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## AN HYBRID FINITE ELEMENT METHOD FOR A QUASI-VARIATIONAL INEQUALITY MODELING A SEMICONDUCTOR

ABDELJALIL NACHAOU<sup>1,\*</sup> AND MOURAD NACHAOU<sup>2</sup>

**Abstract.** A problem of determining the characteristics of a semiconductor can be reduced to the study of the quasi-variational inequality, (J. Abouchabaka, R. Aboulaïch, A. Nachaoui and A. Souissi, *COMPEL* **18** (1999) 143–164.) where the obstacle  $\mathcal{M}(u)$  is the solution of an elliptic problem depending on  $u$ . We present here an hybrid finite element method for the computation of obstacle  $\mathcal{M}(u)$  and we discuss some numerical aspects appearing in its approximation.

**Mathematics Subject Classification.** 65N22, 35Q81, 35J66, 90C53.

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### 1. INTRODUCTION

The stationary distribution of the carrier motions inside a semiconductor device is described by the following system of nonlinear elliptic partial differential equations:

$$\nabla \cdot \varepsilon \nabla u = q(n - p - N) \tag{1}$$

$$\nabla \cdot J_n = 0, \quad J_n = D_n \nabla n - \mu_n n \nabla u \quad \text{in } \Omega \tag{2}$$

$$\nabla \cdot J_p = 0, \quad J_p = -(D_p \nabla p + \mu_p p \nabla u) \tag{3}$$

where (1) is the Poisson equation, (2) is the continuity equations for electrons and (3) is the continuity equations for holes.

The domain  $\Omega$  is an open of  $\mathbb{R}^2$  corresponding to the interior of the semiconductor device, of lipchitzian boundary  $\partial\Omega$  with ,  $\partial\Omega = \Gamma_D \cup \Gamma_N$ . These equations are subject to the following boundary conditions:

$$\nu \cdot \nabla u = \nu \cdot \nabla n = \nu \cdot \nabla p = 0 \quad \text{on } \Gamma_N \tag{4}$$

$$u = u_d, \quad n = n_d, \quad p = p_d; \quad \text{on } \Gamma_D, \tag{5}$$

where  $u_d, n_d, p_d$  are given functions on  $\Gamma_D$  and  $\nu$  denotes the external normal vector of  $\partial\Omega$ .

*Keywords.* Semiconductor device, quasi-variational inequality, finite elements method, topological degree.

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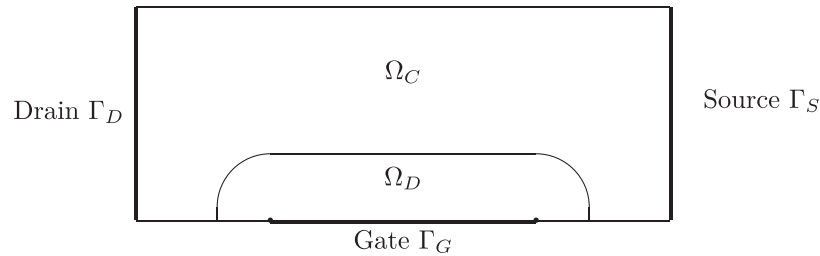


FIGURE 1. Junction field effect transistor.

In the van Roosbroeck model (1)–(3), the potential  $u$ , the electron density  $n$  and the hole density  $p$  are the unknowns; the doping function  $N$ , the charge constant  $q$  and the permittivity  $\varepsilon$  are given. In the expression of the current density  $J_n$  the quantities  $D_n$ ,  $\mu_n$  represent the mobility and the diffusivity of the electrons. This problem has been studied by several authors [19, 20, 30, 31], especially in a fairly general framework, [13, 15, 16, 29] and for more realistic boundary conditions by the author [13, 16]. Uniqueness is not possible in general, it is obtained under restrictive hypotheses [14, 17, 23].

Several authors are interested in the approximation of this system. For example, a finite element approximation is done in [22, 24, 28, 34], mixed finite elements is in [32], and several approximations based on the use of finite volumes can be consulted in [4, 6, 7] and the references therein. A finite-difference approximation is used in [2] for the equilibrium electrostatic potential.

Concerning the convergence of approximations, we refer the reader to [24] where he will find a general framework for the study of the convergence of approximations whatever their types. An application of this study to finite elements can be found in [22].

Under physical assumptions, we show that the system (1)–(2) leads as a limit case to a quasi-variational equation [1, 12, 25], given in the case of a unipolar field-effect semiconductor, by

$$\int_{\Omega} \nabla u \nabla (\varphi - u) dX \geq \xi \int_{\Omega} (\varphi - u) dX \quad \forall \varphi \in K(u) \quad (6)$$

with  $K(u) = \{v \in H^1(\Omega), v = u \text{ on } \Gamma_D \text{ and } v \leq \mathcal{M}(u) \text{ a.e. dans } \Omega\}$  where  $\mathcal{M}(u)$  is the solution of problem

$$\begin{cases} \nabla e^u \nabla e^{-w} = 0 \\ w = w_d \text{ on } \Gamma_D, \\ \frac{\partial w}{\partial \nu} = 0 \text{ sur } \Gamma_N, \end{cases} \quad (7)$$

with  $u_d = w_d = 0$  on the source,  $u_d = w_d = V^+ \geq 0$  on the drain,  $u_d = V^- \leq 0$  and  $w_d = V^+$  on the grid (see Fig. 1).

This IQV model is a formulation of a free boundary problem where we have to find the unknown boundary separating two sets  $\Omega_C$  and  $\Omega_D$ , corresponding to the charge neutrality region where the potential  $u$  and the Fermi quasi-potential  $w = \mathcal{M}(u)$  are identical and the depletion region where  $u < w$  [1, 21, 25].

The existence and uniqueness of the solution to the problem (6)–(7) are discussed in [1] where the authors propose a technique based on topological degree theory which extends the results of [20] to cases where the solution is not regular. The solution of the quasi-variational inequality being obtained as the limit of a series of variational inequalities where the two problems (6) and (7) are decoupled

$$u_m \in K(u_{m-1}), \quad \int_{\Omega} \nabla u_m \nabla (\varphi - u_m) dX \geq \xi \int_{\Omega} (\varphi - u_m) dX \quad \forall \varphi \in K(u_{m-1}). \quad (8)$$

The approximation of (6) poses no problem, any reasonable approximation scheme works. The difficulty appears in the discretization of (7) where it is necessary to take into account the electric field which can be very important in certain regions of the semiconductor. We are interested here in a hybrid discretization, [5], whose main property is the preservation of electric current, which is a particularly important property in practice, and an exact computation of the integral of the exponential. For recent use of this type of approximation in different fields see [3, 9, 18, 26, 27, 33] and the references therein.

## 2. HYBRID METHOD FOR THE OBSTACLE APPROXIMATION

Using the change of variable  $\eta = e^{-\mathcal{M}(u)}$ , we obtain new expressions of the equations verified by the electric current  $J$  :

$$e^{-u} J - \nabla \eta = 0 \text{ and } \nabla \cdot J = 0. \tag{9}$$

The multiplication of the first equation of (9) by  $\varphi \in [L_2(\Omega)]^2$  and the second equation by  $\psi \in H^1(\Omega)$ ,  $\psi = 0$  sur  $\Gamma_D$  and integration on  $\Omega$  gives rise to the following problem: find  $u \in H^1(\Omega)$  and  $J \in [L_2(\Omega)]^2$  such that

$$\begin{cases} \int_{\Omega} e^{-u} J \cdot \varphi \, dX - \int_{\Omega} \nabla \eta \cdot \varphi \, dX = 0 \quad \forall \varphi \in [L_2(\Omega)]^2 \\ \int_{\Omega} J \cdot \nabla \psi \, dX = 0 \quad \forall \psi \in H^1(\Omega), \psi = 0 \text{ on } \Gamma_D. \end{cases} \tag{10}$$

The hybrid discretization will be based on the approximation of (10). For this, consider a regular triangulation  $\mathcal{T}_h$  of  $\Omega$  and define the following finite-dimensional spaces:

$$\begin{aligned} M_h &= \{ \mu_h \in C^0(\bar{\Omega}), \mu_h|_T \in \mathbb{P}_1(T), \forall T \in \mathcal{T}_h \} \\ M_h^\zeta &= \{ \mu_h \in M_h, \mu_h = \zeta \text{ sur } \Gamma_D \} \\ X_h &= \{ q_h \in [L_2(\Omega)]^2, q_h|_T \in [\mathbb{P}_0(T)]^2 \forall T \in \mathcal{T}_h \} \end{aligned}$$

where  $\mathbb{P}_k(T)$  is the set of polynomials in  $x, y$  of degree less than or equal to  $k$ , restricted to  $T$ . The current  $J$  and the function  $\eta$  are then approximated by  $J_h \in X_h$  and  $\eta_h \in M_h^{\eta_d}$  such that

$$\begin{cases} \int_{\Omega} e^{-u} J_h \cdot q_h \, dX - \int_{\Omega} \nabla \eta_h \cdot q_h \, dX = 0 \quad \forall q_h \in X_h \\ \int_{\Omega} J_h \cdot \nabla \mu_h \, dX = 0 \quad \forall \mu_h \in M_h^0. \end{cases} \tag{11}$$

The choice of  $X_h$  and  $M_h^0$  results in the uniqueness of the solution of (11), (see [5]).

On each triangle  $T$ ,  $q_h$  and  $J_h$  are constant, the system (11) is then written

$$\begin{cases} \sum_{T \in \mathcal{T}_h} q_h \cdot (J_h \int_T e^{-u} \, dX - \int_T \nabla \eta_h \, dX) = 0 \\ \sum_{T \in \mathcal{T}_h} J_h \cdot \int_T \nabla \mu_h \, dX = 0. \end{cases} \tag{12}$$

The choice in the first equation of (12) of an element of  $X_h$  not zero on the triangle  $T$  and vanishing elsewhere results in:

$$J_h \int_T e^{-u} \, dX - \int_T \nabla \eta_h \, dX = 0,$$

hence

$$J_h = \frac{\int_T \nabla \eta_h \, dX}{\int_T e^{-u} \, dX}. \tag{13}$$

The introduction of the relation (13) in the second equation of (12) gives a system where the unique unknown is  $\eta_h$

$$\eta_h \in M_h^{\eta_d}, \quad \sum_{T \in \mathcal{T}_h} \alpha_T \int_T \nabla \eta_h \cdot \nabla \mu_h \, dX = 0, \quad \forall \mu_h \in M_h^0. \tag{14}$$

where  $\alpha_T$  is defined by  $\alpha_T = mes(T) \left( \int_T e^{-u} \, dX \right)^{-1}$ .

The following proposition allows us to define an approximation of  $\mathcal{M}(u)$

**Proposition 2.1.** *Suppose that the triangulation  $\mathcal{T}_h$  is such that for all  $T \in \mathcal{T}_h$  the angles of  $T$  are  $\leq \frac{\pi}{2}$ , then  $\min_{\Gamma_D}(\eta_h) \leq \eta_h \leq \max_{\Gamma_D}(\eta_h)$ .*

*Proof.* The matrix  $A$  associated with the system (14) is unreduced, symmetric with strictly dominant diagonal. These coefficients verify the following relation

$$\sum_j A_{ij} \geq 0, \quad \forall i, \quad A_{ij} \leq 0 \text{ pour } i \neq j.$$

As the second member of (14) is zero, we can deduce (cf. Ciarlet [8]) that the principle of discrete maximum can be applied to  $\eta_h$ , that is:

$$\min \eta_d \leq \eta_h \leq \max \eta_d.$$

□

Since  $\eta_d(x) > 0 \quad \forall x \in \Gamma_D$ , we then define an approximation  $M_h$  de  $M$  by  $\mathcal{M}_h(u) = r_h\{-\ln \eta_h(u)\}$ .

### 3. APPROXIMATION OF Q. V. I.

Consider now the convex  $K_h(u)$  defined by

$$K_h(u) = \{\varphi_h \in M_h^{u_d}, \varphi_h \leq \mathcal{M}_h(u)\},$$

the finite element approximation of Q.V.I. is then given by

$$u_h \in K_h(u_h), \quad \int_{\Omega} \nabla u_h \nabla (\varphi_h - u_h) \, dX \geq \xi \int_{\Omega} (\varphi_h - u_h) \, dX \quad \forall \varphi_h \in K_h(u_h). \tag{15}$$

**Proposition 3.1.** *The approximate problem (15) admits a solution in  $M_h^{u_d}$ .*

*Proof.* Let  $w_{hs}$  be the solution of

$$w_{hs} \in M_h^{w_d}, \quad \int_{\Omega} \nabla w_{hs} \nabla \varphi_h \, dX = 0, \quad \forall \varphi_h \in M_h^0,$$

and let  $S_t$  be the operator who with  $z_h \in M_h$  associates  $z'_h$  solution of the variational equation

$$z'_h \in K_{ht}(z_h), \int_{\Omega} \nabla u_h \nabla (\varphi - u_h) dX \geq \xi \int_{\Omega} (\varphi_h - u_h) dX \forall \varphi \in K_{ht}(z_h). \tag{16}$$

where

$$K_{ht}(z_h) = \{\varphi_h \in M_h^{u_d}, \varphi_h \leq w_{ht}(z_h)\},$$

with

$$w_{ht}(z_h) = (1 - t)w_{hs} + t\mathcal{M}_h(z_h) \forall t \in [0, 1].$$

Any fixed point of  $S_1$  is a solution of (15). As for the problem (1)–(5) (see [23]), the fact that  $S_t(z) \in K_{ht}$  and an argument of the principle of the discrete maximum [8] implies that any fixed point  $z_h$  of  $S_t$  is such that  $V^- \leq z_h \leq V^+$ . Let  $B$  be the open defined by:

$$B = \{\varphi_h \in M_h, \|\varphi_h\|_{\infty} < C\},$$

where  $C$  is a constant much greater than  $\max(|V^-|, V^+)$ , it is clear that no fixed point of  $S_t$  belongs to  $\partial B$  therefore  $\text{deg}[I - S_t, B, 0]$  is defined and independent of  $t$ , where  $\text{deg}[\cdot, \cdot, \cdot]$  is the topological degree of Brouwer [10, 11] and  $I$  is the identity of  $M_h$ .

For  $t = 0$ ,  $K_{h0}(z_h) = K_{h0} = \{\varphi_h \in M_h^{u_d}, \varphi_h \leq w_{hs}\}$ . The operator  $S_0$  matches to any  $z_h \in M_h$  the solution of the variational inequality

$$u_h \in K_{h0}, \int_{\Omega} \nabla u_h \nabla (\varphi - u_h) dX \geq \xi \int_{\Omega} (\varphi_h - u_h) dX, \forall \varphi \in K_{h0}.$$

Consequently  $S_0$  is trivial and therefore  $\text{deg}[I - S_0, B, 0] = 1$ , which implies  $\text{deg}[I - S_t, B, 0] = 1, \forall t \in [0, 1]$ . Hence  $S_1$  admits a fixed point in  $B$  which is solution of (15). □

The questions concerning the convergence of  $u_h$  towards  $u$  are natural and will be published in a future work. We will treat the case where the operator  $\mathcal{M}$  is approached by a classical finite element method as well as the case of the new hybrid finite element approach.

#### 4. NUMERICAL ASPECTS

To solve (15) which is also a Q.V.I., we use an iterative scheme based on the operator  $S_1$ , specifically we define the following

$$u_h^m \in B \cap M_h^{u_d}, \text{ telle que } u_h^m = S_1(u_h^{m-1}), \tag{17}$$

$u_h^0$  being chosen as the solution of the variational inequality

$$\int_{\Omega} \nabla u_h^0 \nabla (\varphi - u_h^0) dX \geq \xi \int_{\Omega} (\varphi_h - u_h^0) dX, \varphi \in M_h^{u_d}, \varphi \leq w^0,$$

where  $w^0$  is the Fermi quasi-potential called the Shokley approximation.

### 4.1. Treatment of exponential nonlinearity

The computation of the Fermi quasi-potential  $w_h$ , which serves as an obstacle in the computation of the potential  $u_h$ , solution of the variational inequation (17), is reduced to solving the system of linear equations

$$\sum_{j \in E_{hi}} b_{ij} \eta_{hj} = 0 \tag{18}$$

$$b_{ij} = \sum_{T \in T_h^{ij}} C_{hT} \alpha_{hT} \tag{19}$$

where  $\alpha_{hT} = \text{mes}(T) \left( \int_T e^{-u_h} dX \right)^{-1}$ ,  $C_{hT}$  is the contribution of the triangle  $T$  to the discretization of the Laplacian,  $T_h^i$  is the set of triangles having the node  $i$  as vertex,  $E_{hi}$  is the set of nodes which are vertices of a triangle  $T \in T_h^i$ ,  $T_h^{ij} = T_h^i \cap T_h^j$ , and  $u_{hl}$  the value of  $u_h$  at node  $l$ . In the form (19) the computation of the coefficients  $b_{ij}$  leads to capacity overruns (underflow-overflow), especially for the computation of the diagonal terms.

To eliminate this problem we will describe a technique which does not require any additional storage and which is very efficient for sufficiently small  $h$  steps.

Multiply equation (18) by  $e^{u_{hi}}$  we get

$$\sum_{j \in E_{hi}} b_{ij} e^{u_{hi}} \eta_{hj} = 0,$$

that we write in the form

$$\sum_{j \in E_{hi}} b'_{ij} \eta_{hj} = 0. \tag{20}$$

Typically, the coefficients  $b'_{ij}$  keep the properties of  $b_{ij}$ , *i.e.*, the property of the dominant diagonal. However, the system of linear equation (20) is no longer symmetric. But we can still manipulate this system, by storing only the upper part of the matrix ( $b'_{ij}$ ). Indeed, the non-zero subdiagonal terms can be obtained as follows:

The coefficient  $b'_{ji}$ ,  $j > i$ ; is obtained from the coefficient  $b'_{ij}$ ,  $j > i$ , by the relation

$$b'_{ji} = b'_{ij} e^{(u_{hj} - u_{hi})}.$$

This means that the solution of the system of equation (20) does not require any additional storage than that of the upper part of the matrix ( $b_{ij}$ ) and the vector  $[u_{hi}]_i$ . System (20) therefore preserves the main characteristics of (18), with the following properties:

**Proposition 4.1.** *For  $h$  sufficiently small, the calculation of all the coefficients  $b'_{ij}$  is done without any capacity overrun (neither underflow nor overflow).*

*Proof.* The proof of the proposition is obtained from the relation

$$b'_{ij} = \sum_{T \in T_h^{ij}} C_{hT} \text{mes}(T) \left( \int_T e^{(u_{hi} - u_h)} dX \right)^{-1}.$$

Since  $u_h \in \mathbb{P}_1(T)$ , we can calculate in an exact way the integral  $I$  appearing in the previous relation. For simplicity we assume that

$$I = \sum_{l \in V(T)} a_l e^{(u_{hi} - u_{hl})}$$

where  $V(T)$  is the set of vertices of the triangle  $T$ . Since  $i \in V(T)$ ,  $\forall T \in \mathcal{T}_h^{ij}$ , there is an  $l_0 \in V(T)$  such as  $e^{(u_{hi}-u_{hl_0})} = 1$ , which eliminates any division by zero in the calculation of  $\alpha_{hT}$ . Furthermore,  $u_h \in \mathbb{P}_1(T)$  leads to

$$|u_{hi} - u_{hl}| \leq Ch, \quad \forall i, \forall l \in V(T).$$

It is clear that the sign of  $(u_{hi} - u_{hl})$  can be negative or positive. However, for  $h$  small enough we can ensure that  $-\chi < (u_{hi} - u_{hl}) < \chi$  où  $\chi$  is the lower limit value acceptable by the machine for  $e^x$ , this guarantees that the capacity is not exceeded in the calculation  $e^{(u_{hi}-u_{hl})}$ . □

### 4.2. Computation of Drain current

The following proposition expresses the weak continuity of the electric current.

**Proposition 4.2.** *Let  $(\eta_h, J_h)$  be the solution of the system (11), let  $i$  be a node of  $\mathcal{T}_h$ ,  $i \notin \Gamma_d$  and let  $L_h^i$  be the set of sides having  $i$  as vertex. Then*

$$\sum_{e \in L_h^i} \int_e J_h \cdot \nu \, d\sigma = 0.$$

*Proof.* Integration by parts, the second equation of the system (12) leads to the equation

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} J_h \cdot \nu \mu_h \, d\sigma = 0, \quad \forall \mu_h \in M_h^0.$$

For the basis element  $\varphi_i$  associated with the node  $i$  we obtain  $\sum_{e \in L_h^i} \int_e J_h \cdot \nu \, d\sigma = 0$ . □

In semiconductor device technology the most important feature is the computation of the current flowing along a given contact. For FETs, it is the current of Drain  $J_d$  defined by

$$J_d = \int_{\Gamma_d} J \cdot \nu \, d\sigma,$$

where  $\Gamma_d$  is the Drain contact. To computation of  $J_d$  by this formula requires the computation of the electric field and a numerical integration. In the following we describe how we can get around this problem by using a method that does not require any additional calculation.

Let  $\theta_h$  be the function of  $M_h$  defined by

$$\theta_h = \sum_{l \in V_d} \omega_l$$

where  $\omega_l$  is the base function associated with node number  $l$  and  $V_d$  is the set of nodes belonging to the Drain. By construction  $\theta_h = 1$  on  $\Gamma_d$  and it is identically zero on the remainder of  $\partial\Omega$ , hence

$$\begin{aligned} J_d &= \int_{\partial\Omega} \theta_h J \cdot \nu \, d\sigma \\ &= \int_{\Omega} \theta_h \nabla \cdot J \, dX + \int_{\Omega} J \cdot \nabla \theta_h \, dX. \end{aligned}$$

Since  $J$  has zero divergence, we get

$$J_d = \sum_{l \in V_d} \int_{\Omega} J \cdot \nabla \omega_l \, dX.$$



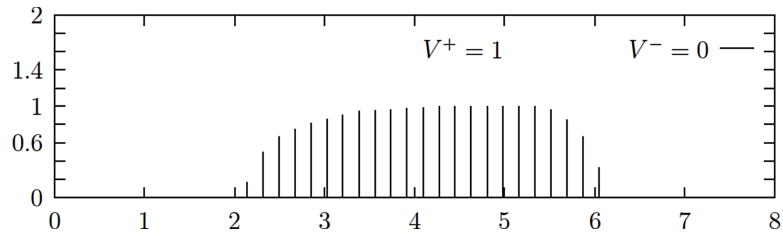


FIGURE 2. Depletion region  $V^- = -0$  and  $V^+ = 1$ .

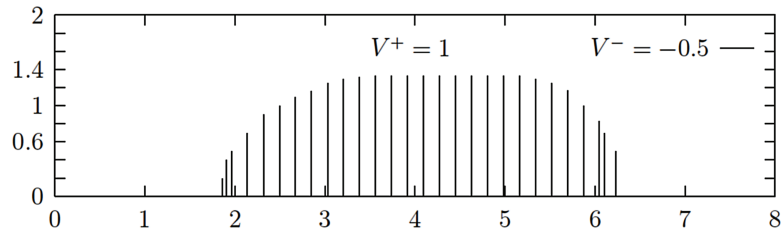


FIGURE 3. Depletion region  $V^- = -0.5$  and  $V^+ = 1$ .

We approach the current  $J$  by  $J_h$ , the current of Drain  $J_d$  is then approached by

$$\begin{aligned}
 J_{dh} &= \sum_{l \in V_d} \sum_{T \in \mathcal{T}_h} \alpha_{hT} \int_T \nabla \eta_h \cdot \nabla \omega_l \, dX \\
 &= \sum_{l \in V_d} \left( \sum_{i=1} b_{li} \eta_{hi} \right)
 \end{aligned} \tag{21}$$

where  $B = (b_{ij})$  is the matrix associated with the system (14).

**Remark 4.3.** The calculation of  $J_{dh}$  by the formula (21) is reduced to the product of the matrix  $B$  by the vector  $[\eta_{hi}]$ , whose elements are already computed and stored.

### 5. NUMERICAL RESULTS

Numerical simulation is made using the following value of the main parameters:  $\frac{Nq}{\epsilon} = 1$ ,  $DA = 8$  microns and  $AB = 2$  microns.

We set the gate voltage to 0 V and we apply a positive voltage to the drain contact thus the current flow through the channel and we can calculate  $J_d$ , the current of the Drain contact.

We increased the drain voltage, the current  $J_d$  increases up to the value 2.8 mA, then it becomes stationary. The same remark when  $Vg = -1$  V the limit value of  $J_d$  approaches 1.2 mA. Then, the limit value only reaches 0.5 mA when  $Vg = -1.5$  V. The current therefore becomes increasingly weak when the absolute value of  $Vg$  becomes increasingly large. This is consistent with the Figures 2-4 which show that the depth of the depletion region increases, when the value of  $Vg$  becomes smaller and smaller, until this zone extends through the active channel which stops the current movement in the channel. This gate-source potential phenomenon is known in physics as ‘‘Pinch-off voltage’’.

We set the drain voltage to +1 V and we vary the Gate voltage from 0 V to  $-1.8$  V. The depth of the depletion region becomes higher when we increase the gate voltage. This continues until the boundary of the depletion

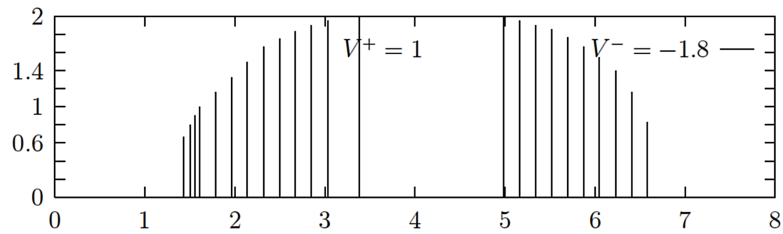


FIGURE 4. Depletion region  $V^- = -1.8$  and  $V^+ = 1$ .

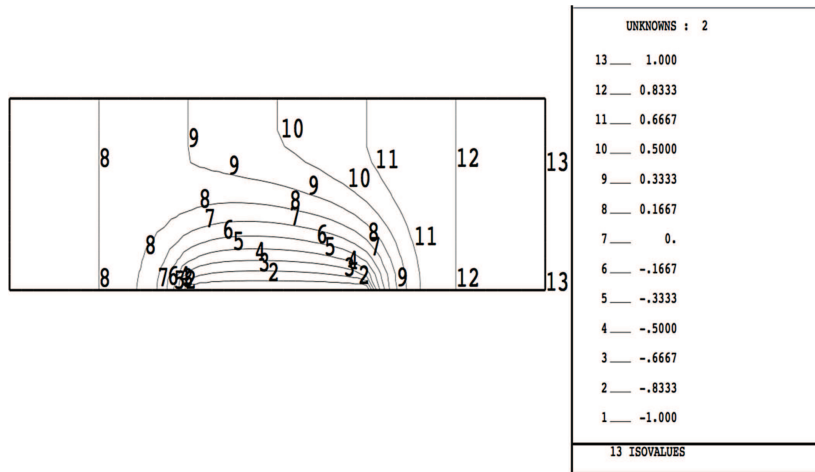


FIGURE 5. MESFET  $n$ -type, equipotentials  $V^- = -1$ ,  $V^+ = 1$ .

layer touches the side opposite the Gate. The channel is then closed, which causes the flow of current to stop (2)–(4).

We observe from Figure 5 that the negative equipotentials do not cross the free boundary. They all remain confined inside line 7, equipotential of value equal to 0 V. The positive equipotentials go around line 7, cross the channel and therefore allow current to flow.

In Figure 6, where only the positive equipotentials have been represented, it can be seen that there are no more lines around the line number 1, where the potential is of value 0 V, which is no longer a closed curve. This shows that the channel is closed and there is no more current flow.

## 6. CONCLUSION

From these numerical results we have observed, (see the following figures), that

- The depletion region is localized around the gate contact.
- As is known from physical experience, when applying a positive potential difference between source and drain contacts, an electric current flows from one end to an other. The gate contact allows us to control the drain current by modifying the thickness of the depletion zone: a sufficiently negative gate-source voltage cuts the conducting channel and thereby switches off the drain current.

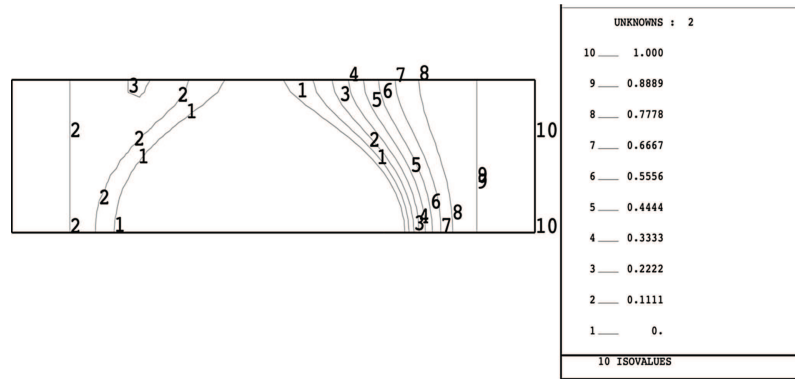


FIGURE 6. MESFET  $n$ -type, equipotentials  $V^- = -1.8$ ,  $V^+ = 1$ .

We can conclude that for the semiconductor device, used in the numerical experiments, the simplified model described in section introduction does not change the behavior of the full system. Also, we can conclude that the proposed hybrid finite element method provides a good alternative for solving free boundary problems arising in the depletion approximation model. This method combined with the scaling technique makes it possible to avoid capacity overruns and leads to a correct calculation of the current.

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## REFERENCES

- [1] J. Abouchabaka, R. Aboulaïch, A. Nachaoui and A. Souissi, Quasi-variational inequality and shape optimization for solution of a free boundary problem. *COMPEL* **18** (1999) 143–164.
- [2] F. Aboud and A. Nachaoui, Single-rank quasi-Newton methods for the solution of nonlinear semiconductor equations. *Adv. Math. Models Appl.* **5** (2020) 70–79.
- [3] Y. Berchenko-Kogan and A. Stern, Constraint-preserving hybrid finite element methods for Maxwell’s equations. *Found. Comput. Math.* **21** (2021) 1075–1098.
- [4] M. Bessemoulin-Chatard, C. Chainais-Hillairet and M.-H. Vignal, Study of a finite volume scheme for the drift-diffusion system. Asymptotic behavior in the quasi-neutral limit. *SIAM J. Numer. Anal.* **52** (2014) 1666–1691.
- [5] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods. Springer serie in Computational Mathematics (1991).
- [6] C. Cancès, C. Chainais-Hillairet, J. Fuhrmann and B. Gaudeul, A numerical-analysis-focused comparison of several finite volume schemes for a unipolar degenerate drift-diffusion model. *IMA J. Numer. Anal.* **41** (2021) 271–314.
- [7] C. Chen and X. Long, Finite volume element approximation and analysis for a kind of semiconductor device simulation. *J. Appl. Math. Comput.* **33** (2010) 155–172.
- [8] P.G. Ciarlet, Basic Error Estimates for Elliptic Problems, in Vol. II *Finite element methodes (Part I)*, *Handbook of numerical analysis*, Edited by P.G. Ciarlet and J.L. Lions (1991).
- [9] M.R. Correa and G. Taraschi, Optimal  $H(\text{div})$  flux approximations from the Primal Hybrid Finite Element Method on quadrilateral meshes. *Comput. Methods Appl. Mech. Engrg.* **400** (2022) 115539.
- [10] J. Cronin, Fixed Points and Topological Degree in Nonlinear Analysis, A. M. S. (1964).
- [11] J. Cronin-Scanlon, Differential Equations. Marcel Dekker, New York (1980).
- [12] A. Ellabib and A. Nachaoui, On the numerical solution of a free boundary identification problem. *Inverse Probl. Sci. Eng.* **9** (2001) 235–260.
- [13] A. Ellabib and A. Nachaoui, Existence de solutions pour le modèle dérive-diffusion avec terme d’avalanche, *CRAS- Ser. I - Math.* **332** (2001) 305–310.
- [14] A. Ellabib and A. Nachaoui, Uniqueness of the stationary solutions for drift-diffusion models with avalanche generation. *Extr. math.* **18** (2003) 13–21.

- [15] A. Glitzky, M. Liero and G. Nika, An existence result for a class of electrothermal drift-diffusion models with Gauss-Fermi statistics for organic semiconductors. *Anal. Appl. (Singap.)* **19** (2021) 275–304.
- [16] N. Hayeck, A. Nachaoui and N.R. Nassif, Existence and Regularity for Van-Roosbroek systems with general mixed boundary conditions. *COMPEL* **9** (1990) 217–228.
- [17] T. Kan and M. Suzuki, Uniform estimates and uniqueness of stationary solutions to the drift-diffusion model for semiconductors. *Appl. Anal.* **98** (2019) 1799–1810.
- [18] T. Li, Q. Wang and X. Jin, A staggered asynchronous step integration algorithm for hybrid finite-element and discrete-element modeling. *Int. J. Comput. Methods* **19** (2022) 2150064.
- [19] P. Markowich, *The Stationary Semiconductor Equations*. Springer (1986).
- [20] M.S. Mock, *Analysis of Mathematical Models of semiconductor devices*. Boole Press, Dublin (1983).
- [21] C.M. Morandi and R. Russo, Numerical approximation of free boundary problem by variational inequalities. Application to semiconductor devices. *Nonconvex Optim. Appl.* **79** (2005) 697–722.
- [22] A. Nachaoui, Iterative solution of the drift-diffusion equations. *Numer. Algorithms* **21** (1999) 323–341.
- [23] A. Nachaoui and N.R. Nassif, On the Uniqueness of the Solution to the Drift-Diffusion model in Semiconductor Analysis. *COMPEL* **11** (1992) 377–390.
- [24] A. Nachaoui and N.R. Nassif, Sufficient conditions for converging drift-diffusion discrete systems. Application to the finite element method. *Math. Methods Appl. Sci.* **19** (1996) 33–51.
- [25] A. Nachaoui, J. Abouchabaka and N. Rafalia, Parallel solvers for the depletion region identification in metal semiconductor field effect transistors. *Numer. Algorithms* **40** (2005) 187–199.
- [26] G. Peralta and K. Kunisch, Mixed and hybrid Petrov-Galerkin finite element discretization for optimal control of the wave equation. *Numer. Math.* **150** (2022) 591–627.
- [27] J. Semmler and M. Stingl, On the efficient optimization of optical properties of particulate monolayers by a hybrid finite element approach. *Struct. Multidiscip. Optim.* **63** (2021) 1219–1242.
- [28] Q. Wang, Z. Ma, S. Bai, L. Zhang, B. Lu and H. Li, Research on parallel finite element methods for the drift-diffusion model in semiconductor device simulations. *J. Numer. Methods Comput. Appl.* **41** (2020) 85–104.
- [29] B. Wu, Existence of weak solutions to a class of degenerate semiconductor equations modeling avalanche generation. *Math. Probl. Eng.* (2009) 873683.
- [30] H. Wu and J. Jiang, Global solution to the drift-diffusion-Poisson system for semiconductors with nonlinear recombination-generation rate. *Asymptot. Anal.* **85** (2013) 75–105.
- [31] G. Xiulan, X. Jiasheng and H. Ling, An existence of stationary solutions for the drift-diffusion semiconductor equations. *Dyn. Syst. Appl.* **11** (2002) 521–529.
- [32] Q. Yang and Y. Yuan, An approximation of three-dimensional semiconductor devices by mixed finite element method and characteristics-mixed finite element method. *Numer. Math. Theory Methods Appl.* **8** (2015) 356–382.
- [33] C. Yu, K. Malakpoor, and J. M. Huyghe, Comparing mixed hybrid finite element method with standard FEM in swelling simulations involving extremely large deformations. *Comput. Mech.* **66** (2020) 287–309.
- [34] Q. Zhang, Q. Wang, L. Zhang and B. Lu, A class of finite element methods with averaging techniques for solving the three-dimensional drift-diffusion model in semiconductor device simulations. *J. Comput. Phys.* **458** (2022) 111086.

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