1, M} + \lambda \|\mu'\|_{TV} \leq \tau + 2\lambda.$$

Set $T' = T/t_0$ and $\hat{T}' = \hat{T}/t_0$. We then have

$$\inf_{t, t' \in T', t \neq t'} |t - t'| \geq \frac{5}{2M},$$

with $M \geq 128$. Then, applying [FG13][Lem 2.1, Thm 1.2, i)] to (28) (the minimization problem is a bit different, but the only required inequality are the ones in (29)), there exist numerical constants $K_1, K_2, K_3, c > 0$ such that for all $t \in T' \cap [-1/2, 1/2]$,

$$(30) \quad \left| a_t - \sum_{\hat{t} \in \hat{T}', |\hat{t} - t| \leq c/M} a_{\hat{t}} \right| \leq K_1(\tau + \lambda),$$

and

$$(31) \quad \sum_{\hat{t} \in \hat{T}'} |a_{\hat{t}}| \min(K_2, K_3 d(\hat{t}, T')^2 M^2) \leq 2(\tau + \lambda),$$

where $d(\hat{t}, T') = \inf_{t \in T'} |\hat{t} - t|$ and $0 < c^2 K_3 < K_2 < 1$. In the sequel, set $\delta = \tau + \lambda$.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz function with Lipschitz constant t_0 and such that $f(0) = 0$. Then, we have by (30) and (31)

$$\begin{aligned}
\left| \int_{\mathbb{R}} f d\mu' - \int_{\mathbb{R}} f d\hat{\mu}' \right| &= \left| \sum_{t \in T'} a_t f(t) - \sum_{\hat{t} \in \hat{T}'} \hat{a}_{\hat{t}} f(\hat{t}) \right| \\
&\leq \sum_{t \in T'} \left| \left(a_t - \sum_{\substack{\hat{t} \in \hat{T}' \\ |t-\hat{t}| < c/M}} \hat{a}_{\hat{t}} \right) f(t) \right| + \sum_{t \in T'} \sum_{\substack{\hat{t} \in \hat{T}' \\ |t-\hat{t}| < c/M}} |\hat{a}_{\hat{t}}| |f(t) - f(\hat{t})| \\
&\quad + \sum_{\substack{\hat{t} \in \hat{T}' \\ d(\hat{t}, T') > c/M}} |\hat{a}_{\hat{t}} f(\hat{t})| \\
&\leq K_1 \delta \sum_{t \in T'} |f(t)| + \sum_{t \in T'} \sum_{\substack{\hat{t} \in \hat{T}' \\ |t-\hat{t}| < c/M}} |\hat{a}_{\hat{t}}| \cdot t_0 \cdot d(\hat{t}, T') + 2/K_2 \|f\|_{\infty, [-1/2, 1/2]} \delta \\
&\leq \|f\|_{\infty, [-1/2, 1/2]} (K_1 \#T' + 2/K_2) \delta + t_0 \sum_{\substack{\hat{t} \in \hat{T}' \\ d(\hat{t}, T') < c/M}} |\hat{a}_{\hat{t}}| \cdot d(\hat{t}, T').
\end{aligned}$$

where we have used the t_0 -Lipschitz property of f in the second sum. Remark that since $\hat{\mu}'$ is the minimizer of (28), we have $\|\hat{\mu}'\|_{TV} \leq \|\mu'\|_{TV} = 1$. Hence, by Cauchy-Schwartz inequality on $L^2(\hat{\mu}')$, we have

$$\sum_{\substack{\hat{t} \in \hat{T}' \\ d(\hat{t}, T') < c/M}} |\hat{a}_{\hat{t}}| \cdot d(\hat{t}, T') \leq \sqrt{\sum_{\substack{\hat{t} \in \hat{T}' \\ d(\hat{t}, T') < c/M}} |\hat{a}_{\hat{t}}| \cdot d(\hat{t}, T')^2} \leq \sqrt{2/K_3} / M \sqrt{\delta},$$

where we have used (31) on the last inequality. Since f is t_0 -Lipschitz and vanishes at 0, $\|f\|_{\infty, [-1/2, 1/2]} \leq t_0/2$, and by the minimal separation property of T , we have $\#T \leq 2M/(5t_0)$. Hence,

$$(32) \quad \left| \int_{\mathbb{R}} f d\mu' - \int_{\mathbb{R}} f d\hat{\mu}' \right| \leq 2(K_1 M/5 + t_0/K_2) \delta + t_0/M \sqrt{2/K_3} \sqrt{\delta}.$$

Likewise, by (30) and (31),

$$(33) \quad |\hat{\mu}(\mathbb{R}) - \mu(\mathbb{R})| = |\hat{\mu}'(\mathbb{R}) - \mu'(\mathbb{R})| \leq K_1 \#T \delta + 2 \frac{\delta}{K_2} \leq 2(K_1 M/(5t_0) + 1/K_2) \delta.$$

Hence, for $f : \mathbb{R} \rightarrow \mathbb{R}$ which is 1-Lipschitz, we have, writing $f_0 = f - f(0)$,

$$\begin{aligned}
\left| \int_{\mathbb{R}} f d\mu - \int_{\mathbb{R}} f \frac{1}{\hat{\mu}(\mathbb{R})} d\hat{\mu} \right| &= \left| \int_{\mathbb{R}} f_0 d\mu - \int_{\mathbb{R}} f_0 \frac{1}{\hat{\mu}(\mathbb{R})} d\hat{\mu} \right| \\
&\leq \left| \int_{[-t_0/2, t_0/2]} f_0 (d\mu - d\hat{\mu}) \right| + \left| \int_{[-t_0/2, t_0/2]} f_0 \left(d\hat{\mu} - \frac{1}{\hat{\mu}(\mathbb{R})} d\hat{\mu} \right) \right| \\
&\leq \left| \int_{[-t_0/2, t_0/2]} f_0 (d\mu - d\hat{\mu}) \right| + |\hat{\mu}(\mathbb{R}) - 1| \left| \int_{[-t_0/2, t_0/2]} f_0 \frac{1}{\hat{\mu}(\mathbb{R})} d\hat{\mu} \right|,
\end{aligned}$$

where we used on the last inequality that the support of $\hat{\mu}$ is $[-t_0, t_0]$. Since f_0 is 1-Lipschitz and satisfies $f_0(0) = 0$, we have by the bound on the p -moment of μ and (33)

$$(\hat{\mu}(\mathbb{R}) - 1) \left| \int_{[-t_0/2, t_0/2]} f_0 \frac{1}{\hat{\mu}(\mathbb{R})} d\hat{\mu} \right| \leq t_0/2 |\mu(\mathbb{R}) - \hat{\mu}(\mathbb{R})| \leq (K_1 M/5 + t_0/K_2) \delta.$$

Moreover, since f_0 is 1-Lipschitz, then $\tilde{f}_0 : t \mapsto f_0(t_0 \cdot t)$ is t_0 -Lipschitz, and by (32) we have

$$\left| \int_{[-t_0/2, t_0/2]} f_0(d\mu - d\hat{\mu}) \right| = \left| \int_{[-1/2, 1/2]} \tilde{f}_0(d\mu' - d\hat{\mu}') \right| \leq 2(K_1M/5 + t_0/K_2)\delta + t_0/M\sqrt{2/K_3}\sqrt{\delta}.$$

Hence, putting all the above bounds together yields

$$\left| \int_{\mathbb{R}} f d\mu - \int_{\mathbb{R}} f \frac{1}{\hat{\mu}(\mathbb{R})} d\hat{\mu} \right| \leq 3(K_1M/5 + t_0/K_2)\delta + t_0/M\sqrt{2/K_3}\sqrt{\delta}.$$

Recalling that $\delta = \lambda + \tau$, we thus have

$$W_1(\hat{\mu}/\hat{\mu}(\mathbb{R}), \mu) \leq 3(K_1M/5 + t_0/K_2)(\lambda + \tau) + t_0/M\sqrt{2/K_3}\sqrt{\lambda + \tau}.$$

□

Aggregating the above estimates yields the proof of Proposition 2.5.

Proof of Proposition 2.5. Let $\lambda_0 > 0$ and suppose that $\mu = \sum_{t \in T} a_t \delta_t$, with $T = \text{Supp}(\mu)$ satisfies

$$\int_{t, t' \in T, t \neq t'} |t - t'| > 5/2 \cdot d\eta,$$

for some $d > 0$, and set $\tilde{\mu} = \frac{1}{\mu([-t_0/2, t_0/2])} \mu|_{[-t_0/2, t_0/2]}$ for some $t_0 > 0$ to tune later. Then, for $t_0 \geq 128d\eta$ and $M = t_0/(d\eta)$, we get by Lemma 5.1

$$\| \exp(\eta|\cdot|) \mathcal{F}[C] - \mathcal{F}[\tilde{\mu}] \|_2^{t_0, M} \leq \frac{\sqrt{t_0}}{\sqrt{2\pi\eta}} \exp(2\pi\eta(M+1)/t_0) \epsilon + \sqrt{2M+1} \frac{3 \cdot 2^p M_p}{t_0^p}.$$

Choose

$$(34) \quad t_0 = \max \left(2\pi\eta, 256d\eta/5, 2 \exp(-2\pi/(pd)) \left[6M_p \sqrt{1 + 4\pi/d/\lambda_0} \right]^{1/p} \right).$$

In particular, given that $M = t_0/(d\eta)$, we get

$$\begin{aligned} \sqrt{2M+1} \frac{3 \cdot 2^p M_p}{t_0^p} &\leq \sqrt{t_0} \sqrt{2/(d\eta) + 1/t_0} \exp(2\pi/d) \frac{3 \cdot 2^p M_p \lambda_0}{6 \cdot 2^p M_p \sqrt{1 + 4\pi/d}} \\ &\leq \frac{\sqrt{t_0}}{\sqrt{2\pi\eta}} \frac{\sqrt{4\pi/d + 2\pi\eta/t_0}}{\sqrt{4\pi/d + 1}} \exp(2\pi\eta(M+1)/t_0) \lambda_0/2 \\ &\leq \frac{\sqrt{t_0}}{\sqrt{2\pi\eta}} \exp(2\pi\eta(M+1)/t_0) \lambda_0/2, \end{aligned}$$

and thus

$$(35) \quad \| \exp(\eta|\cdot|) \mathcal{F}[C] - \mathcal{F}[\tilde{\mu}] \|_2^{t_0, M} \leq \frac{e\sqrt{t_0}}{\sqrt{2\pi\eta}} \exp(2\pi/d) (\epsilon + \lambda_0/2) := \delta_0.$$

Let $\hat{\mu}$ be the minimizer of the problem

$$\arg \min_{\nu \in \mathcal{M}^+([-t_0/2, t_0/2])} \| \mathcal{F}[\nu] - \exp(\eta|\cdot|) \mathcal{F}[C] \|_2^{t_0, M} + \lambda \|\nu\|_{TV},$$

$\|\nu\|_{TV} \leq 1$

with $\lambda = \frac{e\sqrt{t_0}}{\sqrt{2\pi\eta}} \exp(2\pi/d) \lambda_0$. Then, since $5t_0/(2M) = 5/2 \cdot d\eta$, we have

$$\int_{t, t' \in T, t \neq t'} |t - t'| > 5t_0/(2M),$$

and thus by Proposition 5.2 with (35) and Lemma 5.1 we have

$$\begin{aligned} W_1(\mu, \hat{\mu}/\hat{\mu}(\mathbb{R})) &\leq W_1(\mu, \tilde{\mu}) + W_1(\tilde{\mu}, \hat{\mu}/\hat{\mu}(\mathbb{R})) \\ &\leq \frac{3 \cdot 2^{p-1}}{t_0^{p-1}} M_p + 3/2(K_1M/2 + 2t_0/K_2)(\delta_0 + \lambda/2) + t_0/M\sqrt{2/K_3}\sqrt{\delta_0 + \lambda/2} \\ &\leq \frac{t_0}{2\sqrt{2M}} (\delta_0 + \lambda/2) + 3(K_1/(5d\eta) + 1/K_2)t_0(\delta_0 + \lambda/2) + d\eta\sqrt{2/K_3}\sqrt{\delta_0 + \lambda/2}, \end{aligned}$$

where we used (34) and the relation $t_0/M = d\eta$ in the last inequality. Replacing δ_0, λ by their value and using again the relation between M and t_0 , we finally get

$$W_1(\mu, \hat{\mu}/\hat{\mu}(\mathbb{R})) \leq a(t_0^{-1/2})t_0^{3/2} \exp(2\pi/d)(\epsilon + \lambda_0) + bd\eta^{3/4}t_0^{1/4} \exp(\pi/d)\sqrt{\epsilon + \lambda_0},$$

where

$$(36) \quad a(t) = \frac{3e(K_1/(5d\eta) + 1/K_2)}{\sqrt{2\pi\eta}} + \frac{e\sqrt{d}}{4\sqrt{\pi}}t,$$

and

$$(37) \quad b = (2/\pi)^{1/4} \sqrt{e/K_3},$$

where we recall that K_1, K_2, K_3 are numerical constants (see the discussion after the statement of Proposition 5.2 for their exact value). This gives the first statement of the proposition.

As ϵ goes to zero, (34) yields that

$$t_0 = O((M_p/\lambda_0)^{1/p}),$$

and $a(t_0) = O(1), b(t_0) = O(1)$. Hence, choosing $\lambda_0 \sim \epsilon$ yields

$$W_1(\mu, \hat{\mu}/\hat{\mu}(\mathbb{R})) = O\left(\|\mu\|_p^{3/2}\epsilon^{1-3/(2p)}\right) + O\left(\|\mu\|_p^{1/4}\epsilon^{1/2-1/(4p)}\right).$$

In particular, for fixed p with $p \geq 3$, we have

$$W_1(\mu, \hat{\mu}/\hat{\mu}(\mathbb{R})) = O\left(\epsilon^{1/2-1/(4p)}\right),$$

and for $p = \infty$ we get

$$W_1(\mu, \hat{\mu}/\hat{\mu}(\mathbb{R})) = O(\sqrt{\epsilon}).$$

□

6. PROOF OF THE AAS-PROPERTY IN THE UNITARY INVARIANT CASE

The goal of this section is to prove Proposition 3.5: this amounts to prove that for A, B self-adjoint matrices in $\mathcal{M}_N(\mathbb{C})$ and U a Haar unitary and N large enough, the pair (UAU^*, B) satisfies the AAS-property (see Definition 3.4) for some constants $c, C_A, C_B > 0$ which only depend on the first six moments of A and B and on η . Such behavior has already been proven for η close to the real axis by Kargin [Kar15] with constants depending on the bound of the support of A and B (see also [BES17] for optimal result depending also on some analytical properties of μ_A and μ_B). We must thus improve Kargin's approach to obtain bounds which only depend on the first moments of A and B . Let us first review the main tool used by Kargin.

6.1. Matrix subordination. In [PV00], Pastur and Vasilchuk noticed that, since the asymptotic spectral behavior of the addition of matrices is close to a free additive convolution, and since the latter are described by subordination functions, there may exist subordination functions directly at the level of random matrices. They actually found such subordination functions and used them to study the convergence of the spectral distribution of the matrix models towards the free convolution. This approach is in particular fundamental to remove any boundedness assumption on the support of μ_1 and μ_2 in Theorem 3.3. In [Kar12, Kar15], Kargin greatly improved the subordination method of Pastur and Vasilchuk to provide concentration bounds for the additive convolution, when the support of μ_A and μ_B remain bounded. We review here the matricial subordination functions in the additive case. In this paragraph and in the following sections, the symbol \mathbb{E} generally refers to the expectation with respect to the Haar unitary U .

Since $H = UAU^* + B$ with U Haar unitary, we can assume without loss of generalities that A and B are diagonal for any result regarding the spectral distribution of H . Hence, the hypothesis of A and B being diagonal will be kept throughout the rest of the section. Set

$$H' = U^*HU = A + U^*BU,$$

and remark that $m_{H'} = m_H$. For $z \in \mathbb{C}^+$, set $f_A(z) = \text{tr}(AG_{H'}(z))$ and $f_B(z) = \text{tr}(BG_H(z))$. Then, define

$$(38) \quad \omega_A(z) = z - \frac{\mathbb{E}(f_B(z))}{\mathbb{E}(m_H(z))}, \quad \omega_B(z) = z - \frac{\mathbb{E}(f_A(z))}{\mathbb{E}(m_H(z))}.$$

An important point [Kar15, Eq. 11] is that

$$(39) \quad \omega_A(z) + \omega_B(z) = z - \frac{1}{\mathbb{E}m_H(z)},$$

which is the same relation as the one satisfied by the subordination functions for the free additive convolution in (6). After a small modification of Kargin's formulation [Kar15], we get the following approximate subordination relation.

Lemma 6.1. *For $z \in \mathbb{C}^+$,*

$$(40) \quad \mathbb{E}G_{H'}(z) = G_A(\omega_A(z)) + R_A(z),$$

with $R_A(z) := \frac{1}{\mathbb{E}m_H(z)}G_A(\omega_A(z))\mathbb{E}\Delta_A(z)$, and

$$\Delta_A = (m_H - \mathbb{E}m_H)(U^*BUG_{H'} - \mathbb{E}(U^*BUG_{H'})) - (f_B - \mathbb{E}(f_B))(G_{H'} - \mathbb{E}(G_{H'})).$$

Moreover, $\mathbb{E}\Delta_A$ is diagonal and $\text{tr}\mathbb{E}\Delta_A = 0$.

Of course, the same result holds for the expression of $\mathbb{E}G_H$ in terms of $G_B(\omega_B)$ after switching A and B and H and H' .

Proof. By [Kar15, Eqs. (12), (13)],

$$\mathbb{E}G_{H'}(z) = G_A(\omega_A(z)) + R_A(z),$$

with $R_A(z) := \frac{1}{\mathbb{E}m_H(z)}G_A(\omega_A(z))(A - z)\mathbb{E}\tilde{\Delta}_A(z)$, and

$$\tilde{\Delta}_A = -(m_H - \mathbb{E}m_H)G_{H'} - (f_B - \mathbb{E}(f_B))G_AG_{H'}.$$

Since $(A - z)$ is deterministic, $(A - z)\mathbb{E}\tilde{\Delta}_A(z) = \mathbb{E}[(A - z)\Delta_A(z)]$, and we have, forgetting the dependence in z ,

$$\begin{aligned} (A - z)\mathbb{E}\tilde{\Delta}_A &= \mathbb{E}(-(m_H - \mathbb{E}m_H)(A - z)G_{H'} - (f_B - \mathbb{E}(f_B))G_{H'}) \\ &= \mathbb{E}(-(m_H - \mathbb{E}m_H)(1 - U^*BUG_{H'}) - (f_B - \mathbb{E}(f_B))G_{H'}) \\ &= \mathbb{E}((m_H - \mathbb{E}m_H)U^*BUG_{H'} - (f_B - \mathbb{E}(f_B))G_{H'}) \\ &= \mathbb{E}[(m_H - \mathbb{E}m_H)(U^*BUG_{H'} - \mathbb{E}(U^*BUG_{H'})) - (f_B - \mathbb{E}(f_B))(G_{H'} - \mathbb{E}(G_{H'}))] \\ &:= \mathbb{E}\Delta_A. \end{aligned}$$

where we have used on the penultimate step that $\mathbb{E}(X - \mathbb{E}(X)) = 0$ for any random variable X . This proves the first part of the lemma. For the second part, note that if V is any diagonal unitary matrix, noting that UV^* is again Haar distributed and using that $VAV^* = A$ yields that

$$\begin{aligned} V\mathbb{E}((m_H - \mathbb{E}m_H)G_{H'}) &= V\mathbb{E}((\text{tr}((A + U^*BU - z)^{-1}) - \mathbb{E}m_H)(A + U^*BU - z)^{-1}) \\ &= V\mathbb{E}((\text{tr}(V^*(VAV^* + VU^*BUV^* - z)^{-1}V) - \mathbb{E}m_H) \\ &\quad V^*(VAV^* + VU^*BUV^* - z)^{-1}V) \\ &= \mathbb{E}((\text{tr}((A + VU^*BUV^* - z)^{-1}) - \mathbb{E}m_H)(A + VU^*BUV^* - z)^{-1})V \\ &= \mathbb{E}((\text{tr}((A + U^*BU - z)^{-1}) - \mathbb{E}m_H)(A + UBU^* - z)^{-1})V, \end{aligned}$$

where we used the trace property on the third equality. Likewise,

$$V\mathbb{E}((f_B - \mathbb{E}(f_B))G_AG_{H'}) = \mathbb{E}((f_B - \mathbb{E}(f_B))G_AG_{H'})V,$$

and thus V commutes with $\mathbb{E}\tilde{\Delta}_A$. Since $\mathbb{E}\tilde{\Delta}_A$ commutes with any diagonal unitary matrix, it is also diagonal, and so is $\mathbb{E}\Delta_A = (A - z)\mathbb{E}\tilde{\Delta}_A$. Finally,

$$\begin{aligned} \operatorname{tr} \mathbb{E}\Delta_A &= \mathbb{E}[(m_H - \mathbb{E}m_H) \operatorname{tr}(U^* B U G_{H'} - \mathbb{E}(U^* B U G_{H'})) - (f_B - \mathbb{E}(f_B)) \operatorname{tr}(G_{H'} - \mathbb{E}(G_{H'}))] \\ &= \mathbb{E}((m_H - \mathbb{E}m_H)(f_B - \mathbb{E}(f_B)) - (f_B - \mathbb{E}(f_B))(m_H - \mathbb{E}m_H)) = 0. \end{aligned}$$

□

Moreover, an algebraic manipulation of (40) yields

$$(41) \quad \omega_A = A - (\mathbb{E}G_{H'})^{-1} + (-\mathbb{E}G_{H'})^{-1} \frac{1}{\mathbb{E}m_H} \mathbb{E}U\Delta_A,$$

Following [Kar15, Lemma 2.1] (see also Lemma 6.2), remark that we also have

$$(42) \quad -(\mathbb{E}U G_{H'})^{-1} + A - z \in \mathbb{H}(\mathcal{M}_N(\mathbb{C})),$$

where $\mathbb{H}(\mathcal{M}_n(\mathbb{C}))$ denotes the half-space $\{M \in \mathcal{M}_n(\mathbb{C}), \frac{1}{i}(M - M^*) \geq 0\}$.

6.2. AAS-property of the matrix subordination functions. We have seen in the previous section that matrix subordination functions already satisfy similar relations as the one fulfilled by the subordination functions for the free convolutions. In this section we quantify this similarity by estimating the error terms in (40). Namely we show that $\mathbb{E}m_H$ (resp. $\mathbb{E}\tilde{m}_M$) and $m_A(\omega_A)$ or $m_B(\omega_B)$ (resp. \tilde{m}_A or \tilde{m}_B) are approximately the same in the additive case, which will give a proof of Proposition 3.5. In the additive case, this has been already done in [Kar15]; hence the goal of the study of the additive case is just to give precise estimates in the approach of Kargin, without any assumption on the norm of A and B .

The proof of Proposition 3.5 is postponed to the end of the section, and we first prove some intermediary results. Recall notations from Section 6.1, and recall also the notations from 2.1. In particular, we write a_i, b_i for $\operatorname{tr}(A^i), \operatorname{tr}(B^i)$ for $i \geq 1$. First, remark that $m_H = m_{H'}$, where $H' = A + U^* B U$. Hence, we can apply (40) to either H or H' (switching A and B) to deduce information on m_H . Then, by Lemma A.3 and the hypothesis $\operatorname{tr}(A) = \operatorname{tr}(B) = 0$, we have

$$\mathbb{E} \operatorname{tr}((A + U^* B U)^2) = \operatorname{tr}(A^2) + \operatorname{tr}(B^2) + 2 \operatorname{tr}(A) \operatorname{tr}(B) = \operatorname{tr}(A^2) + \operatorname{tr}(B^2) = a_2 + b_2,$$

where we used notations from Section 2.1. Hence, by (3) and the fact that $\mathbb{E} \operatorname{tr}(A + U^* B U) = 0$,

$$(43) \quad |\mathbb{E}(m_H(z))^{-1} + z| \leq \frac{\operatorname{tr}((A + U^* B U)^2)}{\Im(z)} \leq \frac{a_2 + b_2}{\Im(z)}$$

for all $z \in \mathbb{C}^+$. We can obtain a similar bound for $(\mathbb{E}(G_{H'}))^{-1}$, as next lemma shows.

Lemma 6.2. *The matrix $\mathbb{E}(G_{H'})^{-1}$ is diagonal with diagonal entries satisfying the bound*

$$|[\mathbb{E}(G_{H'})^{-1}]_{ii} - \lambda_i^A + z| \leq \frac{b_2}{\eta}.$$

Proof. We know by Lemma 6.1 that $\mathbb{E}(G_{H'})$ is diagonal. Define the map $I : \mathbb{C}^+ \mapsto \mathbb{C}$ by $I(z) = -[\mathbb{E}(G_{H'})^{-1}]_{ii} = -[\mathbb{E}(G_{H'})]_{ii}^{-1}$. By (42), I maps \mathbb{C}^+ to \mathbb{C}^+ . Moreover, as z goes to infinity, $\mathbb{E}G_{H'}(z) = -z^{-1} - \mathbb{E}(A + U^* B U)z^{-2} - \mathbb{E}(A + U^* B U)^2 z^{-3} + o(z^{-3})$. By Lemma A.2, $\mathbb{E}(U^* B U) = \operatorname{tr}(B) = 0$ and

$$\mathbb{E}((A + U^* B U)^2) = A^2 + \mathbb{E}(U^* B U)A + A\mathbb{E}(U^* B U) + \mathbb{E}(U B^2 U^*) = A^2 + b_2.$$

Hence,

$$\mathbb{E}(G_{H'})_{ii} = -z^{-1} - \lambda_i^A z^{-2} - ((\lambda_i^A)^2 + b_2)z^{-3} + o(z^{-3}).$$

Applying Theorem 3.1 to the map I and then using (3) yield the existence of a probability measure ρ on \mathbb{R} such that

$$(-\mathbb{E}(G_{H'})_{ii})^{-1} = z - \lambda_i^A + b_2 m_\rho(t).$$

In particular,

$$|[\mathbb{E}(G_{H'})^{-1}]_{ii} + z - \lambda_i^A| \leq \frac{b_2}{\eta}.$$

□

We now provide a bound on $T\Delta_A$ for $T \in \mathcal{M}_N(\mathbb{C})$, where Δ_A is given in (40). In the following lemma, the dependence in z of Δ_A is omitted.

Lemma 6.3. *For $z \in \mathbb{C}^+$ with $\Im z = \eta$ and for $T \in \mathcal{M}_N(\mathbb{C})$,*

$$\mathbb{E}|\operatorname{tr}(T\Delta_A)| \leq \frac{4b_4^{1/4}}{\eta^4 N^2} \left[\sqrt{b_2} \operatorname{tr}(|TA|^4)^{1/4} + b_4^{1/4} a_4^{1/4} \operatorname{tr}(|T|^4)^{1/4} \right],$$

and

$$\mathbb{E}|\operatorname{tr}(T\Delta_A)| \leq \frac{8b_4^{1/4} \sqrt{b_2} a_4^{1/4} \|T\|_\infty}{\eta^4 N^2}.$$

Proof. Using the definition of Δ_A in (40), we get

$$\operatorname{tr}(T\Delta_A) = (m_H - \mathbb{E}m_H) \operatorname{tr}(TU^*BUG_{H'} - \mathbb{E}(TU^*BUG_{H'})) - (f_B - \mathbb{E}f_B) \operatorname{tr}(TG_{H'} - \mathbb{E}(TG_{H'})).$$

Since $U^*BUG_{H'} = 1 - (A - z)G_{H'}$ and $\operatorname{tr}(T) - \mathbb{E}\operatorname{tr}(T) = 0$, we deduce

$$\begin{aligned} \operatorname{tr}(T\Delta_A) &= - (m_H - \mathbb{E}m_H) \operatorname{tr}(T(A - z)G_{H'} - \mathbb{E}(T(A - z)G_{H'})) \\ &\quad - (f_B - \mathbb{E}f_B) \operatorname{tr}(TG_{H'} - \mathbb{E}(TG_{H'})) \\ &= - (m_H - \mathbb{E}m_H)(f'_{TA} - \mathbb{E}f'_{TA}) + z(m_H - \mathbb{E}m_H)(f'_T - \mathbb{E}f'_T) - (f_B - \mathbb{E}f_B)(f'_T - \mathbb{E}f'_T), \end{aligned}$$

with $f'_X = \operatorname{tr}(XG_{H'})$ for $X \in \mathcal{M}_N(\mathbb{C})$. Using the fact that $zm_H = \operatorname{tr}(UAU^*G_H) + \operatorname{tr}(BG_H) - 1 = f'_A + f_B - 1$ yields finally

$$(44) \quad \operatorname{tr}(T\Delta_A) = -(m_H - \mathbb{E}m_H)(f'_{TA} - \mathbb{E}f'_{TA}) + (f'_A - \mathbb{E}f'_A)(f'_T - \mathbb{E}f'_T).$$

Then, on the first hand, Cauchy-Schwartz inequality and Lemma B.3 with A and B switched give

$$\begin{aligned} \mathbb{E}|\operatorname{tr}(T\Delta_A)| &\leq \sqrt{\operatorname{Var} f'_{TA} \operatorname{Var} m_H} + \sqrt{\operatorname{Var} f'_A \operatorname{Var} f'_T} \\ &\leq \frac{4}{\eta^4 N^2} \left[\sqrt{\operatorname{tr}(B^2)} (\operatorname{tr}(B^4) \operatorname{tr}(|TA|^4))^{1/4} + \sqrt{\operatorname{tr}(B^4)} (\operatorname{tr}(T^4) \operatorname{tr}(A^4))^{1/4} \right], \end{aligned}$$

where in Lemma B.3 we chose $\alpha = \beta = \frac{1}{4}$ for f'_{TA}, f'_A, f'_T and $\alpha = 2, \beta = \infty$ for $m_H = f_{\operatorname{Id}}$. On the second hand, choosing instead $\alpha = \beta = \frac{1}{4}$ for f'_{TA}, f'_A and $\alpha = 2, \beta = \infty$ for $m_H = f_{\operatorname{Id}} f'_T$ in Lemma B.3 gives

$$\begin{aligned} \mathbb{E}|\operatorname{tr}(T\Delta_A)| &\leq \sqrt{\operatorname{Var} f'_{TA} \operatorname{Var} m_H} + \sqrt{\operatorname{Var} f'_A \operatorname{Var} f'_T} \\ &\leq \frac{4}{\eta^4 N^2} \left[\sqrt{\operatorname{tr}(B^2)} (\operatorname{tr}(B^4) \operatorname{tr}(|TA|^4))^{1/4} + \sqrt{\operatorname{tr}(B^2)} \operatorname{tr}(B^4)^{1/4} \|T\|_\infty \operatorname{tr}(A^4)^{1/4} \right] \\ &\leq \frac{8 \operatorname{tr}(B^4)^{1/4} \sqrt{\operatorname{tr}(B^2)} \operatorname{tr}(A^4)^{1/4} \|T\|_\infty}{\eta^4 N^2}. \end{aligned}$$

□

We deduce the following bound on the subordination functions ω_A .

Proposition 6.4. *Let $z \in \mathbb{C}$ with $\Im(z) := \eta$. Then,*

$$|\omega_A - z| \leq \frac{\sigma_B^2}{\eta} + \frac{C_{\text{thres},A}}{N^2} \eta,$$

and

$$\Im \omega_A \geq \eta - \frac{C_{\text{thres},A}}{N^2} \eta,$$

with

$$C_{thres,A}(\eta) = \frac{4\sigma_B^2\sigma_A}{\eta^3} \left(1 + \frac{\sigma_A^2 + \sigma_B^2}{\eta^2}\right) \left(\sqrt{2 \left(1 + \frac{\sigma_A^2 + \sigma_B^2\theta_B}{\eta^2}\right) \cdot \left(1 + \sqrt{\theta_A\theta_B} + \frac{2\sqrt{m_{A^2*B^2}(1^2, 1^2)\theta_A}}{\sigma_B^2\eta^2}\right)} + \sqrt{3 \frac{\sqrt{\theta_B\theta_A}\sigma_A^2}{\eta^2} \left(1 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2}a_4^{1/2} + b_6^{2/3}a_6^{1/3}}{\sigma_A^2\sigma_B^2\eta^2}\right)} + 2 \frac{\theta_B^{1/4}\sigma_B^3\theta_A^{1/4}}{\eta^3} \right).$$

Proof. We modify the original proof of Kargin to get the most explicit bound as possible. From (41), we get

$$\begin{aligned} \omega_A &= A - (\mathbb{E}G_{H'})^{-1} + (-\mathbb{E}G_{H'})^{-1} \frac{1}{\mathbb{E}m_H} \mathbb{E}_U \Delta_A \\ &= A + z - A + \epsilon_1 + \frac{1}{\mathbb{E}m_H} (z - A + \epsilon_1) \mathbb{E}_U \Delta_A, \end{aligned}$$

with $\epsilon_1 \in \mathbb{H}(\mathcal{M}_N(\mathbb{C}))$ by (42) and ϵ_1 is diagonal with $|(\epsilon_1)_{ii}| \leq \frac{b_2}{\Im(z)}$ by Lemma 6.2. Hence, taking the trace yields

$$(45) \quad \omega_A = z + \text{tr}(\epsilon_1) + \delta,$$

with $\delta = \text{tr}[(z - A + \epsilon_1) \frac{1}{\mathbb{E}m_H} \mathbb{E}_U \Delta_A]$ and $\text{tr}(\epsilon_1) \in \mathbb{C}^+$. By (43), $\frac{1}{\mathbb{E}(m_H)} = -z + \epsilon_2$ with $|\epsilon_2| \leq \frac{a_2 + b_2}{\Im(z)}$. Therefore, using $\text{tr}(\mathbb{E}\Delta_A) = 0$ from Lemma 6.1,

$$\begin{aligned} \delta &= \text{tr}((A - z + \epsilon_1)(-z + \epsilon_2) \mathbb{E}_U(\Delta_A)) \\ &= \text{tr}((-z + \epsilon_2)(A + \epsilon_1) \mathbb{E}_U(\Delta_A) - z(-z + \epsilon_2) \mathbb{E}_U(\Delta_A)) \\ &= (-1 + \epsilon_2/z) \mathbb{E}_U[\text{tr}((A + \epsilon_1)(z\Delta_A))]. \end{aligned}$$

First, by (40) we have

$$\begin{aligned} z \text{tr}(A\Delta_A) &= (zm_H - z\mathbb{E}m_H) [\text{tr}(AUBU^*G_{H'}) - \mathbb{E} \text{tr}(AUBU^*G_{H'})] \\ &\quad - [(f_B - \mathbb{E}f_B)(\text{tr}(zAG_{H'}) - \mathbb{E} \text{tr}(zAG_{H'}))]. \end{aligned}$$

Hence, by Cauchy-Schwartz inequality,

$$\mathbb{E}|z \text{tr}(A\Delta_A)| \leq \sqrt{\text{Var}(\tilde{f}_A) \text{Var}(zm_H)} + \sqrt{\text{Var}(f_B) \text{Var}(zf'_A)},$$

with $\tilde{f}_A = \text{tr}(AU^*BUG_{H'})$, $f'_A = \text{tr}(AG_{H'})$. Then, using Lemma B.4 with A and B switched gives

$$\text{Var}(zm_H) \leq \frac{8}{N^2\eta^2} \left(b_2 + \frac{a_2b_2 + b_4}{\eta^2} \right),$$

and using the same lemma with $\alpha_1, \beta_1 = 4$ and $\alpha_2 = 3, \beta_2 = 6$,

$$\begin{aligned} \text{Var}(zf'_A) &\leq \frac{12}{N^2\eta^2} \left(a_2b_2 + \frac{\mathbb{E}(\text{tr}((A\tilde{B}^2A)^2))^{1/2}a_4^{1/2} + b_6^{2/3}a_6^{1/3}}{\eta^2} \right) \\ &\leq \frac{12}{N^2\eta^2} \left(a_2b_2 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2}a_4^{1/2} + b_6^{2/3}a_6^{1/3}}{\eta^2} \right), \end{aligned}$$

where we used Lemma A.3 on the last inequality. Then, by Lemma B.3,

$$\text{Var}(f_B) \leq \frac{4\sqrt{b_4a_4}}{\eta^4N^2},$$

and by Lemma B.5 with A and B switched,

$$\text{Var}(\tilde{f}_A) \leq \frac{4}{N^2\eta^2} \left(a_2b_2 + \sqrt{a_4b_4} + \frac{2\sqrt{m_{A^2*B^2}(1^2, 1^2)}a_4^{1/2}}{\eta^2} \right).$$

Putting all previous bounds together gives then

$$\begin{aligned} \mathbb{E}|z \operatorname{tr}(A\Delta_A)| &\leq \sqrt{\frac{8}{N^2\eta^2} \left(b_2 + \frac{a_2b_2 + b_4}{\eta^2}\right) \cdot \frac{4}{N^2\eta^2} \left(a_2b_2 + \sqrt{a_4b_4} + \frac{2\sqrt{m_{A^2*B^2}(1^2, 1^2)}a_4^{1/2}}{\eta^2}\right)} \\ &\quad + \sqrt{\frac{4\sqrt{b_4a_4}}{\eta^4N^2} \cdot \frac{12}{N^2\eta^2} \left(a_2b_2 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2}a_4^{1/2} + b_6^{2/3}a_6^{1/3}}{\eta^2}\right)} \\ &\leq \frac{4}{N^2\eta^2} \left(\sqrt{2 \left(b_2 + \frac{a_2b_2 + b_4}{\eta^2}\right) \cdot \left(a_2b_2 + \sqrt{a_4b_4} + \frac{2\sqrt{m_{A^2*B^2}(1^2, 1^2)}a_4^{1/2}}{\eta^2}\right)} \right. \\ &\quad \left. + \sqrt{3 \frac{\sqrt{b_4a_4}}{\eta^2} \left(a_2b_2 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2}a_4^{1/2} + b_6^{2/3}a_6^{1/3}}{\eta^2}\right)} \right). \end{aligned}$$

On the other hand, by Lemma 6.3

$$\mathbb{E}|\operatorname{tr}(\epsilon_1\Delta_A)| \leq \frac{8b_4^{1/4}b_2^{1/2}a_4^{1/4}\|\epsilon_1\|_\infty}{\eta^4N^2} \leq \frac{8b_4^{1/4}b_2^{3/2}a_4^{1/4}}{\eta^5N^2},$$

where we used Lemma 6.2 on the last inequality. Therefore,

$$\begin{aligned} &|z\mathbb{E}\operatorname{tr}((A + \epsilon_1)\Delta_A)| \\ &\leq \frac{4}{N^2\eta^2} \left(\sqrt{2 \left(b_2 + \frac{a_2b_2 + b_4}{\eta^2}\right) \cdot \left(a_2b_2 + \sqrt{a_4b_4} + \frac{2\sqrt{m_{A^2*B^2}(1^2, 1^2)}a_4^{1/2}}{\eta^2}\right)} \right. \\ &\quad \left. + \sqrt{3 \frac{\sqrt{b_4a_4}}{\eta^2} \left(a_2b_2 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2}a_4^{1/2} + b_6^{2/3}a_6^{1/3}}{\eta^2}\right)} + 2 \frac{b_4^{1/4}b_2^{3/2}a_4^{1/4}}{\eta^3} \right) \\ &\leq \frac{4b_2\sqrt{a_2}}{N^2\eta^2} \left(\sqrt{2 \left(1 + \frac{a_2 + b_4/b_2}{\eta^2}\right) \cdot \left(1 + \frac{\sqrt{a_4b_4}}{a_2b_2} + \frac{2\sqrt{m_{A^2*B^2}(1^2, 1^2)}a_4^{1/2}}{a_2b_2\eta^2}\right)} \right. \\ &\quad \left. + \sqrt{3 \frac{\sqrt{b_4a_4}}{b_2\eta^2} \left(1 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2}a_4^{1/2} + b_6^{2/3}a_6^{1/3}}{a_2b_2\eta^2}\right)} + 2 \frac{b_4^{1/4}b_2^{1/2}a_4^{1/4}}{\eta^3\sqrt{a_2}} \right). \end{aligned}$$

Since $\operatorname{tr}(B) = \operatorname{tr}(A) = 0$, $b_2 = \sigma_B^2$ and $a_2 = \sigma_A^2$, yielding

$$\begin{aligned} &|z\mathbb{E}\operatorname{tr}((A + \epsilon_1)\Delta_A)| \\ &\leq \frac{4\sigma_B^2\sigma_A}{N^2\eta^2} \left(\sqrt{2 \left(1 + \frac{\sigma_A^2 + \sigma_B^2\theta_B}{\eta^2}\right) \cdot \left(1 + \sqrt{\theta_A\theta_B} + \frac{2\sqrt{m_{A^2*B^2}(1^2, 1^2)}\theta_A}{\sigma_B^2\eta^2}\right)} \right. \\ &\quad \left. + \sqrt{3 \frac{\sqrt{\theta_B\theta_A}\sigma_A^2}{\eta^2} \left(1 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2}a_4^{1/2} + b_6^{2/3}a_6^{1/3}}{\sigma_A^2\sigma_B^2\eta^2}\right)} + 2 \frac{\theta_B^{1/4}\sigma_B^3\theta_A^{1/4}}{\eta^3} \right), \end{aligned}$$

where we recall that $\theta_X = \frac{x_4^0}{\sigma_X^4}$ is the kurtosis of μ_X for X self-adjoint. Finally, taking into account the term $(1 + \epsilon_2/|z|) \leq (1 + \frac{a_2+b_2}{\eta^2})$ (45) yields

$$|\delta| \leq \frac{C_{thres,A}}{N^2}\eta,$$

with

$$C_{thres,A} = \frac{4\sigma_B^2\sigma_A}{\eta^3} \left(1 + \frac{\sigma_A^2 + \sigma_B^2}{\eta^2}\right) \left(\sqrt{2 \left(1 + \frac{\sigma_A^2 + \sigma_B^2\theta_B}{\eta^2}\right) \cdot \left(1 + \sqrt{\theta_A\theta_B} + \frac{2\sqrt{m_{A^2*B^2}(1^2, 1^2)\theta_A}}{\sigma_B^2\eta^2}\right)} \right. \\ \left. + \sqrt{3 \frac{\sqrt{\theta_B\theta_A}\sigma_A^2}{\eta^2} \left(1 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2}a_4^{1/2} + b_6^{2/3}a_6^{1/3}}{\sigma_A^2\sigma_B^2\eta^2}\right)} + 2 \frac{\theta_B^{1/4}\sigma_B^3\theta_A^{1/4}}{\eta^3} \right).$$

The two bounds of the statement are deduced from the latter expressions and (45) with the fact that $\text{tr}(\epsilon_1) \in \mathbb{C}^+$. \square

Proof of Proposition 3.5. We prove Proposition 3.5 with the matricial subordination functions introduced in this section. Let $\eta > 0$ and $z \in \mathbb{C}$ such that $\Im z = \eta$. We already known from (39) that $\omega_A(z) + \omega_B(z) = z - \frac{1}{\mathbb{E}m_H(z)}$, with $H = A + UBU^*$. By Proposition 6.4, we also have $\Im\omega_A(z) \geq \left(1 - \frac{C_{thres,A}}{N^2}\right)\eta$, and the same holds for $\omega_B(z)$ with a constant $C_{thres,B}$ obtained by switching A and B in the constant $C_{thres,A}$. Hence, by setting

$$(46) \quad c = \max(C_{thres,A}, C_{thres,B}),$$

we get $\Im\omega_A(z), \Im\omega_B(z) \geq \left(1 - \frac{c}{N^2}\right)\eta$. Similarly, by Proposition 6.4 we also have that $|\omega_A(z) - z| \leq \frac{\sigma_B}{\Im z} + \frac{c}{N^2}\Im z$ and $|\omega_B(z) - z| \leq \frac{\sigma_A}{\Im z} + \frac{c}{N^2}\Im z$.

It remains to find C_A, C_B such that for $N > \sqrt{3c}$,

$$|\mathbb{E}m_H(z) - m_A(\omega_A(z))| \leq \frac{C_A}{|z|N^2},$$

and

$$|\mathbb{E}m_H(z) - m_B(\omega_B(z))| \leq \frac{C_A}{|z|N^2}.$$

By Lemma 6.1, we have to estimate $\text{tr}(R_A(z)) = \frac{1}{\mathbb{E}m_H} \text{tr}(G_A(\omega_A)\mathbb{E}_U\Delta_A)$. By Proposition 6.4, for $N \geq \sqrt{3c}$, $\Im\omega_A \geq 2\eta/3$, which implies

$$\|G_A(\omega_A)\|_\infty \leq \frac{3}{2\eta}.$$

Hence, (44) and Cauchy-Schwartz inequality yield

$$|\text{tr}(R_A(z))| = \left| \frac{1}{\mathbb{E}m_H} \text{tr}(G_A(\omega_A)\mathbb{E}_U\Delta_A) \right| \\ \leq \frac{1}{|z^2\mathbb{E}m_H(z)|} \left(\sqrt{\text{Var}(zm_H) \text{Var}(zf'_{AG_A(\omega_A)})} + \sqrt{\text{Var}(zf'_A) \text{Var}(zf'_{G_A(\omega_A)})} \right) \\ \leq \frac{2\|G_A(\omega_A)\|_\infty}{|z| \cdot |z\mathbb{E}m_H(z)|} \sqrt{\text{Var}(zf'_A) \text{Var}(zm_H)} \\ \leq \frac{3}{\eta|z| \cdot |z\mathbb{E}m_H(z)|} \sqrt{\text{Var}(zf'_A) \text{Var}(zm_H)}.$$

By Lemma B.4 with A and B switched, we get

$$\text{Var}(zm_H) \leq \frac{8}{N^2\eta^2} \left(b_2 + \frac{b_2a_2 + b_4}{\eta^2} \right),$$

and

$$\text{Var}(zf_A) \leq \frac{12}{N^2\eta^2} \left(a_2b_2 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2}a_4^{1/2} + b_6^{2/3}a_6^{1/3}}{\eta^2} \right).$$

Hence,

$$\sqrt{\text{Var}(zf_A) \text{Var}(zm_H)} \leq \frac{4\sqrt{6}b_2}{N^2\eta^2} \sqrt{1 + \frac{a_2 + b_4/b_2}{\eta^2}} \sqrt{a_2 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2} a_4^{1/2} + b_6^{2/3} a_6^{1/3}}{b_2\eta^2}}.$$

Then, using (43) yields $\frac{1}{|z\mathbb{E}m_H(z)|} \leq 1 + \frac{a_2+b_2}{\eta^2}$. Therefore, since $a_2 = \sigma_A^2$ and $b_2 = \sigma_B^2$,

$$|\text{tr}(R_A(z))| \leq \frac{C_A}{|z|N^2},$$

with

(47)

$$C_A = \frac{12\sqrt{6}\sigma_B^2\sigma_A}{\eta^3} \left(1 + \frac{\sigma_A^2 + \sigma_B^2}{\eta^2}\right) \sqrt{1 + \frac{\sigma_A^2 + \theta_B\sigma_B^2}{\eta^2}} \sqrt{1 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2} a_4^{1/2} + b_6^{2/3} a_6^{1/3}}{a_2 b_2 \eta^2}}.$$

The same holds for B with C_B obtained from C_A by switching the role of A and B . This concludes the proof of Proposition 3.5. \square

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APPENDIX A. INTEGRATION ON THE UNITARY GROUP AND WEINGARTEN CALCULUS

We prove here the integration formulas on the unitary group which are used in the manuscript. The goal is to integrate polynomials in the entries of a random unitary matrix with respect to the Haar measure. We only state the results for polynomials up to order six, which are the useful ones for our problems, and the tedious computations of this section are done using the very efficient software [FKN19]. The fundamental ingredient of the proofs is the Weingarten calculus

developed by Collins and Sniady [Col03, CS06]. In the following theorem, $U = (u_{ij})_{1 \leq i, j \leq N}$ is a Haar unitary matrix.

Theorem A.1 (Weingarten calculus, [Col03]). *Let $\vec{i}, \vec{i}', \vec{j}, \vec{j}' \in \mathbb{N}^r$ with $r \geq 1$. Then,*

$$\int_{U_N} u_{i_1 j_1} \cdots u_{i_r j_r} \bar{u}_{i'_1 j'_1} \cdots \bar{u}_{i'_r j'_r} = \sum_{\substack{\sigma, \tau \in S_r \\ i \circ \sigma = i', j \circ \tau = j'}} W_{N,r}(\sigma \tau^{-1}),$$

where S_r denotes the symmetric group of size r and $W_{N,r} : S_r \rightarrow \mathbb{Q}$ is the Weingarten function whose values at σ only depends on the cycle structure of the permutation. Moreover,

$$W_{N,1}(\text{Id}) = \frac{1}{N},$$

$$W_{N,2}(1^2) = \frac{1}{N^2(1-N^{-2})}, \quad W_{N,2}(2) = \frac{-1}{N^3(1-N^{-2})}.$$

where (1) denotes the permutation identity and (2) a transposition.

Using the latter theorem, we prove the following asymptotic formulas for products of matrices A and UBU^* .

Lemma A.2. *Let $A, B \in \mathcal{M}_N(\mathbb{C})$ and $U \in U_n$ Haar unitary, and suppose that A, B are diagonal. Then, $\mathbb{E}[UBU^*A] = \text{tr}(B)A$,*

$$(1 - 1/N^2)\mathbb{E}(UBU^*AUBU^*) = \left(\text{tr}(A) \text{tr}(B^2) - \text{tr}(A) \text{tr}(B)^2 + A \left(\text{tr}(B)^2 - \frac{1}{N^2} \text{tr}(B^2) \right) \right).$$

Proof. We only explain the proof of the second equality, since the proofs of the first one uses similar pattern. Note first that $\mathbb{E}(UBU^*AUBU^*)$ commutes with A , and thus is diagonal when A has distinct diagonal entries. By a continuity argument, $\mathbb{E}(UBU^*AUBU^*)$ is thus diagonal. Write $U = (u_{ij})_{1 \leq i, j \leq N}$ and expand $\mathbb{E}(UBU^*AUBU^*)_{ii}$ as

$$\begin{aligned} \mathbb{E}(UBU^*AUBU^*)_{ii} &= \sum_{k, j, s=1}^N \mathbb{E}(u_{ik} B_{kk} \bar{u}_{jk} A_{jj} u_{js} B_{ss} \bar{u}_{is}) \\ &= \sum_{k, j, s=1}^N B_{kk} A_{jj} B_{ss} \mathbb{E}(u_{ik} \bar{u}_{jk} u_{js} \bar{u}_{is}). \end{aligned}$$

Let $1 \leq i, j \leq N$ and $1 \leq k, s \leq N$. Then, by Theorem A.1 and summing on permutations of S_2 ,

$$\mathbb{E}(u_{ik} u_{js} \bar{u}_{is} \bar{u}_{jk}) = \begin{cases} -\frac{1}{N(N^2-1)} & \text{if } i \neq j, k \neq s \\ \frac{1}{N(N+1)} & \text{if } i = j, k \neq s \text{ or } i \neq j, k = s \\ \frac{2}{N(N+1)} & \text{if } i = j, k = s \end{cases}$$

Hence, using the latter formula yields

$$\begin{aligned} \mathbb{E}(UBU^*AUBU^*)_{ii} &= \sum_{j \neq i} A_{jj} \left[\sum_{k \neq s} -\frac{1}{N(N^2-1)} B_{kk} B_{ss} + \sum_{k=1}^n \frac{1}{N(N+1)} B_{kk}^2 \right] \\ &\quad + A_{ii} \left[\sum_{k \neq s} \frac{1}{N(N+1)} B_{kk} B_{ss} + \sum_{k=1}^n \frac{2}{N(N+1)} B_{kk}^2 \right] \\ &= (\text{tr}(A) - A_{ii}/N) \left[-\frac{1}{1-1/N^2} \text{tr}(B)^2 + \text{tr}(B^2) \left(\frac{1}{1+1/N} + \frac{1}{N-1/N} \right) \right] \\ &\quad + A_{ii} \left[\frac{1}{1+1/N} \text{tr}(B)^2 + \frac{1}{N+1} \text{tr}(B^2) \right] \\ &= \frac{1}{1-1/N^2} \left[\text{tr}(A) \text{tr}(B^2) - \text{tr}(A) \text{tr}(B)^2 + A_{ii} \left(\text{tr}(B)^2 - \frac{1}{N^2} \text{tr}(B^2) \right) \right] \end{aligned}$$

A similar computation yields the third equality. We used [FKN19] to achieve the computation in the latter case. \square

Lemma A.2 directly yields formulas for expectation of trace of products. For two finite integer sequences s, s' of length $r \geq 1$, set

$$m_{A*B}(s, s') = \mathbb{E} \operatorname{tr}(A^{s_1} U B^{s'_1} U^* \dots A^{s_r} U B^{s'_r} U^*).$$

Lemma A.3. *Suppose that $A, B \in \mathcal{M}_N(\mathbb{C})$. Then,*

$$m_{A*B}(1, 1) = \operatorname{tr}(A) \operatorname{tr}(B),$$

$$m_{A*B}(1^2, 1^2) = \frac{1}{1 - N^{-2}} \left[\operatorname{tr}(A^2) \operatorname{tr}(B)^2 + \operatorname{tr}(A)^2 \operatorname{tr}(B^2) - \operatorname{tr}(A)^2 \operatorname{tr}(B)^2 - \frac{1}{N^2} \operatorname{tr}(A^2) \operatorname{tr}(B^2) \right].$$

APPENDIX B. ANALYSIS ON THE UNITARY GROUP

We provide here concentration inequalities on the unitary group which imply all our concentration results concerning the Stieltjes transform. Proofs are adapted from Kargin's approach in [Kar15] to get bounds only depending on first moments of the matrices involved.

B.1. Poincaré inequality and concentrations results. Several concentrations inequalities exist on the unitary group [AGZ10, BE85]. In this paper, we only use Poincaré inequality, which has the fundamental property of having an error term which is averaged on the unitary group. Poincaré inequalities exist on every compact Riemannian manifolds without boundary, for which the Laplacian operator has a discrete spectrum.

Theorem B.1 (Poincaré inequality). *Suppose that M is a compact manifold without boundary and with volume form μ , and let $\lambda_1 > 0$ be the first non-zero eigenvalue of the Laplacian on M . Then, for all $f \in C^2(M)$ such that $\int_M f d\mu = 0$,*

$$\int_M |f|^2 d\mu \leq \frac{1}{\lambda_1} \int_M \|\nabla f\|^2 d\mu.$$

Proof of this theorem is a direct consequence of the integration by part formula on M . In the case of the unitary group U_N the spectrum of the Laplacian can be explicitly computed using the representation theory of the group (see [Hum72]), and the first eigenvalue of the Laplacian is simply equal to N . Hence, we deduce from Poincaré inequality the following concentration inequality for the unitary group.

Corollary B.2 (Poincaré inequality on U_N). *For all $f \in C^2(U_N)$ such that $\int_{U_N} f d\mu = 0$, where μ denotes the Haar measure on U_N ,*

$$\int_{U_N} |f|^2 d\mu \leq \frac{1}{N} \int_{U_N} \|\nabla f\|^2 d\mu.$$

In the sequel, the functions f we will studied are traces of matrices involved the various resolvents of the manuscript. We will use several times the generalized matrix Hölder inequality for Schatten p -norms. Recall that the Schatten p -norm of a matrix $X \in \mathcal{M}_N(\mathbb{C})$ is defined by

$$\|X\|_p = [N \operatorname{tr}((X^* X)^{p/2})]^{1/p}.$$

Then, if $X_1, \dots, X_k \in \mathcal{M}_N(\mathbb{C})$ and $\alpha_1, \dots, \alpha_k \in [1, +\infty]$, then

$$(48) \quad \|X_1 \dots X_k\|_r \leq \prod_{i=1}^k \|X_i\|_{\alpha_i},$$

where $\frac{1}{r} = \sum_{i=1}^k \frac{1}{\alpha_i}$. Remark that the matrix Hölder is not a trivial consequence of the usual Hölder inequality, and its proof is quite involved (see [Ser10, 7.3]).

B.2. Application to the additive convolutions. For $H = UAU^* + B$, $z \in \mathbb{C}^+$ and $T \in \mathcal{M}_N(\mathbb{C})$, set $G_H = (H - z)^{-1}$ and define the function $f_T(z) = \text{tr}(T(H - z)^{-1}) = \text{tr}(TG_H)$. In the following lemmas, we use the convention $\text{tr}(|T|^\infty)^{1/\infty} = \|T\|_\infty$ for $T \in \mathcal{M}_N(\mathbb{C})$.

Lemma B.3. For $z \in \mathbb{C}^+$ with $\eta = \Im(z)$ and for $T \in \mathcal{M}_N(\mathbb{C})$,

$$\mathbb{E}(|f_T(z) - \mathbb{E}(f_T(z))|^2) \leq \frac{4 \text{tr}(A^\alpha)^{2/\alpha} \text{tr}(|T|^\beta)^{2/\beta}}{\eta^4 N^2},$$

where $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2}$ with $\alpha, \beta \in [2, \infty]$.

Proof. By (B.2), for any function f with zero mean which is C^2 on U_N , $\mathbb{E}(|f|^2) \leq \frac{1}{N} \mathbb{E}(\|\nabla f\|^2)$. Let us apply this to the map f_T . Since $d_X(X - z)^{-1} = (X - z)^{-1}X(X - z)^{-1}$, applying the chain rule for f_T at $U \in U_N$ yields for X anti-Hermitian

$$\nabla_U f_T(X) = \text{tr}(TG_H[X, \tilde{A}]G_H) = \text{tr}([\tilde{A}, G_H T G_H]X),$$

where $\tilde{A} = UAU^*$. Hence,

$$\|\nabla_U f_T\|_2 = \frac{1}{N} \|[\tilde{A}, G_H T G_H]\|_2 \leq \frac{2}{N\eta^2} \|A\|_\alpha \|T\|_\beta$$

with $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2}$, where we applied matrix Hölder inequality in the last inequality. Therefore,

$$\mathbb{E}\|\nabla_U f_T\|_2^2 \leq \frac{4}{N^2 \eta^4} \|A\|_\alpha^2 \|T\|_\beta^2 \leq \frac{4 \text{tr}(A^\alpha)^{2/\alpha} \text{tr}(|T|^\beta)^{2/\beta}}{N \eta^4},$$

so that (B.2) yields

$$\text{Var}(f_T) \leq \frac{4 \text{tr}(A^\alpha)^{2/\alpha} \text{tr}(|T|^\beta)^{2/\beta}}{N^2 \eta^4}.$$

□

Lemma B.4. For $z \in \mathbb{C}^+$ with $\eta = \Im(z)$ and $\text{tr}(B) = 0$,

$$\text{Var}(zm_H) \leq \frac{8}{N^2 \eta^2} \left(\text{tr}(A^2) + \frac{\text{tr}(B^2) \text{tr}(A^2) + \text{tr}(A^4)}{\eta^2} \right),$$

and for $T \in \mathcal{M}_N(\mathbb{C})$,

$$\begin{aligned} & \text{Var}(zf_T) \\ & \leq \frac{12}{N^2 \eta^2} \left(\text{tr}(|T|^2) \text{tr}(A^2) + \frac{\mathbb{E} \left(\text{tr}((B\tilde{A}^2 B)^{\alpha_1/2}) \right)^{2/\alpha_1} \text{tr}(|T|^{\beta_1})^{2/\beta_1} + \text{tr}(A^{2\alpha_2})^{2/\alpha_2} \text{tr}(|T|^{\beta_2})^{2/\beta_2}}{\eta^2} \right) \end{aligned}$$

for any $\alpha_1, \beta_1, \alpha_2, \beta_2 \in [2, \infty]$ satisfying

$$\frac{1}{\alpha_1} + \frac{1}{\beta_1} = \frac{1}{\alpha_2} + \frac{1}{\beta_2} = \frac{1}{2}.$$

Proof. Let us first prove the second statement. As in the latter lemma, taking the derivative of zf_T at $U \in U_N$ yields for X anti-Hermitian

$$\begin{aligned} \nabla_U(zf_T)(X) &= z \text{tr}(TG_H[X, \tilde{A}]G_H) \\ &= \text{tr}([\tilde{A}, zG_H T G_H]X) \\ &= \text{tr} \left(\left[-\tilde{A} T G_H + G_H T \tilde{A} + \tilde{A}(B + \tilde{A})G_H T G_H - G_H T G_H(B + \tilde{A})\tilde{A} \right] X \right), \end{aligned}$$

where $\tilde{A} = UAU^*$ and we used the equality $zG_H = -1 + HG_H$. Hence,

$$\begin{aligned} \|\nabla_U z f_T\|_2^2 &\leq \frac{1}{N^2} \left(2\|T\tilde{A}\|_2 + 2\|\tilde{A}B G_H T G_H\|_2 + 2\|\tilde{A}^2 G_H T G_H\|_2 \right)^2 \\ &\leq \frac{12}{N^2} (\|T\tilde{A}\|_2^2 + \|\tilde{A}B G_H T G_H\|_2^2 + \|\tilde{A}^2 G_H T G_H\|_2^2). \end{aligned}$$

First, $\mathbb{E}(\|T\tilde{A}\|_2^2) = N\mathbb{E}(\text{tr}(TT^*\tilde{A}^2)) = N\text{tr}(TT^*)\text{tr}(A^2)$ by Lemma A.3. Then, we apply the matrix Hölder inequality (48) and then the usual Hölder inequality to get

$$\begin{aligned}\mathbb{E}(\|\tilde{A}BG_HTG_H\|^2) &\leq \frac{1}{\eta^4}\mathbb{E}(\|\tilde{A}B\|_{\alpha_1}^2\|T\|_{\beta_1}^2) \leq \frac{1}{\eta^4}\mathbb{E}(\|\tilde{A}B\|_{\alpha_1}^{\frac{2}{\alpha_1}}\|T\|_{\beta_1}^2) \\ &\leq \frac{N}{\eta^4}\mathbb{E}\left(\text{tr}((B\tilde{A}^2B)^{\alpha_1/2})\right)^{2/\alpha_1}\text{tr}(|T|^{\beta_1})^{2/\beta_1},\end{aligned}$$

and

$$\mathbb{E}(\|\tilde{A}^2G_HTG_H\|^2) \leq \frac{\|A^2\|_{\alpha_2}^2\|T\|_{\beta_2}^2}{\eta^4} \leq \frac{N}{\eta^4}\text{tr}(A^{2\alpha_2})^{2/\alpha_2}\text{tr}(|T|^{\beta_2})^{2/\beta_2}$$

for any $\alpha_1, \beta_1, \alpha_2, \beta_2 \in [2, \infty]$ such that $\frac{1}{\alpha_1} + \frac{1}{\beta_1} = \frac{1}{\alpha_2} + \frac{1}{\beta_2} = \frac{1}{2}$. Hence, using Poincaré inequality yields

$$\begin{aligned}\text{Var}(zf_T) &\leq \frac{12}{N^2\eta^2}\left(\text{tr}(|T|^2)\text{tr}(A^2) + \frac{\mathbb{E}\left(\text{tr}((B\tilde{A}^2B)^{\alpha_1/2})\right)^{2/\alpha_1}\text{tr}(|T|^{\beta_1})^{2/\beta_1} + \text{tr}(A^{2\alpha_2})^{2/\alpha_2}\text{tr}(|T|^{\beta_2})^{2/\beta_2}}{\eta^2}\right)\end{aligned}$$

for such $\alpha_1, \beta_1, \alpha_2, \beta_2$. The proof of the first inequality is similar, since

$$\begin{aligned}\nabla_U(zm_H)(X) &= z\text{tr}(G_H[X, \tilde{A}]G_H) = z\text{tr}([\tilde{A}, G_H^2]X) \\ &= -\text{tr}([\tilde{A}, G_H]X) + \text{tr}((\tilde{A}(B + \tilde{A})G_H^2 - G_H^2(B + \tilde{A})\tilde{A})X),\end{aligned}$$

which yields

$$\mathbb{E}\|\nabla_U zm_H\|^2 \leq \frac{8}{N^2}\left(\frac{\mathbb{E}\|\tilde{A}\|_2^2}{\eta^2} + \frac{\mathbb{E}\|(B + \tilde{A})\tilde{A}\|_2^2}{\eta^4}\right).$$

First $\|\tilde{A}\|_2^2 = N\text{tr}(A^2)$, and then

$$\begin{aligned}\mathbb{E}\|(B + \tilde{A})\tilde{A}\|_2^2 &= N\mathbb{E}\left[\text{tr}\left((B + \tilde{A})\tilde{A}^2(B + \tilde{A})\right)\right] = N\mathbb{E}\left[\text{tr}(B^2\tilde{A}^2) + \text{tr}(\tilde{A}^4) + 2\text{tr}(B\tilde{A}^3)\right] \\ &= N\left(\text{tr}(A^2)\text{tr}(B^2) + \text{tr}(A^4)\right),\end{aligned}$$

where we used Lemma A.3 and $\text{tr}(B) = 0$ on the last equality. The result is then deduced using Poincaré inequality. \square

We give a similar result when the matrix T of the latter lemma also depends on UAU^* .

Lemma B.5. *Let $z \in \mathbb{C}^+$ and for $T \in \mathcal{M}_N(\mathbb{C})$ set $\tilde{f}_T = \text{tr}(TUAU^*G_H)$. Then,*

$$\begin{aligned}\mathbb{E}\left(|\tilde{f}_T(z) - \mathbb{E}(\tilde{f}_T(z))|^2\right) &\leq \frac{4}{N^2\eta^4}\left(\eta^2\left(\text{tr}(|T|^2)\text{tr}(A^2) + \sqrt{\text{tr}(|T|^4)\text{tr}(A^4)}\right)\right. \\ &\quad \left.+ 2\sqrt{\text{tr}(A^4)m_{|T|^2 * A^2}(1^2, 1^2)}\right),\end{aligned}$$

with the formula for $m_{|T|^2 * A^2}(1^2, 1^2)$ given in Lemma A.3.

Proof. Consider the map $\tilde{f}_T : U \mapsto \text{tr}(TUAU^*G_H)$. Then, writing $\tilde{A} = UAU^*$,

$$\nabla_U \tilde{f}_T(X) = \text{tr}(T[X, \tilde{A}]G_H + T\tilde{A}G_H[X, \tilde{A}]G_H) = \text{tr}([\tilde{A}, G_H T]X) + \text{tr}([\tilde{A}, G_H T\tilde{A}G_H]X).$$

Hence, by Hölder inequality,

$$\begin{aligned}\|\nabla_U \tilde{f}_T\|_2^2 &\leq \frac{1}{N^2}(\|\tilde{A}G_H T\|_2 + \|G_H T\tilde{A}\|_2 + \|\tilde{A}G_H T\tilde{A}G_H\|_2 + \|G_H T\tilde{A}G_H\tilde{A}\|_2)^2 \\ &\leq \frac{4}{\eta^2 N^2}\|T\|_4^2\|A\|_4^2 + \frac{4}{\eta^2 N^2}\|T\tilde{A}\|_2^2 + \frac{8}{\eta^4 N^2}\|T\tilde{A}\|_4^2\|A\|_4^2.\end{aligned}$$

Integrating on the unitary group yields then

$$\mathbb{E}\|\nabla_U \tilde{f}_T\|_2^2 \leq \frac{4\sqrt{\text{tr}(|T|^4)\text{tr}(A^4)} + 4\text{tr}(|T|^2)\text{tr}(A^2)}{N\eta^2} + \frac{8\mathbb{E}\left[\text{tr}(|T\tilde{A}|^4)\right]^{1/2}\text{tr}(A^4)^{1/2}}{N\eta^4}.$$

Remark that $\mathbb{E} \left[\text{tr}(|T\tilde{A}|^4) \right] = \mathbb{E} \left[\text{tr}(T\tilde{A}^2 T^* T\tilde{A}^2 T^*) \right] = m_{|T|^2 * A^2}(1^2, 1^2)$, whose formula is given by Lemma A.3. The results then follows by Poincaré inequality. \square

APPENDIX C. LIST OF CONSTANTS

We provide here a list of the constants involved in the main results together with their expressions. Recall the notations from Section 2.1 and Appendix A for notations involving moments of spectral distributions.

C.1. Constant involved in the estimation of $\widehat{\mathcal{C}}_B[\eta]$:

$$\bullet C_{thres,A}(\eta) =$$

$$\frac{12\sigma_B^2\sigma_A}{\eta^3} \left(1 + \frac{\sigma_A^2 + \sigma_B^2}{\eta^2} \right) \left(\sqrt{2 \left(1 + \frac{\sigma_A^2 + \sigma_B^2\theta_B}{\eta^2} \right) \cdot \left(1 + \sqrt{\theta_A\theta_B} + \frac{2\sqrt{m_{A^2*B^2}(1^2, 1^2)\theta_A}}{\sigma_B^2\eta^2} \right)} \right. \\ \left. + \sqrt{3 \frac{\sqrt{\theta_B\theta_A}\sigma_A^2}{\eta^2} \left(1 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2}a_4^{1/2} + b_6^{2/3}a_6^{1/3}}{\sigma_A^2\sigma_B^2\eta^2} \right) + 2 \frac{\theta_B^{1/4}\sigma_B^3\theta_A^{1/4}}{\eta^3}} \right),$$

$$\bullet C_{thres,B}(\eta) =$$

$$\frac{12\sigma_A^2\sigma_B}{\eta^3} \left(1 + \frac{\sigma_B^2 + \sigma_A^2}{\eta^2} \right) \left(\sqrt{2 \left(1 + \frac{\sigma_B^2 + \sigma_A^2\theta_A}{\eta^2} \right) \cdot \left(1 + \sqrt{\theta_A\theta_B} + \frac{2\sqrt{m_{A^2*B^2}(1^2, 1^2)\theta_B}}{\sigma_A^2\eta^2} \right)} \right. \\ \left. + \sqrt{3 \frac{\sqrt{\theta_B\theta_A}\sigma_B^2}{\eta^2} \left(1 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2}b_4^{1/2} + a_6^{2/3}b_6^{1/3}}{\sigma_A^2\sigma_B^2\eta^2} \right) + 2 \frac{\theta_A^{1/4}\sigma_A^3\theta_B^{1/4}}{\eta^3}} \right),$$

$$\bullet C_A(\eta) =$$

$$\frac{12\sqrt{6}\sigma_B^2\sigma_A}{\eta^3} \left(1 + \frac{\sigma_A^2 + \sigma_B^2}{\eta^2} \right) \sqrt{1 + \frac{\sigma_A^2 + \theta_B\sigma_B^2}{\eta^2}} \sqrt{1 + \frac{m_{A^2*B^2}(1^2, 1^2)^{1/2}a_4^{1/2} + b_6^{2/3}a_6^{1/3}}{a_2b_2\eta^2}},$$

$$\bullet C_B(\eta) =$$

$$\frac{12\sqrt{6}\sigma_A^2\sigma_B}{\eta^3} \left(1 + \frac{\sigma_B^2 + \sigma_A^2}{\eta^2} \right) \sqrt{1 + \frac{\sigma_B^2 + \theta_A\sigma_A^2}{\eta^2}} \sqrt{1 + \frac{m_{B^2*A^2}(1^2, 1^2)^{1/2}b_4^{1/2} + a_6^{2/3}b_6^{1/3}}{a_2b_2\eta^2}},$$

$$\bullet C_1(\eta) =$$

$$\left(1 + \frac{2}{\kappa^2} \right) C_B(3\eta/4) + \left(1 + \frac{C_B(3\eta/4) \left(1 + \frac{16(a_2+b_2)}{9\eta^2} \right)}{N^2} \right) \cdot \frac{1 + 2\sigma_B^2/\eta^2}{1 - 4\sigma_1^2/\eta^2} \\ \cdot \left(1 + \frac{4\sigma_1^2}{\eta^2} \right) \cdot \left(\frac{4}{3} + \frac{16\sigma_B^2}{9\eta^2} \right) C_A(3\eta/4) \left(1 + \frac{\sigma_B}{\eta} \right),$$

$$\bullet C_2(\eta) = \left(1 + \frac{C_B(3\eta/4) \left(1 + \frac{16(a_2+b_2)}{9\eta^2} \right)}{N^2} \right) \cdot \frac{1 + 2\sigma_B^2/\eta^2}{1 - 4\sigma_1^2/\eta^2} \cdot \left(1 + \frac{4}{\eta^2} \right) \cdot \left(1 + \frac{\sigma_B}{\eta} \right),$$

$$\bullet C_3(\eta) = 1 + \frac{8\sigma_1^2}{3\eta^2} + \frac{4\sigma_1^2}{\eta^2} \cdot \left(1 + \frac{C_B(\eta) \left(1 + \frac{16(a_2+b_2)}{9\eta^2} \right)}{N^2} \right) \cdot \frac{1 + 2\sigma_B^2/\eta^2}{1 - 4\sigma_1^2/\eta^2} \cdot \left(1 + \frac{\sigma_B}{\eta} \right) \\ \cdot \left(1 + \frac{16\sigma_H^2}{9\eta^2} \right),$$

$$\begin{aligned}
\bullet C_{MSE}(\eta, N) &= \frac{\sqrt{\pi}}{\sqrt{\eta}} \left(\frac{2C_2(\eta)C_{noise} [1 + 2(1 + c/N)(\sigma_1/\eta)]}{\eta} \right. \\
&\quad \left. + \frac{8\sqrt{2}C_3(\eta)}{3\eta} \sqrt{\sigma_A^2 + 4^2 \frac{\sigma_A^2 \sigma_B^2 + a_4}{3^2 \eta^2}} + \frac{C_1(\eta)}{N} \right), \\
\bullet K[\eta] &= \frac{C_1(\eta)}{N} + \left(\frac{2C_2(\eta)C_{noise} \left(1 + \frac{2(1+c/N)\sqrt{\mu_1(2)}}{\eta} \right)}{\eta} + \frac{2^{7/2}C_3(\eta)}{3\eta} \sqrt{\sigma_A^2 + 4^2 \frac{\sigma_A^2 \sigma_B^2 + a_4}{3^2 \eta^2}} \right), \\
\bullet R_0 &= \frac{N(\eta + \sigma_B^2/\eta)}{K[\eta]}, \\
\bullet C_{L^1}(\eta, N) &= \frac{2K[\eta]}{\pi} \left[1 + \frac{1}{\log N} \left(\log \left(\frac{\eta + \sigma_B^2/\eta}{K[\eta]} \right) + 1 \right) \right],
\end{aligned}$$

C.2. Constants regarding the classical deconvolution.

$$\begin{aligned}
\bullet \eta_0 &= 2\sqrt{2}\sigma_1, \\
\bullet t_0 &= \max \left(2\pi\eta_0, 256d\eta_0/5, 2 \exp(-2\pi/(pd)) \left[6M_p \sqrt{1 + 4\pi/dN} / (C_{L^1}(\eta_0, N) \log N) \right]^{1/p} \right), \\
\bullet M &= t_0/(d\eta_0), \\
\bullet C_\lambda &= \frac{e\sqrt{t_0}}{2^{5/4}\sqrt{\pi\sigma_1}} \exp(2\pi/d) C_{L^1}(\eta_0, N), \\
\bullet \lambda &= \frac{e\sqrt{t_0}}{2^{5/4}\sqrt{\pi\sigma_1}} \exp(2\pi/d) C_{L^1}(\eta_0, N), \\
\bullet C_t(N) &= \max \left(\max(2\pi, 256d/5)\eta_0 \left(\frac{\log N}{N} \right)^{1/p}, \right. \\
&\quad \left. 2 \exp(-2\pi/(pd)) \left[6M_p \sqrt{1 + 4\pi/d} / C_{L^1}(\infty, \eta_0) \right]^{1/p} \right), \\
\bullet \lambda &= \frac{e\sqrt{t_0}}{2^{5/4}\sqrt{\pi\sigma_1}} \exp(2\pi/d) C_{L^1}(\eta_0, N), \\
\bullet C_{sparse}(N) &= 2bd\eta_0^{3/4} C_t(N)^{1/4} \exp(\pi/d) \sqrt{C_{L^1}(N, \eta_0)} \\
&\quad + 2a(t_0^{-1/2}) C_t(N)^{3/2} \exp(2\pi/d) C_{L^1}(N, \eta_0) \left(\frac{\log N}{N} \right)^{1/2-5/(4p)}.
\end{aligned}$$

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