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# 1. Fostering inquiry and creativity in Abstract Algebra: the theory of banquets and its reflexive stance on the structuralist methodology

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## 1.1 Introduction

Abstract Algebra designates in this chapter an upper division undergraduate course typically required for mathematics majors and centered on the structures of groups, rings, and fields. It is a subject generally recognized as challenging by students whose difficulties, in particular regarding abstraction processes and the acquisition of “structural sense”, are well documented in university mathematics education research. As far as teaching is concerned, it seems that instructors rarely adopt new pedagogical approaches (Fukawa-Connelly, 2016): lecture is predominant and tutorials are often dedicated to working out standard examples and basic techniques that leave little room to students’ creativity. A few design-based instructional approaches have been experimented, but these focus primarily on the teaching and learning of Group Theory so that research in this area remains limited. Altogether, Abstract Algebra “offers many challenges to researchers in order to develop inquiry-based approaches that may promote adequate conceptualization and understanding” (Hausberger, 2018b).

In fact, the structuralist point of view inherited from German algebraists and systematized in the mid 20<sup>th</sup> century by the Bourbaki group shaped mathematics as a research field to give it its contemporary face (Corry, 1996). As explained, this contrasts with the reality of mathematics classrooms where this powerful and insightful vision struggles to be transposed. What kind of classroom activities may be envisaged to foster inquiry and creativity in Abstract Algebra learning, in line with genuine mathematical research practices?

This chapter centres around the “theory of banquets” (an invented structure outside the standard Abstract Algebra syllabus). This theory was elaborated to facilitate students’ access to structuralist thinking at large through the use of the meta-lever (Hausberger, 2021). The study has been carried out with the methodology of didactic engineering (Artigue, 2014): the activity has been designed, implemented and analyzed using the Theory of Didactic Situations (TDS; Brousseau, 1997) and an epistemological and semio-cognitive framework, the “objects-structures dialectic” (Hausberger, 2017b). The latter has been drawn upon in reference to the interplay between semantic work on mathematical objects and syntactic work on the axiomatic structures that unify these objects. This dynamic is as a fundamental epistemological (and didactical) dialectic that characterizes structuralist thinking.

The results of a classroom experiment with third year Bachelor students are presented and discussed, more synthetically than was done in the author’s RDM paper (2021), written in French. The purpose of this chapter is both to introduce the non-francophone community to this research and to connect and contrast it with selected other studies. In relation to the full paper, less emphasis is given to the debate on the existence of a fundamental situation (in the sense of Brousseau, 1997) for structural concepts, and the point of view of the teacher (how the situation may be managed) is not provided. However, two supplementary lab sessions with more advanced students are discussed, in order to emphasize the inquiry and creativity that the theory of banquets is able to generate. Finally, an early draft of this research was first presented in 2016 at the KHDM conference in Hannover (Hausberger, 2017a). It is a pleasure to take the opportunity of this book in honor of Reinhard Hochmuth to disseminate the advances of this work to the international community of research on university mathematics education.

The chapter is structured as follows: in section 2, main theoretical ideas underlying innovative aspects of the theory of banquets are presented and compared with other instructional approaches. In the next section, mathematical aspects of the didactic engineering are described and a priori

analyses of main tasks are provided as a reference. In section 4, the data from the classroom experimentation and lab sessions are analysed and the learning affordances of the theory of banquets are discussed with respect to inquiry and creativity. A conclusive section summarizes striking elements of this practice-oriented study, and discusses further avenues of research.

## **1.2 Inquiry and creativity in Abstract Algebra teaching and learning**

The goal of this section is twofold: firstly, to situate the theory of banquets among different approaches that draw emphasis on either inquiry, creativity or both; secondly, to develop a few of the main underlying theoretical ideas before the more detailed presentation of the design in the next section.

### *1.2.1 Inquiry*

Good examples of innovation can be found in the Teaching Abstract Algebra for Understanding project (Larsen, 2013), whose ambition was the creation of a research-based inquiry-oriented curriculum for Abstract Algebra. The design followed the Realistic Mathematics Education framework (RME), in the tradition of Freudenthal, and was centered on Group Theory (GT). Under the guidance of the teacher, students investigated the set of symmetries preserving geometric figures, developed a calculus for computing their combinations, and axiomatized the set of rules that governed the algebraic computations. The instructional device culminated with the “reinvention” of the definition of a group and similar processes were used to handle quotient groups and the group isomorphism concept.

Within the Anthropological Theory of the Didactic (ATD), Bosch et al. (2018) also launched a similar program. The global vision is a shift of paradigm, from “monumentalism” (the critical view that contents are rarely questioned and problematized in the current curricula) to a new epistemological and pedagogical paradigm called emblematically “questioning the world” (Chevallard, 2015). With a focus on GT, they looked for problems external to GT that could lead to the reproduction of a substantial part of GT as a means to ascribe some rational to it. They concluded that a counting problem on symmetries of a square might be a suitable candidate for a reconstruction of elementary GT. The main difference with Larsen lies at the implementation level of the inquiry process: ATD proposes a general instructional device called Study and Research Paths (SRP; Chevallard, 2015) and endowed with several theoretical tools (in the form of dialectics) to organize and measure the development of the inquiry process. But the envisaged SRP has not been carried out by the authors and therefore remains hypothetical.

Moreover, Bosch et al. wondered if the counting problem was substantial enough to motivate the study of the isomorphism theorems. Such a question points to the structuralist approach conducted by the author: if groups encode symmetries, a substantial part of the rationale of GT relies in its relationship with the structuralist methodology in general. In other words, some attention must also be paid to the meta-concept of structure itself and to the methodological dimensions of mathematical structuralism, that could also be questioned.

In these lines of thought, the notion of structuralist praxeology was introduced (Hausberger, 2018a) together with the experimentation of a SRP on Ring Theory that used a transcription of an online forum as a crucial component of the milieu. A general interpretation of “questioning the world” in Abstract Algebra was proposed, based on the idea that formalization was both a mathematization of the world (the extra-mathematical reality) and, at a higher level of abstraction, a conceptual rewriting of previous (pre-structuralist) mathematics in terms of structures, usual mathematical objects being taken as the (intra-mathematical) reality. In this context, questioning the world amounts to questioning mathematical objects in such a way that a fruitful dialectic between objects and structures may be developed. Such a vision meets the point of view of RME and in particular its notions of horizontal and vertical mathematization. We will see below that abstraction processes in Abstract Algebra may be distinguished from these two notions.

Before getting into these details, let us introduce a third theoretical framework that inspired the theory of banquets as a second type of inquiry-oriented innovation for the teaching and learning of mathematical structuralism. The banquets have been designed as a problem that may be regarded as a partly a-didactical situation in the sense of TDS. The inquiry thus takes the form of the epistemic actions of the learners who play against an antagonist milieu. Moreover, the meta-lever is used, that is “the use, in teaching, of information or knowledge about mathematics. [...] This information can lead students to reflect, consciously or otherwise, both on their own learning activity in mathematics and the very nature of mathematics” (Dorier et al., 2000, p. 151). Concretely, a meta-discourse is explicitly introduced in the milieu; for example, the worksheet begins in these terms:

A structuralist theory is an abstract theory: it therefore deals with objects whose nature is not specified. They are then noted by symbols:  $x, y, z$  or  $\alpha, \beta, \gamma$ , etc. In the theory of banquets, there is only one type of objects [...] Since the nature of the objects is not specified, it is the *relations* between the objects that are the focus of the theory [...] (Hausberger, 20121, Appendix 2).

The meta-discourse also aims at fostering a level of meta-cognition, along the lines of Piaget's reflective abstraction (Piaget & Beth, 1961), which is viewed as an essential part of the inquiry process. Indeed, the main questions that are, explicitly or implicitly raised by the theory of banquets, are the following: What is a banquet? What does it mean to classify banquets? What do we mean by “structure” in mathematical structuralism? Answers may be found by reflecting on the classification of groups in order to classify banquets using similar structuralist means.

### 1.2.2 Creativity

Discourses on creativity draw back to Poincaré's essay *L'invention mathématique* (1952) on the topic of mathematical discovery, creativity and invention, and to Hadamard's lectures on the psychology of mathematical invention (1945) which emphasized four stages (initiation, incubation, illumination and verification) in the journey to invention in the mathematical field. Those writings influenced the work of Fischbein (1994) on the interaction between the formal and intuitive components of mathematical activity. One of the main point made by Fischbein is that “a world of intuitive models act tacitly and impose their own constraints” (loc. cit. p. 236), even at formal stages of intellectual development. For instance, the abstract notion of set comes with the idea of a collection of objects, with all its connotations. Intuition is often accompanied by figural representations; these lead to idealized mental entities that interplay with axiomatic or deductive systems. Fischbein met here the views of Freudenthal (1983) who advocated in his *didactical phenomenology of mathematical structures* that mathematical concepts should be taught together with their underlying mental images. Fischbein therefore investigated cases of fertile symbiosis - or cases of conflict - between figural/intuitive and conceptual properties of mathematical objects in the elaboration of a mathematical proof, for instance in geometry. As an extension of this work, Kidron (2011) studied means to help students be aware of their tacit models and achieve a complete synthesis between formal and intuitive representations in the sense that the mental structure was flexible and avoided conflicts. A situation in analysis was designed and experimented with students.

To the author's knowledge, there are very few studies on the role of intuition and mental models in Abstract Algebra. In a pilot study, Stewart and Schmidt (2017) used Tall's three worlds (embodied/symbolic/formal; 2013) framework to compare a mathematician and one of his students' mathematical experience as the class was about to prove the Fundamental Theorem of Galois Theory. According to Tall, “natural proof builds on concept imagery involving embodiment and symbolism, which may build on embodiment, symbolism, or a blend of the two” (p. 286). Mathematical activity and access to formal knowledge therefore requires one to navigate between the three worlds, including the embodied world based on gestures and perception of patterns. This raises a didactical issue, since “there are significant problematic changes in meaning that must be addressed to move to another plane of mathematical thinking” (p. 414). In the case of Galois Theory, the student struggled to revisit the concrete examples in the light of the abstract concepts of the theory, and therefore had a quite different mathematical experience than the professional

mathematician. According to Stewart and Schmidt, the difficulty lies in “blending conceptual embodiment and operational symbolism” (p. S47) as a path to formal mathematics.

The theory of banquets is an educational device that aims to facilitate the access to structuralist thinking. As evidenced by Fischbein and Tall, such an access involves the development of mental models and the integration of intuition and logic to build flexible and coherent schemes. The name of the theory (banquets) was chosen in order to facilitate such an integration, as the mental image of guests sitting around tables should be evoked. An important step before designing tasks consists in clarifying the cognitive and epistemological dimensions of structuralist thinking from a theoretical point of view. The objects-structures dialectic aims at providing such a framework.

### 1.2.3 The objects-structures dialectic

According to Cavaillès (1994), two movements of abstraction are at work in structuralist thinking, *idealisation* and *thématisation*, which apply transversally to each other (one is perceived as vertical, the other horizontal). They follow one another dynamically to express a dialectic between form and content, which Cavaillès calls the “dialectic of concepts”. Roughly speaking, idealization consists of extracting a form, which is then thematized into a higher-level object theory. Precise definitions are given in (Hausberger, 2017b). In fact, idealization may be linked to the horizontal mathematization of RME and thematization to vertical mathematization. However, idealization is not centered on real-life phenomena (but on the epistemic action of identifying invariant properties attached to a plurality of heterogeneous situations), and thematization is a particular vertical mathematization, specific to the structuralist project.

Moreover, in the case of Abstract Algebra (unlike elementary school algebra), two levels of organizing principles of phenomena need to be distinguished: on the one hand, the level of the given structure (of group, ring, etc.), which appears as the organizing principle of phenomena involving objects of a lower level; on the other hand, the meta-concept of structure itself, which is playing an architectural role in the elaboration of mathematical theories, in relation to the structuralist methodology. Indeed, similar questions and tools govern the application of the abstract unifying and generalizing point of view of structures and characterize the process of thematization. For instance: which identity principle to adopt (which are the natural morphisms between objects of a given type of structure)? How to classify objects up to isomorphism? Which structuralist theorems govern the decomposition of objects into simpler ones? We recover here key questions that will be used to design the theory of banquets and its reflexive stance (the inquiry dimension).

Let us now develop the cognitive dimension that relates to intuition, creativity and mental models. As mental representations cannot be accessed, semiotic considerations on external representations must also be considered. A first didactical idea is to use contributions of model theory, which offers a fertile point of view to bridge intuition and logic through the distinction between syntax and semantics and the articulation between these two aspects. First of all, a definition by axioms is, from a logical point of view, an open sentence. The models (the instances that satisfy these statements, in other words the objects in the sense of the objects-structures dialectic) constitute the *semantic* content of the structure, in relation to the system of axioms that defines it syntactically. Referring to Fischbein, models may include mental models built from perceptual intuition or embodiment in the sense of Tall. This will be the case with the banquet structure, whence its very name. By contrast, syntactical work with the axioms is carried out in the symbolic world; articulation between syntax and semantics thus amounts to what Fischbein called a fertile symbiosis. This leads us to distinguish between a syntactic point of view on idealization, which consists in abstracting the particular nature of objects and isolating the formal properties of relations (the “logic” of relations), and a semantic point of view which emphasizes the *isomorphism classes of models*. The latter mediate the concrete semantic domain of objects and the abstract syntactic domain of the structure, but the price to pay is the transition from elements to classes. From this point of view, the task of classifying models (up to isomorphism) appears fundamental for the conceptualization of an abstract structure. The conceptual aspects include concept formation but

also the structural horizon of the structuralist theorem of decomposition of objects into simpler ones: in the case of banquets, the decomposition of a banquet in a disjoint union of tables. This is a clear illustration of the role that conceptual embodiment, in the sense of Stewart and Schmidt (2017), may play on the journey to the so-called formal world (in other words, structuralist thinking).

The second idea is to use Duval's theory (2006) to handle representations and work in the symbolic world, in other word the manipulation of signs. According to Duval, the mental model is an internal representation, which serves to objectify the banquet structure; whereas the observables are the external representations produced by learners (*semiosis*), in particular during the conceptualization process (*noesis*). As a means to investigate creativity in students' work, we will pay particular attention to these representations, as well as to the semiotic manipulations (treatments and conversions in the sense of Duval), which are used to determine classes of banquets.

### 1.3 The theory of banquets: a didactic engineering

The theory of banquets was designed according to the methodology of didactic engineering (Artigue, 2014). As stated in the introduction, our main focus in this chapter is on learners' activity that will be analysed in the light of the theoretical elements, centered on inquiry and creativity, that were just presented. The choice of values of didactic variables and the orchestration between didactical and a-didactical dimensions of the situation in the sense of Brousseau (thus the role of the teacher) are discussed in the RDM paper (Hausberger, 2021). We will restrict our account to a brief presentation of mathematical aspects of the theory of banquets and provide an a priori analysis of the tasks that relate to the data discussed in the next section.

#### 1.3.1 Mathematical presentation of the theory of banquets

A banquet is a set  $E$  endowed with a binary relation  $R$  which satisfies the following axioms: (i) No element of  $E$  satisfies  $xRx$ ; (ii) If  $xRy$  and  $xRz$  then  $y = z$ ; (iii) If  $yRx$  and  $zRx$  then  $y = z$ ; (iv) For all  $x$ , there exists at least one  $y$  such that  $xRy$ .

In part I of the worksheet (which has been distributed in one go and may be processed linearly), students are asked the following questions:

- 1 a. *Coherence: is it a valid (non-contradictory) mathematical theory? In other words, does there exist a model?*
- 1 b. *Independence: is any axiom a logical consequence of others or are all axioms mutually independent?*
- 2 a. *Classify all banquets of order  $n \leq 3$*
- 2 b. *Classify banquets of order 4*
- 2 c. *What can you say about  $\mathbb{Z}/4\mathbb{Z}$  endowed with  $xRy \Leftrightarrow y = x+1$ ?*
- 2 d. *How to characterize abstractly the preceding banquet (that is, how to characterize its abstract banquet structure among all classes of banquets, in fact how to characterize its class)?*

The abstract/concrete relationship is reversed in part II, which begins with the empirical definition of a *table* of cardinal number  $n$  to mean a configuration of  $n$  people sitting around a round table. The following questions are raised:

- 1 a. *What relationship between people could be used to abstractly define a table?*
- 1 b. *State a system of axioms abstractly defining a table.*
- 2 a. *Propose a definition of sub-banquet and irreducible banquet. Let  $b=(E,R)$  be a finite banquet and  $x \in E$ . Define and characterize the sub-banquet  $\langle x \rangle$  generated by  $x$ .*
- 2 b. *What is the link between tables and irreducible banquets?*
- 2 c. *Define the operation of union of banquets. State and prove the structure theorem of finite banquets.*
- 2 d. *Apply the theorem to banquets of cardinal number 4.*

The banquet structure possesses a large variety of models since the system of axioms may be

interpreted in quite different worlds, beginning with the empirical interpretation of guests sitting around tables (a component of Tall's embodied world):  $xRy$  if  $x$  is sitting on the left (or right) of  $y$ . Other domains of interpretation include Set Theory (the binary relation is represented by its graph), Functions ( $xRy \Leftrightarrow y = f(x)$  defines a function  $f$  according to axioms (ii) and (iv); the other two axioms mean that it is injective without fixed points), Permutation Groups ( $f$  is a bijection when  $E$  is finite, in other words a permutation without fixed points) or even Matrix Theory (the relation is seen as a function  $E^2 \rightarrow \{0,1\}$  and represented by the corresponding matrix; the axioms express rules on the number of 1 in each row and column) and Graph Theory ( $xRy$  if and only if the vertices  $x$  and  $y$  are connected by an edge oriented from  $x$  to  $y$ ).

The structure theorem of banquets (decomposition in a disjoint union of tables) thus corresponds to the well-known theorem of canonical cycle-decomposition of a permutation, but the analogy remains hidden since the binary relation of banquets is different from binary operations that define groups. A complete rewriting in terms of permutations is not expected from students. These remarks explain why the theory of banquets is mathematically rich but may not be found in any textbook (it is less general than permutation groups). Moreover, it is a simpler theory (in the sense of mathematical technicality) than Group Theory and it carries the underlying intuition and mental image of guests sitting around tables (a wedding banquet).

In the language of TDS, the theory of banquets decomposes into 4 main (sub-)situations:

- the logical analysis of the system of axioms (I 1),
- the classification of banquets of small cardinal numbers (I 2);
- the axiomatic definition of tables (II 1);
- theoretical elaboration and the structure theorem (II 2).

We will now apply the theoretical framework (the objects-structures dialectic, section 2) to the a priori analysis of the tasks dedicated to the classification of banquets of small cardinal numbers (I 2). The main prerequisite is a course in elementary Group Theory.

### *1.3.3 A priori analysis of the classification tasks*

The methods may be divided into two categories: on the one hand a syntactic-dominant approach, which is similar to the reasoning used in the case of the classification of groups of small orders, and on the other hand a semantic-dominant approach, which uses generic models borrowed from matrix or graph theory. It will be necessary, however, in each case to articulate syntax and semantics at a given point in the reasoning.

In the syntactic-dominant approach, let us take the case of three elements  $x, y, z$ . Up to permutation of elements, we can assume  $xRy$  (under i) and iv)); necessarily, ( $yRx$  or  $yRz$ ) and ( $zRx$  or  $zRy$ ), again under i) and iv). Of the four cases, only  $yRz$  and  $zRx$  is possible, by virtue of axioms ii) and iii). The reasoning is similar with four elements, but it requires repeating several times the "up to permutation" argument. This leads to two classes:  $xRy, yRx, zRt, tRz$  and  $xRy, yRz, zRt, tRx$ . One may expect students to stop at this stage, while it remains to justify that these two classes are distinct (and nonempty, by providing a model). The first point requires the notion of isomorphism, in fact the knowledge of properties invariant under isomorphism, which allow to distinguish the two classes. In the case of groups of order 4, well-known to students, the presence or absence of an element of order 4 is usually invoked. Working out the analogy with banquets consists in identifying a pattern of cyclicity: reasoning about the order of an element amounts, in our context, to reasoning about the cardinal of the "chain" generated by an element (by iteration of the relation), which is a closed loop in the case of finite cardinality. Cyclic groups, including those formed by the roots of unity, also rely on this mental image of the circle. While it is unlikely that students will engage in such formalization, except for those who are particularly comfortable with formalism, it is likely that the cyclic pattern will be recognized and emphasized, the more so as it is suggested by the mental image of banquets from the embodied world. The aim of questions c) and d) is to lead students to make this mental image explicit and formalize a notion of cyclic banquet.

In the semantic-dominant approach, matrix or graph theory is used to produce generic models that may represent all possible cases. It is therefore a question of differentiating classes. Graphs allow to quickly deal with the case of 3 elements by replacing analytic reasoning on axioms with a succession of actions, as in a Lego game: there are only two possibilities of endowing three letters  $x, y, z$  with arrows such that the resulting directed graph fulfills the axioms (interpreted in graph theory). To convince oneself that the direction of rotation is not important, a treatment (in the sense of Duval) within the graph symbolic semiotic register may be applied: the first step consists in re-establishing the counter-clockwise direction, which does not change the directed graph; the same cyclic pattern is then recognized up to permutation of  $x$  and  $y$  (step 2). Without formalizing a notion of isomorphism, the principle of abstraction, in its naive sense of abstracting elements, thus allows to figure out that both models lie in the same class, in the etymology of isomorphism (having the same shape).

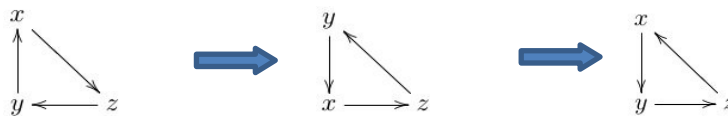


Figure 1: identification of isomorphic models through treatments within the semiotic register of graphs and pattern recognition

The situation is more complex in the case of 4 elements, as the number of configurations is higher. Nevertheless, knowledge from graph theory (treatments to remove the crossings of arrows) makes it easy to come to either the case of the cyclic graph or the case of the graph with two connected components of two elements related by a double arrow. The visual process of pattern recognition allows to conclude, by forgetting the labels of the vertices of the graph.

The formal definition of isomorphism requires to have integrated the syntactic point of view of bijection that preserves relations. In GT, isomorphisms preserve operations, which is conceptually different, but the syntactic proximity of  $x*y$  and  $xRy$  should allow students to easily find the condition  $\forall (x, y) \in E^2, xRy \Rightarrow \varphi(x) R' \varphi(y)$  defining a morphism  $\varphi: (E, R) \rightarrow (E', R')$  of banquets. The actual construction of the isomorphism, for example between the 2 previous banquets of cardinality 3, may be carried out by comparing  $xRy, yRz, zRx$  and  $x'Rz', z'Ry', y'Rx'$ : if  $\varphi$  maps  $x$  to  $x'$ , it will also map  $y$  to  $y'$  and  $z$  to  $z'$ . In fact, in writing relations in such sequences, we have implicitly identified the cyclic pattern. The latter can be made explicit by conversion (in the sense of Duval) to a register that underlines the pattern graphically (such as graphs or empirical banquets from the embodied world).

## 1.4 Learning affordances of the theory of banquets

A classroom experiment with third year Bachelor students took place in 2014 (4 sessions of 1h30 each for the full worksheet). To encourage the meta-cognitive dimension, students worked in small groups of 3-4. Each group had to return its research notebook to the teacher after each session and phases of devolution and institutionalization (in the sense of Brousseau, 1997) took place when appropriate according to the scenario (Hausberger, 2021). The experiment was later supplemented by two lab sessions (outside the classroom), with two pairs of more advanced students called Alice/Bob and Chris/Debby in this chapter. Alice had completed a PhD in mathematical physics and occupied several post-doctoral positions before passing exams to become an upper-high school/upper-secondary teacher. Bob was about to begin a PhD in differential geometry and Chris/Debby were more standard Master students. The data that will be discussed in this section comprise excerpts of students' notebooks and excerpts of transcripts of dialogues among pairs of advanced students.

### 1.4.1 What is a banquet? Students' creative processes in making sense of a formal system of axioms

Unsurprisingly, Alice readily connected the axioms to the mental model of wedding banquets and

blended symbolic manipulation of axioms with intuitive reasoning in the embodied world (figure 2). Alice is indeed close to being a professional mathematician and her work is a wonderful illustration of Tall's claims on symbolism and embodiment.

Alice: Classical, we specify the structure through relations, okay.

Bob: Antisymmetry [about axiom (i)].

Alice: It's not quite like that, it's non-reflexivity; there's one guy on the right and one on the left, that's the idea, [laughter]; there's nobody sitting alone at a table.

Bob: The elements are people? And in relation if together at the table?

Alice: Yes, that's it. The relation is to sit on the right (or left). However, you can have at most one guy on the right and at most one on the left, there is at least one guy on the right. Yes... there is theory and models. To show that it's not contradictory, you can show that there exists a model. I suggest we take one guy. No, one guy doesn't work, 2 guys sitting next to each other. So you take  $E=\{x,y\}$ . You can also put  $\{0,1\}$ .

Bob :  $\{1,2\}$ ?

Alice: Let's take  $E=\{a,b\}$  and for the relations the couples  $(a,b)$  and  $(b,a)$ . So it is indeed a model.

Bob: [after reflection] ok

Alice : Yes, a set with 2 elements, they are sitting opposite each other... obviously, there is at most one on the right and one on the left, they are in relation with the one opposite...

Bob: So this is existence. And consistence?

More surprisingly, Chris and Debby did not relate the axioms to wedding banquets (before part II of the worksheet). Nevertheless, they spontaneously introduced semiotic representations from graph theory through cognitive processes that also relate to embodiment: "Globally, we have a point  $x$  which maps to  $y$  and  $z$ , we necessarily have equality" (discussion of axiom (ii)). The movement of the pencil, from  $x$  to  $y$ , thus the gesture, led them to represent the relation in the form of an oriented arrow. They then borrowed from permutations the notation  $(x y z)$ , more condensed, to designate the resulting directed graph in the case of 3 elements (without linking banquets neither to graph theory nor GT).

In the classroom, nearly every group of students began by representing banquets from the real world in a more or less idealized manner (top of figure 2). In order to solve the assigned tasks, generic models with more affordances towards mathematical treatments had to be produced, therefore the teacher had to introduce the repertoire of either graphs or matrices. Examples of such representations are provided at the bottom of figure 3, which also includes in the middle a purely symbolic representation in Set Theory.

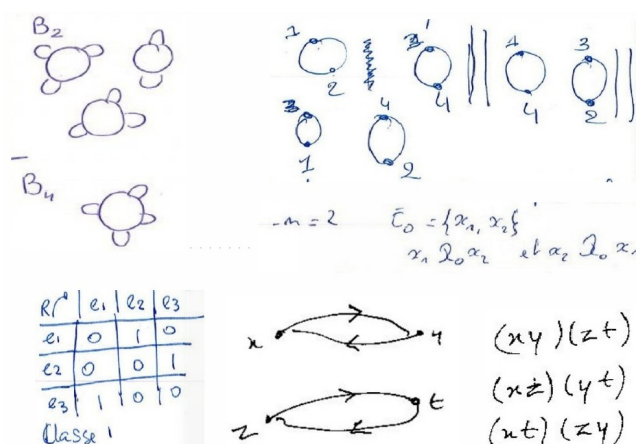


Figure 2: Semiotic representations produced by students throughout part I

#### 1.4.2 What does it mean to classify banquets? Students' creative processes in developing a structuralist point of view

Let us now analyze how the representations from the embodied and symbolic worlds may be used to potentially achieve the journey to the formal world and develop a structuralist sense. We will begin with the expert practice of Alice who is playing the role of teacher towards Bob:

Bob: Cardinal number 3...

Alice: *The circular thing, people  $a, b, c$  around the table.  $(a, b), (b, c), (c, a)$ .* It remains to be seen that this is the only one.  $(a, b)$  by numbering, it is still valid.

Bob:  $(a, c), (c, b), (b, a)$ ?

Alice: It's the same model, up to isomorphism.

Bob: That's true.

Alice:  $(b, a)$ ... there's going to be a problem, because  $c$  is going to be sent on what? If  $c$  is sent on  $a$  or  $b$ , as  $a$  and  $b$  are already reached, we will deny (ii).

Bob: If we had  $(a, b)$  and  $(b, a)$  we wouldn't know what to do with  $c$ ...

Alice: Yes, that's it. *Because his two potential right-wing neighbors already have one neighbor*

Bob: So it's necessarily  $(b, c)$  and we complete.

Alice: Perhaps cardinal number 4 will be more interesting. Shall we say  $\{a, b, c, d\}$  ?

Bob: Yes.

Alice: So there is the circular model... are you following me?

Bob: Always... but, in this case, there can be several *if you put them  $a, b, c, d$  around a table...*

Alice: Yes, but you'll be able to find a bijection, which amounts to a renumbering. If you want, the natural morphisms in there will be... is there a way to send  $E$  on  $E'$  by a bijection that sends  $R$  on  $R'$  ? So if you have a circular model, you're going to be able to send it on a circular model by a permutation.

Bob: Uh, yes...

Alice: So we always have  $(a, b)$ ; we always have  $(b, c)$ ... ah, can  $b$  send itself to  $a$ ? That would make a first case separation.

Bob: *It would make a two-table banquet, so to speak.*

Alice: Yes, this is a possibility. You can have  $(a, b), (b, a), (c, d), (d, c)$ . In fact, we're going back to the previous banquets. We have the circular banquet  $R_{C,4}$ , and we have, one could say, finally a direct sum in fact. It is a direct sum of banquets:  $R_4 = R_2 \oplus R_2$ . Are there others? I don't think so.

Bob: Are there other direct sums possible? No, because there is no one-person banquet.

Alice: In theory, you can have irreducible models, which do not break down into direct sums, and which are a priori different from the circular model. But here, if we have  $(a, b)$  and if we put  $(b, a)$ , then the rest is specified; so we will try to put  $(b, c)$ . If we put  $(c, d)$  we fall back to the circular banquet;  $(c, a)$  we're screwed. So this is the only possibility, I don't know if you follow me...

Bob: OK, so we have our two models.

Striking features of this dialogue include a fertile symbiosis between figural/intuitive and conceptual properties of banquets that result in operative symbolism (salient sentences are underlined in *italics*). Another feature is the conceptual perspective of Group Theory (and Abstract Algebra in general): the direct sum of banquets has not been defined yet (the operation of union of banquets is the focus of part II.2.c), therefore Alice's reasoning cannot be understood but as analogical thinking with, for instance, the decomposition of the Klein group  $V_4$  in a direct product of two cyclic groups of order 2. Students anticipate part II and also introduce a notion of irreducibility (which was not forecasted in the a priori analysis centered on undergraduate students).

Let us now describe how Chris and Debby proceeded within the semiotic register of graphs:

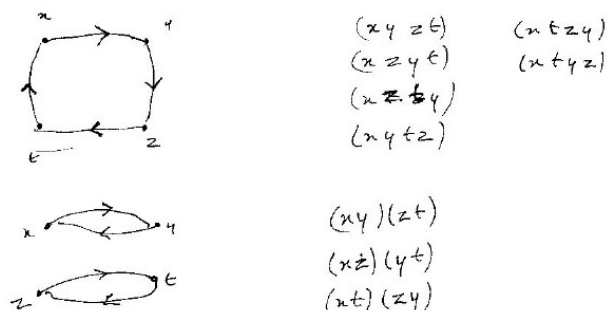


Figure 3: Chris/Debby's classification of banquets of cardinality 4

Chris: There would be 9 of them.

Debby: Nevertheless, we only considered objects that we know. But since the beginning, we have been talking about a structure.

Chris: But wait, the elements can always be numbered. What could go wrong?

Debby: Our own consistency.

Chris: But here, we thought about relationships, we didn't think about the objects themselves, we didn't take a particular relation.

Debby: *Never mind.*

The conclusion they drew is mathematically inaccurate. Nevertheless, the reflexivity shown by the students is remarkable: they emphasized that they abstracted both the nature of elements and the semantics of the relations. However, the algebraic symbolism (the letter) gives the illusion that the process of abstraction is complete. This is not the case, since labels of the graph vertices should also be removed. Although they were visualizing the pattern (left part of figure 3), students did not develop the intuition that the list of 9 models consisted in 2 classes, and as a consequence they did not formalize a notion of isomorphism. The intervention of the instructor (I) was required to achieve this, which proved to be a long journey:

Debby: So there would be 2 classes up to isomorphism, this kind of object and this kind.

Chris: There,  $\mathbb{Z}/4\mathbb{Z}$  and there  $\mathbb{Z}/2\mathbb{Z} \not\cong \mathbb{Z}/2\mathbb{Z}$ , in fact.

I: Are you thinking about the classification of groups?

Debby: Necessarily, we think about the classifications we know.

I: So there are 2 types of objects and here you have listed them all on  $x, y, z, t$  [...] You have listed all the possible oriented graphs on  $x, y, z, t$  that fulfills the axioms. [...] And why do you say there are two classes?

Chris: Two classes? We have put all the permutations behind, anyway.

I: And why would  $(x y z t)$  and  $(x y t z)$  be the same?

Chris: No, not the same, of the same type.

I: What does it mean to be of the same type?

Chris: I am thinking of permutations. One will loop faster than the other. I am clearly thinking about the order behind it.

Debby: A bijection. One can pass from one element of this class to another by a bijection, but not between the 2 classes.

I: Isn't it always possible to find a bijection between two sets of same cardinal number?

Chris: Yes it is!

Debby: Ah yes, but will it respect the structure? [...]

Again, the journey involved Tall's three worlds (a loop is part of the embodied world), but the process of accommodation of knowledge on groups to achieve a structuralist classification of banquets proved to be difficult. The situation is different from the case of Alice who developed a complete mental structure (or scheme) regarding structuralist decompositions that could easily integrate the case of banquets with the support of its mental model. Outside help was therefore needed, but the last question raised by Debby showed that the inquiry process tackled crucial issues in the development of structuralist sense.

Let us finally point out the creative processes generated during the classroom experiment, in particular the role played by the mental model. Unlike Chris/Debby who did not mention wedding banquets at all (maybe, due to a didactical contract that separated real world phenomena from genuine mathematical objects), there were groups who engaged in classifying empirical banquets without questioning abstraction processes (figure 4). The mental model is here an obstacle towards bridging the symbolic world and students did not manipulate the formal axioms at all.

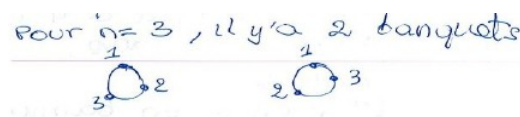


Figure 4: *The mental model as an obstacle: empirical classification of banquets*

To the other end, there were groups who were proficient in syntactical manipulation of axioms but did not produce any synthesis of intuitive and symbolic representations in their semiosis. In between, a few groups made a fertile use of the mental model in identifying the class of the banquet  $(\mathbb{Z}/4\mathbb{Z}, R)$  (figure 5). Students referred to the “table of 4”, which was idealized in their drawing and superposed with another representation in the semiotic register of graphs. In the terms of our

theoretical framework, this is a convincing example of conceptual embodiment. Nevertheless, the notion of isomorphism was not formalized by any group without the intervention of the teacher who had to renegotiate the didactical contract (by emphasizing the need of formal definitions) and clear a path to the formal world as in the Chris/Debby case.

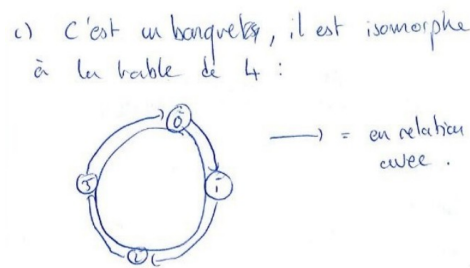


Figure 5: Identification of the isomorphism class by conversion to a graphical register related to embodiment and pattern recognition.

Most of the students perceived analogies with GT, on an intuitive basis (informally) and for various obvious reasons: the notation  $\mathbb{Z}/4\mathbb{Z}$  in the worksheet (whereas it is essential to distinguish the additive group  $\mathbb{Z}/4\mathbb{Z}$  from the banquet  $(\mathbb{Z}/4\mathbb{Z}, R)$ ), the type of task (classifying objects) and the similarities in the results obtained (which is not a coincidence, given the link with permutation groups). However, those who made the analogy most explicit did not manage to expand the mental structure (or schema) from groups to banquets as Alice did. For instance, a group of students used the symbolic representation  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  to designate the 2-table banquet (figure 6) without noticing conflicting aspects with the Cartesian product in Set Theory. Indeed, the disjoint union of tables adds up cardinal numbers whereas the Cartesian product is multiplicative. The same conclusion as Kidron (2011, p. 125) draws, applies here: “it highlights the need for mathematical educators to help students be aware of their tacit models, and to complete the synthesis between the formal and the intuitive into one mental structure”.

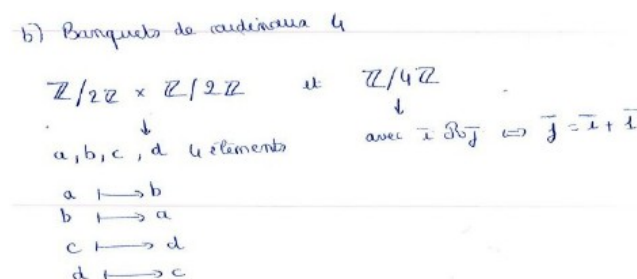


Figure 6: Students' difficulties in expanding the mental structure from groups to banquets

## 1.5 Conclusion and perspectives

The theory of banquets is the fruit of a practice-oriented study that sheds light on mathematical creativity in Abstract Algebra, in the sense of a fertile interplay between intuition and formalism, as it is experienced by professional mathematicians (represented by Alice), and by graduate or undergraduate students. The goal is also to induce a meta-cognitive shift (a dimension of inquiry with a focus on the meta-concept of structure) in order to facilitate the access to structuralist thinking. The mathematical experience varied according to students' personal level of advancement in the mathematical journey through Tall's 3 worlds. However, the 3 dimensions (embodied, symbolic and formal) were almost always present, whatever the level. Most students could take advantage of the cognitive affordances offered by the mental model of the embodied world, to some extent, but only advanced students were able to achieve a complete synthesis between the formal and the intuitive in the development of a structuralist classification of banquets. Unsurprisingly, the intervention of the teacher was needed to point out conflicting aspects between the formal and the

intuitive or stimulate the inquiry process on the meaning of a structuralist classification for students to move forward.

The main novelty of the theory of banquets in comparison with previous didactical designs in Abstract Algebra presented in section 2 is its ambition to tackle the issue of students' access to structuralist thinking at large. GT thus serves as prerequisite, and meta-cognition is crucial in the inquiry process developed by students. If research results may be expressed as above using theoretical constructs from Fischbein and Tall, the framework of the objects-structures dialectic allowed a finer-grained analysis of the interplay between the formal, symbolic and embodied components of mathematical activity than previous accounts (e.g. Stewart and Schmidt, 2017).

The study can still be deepened in different directions, depending on the theoretical framework that is used to complement the analyses in the spirit of networking. Studying structuralist praxeologies (Hausberger, 2018a) in the ATD framework can shed more light on what it means and requires, at the level of praxis and logos, to successfully work out the analogy between groups and banquets in the classification tasks. The inquiry process can also be modeled as a SRP in order to get a finer control on its vitality, economy and ecology. Inside TDS, the different levels of the milieu (its structuration) may be analyzed in order to have a clearer picture of the relationship of the epistemic subject with the milieu during the different phases of the situation, particularly those of meta-cognition, and finally tune didactic variables with higher granularity. Cognitive and semiotic aspects of the analysis may also be deepened, for instance by using semiotic frameworks that pay more attention to embodiment, or by attempting to study the structuralist schemes (or schemas) involved in the theory of banquets using cognitive frameworks.

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