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# **Networking Phenomenology and Didactics: horizons of didactical milieus with a focus on Abstract Algebra**

**Thomas Hausberger<sup>1</sup> and Frédéric Patras<sup>2</sup>**

**Abstract:** Brousseau’s Theory of Didactical Situations (TDS) offers a general model for the analysis of didactical systems: in short, the students act upon a milieu which, in turns, informs or sanctions them. Our main idea is to combine TDS with tools from Philosophy, around the notion of horizon, in order to develop further the hermeneutical and phenomenological dimensions of the learner’s interaction with the milieu. The fertility of combining theoretical constructs from Philosophy and Didactics, in the spirit of networking theories as a research practice in Mathematics Education research, has been evidenced through previous research in which we draw connections between the didactic contract and the notion of horizon of expectation as developed by Jauss and the Constance School. Our analyses lead to a typology of horizons and its pioneering application in the context of the teaching and learning of mathematical structuralism at university level.

**Keywords:** Theory of Didactical Situations, Phenomenology, Husserlian horizons, Networking theories, University Mathematics Education, Mathematical structuralism

## **1 Introduction**

In his broad overview of the philosophy of mathematics education as a sub-field of mathematics education, Ernest (2018) emphasized the following characterization of philosophy as a discipline: “Philosophy is about systematic analysis and the critical examination of fundamental problems. It involves the exercise of the mind and intellect, including thinking, analysis, enquiry, reasoning and its results: judgements, conclusions, beliefs and knowledge”. To wit, philosophy is about knowledge and the mind’s access to knowledge and, as a consequence, there are many ways to apply philosophical

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concepts, results or methods to mathematics education research (MER). Among them, we feature firstly that one should adopt a “critical attitude” to claims, theories, methodologies of MER. Secondly, one should use contributions of philosophical domains (ontology and metaphysics, aesthetics, epistemology, ethics, etc.) and approaches to enhance theoretical development in MER as one cannot disentangle the ambition to offer a secure basis for knowledge from the very analysis of what knowledge is, should be, and how it can be acquired, in mathematical education theories as elsewhere.

Unfortunately, editorial constraints imposed by the main mathematics education journals, notably the standard format of an article, which must include analysis and interpretation of data, rarely allow time for discussion of the foundations of the theories that are applied, and for consideration of potential developments. In other words, the vocation of mathematics education to improve teaching and learning would pull the field towards a form of pragmatism that has little tolerance for the subtleties of language offered by what some reviewers call 'philosophical jargon'. Such use must show the full force of the results produced, or else that the same discourse cannot be held in a more common vocabulary without significant loss of nuance. This creates a strange situation, where the methods used by the science that aims to study the process of knowledge acquisition depart from the way science is usually built. Indeed, science in general does not progress primarily by experiments and data analysis, but by a combination of methods that run from theoretical constructs and research programs to actual experimentation. The interplay of practices, not a dogmatic and uniform approach, is the key to progress, also in MER. From our point of view, MER has a lot to gain from taking more advantage of philosophical writings, which we will try to highlight in this book chapter by taking up Husserl’s theoretical developments on the notion of horizon.

The inclusion of husserlian horizons in MER has already been proposed by Zazkis and Mamolo (2011) as a way to interpret the “knowledge at the mathematical horizon” (KMH; Hill et al., 2008). We will begin by briefly presenting this work and show, in the “critical attitude” of Philosophy, how, although relevant in the context of teacher training, it moves away from Husserl’s project. It also presents limitations when it is a question of carrying out a more advanced analysis of the cognitive processes at stake in learning a topic such as Abstract Algebra.

In the following sections of the paper, we come to the core of our contribution, which is mainly theoretical in nature, and present how we have articulated

phenomenology and mathematics education in order to study the manifestation and acquisition of structuralist thinking in groups of university students. We will rather say “didactics of mathematics” since the main theory considered in the sequel, Brousseau’s (1997) theory of didactical situations (TDS), takes its origin in the French tradition of the field (Artigue, 2019).

On the methodological level, our work may be described as a form of networking of theoretical frameworks (Bikner-Ahsbabs & Prediger, 2014). The same phenomenon in mathematics education, namely how students solve a given problem in Abstract Algebra, can be analyzed from the perspectives of both husserlian phenomenology and Brousseau’s theory. The networking of didactical theories is a research practice allowing the combination of complementary insights. It also leads to the linking of theories at different levels and by means of different strategies (by comparison, contrast, synthesis, local integration or more). In our case, the joint analysis (first stage: comparing/contrasting) of the data led to the identification of common features (second stage: combining/coordinating) between Brousseau’s didactic contract and Jauss’ horizon of expectation in Hermeneutics (Hausberger & Patras, 2019; Hausberger, 2020).

In a third and more advanced stage of networking, *synthetizing* and *integrating locally* are relevant concepts whenever theoretical development is aimed at. This book chapter is a first step in this direction as it aims at a local integration of theoretical constructs of phenomenology to supplement TDS. To do so, we will connect the notion of *milieu* in TDS with that of *world* in Phenomenology, and draw further connections around the notion of horizon. Although the richness of the notion has not been completely taken advantage of in our previous work, we will not come back here to the horizon of expectation, but focus mainly on Husserl.

His philosophy is thus contributing to unraveling the hermeneutical and phenomenological dimensions of the learner’s interaction with the milieu. Key is the learner’s intentionality. Intentionality, in a phenomenological sense, does not refer here merely to intentions (goals such as acquiring understanding or insights, for example), but in a subtler way to the structures of conscience underlying the relationships between the individual and the world. This idea of intentionality comes from a scholastic notion, which Husserl inherited through Brentano. In medieval philosophy, *intentio* referred to the application of the mind to an object. We owe it to Husserl to have made it a foundation of Phenomenology. Intentionality has multiple forms and accounts for

example for both the gaze we direct on the surrounding things and the theoretical gaze we have on mathematical objects. All these dimensions contribute to make it a central idea for the use of phenomenology in didactics – hereafter, we will focus on intentionality in relation to the structures of consciousness underlying the relationships between the learner and the milieu. This philosophical analysis at the level of principles will serve as a background to analyze, in a second step, how these ideas unfold to grasp key aspects of elaborated theoretical knowledge in Abstract Algebra. This is where the key notion of horizon in the sense of Husserl comes into play and supplements the more general one of intentionality, as horizon structures are indeed structures of intentionality.

Our theoretical elaboration will be illustrated in the last section of the paper through the analysis of excerpts of a dialogue between a pair of advanced students (PhD level and beyond) engaged in solving Abstract Algebra tasks. We will unveil a large spectrum of horizon types, without attempting to be exhaustive, featuring in particular a richness and complexity that depart from the descriptions offered by Zazkis and Mamolo's interpretation of horizons in the teacher education context.

## **2 Horizons in teacher education**

Zazkis and Mamolo (2011) focus on the “knowledge at the mathematical horizon” (KMH; Ball & Bass, 2009), a component of the subject-matter knowledge, in the classical sense of Schulman, which designates (primary or secondary) teachers' advanced mathematical knowledge (from university or college) that may prove useful in teaching at school. Their interpretation is driven by the metaphor of horizon as a place “where the land appears to meet the sky” and the distinction between inner and outer horizon, after Husserl. Whereas the inner horizon corresponds to “aspects of an object that are not the focus of attention but are also intended”, the outer horizon represents the “greater world” in which the object exists. Zazkis and Mamolo connected these two types of horizons to the first two components of KMH, respectively: the surroundings of the current topic under study, and “the major disciplinary ideas and structures”. Subsequently, Mamolo and Pali (2014) attempt to add in their descriptions horizon knowledge related to practices and values, in other words to account for the remaining two components of KMH: “key mathematical practices” and “core mathematical values and sensibilities”. But they didn't draw further connections with phenomenology. Their

goal is to study how these horizons may impact teacher's actions in teaching situations.

As a main case study, Mamolo and Taylor (2018) exemplify connections between Abstract Algebra content (a part of the “blue sky”) and secondary school content. Although numerous examples are provided, by pointing out to studies in the volume contributed to, these examples rely on a similar schema: the object attended to is part of the school curriculum and its outer horizon, inside Abstract Algebra, consists of the “generalities which are exemplified in the particular object” (Zazkis and Mamolo, 2011, p. 10). In Philosophy, this is called a type-token relationship and it certainly doesn't exhaust the possible types of relationships, as we will see at the end of this paper.

To summarize, in this approach, the focus is how Abstract Algebra understanding may influence decision-making in teaching situations at school. The analysis of intentionality does not aim at relating abstract structures to lower-level mathematical objects in mathematical practices. Nor does it aim at shedding light on the ways and means a consciousness interacts with abstract mathematical objects (and thus achieves learning in Abstract Algebra). In particular, horizons of objects belonging to Abstract Algebra (horizons *inside* the “blue sky”) are not considered, whereas they will be central in our work that focuses on higher education teaching and learning.

### **3 Modeling teaching-learning phenomena**

The Theory of Didactical Situations (TDS; Brousseau, 1997) offers a general model and tools for the analysis of any didactical system: the main point is that a learner interacts with a *milieu* shaped by the teacher, according to a *didactical contract*. Learning is then asserted when the adequate adaptation to the milieu may be observed in the student.

Precisely, the didactical contract designates the “system of reciprocal obligation” that determines “explicitly to some extent, but mainly implicitly – what each partner, the teacher and the student, will have the possibility for managing and, in some way or another, be responsible to the other person for” (Brousseau 1997, p. 31). It is expected from the milieu to be antagonistic in the sense that it will provide retroactions (to the students' attempts to solve the problem) and allow the target knowledge to emerge due to the “internal logic of the situation”. At this stage of the learning process, the milieu is *a-didactical* in the sense that students shall experience an “absence of [direct]

intentional direction” (didactical intention). The new knowledge acquires the status of a piece of the mathematical text at the later didactic phase of institutionalization by the teacher.

In fact, Brousseau distinguishes different patterns of situations, which are usually integrated in a sequence: an action pattern, in which students act on a material milieu; a formulation pattern, which aims to make explicit the students' “implicit models of action”; and finally a validation pattern, in which a debate is organized to discuss the truth value of the students' findings. Brousseau refers to cognitive psychology when it comes to identifying these implicit models of action and understanding their role in the acquisition of knowledge. He mentions the conceptual field theory (TCF) programme of Vergnaud (1990), and others can be cited (Dubisky's APOS theory, Tall and Vinner's theoretical construct of concept-image, etc.). If such a study falls within the scope of the interactions between psychology and didactics, our aim in this article is to show what an interaction between phenomenology and didactics can bring to shed light on the psychogenesis of concepts when a learner is confronted with an a-didactical milieu.

This is where the notion of horizon comes into play. Before presenting husserlian horizons and their connections to Brousseau's theory, we need to introduce other works that extend TDS on some aspects that may be related to phenomenology. In a pioneering paper, Brousseau and Centeno (1991) investigated how teachers handle the temporary and transient knowledge of pupils to promote learning. They called *didactic memory of the teacher* the knowledge that teachers may evoke on purpose to reactivate and facilitate the transformation of previous knowledge towards the target knowledge. Flückinger (2005) combined the perspective of TDS with TCF to study how students' numerical knowledge on division evolved through the construction of schemes connected to classes of situations partly organized by the teachers and partly emerging as new knowledge in the conceptualization process. She called such a feature the *didactic memory of students* since responsibility for memory processes has been partially devolved to students through a specific didactical contract: for instance, it is the students' responsibility to decide which objects of knowledge are the most pertinent to handle the assigned problems. We argue that the notion of horizon is a tool to capture features of the interaction of the students with a-didactical milieus and will give evidence of its relevance to analyze the evolution of forms of knowledge from implicit models of action to their explicitation (formulation) and then to a path towards a formal proof (validation).

#### 4 The horizon according to Husserl

“We assume, then, that the construction of meaning, as we understand it, implies a constant interaction between the student and problem-situations, a dialectical interaction (because the subject anticipates and directs her actions) in which she engages her previous knowings, submits them to revision, modifies them, completes them or rejects them to form new conceptions. The main object of didactique is precisely to study the conditions that the situations or the problems put to the student must fulfill in order to foster the appearance, the working and the rejection of these successive conceptions” (Brousseau 1997, p. 83).

In the global project summarized by Brousseau’s quote, two questions will retain here our attention and govern our approach to Husserl’s ideas in an a-didactical context. First, how to describe the modalities of interactions between the student and the milieu? This can be done at two levels: firstly, a functional, descriptive level based on experience or observation. For example, a group of students can start playing a game naively to “see how it works” and decide later on a protocol to look for an optimal strategy, or look immediately for a strategy, or mix the two approaches in various ways that the teacher can observe and partially expect. Achieving such a description is important because it can allow the concrete engineering of didactical situations. However, the question can be addressed at the higher level of principles: why is such a thing as the interaction between a student and a milieu possible? What are the available tools to speak of such a thing? How can it be described in a way that will allow didactics to explain and theorize the corresponding processes?

The other question, closely connected in our opinion, as we shall see, is how do previous knowledge play a role in this interaction? Of course, we know practical answers: for example, these knowledge are the tools that will allow her to grasp and analyze the problems. But, once again, at the level of principles, the question is harder to treat: why is it so, for example, that a student will be lead to use induction to solve a counting problem and not a direct argument (a bijection with a set of known cardinality, for example)? How is such a thing as a path of successive guesses, modifications, completions of knowledge possible? Or, more precisely, in what space of cognitive actions, theoretical behaviors, does this path live?

Let us consider the milieu from a phenomenological point of view. Recall from



Brousseau's Glossary (2010) that "a situation is characterized in an institution by a set of relations and reciprocal roles of one or more subjects (pupil, teacher, etc.) with a milieu, aimed at transforming that milieu according to a project. The milieu consists of objects (physical, cultural, social or human) with which the subject interacts in a situation."

In the phenomenological language, the subject interacts with a world. Most of the time, this world is the natural world, the *Lebenswelt* (the world of life). The key role of the *Lebenswelt* for Phenomenology was emphasized by Husserl in various texts, two of the most relevant for us here being the *Krisis* (*The Crisis of European Sciences and Transcendental Phenomenology*, Husserl, 1954) and his contemporary essay on the *Origin of Geometry*. A key thesis, defended in both texts in different forms, is that modern mathematics ultimately refers to a proto-foundation in a system of original evidences whose origin is to be found in our immediate relationship to the world (that is, to the *Lebenswelt*). Mathematical ideas thus have a complex historicity, which is not only the result of their history, but also of this necessary reference to fundamental intuitions. These ideas are extremely important, and of considerable significance for didactics, but we will not go down that road here: instead, we will emphasize the role of the *Lebenswelt* in the constitution of the horizon of mathematical objects and concepts in the classroom.

Another important observation is that, however important the *Lebenswelt*, it is not the only "world" we can be embedded in or interact with: "I can for example also occupy myself with pure numbers and laws of numbers. The world of numbers is also there for me; it constitutes precisely the field of objects where the activity of the arithmetician takes place. During this activity, she will focus on some numbers or numerical constructions surrounded by an arithmetical horizon, partially determined, partially undetermined." (Husserl, 1913, [51], our translation)

A key step when interacting with a world, whatever it is, is to change attitude: the same person can behave naturally and interact with her *Lebenswelt*, her surrounding world, or switch to a theoretical attitude and behave as an arithmetician, or a student in arithmetics. The corresponding world will then be shaped differently. A milieu in the didactical sense can be thought of as a particular kind of world. It exists as a milieu precisely because the student (or the teacher) adopts towards it the right attitude. For example, scissors, a pen and a sheet of paper can constitute a milieu suited for elementary Euclidean geometry, or be simply the tools given to a kid to play.

In Phenomenology, a world cannot be disentangled from its horizon. The horizon is, roughly stated, the configuration of possibilities, meanings, tools, intentions that shape the world/milieu. The horizon is at the same time what makes the dynamical and constructive interactions between the subject and the world possible and the “place” where they occur. A feature of the a-didactical horizons we will consider is that the didactical memory of students is a key ingredient in their constitution and structuration. When the young student considers numbers and properties of numbers, she may already know that there are operations she can perform: addition, subtraction, multiplication. She also maybe knows that there are more complex operations like division or exponentiation that she remembers only vaguely, and she knows that using them would require some care. Lastly, she maybe has learned more advanced ideas, for example the reasoning by induction, but at the moment does not connect this knowledge to numbers, although she could remember it at some stage of a reasoning. These operations, some clearly determined, some still undetermined or under-determined, are one component of the horizon of numbers. They are also tools that I can use to reshape the current milieu. For example, I can transform the problem of computing  $(6+7)*2$  into the problem of computing  $13*2$ . But in the horizon of possibles shaping my interactions with  $(6+7)*2$  other paths of reasoning would be possible, for example its transformation into  $(6+7)+(6+7)$ . Here again we can observe constructive interactions between phenomenology and didactics. The notion of didactic memory, together with its theorization and documentation on classroom experiments can enrich phenomenology by documented examples, where the behavior of students can be analyzed. Conversely, phenomenology enriches the didactical theory with its precise tools of analysis, especially of the theoretical endeavors and ideas formation in the context of interactions between an individual and the world.

Studying the milieu from the phenomenological perspective leads to answers to our initial questions. The horizon is a locus where interesting, dynamical, transforming interactions with the milieu take place. Prior knowledge are some of the components of the milieu, they also contribute to the shaping of the horizon and to the action on the milieu. The horizon and the didactic memory are certainly not exhausting the analysis of the interactions with the milieu, but our thesis is that they are an important constituent that allows us to understand various important didactical phenomena.

## **5 Towards a typology of horizons**

In cognitive sciences, Jorba (2020) argues that the perceptual intentional horizon in Husserl's phenomenology, besides being a general structure of the experience, extends to a viable notion of cognitive horizon that relates to affordances (possibilities of action present in experience). She proposes "to characterize a specific structure of the cognitive horizon – that which presents possibilities for action – as a cognitive affordance. Cognitive affordances present cognitive elements as opportunities for mental action (i.e. a problem affording trying to solve it, a thought affording calculating, an idea affording reflection)." (p. 847) Following Husserl, she also features various types of horizon structures that we will use later to characterize several structures showing up in (a)-didactical experiments. We detail their content, building on her analysis.

The inner horizon accounts, in the phenomenology of perception, for the various ways in which I can have access to an object : "Every experience has its own horizon... this implies that every experience refers to the possibility... of obtaining, little by little as experience continues, new determinations of the same thing. (...) Thus every experience of a particular thing has its internal horizon" (Husserl 1973, §8: 32, quoted in Jorba 2020). In didactics, we propose to use the notion as referring to the various access I can have to an object (a notion, a concept...) that are directly contained, either in the object itself (for example as direct consequences of its definition or as properties of its components), either in a given milieu. Here, "given" refers to the components of the milieu that go immediately with the (a)-didactical situation. Notice that this is a subtle notion. Whereas the inner horizon of a spatio-temporal object or being amounts simply to the various experiences I can make – for example by turning around a building, visiting it, seeing its roof from an airplane, etc. – the inner horizon of a theoretical object such as a mathematical one highly depends on the way this object is given. A sphere defined using the classical axioms of Euclidean geometry can be identified with an object in the space  $\mathbf{R}^3$  equipped with a positive definite quadratic form, but the (technical, conceptual, methodological) horizons that go together with these two definitions are quite different. In other terms, this notion of inner horizon also depends on the learner's background.

Outer horizons refer instead to the possibility of putting an object (notion, concept,...) in relation to other objects or in another context (Jorba 2020, p. 849ff). In our previous example, quadratic forms, metrics, scalar products belong to an outer

horizon of spheres in naive Euclidean geometry. Phenomenology itself enters the scene by providing theoretical tools to analyze how horizons structure the relationships of a consciousness to its objects, whatever they are.

This phenomenology of horizons is further enriched by two families of relations. In our didactical context, we propose to call *associative* outer horizons those relations based on relating two objects, two notions in a non straightforward way (once again, what straightforward means will depend on the learner's background). The example of spheres and quadratic forms can be analyzed that way, for example. Instead, *inferential* outer horizons will denote relations acquired through reasoning, provided new elements, notions, ideas, insights result in this process. For example, the late 19th century insight that a finite set is a set that cannot be put in bijection with a proper subset could be analyzed that way: appealing implicitly to the infinite to define finiteness, besides being counter-intuitive, requires upgrading the horizon of finite collections through a process that, at least at the very beginning, relies more on technicality and reason than intuition.

Turning back to Jorba's general program, we agree with her analyses relating cognitive horizons and cognitive affordances, and point out that the didactic approach to Jauss' notion of horizon of expectation in (Hausberger and Patras, 2019) goes in the same overall direction. Many features of phenomenological horizons of the *Lebenswelt* actually translate into features meaningful in a didactical context. For example : "Every cogito, an external perception or a remembering, and so on, for example, carries with itself in a detectable manner an immanent potentiality: the one of possible life experiences, linked to the same intentional object, that the self can realize [...]. In each cogito, we discover horizons." (Husserl, 1950, p. 18<sup>3</sup>). They induce potentialities, in the natural behavior, for example the possibility to turn my head to the left to discover new components of the countryside, or in arithmetics, the possibility to perform first a sum or a product in a given formula, with some priority constraints on the operations that contribute to shape the horizon of possible arithmetical actions.

An important point, that we will start to develop implicitly in the present article is that the notion of horizon is not a vague concept that would allow to speak of certain phenomena without giving conceptual and methodological tools to investigate their properties and structure: "I can investigate an intentional experience, which means that I can penetrate its horizons, interpret them and, that way, unravel potentialities of my life

<sup>3</sup> On the phenomenological definition of the horizon and its fundamental properties, see also op. cit., p. 82.

and, on another side, clarify, at the objective level, the targeted meaning [Ibid]”.

Taking again the elementary example of arithmetical operations, the expression  $(2+3+4)*(2+1)$  can be transformed into  $(5+4)*(2+1)$ ,  $(2+7)*(2+1)$ ,  $(2+3+4)*3$ , and so on. These potentialities are all open and part of the horizon of the expression. They are the beginning of paths that will lead the student (hopefully) to 27. When I analyze the structures underlying these potentialities, key ideas of arithmetics will show up if I push the analysis to its limit. For example, the equivalence of the first two transformations - which is not obvious – points out at the associativity of addition, an highly sophisticated notion that, in its modern, structural interpretation, appeared relatively late in mathematics (Leibniz, Grassmann,...).

The second question we raised in the beginning of section 4 (how does prior knowledge impact the interaction of the student with the milieu?) has started to be addressed by noticing that knowledge is a key component of the structure of the horizon and by pointing out at the relevance of didactic memory in our context. Let us expand briefly on this and make these observations concrete. If I already know what associativity and distributivity mean, I will be able to devise more complex strategies to solve equations and will be much more confident on their validity. However, the question relates to a very general feature of intentionality that goes beyond the particular case of didactic memory, namely the fact that each life experience has an horizon of anteriority (my past experiences and the memory I have of them). This horizon of anteriority has several components. Short term memory is important, for example, when solving a problem. The ideas and results I just obtained contribute to shape my current understanding of the problem in its present state. In the French educational system, this phenomenon is illustrated by a marked difference between exercises, usually focusing on a few directly related questions, and problems, much longer and where drawing connections between arguments in different parts of the problem is essential to its solution. In such situations (exercise, problem) where didactical and a-didactical components are mixed (depending on the reliance of the solution on already acquired skills), an horizon is constructed largely internally to the situation – in the sense that it is shaped by previous answers. Long term memory impacts differently the interactions of the student with the milieu. For example, recognizing certain prototypical features of a question (for example, to perform a computation involving sums and products) will lead her to use the priority rules and distributivity laws for arithmetical operations that she had learned some time ago and

had remained before one among the many and largely indistinct components of the horizon of the problem.

In conclusion, horizon, didactic memory, and their constructive interactions can be understood and documented in many ways. We will focus now on a specific example in order to illustrate the fertility of our theoretical ideas on concrete empirical data.

## **6 Application to Abstract Algebra**

The purpose of this section is to study, in the spirit of Husserl, the learning processes of advanced students engaged in solving a mathematical problem in Abstract Algebra: the theory of banquets. Cognitive processes will be explored using the lens of phenomenology with the notion of horizon as the main tool: progresses in solving the problem are thus related to changes in the horizon structure which potentially result in new cognitive affordances.

Throughout our analysis, the main questions will therefore be: which is the main intentional object (or noema) that consciousness is focusing on in crucial moments of the mathematical experience? What is the underlying motivation structuring intentionality and, more generally, what is structuring its noetic moment: the way mathematical conscience is conscience of... intuition of... grasping of...? How are inner and outer horizons of intentional objects structured by the learners' interpretation of the milieu and background knowledge (or didactic memory)? We will rely on language and other semiotic representations produced by learners as warrants for our claims; moreover, the chronology of reasonings makes it possible to detect partly implicit features of cognitive anticipation of the horizon through the evidence of how the horizon unfolds in subsequent cogita.

Let us now present the problem. Mathematical structuralism has had a large impact on contemporary mathematical practices (Patras 2001) but also on modern didactics of mathematics. Various members of its founding fathers have indeed been strongly influenced by the problems that arose together with the emergence of "modern maths" where abstract axiomatic structures serve, especially in algebra, as organizing principles in the exposition of mathematical theories and as tools to pose and solve mathematical problems. Pre-structuralist theories about numbers, polynomials and other standard mathematical objects appear as a background to motivate and apply the abstract unifying and generalizing point of view of structures. Structures also give rise

to new questions: which identity principle to adopt (which are the natural morphisms between objects of a given type of structure)? How to classify objects up to isomorphism? Which structuralist theorems govern the decomposition of objects into simpler ones? As a piece of didactic engineering (Artigue, 2009), the theory of banquets (Hausberger, 2020; Hausberger, in press) has been designed in order to tackle these kinds of questions in the context of an Abstract Algebra course at the transition between undergraduate and graduate studies in pure mathematics. The main prerequisite is a course in Group Theory, so that students have already encountered similar structuralist questions and results that will be thematized in the context of banquets.

A banquet is a set  $E$  endowed with a binary relation  $R$  which satisfies the following axioms: (i) No element of  $E$  satisfies  $xRx$ ; (ii) If  $xRy$  and  $xRz$  then  $y = z$ ; (iii) If  $yRx$  and  $zRx$  then  $y = z$ ; (iv) For all  $x$ , there exists at least one  $y$  such that  $xRy$ .

In part I.1 of the worksheet, students are asked the following questions:

1 a. Coherence: is it a valid (non-contradictory) mathematical theory? In other words, does there exist a model?; b. Independence: is any axiom a logical consequence of others or are all axioms mutually independent?

In part I.2, they are asked to classify banquets of small cardinalities and link banquets of order 4 with their knowledge in Group Theory (in particular with the cyclic group of order 4). The abstract/concrete relationship is reversed in part II of the worksheet, which begins with the empirical definition of a table of cardinal number  $n$  as a configuration of  $n$  people sitting around a round table. Its aim is to prove that any banquet decomposes as a disjoint union of tables (the “structure theorem”). We won’t give more details here since excerpts of students’ work that will be analyzed are restricted to part I.1 as we prefer to insist on our method, its significance and concrete use than on all the conclusions that can be drawn from experiments on the theory of banquets.

The banquet structure possesses a large variety of models since the system of axioms may be interpreted in quite different worlds, beginning with the empirical interpretation of guests sitting around tables (whence its name):  $xRy$  if  $x$  is sitting on the left (or right) of  $y$ . Other domains of interpretation include Set Theory (the binary relation is represented by its graph), Functions ( $xRy \Leftrightarrow y = f(x)$  defines a function  $f$  according to axioms (ii) and (iv); the other two axioms mean that it is injective without fixed points), Permutation Groups ( $f$  is a bijection when  $E$  is finite, in other words a permutation without fixed points) or even Matrix Theory and Graph Theory (see

Hausberger, 2021, for a full mathematical analysis). The structure theorem of banquets thus corresponds to the well-known theorem of canonical cycle-decomposition of a permutation, but the analogy remains hidden since the binary relation of banquets is different from binary operations that define groups. These remarks explain why the theory of banquets is mathematically rich but may not be found in any textbook (it is equivalent, in the finite case, to permutation groups). Moreover, it is a simpler theory (in the sense of mathematical technicality) than Group Theory and it carries the underlying intuition and mental image of guests sitting around tables (a wedding banquet).

## **7 Horizons of the abstract structure of banquets**

Question I.1 of the worksheet (coherence; existence of a model; independence of axioms) may be regarded as a first situation in the sense of Brousseau, dedicated to logical analysis. Its milieu contains the axiomatic definition of the banquet structure, the concept of model of a system of axioms and the language of Set Theory.

The intentional object is, in general, the object (of senses, or abstract, theoretical...) towards which consciousness is directed. In this exercise, the main intentional object is the definition of banquet. At any moment during the solution of the exercise, this consciousness and the attention given to the axiom system is embedded into various horizons. The important point is that these horizons are not fixed: every time consciousness is going to be directed towards a particular feature of the axiom system, new horizons will present themselves as surrounding this state of consciousness. On the other hand, taking into account the presence of these horizons will help students to progress and understand the axioms in different ways, so that consciousness itself will evolve accordingly.

What we claim here is simple, but essential and too often forgotten by authors appealing to Phenomenology as a method of philosophical investigation: one can describe the process of thinking by investigating such phenomena. Comprehension of learning in particular is a topic particularly well-suited to such analyses. Our claim is that they help understand the didactical processes and could also be useful in didactical engineering by providing tools to analyze what steps students are expected to perform to reach a satisfactory construction of knowledge.

In the first part of the exercise, the investigation of the meaning of the definition



of banquets goes through the logical investigation of the system of axioms (coherence and independence) using a semantic approach (construction of models). The related work of the two (very advanced) students, called Alice and Bob hereafter, took the form of a dialogue that has been registered and transcribed. It is made of a sequence of 31 speeches. The integrality of the dialogue can be found in Hausberger (2016, annex 4), we use here only some parts to illustrate and support our analysis. Numbers indicated below in front of Alice or Bob statements correspond to the position of the statements in the dialogue: (5) will refer to the fifth speech among 31, and so on, so as to indicate the progression of the argumentation.

Concretely, investigating the meaning of the definition could usually be done in three ways:

- appeal to prior knowledge (their own didactic memory, entangled with the teacher's didactic memory);
- try to grasp directly the meaning of the axioms (with some training it is indeed possible to have a purely formal understanding of algebraic axiomatic systems);
- explore empirically the axioms' content.

In general, mathematical thinking is a blend of several such processes. Each approach goes together with distinct intentional modalities. We will try to account for those that appear in the two students' dialogue. It will appear that several successive horizons may be uncovered and disentangled.

1. First horizon (inner): theoretical memory. As a first attempt, students try to use direct knowledge on binary relations (antisymmetry, irreflexivity) to make sense of the axioms. They appeal therefore to didactic memory in one of its simpler forms, that we may call formal or theoretical: going back to the known properties of the objects and notions under consideration.

(1) Alice: Classical, we specify the structure through relations, okay.

(2) Bob: Antisymmetry [about axiom (i)].

Our memories shape horizons of possibilities and horizons of understanding. In her statement, Alice explicitly acknowledges the idea that a structure can be defined through relations and that such a fact belongs to classroom knowledge. In Brousseau's language, this idea has been already institutionalized, it is contained in the paramathematical<sup>4</sup> concept of structure and is also based on the

<sup>4</sup> Paramathematical concepts are "named objects whose characteristics are studied but which have, for various reasons, not yet been organized and theorized, such as the notion of function in the 19th century, or that of equation in the 16th century, or that of variable in the 20th" (Brousseau 1997, p. 59). The

notion of a binary relation taught in Set Theory. Recognition of institutionalized knowledge is essential, it provides a ground on which further advances can be made.

2. Second horizon (outer): natural semantics. Interestingly, this first (inner) horizon of the banquet structure soon leaves place to a quite different horizon evoked by the name of banquet:

(3) Alice: there's one guy on the right and one on the left, that's the idea; there's nobody sitting alone at a table.

This second horizon is thus driven by natural semantics, empirical knowledge, and more generally our embedding in a *Lebenswelt*: the mental image of banquets acquired from perceptual experience. The theory is embedded in a wider, extra-mathematical, context. Here, didactical engineering is involved since the name of the theory is a main didactical variable (in the sense of TDS) of the situation. By the name « theory of banquets », the instructor has chosen to drive the learners towards a certain type of models and intuitions – he enforced the building of a specific outer horizon.

We emphasize that “natural semantics” refers here to the fact that students give a meaning to the theory of banquets by a “fulfilling of intentions of signification”, in the language of Husserl. By referring to daily life situations, the theory becomes concrete and can be grasped: an element of the set  $E$  is now “a guy at the table”. The desire to associate a meaning to the axioms (intention of signification) starts to be fulfilled.

3. Third horizon (outer): associative. The second horizon is subsequently augmented with knowledge from elementary set theory and logic to give rise to a third horizon with powerful cognitive affordances to construct models and check the validity of statements. As it is based on relating two horizons – the outer one of natural semantics of banquets and the inner one of set theory –, the new horizon is associative:

(5) Alice: To show that it's not contradictory, you can show that there exists a model. I suggest we take one guy. No, one guy doesn't work, 2 guys sitting next to each other. [...]

(7) Alice: Let's take  $E=\{a,b\}$  and for the relationship the couples  $(a,b)$  and  $(b,a)$ . So it is indeed a model. [...]

(9) Alice: Yes, a set with 2 elements, they are sitting opposite each other... obviously, there is at most one on the right and one on the left, they are in relation with the one opposite.

students have not been taught Category Theory, the mathematical framework that aims at theorizing the notion of structure, but Alice has been introduced to Model Theory in her studies.

Here, Alice makes explicitly a move from natural to formal semantics. She uses theoretical memory to relate the non contradiction of axioms with the existence of a model but appeals then to the idea of people around a table to build a model. In speech (7), Alice and Bob have obtained a first mathematical statement: the axiom system has a model and is consistent. Speech (9) interestingly confirms the formal, mathematical, sentence by a translation into natural semantics.

4. Fourth (outer) horizon: inferential. The dialogue proceeds with some easy arguments on independence that will be omitted. Later, as they stumble on a difficulty to deny (ii) while keeping other axioms (that is, when trying to prove the independence of axiom (ii) from the others), Alice feels the need to produce another interpretation of banquets:

(15) Alice: So this thing, it's nice... there are some and at most one, so this thing, it's a function. To  $x$  we associate the unique  $y$  such that  $xRy$ . And we have the injectivity a priori.

The relationship with functions is thus the main component of a fourth horizon that may be qualified as both outer and inferential since it involves several concepts not directly related to the axiomatic system (multivalued functions – Bob mentions for example the possibility of two images of an element –, injectivity) and results (equivalence of injectivity and bijectivity for functional relations between sets of same finite cardinal number), and leads to a breakthrough:

(26) Alice: Perhaps an infinite set is needed, it is possible.

(27) Bob: I have the impression that this is not possible.

(28) Alice: It's a bijectivity thing that makes you need an infinite set.

A formal proof of the necessity of infinite cardinality is not produced, but what they achieve is enough for the production of the counterexample they were looking for.

The experiment we have treated allows us to reach several conclusions. The solution to an exercise is a dynamical process. Understanding it requires the understanding of how the students' thoughts evolve and move forth and back from the object under investigation to a series of insights, some of which are given with the problem (inner horizon: the acquired knowledge directly related in that case to relations and axiomatic systems), some others have to be found in relation to outer horizons that unravel progressively.

We feature once again that, in spite of a common reference to Husserl's horizons, our approach is much more general than a mere type-token analysis, as it

appeared for example in Zazkis and Mamolo. Indeed, whereas the latter is restricted to understanding the subordination of a given mathematical object or problem to a more advanced and general theory, our use of horizons gets into the very dynamic process of knowledge-building in the classroom. The semantical aspects involved in the idea of horizons are a good illustration of the generality of husserlian use of intentionality and horizons.

## **8 Conclusion and perspectives**

The main contribution of this paper is the further development of husserlian horizons, first introduced in a didactical context by Zazkis and Mamolo (2011), as tools to analyze the shifts of attention and interconnectedness of knowledge in learners attending to an abstract structure. Our extension encompasses a larger spectrum of horizons and methods in a pioneering application in the context of university mathematics education, allowing for a fine-grain analysis of the work of learners engaged in the elaboration of a structuralist mathematical theory around the given structure.

The features of horizons that we managed to identify in relation to the manifestation of structural sense among a pair of advanced students are but a first step in understanding the genesis of structuralist thinking in educational contexts. At the theoretical level of frameworks, we contribute by combining/coordinating notions from TDS with the perspective of phenomenology, in the spirit of networking. We believe that such a dual framework may be applied in a large variety of contexts and educational levels. We also point out the coordination with studies in cognitive sciences. These links should be investigated further in subsequent research.

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