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GROTHENDIECK'S THEORY OF SCHEMES AND THE ALGEBRA-GEOMETRY DUALITY

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ABSTRACT. We shall address from a conceptual perspective the duality between *algebra* and *geometry* in the framework of the refoundation of algebraic geometry associated to Grothendieck's theory of schemes. To do so, we shall revisit scheme theory from the standpoint provided by the problem of recovering a mathematical structure A from its representations $A \rightarrow B$ into other similar structures B . This vantage point will allow us to analyze the relationship between the *algebra-geometry duality* and (what we shall call) the *structure-semiotics duality* (of which the *syntax-semantics duality* for propositional and predicate logic are particular cases). Whereas in classical algebraic geometry a certain kind of rings can be recovered by considering their representations with respect to a unique codomain B , Grothendieck's theory of schemes permits to reconstruct general (commutative) rings by considering representations with respect to a category of codomains. The strategy to reconstruct the object from its representations remains the same in both frameworks: the elements of the ring A can be realized—by means of what we shall generally call *Gelfand transform*—as quantities on a topological space that parameterizes the relevant representations of A . As we shall argue, important dualities in different areas of mathematics (e.g. Stone duality, Gelfand duality, Pontryagin duality, Galois-Grothendieck duality, etc.) can be understood as particular cases of this general pattern. In the wake of Majid's analysis of the Pontryagin duality, we shall propose a Kantian-oriented interpretation of this pattern. We shall use this conceptual framework to argue that Grothendieck's notion of *functor of points* can be understood as a "relativization of the *a priori*" (Friedman) that generalizes the relativization already conveyed by the notion of *domain extension* to more general variations of the corresponding (co)domains.

I. INTRODUCTION

In the XVIIth century, Fermat and Descartes independently invented analytic geometry, that is, the study of the correspondence between algebraic expressions and geometric figures by means of coordinate systems (for a history of analytic and algebraic geometry see Refs.[35, 36, 62]). This interplay between algebra and geometry is based on the fact that the solution set of an equation in the variables (x_1, \dots, x_n) defines a subset of the space spanned by these variables, i.e. the subset of points—the “figure”—whose coordinates solve the equation. Now, this correspondence between algebra and geometry was unsettled by the fact that it is not possible to associate a figure to certain equations, such as for instance the equation $x^2 + y^2 + 1 = 0$ (when we search for solutions (x, y) in \mathbb{R}^2). This problem was bypassed by acknowledging that some equations might have solutions that do not belong to the “field of definition” of the equation, i.e. to the field of numbers in which the equation is formulated. The important fact in the present context is that the geometric “figure” that “schematizes” a given algebraic expression, far from being absolutely defined, depends on the field of numbers in which one searches for the solutions of the corresponding equation (see for instance Ref.[91, 385-386]). In this way, the *relativity* of the geometric figures with respect to the *codomains* in which one searches for the solutions obstructs the possibility of establishing a *duality* between algebra and geometry: a single algebraic expression can induce several figures, one for each possible codomain. This situation was described by Barr, McLarty, and Wells in the following terms:

“[...] the unit circle is the set of all solutions in the x - y plane to the equation

$$x^2 + y^2 = 1$$

The set of solutions of this equation depends on the type of number that one understands x and y to refer to. [...] There are many more abstract types of numbers [than the integers, the rationals, the real, and the complex numbers] and each gives rise to its own circle. In the classical set theoretic framework, there is thus an integer circle, a rational circle, a real circle, a complex circle and myriads of others [...] [14]

The theory of schemes has completely refounded the field of algebraic geometry. This theory mainly emerged from the work of A. Weil, C. Chevalley, P. Cartier, M. Nagata, J.-P. Serre, and A. Grothendieck and it was mainly developed by the latter.¹ As we shall see, the theory of affine schemes permits to cope with the aforementioned relativity in a manner that allows to recover an algebra-geometry duality for general (commutative) rings. Grothendieck’s theory accomplishes this task by means of a two-step procedure. Rather than fixing a single codomain, Grothendieck fully embraces the relativity of the figures associated to a family of polynomial equations with respect to the codomains in which one searches for their common solutions. By introducing the category theoretic notion of *functor of solutions* (or, in the dual geometric version, *functor of points*), Grothendieck takes into account the whole family of “figures” at once (Section V). Second, Grothendieck synthesizes the resulting “fan” of figures into a single geometric entity, namely the unique *affine scheme* associated to the original ring. This two-step description of the construction of an affine scheme articulates the two alternative standpoints from which scheme theory can be approached (for a formal demonstration of their equivalence see Ref.[32]). According to the category theoretic description, a scheme is a functor that encodes the solutions to a system of polynomial equations in every possible codomain (or more generally the

¹ According to J.-P. Serre, “[t] his first approximation to the theory of affine schemes was ‘in the air’ in 1954-1955, at least with some restrictions on the rings in question. One of Grothendieck’s contributions was to remove these restrictions: the best category of commutative rings is the category of all commutative rings!” [51, 256]. For a discussion of the genesis of the notion of scheme see Refs.[36, 35, 92].

representations $A \rightarrow B$ of a ring A for every B in the category). From a geometric standpoint, schemes are *ringed spaces*. According to this description, a scheme is constructed by adding structure to an underlying set of points. More precisely, one starts with a set of points, then introduces a notion of locality by means of a topology, enriches this topological space by adding local information (local rings of regular functions) in order to finally glue these local pieces together into a global object (structure sheaf).²

The notion of duality is a central concept in both mathematics and physics (see Ref.[8] for a general survey on the notion of duality, Refs.[65, Chapt.6, Sect.4, 253-266] and [105] for a mathematical approach, Refs.[9, 27, 30, 53, 69, 71, 83, 98] for philosophical-oriented analyses of this notion in modern mathematics).³ Among the different instantiations of the mathematical notion of duality, the duality between *algebra* and *geometry* occupies a privileged place. Now, there are two alternative approaches for understanding the duality between algebra and geometry. In S. Germain's terms, we can either consider that "*algebra is but written geometry*" or that "*geometry is but figured algebra*" [43, 223]. According to the first description (that we shall call *geometric approach*), geometry provides the starting point. This means that we start with some kind of space X and use algebraic devices—such as for instance functions or sections of a fiber bundle on X —to "observe" the space and try to separate its points by means of the properties defined by these "quantities". In this conceptual framework, a ring is an algebraic structure intended to be thought of as a ring of functions on a space. In Nestruev's terms, "smooth manifolds are 'worlds' whose observation can be carried out by means of smooth algebras" [99, ix].⁴

According to the second description (that we shall call *spectral approach*), algebra provides the starting point. This means that we start with an algebraic structure A (e.g. a Boolean algebra, a group, a ring, a C^* -algebra, etc.) and we consider the conditions under which A can be faithfully realized (by means of what we shall generally call *Gelfand transform*) as an algebra of quantities (such as functions or sections) on a suitable space (the *spectrum* of A).⁵ As we shall see, the remarkable fact is that the spectrum of the algebra A is a space that parameterizes a suitable collection of representations $A \rightarrow B$ of A . Hence, the spectral approach establishes a bridge between the *algebra-geometry duality* and what we shall call *structure-semiotics duality*, that is the duality between a *structure* A and the *representations* $A \rightarrow B$ of A in B . It is a striking fact that reconstruction theorems based on the same kind of *algebra/structure-geometry/semiotics duality* can be found in several areas of mathematics such as Boolean algebras (Stone duality), group representation theory (Pontryagin duality), algebraic geometry, Galois-Grothendieck theory, and C^* -algebra theory. As we shall explain in Section N°III, all these dualities can be subsumed under the same general pattern: that of an algebraic structure A that is isomorphic to an algebra of *quantities* over a *space* that parameterizes a certain class of

² In both descriptions, the notion of *sheaf* (introduced by J. Leray in the field of algebraic topology during the second World War [63]) plays a central role, albeit in a different manner: whereas in the case of ringed spaces, a sheaf encodes the global organization of local pieces of information defined on the space (typically functions locally defined), in the functorial approach a sheaf defines a *generalized space* by specifying how this space is "probed" by test objects in a given category ([100, *The basic idea of sheaves*]; for an introduction to sheaf theory see Ref.[124]).

³ It is worth noting that the common use of the term *duality* in physics and mathematics does not necessarily mean that the corresponding formal concepts are the same or even that the term is used in an homogeneous manner within each discipline. Whereas in mathematics a duality usually takes the form of an equivalence of categories [71], in physics the term duality is also used to denote a correspondence between different theories encoding some form of "physical equivalence" between them (e.g. between quantum field theories and string theories in the gauge/gravity dualities). In some cases, the use of the term in physics coincides with the mathematical notion. For instance, the *wave-particle duality* in quantum mechanics relies on the theory of Fourier transform, that is on the Pontryagin duality for abelian locally compact groups. For philosophical discussions of the notion of duality in physics see the different contributions to Ref.[26].

⁴ By following the physics-oriented description proposed in Ref.[99], we can understand the space X as a set of *physical states*, the "*observables*" f as *measuring devices* and the result $f(x)$ of the evaluation as the outcome of the "*observation*" of the state x by means of f .

⁵ For an analysis of both the "geometric" and the "spectral" approaches to the algebra-geometry duality see Ref.[99].

representations of A . As we shall see, the main innovation introduced by Grothendieck’s theory of affine schemes is that the spectrum of the algebra A (in this case a ring) parameterizes representations of A defined with respect to different codomains B . This extension of the semiotics of an algebraic structure justifies the distinction that we shall introduce between *restricted reconstruction theorems* (where the representations $A \rightarrow B$ defined with respect to a single codomain B are enough to recover the structure A) and *general reconstruction theorems* (where the structure A can be recovered by considering representations defined with respect to different codomains B).

Grothendieck’s refoundation of algebraic geometry and the panoply of new notions that he introduced (such as schemes, topoi, motives, and stacks) constitute one of the most important paradigm shifts that occurred in mathematics during the XXth century. Moreover, Grothendieck’s approach is strongly based on—and significantly contributed to develop—category theory and groupoid theory, thereby having a seminal impact on the “foundations of mathematics”. In spite of its importance—and probably due to its technical complexity—the revolution launched by Grothendieck has not really entered as a proper subject into the realm of philosophy of mathematics (a philosophical-oriented introduction to scheme theory can be found in Ref.[91] and references to scheme theory in philosophy of mathematics can be found in Refs.[103, 119, 133]). This paper is—in the wake of the aforementioned references—an attempt to contribute to fill this gap. To do so, we shall provide a conceptual analysis of some aspects of Grothendieck’s theory of affine schemes by mainly following the path described in the introduction to the treatise *Éléments de Géométrie Algébrique* by Grothendieck and Dieudonné [49] (see also Ref.[48]).⁶ The analysis of scheme theory that we shall propose is certainly partial and many important aspects of the subject will not be addressed here (like for instance the role played by scheme theory in bridging the gap between geometry and number theory—which gave rise to the field of *arithmetic geometry*—and the definition of general non-affine schemes). We shall mainly focus on the aspects of scheme theory that are relevant with respect to what we could generally call *philosophy of (mathematical) dualities*. To do so, we shall proceed as follows:

- (1) We shall analyze the relation between the *spectral approach* (where we start with an algebraic structure) and the *geometric approach* (where we start with a geometric object) to the *algebra-geometry duality* (Section N°III).
- (2) We shall analyze the relation between the *algebra-geometry dualities* and (what we shall call) the *structure-semiotics dualities* (of which the syntax-semantics dualities provide particular cases) between an algebraic structure A and a collection of its representations $A \rightarrow B$ (Section N°II).
- (3) We shall characterize the twofold relevance of scheme theory with respect to the debate between syntactic and semantic conceptions of “scientific theories”, namely
 - (a) that it provides the guiding example for the generalization of the Stone duality (encoding the representation theory of propositional logic) to predicate logic (Section N°II),
 - (b) that it entails an extended notion of the “semantics” of a “syntactic” structure A that includes “models” $A \rightarrow B$ defined with respect to different codomains B (Section N°V).
- (4) We shall analyze the general pattern underlying several representation theorems in different areas of mathematics (Boolean algebra theory, algebraic geometry,

⁶ Other introductions to the theory of schemes may be found in Refs.[37, 38, 40, 57, 88, 97, 114]. See also Ref.[90] for a very short introduction to scheme theory and Ref.[132] for an encyclopedic description of Grothendieck’s work (and in particular Chapter N°7 for scheme theory). Ref.[91] provides a nice introduction to the twofold nature of schemes, namely as *functors* (that encode solutions to families of polynomial equations in different codomains) and as *ringed spaces* (topological spaces equipped with sheaves of rings).

- group representation theory, \mathbb{C}^* -algebras, Galois-Grothendieck theory, etc.) (Section N°III).
- (5) We shall establish a distinction between *restricted reconstruction theorems* (like the representation theorem associated to algebraic varieties, Section N°IV) and *general reconstruction theorems* (like the representation theorem associated to affine schemes, Section N°VI).
 - (6) We shall analyze the main formal tool that encodes Grothendieck's functorial extension of the semiotics of a mathematical structure, namely the Yoneda lemma. We shall describe the two-fold conceptual importance of the Yoneda lemma (Section N°V), namely that it encodes
 - (a) a transition from a *substantialist* conception of mathematical objects to a *relational* one,
 - (b) a domain extension from objects in a category (described by representable functors) to general functors (representable or not).
 - (7) We shall stress the innovative aspect of the duality between rings and affine schemes with respect to previous instantiations of the algebra-geometry duality (Section N°VI), namely
 - (a) the presence of *points* corresponding to extended and unextended irreducible subsets (see also Section N°VII),
 - (b) the fact that the corresponding *quantities* on the spectrum of a ring have values in point-dependent codomains,
 - (c) the original treatment of *differential structures*.
 - (d) We shall extend Majid's Kantian-oriented analysis of the Pontryagin duality [85] to the duality between rings and affine schemes and use the resulting conceptual framework to address the difference between the notion of a *(co)domain extension* [15, 20, 87] and the more general variations encoded in Grothendieck's notion of *functor of points* (Sections N°VII and VIII).

In order to make this article readable by the wider possible audience, we have tried to avoid as much technical details as possible by addressing the reader to the corresponding references. In several occasions, we have included some relevant technical precisions in footnotes. We assume a basic knowledge of algebraic notions (e.g. rings, ideals, etc.), the rudiments of algebraic geometry, and an elementary acquaintance with category theory. This material can be found in Ref.[6], Ref.[57, Chap. I] and Refs.[10, 82] respectively. All throughout the article, the term *ring* means *commutative ring with unity*.

II. THE STRUCTURE-SEMIOTICS DUALITY

The so-called *syntax-semantics debate* refers to the tension between the understanding of a scientific theory either in terms of *syntactic structures* or in terms of *classes of models* (see Refs.[54, 81, 131] and references therein). The syntactic approach (also known as the *Received View*) was born in the framework of logical positivism and developed, mainly by Carnap and Hempel, as an attempt to formalize scientific theories in predicate (higher order) logic in a language-independent (or presentation-independent) manner [21, 58]. On the other side, the semantic approach started with the work of Tarski on formal semantics (model theory) and was notably developed from the 60's out of the criticism addressed to the syntactic view of theories by scholars like Putnam [107], Suppe [117], Suppes [118], and Van Fraassen [121, 122].

In Ref.[56, §17.1.3], Halvorson and Tsementzis argued that rather than trying to privilege the syntactic side or the semantic side of the debate, one can focus on the conditions under which the dilemma can be sublated in the form of a *syntax-semantics duality*. This "dualization" of the syntax-semantics dilemma amounts to claim that neither the syntactic side nor the semantic side have any privilege whatsoever over the other side, but that the two sides are related by adjoint functors that we shall call *semantic functor* and

syntactic functor. Whereas the *semantic functor* sends a syntax to a suitable collection of models (i.e. to a collection containing “enough models”), the *syntactic functor* extracts the common syntactic structure from each collection of models. Like the case of the Stone duality for Boolean algebras shows (syntax-semantics duality for propositional theories⁷), a *pre-duality* given by adjoint functors yields a proper duality when the adjunction defines an *equivalence of categories* between certain subcategories (see Ref.[65, §4, 253-269] and Ref.[105]).⁸ This means that the the two possible compositions of the syntactic and the semantic functors yield the identity on objects modulo isomorphisms. This category theoretic description of the notion of duality can be summarized by means of the description according to which *a duality is given by the fixed points of an adjunction* [100, fixed point of an adjunction].⁹

As Halvorson and Tsementzis have stressed, the example provided by the Stone duality for Boolean algebras shows that the corresponding collections of models have to be endowed with a suitable geometric structure encoding the “topological relations between models” [56, 411]. In this framework, as Lawvere writes, “models *are* points” in a topological space [77, 137]. The conceptual importance of this geometrization of the semantics is that it establishes a bridge between the *syntax-semantics duality* and the *algebra-geometry duality*, where the algebra is on the syntactic side and the geometry on the geometric side. Of course, we could legitimately question the generality of this relation between these two kinds of duality. As Halvorson and Tsementzis write, “*we do not want to fall into the trap—all too common in twentieth-century philosophy—of being blinded by the glow of a shiny new piece of formal apparatus*” [56, 412]. In order to counterbalance this healthy skepticism, we can propose two pieces of evidence that suggest that this relation between the *syntax-semantics duality* and the *algebra-geometry duality* encodes some kind of fundamental meta-mathematical pattern or—in Lawvere’s terms—“*a kind of philosophical theorem in a soft mathematical guise.*” [79, 16] (see also [79, Th.2, 78]).

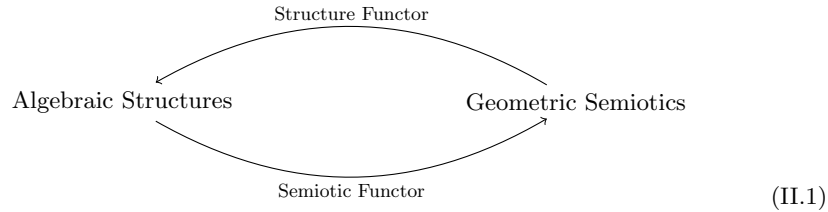
First, the attempts to extend the Stone duality to predicate logic by using topos theory also rely on a geometrization of the semantics [9, 11, 12, 13, 19, 72]. Second, if we extend the logical meaning of the terms *syntax* and *semantics* to any morphism $p : A \rightarrow B$ in a category (where p might be loosely understood as a “model” in B of the “syntax” A), then we find similar reconstruction theorems entangling both dualities in several areas of mathematics like algebraic geometry, group-representation theory, C^* -algebras theory, and Galois-Grothendieck theory. In order to introduce some terminological precision, we shall avoid using the logical terms *syntax* and *semantics* for the general case of an algebraic structure A and its representations $A \rightarrow B$ respectively. Instead, we shall use the term *structure* to denote any object A in a category \mathcal{C} and the term *semiotics* to denote the representations of A with respect to other objects in \mathcal{C} . Whereas the logical term *semantics* denotes the truth valuations of the corresponding propositional constants, the more general term *semiotics* denotes the representation of a structure A by means of morphisms into another structure B of the same kind. This means that the “values” that define each representation of A are not necessarily truth values, but rather values in more general mathematical structures (e.g. fields, rings, groups, etc.; see the column *Representations* in Table N^o1). In this sense, truth values provide a particular instantiation of the more general notion of *quantity*. It follows that the notion of *syntax-semantics*

⁷ A Boolean algebra A can always be understood as the Lindenbaum-Tarski algebra of a propositional theory T . In this case, the $\{0, 1\}$ -models of T are the truth valuations of A .

⁸ Regarding this category theoretic understanding of the notion of duality, Lambek and Scott summarize (what they call in the wake of Lawvere) “the unity of opposites’ principle’ by means of the following ‘slogan’: “*Many equivalence and duality theorems in mathematics arise as an equivalence of fixed subcategories induced by a pair of adjoint functors*” [73, 18] (see also Ref.[71]). Regarding the *syntactic-semantics adjunction* (sometimes called *structure-semantics adjunction*) see Ref.[78].

⁹ As we shall see at the end of Section N^oIII, it might be the case that only the unit or the counit of the adjunction is a natural isomorphism. In such an intermediate case, the pre-duality is not strictly speaking a duality even if one of the two possible compositions of the adjoint functors does yield the identity modulo isomorphisms.

duality in the strict logical sense of the term can be understood as a particular case of the *structure-semiotics duality*. In particular, the Stone representation theorem for Boolean algebras that encodes the duality between syntax and semantics for propositional theories (as well as its generalizations to predicate logic) can be understood as a particular case of what we could generally call *representation theory* (this point has been stressed by Halvorson in Ref.[53, 165]). In Fig. N°II.1, we have schematized the general duality between algebraic structures and their geometric semiotics, where the two adjoint functors are called *structure functor* and *semiotic functor*.



In what follows, we shall consider the relevance of Grothendieck's theory of affine schemes with respect to this entanglement between the structure-semiotics duality and the algebra-geometry duality. First, the theory of schemes provided the guiding example for the generalization of the Stone duality for propositional logic to predicate logic [9, 11, 12, 13, 19, 72]. Whereas Grothendieck's representation theorem states (as we shall see) “*that every ring is isomorphic to the ring of global sections of a sheaf of local rings,*” the sheaf representation theorem for toposes—in the version demonstrated by Lambek and Moerdijk [72]—states that “*every topos is isomorphic to the topos of global sections of a sheaf of local toposes*” [9, 42 & 44]. In this sense, an understanding of Grothendieck's representation theorem for rings seems to be a necessary intermediate step to address the syntax-semantics duality for predicate logic. It is also worth noting—as Awodey and Forsell have stressed—that the relation between the syntax-semantics duality and the algebra-geometry duality provided the “leading idea” to extend the Stone duality from propositional to predicate logic.¹⁰

Second, the duality between rings and affine schemes encodes an important enrichment in the notion of semiotics. Stone representation theorem shows that it is possible to recover a Boolean algebra A from a semantics containing models *defined with respect to a unique codomain*, namely the set $\{0, 1\}$ endowed with the discrete topology [65]. By contrast, the reconstruction of a ring A requires to consider representations $A \rightarrow B$ of A *defined with respect to different codomains* B . As we shall see, the definition of a scheme as a *functor of solutions* (of a family of polynomial equations) explicitly formalizes this extension in the definition of the semiotics of a mathematical structure. The remarkable fact is that the resulting semiotics can still be described by a unique geometric object, namely the corresponding affine scheme *qua* ringed space. In this sense, the equivalence between the definition of a scheme as a functor and its definition as a ringed space [32] shows that the entanglement between the algebra-geometry duality and the structure-semiotics duality is still valid in the framework of Grothendieck's extended definition of the semiotic side.

It is also worth noting that in the geometric approach to the algebra-geometry duality, this functorial extension of the semiotics takes the form of a generalization of the notion of *quantity* on the space. Whereas in a restricted semiotics (defined by representations into a unique codomain) the quantities on the space takes their values in a structure that is constant all over the space, in the extended semiotic the quantities take their values in

¹⁰ In Awodey and Forsell's own terms: “*In broad strokes, the leading idea [to obtain an extension of Stone duality for Boolean algebras (Stone duality) from classical propositional logic to classical first-order logic] is to take the traditional logical distinction between syntax and semantics and analyze it in terms of the classical mathematical distinction between algebra and geometry, with syntax corresponding to algebra and semantics to geometry*” [13].

structures that continuously vary over the space. This means that it is not only the value of a quantity that might change from point to point, but also the very structure to which these values belong. In other terms, the notion of continuous variation of a quantity on a space is enhanced to the notion of a continuously varying quantity valued in varying structures (we shall come back to this point in Section N°VI).

III. ON RESTRICTED AND GENERAL RECONSTRUCTION THEOREMS

In philosophical terms, we can define the *problem of representation* as the problem of reconstructing a given structure from a multiplicity of representations, interpretations, or “models” of the structure. In figurative terms, we could say that the different representations provide different partial “perspectives” on the corresponding structure and that the latter can be faithfully recovered only when one considers a sufficient number of such perspectival representations. In order to cover the variety of mathematical examples we are interested in, we shall use a sufficiently general notion of representation, namely we shall understand any morphism $p : A \rightarrow B$ in a category \mathcal{C} as a *B-representation* of the structure A .¹¹ By means of such maps the elements and operations of A are “represented” so to speak as elements and operations in B . By using Lawvere’s terminology, we shall call the object B *dualizing object* of the set of representations $A \rightarrow B$.¹² In turn, the set of B -representations of A defines what we shall call the *B-spectrum of A* (denoted $\text{Spec}_B(A)$). The fact that all the elements of $\text{Spec}_B(A)$ are morphisms $A \rightarrow B$ allows us to introduce what we shall call *Gelfand transform*. The Gelfand transform is a map that associates to each element $a \in A$ a B -valued function a_B (a *quantity* in Lawvere’s terms) on $\text{Spec}_B(A)$ defined by the Gelfand inversion formula

$$a_B(p) = p(a), \quad (\text{III.1})$$

for every $(p : A \rightarrow B) \in \text{Spec}_B(A)$. The Gelfand transform allows us to present the elements a of the mathematical structure A as *quantities* a_B on the *space* $\text{Spec}_B(A)$ that parameterizes its B -representations $p : A \rightarrow B$. In conceptual terms, the quantity a_B encodes the different representations of a in all the B -representations of A . This means that the evaluation of a quantity a_B on an element $p : A \rightarrow B$ of the spectrum $\text{Spec}_B(A)$ yields the representation of a in the B -representation p .

For the moment, the B -spectrum $\text{Spec}_B(A)$ of A is a set containing the different representations $p : A \rightarrow B$ of A in B . As it was stressed by Halvorson and Tsementzis for the propositional case given by the Stone duality (as well as its generalizations to predicate logic [9, 86]), the formulation of a duality between a mathematical structure and its representations might require to equip the set of representations with some extra structure encoding the relations between the latter, typically the structure of a geometric object (see Refs.[56, §17.1.3], [55, §7]).¹³ By doing so, the spectrum $\text{Spec}_B(A)$ containing the B -representations of A becomes a space. This geometrization of (what we could call) the *B-semiotics* of the structure A relies on the identification of the maps $A \rightarrow B$ (in the case of affine schemes, modulo an equivalence relation) to points of a space. By means of this identification between the semiotic notion of representation and the geometric notion of

¹¹ In Cartier’s terms, “[c]ategory theory is the mathematical expression of the idea of representation [...]” [23, 398].

¹² Lawvere and Rosebrugh called *formal dualization* the involution operation $(-)^{op} : \text{Cat} \rightarrow \text{Cat}$ that consists in reversing the arrows of a category in a merely formal manner (which means that it might not be always possible to understand the reversed arrows in terms of specific mappings). By fixing a *dualizing object* C , it is possible to define a “concrete duality” that assigns a reversed arrow $C^B \rightarrow C^A$ to each arrow $A \rightarrow B$ [80, 121-122].

¹³ In Halvorson’s terms: “[...] the semantic view was not wrong to treat theories as collections of models; rather, it was wrong to treat theories as nothing more than collections of models. Beginning with a syntactically formulated theory T , we can construct its class $\text{Mod}(T)$ of models. But we have more information than just the collection of models: in particular, we have information about relations between these models. [...] [as the case of the Stone’s duality theorem for Boolean algebras shows] we could rehabilitate the semantic view of theories by taking a theory to be a structured set of models, namely, a topological space of models” [55, 204-205].

point, the *algebra-geometry dualities* and the *structure-semiotics dualities* (of which the *syntax-semantics dualities* provide particular cases) get entangled. While the algebra is on the “syntactic” side of the duality, the geometry is on the “semantic” side.

Let's consider now the following *reconstruction problem*: is it possible to recover an algebraic structure A from its representations defined with respect to other (one or many) algebraic structures? Conceptually, a structure A can be reconstructed from its B -representations if the B -spectrum has enough points (or representations), i.e. if for any two elements $a, a' \in A$ there is at least a representation p in $\text{Spec}_B(A)$ such that $a_B(p) \neq a'_B(p)$.¹⁴ The reconstruction theorems according to which a structure A can be reconstructed from the representations defined with respect to a *unique* dualizing object B will be called *restricted reconstruction theorems*. Examples of such restricted reconstruction theorems are provided by:

- the algebra-geometry duality for algebraic varieties [57, Th.3.2 (a), 17],¹⁵
- the Stone duality for Boolean algebras [65],¹⁶
- the Gelfand duality for commutative C^* -algebras [39, 74],¹⁷
- the Pontryagin duality for abelian locally-compact groups and the associated theory of Fourier transform [104] (see also [29]),¹⁸
- the Galois-Grothendieck duality ([18, Th.2.4.3, 28], see also [27] for a conceptual-oriented analysis of this result).¹⁹

In all these examples, the structures at stake are reconstructed by just considering their representations with respect to a *unique* dualizing object.²⁰ Now, it is not always possible to reconstruct an algebraic structure from its representations into a fixed codomain. Grothendieck's fundamental insight is that this problem can be bypassed by considering representations of A with respect to a whole category of dualizing objects B . Examples of such *general reconstruction theorems* are provided by:

- the algebra-geometry duality for affine schemes (that we shall analyze in Section N°IV),²¹
- the sheaf representation theorems for toposes [9, 12, 13, 19, 70, 72].

In all these examples the codomain of the representations parameterized by the corresponding spectra might be different for different points. Whereas in the case of restricted reconstruction theorems (e.g. Stone duality or Gelfand duality) the quantities defined by the elements of the algebra have their values in a constant algebraic structure (e.g. $\{0, 1\}$ for the Stone duality and \mathbb{R} for the Gelfand duality), in the theory of affine schemes the quantities have values in local rings that vary continuously on the spectrum.

¹⁴ From a logical standpoint, the property of having enough points is related to *semantic completeness* (i.e. that tautologies $\vDash \phi$ are theorems $\vdash \phi$) [9]. See in particular the relation between Deligne's theorem and Gödel's completeness theorem for first-order logic [84, 523].

¹⁵ That is, the isomorphism between a reduced and finitely generated commutative algebra A over an algebraically closed field K and the affine ring of K -valued regular functions on the algebraic variety defined by the K -algebra homomorphisms $A \rightarrow K$ [57, Th.3.2 (a), 17].

¹⁶ That is, the isomorphism between a Boolean algebra B and the algebra of $\{0, 1\}$ -valued functions on a (compact, totally disconnected, and Hausdorff) space $S(B)$ called *Stone space*. The Stone space parameterizes the Boolean algebra morphisms $B \rightarrow \{0, 1\}$ (that we can understand as models or truth-valuations of the theory B) [65].

¹⁷ That is, the isomorphism between a (resp. unital) commutative C^* -algebra A and the algebra of \mathbb{C} -valued functions on a locally compact (resp. compact) and Hausdorff space $Sp(A)$ called *Gelfand spectrum*. The Gelfand spectrum parameterizes the non-zero homomorphisms $A \rightarrow \mathbb{C}$ (or characters) [39, 74].

¹⁸ That is, the isomorphism between a locally-compact abelian group G and the algebra of $U(1)$ -valued functions on the space \hat{G} that parameterizes the characters $G \rightarrow U(1)$ [29].

¹⁹ That is, the isomorphism between a finite dimensional k -algebra A split by an algebraically closed field L (where $L : k$ is a finite dimensional Galois extension of fields) and the algebra of L -valued functions on the $Gal(L : k)$ -space that parameterizes the k -algebra homomorphisms $A \rightarrow L$ ([18, Th.2.4.3, 28], [27, Sec.6]).

²⁰ The dualizing objects that allows us to formulate the corresponding isomorphisms are K , $\{0, 1\}$, \mathbb{C} , $U(1)$, and L respectively.

²¹ That is, the isomorphism between a general ring A and the ring of global sections of the structure sheaf of local rings over the prime spectrum $\text{Spec}(A)$ of A .

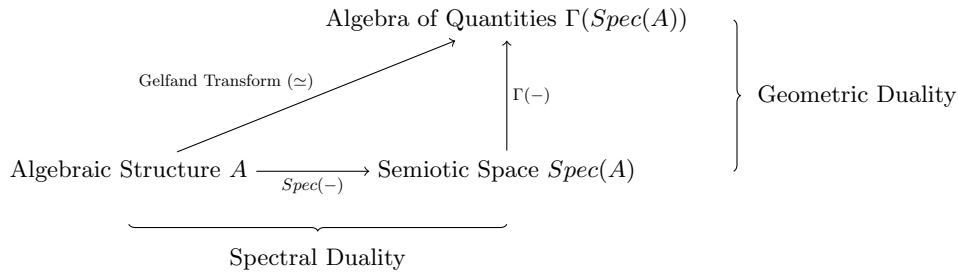
TABLE 1.

| | Duality | Object (algebraic structure) | Representations (geometric semiotics) |
|---|--|--|---|
| Restricted Reconstruction Theorems | Stone (syntax-semantics duality for propositional logic) | Bool. Alg. A | Bool. alg. morph. $A \rightarrow \{0, 1\}$ |
| | Classical Algebraic Geometry | Ring $k[x_1, \dots, x_n]/\mathfrak{t}$ (reduced and finitely generated over an alg. closed field K) | k -alg. homo. $k[x_1, \dots, x_n]/\mathfrak{t} \rightarrow K$ |
| | Gelfand duality (observables-states duality) | Commutative C^* -algebra A | Characters $A \rightarrow \mathbb{C}$ |
| | Pontryagin duality (Fourier theory, position-momentum duality in quantum mechanics) | Locally Compact Abelian Group G | Characters $G \rightarrow U(1)$ |
| | Galois-Grothendieck | k -alg. A split by K | k -alg. homo. $A \rightarrow K$ |
| General Reconstruction Theorems | Affine Schemes | Ring A (with no restrictions) | $A \rightarrow A_{\mathfrak{p}}$ ($A_{\mathfrak{p}}$ local ring) |
| | Sheaf representations for toposes (syntax-semantics dualities for predicate logic) | Topos \mathcal{E} | Functor $\mathcal{E} \rightarrow \mathcal{E}_{\mathfrak{p}}$ ($\mathcal{E}_{\mathfrak{p}}$ local topos) |

In the Table N°1 we include all the dualities mentioned thus far and specify both the mathematical structure that is being represented and the kind of representations used to recover it (see also Refs.[9] and [5, Table 4.3, 160])²². It is a remarkable fact that all these important mathematical results are instances—up to contextual differences depending on each particular mathematical framework—of a common reconstruction protocol. In Figure N°III.2, we have summarized the general meta-mathematical pattern instantiated by these different algebra/structure-geometry/semiotics dualities. Given an algebraic structure A we can consider a certain class of representations of A , that is morphisms of the form $A \rightarrow B$ (for a fixed or a variable codomain B). In order to obtain a restricted (respectively, general) reconstruction theorem, the corresponding set (respectively, structured set) of representations has to be endowed with extra geometric structure. In general, there is a

²² For a unified discussion of some of these dualities see Ref.[125, §5.8, 171-184].

“semiotic” functor—that we have noted $Spec(-)$ ²³—from the (opposite of the) category of algebraic structures to the corresponding category of geometric objects. Since any point in $Spec(A)$ is a map of the form $p : A \rightarrow B$, each element a in the algebraic structure A defines a quantity \hat{a} on $Spec(A)$ given by the Gelfand inversion formula $\hat{a}(p) = p(a)$. The quantity \hat{a} encodes the representations of the element $a \in A$ in all the representations parameterized by $Spec(A)$. The reconstruction theorem takes the form of an isomorphism between the original algebraic structure A and an algebra of quantities on $Spec(A)$ defined by means of a *structure functor* $\Gamma(-)$. We could say that the elements of the structure A to be reconstructed are realized—by means of a Gelfand transform—as quantities on the “semiotic” space $Spec(A)$ that parameterizes certain representations of A . The fact that this general pattern is at the basis of important reconstruction theorems in different areas of mathematics gives support to the thesis according to which this *algebra/structure-geometry/semiotics* duality is (as Lawvere states) “*fundamental to mathematics*” [75, 17] [79, 16].²⁴



(III.2)

In order to conclude this section, let’s revisit the relation between the two approaches to the algebra-geometry duality, namely the *spectral approach* (in which one starts with an algebra) and the *geometric approach* (in which one starts with a geometric object). In Manin’s terms, the “*fundamental [algebra-geometry] duality that increasingly manifests itself in modern mathematics*” assumes the following two symmetric forms (“*with a different emphasis*”) [88, 12]:

“The space \longleftrightarrow the ring of functions on the space” (geometric approach)

“The ring \longleftrightarrow the spectrum of its ideals of certain type” (spectral approach)

(III.3)

We could say that these two descriptions of the algebra-geometry duality differ in the characterization of which structure is being represented or “observed”. In the spectral approach, the algebra A plays the role of the represented object and each point of the space $Spec(A)$ defines a particular representation of A . In the geometric approach, the points

²³ Since we are also considering here the general case in which the codomain B can vary, we omit the subscript indicating the codomain of the representations.

²⁴ In the article *Adjointness in foundations*, Lawvere states that the duality between commutative rings and the corresponding geometric entities is a particular case of a more general duality between what he calls the “Formal” (the side of the duality corresponding to the algebraic structure) and the “Conceptual” (the semiotic side of the duality) [78, Sect.4] (see the discussions of this distinction in Refs.[111, Sect.5.7] and [94, Sect.2.4]). In Lawvere’s terms: “*That pursuit of exact knowledge which we call mathematics seems to involve in an essential way two dual aspects, which we may call the Formal and the Conceptual. For example, we manipulate algebraically a polynomial equation and visualize geometrically the corresponding curve. [...] any attempt to formalize Foundations [‘the study of what is universal in mathematics’] would be a description of this claimed ‘duality’ between the Formal and the Conceptual*” [78, 3-4].

of the space $\text{Spec}(A)$ are the “states” being “observed” by the “observables” (functions or sections) in the corresponding algebra of quantities. We could say that the Gelfand transform transforms each *represented* element a in A into an “observable” \hat{a} on the corresponding space.²⁵ Now, the apparent symmetry between the two descriptions should not lead us to believe that they are strictly equivalent. Briefly, there exist geometric objects that cannot be constructed as the spectrum of an algebra. Let’s consider this point in some detail.

A reconstruction theorem encodes the fact that the corresponding *moduli space* of representations contains “enough points/representations” to recover the represented algebraic structure. Technically, this means that the composition *structure functor* $\Gamma(-) \circ$ *semiotic functor* $\text{Spec}(-)$ in Figure N°II.1 yields the identity on the algebraic structures modulo isomorphisms (i.e. $\Gamma(\text{Spec}(A)) \simeq A$). But we could start with a geometric object X that is not in the image of the semiotic functor $\text{Spec}(-)$ and consider the effect of the other possible composition of the two functors—i.e. the composition *semiotic functor* $\text{Spec}(-) \circ$ *structure functor* $\Gamma(-)$ —acting on this geometric object. So, we start with the geometric object X , we extract an algebraic structure $\Gamma(X)$ out of it and then consider the space $\text{Spec}(\Gamma(X))$ that parameterizes the corresponding representations of $\Gamma(X)$. Now, even if we have a representation theorem of the form $\Gamma(\text{Spec}(A)) \simeq A$, it might not be necessarily the case that $\text{Spec}(\Gamma(X)) \simeq X$. In other terms, we can always use the functor $\Gamma(-)$ to extract an algebraic structure $\Gamma(X)$ out of the geometric object X , but X might not coincide with the moduli space $\text{Spec}(\Gamma(X)) \simeq X$ that parameterizes the representations of $\Gamma(X)$. In that case, the pre-duality given by the adjunction between the structure functor and the semiotic functor is not strictly speaking a duality (i.e. an equivalence of categories), even if we have a reconstruction theorem of the form $\Gamma(\text{Spec}(A)) \simeq A$.²⁶ Hence, the spectral approach (that starts with an algebra) and the geometric approach (that starts with a geometric object) are equivalent only when the pre-duality between algebra and geometry is a duality. An instance of this scenario is provided by the notion of *general* (i.e. not necessarily affine) *scheme* (e.g. the scheme associated to a projective variety). Roughly speaking a scheme X is a *locally ringed space* such such each point has an open neighborhood where the scheme looks like an affine scheme (that is, like a scheme obtained as the spectrum of a ring) [57, 74]. The composition $\text{Spec}(\Gamma(X))$ is related to the original scheme X by a unique morphism $\psi : X \rightarrow \text{Spec}(\Gamma(X))$ that is an isomorphism only when X is affine [32, 4 & 7]. We could say that in this case there exists a pre-duality between the (opposite of the) category of rings and the category of schemes that restricts to a duality on affine schemes.

IV. RESTRICTED RECONSTRUCTION THEOREM IN ALGEBRAIC GEOMETRY

We shall now consider in more detail the restricted reconstruction theorem in classical algebraic geometry. This theorem only allows to recover certain kind of rings from a collections of their representations, namely reduced and finitely generated rings over algebraically closed fields (for the discussion proposed in this section we recommend the reader the introduction to Ref.[37] and Ref.[88]). A polynomial ring $P \doteq k[x_1, \dots, x_n]$ in n variables over a field k can be understood as a sort of “linguistic universe” in which it is possible to define predicates of the form

$$\exists n\text{-tuples } (a_1, \dots, a_n) \text{ such that } p(a_1, \dots, a_n) = 0 \quad (\text{IV.1})$$

²⁵ Majid describes this transformation from an element being “observed” into an “observable” in the following terms: “This map [the Gelfand transform] literally turns the tables and says that the object a being seen by p can instead be viewed as an observer \hat{a} (corresponding directly to a) seeing p with the same value $p(a) = \hat{a}(p)$ ” [85, 397].

²⁶ In mathematical terms, both the unit and the counit of an adjunction are natural isomorphisms when the adjunction defines an equivalence of categories [82, 93]. But we can have an intermediate situation in which only the unit or the counit of the adjunction is a natural isomorphism.

with $p \in P$.²⁷ The field k plays here the role of what we could call in the wake of Kronecker *domain of rationality* [*Rationalitätsbereich*] (also called *coefficient domain* or *domain of definition*).

We shall now consider the truthmakers of such predicates, i.e. the values of the x_i 's that make the predicates true. In a first approach, it is natural to look for the truthmakers of a predicate in the domain of definition of the latter. However, there is no reason for such a restriction: the fact that a predicate is defined over a domain k does not mean that its truthmakers only exist in k . This distinction between the domain of rationality k in which a problem is formulated and the possible codomains of rationality wherein one can look for its solutions made possible the successive extensions of the number systems (from \mathbb{N} to \mathbb{Z} to \mathbb{Q} to \mathbb{R} to \mathbb{C}) (see Refs.[15, 20, 87] for a conceptual analysis of the notion of *domain extension*).²⁸

The minimal condition imposed on the possible codomains of definition is that they must be k -algebras, i.e. that their elements can be multiplied, added, and multiplied by scalars in k (otherwise, the polynomial expressions in the corresponding variables could not even be defined). In what follows, we shall restrict the possible codomains to k -algebras k' that are fields. If T is a subset of P (i.e. a particular family of predicates defined over k), the n -tuples (a_1, \dots, a_n) with $a_i \in k'$ such that $p(a_1, \dots, a_n) = 0$ for all $p \in T$ will be called *k' -truthmakers* of the family T . By using Grothendieck and Dieudonné's terminology, we shall also say that the solutions in k' are *k' -valued points* of the *variety over k* defined by the family T [49, 7]. The set of k' -truthmakers of the family T defines a subset (called an *algebraic affine variety*) of the affine n -space $A_{k'}^n$ over k' (i.e. of the set of n -tuples of elements in k'). Now, the points in the variety also satisfies equations of the form $\sum_i h_i p_i = 0$, where $p_i \in T$ and h_i are polynomials in P . In order to get rid of this ambiguity regarding the polynomials that define the variety, we have to substitute the family T by the *ideal* \mathfrak{t} generated by T . The affine variety in $A_{k'}^n$ defined by an ideal \mathfrak{t} will be denoted $V_i(k')$. Now, an ideal \mathfrak{t} does not define an algebraic variety as such but rather an algebraic variety immersed in $A_{k'}^n$ (see for instance the example in Ref.[37, 236]). It is possible to "factor out" this immersion and pass to an intrinsic description by considering the quotient rings P/\mathfrak{t} .²⁹ Indeed, the quotient rings defined by different immersions of the same abstract variety are isomorphic. Passing from P to P/\mathfrak{t} amounts to identify different polynomials in P that coincide when evaluated on the variety, i.e. polynomials that differ in a function vanishing on the variety. In other terms, we are making abstraction from the fact that these polynomials might differ elsewhere.³⁰

Given an ideal \mathfrak{t} in P and a k -algebra k' there is a natural bijection between the algebraic variety $V_i(k')$ and the k -algebras morphisms $P/\mathfrak{t} \rightarrow k'$ defined by evaluating the polynomials in P/\mathfrak{t} on the corresponding point of the variety [49, 8]. This means that each point of the variety $V_i(k')$ can be interpreted as a k' -representation of P/\mathfrak{t} . We could say that the variety $V_i(k')$ encodes the *k' -representation theory* of the algebraic structure P/\mathfrak{t} . We can now pose the following reconstruction problem: is it possible to recover the quotient ring P/\mathfrak{t} from its k' -representations? As we have explained in Section N°III, the reconstruction of an algebraic structure from a collection of representations requires to

²⁷More precisely, a polynomial $p \in P$ can be written in the form $p(x_1, \dots, x_n) = \sum_i c_i x^i$, where $c_i \in k$ and x^i are monomials of the form $x^i = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$ with $i = (i_1, \dots, i_n)$ an n -tuple of non-negative integers.

²⁸By working with polynomial equations in one variable (which is a particular case of 0-dimensional algebraic geometry), Galois theory is essentially based on the distinction between the *domain of definition* (or *coefficient domain*) of the polynomial in question (i.e. a field containing the coefficients of the polynomial) and the possible *codomains* in which one can search for its solutions [27]. According to Cartier, "*Galois was certainly the first person to notice the polarity between equations and their solutions*" [22, 290].

²⁹For a discussion of the difference between extrinsic and intrinsic descriptions in terms of the functorial approach to algebraic geometry that we shall introduce below see Ref.[48, 29–37].

³⁰For instance, the polynomials $x+y$ and $x+x^2$ coincide on the variety defined by the equation $y = x^2$. In other terms, their difference is "invisible" on the parabola.

introduce a Gelfand transform between the algebraic structure at stake (in this case the ring P/\mathfrak{t}) and an algebra of quantities on a *moduli space* of representations. In this case, the Gelfand transform assigns to every element $[p]$ of the ring P/\mathfrak{t} a k' -valued quantity $p_{k'}$ on the variety $V_{\mathfrak{t}}(k')$ defined by the expression $p_{k'}(\alpha) = p(\alpha)$ for $\alpha \in V_{\mathfrak{t}}(k')$. The quantity $p_{k'}$ encodes all the k' -representations $p(\alpha)$ of $[p] \in P/\mathfrak{t}$. We could say that the Gelfand transform acts as a sort of *prism*, in the sense that it permits to associate to each element $[p] \in P/\mathfrak{t}$ its k' -*spectrum*, i.e. the set of k' -representations of $[p]$.

The ring P/\mathfrak{t} can be reconstructed from its k' -representations if the Gelfand transform is injective. Now, it might be the case that the codomain k' is such that the variety $V_{\mathfrak{t}}(k')$ does not have “enough points” (i.e. enough k' -representations of P/\mathfrak{t}) to faithfully reconstruct P/\mathfrak{t} . For instance, if the field k' does not contain all the roots of a polynomial $p \in k[x]$, the algebra $k[x]/(p)$ may not be faithfully represented as an algebra of quantities on the space $V_{\mathfrak{t}}(k')$. This means that different elements $[f]$ and $[g]$ in P/\mathfrak{t} might define the same quantity on the space $V_{\mathfrak{t}}(k')$, thereby having the same k' -spectrum. In such a case, the variety $V_{\mathfrak{t}}(k')$ does not “separate” $[f]$ and $[g]$. In the extreme case, there might be no k' -representations of P/\mathfrak{t} .³¹ In order to guarantee that the codomain provides enough representations of P/\mathfrak{t} so that this ring can be faithfully reconstructed, the codomain must be an algebraically closed field extension K of k .³² If in addition we restrict the ideals under consideration to *radical ideals* (i.e. to ideals \mathfrak{t} such that $\mathfrak{t} = \sqrt{\mathfrak{t}}$)³³, then we obtain a bijective correspondence between algebraic objects (radical ideals of polynomials) and geometric objects (algebraic varieties) [57, Cor.1.4, 4].

A restricted reconstruction theorem can now be obtained by comparing two algebraic objects, namely the quotient ring P/\mathfrak{t} and a ring of quantities on the variety $V_{\mathfrak{t}}(K)$ defined by means of the Gelfand transform. The corresponding reconstruction theorem states that the Gelfand transform establishes an isomorphism

$$P/\mathfrak{t} \simeq \mathcal{O}(V_{\mathfrak{t}}(K)) \quad (\text{IV.2})$$

between the ring P/\mathfrak{t} that one wants to reconstruct and the ring $\mathcal{O}(V_{\mathfrak{t}}(K))$ of *regular functions*³⁴ on the variety $V_{\mathfrak{t}}(K)$ [57, Th.3.2 (a), 17]. Since P/\mathfrak{t} is reconstructed from the representations of P/\mathfrak{t} into a *unique* codomain (the algebraically closed field K), this isomorphism provides an example of a *restricted* reconstruction theorem. We could say that the algebraically closed field K provides enough representations of the ring P/\mathfrak{t} (i.e. enough common roots of the polynomials in \mathfrak{t}) so that P/\mathfrak{t} can be faithfully reconstructed from the collection of its K -representations.

It is worth noting that this algebra-geometry (or structure-semiotics) duality encodes a sort of “indeterminacy principle” that presents an analogy with the indeterminacy principle in quantum mechanics. Indeed, there is an inverse correlation between the unambiguous (or sharp) determination of an element in P/\mathfrak{t} and the unambiguous determination of a particular K -representation in the variety $V_{\mathfrak{t}}(K)$. If one just considers a single K -representation of P/\mathfrak{t} —i.e. a single point in $V_{\mathfrak{t}}(K)$ —, then the elements in the ring P/\mathfrak{t} remain maximally undetermined. Indeed, different elements $[f], [g] \in P/\mathfrak{t}$ might be

³¹ For example, there are no \mathbb{R} -representations of the ring $\mathbb{R}[x]/(x^2 + 1)$, which means that there are no \mathbb{R} -truthmakers of the predicate $x^2 + 1 = 0$. Of course, this does not mean that this predicate is false, but rather that its truthmakers have to be searched in an algebraic extension of the codomain \mathbb{R} .

³² The *Hilbert Weak Nullstellensatz* (*weak theorem of zeros*) states that if the codomain K is algebraically closed, then the only ideal \mathfrak{t} such that $V_{\mathfrak{t}}(K) = \emptyset$ is $\mathfrak{t} = P$. This theorem may be understood as the fundamental theorem of algebra for polynomials in several variables in the sense that it states that every family of polynomials generating an ideal smaller than the whole ring has at least one common zero in K^n , i.e. it does not define the empty variety.

³³ Let's recall that the radical $\sqrt{\mathfrak{t}}$ of an ideal \mathfrak{t} is the set of elements $f \in P$ such that $f^n \in \mathfrak{t}$ for some natural number n . The *Hilbert Nullstellensatz* states that two ideals \mathfrak{t}_1 and \mathfrak{t}_2 define the same K -variety (being K an algebraically closed field) if and only if $\sqrt{\mathfrak{t}_1} = \sqrt{\mathfrak{t}_2}$. Therefore, ideals \mathfrak{t}_1 and \mathfrak{t}_2 with the same radical cannot be distinguished by just considering their K -representations. This explains why we have to restrict the algebraic side to radical ideals in order to obtain a reconstruction theorem.

³⁴ A function $f : X \rightarrow K$ is *regular* at a point $p \in X$ if f can be expressed as a rational function g/h on a neighborhood of p such that h is nowhere zero in U . A function f is regular on X if it is regular at every $p \in X$ [57, 15].

represented by the same “image” in a K -representation $x \in V_t(K)$. Conversely, if the K -representation is completely “undetermined” (i.e. if one considers all the possible K -representations of the elements in P/t at once), then the elements in the ring may be faithfully determined. This remark permits to understand why each single $[f] \in P/t$ is in correspondence—via the Gelfand transform—with a quantity f_K on the set of *all* possible K -representations of P/t . This leads to what we could call the “indeterminacy principle” associated to the algebra-geometry duality: whereas the sharp determination of a point in the space requires to consider a whole family of quantities, the sharp determination of a quantity requires to consider the values that it takes on all the points of the space. This “indeterminacy principle” could be understood as the mathematical incarnation of the phenomenological fact that an empirical object cannot be faithfully reconstructed from one of its profiles, i.e. from the single aspect that it offers to a subject placed in a particular position. Roughly speaking, a faithful determination of an object requires to turn around the object, i.e. to “undetermine” the viewpoint of observation.³⁵

V. FROM SETS OF POINTS TO FUNCTORS OF POINTS

As we have explained before, the distinction between the *domain of definition* containing the coefficients of a family of polynomial predicates and the different possible *codomains* that might contain truthmakers of the former elicits a sort of *relativity* in algebraic geometry: we can associate to a quotient ring P/t different algebraic varieties, namely one algebraic variety $V_t(k')$ for each k -algebra k' . As we have seen in the previous Section, it is possible to reconstruct certain rings from the algebra of regular functions on the corresponding varieties by choosing a suitable codomain, namely a k -algebra given by an algebraically closed field K . The decision to privilege a (well-chosen) particular codomain was notably fostered by A. Weil in his treatise *Foundations of Algebraic Geometry* in the form of what he called “*universal domains*” (algebraically closed fields of infinite transcendence degree) [128] (regarding Weil’s treatise see also Ref.[108]).³⁶ Roughly speaking, the idea is to work within a codomain that is big enough so that it contains all that one might need for the purposes at stake.

The problem is that the reconstruction protocol used in classical (i.e. before Grothendieck) algebraic geometry only works for certain rings, namely reduced and finitely generated rings over algebraically closed fields. Now, in the framework of the refoundation of algebraic geometry associated to scheme theory, Grothendieck proposed a more general reconstruction protocol that works for general rings. The key point of Grothendieck’s innovative strategy is that it does not “break” the aforementioned relativity by privileging a particular codomain, but rather fully embraces it *by considering all the codomains at once* [48].³⁷ According to Grothendieck’s philosophy, a family of polynomial equations over k does not have a solution set defined in a privileged codomain, but rather (what we could call in the wake of Ref.[97, p.113]) a *solution structured set* composed of the sets of solutions in each k -algebra B for every B (also called *variable set of solutions* in Ref.[91,

³⁵ This heuristic relation between the algebra-geometry duality and the indeterminacy principle in quantum mechanics has been explored in the series of papers [27, 101, 102]. In order to argue that this might be more than a mere analogy, it is worth noting that the position-momentum duality in quantum physics is provided by Fourier theory, that is—in group-theoretic terms—by the Pontryagin duality for locally-compact abelian groups [29, 104]. Moreover, the symplectic structure of the corresponding phase space encodes the “internal” self-dual structure associated to the “external” duality between the position and the momentum representations (see Refs.[85, 102, 101] for further discussions of this point).

³⁶ Grothendieck and Dieudonné describe Weil’s approach in the following terms: “*C’est seulement à partir de 1940, avec la Géométrie algébrique ‘abstraite’ (c’est-à-dire sur un corps de base k quelconque, pouvant être de caractéristique $\neq 0$) développée surtout par Weil, Chevalley et Zariski, que l’idée du changement de base prend de l’importance dans un contexte plus général [...] Toutefois [...] chez Weil [la généralité de cette opération] est quelque peu masquée par le parti pris de se restreindre une fois pour toutes à n’envisager que des sous-corps d’un corps algébriquement clos ‘assez grand’ (le ‘corps universel’), restant donc en apparence assez proche du point de vue classique où le corps \mathbb{C} tenait ce rôle*” [49, 6].

³⁷ See Ref.[109, Sec.5.4] for a comparison between Weil’s universal domains and Grothendieck’s functorial approach.

14.3.2]).³⁸ More formally, the “solution” to a family of polynomial equations over k will be defined as the “application” that sends any k -algebra B to the solution set in B in a functorial manner.

The central mathematical object in Grothendieck’s approach is given by the so-called *functor of solutions* [48] or (in the dual geometric version) *functor of points*. So, let us consider this notion with some detail (see also Refs.[27, 40, 124]). To do so, let’s consider a general ring A . The idea is to assign to each possible codomain given by a ring B a “space” $\text{Spec}_B(A)$ whose underlying set will be given by $\text{Hom}_{\text{Ring}}(A, B)$. More precisely, we can formalize the fact that a ring A defines a different space $\text{Spec}_B(A)$ of B -representations $A \rightarrow B$ for each possible codomain B by associating to A a covariant functor

$$\text{Spec}_{(-)}(A) : \text{Ring} \rightarrow \text{Set} \quad (\text{V.1})$$

sending each ring B to the B -spectrum $\text{Spec}_B(A) = \text{Hom}_{\text{Ring}}(A, B)$ of A .³⁹

By definition, the functor of solutions $\text{Spec}_{(-)}(A)$ of A is an object in the category $[\text{Ring}, \text{Set}]$ of all functors from Ring to Set (where the morphisms are given by the natural transformations of functors). The *Yoneda embedding* associates to any ring A the functor (V.1) “represented” by A (regarding the Yoneda embedding and the Yoneda lemma, see for instance Ref.[10, §8.2 & 8.3, 160-166]). This means that the functor (V.1) can also be described as the covariant *representable functor* represented by the ring A (see for instance the discussion of representable functors proposed in Ref.[93, Sect.3.2.2]). In particular, the ring P/\mathfrak{t} is the object that represents the “functor of solutions” $\text{Spec}_{(-)}(P/\mathfrak{t})$, that is, the object that “classifies” the solutions of the equations defined by the ideal \mathfrak{t} in all the k -algebras k' . The Yoneda lemma states that the set $F(B)$ is in bijective correspondence with the set of natural transformations $F(-) \rightarrow \text{Spec}_{(-)}(B)$, where $F(-)$ is any (i.e. not necessarily representable) Set -valued functor on Ring . It follows from the Yoneda lemma that the Yoneda embedding is *fully faithful*, i.e. that this embedding defines a full subcategory of the category of Set -valued functors on Ring .

The conceptual importance of the Yoneda lemma is (at least) twofold. Firstly, the Yoneda lemma entails a shift of perspective from a *substantialist* conception according to which the fundamental mathematical entities (of algebraic geometry in this case) are given by the objects of the corresponding category to a *structuralist, relational, behavioral*, or (in Manin’s words [89]) *sociological* perspective according to which the fundamental entities are functors. As Mazur puts it, “[t]he lights are dimmed on mathematical objects and beamed rather on the corresponding functors; that is, on the networks of relationships entailed by the objects” [95]. The main conceptual point here is that the functor $\text{Spec}_{(-)}(A)$ provides a good ‘semiotic’ substitute for the ring A . This means that $\text{Spec}_{(-)}(A)$ describes A in terms of the representations $A \rightarrow B$ of A defined with respect to all the other objects B of the category. If two objects have equal (i.e. isomorphic) functors encoding their representations, then the two objects are also equal (i.e. isomorphic).

The second important conceptual consequence of the Yoneda lemma is that it introduces an *ontological extension* of the domain of entities at stake, namely an extension from the objects of the initial category to *any* functor in $[\text{Ring}, \text{Set}]$, be it representable or not. Indeed, the Yoneda embedding is not *essentially surjective*, i.e. not every functor

³⁸ In Grothendieck and Dieudonné’s own terms: “[o]n y fait abstraction des propriétés spéciales aux solutions du système $[p(x_1, \dots, x_n) = 0 \text{ for all } p \in T]$ dans l’espace particulier k^n d’où l’on est parti, pour considérer, pour *chaque* k -algèbre k' , l’ensemble des solutions de [ce système] *dans* k'^n , et la façon dont cet ensemble *varie avec* k' [...] L’idée de ‘variation’ de l’anneau de base que nous venons d’introduire s’exprime mathématiquement sans peine grâce au langage fonctoriel [...]” [49, 6].

³⁹ On morphisms, the functor $\text{Spec}_{(-)}(A)$ sends each morphism $B \xrightarrow{f} C$ to the morphism $\text{Spec}_f(A) : \text{Spec}_B(A) \rightarrow \text{Spec}_C(A)$ given by $(A \xrightarrow{g} B) \mapsto (A \xrightarrow{f \circ g} C)$. In particular, when the ring A is the k -algebra P/\mathfrak{t} defined by an ideal \mathfrak{t} of polynomials, the functor (V.1) sends each k -algebra B to the solution set in B of the corresponding polynomials (i.e. $B \mapsto V_{\mathfrak{t}}(B)$) and each morphism $h : B \rightarrow B'$ is sent into $V_{\mathfrak{t}}(h) : V_{\mathfrak{t}}(B) \rightarrow V_{\mathfrak{t}}(B')$ where $V_{\mathfrak{t}}(h)(a_1, \dots, a_n) = (h(a_1), \dots, h(a_n))$.

in $[Ring, Set]$ is representable.⁴⁰ Now, the Yoneda lemma guarantees that even non-representable functors $F(-)$ can be interpreted as functors of points, i.e. in terms of morphisms to other objects. More precisely, the Yoneda lemma states that the elements in the set $F(B)$ are in bijection with the natural transformations $F(-) \rightarrow \text{Spec}_{(-)}(B)$, i.e. with (what we could call) the $\text{Spec}_{(-)}(B)$ -representations of $F(-)$. Even if the elements in $F(B)$ for F a non-representable functor cannot be interpreted in terms of morphisms in $Ring$, the Yoneda lemma guarantees that these elements can still be interpreted as representations of $F(-)$ in the functor category $[Ring, Set]$. The trick to do so, is to substitute B by its functor of points $\text{Spec}_{(-)}(B)$. It is also worth noting that the description of the functor F in terms of its representations only uses morphisms to representable functors. Roughly speaking, the representable functors “generate” the whole functor category $[Ring, Set]$.⁴¹ In the terms used by Schreiber, the representable functors $\text{Spec}_{(-)}(B)$ can be understood as “test objects” out of which “generalized objects” can be constructed. A “generalized object” $F(-)$ is a non-representable element of the functor category $[Ring, Set]$ that is completely defined by its “representations” into the test objects, i.e. by the sets $\text{Hom}_{[Ring, Set]}(F(-), \text{Spec}_{(-)}(B))$ of $\text{Spec}_{(-)}(B)$ -representations of $F(-)$ for every $\text{Spec}_{(-)}(B)$ (see for instance Refs.[112, 61] and [75, 17]).

VI. GENERAL RECONSTRUCTION THEOREM IN ALGEBRAIC GEOMETRY

As discussed in the previous section, the functor of points $\text{Spec}_{(-)}(A)$ on the category of rings is a fully faithful substitute for the ring A . Since each set $\text{Spec}_B(A)$ may be thought of as a space, the functor of points establishes a correspondence between rings and a whole family of spaces. In Grothendieck and Dieudonné’s terms, “[...] *on n’a plus affaire à un ‘objet’ bien déterminé comme dans la Géométrie algébrique classique, mais à une ‘famille d’objets’ variable avec $[B]$* ” [49, 12]. Now, is it possible to recover an algebra-geometry duality between the ring A and a *single* space? Is it possible to establish an isomorphism between A and a ring of quantities on a *single* “moduli space” parameterizing representations of A ? To do so, one should be able to unify the family of B -spectra $\text{Spec}_B(A)$ for all B into a single topological space (that we shall denote) $\text{Spec}(A)$. Then, and in order to obtain an algebra-geometry duality, we have to introduce quantities on this space. As we shall see, these quantities are given by sections of a sheaf on $\text{Spec}(A)$ endowed with a suitable topology (called *structure sheaf* and denoted $\mathcal{O}_{\text{Spec}(A)}$). The geometric object $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ resulting from this construction is called *affine scheme*.⁴² Briefly, an affine scheme associated to a ring A is an underlying set $\text{Spec}(A)$ endowed with a topology and a sheaf of quantities $\mathcal{O}_{\text{Spec}(A)}$ on it. As we shall see, the stalks of the sheaf are local rings. Hence, an affine scheme is a *locally ringed space*.⁴³ We shall now briefly describe the salient features of this construction.

First, the underlying set of points of an affine scheme is defined by the field-valued representations of the ring, i.e. by representations of the form $A \rightarrow k$ for k a field. Now,

⁴⁰ The utilization of the Yoneda embedding sometimes requires to study the conditions under which a given functor is representable [46]. One important feature of representable functors is that they have the sheaf property with respect to certain Grothendieck topologies (the *subcanonical* topologies) on the corresponding category. These ideas were used by Grothendieck in Ref.[46] for the construction of various important spaces, like for instance the Hilbert and Picard schemes. The strategy used by Grothendieck was to first study the relevant functor, and to prove then that the functor is representable.

⁴¹ This characterization of the functor category $[Ring, Set]$ as “generated” by the representable functors is formalized by the result according to which any non-representable functor F is a colimit (i.e., roughly speaking, the result of a pasting) of representable functors (see for instance Ref.[84, 41]). The category $[Ring, Set]$ can then be understood as the *free co-completion* of the category of rings. We could also say that the representable functors are “dense” so to speak in $[Ring, Set]$.

⁴² In Refs.[40, 57], the term *spectrum* is used to denote affine schemes $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$, that is topological spaces of the form $\text{Spec}(A)$ endowed with sheaves of quantities $\mathcal{O}_{\text{Spec}(A)}$. Here we prefer to use the term *spectrum* only for the underlying set $\text{Spec}(A)$ and called the corresponding ringed space *affine scheme* (like Dieudonné does in Ref.[35]).

⁴³ More generally, a *scheme* (not necessarily affine) is a ringed space (X, \mathcal{O}_X) that is *locally* isomorphic to an affine scheme.

the field-valued representations of a ring are in a certain sense redundant since for instance $t : A \rightarrow k$ and $t' : A \xrightarrow{t} k \hookrightarrow K$ (where K is a field extension of k) would count as different points. This redundancy can be removed by introducing an equivalence relation for k -valued representations $A \rightarrow k$ for variable k , namely that two such representations are equivalent if they have the same kernel [49, 12-13]. Since a field is an integral domain, the kernel $\mathfrak{p} = \ker(t)$ of a k -representation $t : A \rightarrow k$ is a prime ideal of A and the integral domain A/\mathfrak{p} is isomorphic to a subring of k . All in all, the underlying set of the affine scheme associated to a ring A will be given by the set of *prime ideals* of A , where the latter encode the field-valued representations of A modulo the aforementioned redundancy.

This definition of the underlying set of points of an affine scheme constitutes a first surprising feature of Grothendieck’s construction. In the classical theory of algebraic varieties, the underlying set of points of an algebraic variety is in bijective correspondence with the *maximal ideals* of the coordinate ring [57, Th.3.2(b), 17]. In turn, prime ideals are in bijective correspondence with the irreducible closed subsets of the algebraic variety. Therefore, the underlying set $\text{Spec}(A)$ of an affine scheme includes not only the standard points corresponding to maximal ideals but also the so-called *generic points* (introduced by the Italian algebraic geometers at the beginning of the XXth century) that correspond to irreducible closed subsets (see for instance [57, 75]).⁴⁴ In a certain sense, Grothendieck’s theory of schemes bridges the gap between what De Risi calls the *classical* (Greek) *geometry of figures* on the one hand and the *modern geometry of space* on the other [33]. According to De Risi, Greek geometry deals with individual figures that “*are not embedded in any spatial background*” since “*the concept of space itself is altogether missing from the cultural background of Antiquity*” [33, 3-4].⁴⁵ In this context, no figure plays a privileged role to the detriment of the others. In Cartier’s terms,

“For Euclid there are *geometric figures*, and a point is merely one element of a figure, the most elementary, perhaps, since it is assumed to have no dimension: neither length, nor breadth, nor thickness. Figures generate one another. A given line D is defined by the property of passing through two given points P and Q , but a point P is defined as the intersection of two lines D and Δ . [...] it is not in the spirit of Euclid to regard a line as a set of points. A figure in the sense of Euclid is more than a simple set of points.” [23]

By contrast—and according to the modern conception that began with Leibniz’s *analysis situs* [34] and became the common definition by the dawn of the XIXth century—, modern geometry is the *science of space*.⁴⁶ In this last framework, a space is understood as a set of points equipped with additional geometric structures (e.g. a topology, a connection, a metric, etc.) and the geometric figures are now understood as subsets of such space. In this framework, a particular kind of irreducible figure plays a privileged role to the detriment of the other irreducible figures, namely the unextended points that belong to the underlying set of the space. In a certain sense, the notion of affine scheme is diagonal to De Risi’s distinction. On the one hand, an affine scheme is a set equipped with

⁴⁴ A standard argument to justify the identification between the points of an affine scheme and the prime ideals of the corresponding ring (rather than just the maximal ideals) is that a ring homomorphism $\varphi : A \rightarrow B$ does not induce a well-define map $\text{Specm}(B) \rightarrow \text{Specm}(A)$ since $\varphi^{-1}(\mathfrak{b})$ is not necessarily a maximal ideal in A (where $\text{Specm}(B)$ denotes the set of maximal ideals of B). However, the inverse image of a prime ideal under a ring homomorphism $\varphi : A \rightarrow B$ is a prime ideal of A .

⁴⁵ According to Grant, “[t]here is nothing in Euclid’s geometry to suggest that he assumed an independent, infinite, three-dimensional, homogeneous space in which the figures of his geometry were located. In a purely geometric sense, such a space would have been superfluous because every geometric figure has its own internal space” [45, 16]. In *Space-Time-Matter*, also Weyl establishes a contrast between “a true geometry, [understood as] a doctrine of space itself” and “almost everything else that has been done under the name of geometry, a doctrine of the configurations that are possible in space” [130, 102].

⁴⁶ Regarding the difference—for instance—between Euclid’s *Elements* and Hilbert’s *Grundlagen der Geometrie* see Ref.[96, 14-15].

additional structure (namely, a topology and a sheaf of quantities), thereby providing an example of the modern understanding of the notion of space. However, the underlying set of “points” does not only include the standard unextended points associated to maximal ideals, but also one generic point for each irreducible closed subset. Hence, the underlying set of the space is a set of both extended and unextended irreducible figures. An affine scheme can therefore be understood as an underlying set of irreducible figures endowed with an additional geometric structure. The irreducible figures—rather than being understood as subsets of a preexisting space *qua* set of unextended points—form the “*skeleton*” of the space.⁴⁷

In order to obtain a reconstruction theorem for a ring A we need to define a Gelfand transform sending each element in A to a quantity on $\text{Spec}(A)$. As we have explained in Section N°III, the Gelfand transform is defined by an inversion formula (III.1) that defines the value of a quantity \hat{a} associated to a ring element $a \in A$ at a point \mathfrak{p} by using the fact that this point is itself a representation of A . So, we shall define a Gelfand transform that sends each element $a \in A$ to the quantity \hat{a} on $\text{Spec}(A)$ given by $\hat{a}(\mathfrak{p}) = t_{\mathfrak{p}}(a)$ where $t_{\mathfrak{p}} : A \rightarrow A/\mathfrak{p}$ is the quotient map associated to the point \mathfrak{p} . As we said before, the map $t_{\mathfrak{p}}$ can be redundantly understood as a field-valued representation of A with respect to different fields. Here we shall consider $t_{\mathfrak{p}}$ as a field-valued representation of A with values in the smallest field in which A/\mathfrak{p} can be embedded, namely the fraction field $\kappa_{\mathfrak{p}}$ (also called *residue field*) of the integral domain A/\mathfrak{p} . Since the $A/\mathfrak{p} \rightarrow \kappa_{\mathfrak{p}}$ is an inclusion, we shall consider that the quantity \hat{a} is a $\kappa_{\mathfrak{p}}$ -valued quantity on $\text{Spec}(A)$. The fact that the fields $\kappa_{\mathfrak{p}}$ depend on the points \mathfrak{p} in $\text{Spec}(A)$ on which we evaluate the quantity \hat{a} encodes the fact that the space $\text{Spec}(A)$ parameterizes representations of A into different codomains.⁴⁸

According to the general pattern schematized in Figure III.2, we want to use this Gelfand transform to define an isomorphism between the ring A and an algebra of quantities on $\text{Spec}(A)$. The main obstruction is that the Gelfand transform $a \mapsto \hat{a}$ that we have just defined is not injective if A has nilpotent elements (i.e. elements a such that $a^n = 0$ for some n). Since a nilpotent element a is contained in every prime ideal [7, Prop.1.8, 5], it defines a constant zero function on $\text{Spec}(A)$, i.e. $\hat{a}(\mathfrak{p}) = t_{\mathfrak{p}}(a) = 0$ for every $\mathfrak{p} \in \text{Spec}(A)$. This means that all the nilpotent elements are in the kernel of the Gelfand transform. It follows that the quantities \hat{a} only allows us to recover the reduced ring A/\mathcal{N} associated to A , namely A quotiented by the ideal \mathcal{N} of nilpotent elements (called *nilradical*).

In standard differential calculus, a function $f(x)$ on a space M is completely determined by the values it takes at every $x \in M$. This means that the information encoded in its derivatives is (as the name indicates) “derived” from the functional expression that defines its values at every point. Now, the obstruction to the injectivity to the Gelfand transform given by the nilpotent elements in A can be bypassed by providing an enhanced definition of the quantities on $\text{Spec}(A)$. In scheme theory, this is done by defining a quantity not only by means of its punctual values, but also by specifying from scratch (rather than “deriving” it) the differential behavior of the “quantity” around each point. This can be formally done by introducing quantities on $\text{Spec}(A)$ with values in *germs* of sections of a sheaf.

In order to define such a sheaf, we have to introduce a topology on the set $\text{Spec}(A)$. The coarsest topology that does the job is the so-called *Zariski topology*.⁴⁹ The open sets

⁴⁷ It is worth noting that the set of ideals of a ring has the structure of a lattice (where the order relation is given by the inclusion), i.e. of an ordered set with finite *join* (or operation or sum) and *meet* (and operation or intersection). This lattice yields the “*skeleton*” of the variety in the sense that it encodes the relations of *incidence* and *specialization* between the different subvarieties.

⁴⁸ Let's consider for instance $\text{Spec}(\mathbb{Z})$ and let's denote $\mathbb{Z}p$ a prime ideal of \mathbb{Z} . Any integer $n \in \mathbb{Z}$ defines a quantity \hat{n} on $\text{Spec}(\mathbb{Z})$ such that $\hat{n}(\mathbb{Z}p) = \text{class of } n \text{ modulo } p$, which is an element of the field $\mathbb{F}_p = \mathbb{Z}/\mathbb{Z}p$. Hence, the function \hat{n} takes values in a different field \mathbb{F}_p for each prime p (see for instance Ref.[88, Fig.1.4, 12]).

⁴⁹ For certain purposes it is necessary to refine the topology by passing for instance to the *étale topology* or the *fppf* (*fidèlement plat de présentation finie*) topology, which are Grothendieck topologies. This

of this topology are given by the complement of algebraic sets.⁵⁰ The so-called *principal open sets* $U_f = \text{Spec}(A) - V(f)$ obtained by removing the biggest possible algebraic sets (i.e. the algebraic sets $V(f)$ defined by the zeros of a single function $f \notin \mathfrak{p}$) are the smallest neighborhoods of the point \mathfrak{p} and form a basis of the Zariski topology. Now that the set $\text{Spec}(A)$ was endowed with a topology, the next step is to define a sheaf of quantities on $\text{Spec}(A)$. The corresponding sheaf is called *structure sheaf* and denoted $\mathcal{O}_{\text{Spec}(A)}$.⁵¹ It can be shown that the stalk $\mathcal{O}_{\text{Spec}(A)}(\mathfrak{p})$ of the structure sheaf at \mathfrak{p} (i.e. the set of germs of sections of the sheaf at \mathfrak{p}) is given by the *local ring* $A_{\mathfrak{p}}$ [57, Prop.2.2, 71] (regarding the notion of *germ* see for instance Ref.[84, Sect.II.5, 83-85]).⁵² An *affine scheme* is a *locally ringed space* $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$, i.e. a topological space $\text{Spec}(A)$ endowed with a structure sheaf of rings $\mathcal{O}_{\text{Spec}(A)}$ such that the stalk at each point \mathfrak{p} is the local ring $A_{\mathfrak{p}}$.

It is worth unpacking here the geometric meaning of the algebraic notion of *local ring*. The spectrum of the local ring $A_{\mathfrak{p}}$ is given by $\text{Spec}(A_{\mathfrak{p}}) = \{\mathfrak{q} \in \text{Spec}(A)/\mathfrak{q} \subseteq \mathfrak{p}\}$ [44, 45]. This means that the localization from $\text{Spec}(A)$ to $\text{Spec}(A_{\mathfrak{p}})$

- (1) makes abstraction of the inner structure of the irreducible variety \mathfrak{p} , i.e. it considers \mathfrak{p} as if it were just an unextended point lacking any internal structure,⁵³
- (2) and only focus on the subvarieties of $\text{Spec}(A)$ containing \mathfrak{p} (i.e. it only keeps the ideals contained in \mathfrak{p}).

In Mumford's terms, the localization $A \rightarrow A_{\mathfrak{p}}$ allows us to focus on the directions "normal" to $V_{\mathfrak{p}}$ [97, 107]. Since we have only kept the prime ideals contained in \mathfrak{p} , the ideal \mathfrak{p} is the unique maximal ideal of $A_{\mathfrak{p}}$ (this explains why $A_{\mathfrak{p}}$ is called a *local ring*). Roughly speaking, $\text{Spec}(A_{\mathfrak{p}})$ contains one unextended point (the maximal ideal \mathfrak{p}) and all the extended subschemes containing it.⁵⁴

We shall now define an enhanced Gelfand transform $\tilde{\mathcal{G}}$ that is injective, i.e. that distinguishes the different nilpotent elements. To do so, let's note that the field-valued representations $A \rightarrow A/\mathfrak{p} \hookrightarrow \kappa_{\mathfrak{p}}$ associated to each point $\mathfrak{p} \in \text{Spec}(A)$ coincides with the map $A \rightarrow A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \kappa_{\mathfrak{p}}$ (i.e., that the operations of *localization* and *passing to the quotient* commute [44, Rem.2.13, 45]). We shall now define an enhanced Gelfand transform that only retains the localization map $\lambda_{\mathfrak{p}} : A \rightarrow A_{\mathfrak{p}}$ in the last composition. In other terms, the enhanced Gelfand transform sends each $a \in A$ to an enhanced quantity \tilde{a} on $\text{Spec}(A)$ given by $\tilde{a}(\mathfrak{p}) = \lambda_{\mathfrak{p}}(a)$. Since the local ring $A_{\mathfrak{p}}$ is the stalk of the structure sheaf at \mathfrak{p} , the enhanced Gelfand transform $\tilde{\mathcal{G}}$ sends each element $a \in A$ to a quantity on

means that the construction of a space associated to a given ring is not canonical, since requires to choose a particular topology on the set of points.

⁵⁰ The closed set $V(S)$ of the Zariski topology defined by a subset $S \subset A$ is given by the algebraic "figure" defined by S (i.e. by the points $\mathfrak{p} \in \text{Spec}(A)$ such that $\hat{f}(\mathfrak{p}) = 0$ for all $f \in S$).

⁵¹ For each open set $U \subset \text{Spec}(A)$, we take $\mathcal{O}_{\text{Spec}(A)}(U)$ as the set of *regular sections* $s : U \rightarrow \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$, where $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$ and s is locally induced by fractions a/b with $a, b \in A$ and $b \neq 0$ on the corresponding open sets [57, 70]. Formally, a regular section is given by pairs $(U, x/y)$ modulo the equivalence relation $(U, x/y) \sim (V, z/h)$ if $x/y = z/h$ on $U \cap V$. In order to understand the restriction to regular sections, it might be useful to remember that not every possible family of germs defines a section of a sheaf. The extra needed condition that a family of germs $\{u_x \in \mathcal{O}_x\}_{x \in U}$ has to satisfy is that for all $x \in U$ there exists a neighborhood $W \subset U$ containing x and a section $t \in \mathcal{O}(W)$ such that the germs defined by the section t on every $y \in W$ are the elements u_y in the family [114, 24]. It is worth noting that this is also the extra condition used to define the sheaf associated to a given pre-sheaf [57, 64].

⁵² The local ring $A_{\mathfrak{p}}$ is the ring obtained by adding inverses to all the ring elements $f \in A$ that are not in \mathfrak{p} , i.e. the ring $A_{\mathfrak{p}} = \left\{ \frac{a}{f}, f \notin \mathfrak{p} \right\}$ of rational functions well-defined on \mathfrak{p} . Conceptually, the *algebraic localization* $A \rightarrow A_{\mathfrak{p}}$ entails a *geometric localization*, namely a localization to the locus in which it is not possible to divide by elements in \mathfrak{p} .

⁵³ This results from the fact that the localization removes the prime ideals containing \mathfrak{p} , i.e. defining subvarieties of \mathfrak{p} .

⁵⁴ It is worth noting that the notion of *local ring* is "orthogonal" to the notion of *coordinate ring*: "[...] for a prime ideal $\mathfrak{p} \subset A$ the passage from A to $A_{\mathfrak{p}}$ cuts out all prime ideals except those contained in \mathfrak{p} . The passage from A to A/\mathfrak{p} cuts out all prime ideals except those containing \mathfrak{p} " [44, 45]. Conceptually, whereas the notion of *local ring* makes abstraction of the inner structure of $V_{\mathfrak{p}}$ and only focuses on its immersion in the ambient variety (the "normal" directions), the notion of *coordinate ring* makes abstraction of the immersion and only focuses on the regular functions on the resulting "abstract" variety $V_{\mathfrak{p}}$ (the "internal" directions).

$\text{Spec}(A)$ whose possible values at \mathfrak{p} are given by the germs of the sections of the structure sheaf. The specification of a germ of a section at a point \mathfrak{p} contains much more information than the mere value of the section at \mathfrak{p} since two sections having the same germ at a point \mathfrak{p} necessarily coincide on an open neighborhood of \mathfrak{p} (which is not necessarily true if the two sections only have the same value at \mathfrak{p}). The important point is that the enhanced Gelfand transform $\tilde{\mathcal{G}}$ is injective.⁵⁵ This means that the enhanced quantities \tilde{a} distinguish the nilpotent elements in the ring A . This construction shows that the nilpotent elements encode differential (would could also say “germinal”) structures around the points in $\text{Spec}(A)$. Conceptually, the fact that the enhanced Gelfand transform keeps the localization operation $A \rightarrow A_{\mathfrak{p}}$ and discards the quotient by the prime ideal \mathfrak{p} means that one is making abstraction from the internal structure of the corresponding irreducible set but not of its immersion in the ambient space (see footnote N°54). The idea of infinitesimal structures corresponding to non-reduced rings (i.e. rings with nilpotent elements) is central in scheme theory and was used by Grothendieck to develop deformation theory and infinitesimal calculus on schemes (see Refs.[46] and [50, Exp.14]).⁵⁶ In particular, the consideration of non-reduced rings provided the formal tools required to deal with important geometric notions such as *tangency*, *multiplicity of an intersection*, and *infinitesimal deformations* [88, 13-15].

The construction of an affine scheme $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ associated to a ring A allows to prove a reconstruction theorem for general rings. It can be shown that the restriction of the structure sheaf to a distinguished open set U_f is $\mathcal{O}_{\text{Spec}(A)}(U_f) = A_f$ [57, Prop.2.2, 71].⁵⁷ If in particular we consider the element $f = 1$, then $U_f = \text{Spec}(A) - V(1) = \text{Spec}(A)$ and $A_1 = A$. It follows that $\mathcal{O}_{\text{Spec}(A)}(\text{Spec}(A)) \simeq A$, i.e. that A is isomorphic to the ring of global sections of the structure sheaf. The central difference with respect to the restricted reconstruction theorem of classical algebraic geometry (for reduced and finitely generated rings over algebraically closed fields) is that the reconstruction of a general ring A requires to consider representations of A into different codomains $A_{\mathfrak{p}}$.

The algebra-geometry duality for general rings takes the form of an equivalence of categories between the opposite of the category of rings and the category of affine schemes⁵⁸:

$$\begin{array}{ccc}
 & \Gamma(-) & \\
 & \curvearrowright & \\
 \text{Ring}^{op} & & \text{Aff} \\
 & \curvearrowleft & \\
 & \text{Sch}(-) &
 \end{array}
 \tag{VI.1}$$

This equivalence of categories is defined by the functor $\text{Sch}(-)$ that sends a ring A to the affine scheme $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ and by the global section functor $\Gamma(-)$ that sends an affine scheme $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ to the ring $\mathcal{O}_{\text{Spec}(A)}(\text{Spec}(A))$ of global sections of its structure sheaf. In the terms introduced in Section N°II (see Fig. N°II.1), the functor $\text{Sch}(-)$ is the *semiotic functor* that sends a ring to the locally ringed space that encodes its general representation theory. In turn, the global section functor $\Gamma(-)$ is the *structure*

⁵⁵ If $\tilde{\mathcal{G}}(a) = 0$ then $a/1 = 0 \in A_{\mathfrak{p}}$ for every prime \mathfrak{p} . This means that for each \mathfrak{p} there exists $b_{\mathfrak{p}} \in A - \mathfrak{p}$ such that $a b_{\mathfrak{p}} = 0$. It follows that $a = 0$ since the ideal of all $b \in A$ such that $a b = 0$ is not contained in any prime ideal and hence has to contain $1 \in A$.

⁵⁶ From a historical standpoint, it is worth noting that an important antecedent of Grothendieck's work on infinitesimal structures was provided by Weil's article *Théorie des points proches sur les variétés différentiables* [129].

⁵⁷ This means that the notion of locality encoded by the notion of local ring A_f obtained by localizing A at $f \in A$ is the algebraic counterpart of the topological notion of locality encoded by the distinguished open sets U_f of the Zariski topology. It is worth noting that the notations A_f and $A_{\mathfrak{p}}$ are somehow confusing since A_f localizes to the open set U_f where we can divide by f and $A_{\mathfrak{p}}$ localizes to the point \mathfrak{p} where we cannot divide by the elements f in \mathfrak{p} .

⁵⁸ A morphism of affine schemes $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is given by (i) a map of topological spaces $\pi : X \rightarrow Y$ and (ii) a morphism of sheaves $\pi^{\#} : \mathcal{O}_Y \rightarrow \pi_{*} \mathcal{O}_X$, where the pushforward $\pi_{*} \mathcal{O}_X$ is given by $\pi_{*} \mathcal{O}_X(U) = \mathcal{O}_X(\pi^{-1}(U))$ for U an open set of Y .

functor that extracts the algebraic structure (the ring) whose representation theory is encoded by the corresponding locally ringed space. As we have briefly explained in this section, this *algebraic structure-geometric semiotics* duality relies on a precise dictionary between algebraic and geometric notions (see Table N°2).

TABLE 2.

| Algebra | Geometry |
|-------------------------------|--|
| Ring A | Affine scheme $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ |
| Prime ideal \mathfrak{p} | Point in $\text{Spec}(A)$ corresponding to an irreducible subscheme (generic points) |
| Maximal ideal \mathfrak{m} | Point in $\text{Spec}(A)$ corresponding to an irreducible unextended subscheme (standard points) |
| Local ring $A_{\mathfrak{p}}$ | Stalk $\mathcal{O}_{\text{Spec}(A)}(\mathfrak{p})$ of germs of sections on \mathfrak{p} |
| Ring element $a \in A$ | Global section \tilde{a} of the sheaf $\mathcal{O}_{\text{Spec}(A)}$ with values in point-dependent codomains $A_{\mathfrak{p}}$ |

VII. THE MAGIC FAN

In the last section, we have obtained an algebra-geometry duality given by an equivalence of categories between a category of algebraic objects—rings A —and a category of geometric objects—affine schemes $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ —. Equipped as we are with these conceptual and formal tools, let's come back now to the starting point of the whole discussion, namely to the fact that a system of polynomial equations over a field of definition k defines different figures depending on the codomain in which one searches for their solutions (see also the discussion in Ref.[22, 290-292] and Ref.[31, 13-14]).

As we have shown in Section N°V, an ideal \mathfrak{t} of polynomial equations in $P \doteq k[x_1, \dots, x_n]$ defines a *functor of solutions* $\text{Spec}_{(-)}(P/\mathfrak{t})$ that sends each k -algebra B to the set of solutions of the corresponding equations in B . As we have seen in Section N°IV, each solution defines a morphism $P/\mathfrak{t} \rightarrow B$, which means that the functor of solutions $\text{Spec}_{(-)}(P/\mathfrak{t})$ is a representable functor. In other terms, the quotient ring P/\mathfrak{t} is the object that *classifies* the solutions of the equations in the ideal \mathfrak{t} (see Section N°V). Now, the algebra-geometry duality between rings and affine schemes—i.e. the equivalence of categories (VI.1)—implies that $\text{Hom}_{\text{Ring}}(P/\mathfrak{t}, B) \simeq \text{Hom}_{\text{Aff}}(\text{Sch}(B), \text{Sch}(P/\mathfrak{t}))$. In other terms, there is a bijective correspondence between each B -solution of the corresponding equations and each morphism of schemes $\varphi : \text{Sch}(B) \rightarrow \text{Sch}(P/\mathfrak{t})$. As Cartier writes:

“A solution of the ‘system of equations’ [t], with the ‘domain of constants’ [k], with values in the ‘universal domain’ [B] corresponds to a morphism φ from $[Sch(B)]$ to $[Sch(P/t)]$. [...] Admirable simplicity—and a very fruitful point of view—but a complete change of paradigm!”
[22, 291]

Hence, the existence of an algebra-geometry duality between rings and affine schemes means that the *functor of solutions* that encodes the solutions of an ideal of polynomial equations in the different possible codomains B can be dually understood as a *functor of points* that sends each affine scheme $Sch(B)$ to the set $Hom_{Aff}(Sch(B), Sch(P/t))$ of (what we shall call) *Sch(B)-shaped generalized points* of $Sch(P/t)$. A morphism of affine schemes $\varphi : Sch(B) \rightarrow Sch(P/t)$ can be understood as a generalization of the characteristic functions $\chi_a : \{*\} \rightarrow X$ defined by each element a in a set X given by $* \mapsto a$, that is of the functions that “point” to the different elements or “points” of X (or, in figurative terms, the functions that present each “point” of the set X as a concrete incarnation of the “universal set-theoretic point” given by the singleton $\{*\}$). When the codomain B is a field extension k' of k , the space $Spec(k')$ contains a single point and the maps $Sch(k') \rightarrow Sch(P/t)$ are indeed similar to the characteristic functions $\chi_a : \{*\} \rightarrow X$. The difference with respect to the set-theoretic situation is that now we have different “punctual” schemes of the form $Sch(k)$ corresponding to different fields k .⁵⁹ More generally, a B -representation of P/t (where the k -algebra B is not necessarily a field) can be geometrically “schematized” as a morphism $Sch(B) \rightarrow Sch(P/t)$ that “points” to a $Sch(B)$ -shaped generalized point of $Sch(P/t)$.

We could say that in the theory of schemes the notion of *point* is subjected to two generalizations. Firstly, the notion of *point* is generalized from *unextended* irreducible subsets (maximal ideals) to general irreducible subsets (prime ideals). As we have explained in Section N°VI, the underlying set of an affine scheme $Sch(A)$ contains all the prime ideals of A , be them maximal (unextended points) or not (generic points). Now, both kinds of points (unextended and generic) describe representations of the original ring A into fields. The second generalization in the definition of a point amounts to pass from points associated to such field-valued representations to generalized points associated to more general (k -algebra)-valued representations. This means that the generalized points of a scheme $Sch(P/t)$ are not only given by morphisms $Sch(k') \rightarrow Sch(P/t)$ from “punctual” schemes of the form $Sch(k')$, but also by more general $Sch(B)$ -shaped generalized points $Sch(B) \rightarrow Sch(P/t)$ where B is not necessarily a field. It is worth noting that whereas a general scheme $Sch(A)$ is not defined by the underlying set of points of the corresponding topological space (e.g. the different schemes $Sch(k)$ and $Sch(k')$ have the same underlying set of points), a scheme is defined by its functor of points, i.e. by the structured set of its $Sch(B)$ -shaped generalized points for every ring B .

The scheme $Sch(P/t)$ over the field of definition k encodes (what Grothendieck calls) a “fan” of schemes over field extensions k' of k . Let's note first that $Sch(P/t)$ is endowed with a morphism to the scheme $Sch(k)$ dual to the inclusion $k \hookrightarrow P/t$. Schemes X equipped with a morphism to another scheme S are called *S-schemes* or *relative schemes over S* (and the morphism $X \rightarrow S$ is called *structure morphism*) [49, 226]. Given another morphism $S' \rightarrow S$ over S , we can pullback a relative scheme X over S to a relative scheme $X \times_S S'$ over S' . This fundamental operation of the theory of schemes is called *base change*. In particular, one can pullback $Sch(P/t)$ along the morphism $Sch(k') \rightarrow Sch(k)$ dual to the inclusion $k \hookrightarrow k'$, thereby obtaining a scheme over $Sch(k')$. In this description, the $Sch(k')$ -points of $Sch(P/t)$ can be identified with the sections of the projection $Sch(P/t) \times_{Sch(k)} Spec(k') \rightarrow Sch(k')$ [31, 13]. All in all, a scheme $Sch(P/t)$

⁵⁹ Even if the underlying sets of such schemes contain all a single point, the structure sheaf on the point is different for different k . In Ref.[49, 12], the points associated to field-valued representations are called *geometric points*.

over $Sch(k)$ (or, equivalently, a scheme associated to a ring that is a k -algebra) can be understood as a “fan” of schemes, one for each field extension k' of k . In this way, the scheme $Sch(P/t)$ defined by a system of polynomial equations over k unifies the different “figures” associated to the solutions of these equations in the different field extensions k' of k into a single geometric object.

As it is noted in Ref.[49, §17, 17-18], the scheme $Sch(P/t)$ only depends on the ring structure of the k -algebra P/t . The fact that P/t is also a k algebra translates geometrically into the fact that the scheme $Sch(P/t)$ is endowed with an additional structure, namely the structure morphism to $Sch(k)$. Hence, we can associate an affine scheme $Sch(A)$ to any ring A , be it a k -algebra or not (as we have done in Section N°VI). Now, for any ring A there is a canonical morphism $\mathbb{Z} \rightarrow A$ given by $n \mapsto 1 + 1 + \dots + 1$ where the unity $1 \in A$ is added n times (which means that \mathbb{Z} is an initial object in the category of rings). Therefore, every scheme $Sch(A)$ is canonically endowed with a structure morphism $Sch(A) \rightarrow Sch(\mathbb{Z})$ dual to $\mathbb{Z} \rightarrow A$. An *absolute scheme* is a $Sch(\mathbb{Z})$ -scheme, that is a scheme over (what Grothendieck calls) the *absolute base* $Sch(\mathbb{Z})$ [47]. In turn, the morphism of rings $\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ (with p a prime number) induces a morphism of affine schemes $Sch(\mathbb{Z}/p\mathbb{Z}) \rightarrow Sch(\mathbb{Z})$. If we take the pullback of $Sch(A) \rightarrow Sch(\mathbb{Z})$ along this last map we obtain a scheme over $Sch(\mathbb{Z}/p\mathbb{Z})$, that is a *scheme of characteristic p* [44, 89]. In this way, an absolute scheme $Sch(A)$ can be understood—in Grothendieck’s terms—as a “magic fan” that “gives birth” to a scheme of characteristic p for each prime number p . It is worth quoting here Grothendieck’s own description of this “magic fan”:

“La notion de schéma est la plus naturelle, la plus ‘évidente’ imaginable, pour englober en une notion unique la série infinie de notions de ‘variété’ (algébrique) qu’on maniait précédemment (*une* telle notion pour chaque nombre premier). [...] De plus, un seul et même ‘schéma’ (ou ‘variété’ nouveau style) donne naissance, pour *chaque* nombre premier p , à une ‘variété (algébrique) de caractéristique p ’ bien déterminée. La collection de ces différentes variétés des différentes caractéristiques peut alors être visualisée comme une sorte d’‘éventail (infini) de variétés’ (une pour chaque caractéristique). Le ‘schéma’ est cet éventail magique, qui relie entre eux, comme autant de ‘branches’ différentes, ses ‘avatars’ ou ‘incarnations’ de toutes les caractéristiques possibles. Par là-même, il fournit un efficace ‘principe de passage’ pour relier entre elles des ‘variétés’, ressortissant de géométries qui jusque là étaient apparues comme plus ou moins isolées, coupées les unes des autres. A présent, elles se trouvent englobées dans une ‘géométrie’ commune et reliées par elle” [47].

Regarding the distinguished role played by the ring \mathbb{Z} in the category of rings, let us conclude this section with an important remark (see for instance Ref.[124, 445] and Ref.[32, 6]). As we have argued all throughout this article, the reconstruction of a ring A from its representations $A \rightarrow B$ requires to consider representations with respect to a varying codomain B . This “generalized semiotics” of the ring A can still be encoded in a single geometric object, namely the affine scheme $(Spec(A), \mathcal{O}_{Spec(A)})$. As we have explained, the fact that the codomain of the representations can vary translates geometrically into the fact that the elements in A define global sections of the structure sheaf *with values in point-dependent codomains*. Now, *once we have the geometric notion of affine scheme at our disposal*, it is possible to define a unique codomain given by an affine scheme—namely, the so-called *affine line* $\mathbb{A}^1 := Sch(\mathbb{Z}[x])$ —such that the elements in A are in bijective correspondence with the \mathbb{A}^1 -valued functions on $Spec(A)$.⁶⁰ In this way, we have described a sort of dialectical process that moves back and forward from the presupposition

⁶⁰ To see that this is indeed the case, let’s note that the ring $\mathbb{Z}[x]$ of polynomials in one variable with entire coefficients is such that $A \simeq Hom_{Ring}(\mathbb{Z}[x], A)$, where each $a \in A$ defines a morphism $\phi_a : \mathbb{Z}[x] \rightarrow A$ given by $\phi_a(p) = p(a)$. The duality between rings and affine schemes—i.e. the fact

of a unique codomain (like the algebraically closed fields in the algebraic geometry of affine varieties; see Section N°IV) to the consideration of a whole category of codomains, and then back to the realization that in the end any ring A is in bijective correspondence with the \mathbb{A}_1 -valued functions on its prime spectrum $\text{Spec}(A)$. As we have stressed above, this last step can be accomplished only when we already have at our disposal the notion of affine scheme.

VIII. A KANTIAN INTERPRETATION OF THE RECONSTRUCTION THEOREMS

In the wake of the Kantian analysis of the Pontryagin duality for locally-compact abelian groups proposed by the mathematician Majid [85], we shall now consider the mathematical problem of representation from a perspective inspired by Kant's *phenomenon-noumenon distinction*.⁶¹ The interpretation of this distinction is a central topic in Kant's scholarship (see for instance Refs.[2, 17, 52, 106, 116]). In what follows we do not intend to take a stand in these debates. We shall rather use the interpretation of Kant's distinction that is best adapted to the conceptual analysis of the problem at stake, that is the problem of reconstructing a mathematical structure A from its representations $A \rightarrow B$.

In *The Critique of Pure Reason* [66, B295/A236—B315/A260], Kant introduced the central difference between the *things in themselves* or *noumena* and the *phenomena* of human experience. According to the reading of Kant's distinction that we shall endorse here (which follows to a certain extent the so called *two-aspect interpretation* defended by Allison [2], Bird [17], and Prauss [106] among others⁶²), the noumena and the phenomena do not refer to two kinds of entities but rather to the same entity considered from two different standpoints, namely the entity as it appears to subjects equipped with a particular cognitive (perceptual and conceptual) apparatus and the entity "considered as it is in itself" [*Ding an sich selbst betrachten*], that is, independently of any subjective cognitive framing. Whereas a thing in itself is independent of any subject, its *phenomenalisation* (i.e. the way according to which it appears for a subject) depends on the particular cognitive apparatus of the subject, that is, on what we could call its *transcendental type*. We could then say that subjects of different transcendental types B and B' (e.g. a human being and a tick) constitute different phenomena A_B and $A_{B'}$ out of the same thing in itself A .

In order to adapt this conceptual framework to the subject of this article, we shall endorse the readings of Kant's terminology (mainly associated to the two aspect interpretations) that enrich the phenomenon-noumenon distinction with a further distinction, namely the distinction between the *phenomena* of experience and the subjective *representations* [*Vorstellungen*] of these phenomena (regarding this distinction see Ref.[115, Sect.3.1] and references therein). We could say, in Husserl's terms (see for instance Ref.[64, §41, 74; §44, 82]), that the phenomena of experience are not given to a subject all at once, but rather "onesidedly", that is through a multiplicity of perspectival profiles, aspects or "*adumbrations*" [*Abschaltungen*]. According to these definitions, the subjective experience of a phenomenon is given by one-sided representations of the phenomenon's adumbrations. By changing the relative position between the phenomenon and the subject, the latter can gain access to the different adumbrations of the phenomenon. All in all (and according to the particular interpretative framework endorsed here) the Kantian analysis of experience gives rise to a threefold structure composed of:

that $\text{Hom}_{\text{Ring}}(\mathbb{Z}[x], A) \simeq \text{Hom}_{\text{Aff}}(\text{Spec}(A), \mathbb{A}^1)$ —allows us to translate this bijection into a bijection between A and the \mathbb{A}^1 -valued functions on $\text{Spec}(A)$.

⁶¹ In Ref.[76, Sect.1], Lawvere provided a related Hegelian-oriented attempt to provide a philosophical interpretation of the duality between "space" and "quantity".

⁶² The two-aspect interpretation arised as an alternative with respect to the so-called *two-world interpretation* mainly defended by Strawson [116] (see also related stances in Ref.[52, V.15, 333] and Ref.[123]). Regarding the tension between these two interpretational frameworks see notably Refs.[1, 3, 110, 127].

- (1) the things in themselves (i.e. considered independently of the subjective conditions under which they appear),
- (2) the phenomena or the things as they appear to subjects of a given transcendental type,
- (3) the sets of perspectival representations that a subject can have of the corresponding phenomenon.

In order to apply this Kantian-oriented conceptual framework to the problem of reconstructing a mathematical structure from its representations, let's establish a precise dictionary between philosophical concepts and mathematical notions (see Table N°VIII). Each map $A \rightarrow B$ will be understood—in agreement with the terminology that we have used thus far—as a “representation” of the “thing in itself” A defined with respect to the “transcendental” codomain B . In this sense, the codomains B provide what we could metaphorically call the “subjective” side of the representations. In order to avoid any subjectivist overtones, we shall refer to B by means of the term (inspired by both Kant and Kronecker) *transcendental codomain of rationality*. We could say that each map $A \rightarrow B$ represents A by means of a particular “image” in B , namely the image (in the mathematical sense of the term) of the map. For each codomain B , there is a set $\text{Spec}_B(A) = \text{Hom}_C(A, B)$ of B -representations of A . By taking into account that the term *spectrum* comes for the Latin for *revelation*, *apparition*, or *manifestation*, we could say that $\text{Spec}_B(A)$ encodes all the ways according to which A can “appear” (i.e. it can be re-presented or modeled) in the transcendental codomain of rationality B . According to this description, each thing in itself A defines a multiplicity of spectra $\text{Spec}_B(A)$, $\text{Spec}_{B'}(A)$, $\text{Spec}_{B''}(A)$... (one for each transcendental codomain B, B', B'' ...). We could then say that each thing in itself A defines a *structured set of representations* or a *variable set* (where we are here borrowing the mathematical terminology used in Refs.[97, 113], [91, 386], and [14]). This means that the set of representations induced by a given thing in itself A is a collection of sets of the form $\text{Spec}_B(A)$ parameterized by the objects B in the category \mathcal{C} , where each set $\text{Spec}_B(A)$ in the collection includes all the B -representations $A \rightarrow B$ of A . As we have seen in Section N°V, this assignation of a set $\text{Spec}_B(A)$ to each B in the category \mathcal{C} is functorial. This means that morphisms $B \rightarrow C$ between transcendental codomains of rationality in \mathcal{C} translate into morphisms $\text{Spec}_B(A) \rightarrow \text{Spec}_C(A)$ between the corresponding sets of representations.

Here we have two possible scenarios corresponding to what we have called *restricted* and *general reconstruction theorems*. First, it might be possible to faithfully reconstruct A —i.e. to reconstruct A as “it is in itself”—by only considering representations of A defined with respect to a single transcendental codomain of rationality B (restricted reconstruction theorem). In philosophical terms, this means that nothing is lost when passing from the thing in itself A to its B -phenomenalisation, i.e. to the phenomenalization of A with respect to the transcendental codomain of rationality B .

Now, representations defined with respect to a single transcendental codomain might not be enough to reconstruct A as “it is in itself”. If the space $\text{Spec}_B(A)$ does not have “enough points”, it might be the case that different elements $a, a' \in A$ define—via the Gelfand transform—the same quantity on $\text{Spec}_B(A)$. In that case, the representations parameterized by the space $\text{Spec}_B(A)$ cannot “separate” the elements a and a' . Since—as we have seen—the obstruction to such a reconstruction is given by the kernel of the Gelfand transform, the quotient $A_B = A/\ker(\mathcal{G}_{A,B})$ could be called *B-phenomenalization of A*. Conceptually, the B -phenomenalization of A encodes what can be reconstructed from A when one only considers representations of A with respect to the transcendental domain of rationality B . In classical algebraic geometry, the lack of “enough points” can be bypassed by adding more points, that is by considering representations into an algebraically closed extension K of the domain of definition k of the corresponding system of equations (Section

TABLE 3.

| Kantian Concepts | Mathematical Notions |
|--|--|
| Thing in Itself | A |
| Transcendental Codomain of Rationality | B |
| B -Representations of A | $A \rightarrow B$ |
| Space of B -Representations | $\text{Spec}_B(A) = \text{Hom}_C(A, B)$ endowed with a topology |
| B -Phenomenalization of A | $A_B = A/\ker(\mathcal{G}_{A,B})$ |

N°IV). In the terms of model theory, this algebraically closed extension of the domain of definition k can be understood as an *existential closure* of k [61, 362] (see also Refs.[15, 87]).

The *extension* of a given (co)domain of rationality by the adjunction of new “ideal elements” provides an important mechanism in the development of mathematics. We could say that these adjunctions elicit an expansion of the different transcendental regimes of “pure intuition”⁶³, like for instance the regime provided by the understanding of numbers as formal entities intended to count an aggregate of things—or, as Kant writes in the *Prolegomena*, as concepts formed “through successive additions of units in time” [67, 4:283, §10, 35].⁶⁴ According to Hilbert, “[t]his procedure of introducing ideal elements is one of the most important mathematical methods, the application of which is always repeated up to the highest parts of mathematics” [60, 90]. The possibility of extending a given domain of rationality and enriching the corresponding forms of “pure intuition” entails a particular form of (what Friedman calls in his reading of Carnap, Reichenbach, and Schlick) “relativization of the *a priori*” [42, I.3]. In the particular case treated here, this means that the arithmetic and geometric forms of “pure intuition” (e.g. natural numbers, Euclidean space), far from being fixed once and for all (i.e. far from being endowed with a perennial form of aprioricity as it seems to be the case in Kant⁶⁵), are subjected to changes elicited by the very evolution of mathematics. The fact that domain extensions force to modify what in a given historical context takes the form of an *a priori* form of “pure intuition” explains the difficulties encountered in performing such extensions.⁶⁶ These difficulties left their traces in the denominations chosen to design the new elements

⁶³ Regarding the Kantian notion of *pure intuition* see Ref.[41].

⁶⁴ Regarding the concept of *domain extension*, see the corresponding texts by Hilbert [59, 60], the discussion of “ideal elements” proposed by Cassirer [24, III.4.4], and the discussion of this topic in contemporary philosophy of mathematics [15, 20, 87].

⁶⁵ For instance, Kant expresses in a letter to A.W. Rehberg the “puzzlement” he feels about an irrational number like $\sqrt{2}$: “The understanding is not even in a position to assume the possibility of an object $\sqrt{2}$, since it cannot adequately present the concept of such a quantity in an intuition of number, and would even less anticipate that such a quantity could be given *a priori*.” [68, 168-169].

⁶⁶ As Cassirer writes in *Substance and Function*: “The difficulties encountered in the introduction of every new type of number—of the negatives and the irrationals as well as the imaginaries,—are easily explained if we consider that, in all these transformations, the real basis of numerical assertions seems more and more to disappear. Enumeration, in its most fundamental sense, could be immediately shown to be ‘real’ by means of sensible objects and therefore valid” [25, 55].

(e.g. irrational numbers, imaginary numbers, ideal elements). Now—in Hilbert’s terms—, “[t]he expression ‘ideal elements’ is only justified from the point of view of the original system. In the new system, we no longer distinguish between real [wirklichen] and ideal [idealen] elements” [60, 91]. It follows—as Bernays states with respect to “the problem of infinity”—that “the sharp distinction between the intuitive [Anschaulichen] and the non-intuitive [Nicht-Anschaulichen] [...] can apparently not be drawn so strictly [...]” [16, 61]. In particular, the extensions of the successive number systems and the projectivization of Euclidean space by adding new points at infinity constitute two paradigmatic examples of domain extensions that entail new forms of (arithmetic and geometric) intuition.

Now, Grothendieck’s groundbreaking insight is that the reconstruction of a general ring A might require to consider a notion of variation of the codomain of the representations of A more general than the notion of *field extension*. The “relativization of the *a priori*” encoded in the notion of domain extension remains limited and the reconstruction of a general ring from its representations makes necessary to perform a more general relativization of the *a priori*. We could say in Kantian terms that the reconstruction of a thing A as “it is in itself” might require not only to consider representations defined with respect to a single transcendental codomain of rationality (or with respect to a suitable extension of it), but also to consider representations defined with respect to a whole category of codomains. As Grothendieck and Dieudonné write in the introduction to the *Éléments de Géométrie Algébrique I*, the strategy that consists in considering solutions in an extension of the field of definition of the system of equations worked “too well” since it caused mathematicians to lose sight of the fundamental idea of a “change of the base field” in all its generality.⁶⁷ Grothendieck’s notion of a *functor of solutions* associated to a ring P/t —or the dual notion of a *functor of points* represented by the affine scheme $Sch(P/t)$ —introduced for the first time an extended kind of variations that includes a “fan” of transcendental codomains of rationality not necessarily related by domain extensions. Whereas the relativity associated to domain extensions ultimately leads to the consideration of a single domain that is “big enough” (like Weil’s *universal domains*), the functorial relativity envelops domains that cannot be nested into one another.

Galois theory can be understood as a sort of intra-mathematical reflection on the “relativization of the *a priori*” encoded in the notion of *domain extension* for the particular case given by sequences of (normal and separable) field extensions $k \hookrightarrow k'$ of a field k defined by the adjunctions of new “ideal elements” (e.g., the extension $\mathbb{C} = \mathbb{R}[i]$) [18, 27]. As we have just argued, Grothendieck’s “relative point of view” pushes even further the Galoisian “relativization of the *a priori*”. In this sense, we could say that Grothendieck can indeed be considered (as he claims) as a legitimate heir of “Galois’ heritage” (regarding the influence of Galois on Grothendieck see Ref.[28]).⁶⁸

⁶⁷ In Grothendieck and Dieudonné’s own terms, “[...] dès le début du XVIII^e siècle, et systématiquement à partir de Monge et de Poncelet, on associe à un système [d’équations polynomiales] à coefficients réels le même système dont on ne cherche plus seulement les solutions dans \mathbb{R}^1 , mais bien dans l’espace complexe correspondant \mathbb{C}^1 , utilisant le fait que \mathbb{R} est un sous-corps de \mathbb{C} . Cette idée se montra très féconde, du fait que les propriétés des êtres algébriques étudiés se simplifiaient considérablement par cette ‘extension’ du corps de base; en fait on peut même dire que cette ‘extension’ réussit en un certain sens trop bien, car l’avantage additionnel de disposer, sur le corps \mathbb{C} , de la puissante théorie des fonctions analytiques fut cause que pendant tout le XIX^e siècle on cessa pratiquement de considérer d’autres systèmes [d’équations polynomiales] que ceux à coefficients complexes (ou dans des sous-corps de \mathbb{C} tels que les corps de nombres algébriques); ce qui conduisit à perdre de vue l’idée fondamentale du ‘changement du corps de base’ sous sa forme générale [...]” [49, 5-6].

⁶⁸ In *Recoltes et Semailles*, Grothendieck writes: “[...] le premier dans la lignée de mes ‘frères de tempérament’ [...] c’est Evariste Galois. Dans sa courte et fulgurante vie, je crois discerner l’amorce d’une grande vision—celle justement des ‘épousailles du nombre et de la grandeur’, dans une vision géométrique nouvelle. J’évoque ailleurs dans *Recoltes et Semailles* comment, il y a deux ans, est apparu en moi cette intuition soudaine: que dans le travail mathématique qui à ce moment exerçait sur moi la fascination la plus puissante, j’étais en train de ‘reprendre l’héritage de Galois’. Cette intuition, rarement évoquée depuis, a pourtant eu le temps de mûrir en silence. La réflexion rétrospective sur mon œuvre que je poursuis depuis trois semaines y aura sûrement encore contribué. La filiation la plus directe que je crois reconnaître à présent avec un mathématicien du passé, est bien celle qui me relie à Evariste Galois” [47].

IX. CONCLUSION

In this article we have tried to contribute to what we could call the “philosophy of (mathematical) duality” [27, 30, 53, 69, 71] by analyzing a major event in the history of this notion, namely the construction of a geometric object (an affine scheme) associated to any (commutative) ring. Let’s stress the main argumentative thread of the proposed analysis:

- (1) In Section N°III, we have described a general meta-mathematical pattern that reappears in different algebra-geometry dualities in many areas of mathematics (e.g. mathematical logic, algebraic geometry, group representation theory, C^* -algebra theory, Galois-Grothendieck theory). In all these results, a mathematical structure A (e.g. a Boolean algebra, a ring, a group, a C^* -algebra, etc.) is reconstructed from its representations $A \rightarrow B$ with respect to other similar structures B . The underlying reconstruction protocol (schematized in Figure III.2) is always the same: the mathematical structure A is recovered as (i.e. is isomorphic to) an algebra of quantities $\Gamma(\text{Spec}(A))$ defined by means of a Gelfand transform $A \rightarrow \Gamma(\text{Spec}(A))$ on a *moduli space* $\text{Spec}(A)$ that parameterizes some of its representations.
- (2) We have argued that this general meta-mathematical pattern exhibits an entanglement between an *algebra-geometry duality* and (what we have called) a *structure-semiotics duality* (of which the syntax-semantics dualities for propositional and predicate logic are particular cases) (Section N°II). This entanglement between these two kinds of duality results from the fact (already stressed by Halvorson and Tsementzis for the case of the Stone duality [55, 56]) that the formulation of a duality between a “theory” and its “models” requires to endow the set of “models” with an extra geometric structure encoding their relations (which means that the geometry is on the semiotic side of the duality).
- (3) In Section N°IV, we have reviewed a particular case of the algebra/structure-geometry/semiotics duality, namely the (restricted) reconstruction theorem of classical algebraic geometry. This result states that certain kind of rings (reduced and finitely generated rings over algebraically closed fields) can be reconstructed from its representations *into a unique codomain*.
- (4) In Section N°V, we have introduced Grothendieck’s theory of affine schemes as an attempt to obtain a reconstruction theorem in algebraic geometry that works for general commutative rings. This can be done by extending the semiotic side of the structure-semiotics duality to representations *defined with respect to a whole family of codomains* (general reconstruction theorems). This extension of the semiotic side of the duality is encoded in the category theoretic notion of *functor of points*.
- (5) As we have explained in Section N°VI, Grothendieck’s functorial extension of the structure-semiotics duality threatens the possibility of casting this duality into the form of an algebra-geometry duality since we have now a different space $\text{Spec}_B(A)$ for each codomain of rationality B . Now, Grothendieck showed that it is still possible to construct a single geometric object encoding the generalized semiotics of the original ring A , namely the affine scheme $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$.
- (6) In Section N°VI, we have proposed a conceptual-oriented introduction to the construction of the affine scheme $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ associated to a ring A . The proposed presentation stresses the salient features that differentiates Grothendieck’s construction from previous forms of the algebra-geometry duality, namely:
 - that the “points” of the underlying space $\text{Spec}(A)$ encode all the irreducible sub-schemes (and not just the irreducible sub-schemes given by the unextended points),

- that the quantities \tilde{a} on the space $\text{Spec}(A)$ defined by the ring elements $a \in A$ have values *in point-dependent codomains*,
 - that these values are given by *germs* of sections that encode differential structures around the points.
- (7) We have finally proposed a Kantian-oriented interpretation of both the restricted and the general reconstruction theorems. According to this interpretation, Grothendieck’s construction can be understood as a further generalization of the “relativization of the *a priori*” encoded in the notion of *domain extension* to more general variations of the “transcendental codomains of rationality”.

In order to conclude, let us note that this transition from a restricted *set of representations* to a general *sheaf of representations* defined on a whole category of transcendental codomains is just the first step in a series of far-reaching generalizations. These generalizations proceed by modifying the domain category and/or the codomain category of the functor of points. First, we can pass from functors of points with values in the category of sets, i.e. from functors of the form

$$\text{Spec}_{(-)}(A) : \text{Ring} \rightarrow \text{Set},$$

to functors with values in the category of groupoids (stacks) or in the category of simplicial sets (higher stacks). The next generalization—which is the grounding idea of the field of *derived geometry*—amounts to pass from functors over the category of rings to functors over the category of differential graded algebras, thereby giving rise to the theory of *derived stacks* (see for instance Refs.[4, 120, 126]). This final note suggests a possible direction for the continuation of the research project started in this article, namely that of upgrading the proposed analysis of the philosophical (and in particular mathematical) problem of reconstructing a structure from its representations to the formal framework defined by the two aforementioned generalizations.

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