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ON THE ASYMPTOTICS OF EXTREMAL ℓ^p -BLOCKS CLUSTER INFERENCE

GLORIA BURITICÁ AND OLIVIER WINTENBERGER

ABSTRACT. Extremes occur in stationary regularly varying time series as short periods with several large observations, known as extremal blocks. We study cluster statistics summarizing the behavior of functions acting on these extremal blocks. Examples of cluster statistics are the extremal index, cluster size probabilities, and other cluster indices. The purpose of our work is twofold. First, we state the asymptotic normality of block estimators for cluster inference based on consecutive observations with large ℓ^p -norms, for $p > 0$. The case $p = \alpha$, where $\alpha > 0$ is the tail index of the time series, has specific nice properties thus we analyze the asymptotics of block estimators when approximating α using the Hill estimator. Second, we verify the conditions we require on classical models such as linear models and solutions of stochastic recurrence equations. Regarding linear models, we prove that the asymptotic variance of classical index cluster-based estimators is null as first conjectured in [26]. We illustrate our findings on simulations.

1. INTRODUCTION

We study stationary heavy-tailed time series (\mathbf{X}_t) in $(\mathbb{R}^d, |\cdot|)$, with regularly varying distributions, and tail index $\alpha > 0$; cf. [4], a formal definition is conferred to Section 2.2. In this framework, extremal observations cluster: an extreme value triggers a short period with numerous large observations. This behavior is known to perturb classical inference procedures tailored for independent observations like high quantile inference; see [21]. This clustering effect can be summarized by the *extremal index*, initially introduced in [32] and [33]. We can interpret it as the inverse of the mean number of consecutive exceedances above a high threshold in a short period of time. In this article, we aim to infer statistics of the clustering effect by letting functionals act on consecutive observations with extremal behavior. For example, we can recover the extremal index from this setting and also other important indices of the extremes of the series.

For extremal cluster inference, we consider a sample $\mathbf{X}_{[1,n]}$ together with a sequence (b_n) , and we define the sample of disjoint blocks $(\mathcal{B}_j)_{j=1,\dots,m_n}$ as blocks of consecutive observations:

$$(1.1) \quad \mathcal{B}_j := (\mathbf{X}_{(j-1)b_n+1}, \dots, \mathbf{X}_{jb_n}) = \mathbf{X}_{(j-1)b_n+[1,b_n]},$$

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such that $b_n \rightarrow \infty$, $m_n = \lfloor n/b_n \rfloor \rightarrow \infty$, as $n \rightarrow \infty$. Following the p -clusters theory developed in [12], the extremal behavior of the series is modeled by the conditional behavior of a block \mathcal{B}_j given that its ℓ^p -norm is large:

$$(1.2) \quad \mathbb{P}(\mathcal{B}_1/x_{b_n} \in A \mid \|\mathcal{B}_1\|_p > x_{b_n}) \xrightarrow{w} \mathbb{P}(Y\mathbf{Q}^{(p)} \in A), \quad n \rightarrow \infty,$$

such that Y is independent of the p -cluster $\mathbf{Q}^{(p)} \in \ell^p$, $\mathbb{P}(Y > y) = y^{-\alpha}$, for $y > 1$, and $\|\mathbf{Q}^{(p)}\|_p = 1$ a.s., for $p \in (0, \infty]$. The weak convergence holds for a family of shift-invariant continuity sets $A \subset \ell^p$, and (x_n) is a suitable sequence satisfying $\mathbb{P}(\|\mathcal{B}_1\|_p > x_{b_n}) \rightarrow 0$, as $n \rightarrow \infty$. The limit distribution in the right-hand side of (1.2) summarizes the extremal behavior of regularly varying ℓ^p -blocks. In a nutshell, the (α) -Pareto component Y models the magnitude of the rescaled extremal block $\|\mathcal{B}_1\|_p/x_{b_n}$, and the p -cluster $\mathbf{Q}^{(p)}$ describes the extremal dependencies from the normalized block $\mathcal{B}_1/\|\mathcal{B}_1\|_p$ when its ℓ^p -norm reaches extreme levels. In this way, $\mathbf{Q}^{(p)}$ allows us to describe the appearance of consecutive extremes in regularly varying time series.

In this article we study the inference of p -cluster statistics of the form

$$(1.3) \quad f^{\mathbf{Q}(p)} = \mathbb{E}[f(Y\mathbf{Q}^{(p)})],$$

for suitable ℓ^p -continuity functions $f : \ell^p \rightarrow \mathbb{R}$ which are invariant to the shift operator of sequences. Examples of shift-invariant functions are p -norms, and also coordinate-wise averages. These functions are typically good summaries of the extremal clustering behavior of blocks. To stress the relation of the functional f with the p -cluster statistic in (1.3) it is convenient to write $f = f(p)$. To infer these statistics, we use the disjoint block estimators proposed in [12] defined as

$$(1.4) \quad \widehat{f^{\mathbf{Q}(p)}} := \frac{1}{k_n} \sum_{t=1}^{m_n} f(\mathcal{B}_t/\|\mathcal{B}\|_{p,(k+1)}) \mathbf{1}(\|\mathcal{B}_t\|_p > \|\mathcal{B}\|_{p,(k+1)}),$$

where $\|\mathcal{B}\|_{p,(1)} \geq \|\mathcal{B}\|_{p,(2)} \geq \dots \geq \|\mathcal{B}\|_{p,(m_n)}$, denotes the sequence of order statistics of the ℓ^p -norms of blocks defined in (1.1), and (k_n) is an integer sequence determining the number of extremal blocks that we select for inference, and $k_n \rightarrow \infty$, as $n \rightarrow \infty$.

The case $p = \alpha$ is particularly relevant for two reasons. The first reason is that under mixing and anti-clustering conditions, choosing (a_n) satisfying $n \mathbb{P}(|\mathbf{X}_1| > a_n) \rightarrow 1$, as $n \rightarrow \infty$, [13] prove that

$$(1.5) \quad \sum_{t=1}^n \varepsilon_{a_n^{-1}} \mathbf{x}_t \xrightarrow{d} \sum_{i=1}^{\infty} \sum_{j=-\infty}^{\infty} \varepsilon_{\Gamma_i^{-1/\alpha}} \mathbf{Q}_{ij}, \quad n \rightarrow \infty,$$

where $\Gamma_1 \leq \Gamma_2 \leq \dots$ are the points of an homogeneous Poisson process, $(\mathbf{Q}_{ij})_j$ are independent copies of the α -cluster process $\mathbf{Q} := \mathbf{Q}^{(\alpha)}$ independent of (Γ_i) . The extremal cluster dependencies of the series are fully modeled using the spectral cluster process \mathbf{Q} , and from it one can recover the

distribution of $\mathbf{Q}^{(p)}$, for every $p \in (0, \infty]$, using the change-of-norms equation given by [12] and recalled in (2.13). In practice this means α -cluster inference allows us to infer any statistic in the form of (1.3). Moreover, the choice $p = \alpha$ is also ideal for selecting extremal blocks in (1.4) as it is less susceptible to local serial dependencies; see [12] for a broader discussion.

The main goal of this article is to establish the asymptotic normality of the block estimators from Equation (1.4), tailored for cluster inference. We state moment, mixing and bias assumptions yielding the existence of a sequence (k_n) , satisfying $k = k_n \rightarrow \infty$, $m_n/k_n \rightarrow \infty$ such that

$$(1.6) \quad \sqrt{k}(\widehat{f}^{\mathbf{Q}}(p) - f^{\mathbf{Q}}(p)) \xrightarrow{d} \mathcal{N}(0, \text{Var}(f(Y^{\mathbf{Q}}(p)))) , \quad n \rightarrow \infty,$$

and the limit is a centered Gaussian distribution. Our inference methodology can be viewed as a Peak Over Threshold over order statistics of blocks. Moreover, for $p = \alpha$, fixing k and letting first $n \rightarrow \infty$ in (1.4) implies heuristically from (1.5) that

$$\widehat{f}^{\mathbf{Q}}(\alpha) \approx \frac{1}{k} \sum_{i=1}^k f((\Gamma_i/\Gamma_{k+1})^{-1/\alpha}(\mathbf{Q}_{it})_t).$$

Then, the simple expression of the asymptotic variance in (1.6) follows as $(\Gamma_i/\Gamma_{k+1}) \stackrel{d}{=} (U_{k,i})$ where $U_{k,1} < \dots < U_{k,k}$ are the ordered statistics of iid uniformly distributed U_j , $1 \leq j \leq k$, and $U^{-1/\alpha} \stackrel{d}{=} Y$. This heuristic is extended to any p -cluster $\mathbf{Q}^{(p)}$, $p \in (0, \infty]$, via the change-of-norms in (2.13), and rigorously proved in this article using the theory of the tail empirical process for time series; cf. [31].

In general, for p -cluster inference, the function $f = f(p)$ can involve the tail-index α in its expression, meaning $f(p) = f_\alpha(p)$. Furthermore, we already mentioned that the choice of $p = \alpha$ has the advantage of being robust to serial dependencies and to fully characterize clusters of extreme values (cf. [12]), thus to implement these procedures we replace α with an estimate $\widehat{\alpha}$. We then show the asymptotic normality of the p -cluster estimator $\widehat{f}_{\widehat{\alpha}}^{\mathbf{Q}}(p)$ when we let $1/\widehat{\alpha}$ equal the classical Hill estimator, and we extend the analysis to cover $\widehat{\alpha}$ -cluster inference. Furthermore, we conduct simulations to illustrate that $\ell^{\widehat{\alpha}}$ -block estimators are competitive both in terms of bias and variance for finite sample sizes.

Our asymptotic results highlight how introducing ℓ^p -norm block order statistics in (1.4), instead of order statistics of the sample $(|\mathbf{X}_t|)$ as in [20, 14], can lead to a better asymptotic variance for cluster inference. We give examples of variance reduction in the case of linear models with short-range dependence, for inference of classical indices. In our examples, the asymptotic variance of linear models $\text{Var}(f(Y^{\mathbf{Q}}(p)))$ is null because of the deterministic properties of the spectral p -cluster process of linear models. For linear models, the advantage of replacing thresholds with block maxima records was previously investigated in [26]. Existing works [18, 20, 14, 31] following [26] focus on cluster of exceedances inference such that $p = \infty$.

Our asymptotic result comforts and extends the heuristics presented in [26] for $p = \infty$ and linear models to the case $p < \infty$ and general models. To prove the asymptotic normality of block estimators, we rely on the theory of empirical processes [42], but adapted to block estimators. For this purpose, we build on the previous work of [18], and the modern overview in [31]. To handle the asymptotics of extremal ℓ^p -blocks, we build on the large deviation principles studied in [12], and appeal to the p -cluster processes theory therein.

The article is organized as follows. Preliminaries on mixing coefficients, regular variation, and the p -cluster theory of stationary time series are compiled in Section 2. In Section 3 we present our main result in Theorem 3.2, stating the asymptotic normality of the block estimators introduced in Equation (1.4). We work under mixing, moment, and bias conditions on the series that we also present in Section 3. Section 4 studies examples of extremal cluster inference such as estimation of the extremal index, the cluster size probabilities, and the cluster index for sums. We conclude by verifying our conditions on classical models such as linear processes and stochastic recurrence equations in Section 5. In the case of linear models with short-range dependence, Theorem 5.6 states that the ℓ^p -block estimators of all the aforementioned quantities have null-asymptotic variance. Thereby, they are super-efficient for cluster inference of important indices as conjectured by [26] for $p = \infty$. We illustrate the finite-sample performances of our estimators in Section 6. All proofs are deferred to Appendices A, B, C, D, E, and F.

1.1. Notation. We consider stationary time series (\mathbf{X}_t) taking values in \mathbb{R}^d , that we endow with a norm $|\cdot|$. Let $p \in (0, \infty]$, and $(\mathbf{x}_t) \in (\mathbb{R}^d)^{\mathbb{Z}}$. Define the p -modulus function $\|\cdot\|_p : (\mathbb{R}^d)^{\mathbb{Z}} \rightarrow [0, \infty]$ as

$$\|(\mathbf{x}_t)\|_p^p := \sum_{t \in \mathbb{Z}} |\mathbf{x}_t|^p,$$

and define the sequential space ℓ^p as $\ell^p := \{(\mathbf{x}_t) \in (\mathbb{R}^d)^{\mathbb{Z}} : \|(\mathbf{x}_t)\|_p^p < \infty\}$, with the convention that, for $p = \infty$, the space ℓ^∞ refers to sequences with finite supremum norm. For any $p \in (0, \infty]$, the p -modulus functions induce a distance d_p in ℓ^p , and for $p \in [1, \infty]$, it defines a norm. Abusing notation, we call them all ℓ^p -norms. Let $\tilde{\ell}^p = \ell^p / \sim$ be the shift-invariant quotient space where $(\mathbf{x}_t) \sim (\mathbf{y}_t)$ if and only if there exists $k \in \mathbb{Z}$ such that $\mathbf{x}_{t-k} = \mathbf{y}_t$, $t \in \mathbb{Z}$. We also consider the metric space $(\tilde{\ell}^p, \tilde{d}_p)$ such that for $[\mathbf{x}], [\mathbf{y}] \in \tilde{\ell}^p$,

$$\tilde{d}_p([\mathbf{x}], [\mathbf{y}]) := \inf_{k \in \mathbb{Z}} \{d_p(\mathbf{x}_{t-k}, \mathbf{y}_t), (\mathbf{x}_t) \in [\mathbf{x}], (\mathbf{y}_t) \in [\mathbf{y}]\},$$

and without loss of generality, we write an element $[\mathbf{x}]$ in $\tilde{\ell}^p$ also as (\mathbf{x}_t) . Further details on the shift-invariant spaces are deferred to [12, 3].

The operator norm for $d \times d$ matrices, $\mathbf{A} \in \mathbb{R}^{d \times d}$, is defined as $|\mathbf{A}|_{op} := \sup_{|\mathbf{x}|=1} |\mathbf{A}\mathbf{x}|$. The truncation operations of (\mathbf{x}_t) at the level ϵ , for $\epsilon > 0$, are

defined by

$$(\bar{\mathbf{x}}_t^\varepsilon) := (\mathbf{x}_t \mathbb{1}_{|\mathbf{x}_t| \leq \varepsilon}), \quad (\mathbf{x}_{t_\varepsilon}) := (\mathbf{x}_t \mathbb{1}_{|\mathbf{x}_t| > \varepsilon}).$$

The notation $a \wedge b$ denotes the minimum between two constants $a, b \in \mathbb{Z}$, and $a \vee b$ denotes its maximum. We write $\log^+(x) := \log(x) \vee 0$, for $x \in (0, \infty)$. We sometimes write \mathbf{x} for the sequence $\mathbf{x} := (\mathbf{x}_t) \in (\mathbb{R}^d)^\mathbb{Z}$. Furthermore, for $a, b \in \mathbb{R}$, and $a \leq b$, we write as $\mathbf{x}_{[a,b]}$ the vector $(\mathbf{x}_t)_{t=a, \dots, b}$ taking values in $(\mathbb{R}^d)^{b-a+1}$. We write $\mathbf{x}_{[a,b]} \in \tilde{\ell}^p$, which means we take the natural embedding of $\mathbf{x}_{[a,b]}$ in $\tilde{\ell}^p$ defined by assigning zeros to undefined coefficients. It will be convenient to write $\mathcal{G}_+(\tilde{\ell}^p)$ for the continuous non-negative functions on $(\tilde{\ell}^p, \tilde{d}_p)$ which vanish in a neighborhood of the origin.

2. PRELIMINARIES

2.1. Mixing coefficients. Let (\mathbf{X}_t) be an \mathbb{R}^d -valued strictly stationary time series defined over a probability space $((\mathbb{R}^d)^\mathbb{Z}, \mathcal{A}, \mathbb{P})$. The properties of stationary sequences are usually studied through mixing coefficients. Denote the past and future σ -algebras by

$$\mathcal{F}_{t \leq 0} := \sigma((\mathbf{X}_t)_{t \leq 0}), \quad \mathcal{F}_{t \geq h} := \sigma((\mathbf{X}_t)_{t \geq h}), \quad h \geq 1,$$

respectively. We recall the definition of the mixing coefficients (β_h) below

$$\beta_h := d_{TV}(\mathbb{P}_{\mathcal{F}_{t \leq 0} \otimes \mathcal{F}_{t \geq h}}, \mathbb{P}_{\mathcal{F}_{t \leq 0}} \otimes \mathbb{P}_{\mathcal{F}_{t \geq h}}),$$

where $d_{TV}(\cdot, \cdot)$ is the total variation distance between two probability measures: $((\mathbb{R}^d)^\mathbb{Z}, \mathcal{A}, \mathbb{P}_1)$, $((\mathbb{R}^d)^\mathbb{Z}, \mathcal{A}, \mathbb{P}_2)$, and $\mathbb{P}_1 \otimes \mathbb{P}_2(A \times B) := \mathbb{P}_1(A)\mathbb{P}_2(B)$, for $A, B \in \mathcal{A}$. For a summary on mixing conditions see [8, 17, 39].

Remark 2.1. A detailed interpretation of the β -mixing coefficients (β_t) in terms of the total variation distance can be found in Chapter 1.2 in [17]. These mixing coefficients are well adapted while working with Markov processes. Indeed, a strictly stationary Harris recurrent Markov chain (\mathbf{X}_t) , satisfies $\beta_t \rightarrow 0$ exponentially fast as $t \rightarrow \infty$; see Theorem 3.5 in [8].

2.2. Regular variation. We consider stationary time series (\mathbf{X}_t) taking values in $(\mathbb{R}^d, |\cdot|)$ and regularly varying with tail index $\alpha > 0$: all its finite-dimensional vectors are multivariate regularly varying of the same index. In this case we write (\mathbf{X}_t) satisfies \mathbf{RV}_α . Borrowing the ideas in [4], (\mathbf{X}_t) satisfies \mathbf{RV}_α if and only if, for all $h \geq 0$, there exists a vector $(\Theta_t)_{|t| \leq h}$, taking values in $(\mathbb{R}^d)^{2h+1}$ such that

$$(2.7) \quad \mathbb{P}(x^{-1}(\mathbf{X}_t)_{|t| \leq h} \in \cdot \mid |\mathbf{X}_0| > x) \xrightarrow{d} \mathbb{P}(Y(\Theta_t)_{|t| \leq h} \in \cdot), \quad x \rightarrow \infty,$$

where Y is independent of $(\Theta_t)_{|t| \leq h}$ and $\mathbb{P}(Y > y) = y^{-\alpha}, y > 1$. We call the sequence (Θ_t) , taking values in $(\mathbb{R}^d)^\mathbb{Z}$, the spectral tail process.

The time series (Θ_t) does not inherit the stationarity property of the series. Instead, the time-change formula of [4] holds: for any $s, t \in \mathbb{Z}, s \leq$

$0 \leq t$ and for any measurable bounded function $f : (\mathbb{R}^d)^{t-s+1} \rightarrow \mathbb{R}$,

(2.8)

$$\mathbb{E}[f(\boldsymbol{\Theta}_{s-i}, \dots, \boldsymbol{\Theta}_{t-i}) \mathbb{1}(|\boldsymbol{\Theta}_{-i}| \neq 0)] = \mathbb{E}[|\boldsymbol{\Theta}_i|^\alpha f(\boldsymbol{\Theta}_s/|\boldsymbol{\Theta}_i|, \dots, \boldsymbol{\Theta}_t/|\boldsymbol{\Theta}_i|)].$$

2.3. ℓ^p -cluster processes. Let (\mathbf{X}_t) be a stationary time series satisfying \mathbf{RV}_α . For $p \in (0, \infty]$, we say the series admits a p -cluster process $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$ if there exists a sequence (x_n) , satisfying

$$(2.9) \quad \mathbb{P}(\|\mathbf{X}_{[1,n]}\|_p > x_n) \sim n c(p) \mathbb{P}(|\mathbf{X}_1| > x_n), \quad n \rightarrow \infty,$$

with $c(p) \in (0, \infty)$, $n\mathbb{P}(|\mathbf{X}_1| > x_n) \rightarrow 0$, and

$$(2.10) \quad \mathbb{P}(\mathbf{X}_{[1,n]}/x_n \in \cdot \mid \|\mathbf{X}_{[1,n]}\|_p > x_n) \xrightarrow{w} \mathbb{P}(Y\mathbf{Q}^{(p)} \in \cdot), \quad n \rightarrow \infty,$$

where Y is independent of $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$, $\mathbb{P}(Y > y) = y^{-\alpha}$, for $y > 1$, $\|\mathbf{Q}^{(p)}\|_p = 1$ a.s., and the limit in (2.10) holds in $(\tilde{\ell}^p, \tilde{d}_p)$. We study below the anti-clustering and vanishing-small values conditions denoted \mathbf{AC} , \mathbf{CS}_p , respectively, which guarantee the existence of ℓ^p -clusters. We rephrase next the Theorem 2.1. of [12].

Proposition 2.2. *Let (\mathbf{X}_t) be a stationary time series satisfying \mathbf{RV}_α . Let (x_n) be a sequence such that $n\mathbb{P}(|\mathbf{X}_1| > x_n) \rightarrow 0$, as $n \rightarrow \infty$, and $p > 0$. For all $\delta > 0$, assume*

$$\mathbf{AC}: \lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(\|\mathbf{X}_{[s,n]}\|_\infty > \epsilon x_n \mid |\mathbf{X}_1| > \epsilon x_n) = 0, \quad \epsilon > 0,$$

$$\mathbf{CS}_p: \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\overline{\|\mathbf{X}_{[1,n]}/x_n\|_p^\epsilon} > \delta)}{n\mathbb{P}(|\mathbf{X}_1| > x_n)} = 0.$$

Then, if $p \geq \alpha$, Equation (2.9) holds with $c(\infty) \leq c(p) \leq c(\alpha) = 1$, and (\mathbf{X}_t) admits a p -cluster process $\mathbf{Q}^{(p)}$ in the sense of (2.10). If $p < \alpha$, existence of the p -cluster process holds if $\mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_p^\alpha] < \infty$. In this case, Equation (2.9) holds with $c(p) = \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_p^\alpha]$.

We see from Proposition 2.2 assuming \mathbf{AC} and \mathbf{CS}_α implies the time series (\mathbf{X}_t) admits an α -cluster $\mathbf{Q}^{(\alpha)}$, where $\alpha > 0$, denotes the tail index. In this case, appealing to Proposition 3.1. in [12], we have

$$(2.11) \quad \mathbf{Q} := \mathbf{Q}^{(\alpha)} \stackrel{d}{=} \boldsymbol{\Theta} / \|\boldsymbol{\Theta}\|_\alpha, \quad \in \tilde{\ell}^\alpha,$$

where $(\boldsymbol{\Theta}_t)$ is the spectral tail process from Equation (2.7). Moreover, if \mathbf{CS}_p , $\mathbf{CS}_{p'}$, and $\mathbb{E}[\|\mathbf{Q}\|_p^\alpha] + \mathbb{E}[\|\mathbf{Q}\|_{p'}^\alpha] < \infty$ also hold, then the p, p' -clusters exist and are related by the change-of-norms formula below

$$(2.12) \quad \mathbb{P}(\mathbf{Q}^{(p)} \in \cdot) = c(p)^{-1} \mathbb{E}[\|\mathbf{Q}\|_p^\alpha \mathbb{1}(\mathbf{Q}/\|\mathbf{Q}\|_p \in \cdot)]$$

$$(2.13) \quad = \frac{c(p')}{c(p)} \mathbb{E}[\|\mathbf{Q}^{(p')}\|_p^\alpha \mathbb{1}(\mathbf{Q}^{(p')}/\|\mathbf{Q}^{(p')}\|_p \in \cdot)].$$

Since $\|\mathbf{Q}^{(p)}\|_p = 1$ a.s. for any $p \in (0, \infty]$, then $c(\alpha) = 1$, and $\mathbb{E}[\|\mathbf{Q}^{(p')}\|_p^\alpha] = c(p)/c(p')$, where $c(p), c(p')$, are as in Equation (2.9). In the following we denote by \mathbf{Q} the α -cluster as in (2.11).

Remark 2.3. Using the monotonicity of norms, we see \mathbf{CS}_p implies $\mathbf{CS}_{p'}$, for $p' > p > 0$. If $p > \alpha$, condition \mathbf{CS}_p is always satisfied for sequences (x_n) such that $n\mathbb{P}(|\mathbf{X}_1| > x_n) \rightarrow 0$, as $n \rightarrow \infty$. If $\alpha/2 < p \leq \alpha$, \mathbf{CS}_p still holds for short-range dependence models and sequences (x_n) such that there exists $\kappa' > 0$, satisfying $n/x_n^{p \wedge (\alpha - \kappa')} \rightarrow 0$, as $n \rightarrow \infty$ (see remarks 5.1. and 5.2. in [12]). We verify this condition for classical models in Section 5.

2.4. Tail-index estimation. To estimate the tail-index α of the regularly varying series (see (2.7)), we consider the Hill estimator:

$$(2.14) \quad \frac{1}{\widehat{\alpha}^n} := \frac{1}{\widehat{\alpha}^n(k')} := \frac{1}{k'} \sum_{t=1}^n \log^+(|\mathbf{X}_t|/|\mathbf{X}|_{(k'+1)}),$$

where $|\mathbf{X}|_{(1)} \geq |\mathbf{X}|_{(2)} \geq \dots \geq |\mathbf{X}|_{(n)}$, and $k' = k'_n$ is a tuning sequence for (2.14) satisfying $k' \rightarrow \infty$, $n/k' \rightarrow \infty$, as $n \rightarrow \infty$. To study the asymptotic properties of the Hill estimator, we write it as a cluster statistic and consider the functional $h : \tilde{\ell}^\infty \rightarrow [0, \infty)$ given by

$$(2.15) \quad h(\mathbf{x}) = \sum_{t \in \mathbb{Z}} \log(|\mathbf{x}_t|) \mathbb{1}(|\mathbf{x}_t| > 1).$$

It is easy to see $h(\mathbf{x}) = h(\mathbf{x}) \mathbb{1}(\|\mathbf{x}\|_p > 1)$ for every $p \in (0, \infty]$, and

$$h^{\mathbf{Q}(p)} = \int_0^\infty \mathbb{E}[\sum_{t \in \mathbb{Z}} \log(y|\mathbf{Q}_t|) \mathbb{1}(y|\mathbf{Q}_t| > 1)] d(-y^{-\alpha}) = \frac{1}{\alpha}.$$

We also introduce the counts of exceedances functional $e : \tilde{\ell}^\infty \rightarrow [0, \infty)$ given by

$$(2.16) \quad e(\mathbf{x}) := \sum_{j \in \mathbb{Z}} \mathbb{1}(|\mathbf{x}_j| > 1),$$

which also satisfies $e(\mathbf{x}) = e(\mathbf{x}) \mathbb{1}(\|\mathbf{x}\|_p > 1)$ for every $p \in (0, \infty]$, and

$$e^{\mathbf{Q}(p)} = \int_0^\infty \mathbb{E}[\sum_{t \in \mathbb{Z}} \mathbb{1}(y|\mathbf{Q}_t| > 1)] d(-y^{-\alpha}) = 1.$$

3. ASYMPTOTICS OF p -CLUSTER BLOCK ESTIMATORS

3.1. Block estimators. Let (\mathbf{X}_t) be an \mathbb{R}^d -valued stationary time series satisfying \mathbf{RV}_α . For $p > 0$ fixed, assume the conditions of Proposition 2.2 hold for p , thus the series admits a p -cluster process $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$, and (2.10) holds for a sequence of high levels (x_n) satisfying $\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_p > x_n) \rightarrow 0$, as $n \rightarrow \infty$. Recall the ℓ^p -block estimator in (1.4) is tuned with the block lengths (b_n) , and the number (k_n) of extremal blocks. The total number of disjoint blocks in a sample is denoted (m_n) with $m_n = \lfloor n/b_n \rfloor$. To study the asymptotics of block estimators we assume the following relation between (k_n) , (x_n) and (b_n) :

$$(3.17) \quad \begin{aligned} k &:= k_n(p) = \lfloor m_n \mathbb{P}(\|\mathcal{B}_1\|_p > x_{b_n}) \rfloor \\ &\sim n c(p) \mathbb{P}(|\mathbf{X}_1| > x_{b_n}), \quad n \rightarrow \infty, \end{aligned}$$

holds, where $c(p) \in (0, \infty)$ are as in (2.9), and $c(\alpha) = 1$. Similarly we consider a sequence (x'_n) satisfying (2.10) for $p = \infty$, and we assume the sequence (k'_n) satisfies the following relation

$$(3.18) \quad k' := k'_n = \lfloor n \mathbb{P}(|\mathbf{X}_1| > x'_{b_n}) \rfloor, \quad n \rightarrow \infty.$$

Implementing the Hill estimator in (2.14) for $p = \infty$ imposes minimal restrictions on the sequences (k'_n) and (x'_n) compared to the case $p < \infty$ in view of the conditions from Proposition 2.2. In what follows, the sequences (x_n) , (b_n) , (m_n) , (k_n) , (x'_n) , (k'_n) that appear in this article satisfy conditions (3.17), (3.18) above.

Let $f : \tilde{\ell}^p \rightarrow \mathbb{R}$ be a functional defining the p -cluster statistic in (1.3). To stress its relation with p it will be convenient to write $f = f(p)$. In numerous examples, the functional $f(p)$ might also depend on α in its expression, meaning $f(p) = f_\alpha(p)$. Hence, it will also be useful to consider the family of functions $f_q(p) : \tilde{\ell}^p \rightarrow \mathbb{R}$, indexed by q , for q in a neighborhood of α . In addition, we define the functional $1(p)$ as

$$(3.19) \quad 1(p)(\mathbf{x}) := \mathbb{1}(\|\mathbf{x}\|_p > 1),$$

which satisfies $1^{\mathbf{Q}(p)} = \mathbb{P}(Y \|\mathbf{Q}^{(p)}\|_p > 1) = 1$. Recall also the functionals h, e in (2.15) and (2.16) defining the Hill and exceedances estimator, respectively.

3.2. Consistency of block estimators. We start by stating the consistency of the $\ell^{\hat{\alpha}}$ -block estimators in (1.4) where we choose extremal blocks as those with the largest $\ell^{\hat{\alpha}}$ -norm, and $\hat{\alpha}$ is an estimate of the tail-index α . This is the purpose of Lemma 3.1 for which we require the assumptions below. The consistency of ℓ^p -block estimators has already been shown in Theorem 4.2. in [12].

C: Let $f = f_\alpha(p) \in \mathcal{G}_+(\tilde{\ell}^p)$. Assume there exists $\epsilon, \delta > 0$, such that

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E} \left[\sup_{\substack{q \in [\alpha - \epsilon, \alpha + \epsilon] \\ u \in [1 - \epsilon, 1 + \epsilon]}} (f_q(\mathbf{X}_{[1,n]}/(ux_n)))^{1+\delta} \mathbb{1}(\|\mathbf{X}_{[1,n]}\|_p > ux_n) \right]}{\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_p > x_n)} < \infty,$$

and $\sup_{\substack{q \in [\alpha - \epsilon, \alpha + \epsilon] \\ u \in [1 - \epsilon, 1 + \epsilon]}} \mathbb{E}[f_q(Y\mathbf{Q}/u)] < \infty$, with p replaced by q in the numerator if $p = \alpha$.

S: Let $f = f_\alpha(p) \in \mathcal{G}_+(\tilde{\ell}^p)$. Assume there exists $\epsilon > 0$ such that, for $q \in [\alpha - \epsilon, \alpha + \epsilon]$, for all $\mathbf{x} \in \tilde{\ell}^p \setminus \{0\}$, f_q admits the Taylor development:

$$(3.20) \quad f_q(\mathbf{x}) = f_\alpha(\mathbf{x}) + (q - \alpha) \frac{\partial f_q}{\partial q} \Big|_{q=\alpha}(\mathbf{x}) + \frac{1}{2} (q - \alpha)^2 \frac{\partial^2 f_q}{\partial q^2} \Big|_{q=\xi}(\mathbf{x}),$$

where $\xi = \xi(\mathbf{x})$ is a real value in $[\alpha - q, \alpha + q]$, and here $\partial f_q / \partial q$ and $\partial^2 f_q / \partial q^2$ denote the first and second-order derivatives of the function $q \mapsto f_q$.

In the following, if we assume that the consistency and smoothness assumptions: **C**, **S** hold, then we let $\epsilon > 0$, $\delta > 0$ be such that both statements stay true. The proof of the next Lemma is deferred to Appendix A.

Lemma 3.1. *Let (\mathbf{X}_t) be a stationary time series satisfying \mathbf{RV}_α . Assume the conditions of Proposition 2.2 hold for $p = \alpha$ and the series admits an α -cluster process $\mathbf{Q} \in \tilde{\ell}^\alpha$. Consider the mixing-coefficients (β_t) and assume there exists a sequence (ℓ_n) satisfying $\ell_n \rightarrow \infty$, $m_n \beta_{\ell_n} / k_n \rightarrow 0$, and $\ell_n / b_n \rightarrow 0$, as $n \rightarrow \infty$. Let $f = f_\alpha(\alpha) : \ell^\alpha \rightarrow \mathbb{R}$, assume **C** holds for $\epsilon, \delta > 0$, and $c(q) = \mathbb{E}[\|\mathbf{Q}\|_q^\alpha]$ satisfies $c(q) < \infty$, for $q \in [\alpha - \epsilon, \alpha + \epsilon]$. Moreover, assume $\hat{\alpha} \xrightarrow{\mathbb{P}} \alpha$, and $\|\mathcal{B}_t\|_{\hat{\alpha}, (k+1)} / x_{b_n} \xrightarrow{\mathbb{P}} 1$, as $n \rightarrow \infty$. Then,*

$$\widehat{f}^{\mathbf{Q}}(\hat{\alpha}) \xrightarrow{\mathbb{P}} f^{\mathbf{Q}}(\alpha), \quad n \rightarrow \infty.$$

Furthermore, if **S** holds and $\partial f_q / \partial q$ satisfies **C**, then $\widehat{f}_{\hat{\alpha}}^{\mathbf{Q}}(\hat{\alpha}) \xrightarrow{\mathbb{P}} f_\alpha^{\mathbf{Q}}(\alpha)$, as $n \rightarrow \infty$.

3.3. Assumptions for asymptotic normality. Our main result is presented in Theorem 3.2 stating the asymptotic normality of the ℓ^p -block estimators in (1.4) under the Lindeberg, bias, moment and mixing assumptions below. We extend the result to $\ell^{\hat{\alpha}}$ -block estimators where $1/\hat{\alpha}$ is the Hill estimator. We also cover the implementation of block estimators for functionals $f_\alpha(p)$ where we plug-in $\hat{\alpha}$ at the place of α .

L: Let $f = f_\alpha(p) \in \mathcal{G}_+(\tilde{\ell}^p)$ such that $u \mapsto f(\mathbf{x}_t)/u$ is non-increasing, and there exists $\delta > 0$ such that, for all $u > 0$, the following Lindeberg-type condition holds

$$(3.21) \quad \limsup_{n \rightarrow \infty} \frac{\mathbb{E}[(f(\mathbf{X}_{[1,n]}/(u x_n)))^{2+\delta} \mathbf{1}(\|\mathbf{X}_{[1,n]}/x_n\|_p > u)]}{\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_p > x_n)} < \infty.$$

Assume $\mathbb{E}[(f(Y^{\mathbf{Q}(p)}))^{2+\delta}] < \infty$, and if $p = \alpha$ assume there exists $\epsilon > 0$ such that $\sup_{q \in [\alpha - \epsilon, \alpha + \epsilon]} \mathbb{E}[(f(Y^{\mathbf{Q}(q)}))^{2+\delta}] < \infty$.

B(k_n): Consider f satisfying **L**, and assume there exists $\epsilon > 0$ such that the bias condition:

$$(3.22) \quad \lim_{n \rightarrow \infty} \sqrt{k_n} \sup_{u \in [1-\epsilon, 1+\epsilon]} \left| \frac{\mathbb{E}[f(\mathcal{B}_1/(u x_{b_n})) \mathbf{1}(\|\mathcal{B}_1/x_{b_n}\|_p > u)]}{\mathbb{P}(\|\mathcal{B}_1\|_p > x_{b_n})} - u^{-\alpha} f^{\mathbf{Q}}(p) \right| = 0,$$

holds where $f^{\mathbf{Q}}(p)$ is as in (1.3).

$\mathbf{B}_\alpha(k_n)$: Consider f satisfying \mathbf{L} , and assume there exists $\epsilon > 0$ such that

$$(3.23) \quad \lim_{n \rightarrow \infty} \sqrt{k_n} \sup_{\substack{u \in [1-\epsilon, 1+\epsilon], \\ q \in [\alpha-\epsilon, \alpha+\epsilon]}} \left| \frac{\mathbb{E}[f(\mathcal{B}_1/(u x_{b_n})) \mathbf{1}(\|\mathcal{B}_1/x_{b_n}\|_q > u)]}{\mathbb{P}(\|\mathcal{B}_1\|_q > x_{b_n})} - u^{-\alpha} f^{\mathbf{Q}}(q) \right| = 0.$$

\mathbf{M} : Assume there exists $\epsilon > 0$ such that the moment condition below holds

$$\mathbb{E}[\|\mathbf{Q}\|_{\alpha-\epsilon}^{2\alpha}] < \infty.$$

\mathbf{MX}_β : Consider f satisfying \mathbf{L} , and let δ be such that (3.21) hold. Assume that the mixing coefficients (β_t) satisfy for some sequence (ℓ_n) satisfying $\ell_n \rightarrow \infty$, and $\ell_n/b_n \rightarrow 0$, $m_n \beta_{\ell_n}/k_n \rightarrow 0$, as $n \rightarrow \infty$,

$$(3.24) \quad \lim_{n \rightarrow \infty} \sum_{t=2}^{m_n} (m_n \beta_{t b_n} / k_n)^{\frac{\delta}{2+\delta}} = 0.$$

If f is bounded, note it is enough to assume $\sum_{t=2}^{m_n} m_n \beta_{t b_n} / k_n \rightarrow 0$, $n \rightarrow \infty$.

In the remainder of the article, if we assume that certain of the consistency and smoothness assumptions: \mathbf{C}, \mathbf{S} hold or if the Lindeberg, bias, moment, and mixing assumptions: $\mathbf{L}, \mathbf{B}, \mathbf{M}, \mathbf{MX}_\beta$ hold, then we let $\epsilon > 0$, $\delta > 0$ be such that all previous statements stay true.

3.4. Asymptotic normality of block estimators. We state next the asymptotic normality of the ℓ^p -block estimators in (1.4). We defer its proof to Section B.

Theorem 3.2. *Let (\mathbf{X}_t) be a stationary time series satisfying \mathbf{RV}_α . Assume the conditions of Proposition 2.2 hold and the series admits a p -cluster process $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$. Consider $f = f(p) : \tilde{\ell}^p \rightarrow \mathbb{R}$ satisfying \mathbf{L} , and if $f(p) = f_\alpha(p)$, assume \mathbf{M} and \mathbf{S} hold, and $\partial f_q / \partial q$, and $\partial^2 f_q / \partial q^2$ satisfy \mathbf{C} , for values $\epsilon, \delta > 0$. Assume $f(p)$ and $1(p)$ satisfy $\mathbf{B}(k_n)$, and h, e , in (2.15), (2.16) satisfy \mathbf{L} and $\mathbf{B}(k'_n)$. Moreover, assume \mathbf{MX}_β holds. Then, if $k_n/k'_n \rightarrow \kappa$, with $\kappa \geq 0$,*

$$(3.25) \quad \sqrt{k_n} \left(\widehat{f_{\hat{\alpha}}^{\mathbf{Q}}}(p) - f_{\alpha}^{\mathbf{Q}}(p) \right) \xrightarrow{d} \mathcal{N}(0, \sigma^2(\kappa)), \quad n \rightarrow \infty.$$

The limiting variance can be computed in terms of κ , $\mathbf{Q}^{(p)}$, as

$$\sigma^2(\kappa) := \text{Var}(f_\alpha(Y \mathbf{Q}^{(p)})) + \kappa \alpha^2 (f'_\alpha)^2 \sigma_\alpha^2 + 2\kappa \alpha f'_\alpha \sigma_{f, \alpha}(\kappa),$$

where

$$\begin{aligned} \sigma_\alpha^2 &:= \mathbb{E}[(\alpha h(Y \mathbf{Q}^{(p)}) - e(Y \mathbf{Q}^{(p)}))^2], \\ \sigma_{f, \alpha}(\kappa) &:= \mathbb{E}[f_\alpha(Y \mathbf{Q}^{(p)}) (\alpha h(Y \mathbf{Q}^{(p)}) \kappa^{1/\alpha} - e(Y \mathbf{Q}^{(p)}) \kappa^{1/\alpha})], \end{aligned}$$

and Y is independent of $\mathbf{Q}^{(p)}$, and $\mathbb{P}(Y > y) = y^{-\alpha}$, for $y > 1$. Moreover, if $f_\alpha(\mathbf{x}) = f_\alpha(\mathbf{x}/\|\mathbf{x}\|_\alpha)$, then, for all $\kappa > 0$, $\sigma_{f, \alpha}(\kappa) = 0$, and in all cases, $\sigma_{f, \alpha}(\kappa) \rightarrow 0$, as $\kappa \downarrow 0$. Furthermore, if $p = \alpha$, assume \mathbf{M} holds and assume

$f(\alpha)$ and $1(\alpha)$ satisfy $\mathbf{B}_\alpha(k_n)$. Then, (3.25) continues to hold replacing p by $\hat{\alpha}$ in the block estimator.

The choice of k_n, k'_n are subject to the bias conditions $\mathbf{B}(k_n)$ and $\mathbf{B}(k'_n)$. Actually, it is common practice to choose k'_n larger than k_n , and the numerical results from Section 6 support this practice. When we use fewer blocks k_n for p -cluster inference, compared to the number of records k'_n we use to tune the Hill estimator, precisely if $k_n/k'_n \rightarrow 0$, as $n \rightarrow \infty$, the variance term simplifies to

$$\sqrt{k} \left(\widehat{f_{\hat{\alpha}}^{\mathbf{Q}}}(p) - f_{\alpha}^{\mathbf{Q}}(p) \right) \xrightarrow{d} \mathcal{N}(0, \text{Var}(f_{\alpha}(Y\mathbf{Q}^{(p)}))), \quad n \rightarrow \infty.$$

This expression also holds when the functional f doesn't include α in its expression.

Remark 3.3. To plug-in $\hat{\alpha}$ in the place of α for p -cluster inference we require the bias condition $\mathbf{B}(k'_n)$ for h, e . To do so, the Hill estimator is seen as a block estimator as in (1.4) that evaluates the block functional h on (\mathcal{B}_t/x'_{b_n}) , $t = 1, \dots, m_n$, replacing the high threshold level (x'_{b_n}) by k' -order statistic from the sample $(|\mathbf{X}_t|)$ where

$$k'_n(\alpha) \sim n \mathbb{P}(|\mathbf{X}_1| > x'_{b_n}), \quad n \rightarrow \infty.$$

Then the bias condition $\mathbf{B}(k'_n)$ can be rewritten as

$$\lim_{n \rightarrow \infty} \sqrt{k'_n} \sup_{u \in [1-\epsilon, 1+\epsilon]} \left| \frac{\mathbb{E}[\log(|\mathbf{X}_1|/(u x'_{b_n})) \mathbf{1}(|\mathbf{X}_1| > (u x'_{b_n}))]}{\mathbb{P}(|\mathbf{X}_1| > x'_{b_n})} - u^{-\alpha} \frac{1}{\alpha} \right| = 0,$$

which is no longer a condition on blocks, but on $|\mathbf{X}_1|$. This type of condition was considered in [19, 15]. Notice that the dependence of the threshold x'_{b_n} with b_n is an artifact of our notation for \mathbf{B} which is common for (k_n) extremal blocks or (k'_n) extremal components.

Remark 3.4. The mixing condition \mathbf{MX}_β is comparable to the classical mixing conditions for central limit theory for block estimators, notably, (B1) and (B2) in [18]. In our notation we can write $k_n = nv_n$, $b_n = r_n/k_n$, and in both cases we can take $\ell_n = o(b_n)$, as $n \rightarrow \infty$. Then, their assumption reads as $n\beta_{\ell_n}/(k_n b_n) \rightarrow 0$, as $n \rightarrow \infty$, which is the first part of \mathbf{MX}_β . Actually, in the models we consider in Section 5, this is the leading term of our mixing assumption. Instead, our gain comes from the way we deal with the entropy of the cluster functionals, which is novel here. In [18], the authors require additionally (D2) or (D2'). In this case, for example, to deal with the extremal index (see e.g. [20]), the authors also require $b_n \sqrt{k_n} \rightarrow 0$, as $n \rightarrow \infty$, which is no longer an assumption in our theory. Instead, we show in Section 5 that in the classical model, we can take k_n as in (5.38), provided that the bias assumptions are met.

4. CLUSTER STATISTICS

In view of Theorem 3.2, we derive asymptotic normality of the classical cluster index estimators in extreme value theory.

4.1. The extremal index. Let (\mathbf{X}_t) be a stationary time series in $(\mathbb{R}^d, |\cdot|)$ satisfying \mathbf{RV}_α . The extremal index $\theta_{|\mathbf{X}|}$ of the series $(|\mathbf{X}_t|)$ is a measure of serial clustering introduced in [32] and [33]. We recall the extremal index estimator proposed in [13], based on extremal ℓ^α -blocks.

Corollary 4.1. *Consider $f_\alpha : \tilde{\ell}^\alpha \rightarrow \mathbb{R}$ to be the function $\mathbf{x} \mapsto \|\mathbf{x}\|_\infty^\alpha / \|\mathbf{x}\|_\alpha^\alpha$. Assume the conditions of Theorem 3.2 hold for $p = \alpha$, and $k/k' \rightarrow 0$, as $n \rightarrow \infty$. Let $\theta_{|\mathbf{X}|} = \mathbb{E}[\|\mathbf{Q}\|_\infty^\alpha]$, hence we deduce an estimator*

$$(4.26) \quad \hat{\theta}_{|\mathbf{X}|} := \frac{1}{k} \sum_{t=1}^m \frac{\|\mathcal{B}_t\|_\infty^{\hat{\alpha}}}{\|\mathcal{B}_t\|_{\hat{\alpha}}^{\hat{\alpha}}} \mathbf{1}(\|\mathcal{B}_t\|_{\hat{\alpha}} > \|\mathcal{B}\|_{\hat{\alpha},(k+1)}),$$

such that

$$\sqrt{k}(\hat{\theta}_{|\mathbf{X}|} - \theta_{|\mathbf{X}|}) \xrightarrow{d} \mathcal{N}(0, \text{Var}(\|\mathbf{Q}\|_\infty^\alpha)), \quad n \rightarrow \infty.$$

Proof. The proof of Corollary 4.1 follows directly from an application of Theorem 3.2 to the function $\tilde{f}_\alpha(\mathbf{x}) = f_\alpha(\mathbf{x})\mathbf{1}(\|\mathbf{x}\|_\alpha > 1)$ satisfying $\tilde{f}_\alpha \in \mathcal{G}_+(\tilde{\ell}^\alpha)$ and \tilde{f}_α is a bounded a.s. continuous function satisfying **L**. \square

For comparison, we also review the block estimator based on extremal ℓ^∞ -blocks proposed in [25]:

$$(4.27) \quad \hat{\theta}_{|\mathbf{X}|}^B := \frac{1}{k} \sum_{t=1}^{m_n} \mathbf{1}(\|\mathcal{B}_t\|_\infty > |\mathbf{X}|_{(k+1)}).$$

Direct computations from Example 10.4.2 in [31] yield

$$\sqrt{k}(\hat{\theta}_{|\mathbf{X}|}^B - \theta_{|\mathbf{X}|}) \xrightarrow{d} \mathcal{N}(0, \sigma_\theta^2), \quad n \rightarrow \infty,$$

where $\sigma_\theta^2 \in [0, \infty)$, and

$$(4.28) \quad \begin{aligned} \sigma_\theta^2 &:= \theta_{|\mathbf{X}|}^2 \sum_{j \in \mathbb{Z}} \mathbb{E}[|\Theta_j|^\alpha \wedge 1] - \theta_{|\mathbf{X}|} \\ &= \theta_{|\mathbf{X}|}^2 \sum_{j \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \mathbb{E}[|\mathbf{Q}_{j+t}|^\alpha \wedge |\mathbf{Q}_t|^\alpha] - \theta_{|\mathbf{X}|}. \end{aligned}$$

The last equality follows appealing to the time-change formula in (2.8) and Equation (2.11). As a result, we can compare the asymptotic variances of $\hat{\theta}_{|\mathbf{X}|}$ and $\hat{\theta}_{|\mathbf{X}|}^B$ in the cases where \mathbf{Q} is known. This is the topic of Section 5.

Remark 4.2. *An alternative α -cluster estimator of the extremal index corresponds to the block functional $f'(\mathbf{x}) = \mathbf{1}(\|\mathbf{x}\|_\infty > 1)$. A similar asymptotic normality result applies but with an asymptotic variance $\text{Var}(f'(Y\mathbf{Q}))$ larger*

than $\text{Var}(f_\alpha(Y\mathbf{Q}))$. It motivates the use of $\hat{\theta}_{|\mathbf{x}|}$ although it requires the estimation of α . The latter using Hill's estimator is harmless choosing k' sufficiently large with respect to k , which is always the case in practice.

Remark 4.3. To check condition **S** for the functional $f_\alpha(\mathbf{x}) = \|\mathbf{x}\|_\infty^\alpha / \|\mathbf{x}\|_\alpha^\alpha$, note the Taylor expansions:

$$\begin{aligned} & \|\mathbf{x}\|_q^q - \|\mathbf{x}\|_\alpha^\alpha \\ &= (q - \alpha) \sum_{t \in \mathbb{Z}} |\mathbf{x}_t|^\alpha \log(|\mathbf{x}_t|) + \frac{1}{2}(q - \alpha)^2 \sum_{t \in \mathbb{Z}} |\mathbf{x}_t|^{q'} \log^2(|\mathbf{x}_t|), \\ & \|\mathbf{x}\|_\infty^q - \|\mathbf{x}\|_\infty^\alpha \\ &= (q - \alpha) \|\mathbf{x}\|_\infty^\alpha \log(\|\mathbf{x}\|_\infty) + \frac{1}{2}(q - \alpha)^2 \|\mathbf{x}\|_\infty^{q''} \log^2(\|\mathbf{x}\|_\infty), \end{aligned}$$

hold for some $q', q'' \in [\alpha \wedge q, \alpha \vee q]$. Hence, f_q satisfies

$$\begin{aligned} & f_q(\mathbf{x}) - f_\alpha(\mathbf{x}) \\ &= (q - \alpha) \underbrace{\frac{\|\mathbf{x}\|_\infty^\alpha}{\|\mathbf{x}\|_\alpha^\alpha} \sum_{t \in \mathbb{Z}} \frac{|\mathbf{x}_t|^\alpha}{\|\mathbf{x}\|_\alpha^\alpha} \log(\|\mathbf{x}\|_\infty / |\mathbf{x}_t|)}_{=\partial f_q / \partial q|_{q=\alpha}} \\ & \quad + \frac{1}{2}(q - \alpha)^2 \underbrace{\frac{\|\mathbf{x}\|_\infty^{q'}}{\|\mathbf{x}\|_{q'}^{q'}} \sum_{t \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \frac{|\mathbf{x}_t|^{q'} |\mathbf{x}_j|^{q'}}{\|\mathbf{x}\|_{q'}^{q'} \|\mathbf{x}\|_{q'}^{q'}} \log(\|\mathbf{x}\|_\infty / |\mathbf{x}_j|) \log(|\mathbf{x}_j| / |\mathbf{x}_t|)}_{=\partial^2 f_q / \partial q^2|_{q=q'}}. \end{aligned}$$

As mentioned, this expansion is helpful to verify condition **S** on the models from Section 5.

4.2. The cluster index for sums. Let (\mathbf{X}_t) be a stationary time series with values in $(\mathbb{R}^d, |\cdot|)$ satisfying \mathbf{RV}_α . We recall that when $\alpha < 2$ [37] coined the constant $c(1)$ in (2.9) as the cluster index for sums. We review a cluster-based estimator of it, introduced in [12], based on extremal ℓ^α -blocks.

Corollary 4.4. Consider $f_\alpha : \tilde{\ell}^\alpha \rightarrow \mathbb{R}$ to be the function $\mathbf{x} \mapsto \|\mathbf{x}\|_1^\alpha / \|\mathbf{x}\|_\alpha^\alpha$. Assume the conditions of Theorem 3.2 hold for $p = \alpha \wedge 1$, and $k/k' \rightarrow 0$, as $n \rightarrow \infty$. Let $c(1) = \mathbb{E}[\|\mathbf{Q}\|_1^\alpha] < \infty$, hence one deduces an estimator

$$(4.29) \quad \hat{c}(1) := \frac{1}{k} \sum_{t=1}^m \frac{\|\mathcal{B}_t\|_1^{\hat{\alpha}}}{\|\mathcal{B}_t\|_{\hat{\alpha}}^{\hat{\alpha}}} \mathbf{1}(\|\mathcal{B}_t\|_{\hat{\alpha}} > \|\mathcal{B}\|_{\hat{\alpha}, (k+1)}),$$

such that

$$\sqrt{k}(\hat{c}(1) - c(1)) \xrightarrow{d} \mathcal{N}(0, \text{Var}(\|\mathbf{Q}\|_1^\alpha)), \quad n \rightarrow \infty,$$

and $c(1)$ is as in (2.9) with $p = 1$.

Proof. The proof of Corollary 4.4 follows directly from Theorem 3.2 the function $\tilde{f}_\alpha(\mathbf{x}) = f_\alpha(\mathbf{x}) \mathbf{1}(\|\mathbf{x}\|_p > 1)$ satisfies $\tilde{f}_\alpha \in \mathcal{G}_+(\tilde{\ell}^\alpha)$ where \tilde{f}_α is a bounded a.s. continuous function satisfying **L**. \square

Another sums index cluster-based estimator we can consider is the one proposed in [31] based on extremal ℓ^∞ -blocks:

$$(4.30) \quad \widehat{c}^B(1) = \frac{1}{kb_n} \sum_{t=1}^{m_n} \mathbb{1}(\|\mathcal{B}_t\|_1 > |\mathbf{X}|_{(k+1)}).$$

Then, relying on Example 10.4.2 in [31],

$$\sqrt{k}(\widehat{c}^B - c(1)) \xrightarrow{d} \mathcal{N}(0, \sigma_{c(1)}^2), \quad n \rightarrow \infty,$$

for a constant $\sigma_{c(1)}^2 \in [0, \infty)$ defined by

$$(4.31) \quad \sigma_{c(1)}^2 = c(1)^2 \sum_{j \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \mathbb{E}[|\mathbf{Q}_{j+t}|^\alpha \wedge |\mathbf{Q}_t|^\alpha] - c(1).$$

Similarly as in Example 4.1, whenever \mathbf{Q} is known, we can directly compare the asymptotic variances relative to the estimators $\widehat{c}(1)$ and $\widehat{c}^B(1)$. Section 5 covers this topic for classical models where the cluster process is known.

Moreover, we can use the computations in Remark 4.3 to verify condition **S** holds. In this case it suffices to replace $\|\mathbf{x}\|_\infty$ by $\|\mathbf{x}\|_1$ in all its appearances.

4.3. The cluster sizes. In general, a classical approach to model serial exceedances is using point processes as in [33] and [26]. For the levels (a_n) , satisfying $n\mathbb{P}(|\mathbf{X}_1| > a_n) \rightarrow 1$, as $n \rightarrow \infty$, and for every fixed $x > 0$ consider the point process of exceedances with state space $(0, 1]$:

$$\eta_{n,x}(\cdot) := \overline{N}_n(\{\mathbf{y} \in \tilde{\ell}^\infty : |\mathbf{y}| > x\} \times \cdot) = \sum_{i=1}^n \varepsilon_{i/n}(\cdot) \mathbb{1}(|\mathbf{X}_i| > x a_n).$$

Under mixing and anti-clustering conditions, for fixed $x > 0$, we can express the limiting point process in [26] such as

$$\begin{aligned} \eta_{n,x}(\cdot) &\xrightarrow{d} \eta_x(\cdot) := \overline{N}(\{\mathbf{y} \in \tilde{\ell}^\infty : |\mathbf{y}| > x\} \times \cdot) \\ &= \sum_{i=1}^{\infty} \sum_{j \in \mathbb{Z}} \mathbb{1}(\Gamma_i^{-1/\alpha} |\mathbf{Q}_{ji}| > x) \varepsilon_{U_i}(\cdot), \end{aligned}$$

where the points (U_i) are iid uniformly distributed on $(0, 1)$, (Γ_i) are the points of a standard homogeneous Poisson process, and (\mathbf{Q}_i) are iid copies of the cluster process \mathbf{Q} . Using the independence among these three processes, one can easily rewrite the limit as

$$(4.32) \quad \eta_x((0, t]) := \sum_{i=1}^{N_x(t)} \xi_i, \quad 0 < t \leq 1,$$

where

- N_x is a homogeneous Poisson process on $(0, 1]$ with intensity $x^{-\alpha}$,

- for an iid sequence (Y_i) of Pareto(α)-distributed random variables which is also independent of (\mathbf{Q}_i) ,

$$\xi_i := \sum_{j \in \mathbb{Z}} \mathbb{1}(Y_i | \mathbf{Q}_{ji}| > 1),$$

- $N_x, (\xi_i)$ are independent.

Relying on the point process of exceedances representation in (4.32), the random variables (ξ_i) can be interpreted as counts of serial exceedances from one cluster. Furthermore, we deduce the relation $\mathbb{P}(\xi_1 > 0) = \mathbb{E}[\|\mathbf{Q}\|_\infty^\alpha] = \theta_{|\mathbf{x}|}$, and also get an expression for the cluster size probabilities

$$(4.33) \quad \mathbb{P}(\xi_1 = j) = \mathbb{E}[|\mathbf{Q}|_{(j)}^\alpha - |\mathbf{Q}|_{(j+1)}^\alpha] = \pi_j, \quad j \geq 1.$$

The statistic π_j can be understood as the probability of recording a cluster of length j . The block estimators provide natural estimators of these quantities

$$(4.34) \quad \hat{\pi}_j := \frac{1}{k} \sum_{t=1}^m \frac{|\mathcal{B}_t|_{(j)}^{\hat{\alpha}} - |\mathcal{B}_t|_{(j+1)}^{\hat{\alpha}}}{\|\mathcal{B}_t\|_{\hat{\alpha}}^{\hat{\alpha}}} \mathbb{1}(\|\mathcal{B}_t\|_{\hat{\alpha}} > \|\mathcal{B}\|_{\hat{\alpha},(k+1)}),$$

$|\mathcal{B}_t|_{(1)} \geq |\mathcal{B}_t|_{(2)} \geq \dots \geq |\mathcal{B}_t|_{(m)}$ are the order statistics of \mathcal{B}_t , the t -th block.

Corollary 4.5. *Consider the function $\pi_j : \tilde{\ell}^\alpha \rightarrow \mathbb{R}$ defined by $\pi_j(\mathbf{x}) := (|\mathbf{x}|_{(j)}^\alpha - |\mathbf{x}|_{(j+1)}^\alpha) / \|\mathbf{x}\|_\alpha^\alpha$, where $|\mathbf{x}|_{(1)} \geq |\mathbf{x}|_{(2)} \geq \dots$. Assume the conditions of Theorem 3.2 hold for $p = \alpha$ and $k/k' \rightarrow 0$, as $n \rightarrow \infty$. Then, for all $j \geq 1$ we have*

$$(4.35) \quad \sqrt{k}(\hat{\pi}_j - \pi_j) \xrightarrow{d} \mathcal{N}(0, \text{Var}(\pi_j^{\mathbf{Q}}(\mathbf{Q}))), \quad n \rightarrow \infty.$$

Proof. The proof of Corollary 4.4 follows directly from Theorem 3.2 the function $\tilde{\pi}_j(\mathbf{x}) = \pi_j(\mathbf{x}) \mathbb{1}(\|\mathbf{x}\|_\alpha > 1)$ satisfies $\tilde{\pi}_j \in \mathcal{G}_+(\tilde{\ell}^\alpha)$ and $\tilde{\pi}_j$ is a bounded a.s. continuous function satisfying **L**. \square

Corollary 4.5 provides a novel procedure for estimating cluster size probabilities based on extremal ℓ^α -blocks. As in the previous examples, the asymptotic variance can be computed as long as \mathbf{Q} is known. This allows for comparison with the other cluster-based inference procedures provided in [25, 22, 41]. One advantage of our methodology is that we can straightforwardly infer the asymptotic variances of cluster sizes since we express them as cluster statistics in (4.35). Moreover, inference through extremal ℓ^α -blocks has already proven to be useful in [12] for fine-tuning the hyperparameters of the estimators, see also the discussion in Section 6.

As before, we can use the computations in Remark 4.3 to verify condition **S**, it suffices to replace $\|\mathbf{x}\|_\infty$ by $(|\mathbf{x}|_{(j)} - |\mathbf{x}|_{(j+1)})$ in the equations therein.

5. MODELS

5.1. Linear m_0 -dependent sequences. We consider (\mathbf{X}_t) to be a m_0 -dependent time series with values in $(\mathbb{R}^d, |\cdot|)$ satisfying **RV** $_\alpha$.

Example 5.1. *The time series (\mathbf{X}_t) is a linear moving average of order $m_0 \geq 1$ if it satisfies*

$$(5.36) \quad \mathbf{X}_t := \mathbf{Z}_t + \varphi_1 \mathbf{Z}_{t-1} + \cdots + \varphi_{m_0} \mathbf{Z}_{t-m_0}, \quad t \in \mathbb{Z},$$

with \mathbb{R}^d -variate iid innovations (\mathbf{Z}_t) satisfying \mathbf{RV}_α , and $(\varphi_j) \in \mathbb{R}^{m_0}$.

Alternatively, the max-moving average of order $m_0 \geq 1$ satisfies

$$(5.37) \quad X_t := \max\{Z_t, \varphi_1 Z_{t-1}, \dots, \varphi_{m_0} Z_{t-m_0}\}, \quad t \in \mathbb{Z},$$

with \mathbb{R}_+ -variate iid innovations (Z_t) satisfying \mathbf{RV}_α , and $(\varphi_j) \in \mathbb{R}_+^{m_0}$.

Then both moving averages satisfy \mathbf{RV}_α with $|\mathbf{Q}|$ admitting the same deterministic expression $(|\varphi_t|/\|(\varphi_j)\|_\alpha)$ in $\tilde{\ell}^\alpha$, see for instance Proposition 3.1. in [12] and Chapter 5 of [31].

Let $p > \alpha/2$. For all $\kappa' > 0$, a sequence satisfying $x_n = O(b_n^{\kappa'+1/(p \wedge \alpha)})$ verifies \mathbf{AC} and \mathbf{CS}_p . This is a consequence of Remark 2.3. Choosing (x_n) in this way implies there exist $\kappa'' > 0$, and (k_n) satisfying

$$(5.38) \quad k_n = O(n b_n^{-\kappa'' - \alpha/(p \wedge \alpha)}),$$

as $n \rightarrow \infty$, such that Equation (3.17) holds from an application of Potter's bound. Since κ' can be chosen arbitrarily small, κ'' can also be arbitrarily close to zero.

Keeping this in mind, we can state the Proposition below. The proof is postponed to Section F.

Proposition 5.2. *Consider (\mathbf{X}_t) to be an m_0 -dependent time series with values in $(\mathbb{R}^d, |\cdot|)$. Consider $p > \alpha/2$, and sequence (k_n) and (k'_n) satisfying (5.38), such that $m_n/k_n \rightarrow \infty$, and $k_n/k'_n \rightarrow 0$. Consider $f_\alpha(p) : \tilde{\ell}^p \rightarrow \mathbb{R}$, and assume \mathbf{L} , \mathbf{S} hold. Then,*

$$\sqrt{k} (\widehat{f}_\alpha^{\mathbf{Q}}(p) - f_\alpha^{\mathbf{Q}}(p)) \xrightarrow{d} \mathcal{N}(0, \text{Var}(f_\alpha(Y\mathbf{Q}^{(p)}))), \quad n \rightarrow \infty,$$

under the bias conditions $\mathbf{B}_\alpha(k)$, $\mathbf{B}(k')$, and the result extends to $\widehat{\alpha}$ -cluster inference. In particular the $\widehat{\alpha}$ -cluster based estimators from Section 4 in (4.26) (4.29), and (4.34) are asymptotically normally distributed, and in the case of the moving averages of Example 5.1

$$\sqrt{k} (\widehat{f}_\alpha^{\mathbf{Q}}(\widehat{\alpha}) - f_\alpha^{\mathbf{Q}}(\alpha)) \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty.$$

5.2. Linear processes. In this section we consider stationary linear processes (\mathbf{X}_t) with values in $(\mathbb{R}^d, |\cdot|)$ satisfying \mathbf{RV}_α .

Example 5.3. *Consider (\mathbf{X}_t) to be an \mathbb{R}^d -variate sequence satisfying*

$$(5.39) \quad \mathbf{X}_t = \sum_{j \in \mathbb{Z}} \varphi_j \mathbf{Z}_{t-j}, \quad t \in \mathbb{Z},$$

for a sequence of iid innovations (\mathbf{Z}_t) satisfying \mathbf{RV}_α , and a sequence (φ_j) in $\mathbb{R}^{\mathbb{Z}}$. Moreover, assume there exists $\kappa' > 0$ such that $\|(\varphi_j)\|_{(\alpha - \kappa') \wedge 2} < \infty$.

In the setting of Example 5.3, a stationary solution (\mathbf{X}_t) exists and satisfies \mathbf{RV}_α (c.f. [16, 35, 27]). Proposition 5.4 below demonstrates conditions \mathbf{AC} , \mathbf{CS}_p hold for $p > \alpha/2$, and a suitable sequence (x_n) such that $n\mathbb{P}(|\mathbf{X}_1| > x_n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, the time series (\mathbf{X}_t) admits an α -cluster process \mathbf{Q} , which we can compute in terms of the filter (φ_j) , and the spectral measure of the random variable \mathbf{Z}_0 , denoted by $\boldsymbol{\Theta}_0^{\mathbf{Z}}$, with $|\boldsymbol{\Theta}_0^{\mathbf{Z}}| = 1$ a.s. We obtain the expression, cf. Chapter 5 of [31],

$$(5.40) \quad \mathbf{Q} \stackrel{d}{=} (\varphi_t / \|(\varphi_j)\|_\alpha) \boldsymbol{\Theta}_0^{\mathbf{Z}}, \quad \in \tilde{\ell}^\alpha.$$

Note again that the norm of the α -cluster process, i.e., $|\mathbf{Q}|$, is deterministic in $\tilde{\ell}^\alpha$. Assuming $\|(\varphi_j)\|_p < \infty$, we can compute the indices $c(p)$ in (2.9) by

$$(5.41) \quad c(p) = \mathbb{E}[\|\mathbf{Q}\|_p^\alpha] = \|(\varphi_j)\|_p^\alpha / \|(\varphi_j)\|_\alpha^\alpha < \infty.$$

Classic examples of these heavy-tailed linear models are auto-regressive moving averages, i.e., ARMA processes, with iid regularly varying noise; cf. [10].

The proposition below guarantees that the assumptions of Proposition 2.2 hold. We defer its proof to Section F.2.

Proposition 5.4. *Consider (\mathbf{X}_t) to be a linear process with values in $(\mathbb{R}^d, |\cdot|)$, as in Example 5.3. Consider $p > \alpha/2$, and a sequence (x_n) such that $n/x_n^{p \wedge (\alpha - \kappa')} \rightarrow 0$, $n \rightarrow \infty$, for some $\kappa' > 0$. Then it holds for all $\delta > 0$*

$$(5.42) \quad \lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}_{[1,n]}/x_n - \mathbf{X}_{[1,n]}^{(s)}/x_n\|_p^p > \delta)}{n\mathbb{P}(|\mathbf{X}_1| > x_n)} = 0,$$

where $\mathbf{X}_t^{(s)} := \sum_{|j| \leq s} \varphi_j \mathbf{Z}_{t-j}$. Thus \mathbf{AC} and \mathbf{CS}_p are satisfied.

We now review the mixing properties of a linear process. We recall below the statement in Theorem 2.1. in [38] (see Lemma 15.3.1. in [31]).

Proposition 5.5. *Consider (\mathbf{X}_t) to be a causal linear process with values in $(\mathbb{R}^d, |\cdot|)$, as in Example 5.3 with $\varphi_j = 0$, for $j < 0$. Assume the distribution of \mathbf{Z}_0 is absolutely continuous with respect to the Lebesgue measure in \mathbb{R}^d , and has a density $g_{\mathbf{Z}}$ satisfying*

- i) $\int |g(\mathbf{x} - \mathbf{y}) - g(\mathbf{x})| d\mathbf{x} = O(|\mathbf{y}|)$, for all $\mathbf{y} \in \mathbb{R}^d$,
- ii) $\varphi_t = O(t^{-\rho})$, for $t \geq 0$, and $\rho > 2 + 1/\alpha$,
- iii) $\sum_{j=0}^{\infty} \varphi_j \mathbf{x}^j \neq 0$, for all $\mathbf{x} \in \mathbb{R}^d$ with $|\mathbf{x}| < 1$,

Then, for all $0 < \varepsilon < \alpha$, the mixing coefficients (β_t) satisfy

$$(5.43) \quad \beta_t = O\left(t^{1 - \frac{(\rho-1)(\alpha-\varepsilon)}{1+\alpha-\varepsilon}}\right).$$

Combining Propositions 5.4 and 5.5, we state below the asymptotic normality of the p -cluster based estimators for linear processes in Theorem 5.6. We defer its proof to Section F.3.

Theorem 5.6. *Consider (\mathbf{X}_t) to be a causal linear process with values in $(\mathbb{R}^d, |\cdot|)$, as in Example 5.3. Let $\rho > 0$, and assume the conditions of*

Proposition 5.5 hold with $\varphi_t = O(t^{-\rho})$, for $t > 0$. Consider $p > \alpha/2$, and sequences (k_n) and (k'_n) satisfying (5.38), such that $m_n/k_n \rightarrow \infty$, and $k_n/k'_n \rightarrow 0$, as $n \rightarrow \infty$. Consider $f_\alpha(p) : \tilde{\ell}^p \rightarrow \mathbb{R}$, and assume \mathbf{L}, \mathbf{S} hold. Assume that for $\delta > 0$ as in (3.21),

$$(5.44) \quad \rho > 3 + \frac{2}{\alpha} + \frac{2}{\delta}(1 + \frac{1}{\alpha}).$$

If $f_\alpha(p)$ is bounded, condition (5.44) can be replaced by $\rho > 3 + 2/\alpha$. Then,

$$\sqrt{k} (\widehat{f_\alpha^{\mathbf{Q}}}(p) - f_\alpha^{\mathbf{Q}}(p)) \xrightarrow{d} \mathcal{N}(0, \text{Var}(f_\alpha(Y^{\mathbf{Q}}(p)))) , \quad n \rightarrow \infty,$$

under the bias conditions $\mathbf{B}_\alpha(k), \mathbf{B}(k')$, and the result extends to $\widehat{\alpha}$ -cluster inference. In particular the $\widehat{\alpha}$ -cluster based estimators from Section 4 in (4.26) (4.29), and (4.34), satisfy

$$\sqrt{k} (f_{\widehat{\alpha}}^{\mathbf{Q}}(\widehat{\alpha}) - f_\alpha^{\mathbf{Q}}(\alpha)) \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty.$$

Regarding cluster inference in the case of linear models, the α -cluster approach has an optimal asymptotic variance for shift-invariant functionals since we use the ℓ^α -norm order statistics. For this reason, it compares favourably with state-of-the-art block estimator. For example, for the extremal index, the super-efficient estimator in (4.26) has a lower asymptotic variance than the block estimator in (4.27). Indeed the asymptotic variance σ_θ^2 of the latter, computed in (4.28), is not necessarily null. For example, for the autoregressive process of order one AR(1) one has $\sigma_\theta^2 = 1 - \theta_{|\mathbf{X}|} > 0$.

5.3. Affine stochastic recurrence equation solution under Kesten's conditions. In this section we focus on the causal solution to the affine stochastic recurrence equation SRE under Kesten's conditions. To guarantee the existence of a solution (\mathbf{X}_t) , with values in $(\mathbb{R}^d, |\cdot|)$ as in (5.45) satisfying \mathbf{RV}_α , we rely on Theorem 2.1. and Theorem 2.4 in [1]. For an overview, we refer to [11]. In what follows, we study time series (\mathbf{X}_t) as in the Example 5.7 below.

Example 5.7. Consider (\mathbf{X}_t) to be a sequence with values in \mathbb{R}^d satisfying

$$(5.45) \quad \mathbf{X}_t = \mathbf{A}_t \mathbf{X}_{t-1} + \mathbf{B}_t, \quad t \in \mathbb{Z},$$

where $((\mathbf{A}_t, \mathbf{B}_t))$ is an iid sequence of non-negative random $d \times d$ matrices with generic element \mathbf{A} , and non-negative random vectors with generic element \mathbf{B} taking values in \mathbb{R}^d . For the existence of a causal stationary solution, we assume

- i) $\mathbb{E}[\log^+ |\mathbf{A}|_{op}] + \mathbb{E}[\log^+ |\mathbf{B}|] < \infty$,
- ii) under i), assume the Lyapunov exponent of (\mathbf{A}_t) , denoted γ , satisfies

$$\gamma := \lim_{n \rightarrow \infty} n^{-1} \log |\mathbf{A}_n \cdots \mathbf{A}_1|_{op} < 0, \quad a.s.$$

To guarantee the heavy-tailedness condition \mathbf{RV}_α , we also assume

- iii) $\mathbf{B} \neq \mathbf{0}$ a.s., and \mathbf{A} has no zero rows a.s.
- iv) there exists $\kappa > 0$ such that $\mathbb{E}[|\mathbf{A}|_{op}^\kappa] < 1$,

v) the set Γ from Equation (5.46) generates a dense group on \mathbb{R} ,

$$(5.46) \quad \Gamma = \{ \log |\mathbf{a}_n \cdots \mathbf{a}_1|_{op} : n \geq 1, \mathbf{a}_n \cdots \mathbf{a}_1 > 0, \\ \mathbf{a}_n, \dots, \mathbf{a}_1 \text{ are in the support of } \mathbf{A}'\text{'s distribution} \},$$

vi) there exists $\kappa_1 > 0$ such that $\mathbb{E}[(\min_{i=1, \dots, d} \sum_{t=1}^d A_{ij})^{\kappa_1}] \geq d^{\kappa_1/2}$, and $\mathbb{E}[|\mathbf{A}|_{op}^{\kappa_1} \log^+ |\mathbf{A}|_{op}] < \infty$.

vii) under i) – vi), there exists a unique $\alpha > 0$ such that

$$(5.47) \quad \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{E}[|\mathbf{A}_n \cdots \mathbf{A}_1|_{op}^\alpha] = 0,$$

and $\mathbb{E}[|\mathbf{B}|^\alpha] < \infty$. If $d > 1$ assume α is not an even integer.

The \mathbb{R}^d -variate series (\mathbf{X}_t) , satisfying (5.45) and i) – vii), admits a causal stationary solution and satisfies \mathbf{RV}_α , with $\alpha > 0$ as in Equation (5.47).

The previous example is motivated by the seminal Kesten's paper [30]. We follow Theorem 2.1. in [1] to state conditions i) – ii) of Example 5.7. Under the conditions i) – ii), the unique solution (\mathbf{X}_t) of (5.45) has the a.s. causal representation

$$(5.48) \quad \mathbf{X}_t = \sum_{i \geq 0} \mathbf{A}_{t-i+1} \cdots \mathbf{A}_t \mathbf{B}_{t-i}, \quad t \in \mathbb{Z},$$

where the first summand is \mathbf{B}_t for $i = 0$ by convention; for an overview see [11].

One of the main reasons why the solutions to SRE as in Example 5.7 have received strong interest, is because (\mathbf{X}_t) satisfies \mathbf{RV}_α even when the innovations $((\mathbf{A}_t, \mathbf{B}_t))$ are light-tailed. This feature was first noticed in [30] where the original Kesten's assumptions were introduced. In Kesten's framework, a causal stationary solution to the SRE exists as in (5.48), and the extremes of the series occur due to the sums of infinitely many terms of growing length products appearing in (5.48); see [5] for a review. Further, the community adopted the simplified Kesten's conditions stated by Goldie in [23] for univariate SRE. These conditions also aim to capture the heavy-tailed feature under lighter-tailed innovations. In Example 5.7, we borrow the conditions iii) – vii) established for the multivariate setting from Theorem 2.4 and Corollary 2.7. in [1]; see also [2]. Then, a solution (\mathbf{X}_t) as in Example 5.7 satisfies \mathbf{RV}_α , for $\alpha > 0$, and the index of regular variation α is the unique solution to the Equation (5.47). We are also interested in Example 5.7 because it models classic econometric time series such as the squared ARCH(p), and the volatility of GARCH(p, q) processes; see [11].

Concerning the extremes of (\mathbf{X}_t) in Example 5.7, the forward spectral tail process satisfies the relation

$$\boldsymbol{\Theta}_t = \mathbf{A}_t \cdots \mathbf{A}_1 \boldsymbol{\Theta}_0, \quad t \geq 0,$$

where (\mathbf{A}_t) is an iid sequence distributed as \mathbf{A} ; see [29]. The backward spectral tail process has a cumbersome representation that we omit here; c.f. [29]. We state in Proposition 5.8 sufficient conditions on (\mathbf{A}, \mathbf{B}) yielding assumptions \mathbf{AC} , \mathbf{CS}_p hold for $p > \alpha/2$, and a suitable sequence (x_n) such

that $n\mathbb{P}(|\mathbf{X}_1| > x_n) \rightarrow 0$ as $n \rightarrow \infty$. In this case the time series (\mathbf{X}_t) admits an α -cluster process \mathbf{Q} . We recall the identity from Equation (8.6) of [12]: $c(p) = \mathbb{E}[\|\mathbf{Q}\|_p^\alpha] = \mathbb{E}[\|(\boldsymbol{\Theta}_t)_{t \geq 0}\|_p^\alpha - \|(\boldsymbol{\Theta}_t)_{t \geq 1}\|_p^\alpha]$, for $c(p)$ as in (2.9). Then, letting $p = \alpha/2$, a straightforward computation yields

$$\begin{aligned} c(p) &\leq 2 \mathbb{E}[\|(\boldsymbol{\Theta}_t)_{t \geq 0}\|_p^{\alpha-p}] = 2 \mathbb{E}[\sum_{t \geq 0} |\mathbf{A}_t \cdots \mathbf{A}_1 \boldsymbol{\Theta}_0|^p] \\ &\leq 2s \sum_{t \geq 0} (\mathbb{E}[\|\mathbf{A}_s \cdots \mathbf{A}_1\|_{op}^p])^t, \end{aligned}$$

and $\mathbb{E}[\|\mathbf{A}_s \cdots \mathbf{A}_1\|_{op}^p] < 1$, for $p < \alpha$ and $s \geq 1$ fixed sufficiently large in the setting of Example 5.7. Hence, for $p \in (\alpha/2, \alpha)$, $c(p) < \infty$ in (2.9), and then the series admits a p -cluster process $\mathbf{Q}^{(p)}$.

We state now Proposition 5.8 which verifies conditions **AC**, **CS_p** for the SRE equation. The proof is postponed to Section F.4.

Proposition 5.8. *Let (\mathbf{X}_t) be a stationary time series with values in $(\mathbb{R}^d, |\cdot|)$, as in Example 5.7. Let $p > \alpha/2$, and consider (x_n) such that there exists $\kappa' > 0$ satisfying $n/x_n^{p \wedge (\alpha - \kappa')} \rightarrow 0$, as $n \rightarrow \infty$. Then, (x_n) satisfies conditions **AC** and **CS_p**.*

In the setting of SRE equations, condition **AC** has been shown in Theorem 4.17 in [36]. In [36], the authors already considered a condition similar to **CS_p**. Parallel to their setting, we propose a proof of Proposition 5.8 which shows **CS_p** holds over uniform regions $\Lambda_n = (x_n, \infty)$ such that $n/x_n^p \rightarrow 0$, as $n \rightarrow \infty$, in the sense of (F.114). Thereby, our proof extends Theorem 4.17 in [36] to uniform regions Λ_n not having an upper bound.

Concerning the mixing properties of $(\mathbf{X}_t)_{t \geq 0}$ as in Example 5.7, we use that it is a Markov chain and that \mathbf{X}_0 has the stationary distribution. As mentioned in Remark 2.1, we can then use Markov chain's theory to compute its mixing coefficients; cf. [34]. We review Theorem 2.8. in [1], yielding an exponential decay of the mixing-coefficients (β_t) of the series. For a general treatment see Chapter 4.2 in [11].

Proposition 5.9. *Consider a time series (\mathbf{X}_t) with values in $(\mathbb{R}^d, |\cdot|)$, as in Example 5.7. Assume there exists a Borel measure μ on $(\mathbb{R}^d, |\cdot|)$, such that the Markov chain $(\mathbf{X}_t)_{t \geq 0}$ is μ -irreducible, i.e., for all $C \subset \mathbb{R}^d$ with $\mu(C) > 0$,*

$$(5.49) \quad \sum_{t=0}^{\infty} \mathbb{P}(\mathbf{X}_t \in C \mid \mathbf{X}_0 = \mathbf{x}) > 0, \quad \mathbf{x} \in \mathbb{R}.$$

Then (\mathbf{X}_t) has mixing coefficients (β_t) satisfying $\beta_t = O(\rho^t)$ for some $\rho \in (0, 1)$, and we say it is strongly mixing with geometric rate. Moreover, $(\mathbf{X}_t)_{t \geq 0}$ is irreducible with respect to the Lebesgue measure if (\mathbf{A}, \mathbf{B}) admits a density.

We can now state the asymptotic normality of cluster-based estimator for SRE solutions in Theorem 5.10 below. The proof is postponed to Section F.5.

Theorem 5.10. Consider (\mathbf{X}_t) to be the causal solution to the SRE in (5.45) with values in $(\mathbb{R}^d, |\cdot|)$, as in Example 5.7. Assume the conditions of Proposition 5.9 hold. Consider $p > \alpha/2$, and sequences (k_n) and (k'_n) satisfying (5.38), such that $m_n/k_n \rightarrow \infty$, and $k_n/k'_n \rightarrow 0$, as $n \rightarrow \infty$. Consider $f_\alpha(p) : \tilde{\ell}^p \rightarrow \mathbb{R}$, and assume \mathbf{L}, \mathbf{S} hold. Then,

$$\sqrt{k} \left(\widehat{f_{\hat{\alpha}}^{\mathbf{Q}}}(p) - f_{\alpha}^{\mathbf{Q}}(p) \right) \xrightarrow{d} \mathcal{N}(0, \text{Var}(f_{\alpha}(Y^{\mathbf{Q}}(p)))) , \quad n \rightarrow \infty,$$

under the bias conditions $\mathbf{B}_{\alpha}(k), \mathbf{B}(k')$, and the result extends to $\hat{\alpha}$ -cluster inference. In particular, the $\hat{\alpha}$ -cluster based estimators from Section 4 in (4.26) (4.29), and (4.34), are asymptotically normally distributed.

Remark 5.11. In this example, the asymptotic variances of the α -cluster based estimators from Section 4 in (4.26) (4.29), and (4.34) are non-null. The limiting variances in Theorem 5.10 are difficult to compare with the existing ones in the literature because of the complexity of the distribution of $\mathbf{Q}^{(p)}$. However, we provide simple ℓ^α -block estimators of the asymptotic variances in Section 6.

6. NUMERICAL EXPERIMENTS

This section aims to illustrate the finite-sample performance of the $\hat{\alpha}$ -cluster estimators on time series (\mathbf{X}_t) with tail-index $\alpha > 0$. In all the models we consider in Section 5, we work under the assumption that the tuning parameters of the α -cluster satisfy (5.38). We take $\kappa' = 1$ in (5.38) which yields $b = \sqrt{n/k}$. In this case, the implementation of our estimators can be written solely as a function of k and k' . Recall $k = k_n$ must satisfy $k \rightarrow \infty$, $m/k \rightarrow \infty$ with $m = \lfloor n/b \rfloor$, $n/k' \rightarrow \infty$, and $k/k' \rightarrow 0$ as $n \rightarrow \infty$. Numerical comparisons of our $\hat{\alpha}$ -cluster based approach with other existing estimators for the extremal index and the cluster index are at the advantage of our approach; see [13] and [12]. The code of all numerical experiments is available at: https://github.com/GBuritica/cluster_functionals.git.

6.1. Tuning the Hill estimator. We recommend choosing the tuning sequence of the tail-index and of the cluster estimators as $(k'_n), (k_n)$, respectively, such that $k/k' \rightarrow 0$. Roughly speaking, the cluster statistics capture the block extremal behavior whereas the tail-index recovers an extremal property of margins. In this section, we illustrate that the α cluster-based estimators perform well in simulation when we use the Hill estimator $1/\hat{\alpha}(k')$ with k' larger than k . To illustrate this point, we simulate 500 samples $(X_t)_{t=1, \dots, n}$ of an $\text{AR}(\varphi)$ model with absolute value student(α) noise for $n = 12000$, $\alpha = 1$ and $\varphi \in \{0.5, 0.7\}$, and for samples of Example 5.7. We estimate the extremal index $\hat{\theta}_X(k)$ as in (4.26) where we replace $\hat{\alpha}$ by $\hat{\alpha}(k')$. Recall that for an $\text{AR}(\varphi)$ model the asymptotic variances of the extremal index estimator are asymptotically null when $k/k' \rightarrow 0$ as $n \rightarrow \infty$. We see in Figures 1, 2 and 3 that in practice we have to choose k small to reduce the bias of the estimator. Moreover, the estimation procedure is robust with

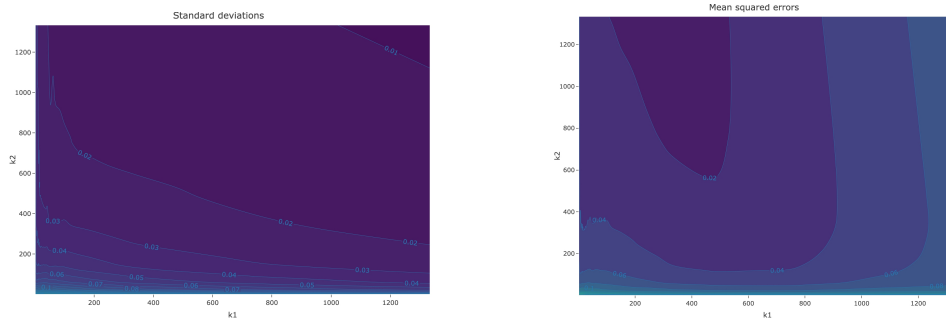


FIGURE 1. Heatmap with contour curves of standard deviations and mean squared errors for estimates of the extremal index $k1 = k \mapsto \hat{\theta}_X(k)$ in (4.26), using a Hill-type estimator $k2 = k' \mapsto \hat{\alpha}(k')$. We simulate 500 samples $(X_t)_{t=1,\dots,n}$ of an $AR(\varphi)$ model with absolute value student(α) noise for $n = 12\,000$, $\varphi = 0.5$, $\alpha = 1$, such that $\theta_X = 0.5$.

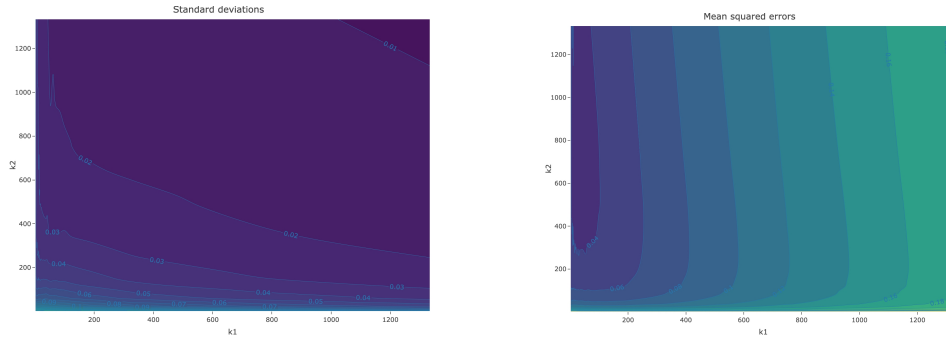


FIGURE 2. Heatmap with contour curves as in Figure 1. Here we simulate 500 samples $(X_t)_{t=1,\dots,n}$ of an $AR(\varphi)$ model with absolute value student(α) noise for $n = 12\,000$, $\varphi = 0.7$, $\alpha = 1$, such that $\theta_X = 0.3$.

respect to k' therefore we recommend taking k' large to reduce variance. Similar results were found for $n = 3\,000$, $n = 5\,000$, and $n = 8\,000$ and these are available upon request. To conclude, we see in Figures 1, 2 and 3 that standard deviations are small, and thus the error of cluster inference is mainly due to bias. For this reason, we recommend choosing k small and k' larger in all settings.

6.2. Cluster size probabilities. We reviewed in Section 4.3 how cluster sizes play a key role to model the serial behavior of exceedances. In this section, we implement the cluster size probabilities estimation procedure from Equation (4.34) in an example of a solution to the SRE under Kesten's conditions.

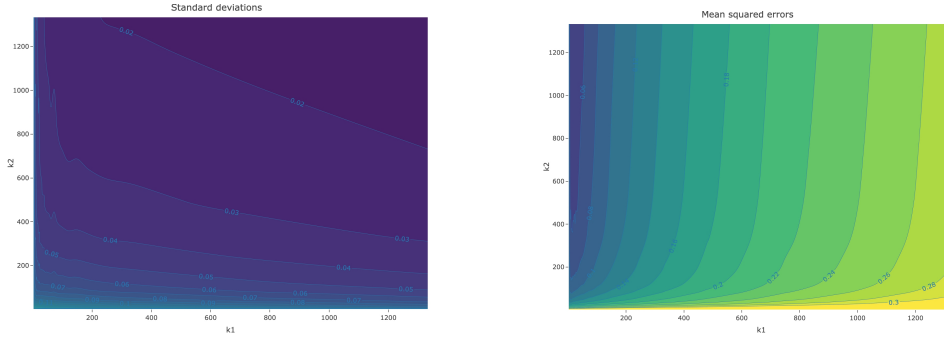


FIGURE 3. Heatmap with contour curves as in Figure 1. Here we simulate 500 samples $(X_t)_{t=1,\dots,n}$ of Example (6.1) for $n = 12\,000$ such that $\theta_X \approx 0.2792$.

Example 6.1. Consider the non-negative univariate random variables A , B , defined by $\log A = N - 0.5$, where N denotes a standard Gaussian random variable, and B is uniformly distributed in $[0, 1]$. Let (X_t) be the solution to the SRE in (5.45). Then, (X_t) satisfies \mathbf{RV}_α with $\alpha = 1$. If (A_j) is a sequence of iid random variables with generic element A , then

$$Q_t \stackrel{d}{=} \Pi_t / \|(\Pi_j)\|_\alpha, \quad t \in \mathbb{Z},$$

with

$$\Pi_t \stackrel{d}{=} \begin{cases} A_t \cdots A_1 & \text{if } t \geq 1, \\ A_t \cdots A_{-1} & \text{if } t \leq -1, \\ 1 & \text{if } t = 0. \end{cases}$$

This follows by Example 6.1 in [29], and Proposition 3.1 in [12]. Then, for $p > \alpha/2$, the p -cluster based estimators (4.26) (4.29), and (4.34) are asymptotically normally distributed.

Recall the cluster sizes π_1, π_2, \dots , defined in (4.33). We infer the cluster sizes of Example 6.1 using α -cluster estimates. To illustrate Theorem 5.10, we run a Monte-Carlo simulation experiment based on 1 000 samples $(X_t)_{t=1,\dots,n}$ of length $n = 12\,000$ from Example 6.1. For each sampled trajectory, we obtain estimates $\hat{\pi}_1, \hat{\pi}_2, \dots$, letting $k = 10$ and $b = 38$ in (4.34). For the implementation we use Hill-based estimates of the tail-index $\hat{\alpha}(k')$ with $k' = 1\,000$. We also estimate the extremal index θ_X of the series from Equation (4.26). Theorem 5.10 yields, for $j \geq 0$,

$$(6.50) \quad \text{Var}(\sqrt{k}(\hat{\pi}_j - \pi_j^{\mathbf{Q}})) \rightarrow \text{Var}(\pi_j^{\mathbf{Q}}(\mathbf{Q})), \quad n \rightarrow \infty,$$

where $\pi_j^{\mathbf{Q}}$ are the cluster functional yielding the cluster sizes π_j with the notation in (4.35). Notice that the asymptotic variance of our cluster sizes estimate is again a cluster statistic that we can infer. We compute an estimate of the asymptotic variance in (6.50) using cluster-based estimates, and compare this estimate with the empirical variance obtained from the

Monte-Carlo simulation study. Figure 4 plots the profile of the limit Gaussian distribution where the asymptotic variance is computed in these two ways. As expected from Equation (6.50), the curves overlap, even if k is small. In our simulation, a clear bias appears when we choose k larger.

In the case of SRE equations, the cluster sizes were studied in detail in [24]. The authors proposed a method to approximate the true values when the tail-index α , and the random variable A are known. We approximate true values using Equation 3.5 in [24], and a Monte-Carlo study with 10 000 samples of length 500 000. The obtained values are pointed out in red in Figure 4. We see that this choice of k yields estimates centered around the true value.

6.3. Conclusion. Our main theoretical result in Theorem 3.2 states asymptotic normality of α cluster-based disjoint block estimators $\hat{f}_\alpha^{\mathbf{Q}}(\hat{\alpha})$, based on k extremal $\ell^{\hat{\alpha}}$ -blocks, where $\hat{\alpha}$ is an estimate of the tail index of the series. The advantage of $\hat{\alpha}$ cluster-based methods is that the choice of k is robust to time dependencies and that it fully describes clusters of extreme values; see [12]. Equation (3.25) characterizes their asymptotic variance in terms of a cluster statistic that we can also infer. We further show in Section 4 that many important indices in extremes can be written in terms of an α -cluster statistic, e.g., the extremal index and cluster sizes. Section 5 verifies that our assumptions hold for numerous models like causal linear models and SRE solutions under Kesten's conditions. For linear models, we obtain estimators with null asymptotic variance for classical indices as first conjectured in [26]. In the examples we considered, our estimators have a small variance that can also be estimated. To illustrate the performance of our α -cluster inference methodology, we run finite-sample simulations in Section 6. Our simulations support that replacing α by $\hat{\alpha}(k')$ as in Section 6.1 does not have a big impact on the asymptotic variance. This is because in practice k needs to be chosen small to obtain unbiased estimates, whereas k' can be chosen larger. Then, even if we choose k small, the uncertainty of our procedure is well quantified by plugging an estimate of the asymptotic variance in the Gaussian limit. Finally, a complete study of the tuning parameters k, k' requires a careful analysis of the bias conditions for blocks: $\mathbf{B}_\alpha, \mathbf{B}$, as we pointed out in Remark 3.3, which we see as a road for future research.

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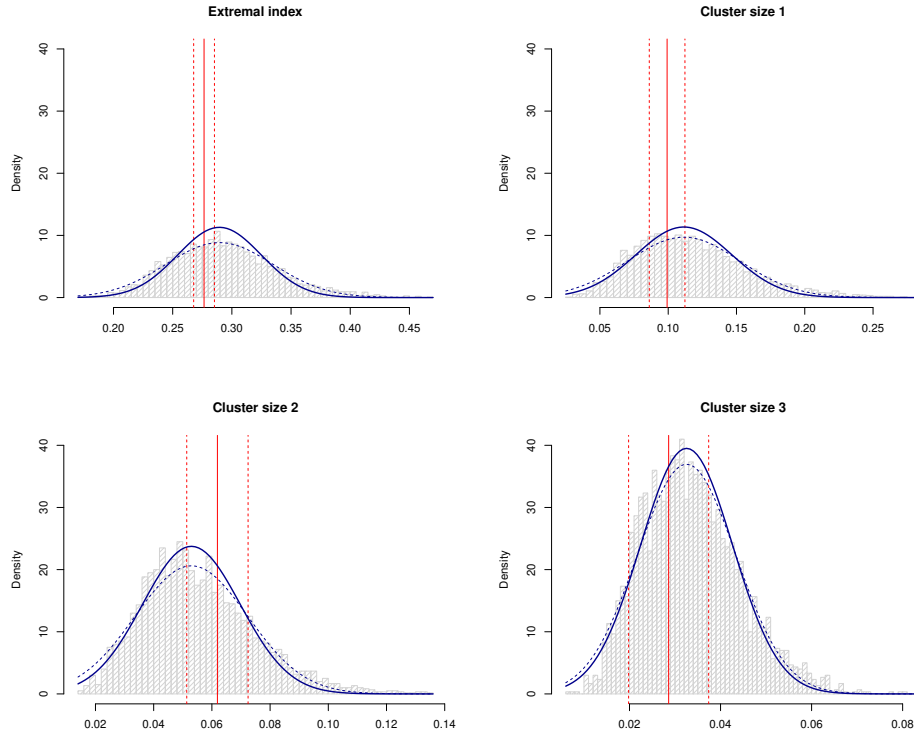


FIGURE 4. Histogram of estimates $\hat{\theta}_X$ of the extremal index using (4.26), and the cluster size probability $\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3$, using (4.33). We simulate 1 000 samples $(X_t)_{t=1, \dots, n}$ of Example 6.1 with $n = 12\,000$. The Gaussian density curves are centered in the median of the estimators. Their variances are estimated by Monte-Carlo (dotted curve) or using the average of the cluster-based estimate of the asymptotic variance defined in (6.50) (solid curve). The red lines point to the Monte-Carlo approximation of the real values with standard deviation. These were computed using Equation 3.5 in [24], and a simulation study with 10 000 samples of length 500 000.

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APPENDIX A. CONSISTENCY OF BLOCK ESTIMATORS

In the following proofs, we assume the conditions of Proposition 2.2 hold. In this setting, the time series (\mathbf{X}_t) admits a p -cluster $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$ and (2.9), (2.10) hold for (x_n) . For inference purposes, we fix the sequence of block lengths (b_n) , and we write $m_n = \lfloor n/b_n \rfloor$, such that $b_n \rightarrow \infty$, $m_n \rightarrow \infty$. Given $p \in (0, \infty]$ fixed, we assume that the relation

$$(A.51) \quad k := k_n(p) = \lfloor m_n \mathbb{P}(\|\mathcal{B}_1\|_p > x_{b_n}) \rfloor$$

$$(A.52) \quad \sim n c(p) \mathbb{P}(|\mathbf{X}_1| > x_{b_n}), \quad n \rightarrow \infty,$$

holds, where $c(p) \in (0, \infty)$ are as in (2.9). We can verify $m_n/k_n \rightarrow \infty$ using the relation $n\mathbb{P}(|\mathbf{X}_1| > x_n) \rightarrow 0$, as $n \rightarrow \infty$.

In addition, to infer the tail-index of the series α we implement the Hill estimator in (2.14), hence we consider sequences (x'_n) and (k'_n) such that (x'_n) verifies (2.9) for $p = \infty$, and we assume

$$(A.53) \quad k' := k'_n = \lfloor n \mathbb{P}(|\mathbf{X}_1| > x'_{b_n}) \rfloor,$$

holds. Similarly as before we can verify $m_n/k'_n \rightarrow \infty$, as $n \rightarrow \infty$. The sequences (b_n) , (m_n) , (x_n) , (k_n) , and (x'_n) , (k'_n) that we consider henceforth are the ones fixed here.

A.1. Proof of Lemma 3.1. We start by showing that the conditions in Lemma 3.1 on the mixing coefficients of the series (see (A.54)) entail (A.55) below. This last condition was already introduced in Equation (4.1) in [12] to show consistency of ℓ^p -block estimators. The proof of the next lemma is provided in Appendix C.1.

Lemma A.1. *Let $p \in (0, \infty]$, let $f = f(p) \in \mathcal{G}_+(\tilde{\ell}^p)$ be a bounded Lipschitz continuous function. If there exists a sequence (ℓ_n) , satisfying $\ell_n \rightarrow \infty$, as*

$n \rightarrow \infty$, such that the mixing coefficients (β_t) satisfy

$$(A.54) \quad \lim_{n \rightarrow \infty} m_n \beta_{\ell_n} / k_n = \lim_{n \rightarrow \infty} \ell_n / b_n = 0,$$

then the sequences (x_n) and (b_n) satisfy, for all $u > 0$, the relation

$$(A.55) \quad \left| \mathbb{E} \left[e^{-k^{-1} \sum_{t=1}^m f((ux_b)^{-1} \mathcal{B}_t) \mathbf{1}(\|\mathcal{B}_t\|_p > ux_b)} \right] - \mathbb{E} \left[e^{-k^{-1} f((ux_b)^{-1} \mathcal{B}_1) \mathbf{1}(\|\mathcal{B}_1\|_p > ux_b)} \right]^m \right| \rightarrow 0,$$

as $n \rightarrow \infty$, where (k_n) is chosen as in (A.52).

Next, we state the consistency of ℓ^p -block estimators for arbitrary p , and we cover the case of $\ell^{\hat{\alpha}}$ -block estimators. The proof of the next Lemma is deferred to Appendix C.2.

Lemma A.2. *Assume the mixing condition in (A.55) holds. Consider a function $f = f_\alpha(p) \in \mathcal{G}_+(\tilde{\ell}^p)$, not necessarily bounded, and assume $\mathbb{E}[f(Y\mathbf{Q}^{(p)})] < \infty$. Moreover, assume*

$$(A.56) \quad \limsup_{n \rightarrow \infty} \frac{\mathbb{E}[(f(\mathbf{X}_{[1,n]}/x_n))^{1+\delta} \mathbf{1}(\|\mathbf{X}_{[1,n]}\|_p > x_n)]}{\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_p > x_n)} < \infty.$$

Then, the ℓ^p -block estimator in (1.4) satisfies

$$\widehat{f}^{\mathbf{Q}}(p) \xrightarrow{\mathbb{P}} f^{\mathbf{Q}}(p), \quad n \rightarrow \infty.$$

Moreover, for $p = \alpha$ assume $\hat{\alpha} \xrightarrow{\mathbb{P}} \alpha$, and $\|\mathcal{B}_t\|_{\hat{\alpha},(k+1)}/x_{b_n} \xrightarrow{\mathbb{P}} 1$, as $n \rightarrow \infty$, hold. Furthermore, assume **C** holds for $\epsilon, \delta > 0$, and $c(q) = \mathbb{E}[\|\mathbf{Q}\|_q^\alpha]$ satisfies $c(q) < \infty$, for $q \in [\alpha - \epsilon, \alpha + \epsilon]$. Then, we also have

$$\widehat{f}^{\mathbf{Q}}(\hat{\alpha}) \xrightarrow{\mathbb{P}} f^{\mathbf{Q}}(\alpha), \quad n \rightarrow \infty.$$

Assuming **S** and assuming $\partial f_q / \partial q$ satisfies **C** in addition, yields

$$\widehat{f}_{\hat{\alpha}}^{\mathbf{Q}}(\hat{\alpha}) \xrightarrow{\mathbb{P}} f_{\alpha}^{\mathbf{Q}}(\alpha), \quad n \rightarrow \infty.$$

Proof of Lemma 3.1. The proof consists in applying Lemma A.2. For this purpose, notice Lemma 3.1 already assumes **C** holds for $\epsilon, \delta > 0$ and $c(q) < \infty$, for $q \in [\alpha - \epsilon, \alpha + \epsilon]$. In addition, Lemma A.1 entails condition (A.55) which we require to apply Lemma A.2. This verifies all the assumptions of Lemma A.2 and thus this concludes the proof. \square

APPENDIX B. ASYMPTOTIC NORMALITY OF BLOCK ESTIMATORS

In the following sections it will be useful to consider the deterministic threshold estimators defined by

$$(B.57) \quad \tilde{f}_{\alpha}^{\mathbf{Q}}(u, q) := \frac{1}{k_n} \sum_{t=1}^{m_n} f_{\alpha}(\mathcal{B}_t / (u x_{b_n})) \mathbf{1}(\|\mathcal{B}_t\|_q > u x_{b_n}), \quad u, q > 0,$$

$$(B.58) \quad \tilde{h}^{\mathbf{Q}}(u) := \frac{1}{k'_n} \sum_{t=1}^{m_n} h(\mathcal{B}_t / (u x'_{b_n})) \mathbf{1}(\|\mathcal{B}_t\|_{\infty} > u x'_{b_n}), \quad u > 0,$$

where $f_\alpha(p) : \tilde{\ell}^p \rightarrow \mathbb{R}$ is a cluster functional that can depend on α in its expression and $h : \tilde{\ell}^\infty \rightarrow \mathbb{R}$ is as in (2.15). The sequences (x_n) , (b_n) , (m_n) , (k_n) that we consider, defining the block estimator are the ones fixed in (A.52), and (x'_n) , (k'_n) are as in (A.53). Similarly to (B.57), we define $\tilde{\Gamma}^{\mathbf{Q}}(u, q)$ to be the deterministic threshold estimator for the cluster functional $\mathbf{x} \mapsto 1(q)(\mathbf{x})$ of (3.19), and we define $\tilde{e}^{\mathbf{Q}}(u)$ as in (B.58) where $e : \tilde{\ell}^\infty \rightarrow \mathbb{R}$ is the exceedances functional in (2.16).

In addition, we assume $k/k' \rightarrow \kappa$, for $\kappa \geq 0$, or equivalently $x'_{b_n}/x_{b_n} \rightarrow \kappa^{1/\alpha}$, as $n \rightarrow \infty$. Indeed, if we denote $|\mathbf{X}'|$'s distribution by $F_{|\mathbf{X}'|}$ and its left inverse by $F_{|\mathbf{X}'|}^{\leftarrow}$, then the relations (A.52) and (A.53) imply

$$(B.59) \quad \frac{x'_{b_n}}{x_{b_n}} \sim \frac{F_{|\mathbf{X}'|}^{\leftarrow}\left(1 - \frac{k'_n}{n}\right)}{F_{|\mathbf{X}'|}^{\leftarrow}\left(1 - \frac{k_n}{n}\right)} \sim \left(\frac{k'_n}{k_n}\right)^{-1/\alpha} \rightarrow \kappa^{1/\alpha},$$

as $n \rightarrow \infty$, such that the last relation follows by regular variation of $|\mathbf{X}'|$ and [40, Proposition 2.6 (v)].

Let $\mathcal{F}_f \subseteq \mathcal{G}_+(\tilde{\ell}^p)$ be the set including the functions: $\mathbf{x} \mapsto f_\alpha(\mathbf{x})$, $\mathbf{x} \mapsto 1(\mathbf{x})$, and $\mathcal{F}_h \subseteq \mathcal{G}_+(\tilde{\ell}^\infty)$ the set containing $\mathbf{x} \mapsto h(\mathbf{x})$ and $\mathbf{x} \mapsto e(\mathbf{x})$ as in (2.15), (2.16), and define

$$(B.60) \quad \mathcal{F} = \mathcal{F}_f \cup \mathcal{F}_h.$$

Recall the deterministic threshold estimators in (B.57), then, let $\epsilon, \delta > 0$ be such that the conditions of Theorem 3.2 are satisfied and define

$$\begin{aligned} \mathcal{T}_f &:= \{\tilde{g}^{\mathbf{Q}}(u, q)\}_{\{g(\cdot/u): g \in \mathcal{F}_f, u \in [1-\epsilon, 1+\epsilon], q \in [\alpha-\epsilon, \alpha+\epsilon]\}}, \\ \mathcal{T}_h &:= \{\tilde{g}^{\mathbf{Q}}(u)\}_{\{g(\cdot/u): g \in \mathcal{F}_h, u \in [1-\epsilon, 1+\epsilon]\}}. \end{aligned}$$

We are interested in the family

$$(B.61) \quad \mathcal{T} = \mathcal{T}_f \cup \mathcal{T}_h,$$

indexed by four functions $f_\alpha, 1, h, e$, and the values of $(u, q) \in [1-\epsilon, 1+\epsilon] \times [\alpha-\epsilon, \alpha+\epsilon]$. Here

$$(B.62) \quad (u, q) \mapsto \tilde{g}^{\mathbf{Q}}(u, q), \quad g \in \mathcal{F}_f \cup \mathcal{F}_h,$$

is a stochastic process taking values in the multivariate Skorohod space $\mathbb{D}([1-\epsilon, 1+\epsilon] \times [\alpha-\epsilon, \alpha+\epsilon])$ endowed with the product J_1 -topology of the metric space of càdlàg functions.

Let (\mathcal{B}_t^*) denote a triangular array of iid blocks distributed as \mathcal{B}_1 , and define the independent-block estimators on this sequence as

$$(B.63) \quad \begin{aligned} \tilde{g}_{*,*}^{\mathbf{Q}}(u, q) &= \frac{1}{k_n} \sum_{t=1}^{m_n} g(\mathcal{B}_t^*/(u x_b)) \mathbb{1}(\|\mathcal{B}_t^*\|_q > u x_b), \quad g \in \mathcal{F}_f, \\ \tilde{g}_{*,*}^{\mathbf{Q}}(u) &= \frac{1}{k'_n} \sum_{t=1}^{m_n} g(\mathcal{B}_t^*/(u x'_b)) \mathbb{1}(\|\mathcal{B}_t^*\|_\infty > u x'_b), \quad g \in \mathcal{F}_h. \end{aligned}$$

Also, define $\mathcal{T}_{f,*}$ and $\mathcal{T}_{h,*}$ consisting of the union of these independent-block estimators on \mathcal{F}_f and \mathcal{F}_h , respectively, and similarly as in (B.61) we define

$$(B.64) \quad \mathcal{T}_{,*} = \mathcal{T}_{f,*} \cup \mathcal{T}_{h,*}.$$

B.1. Proof sketch. To prove the uniform central limit theorem on the family \mathcal{T} in (B.61) we follow the plan below:

- (1) We start by showing that, asymptotically, we can replace (\mathcal{B}_t) by an array (\mathcal{B}_t^*) of iid blocks meaning it suffices to show the uniform central limit theorem on $\mathcal{T}_{,*}$,
- (2) we then compute the covariance structure of the independent-block estimators (B.63) evaluated on (\mathcal{B}_t^*) ,
- (3) and finally we establish how to control the complexity of the family \mathcal{F} defined in (B.60).

Bearing this in mind, Section B.2 tackles (1), then Section B.3 studies (2), and Section B.4 discusses the uniform entropy theory to assess (3). The proof of Theorem 3.2 is deferred to Section B.5. The proofs of auxiliary results are deferred to Appendix D.

B.2. Coupling theory. The following Proposition tackles part number (1) from the proof sketch in Section B.1, and its proof is deferred to Section D.2.

Proposition B.1. *Consider a cluster functional $g \in \mathcal{G}_+(\tilde{\ell}^p)$ satisfying \mathbf{L} , and assume \mathbf{M} , \mathbf{MX}_β are satisfied. Then, there exists an array $(\mathcal{B}_t^*)_{1 \leq t \leq m_n}$ of iid blocks distributed as \mathcal{B}_1 such that, for all $\delta > 0$,*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\sup_{\substack{u \in [1-\epsilon, 1+\epsilon], \\ q \in [\alpha-\epsilon, \alpha+\epsilon]}} \sqrt{k_n} |\tilde{g}^{\mathbf{Q}}(u, q) - \tilde{g}_{*,*}^{\mathbf{Q}}(u, q)| > \delta \right) = 0, \quad g \in \mathcal{F}_f,$$

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\sup_{u \in [1-\epsilon, 1+\epsilon]} \sqrt{k'_n} |\tilde{g}^{\mathbf{Q}}(u) - \tilde{g}_{*,*}^{\mathbf{Q}}(u)| > \delta \right) = 0, \quad g \in \mathcal{F}_h.$$

We conclude from the Proposition above that it suffices to show the uniform central limit theorem of the independent-block estimators (B.63) on $\mathcal{T}_{,*}$. Indeed, since the family is indexed by only a finite number of functions, it is enough to check it separately, for each $g \in \mathcal{F}$ (see (B.60)).

B.3. Covariance structure. The covariance structure of the deterministic threshold estimators is determined by the Lemma below whose proof is postponed to Section D.1. Moreover, this Lemma studies the step (2) of the proof plan in Section B.1.

Lemma B.2. *Consider $g, f : \tilde{\ell}^p \rightarrow \mathbb{R}$ to be cluster functional satisfying \mathbf{L} , and let $h, e : \tilde{\ell}^\infty \rightarrow \mathbb{R}$ be the cluster functional for the Hill estimator and the exceedances estimator in (2.15) and (2.16). Then, for $q, q' \in [p - \epsilon, p + \epsilon]$*

and $u, v \in [1 - \epsilon, 1 + \epsilon]$,

$$(B.65) \quad \begin{aligned} & k_n \text{Cov}(\tilde{g}_{*,*}^{\mathbf{Q}}(v, q), \tilde{f}_{*,*}^{\mathbf{Q}}(u, q')) \\ & \rightarrow v^{-\alpha} \frac{c(q)}{c(p)} \mathbb{E}[g(Y\mathbf{Q}^{(q)})f(Y\mathbf{Q}^{(q)}v/u)\mathbf{1}(\|Y\mathbf{Q}^{(q)}\|_{q'} > v/u)], \end{aligned}$$

as $n \rightarrow \infty$. Moreover, if $k_n/k'_n \rightarrow \kappa > 0$, then

$$(B.66) \quad \text{Cov}(\tilde{g}_{*,*}^{\mathbf{Q}}(v, q), \tilde{h}_{*,*}^{\mathbf{Q}}(u)) \rightarrow v^{-\alpha} \kappa \frac{c(q)}{c(p)} \mathbb{E}[g(Y\mathbf{Q}^{(q)})h(Y\mathbf{Q}^{(q)}\kappa^{1/\alpha}v/u)],$$

$$(B.67) \quad \text{Cov}(\tilde{h}_{*,*}^{\mathbf{Q}}(v), \tilde{e}_{*,*}^{\mathbf{Q}}(u)) \rightarrow v^{-\alpha} \kappa \frac{c(q)}{c(p)} \mathbb{E}[h(Y\mathbf{Q}^{(q)})e(Y\mathbf{Q}^{(q)}v/u)],$$

as $n \rightarrow \infty$. Furthermore, the previous limits continue to hold replacing h by e in the previous equalities, or e by h .

B.4. Uniform entropy theory. In this section we discuss how we assess the step (3) of the proof plan (see Section B.1 for details). Theorem 3.2 states asymptotic normality of $\ell^{\hat{\alpha}}$ -block estimators in (1.4). Our proof considers a family of independent-block estimators indexed by q , for q in a neighborhood of α , and relies on Lemma B.3 below showing this family has low complexity in terms of entropy numbers. We review the classical results of the theory of Vapnik-Cervonenkis below to measure the complexity of classes of functions. We refer to [42, Section 2.6] for a detailed treatment.

Let \mathcal{X} be a measurable space and let \mathcal{V} be a collection of sets from this space. The VC-dimension of \mathcal{V} is the smallest number s such that for every set containing s elements, we can find a subset that is not picked out by the class \mathcal{V} . We say that \mathcal{V} is a VC-class if its VC-dimension is finite. A VC-class of functions \mathcal{F} is such that the collections of all the subgraphs $\{(\mathbf{x}, u) : \phi(\mathbf{x}) > u\}$ of real-valued functions $\phi \in \mathcal{F}$ is a VC-class. The entropy number of a VC-class has a polynomial expression on the VC-dimension (see Theorem 2.6.7. in [42]). Moreover, given a VC-class of functions \mathcal{F} , the VC-Hull of \mathcal{F} is the collection of functions \mathcal{G} such that for every $g \in \mathcal{G}$ there exists a symmetric convex combination $f_m = \sum_{i=1}^m \alpha_i f_i$, with $\sum_{i=1}^m |\alpha_i| \leq 1$, $f_i \in \mathcal{F}$, such that g is the pointwise limit of the sequence $(f_m)_{m \in \mathbb{N}}$. Moreover, by Corollary 2.6.12 in [42] the entropy number of a VC-hull also has a polynomial expression on the VC-dimension of the underlying VC-class.

It is often easier to check that \mathcal{F} is a VC-major class of functions \mathcal{F} , i.e., that $\{\mathbf{x} : \phi(\mathbf{x}) > u\}$, for every $u \in \mathbb{R}$, is a VC-class. We can construct new VC-major classes using classical operations: addition, products. If \mathcal{F} is VC-major, then the class of function $h \circ \phi$, with h ranging over the monotone functions $h : \mathbb{R} \rightarrow \mathbb{R}$, with $\phi \in \mathcal{F}$, is VC-major. This is Lemma 2.6.19 in [42]. One example are functions $\mathbf{x} \mapsto f(\mathbf{x}/u)\mathbf{1}(\phi(\mathbf{x}) > u)$, where $u \mapsto f(\mathbf{x}/u)$ is a non-increasing function. Moreover, a bounded VC-major class satisfies the uniform entropy condition by Lemma 2.6.13 in [42].

In particular, we show that ℓ^q -norms constitute a VC-major class. This is the purpose of the next Lemma whose proof is postponed to Section D.3.

Lemma B.3. *Consider the class containing all sets of the form $\{\mathbf{x} \in \tilde{\ell}^{q_0} : \|\mathbf{x}\|_q > u\}$, where $q \in (q_0, q'_0)$, and $u \in \mathbb{R}$. Then, this is a VC-class of dimension 3. This implies that the classes of functions*

$$\mathcal{F} := \{\|\cdot\|_q : \tilde{\ell}^{q_0} \mapsto [0, \infty), q \in (q_0, q'_0)\},$$

are VC-major for every $q'_0 > q_0 > 0$.

B.5. Proof of Theorem 3.2. We focus on showing the asymptotic result for the block estimator implemented with $p = \alpha$. The general result on p -cluster inference, for $p \in (0, \infty]$, follows the same lines of this proof by fixing $q = p$ during the proof, instead of considering q in a neighborhood of α , thus we omit the details.

Let $g \in \mathcal{G}_+(\tilde{\ell}^q)$ be a cluster functional, let $g^{\mathbf{Q}}(q)$ be its q -cluster statistic given by

$$(B.68) \quad g^{\mathbf{Q}}(q) = \mathbb{E}[g(Y\mathbf{Q}^{(q)})],$$

and to simplify notation we write $g^{\mathbf{Q}}(\alpha) = g^{\mathbf{Q}}$. We consider the centered Gaussian process

$$(B.69) \quad \mathbb{G} = (\mathbb{G}_q(g(\cdot/u)))$$

for $q, q' \in [\alpha - \epsilon, \alpha + \epsilon]$ and $u, v \in [1 - \epsilon, 1 + \epsilon]$, $g \in \mathcal{F}$ as in (B.60) with covariance structure

$$(B.70) \quad \begin{aligned} & \text{Cov}(\mathbb{G}_q(g(\cdot/u)), \mathbb{G}_{q'}(f(\cdot/v))) \\ &= c(q)u^{-\alpha} \mathbb{E}[g(Y\mathbf{Q}^{(q)})f(Y\mathbf{Q}^{(q)}u/v)\mathbb{1}(\|Y\mathbf{Q}^{(q)}u/v\|_{q'} > 1)], \end{aligned}$$

$$(B.71) \quad \begin{aligned} & \text{Cov}(\mathbb{G}_q(g(\cdot/u)), \mathbb{G}(h(\cdot/v))) \\ &= \kappa c(q)u^{-\alpha} \mathbb{E}[g(Y\mathbf{Q}^{(q)})h(Y\mathbf{Q}^{(q)}\kappa^{1/\alpha}u/v)], \end{aligned}$$

$$(B.72) \quad \begin{aligned} & \text{Cov}(\mathbb{G}(e(\cdot/u)), \mathbb{G}(h(\cdot/v))) \\ &= \kappa u^{-\alpha} \mathbb{E}[e(Y\mathbf{Q})h(Y\mathbf{Q}u/v)], \end{aligned}$$

for $u, v \in [1 - \epsilon, 1 + \epsilon]$, $q, q' \in [\alpha - \epsilon, \alpha + \epsilon]$, and $g, f \in \mathcal{F}$, such that $\mathbf{Q}^{(q)} \in \tilde{\ell}^q$, is the q -cluster process of the series. We can replace h by e in the previous equalities or e by h to obtain the full covariance structure of this Gaussian process.

B.5.1. Asymptotic equicontinuity. In view of the discussion after Proposition B.1, it suffices to establish asymptotic equicontinuity for the independent-block estimators in (B.63). To simplify notation, we denote

$$g(\mathcal{B}_t^*/(u x_{b_n}))\mathbb{1}(\|\mathcal{B}_t^*\|_q > (u x_{b_n})) = g(\mathcal{B}_t^*/x_{b_n})(u, q),$$

where $\tilde{g}_{*,*}^{\mathbf{Q}}(u, q)$ is as in (B.63), and we consider the family $\mathcal{T}_{g,*}$ of stochastic processes defined by the independent-block estimators in (B.63), namely,

$$\mathcal{T}_{g,*} = \{\tilde{g}_{*,*}^{\mathbf{Q}}(u, q)\}_{u \in [u_0, s_0], q \in [q_0, q'_0]}.$$

As we mentioned, this process is indexed by $[u_0, s_0] \times [q_0, q'_0]$. Define the random metric $d_n(\cdot, \cdot)$ on this family by

$$(B.73) \quad \begin{aligned} & (d_n((u, q), (v, q')))^2 \\ &= \frac{1}{k_n} \sum_{t=1}^{m_n} [g(\mathcal{B}_t^*/x_{b_n})(u, q) - g(\mathcal{B}_t^*/x_{b_n})(v, q')]^2. \end{aligned}$$

In the remaining of the proof, we verify the sequence of processes $\mathcal{T}_{g, *}$ satisfies the Lindeberg condition (i), continuity condition (ii), and entropy condition (iii) from Theorem C.4.5 in [31] hold.

B.5.2. Lindeberg condition (i). Since $u \mapsto g(\mathbf{x}/u)$ is a non-increasing function, then we it suffices to verify, for every $\eta > 0$, $q \in [q_0, q'_0]$, $u \in [u_0, s_0]$,

$$\begin{aligned} I &= \frac{m_n}{k_n} \mathbb{E} \left[(g(\mathcal{B}_1/x_{b_n})(u, q))^2 \mathbb{1}(g(\mathcal{B}_1/x_{b_n})(u, q) > \sqrt{\eta k_n}) \right] \\ &\rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Indeed, we have

$$\begin{aligned} I &\leq \frac{m_n}{k_n} \mathbb{E} \left[(g(\mathcal{B}_1/x_{b_n})(u, q))^{2+\delta} \right]^{\frac{2}{2+\delta}} \mathbb{P} \left(g(\mathcal{B}_1/x_{b_n})(u, q) > \sqrt{\eta k_n} \right)^{\frac{\delta}{2+\delta}} \\ &\leq (\eta k_n)^{-\delta/2} \frac{m_n}{k_n} \mathbb{E} \left[(g(\mathcal{B}_1/x_{b_n})(u, q))^{2+\delta} \right]. \end{aligned}$$

where δ here follows the notation in (3.21). Then, appealing to assumption **L**, we deduce $I \rightarrow 0$, as $n \rightarrow \infty$.

B.5.3. Continuity condition (ii). Let $s > u$, q, q' , and denote

$$(B.74) \quad c((u, q), (s, q')) = \text{Cov}(\mathbb{G}_q(g(\cdot/u)), \mathbb{G}_{q'}(g(\cdot/v))),$$

and $c(u, q) = c((u, q), (u, q))$, and $c(q)$ is as in (2.9). Then, applying Lemma B.2,

$$\begin{aligned} & \mathbb{E}[(d_n((u, q), (s, q')))^2] \\ &= \frac{m_n}{k_n} \mathbb{E} \left[(g(\mathcal{B}_1/x_{b_n})(u, q) - g(\mathcal{B}_1/x_{b_n})(s, q'))^2 \right] \\ &\rightarrow c(u, q) + c(s, q') - 2c((u, q), (s, q')) \\ &= u^{-\alpha} c(q) (g^2)^{\mathbf{Q}}(q) + s^{-\alpha} c(q') (g^2)^{\mathbf{Q}}(q') \\ &\quad - 2u^{-\alpha} c(q) \mathbb{E}[g(Y\mathbf{Q}^{(q)})g(Y\mathbf{Q}^{(q)}u/v) \mathbb{1}(\|Y\mathbf{Q}^{(q)}u/v\|_{q'} > 1)]. \end{aligned}$$

We now use the fact that $v \mapsto g(\cdot/v)$ is a non-increasing function, for $u > v$,

$$(B.75) \quad \begin{aligned} & c(u, q) + c(s, q') - 2c((u, q), (s, q')) \\ &\leq u^{-\alpha} c(q) (g^2)^{\mathbf{Q}}(q) + s^{-\alpha} c(q') (g^2)^{\mathbf{Q}}(q') \\ &\quad - 2u^{-\alpha} c(q) \mathbb{E}[g(Y\mathbf{Q}^{(q)})^2 \mathbb{1}(\|Y\mathbf{Q}^{(q)}\|_{q'} > 1)] \\ &= s^{-\alpha} c(q') (g^2)^{\mathbf{Q}}(q') - u^{-\alpha} (g^2)^{\mathbf{Q}}(q) \\ &\quad + 2u^{-\alpha} c(q) \mathbb{E}[g(Y\mathbf{Q}^{(q)})^2 \mathbb{1}(\|Y\mathbf{Q}^{(q)}\|_{q'} \leq 1)]. \end{aligned}$$

We now focus on the last term. Notice that for $q \geq q'$, the last term equals zero. We consider the case $q < q'$,

$$(B.76) \quad \begin{aligned} & \mathbb{E}[g(Y\mathbf{Q}^{(q)})^2 \mathbb{1}(\|Y\mathbf{Q}^{(q)}\|_{q'} \leq 1)] \\ & \leq \mathbb{E}[g(Y\mathbf{Q}^{(q)})^{2+\delta}]^{2/(2+\delta)} \mathbb{P}(\|Y\mathbf{Q}^{(p)}\|_{q'} \leq 1)^{\delta/(2+\delta)}. \end{aligned}$$

Furthermore, for $q < q'$,

$$\begin{aligned} & \mathbb{P}(Y\|\mathbf{Q}^{(q)}\|_{q'} \leq 1) \\ & \leq \mathbb{P}(Y^q(\|\mathbf{Q}^{(q)}\|_q^q - (q - q')\sum_{t \in \mathbb{Z}} |\mathbf{Q}_t^{(q)}|^{q'} \log 1/|\mathbf{Q}_t^{(q)}|) \leq 1) \\ & = \mathbb{P}(Y^q(1 - (q - q')\sum_{t \in \mathbb{Z}} |\mathbf{Q}_t^{(q)}|^q \log 1/|\mathbf{Q}_t^{(q)}|) \leq 1) \\ & \leq 1 - \mathbb{E}[(1 - (q - q')\sum_{t \in \mathbb{Z}} |\mathbf{Q}_t^{(q)}|^q \log 1/|\mathbf{Q}_t^{(q)}|)^{\alpha/q}] \\ & \leq (q - q')\alpha q^{-1} \mathbb{E}[\sum_{t \in \mathbb{Z}} |\mathbf{Q}_t^{(q)}|^{q'} \log 1/|\mathbf{Q}_t^{(q)}|]. \end{aligned}$$

Therefore, as $q < q'$, we obtain

$$\mathbb{P}(Y\|\mathbf{Q}^{(q)}\|_{q'} \leq 1) \leq (q - q')\alpha q^{-1} \mathbb{E}\left[\sum_{t \in \mathbb{Z}} |\mathbf{Q}_t^{(q)}|^q \log 1/|\mathbf{Q}_t^{(q)}|\right].$$

Then, notice that by the change-of-norms in Equation (2.12)

$$\begin{aligned} \mathbb{E}\left[\sum_{t \in \mathbb{Z}} |\mathbf{Q}_t^{(q)}|^q \log 1/|\mathbf{Q}_t^{(q)}|\right] & = c(q)^{-1} \mathbb{E}\left[\frac{\|\mathbf{Q}\|_q^\alpha}{\|\mathbf{Q}\|_q^q} \sum_{t \in \mathbb{Z}} |\mathbf{Q}_t|^q \log \frac{\|\mathbf{Q}\|_q}{|\mathbf{Q}_t|}\right] \\ & \leq (\epsilon c(q))^{-1} \mathbb{E}[\|\mathbf{Q}\|_{\alpha-\epsilon}^{\alpha+\epsilon}]. \end{aligned}$$

and the last relation follows by the monotonicity of ℓ^p -norms and the fact that for every chosen $0 < \eta < 1$ we have

$$(B.77) \quad \log(1/x) = 1/\eta \log(1/x^\eta) \leq 1/\eta x^{-\eta}, \quad 0 < x \leq 1,$$

Hence, appealing to conditions **M** and **L** the term in (B.76) is bounded by constant $C < \infty$. We can now conclude the following bound for (B.75)

$$\begin{aligned} & c(u, q) + c(s, q') - 2c((u, q), (s, q')) \\ & \leq s^{-\alpha} c(q')(g^2)^{\mathbf{Q}(q')} - u^{-\alpha} c(q)(g^2)^{\mathbf{Q}(q)} + 2u^{-\alpha} c(q)(q - q')\alpha q^{-1} C. \end{aligned}$$

Finally, recall from Proposition 2.2 that

$$c(q) = \mathbb{E}[\|\mathbf{Q}\|_q^\alpha] = (\mathbb{E}[1/\|\mathbf{Q}^{(q)}\|_\alpha^\alpha])^{-1},$$

and then it is easy to see by monotone convergence that $q \mapsto c(q)$ is a continuous function at α . Finally, this implies

$$\lim_{\eta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{\substack{u, s \in [u_0, s_0], \\ q, q' \in [q_0, q'_0], \\ d((u, q), (s, q')) < \eta}} \mathbb{E}[(d_n((u, q), (s, q')))^2] = 0.$$

From this we conclude that (ii) holds.

B.5.4. Entropy condition (iii). Recall \mathcal{F} is the set with four functions: $\mathbf{x} \mapsto f(\mathbf{x})$, $\mathbf{x} \mapsto 1$, $\mathbf{x} \mapsto h(\mathbf{x})$ and $\mathbf{x} \mapsto e(\mathbf{x})$ (see (3.19), (2.15), (2.16)). We can

assume without loss of generality that g takes non-negative values. Denote \mathcal{T}_g the class of functions

$$(\mathbf{x}, u, q) \mapsto g(\mathbf{x}/u)\mathbb{1}(\|\mathbf{x}\|_q > u),$$

indexed by $[u_0, s_0] \times [q_0, q'_0]$. It is sufficient to show that for each $g \in \mathcal{F}$, the class \mathcal{T}_g satisfies the entropy condition in (iii) with respect to the random metric introduced in (B.73). We argue separately for each function $g \in \mathcal{F}$. In what follows we denote the envelope of the class \mathcal{T}_g as

$$\begin{aligned} \mathbf{G}_g(\mathbf{x}) &:= \sup_{(u,q) \in [u_0, s_0] \times [q_0, q'_0]} |g(\mathbf{x}/u)\mathbb{1}(\|\mathbf{x}\|_q > u)| \\ &= g(\mathbf{x}/u_0)\mathbb{1}(\|\mathbf{x}\|_{q_0} > u_0). \end{aligned}$$

Moreover, notice we can apply Lemma C.4.8 in [31]. Indeed, we can verify condition C.4.8 using Lemma (B.2), as $n \rightarrow \infty$,

$$\begin{aligned} &\frac{m_n}{k_n} \mathbb{E} \left[(g(\mathcal{B}_1/x_{b_n})(u, q) - g(\mathcal{B}_1/x_{b_n})(s, q'))^2 \right] \\ &\rightarrow c(u, q) + c(s, q') - 2c((u, q), (s, q')) < \infty. \end{aligned}$$

This means that it is enough to check that \mathcal{T}_g is a VC-hull class, as introduced in Section B.4, and then apply Corollary 2.6.12 in [42] giving a satisfactory bound on the entropy. In the following we treat separately the case g equal to f_α or 1 and the case g equal to h or e .

Case $g = f_\alpha$ and $g = 1$. Consider the class of functions $\|\mathbf{x}\|_q : \mathbf{x} \mapsto \|\mathbf{x}\|_q$, for elements $\mathbf{x} \in \tilde{\ell}^{q_0}$, and $q \in [q_0, q'_0]$. By Lemma B.3, this class of functions is a VC-major class, as the sets $\{\mathbf{x} \in \tilde{\ell}^{q_0} : \|\mathbf{x}\|_q > u\}$, for $q \in [q_0, q'_0]$, and $u > 0$, forms a VC-class of dimension 3. Finally, applying Lemma 2.6.19 for the monotone functions $\psi_u : \mathbb{R} \rightarrow \mathbb{R}$ defined by:

$$u \mapsto g(\mathbf{x}/u)\mathbb{1}(\|\mathbf{x}\|_q > u),$$

indexed by u , we see that \mathcal{T}_g is a VC-major. Finally, Lemma 2.6.13 states that bounded VC-major classes are VC-hull classes and this yields the desired result.

Case $g = h, e$. This case has been studied previously, for example, we can borrow the results in [31]. Here by an applications of Lemma C.4.18 in [31] and Lemma c.4.19 we conclude the entropy condition is satisfied.

To sum up, we have verified the sequence of processes \mathcal{T}_g^* satisfies the Lindeberg condition (i), the continuity condition (ii), and the entropy condition (iii) from Theorem C.4.5. in [31]. Therefore, we conclude the uniform asymptotic normality of the estimators indexed by \mathcal{T} and this concludes the proof of Theorem 3.25.

B.5.5. Uniform central limit theorem. Consider a function $f_\alpha : \tilde{\ell}^\alpha \rightarrow \mathbb{R}$ satisfying **L**. Recall the family \mathcal{F} in (B.60). Note that by the assumptions on Theorem 3.2 we have that Proposition B.1 and Lemma B.2 hold. Asymptotic

equicontinuity of the family

$$(B.78) \quad \left\{ \sqrt{k} (\tilde{g}^{\mathbf{Q}}(u, q) - c(q)u^{-\alpha}g^{\mathbf{Q}}(q)) \right\}_{\substack{u \in [u_0, s_0], \\ q \in [q_0, q'_0], \\ g \in \mathcal{F}}},$$

holds by the calculations from Section (B.5.1), then it remains to verify the asymptotic normality of the finite-dimensional parts of the family in (B.78). Applying the Wold device it is enough to check that every linear combination of deterministic threshold estimators in (B.57) is asymptotically normal distributed. Moreover, note any such a linear combination g is again a cluster functional satisfying **L**. We thus apply Proposition B.1 to g .

Hence, for such a linear combination g denote by $\tilde{g}_{,*}^{\mathbf{Q}}$ its independent-block estimator corresponding to the linear combination of independent-block estimators in (B.63), and denote $g^{\mathbf{Q}}(p)$ its cluster-statistic with the notation in (B.68). Then, it is enough to check that the real-valued variable:

$$(B.79) \quad \sqrt{k} (\tilde{g}_{,*}^{\mathbf{Q}} - g^{\mathbf{Q}}(p)),$$

admits a Gaussian limit, as $n \rightarrow \infty$. Moreover, note we can replace $g^{\mathbf{Q}}(p)$ by the expectation of $\tilde{g}_{,*}^{\mathbf{Q}}$ in (B.79) thanks to the bias assumptions $\mathbf{B}_{\alpha}(k_n)$ and $\mathbf{B}(k'_n)$. In addition, note that (B.79) is the sum of independent random variables, hence we can apply the Lindeberg central limit theorem for triangular arrays [6, Section 18]. Finally, Lemma B.2 allow us to compute the asymptotic variance of (B.79) recentering by its expectation, and this is enough for the convergence of (B.79) to a normal distribution as $n \rightarrow \infty$. Then, we conclude the uniform asymptotic normality of the family in (B.78) towards the Gaussian process \mathbb{G} defined in (B.69).

B.6. Variance calculations. We now focus on establishing the asymptotic variance in (3.25). Recall that when f depends on α , i.e., $f = f_{\alpha}$, we impose a smoothness assumption **S** on the function $q \mapsto f_q$. More precisely, we assume

$$f_q(\mathbf{x}) = f_{\alpha}(\mathbf{x}) + (q - \alpha)f'_{\alpha}(\mathbf{x}) + \frac{1}{2}(q - \alpha)^2 R_{\alpha}(\mathbf{x}),$$

where f'_{α} is the derivative of f_{α} with respect to a , i.e., $f'_{\alpha} = \frac{\partial f_q}{\partial q} \Big|_{q=\alpha}$, and

$$R_{\alpha}(\mathbf{x}) \leq \sup_{q \in (\alpha - \epsilon, \alpha + \epsilon)} \left| \frac{\partial^2 f_q}{\partial q^2}(\mathbf{x}) \right|.$$

Then, we see by Lemma 3.1 that $f'_{\alpha}, \sup_{q \in (\alpha - \epsilon, \alpha + \epsilon)} |\partial^2 f_q / \partial q^2|$ admit consistent $\ell^{\hat{\alpha}}$ -cluster estimates. Moreover we have

$$(B.80) \quad \begin{aligned} \sqrt{k} (\hat{f}_{\hat{\alpha}}^{\mathbf{Q}}(\hat{\alpha}) - f_{\alpha}^{\mathbf{Q}}) &= \sqrt{k} (\hat{f}_{\hat{\alpha}}^{\mathbf{Q}}(\hat{\alpha}) - f_{\alpha}^{\mathbf{Q}}) + \frac{\sqrt{k}}{\sqrt{k'}} \sqrt{k'} (\alpha - \hat{\alpha}) \hat{f}'_{\alpha}{}^{\mathbf{Q}}(\hat{\alpha}) \\ &\quad + \frac{\sqrt{k}}{\sqrt{k'}} \sqrt{k'} (\alpha - \hat{\alpha})^2 \hat{R}_{\alpha}^{\mathbf{Q}}(\hat{\alpha}). \end{aligned}$$

as $n \rightarrow \infty$, where k' is the tuning parameter for the Hill estimator whereas k is used to tune the extremal cluster estimator.

Recall $k/k' \rightarrow \kappa \geq 0$. Then, we consider separately the cases $\kappa = 0$ and $\kappa > 0$. We start with the case $\kappa > 0$. Applying Lemma E.1 we obtain

$$\sqrt{k}(\alpha - \hat{\alpha}) \xrightarrow{d} \alpha \mathbb{G}_\alpha(\alpha h(\cdot/1) - e(\cdot/1)),$$

as $n \rightarrow \infty$. Moreover, an application of Lemma E.2 yields

$$\sqrt{k}(\widehat{f_\alpha^\mathbf{Q}}(\hat{\alpha}) - f_\alpha^\mathbf{Q}) \xrightarrow{d} f_\alpha^\mathbf{Q} \mathbb{G}_\alpha(f_\alpha(\cdot/1)/f_\alpha^\mathbf{Q} - 1(\cdot/1)).$$

Then, we can conclude from Equation (B.80) that

$$\begin{aligned} & \sqrt{k}(\widehat{f_\alpha^\mathbf{Q}}(\hat{\alpha}) - f_\alpha^\mathbf{Q}) \\ & \xrightarrow{d} \mathbb{G}_\alpha((f_\alpha(\cdot/1) - f_\alpha^\mathbf{Q}1(\cdot/1)) + (f_\alpha^\mathbf{Q})\alpha(\alpha h(\cdot/1) - e(\cdot/1))) \\ & = \mathcal{N}(0, \sigma^2(\kappa)). \end{aligned}$$

The limit variance $\sigma^2(\kappa)$ can be computed from the covariance structure in (B.70) and is given explicitly in (B.81). Notice that it depends on the parameter $\kappa > 0$. More precisely,

$$(B.81) \quad \sigma^2(\kappa) = \text{Var}(f_\alpha(Y\mathbf{Q})) + \kappa\alpha^2(f_\alpha^\mathbf{Q})^2\sigma_\alpha^2 + 2\kappa\alpha f_\alpha^\mathbf{Q}\sigma_{f,\alpha}(\kappa),$$

where

$$\begin{aligned} \sigma_f^2 &= \mathbb{E}[(f_\alpha(\cdot/1) - f_\alpha^\mathbf{Q}1(\cdot/1))^2] = \text{Var}(f_\alpha(Y\mathbf{Q})), \\ \sigma_\alpha^2 &= \mathbb{E}[(\alpha h(Y\mathbf{Q}) - e(Y\mathbf{Q}))^2], \\ \sigma_{f,\alpha}(\kappa) &= \mathbb{E}[f_\alpha(Y\mathbf{Q})(\alpha h(Y\mathbf{Q}\kappa^{1/\alpha}) - e(Y\mathbf{Q}\kappa^{1/\alpha}))]. \end{aligned}$$

Furthermore, notice that if $f_\alpha(\mathbf{x}) = f_\alpha(\mathbf{x}/\|\mathbf{x}\|_\alpha)$, then $\sigma_{f,\alpha}(\kappa) = 0$, for all $\kappa > 0$. Otherwise, notice that by Jensen's inequality we have

$$(\sigma_{f,\alpha}(\kappa))^2 \leq \mathbb{E}[(f_\alpha(Y\mathbf{Q}) - f_\alpha^\mathbf{Q})^2]\mathbb{E}[(\alpha h(Y\mathbf{Q}\kappa^{1/\alpha}) - e(Y\mathbf{Q}\kappa^{1/\alpha}))^2],$$

such that $\mathbb{E}[f_\alpha(Y\mathbf{Q})^2] < \infty$. Then, we focus on the right-hand term in the previous equation. Then, relying on the properties of h and e as defined in (2.15), (2.16), we obtain that for $\kappa < 1$,

$$\begin{aligned} & (\sigma_{f,\alpha}(\kappa))^2 / \text{Var}(f_\alpha(Y\mathbf{Q})) \\ & \leq \mathbb{E}[(\alpha h(Y\mathbf{Q}\kappa^{1/\alpha}) - e(Y\mathbf{Q}\kappa^{1/\alpha}))^2] \\ & = \mathbb{E}[(\alpha h(Y\mathbf{Q}\kappa^{1/\alpha}) - e(Y\mathbf{Q}\kappa^{1/\alpha}))^2 \mathbb{1}(\|Y\mathbf{Q}\kappa^{1/\alpha}\|_\alpha > 1)] \\ & = \kappa \int_0^\infty \mathbb{E}[(\alpha h(y\mathbf{Q}) - e(y\mathbf{Q}))^2 \mathbb{1}(y > \max\{k^{1/\alpha}, 1\})] d(-y^{-\alpha}) \\ & = \kappa\sigma_\alpha^2. \end{aligned}$$

Hence, we conclude that, for $\kappa < 1$,

$$(\sigma_{f,\alpha}(\kappa))^2 \leq \kappa\sigma_\alpha^2 \text{Var}(f_\alpha(Y\mathbf{Q})).$$

In particular this implies $\sigma^2(\kappa) \rightarrow \text{Var}(f_\alpha(Y\mathbf{Q}))$, as $\kappa \rightarrow 0$.

If $\kappa = 0$, then under the assumption of Theorem 3.2, similar steps as for the proof provided in Section B.5, but now restricting the family \mathcal{F} to \mathcal{F}_h with the notation in (B.61), allow us to conclude the following

$$\sqrt{k'}(\alpha - \hat{\alpha}) \xrightarrow{d} \alpha \mathcal{N}(0, \mathbb{E}[(\alpha h(Y\mathbf{Q}) - e(Y\mathbf{Q}))^2]),$$

as $n \rightarrow \infty$. Hence, we see $\frac{\sqrt{k}}{\sqrt{k'}}\sqrt{k'}(\alpha - \hat{\alpha}) \xrightarrow{\mathbb{P}} 0$, as $n \rightarrow \infty$. Then, Equation (B.80) together with the previous limit implies

$$\begin{aligned} \sqrt{k}(\widehat{f}_{\hat{\alpha}}^{\mathbf{Q}}(\hat{\alpha}) - f_{\alpha}^{\mathbf{Q}}) &= \sqrt{k}(f_{\hat{\alpha}}^{\widehat{\mathbf{Q}}}(\hat{\alpha}) - f_{\alpha}^{\mathbf{Q}}) + o(1) \\ &\xrightarrow{d} \mathbb{G}_{\alpha}((f_{\alpha}(\cdot/1) - f_{\alpha}^{\mathbf{Q}}1(\cdot/1))) \\ &= \mathcal{N}(0, \text{Var}(f_{\alpha}(Y\mathbf{Q}))), \end{aligned}$$

as $n \rightarrow \infty$, where the last limit follows again from Lemma E.2.

Overall, this calculations demonstrate the expression of the asymptotic variance in (3.25), and this concludes the proof of Theorem 3.2.

APPENDIX C. PROOFS OF APPENDIX A

C.1. Proof of Lemma A.1. We start by denoting disjoint blocks as

$$(C.82) \quad \mathcal{B}_t := \mathbf{X}_{(t-1)b+[1,b]}, \quad \mathcal{B}_t^* := \mathbf{X}_{(t-1)b+[1,b]}^*,$$

$t = 1, \dots, m$, such that $(\mathcal{B}_t^*)_{t=1, \dots, m}$ is a sequence of iid blocks, distributed as \mathcal{B}_1 , independent of $(\mathcal{B}_t)_{t=1, \dots, m}$. We also denote $\ell := \ell_n \rightarrow \infty$, and disjoint blocks as

$$\mathcal{B}_{t,\ell} := \mathbf{X}_{(t-1)b+[1,b-\ell]}, \quad t = 1, \dots, m.$$

such that for $\ell = 0$ we keep the notation in (C.82). Notice that for all $\delta > 0, \epsilon > 0$, and for every bounded Lipschitz-continuous function $f \in \mathcal{G}_+(\tilde{\ell}^p)$

$$\begin{aligned} &|\mathbb{E}[\exp\{-\frac{1}{k}\sum_{t=1}^m f(x_b^{-1}\mathcal{B}_t)\}] - \mathbb{E}[\exp\{-\frac{1}{k}\sum_{t=1}^m f(x_b^{-1}\mathcal{B}_{t,\ell})\}]| \\ &\leq \mathbb{E}[|\frac{1}{k}\sum_{t=1}^m f(x_b^{-1}\mathcal{B}_t) - \frac{1}{k}\sum_{t=1}^m f(x_b^{-1}\mathcal{B}_{t,\ell})|] \\ &\leq \mathbb{E}[\frac{1}{k}\sum_{t=1}^m |f(x_b^{-1}\mathcal{B}_t) - f(x_b^{-1}\mathcal{B}_{t,\ell})|] \\ &= o(m\mathbb{P}(\|\overline{\mathcal{B}_1}/x_b\|_p > \delta)/k), \quad n \rightarrow \infty. \end{aligned}$$

This term vanishes by condition \mathbf{CS}_p . Moreover, define

$$I = |\mathbb{E}[\exp\{-\frac{1}{k}\sum_{t=1}^m f_{\epsilon}(x_b^{-1}\mathcal{B}_t)\}] - \mathbb{E}[\exp\{-\frac{1}{k}\sum_{t=1}^m f_{\epsilon}(x_b^{-1}\mathcal{B}_{t,\ell})\}]|,$$

where $f_{\epsilon}(\mathbf{x}_t) := f(\mathbf{x}_{t,\epsilon})$. Then, there exists a constant $c > 0$ such that

$$\begin{aligned} I &\leq c \frac{1}{k} \mathbb{P}(\max_{1 \leq j \leq m} \max_{1 \leq i \leq \ell} |\mathbf{X}_{(j-1)b-i+1}| > \epsilon x_b) \\ &\leq c \frac{m}{k} \mathbb{P}(\|\mathcal{B}_{1,\ell}\|_{\infty} > \epsilon x_b) \\ &\leq c \frac{m\ell}{k} \mathbb{P}(|\mathbf{X}_0| > \epsilon x_b) = O(\ell/b), \end{aligned}$$

as $n \rightarrow \infty$. Thus, we conclude that $\lim_{n \rightarrow \infty} \ell_n/b_n = 0$ is a sufficient condition yielding $I \rightarrow 0$, as $n \rightarrow \infty$. Recall the definition of the mixing coefficients (β_t) in Section 2.1. Moreover, applying the mean value theorem we have $|e^{-x} - e^{-y}| \leq |x - y|$, thus

$$\begin{aligned} & \left| \mathbb{E} \left[\exp \left\{ -\frac{1}{k} \sum_{t=1}^m f_\varepsilon(x_b^{-1} \mathcal{B}_t) \right\} \right] - \mathbb{E} \left[\exp \left\{ -\frac{1}{k} \sum_{t=1}^m f_\varepsilon(x_b^{-1} \mathcal{B}_t^*) \right\} \right] \right| \\ & \leq \frac{1}{k} \sum_{t=1}^m \mathbb{E} \left[|f_\varepsilon(x_b^{-1} \mathcal{B}_t) - f_\varepsilon(x_b^{-1} \mathcal{B}_t^*)| \right] \\ & \leq \frac{m}{k} \|f\|_\infty 2 d_{TV} \left(\mathcal{L}(\mathcal{B}_{t,l}) \otimes \underbrace{\mathcal{L}(\mathbf{X}_1) \otimes \cdots \otimes \mathcal{L}(\mathbf{X}_1)}_{\ell \text{ times}}, \mathcal{L}(\mathcal{B}_t) \right) \\ & \leq 2 \frac{m}{k} \|f\|_\infty \beta_{\ell_n} \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

We use first the definition of the total variation distance, and second a reformulation of the distance in terms of the mixing coefficients (β_t) . This last relation allows to conclude that (A.55) holds under the conditions of Lemma A.1.

C.2. Proof of Lemma A.2. Consider a non-negative continuous function $f_q : \tilde{\ell}^p \rightarrow \mathbb{R}$ satisfying the assumptions of Lemma A.2. We follow a similar argument as in the proof of Lemma 8.5 in [12] to show

$$|\tilde{f}_q^{\mathbf{Q}}(u, q) - f_\alpha^{\mathbf{Q}}(\alpha)| \xrightarrow{\mathbb{P}} 0,$$

as $(n, u, q) \rightarrow (\infty, 1, \alpha)$. Fix $\eta > 0$, and denote $g = f_\alpha \wedge \eta$, then

$$\begin{aligned} & |\tilde{f}_q^{\mathbf{Q}}(u, q) - f_\alpha^{\mathbf{Q}}(\alpha)| \\ & \leq |\tilde{f}_q^{\mathbf{Q}}(u, q) - \tilde{f}_\alpha^{\mathbf{Q}}(u, q)| + |f_\alpha^{\mathbf{Q}}(u, q) - \tilde{g}^{\mathbf{Q}}(u, q)| + |\tilde{g}^{\mathbf{Q}}(u, q) - g^{\mathbf{Q}}(\alpha)| \\ & \quad + |g^{\mathbf{Q}}(\alpha) - f_\alpha^{\mathbf{Q}}(\alpha)| \end{aligned} \tag{C.83} \quad I + II + III + IV.$$

We treat the four terms in (C.83) separately.

Regarding term III in (C.83), note that

$$\begin{aligned} & |\tilde{g}^{\mathbf{Q}}(u, q) - g^{\mathbf{Q}}(\alpha)| \\ & \leq |\tilde{g}^{\mathbf{Q}}(u, q) - \tilde{g}^{\mathbf{Q}}(1, q)| + |\tilde{g}^{\mathbf{Q}}(1, q) - \tilde{g}^{\mathbf{Q}}(1, \alpha)| + |\tilde{g}^{\mathbf{Q}}(1, \alpha) - g^{\mathbf{Q}}(\alpha)| \\ & = |\tilde{g}^{\mathbf{Q}}(u, q) - \tilde{g}^{\mathbf{Q}}(1, q)| + |\tilde{g}^{\mathbf{Q}}(1, q) - \tilde{g}^{\mathbf{Q}}(1, \alpha)| + o_{\mathbb{P}}(1), \quad n \rightarrow \infty, \end{aligned}$$

such that the last relation holds by an application of Lemma 8.5. in [12]. Moreover, note that g is a bounded function and with probability zero $Y^{\mathbf{Q}}$ belongs to the points of discontinuity of g . Moreover, this implies that g can be approximated by a monotone sequence of Lipschitz-continuous functions. Therefore, we can assume without loss of generality that g is

Lipschitz-continuous. In this case note

$$\begin{aligned} & |\tilde{g}^{\mathbf{Q}}(u, q) - g^{\mathbf{Q}}(\alpha)| \\ & \leq (\tilde{g}^{\mathbf{Q}}(u, q) - \tilde{g}^{\mathbf{Q}}(1, q)) + (\tilde{g}^{\mathbf{Q}}(1, q) - \tilde{g}^{\mathbf{Q}}(1, \alpha)) + |\tilde{g}^{\mathbf{Q}}(u, q) - \tilde{g}^{\mathbf{Q}}(1, q)| \\ & \leq \eta(u_0^{-\alpha} - 1)\tilde{\mathbf{I}}^{\mathbf{Q}}(u_0, q_0) + (\tilde{g}^{\mathbf{Q}}(1, q_0) - \tilde{g}^{\mathbf{Q}}(1, \alpha)) + |\tilde{g}^{\mathbf{Q}}(u, q) - \tilde{g}^{\mathbf{Q}}(1, q)|, \end{aligned}$$

where we have also used here the monotonicity of the function $q \mapsto \mathbb{1}(\|\mathbf{x}_t\|_q > 1)$. Then there exists u_0, q_0 , such that $0 < u_0^{-\alpha} - 1 < \eta^{-1}\epsilon/3$ and $0 < c(\alpha) - c(q_0) < \epsilon/3$, $u_0^{-\alpha}c(q_0) < 2$, and $1 > u > u_0, \alpha > q > q_0$. This last assertion follows by continuity and monotonicity of the functions $u \mapsto u^{-\alpha}$, and also of $q \mapsto c(q)$. This last is granted by the fact that $c(q) = \mathbb{E}[\|\mathbf{Q}\|_q^\alpha] < \infty$ and monotonicity of the ℓ^p -norms yields the continuity of this last function. Similarly, we can find $n_0 \in \mathbb{N}$ such that, for $1 > u > u_0, 1 > q > q_0$, and $n > n_0$,

$$\begin{aligned} & |\tilde{g}^{\mathbf{Q}}(u, q) - g^{\mathbf{Q}}(\alpha)| \\ & \leq \eta(u_0^{-\alpha} - 1)\tilde{\mathbf{I}}^{\mathbf{Q}}(u_0, q_0) + (g^{\mathbf{Q}}(1, q_0) - g^{\mathbf{Q}}(1, \alpha)) + \epsilon/3 \\ & \leq \eta(u_0^{-\alpha} - 1)(u_0^{-\alpha}c(q_0) + o_{\mathbb{P}}(1)) + (c(q_0) - 1) + \epsilon/2 \\ & \leq \epsilon. \end{aligned}$$

where the second inequality holds by an application of Lemma 8.5 in [12]. Since ϵ was chosen arbitrarily, then we conclude $|\tilde{g}^{\mathbf{Q}}(u, q) - g^{\mathbf{Q}}(\alpha)| \xrightarrow{\mathbb{P}} 0$, as $(n, u, q) \rightarrow (\infty, 1, \alpha)$.

We now turn our attention to the approximations in *II* and *IV* in (C.83), which are the two of them of a similar nature. In this case, note

$$\begin{aligned} & \mathbb{E} \left[\sup_{u \geq u_0, q \geq q_0} |f_\alpha^{\tilde{\mathbf{Q}}}(u, q) - \tilde{g}^{\mathbf{Q}}(u, q)| \right] \\ & \leq \frac{m_n}{k_n} \mathbb{E} \left[\sup_{u, q} f_\alpha(\mathcal{B}_t/(ux_{b_n})) \mathbb{1}(\|\mathcal{B}_t\|_q > (ux_{b_n})) \right. \\ & \quad \left. \times \mathbb{1}(f_\alpha(\mathcal{B}_t/(ux_{b_n})) \mathbb{1}(\|\mathcal{B}_t\|_q > (ux_{b_n})) > \eta) \right] \\ & \leq \frac{m_n}{k_n} \mathbb{E} \left[(f_\alpha(\mathcal{B}_t/(u_0x_{b_n})))^{1+\delta} \mathbb{1}(\|\mathcal{B}_t\|_{q_0} > (u_0x_{b_n})) \right]^{\frac{1}{1+\delta}} \\ & \quad \times \mathbb{P}(f_\alpha(\mathcal{B}_t/(u_0x_{b_n})) \mathbb{1}(\|\mathcal{B}_t\|_{q_0} > (u_0x_{b_n})) > \eta)^{\frac{\delta}{1+\delta}} \\ & \leq \frac{m_n}{k_n} \eta^{-\delta} \mathbb{E} \left[(f_\alpha(\mathcal{B}_t/(u_0x_{b_n})))^{1+\delta} \mathbb{1}(\|\mathcal{B}_t\|_{q_0} > (u_0x_{b_n})) \right]. \end{aligned}$$

This implies *II* $\xrightarrow{\mathbb{P}} 0$ letting $(n, u, q) \rightarrow (\infty, 1, \alpha)$, and then letting $\eta \rightarrow \infty$ in the previous inequality by the Lindeberg-type condition in (A.56). A similar argument implies *IV* $\rightarrow 0$ letting $(n, u, q) \rightarrow (\infty, 1, \alpha)$, and then letting $\eta \rightarrow \infty$.

To conclude, we now show that I in (C.83) also converges to zero. In this case, notice that by assumption **S** we have

$$I = |\tilde{f}_q^{\mathbf{Q}}(u, q) - \tilde{f}_\alpha^{\mathbf{Q}}(u, q)| \leq |q - \alpha| \sup_{\substack{q \in [\alpha - \epsilon, \alpha + \epsilon] \\ u \in [1 - \epsilon, 1 + \epsilon]}} \frac{\partial g_q}{\partial q} \Big|_{q=q}^{\mathbf{Q}}(u, q).$$

Therefore,

$$\begin{aligned} \mathbb{E}[I] &= \mathbb{E}[|\tilde{f}_q^{\mathbf{Q}}(u, q) - \tilde{f}_\alpha^{\mathbf{Q}}(u, q)|] \\ &\leq \frac{m_n}{k_n} |q - \alpha| \mathbb{P}(\|\mathcal{B}_t\|_{q_0} > u_0)^{\frac{\delta}{1+\delta}} \\ &\quad \times \mathbb{E} \left[\sup_{\substack{q \in [\alpha - \epsilon, \alpha + \epsilon] \\ u \in [1 - \epsilon, 1 + \epsilon]}} \left(\frac{\partial g_q}{\partial q} \Big|_{q=q} \right)^{1+\delta} (\mathcal{B}_t / (ux_b)) \mathbb{1}(\|\mathcal{B}_t\|_q > ux_b) \right]^{\frac{1}{1+\delta}}. \end{aligned}$$

Then, applying assumption **C** we conclude $I \xrightarrow{\mathbb{P}} 0$ as $(n, u, q) \rightarrow (\infty, 1, \alpha)$. Then, this shows,

$$|\tilde{f}_q^{\mathbf{Q}}(u, q) - f_\alpha^{\mathbf{Q}}(\alpha)| \xrightarrow{\mathbb{P}} 0,$$

as $(n, u, q) \rightarrow (\infty, 1, \alpha)$. Recall we assumed that $\hat{\alpha} \xrightarrow{\mathbb{P}} \alpha$, and $\|\mathcal{B}_t\|_{\hat{\alpha}, (k+1)} / x_{b_n} \xrightarrow{\mathbb{P}} 1$. Then, the continuous mapping Theorem implies

$$|\tilde{f}_{\hat{\alpha}}^{\mathbf{Q}}(\|\mathcal{B}_t\|_{\hat{\alpha}, (k+1)} / x_{b_n}, \hat{\alpha}) - f_\alpha^{\mathbf{Q}}(\alpha)| = |\hat{f}_{\hat{\alpha}}^{\mathbf{Q}} - f_\alpha^{\mathbf{Q}}| \xrightarrow{\mathbb{P}} 0,$$

as $n \rightarrow \infty$, and this gives the desired result. \square

APPENDIX D. PROOFS OF AUXILIARY RESULTS

D.1. Proof of Proposition B.2. We start by showing that if $g : \tilde{\ell}^p \rightarrow \mathbb{R}$ satisfies **L** then

$$(D.84) \frac{m_n}{k_n} \mathbb{E}[g(\mathcal{B}_1 / (vx_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n})] \rightarrow v^{-\alpha} \frac{c(q)}{c(p)} \mathbb{E}[g(Y\mathbf{Q}^{(q)})],$$

as $n \rightarrow \infty$. Note that if g is a bounded function, then the previous relation follows straightforwardly from Proposition 2.2. If g is not bounded, then note that for $\delta > 0$ as in (3.21), and for all $\eta > 0$,

$$\begin{aligned} &\frac{m_n}{k_n} \mathbb{E}[g(\mathcal{B}_1 / (vx_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n})] \\ &= \frac{m_n}{k_n} \mathbb{E}[g(\mathcal{B}_1 / (vx_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) \mathbb{1}(g(\mathcal{B}_1 / (vx_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) > \eta)] \\ &\quad + \eta \frac{m_n}{k_n} \mathbb{P}(g(\mathcal{B}_1 / (vx_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) > \eta) \\ &\quad + \frac{m_n}{k_n} \mathbb{E}[g \wedge \eta(\mathcal{B}_1 / (vx_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n})] \\ &= (D.85) I + III. \end{aligned}$$

Applying Proposition 2.2 we have $III \rightarrow c(q)v^{-\alpha} \mathbb{E}[g \wedge \eta(\mathbf{Q}^{(q)})]$, as $n \rightarrow \infty$, and letting $\eta \rightarrow \infty$ we obtain the right-hand side term in (D.84). Hence, it remains to show that $I + II \rightarrow 0$, letting $n \rightarrow \infty$, and then $\eta \rightarrow \infty$.

Regarding term I in (D.85),

I

$$\begin{aligned} &= \mathbb{E}[g(\mathcal{B}_1/(vx_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) \mathbb{1}(g(\mathcal{B}_1/(vx_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) > \eta)] \\ &\leq \frac{m_n}{k_n} \mathbb{E}[(g(\mathcal{B}_1/(vx_{b_n})))^{1+\delta} \mathbb{1}^{\frac{1}{1+\delta}} \mathbb{P}(g(\mathcal{B}_1/(vx_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) > \eta)^{\frac{\delta}{1+\delta}}] \\ &\leq \frac{m_n}{k_n} \eta^{-\delta} \mathbb{E}[(g(\mathcal{B}_1/(vx_{b_n})))^{1+\delta} \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n})], \end{aligned}$$

and we see that by assumption **L** that $I \rightarrow 0$ letting first $n \rightarrow \infty$ and lastly $\eta \rightarrow \infty$. Finally, for term II in (D.85), we apply a Markov inequality which yields

$$\begin{aligned} II &= \eta \frac{m_n}{k_n} \mathbb{P}(g(\mathcal{B}_1/(vx_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) > \eta) \\ &\leq \eta^{-\delta} \frac{m_n}{k_n} \mathbb{E}[(g(\mathcal{B}_1/(vx_{b_n})))^{1+\delta} \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n})], \end{aligned}$$

and we conclude also for term II that $II \rightarrow 0$ letting $n \rightarrow \infty$ and then $\eta \rightarrow \infty$. Overall this shows (D.84) holds.

We now consider the functional h in (2.15) defining the Hill estimator. Assume that $k_n/k'_n \rightarrow \kappa$ and $\kappa > 0$. In this case notice that since (k_n) and (k'_n) satisfy (A.51) and (3.18), respectively, then

$$\begin{aligned} \frac{m_n}{k'_n} &\sim \frac{k_n}{k'_n \mathbb{P}(\|\mathcal{B}_1\|_p > x_{b_n})} \sim \frac{k_n}{k'_n} \frac{\mathbb{P}(|\mathbf{X}_1| > x'_{b_n})}{c(p) \mathbb{P}(|\mathbf{X}_1| > x_{b_n})} \frac{1}{b_n \mathbb{P}(|\mathbf{X}_1| > x'_{b_n})} \\ &\sim \frac{1}{b_n c(p) \mathbb{P}(|\mathbf{X}_1| > x'_{b_n})}, \quad n \rightarrow \infty, \end{aligned}$$

and this holds since $k_n/k'_n \rightarrow \kappa$, and $\mathbb{P}(|\mathbf{X}_1| > x'_{b_n})/\mathbb{P}(|\mathbf{X}_1| > x_{b_n}) \rightarrow \kappa^{-1}$, as $n \rightarrow \infty$, due to (B.59), and regular variation of the variable $|\mathbf{X}_1|$. Then,

$$\begin{aligned} \frac{m_n}{k'_n} \mathbb{E}[h(\mathcal{B}_1/(vx'_{b_n}))] &= \frac{m_n}{k'_n} \mathbb{E}[h(\mathcal{B}_1/(vx'_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_\infty > vx'_{b_n})] \\ &\rightarrow v^{-\alpha} \frac{c(\infty)}{c(p)} \mathbb{E}[h(Y\mathbf{Q}^{(\infty)})], \quad n \rightarrow \infty, \\ \text{(D.86)} \quad &= \frac{v^{-\alpha}}{c(p)} \mathbb{E}[h(Y\mathbf{Q})]. \end{aligned}$$

such that the first limit holds applying Proposition 2.2, and the last equality follows from the change-of-norms equation in (2.12). Moreover by similar arguments we can also show that $m_n \mathbb{E}[e(\mathcal{B}_1/(vx'_{b_n}))]/k'_n \rightarrow v^{-\alpha} \mathbb{E}[e(Y\mathbf{Q})]/c(p)$, for the exceedances functional in (2.16), and also $m_n \mathbb{P}(\|\mathcal{B}_1\|_q > (vx_{b_n}))/k'_n \rightarrow v^{-\alpha} c(q)/c(p)$, as $n \rightarrow \infty$.

We now focus on computing the covariance of blocks. Denote $\kappa_n = k_n/k'_n$, and denote $c_n = x_{b_n}/x'_{x_n}$. Note that by our assumptions $\kappa_n \rightarrow \kappa > 0$ and by

Equation (B.59) we also have $c_n \rightarrow \kappa^{-1/\alpha}$. We start by considering the case where $g : \tilde{\ell}^p \rightarrow \mathbb{R}$ is a functional satisfying **L**, and h is the functional defining the Hill estimator in (2.15). Now, since (\mathcal{B}_t^*) is a sequence of independent blocks distributed as \mathcal{B}_1 we have

$$\begin{aligned}
 & k_n \operatorname{Cov}(\tilde{g}_{*,*}^{\mathbf{Q}}(v, q), \tilde{h}_{*,*}^{\mathbf{Q}}(v)) \\
 &= \frac{k_n m_n}{k_n k'_n} \mathbb{E} \left[g(\mathcal{B}_1/(vx_{b_n})) h(\mathcal{B}_1/(ux'_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) \right] \\
 &\quad - \frac{k_n m_n}{k_n k'_n} \mathbb{E} \left[g(\mathcal{B}_1/(vx_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) \right] \mathbb{E} \left[h(\mathcal{B}_1/(ux'_{b_n})) \right], \\
 \text{(D.87)} &= \frac{m_n \kappa_n}{k_n} \mathbb{E} \left[g(\mathcal{B}_1/(vx_{b_n})) h(\mathcal{B}_1/(ux'_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) \right] + o(1),
 \end{aligned}$$

as $n \rightarrow \infty$, where the last equality follows from (D.84) and (D.86) and the fact that $k_n/m_n \sim \mathbb{P}(\|\mathcal{B}\|_p > x_{b_n}) \rightarrow 0$, as $n \rightarrow \infty$. Moreover, by similar calculations as in (D.85) we can see that it suffices to show, for all $\eta > 0$,

$$\begin{aligned}
 & \frac{m_n \kappa_n}{k_n} \mathbb{E} \left[g \wedge \eta(\mathcal{B}_1/(vx_{b_n})) h \wedge \eta(\mathcal{B}_1/(ux'_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) \right] \\
 \text{(D.88)} & \rightarrow v^{-\alpha} \kappa \frac{c(q)}{c(p)} \mathbb{E} [g \wedge \eta(Y\mathbf{Q}^{(q)}) h \wedge \eta(Y\mathbf{Q}^{(q)} \kappa^{1/\alpha} v/u)],
 \end{aligned}$$

as $n \rightarrow \infty$, and then we conclude by monotone convergence by letting $\eta \rightarrow \infty$ in the right-hand side of the previous equation.

We now focus on showing (D.88) holds. For this note that the function $c \mapsto h \wedge \eta(\mathbf{x}_t)/c$ is a non-increasing function. Then, relying on (B.59) we obtain that, for all $\epsilon > 0$, and for n sufficiently large,

$$\begin{aligned}
 & \frac{m_n}{k_n} \mathbb{E} \left[g \wedge \eta(\mathcal{B}_1/(vx_{b_n})) h \wedge \eta(\mathcal{B}_1/(ux'_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) \right] \\
 &= \frac{m_n}{k_n} \mathbb{E} \left[g \wedge \eta(\mathcal{B}_1/(vx_{b_n})) h \wedge \eta(\mathcal{B}_1/(uc_n x_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) \right] \\
 &\leq \frac{m_n}{k_n} \mathbb{E} \left[g \wedge \eta(\mathcal{B}_1/(vx_{b_n})) h \wedge \eta(\mathcal{B}_1(\kappa^{1/\alpha} + \epsilon)/(ux_{b_n})) \mathbb{1}(\|\mathcal{B}_1\|_q > vx_{b_n}) \right] \\
 &\rightarrow v^{-\alpha} \frac{c(q)}{c(p)} \mathbb{E} [g \wedge \eta(Y\mathbf{Q}^{(q)}) h \wedge \eta(Y\mathbf{Q}^{(q)} v(\kappa^{1/\alpha} + \epsilon)/u)],
 \end{aligned}$$

as $n \rightarrow \infty$, where the last limit is again an application of Proposition 2.2. In a similar manner we can also obtain a lower bound with respect to ϵ . Finally, notice these bounds hold for arbitrary ϵ , thus, letting $\epsilon \downarrow 0$ we can conclude that (D.88) holds. To sum up, the relation in (D.87) together with similar calculations as in (D.85) allow us to establish that (B.66) holds. Furthermore, the calculations for (B.65) and (B.67) follow the same line of arguments thus we omit the details. \square

D.2. Proof of Proposition B.1. The proof of Proposition B.1 relies on the following auxiliary results.

Lemma D.1. Consider functions $g, h : \tilde{\ell}^p \rightarrow \mathbb{R}$ satisfying **L**. Assume that there exists a sequences (ℓ_n) , satisfying $\ell_n \rightarrow \infty$, and

$$\lim_{n \rightarrow \infty} m_n \beta_{\ell_n} / k_n = \lim_{n \rightarrow \infty} \ell_n / b_n = 0.$$

Then, for all $t = 2, 3, \dots$, the relation below holds

$$(D.89) \quad \lim_{n \rightarrow \infty} \frac{m}{k} \text{Cov}(g(x_b^{-1} \mathcal{B}_1), f(x_b^{-1} \mathcal{B}_t)) = 0.$$

Proposition D.2. Consider $g, f : \tilde{\ell}^\alpha \rightarrow \mathbb{R}$ such that **L** holds. Assume $\mathbf{B}_\alpha(k_n)$, $\mathbf{B}(k'_n)$, and condition \mathbf{MX}_β hold. Then,

$$k \text{Cov}(\tilde{g}^{\mathbf{Q}}(u, q), \tilde{f}^{\mathbf{Q}}(v, q')) \rightarrow \text{Cov}(\mathbb{G}_q(g(\cdot/u)), \mathbb{G}_{q'}(f(\cdot/v))),$$

as $n \rightarrow \infty$.

Proof of Lemma D.1. We assume without loss of generality that the functions $g, f : \tilde{\ell}^p \rightarrow \mathbb{R}$ satisfying **L** take non-negative values. Notice that for all $t > 0$

$$(D.90) \quad \frac{m}{k} \text{Cov}(g(x_b^{-1} \mathcal{B}_1) f(x_b^{-1} \mathcal{B}_t)) = \frac{m}{k} \mathbb{E}[g(x_b^{-1} \mathcal{B}_1) f(x_b^{-1} \mathcal{B}_t)] + o(1),$$

as $n \rightarrow \infty$, since $m \mathbb{E}[f^2(x_b^{-1} \mathcal{B}_1)]/k \rightarrow \mathbb{E}[f^2(Y^{\mathbf{Q}(p)})]$, as $n \rightarrow \infty$, by the moment assumption in (3.21). Moreover,

$$\begin{aligned} I &= \frac{m}{k} \mathbb{E}[g(x_b^{-1} \mathcal{B}_1) \mathbb{1}(g(x_b^{-1} \mathcal{B}_1) > \eta) f(x_b^{-1} \mathcal{B}_t)] \\ &\leq \frac{m}{k} \mathbb{E}[g(x_b^{-1} \mathcal{B}_1)^2 \mathbb{1}(g(x_b^{-1} \mathcal{B}_1) > \eta)]^{1/2} \mathbb{E}[f(x_b^{-1} \mathcal{B}_1)^2]^{1/2} \\ &\leq \frac{m}{k} (\eta)^{-\delta/2} \mathbb{E}[g(x_b^{-1} \mathcal{B}_1)^{2+\delta}]^{\frac{1}{2+\delta}} \mathbb{E}[g(x_b^{-1} \mathcal{B}_1)^{2+\delta}]^{\frac{\delta}{2(2+\delta)}} \mathbb{E}[f(x_b^{-1} \mathcal{B}_1)^2]^{1/2}. \end{aligned}$$

We deduce from Equation (3.21) that $I = O(\eta^{-\delta/2})$, as $n \rightarrow \infty$, thus this term is negligible letting $n \rightarrow \infty$, and then $\eta \rightarrow \infty$. Therefore,

$$(D.91) \quad \begin{aligned} &\lim_{n \rightarrow \infty} \frac{m}{k} \mathbb{E}[g(x_b^{-1} \mathcal{B}_1) f(x_b^{-1} \mathcal{B}_t)] \\ &= \lim_{\eta \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{m}{k} \mathbb{E}[(g \wedge \eta)(x_b^{-1} \mathcal{B}_1) (f \wedge \eta)(x_b^{-1} \mathcal{B}_t)]. \end{aligned}$$

We conclude that it suffices to establish (D.89) for continuous bounded functions. We consider Lipschitz-continuous bounded functions $f, f' : \tilde{\ell}^p \rightarrow \mathbb{R}$ in $\mathcal{G}_+(\tilde{\ell}^p)$. The extension to continuous bounded functions then holds following a Portmanteau argument. Now notice

$$\begin{aligned} &\frac{m}{k} \mathbb{E}[f(x_b^{-1} \mathcal{B}_1) f'(x_b^{-1} \mathcal{B}_t)] \\ &= \frac{m}{k} \mathbb{E}[f(\underline{x_b^{-1} \mathcal{B}_1}_\epsilon) f'(x_b^{-1} \mathcal{B}_t)] + \frac{m}{k} \mathbb{E}[(f(x_b^{-1} \mathcal{B}_1) - f(\underline{x_b^{-1} \mathcal{B}_1}_\epsilon)) f'(x_b^{-1} \mathcal{B}_t)] \end{aligned}$$

For the second term, we rely on condition \mathbf{CS}_p since f, f' are bounded Lipschitz-continuous functions. In this case, the second term is negligible letting first $n \rightarrow \infty$ and then $\epsilon \downarrow 0$. Similarly, we deduce from condition

\mathbf{CS}_p that for $t = 2, 3, \dots$, and for all $\epsilon > 0$,

$$(D.92) \quad \frac{m}{k} \mathbb{E}[f(x_b^{-1} \mathcal{B}_1) f'(x_b^{-1} \mathcal{B}_t)] \sim \frac{m}{k} \mathbb{E}[f(x_b^{-1} \mathcal{B}_{1_\epsilon}) f'(x_b^{-1} \mathcal{B}_{t_\epsilon})],$$

if we let $n \rightarrow \infty$, and then $\epsilon \downarrow 0$. Thus it suffices to show that

$$\lim_{n \rightarrow \infty} \frac{m}{k} \text{Cov}(f_\epsilon(x_b^{-1} \mathcal{B}_1), f'_\epsilon(x_b^{-1} \mathcal{B}_t)) = 0, \quad t = 2, 3, \dots$$

Consider the sequence (ℓ_n) satisfying the condition of Lemma D.1, and recall the notation $\mathcal{B}_{t,\ell} = \mathbf{X}_{(t-1)b+[1,b-\ell]}$. We can use similar steps as in the proof of Lemma A.1 and replace \mathcal{B}_1 by $\mathcal{B}_{1,\ell}$ inside the covariance term. For this step we require $\ell_n/b_n \rightarrow 0$ as $n \rightarrow \infty$. Moreover,

$$\frac{m}{k} \text{Cov}(f_\epsilon(x_b^{-1} \mathcal{B}_{1,\ell}), f'_\epsilon(x_b^{-1} \mathcal{B}_t)) \leq 2 \|f\|_\infty \|f'\|_\infty \frac{m}{k} \beta_{\ell+(t-2)b}.$$

Finally, we have $m_n \beta_{\ell_n}/k_n \rightarrow 0$, as $n \rightarrow \infty$. In the case where g or both g, h , equal one of the functionals defining the Hill and exceedances estimator, i.e., h, e , in (2.15) and (2.16), the proof follows similar steps and thus we omit the details. In this case we use (D.88) instead of (D.91) in the first step of the proof. \square

Proof of Proposition D.2. Let $g, f : \tilde{\ell}^p \rightarrow \mathbb{R}$, be two functions verifying \mathbf{L} . Then,

$$\begin{aligned} & k \text{Cov}\left(\frac{1}{k} \sum_{t=1}^m g(x_b^{-1} \mathcal{B}_t), \frac{1}{k} \sum_{t=1}^m f(x_b^{-1} \mathcal{B}_t)\right) \\ &= \frac{m}{k} \text{Cov}[g(x_b^{-1} \mathcal{B}_1), f(x_b^{-1} \mathcal{B}_1)] + \frac{2}{k} \sum_{1 \leq t < j \leq m} \text{Cov}(g(x_b^{-1} \mathcal{B}_t), f(x_b^{-1} \mathcal{B}_j)) \\ &=: I + II. \end{aligned}$$

A direct corollary of Theorem 1.1. in [39], directly stated in equation (1.12b) therein, yields for $j = 3, 4, \dots$, and $\delta > 0$,

$$\begin{aligned} & \frac{m}{k} \text{Cov}(g(x_b^{-1} \mathcal{B}_1) f(x_b^{-1} \mathcal{B}_j)) \\ & \leq 2 \frac{m}{k} \beta_{(j-2)b_n}^{\frac{\delta}{(2+\delta)}} \mathbb{E}[g(x_b^{-1} \mathcal{B}_1)^{2+\delta}]^{\frac{1}{2+\delta}} \mathbb{E}[f(x_b^{-1} \mathcal{B}_1)^{2+\delta}]^{\frac{1}{2+\delta}}, \\ & = O\left(\left(\frac{m}{k} \beta_{(j-2)b_n}\right)^{\frac{\delta}{(2+\delta)}}\right), \quad n \rightarrow \infty. \end{aligned}$$

The last equality holds by condition (3.21). The above inequality can be extended to bounded random variables letting $\delta \uparrow \infty$. Finally, for $j = 2$ we use the result in Lemma D.1. To sum up, we have also shown that if $\sum_{t=1}^{m_n} (m_n \beta_{tb_n}/k_n)^{\delta/(2+\delta)} \rightarrow 0$ then $II \rightarrow 0$, as $n \rightarrow \infty$.

Therefore this shows

$$\begin{aligned} I &= \frac{m}{k} \text{Var}(g(x_b^{-1} \mathcal{B}_1) f(x_b^{-1} \mathcal{B}_1)) \\ &\rightarrow c(g, h) := \int_0^\infty g(y \mathbf{Q}^{(p)}) f(y \mathbf{Q}^{(p)}) d(-y^{-\alpha}), \quad n \rightarrow \infty. \end{aligned}$$

To conclude, we now consider the functions $g(\mathbf{x}) = g(\mathbf{x}_t/u)\mathbb{1}(\|\mathbf{x}_t\|_q > u)$ and $f(\mathbf{x}) = f(\mathbf{x}_t/v)\mathbb{1}(\|\mathbf{x}_t\|_q > v)$. Then, an application of the change-of-norms formula in Equation (2.12) yields the limit covariance in Equation (B.70).

Finally, the case where g or both g, h , equal one of the functionals defining the Hill and exceedances estimator, i.e., h, e , in (2.15) and (2.16), follows in a similar manner now relying on (D.88) thus we omit the details. \square

Proof of Proposition B.1. We show the result for $g \in \mathcal{F}_f$ as the proof in the case $g \in \mathcal{F}_h$ follows the same line of arguments. Let $u_0 = 1 - \epsilon < 1 < 1 + \epsilon = s_0$ and $q_0 = \alpha - \epsilon < \alpha < \alpha + \epsilon = q'_0$. We argue using a copuling argument and design recursively coupled blocks $(\mathcal{B}_t^*)_{1 \leq t \leq m_n}$ as follows: for every $t = 1, \dots, m_n$, we apply the maximal coupling Theorem 5.1 in [39] to create the block (\mathcal{B}_t^*) independent of the past blocks $(\mathcal{B}_j, \mathcal{B}_j^*)_{j < t}$, distributed as \mathcal{B}_1 , and such that

$$(D.93) \quad \mathbb{P}(\mathcal{B}_{t,\ell} \neq \mathcal{B}_{t,\ell}^*) = \beta_{\ell_n}, \quad t = 1, 2, \dots, m_n,$$

with the notation $\mathcal{B}_{t,\ell} = \mathbf{X}_{[tb+\ell, (t+1)b]}$.

Now, applying the Markov inequality of order 2 yields

$$\begin{aligned} & \mathbb{P}\left(\sup_{u,q} \sqrt{k_n} |\tilde{g}^{\mathbf{Q}}(u, q) - \tilde{g}_{*,*}^{\mathbf{Q}}(u, q)| > \delta\right) \\ & \leq \frac{1}{k_n \delta^2} \mathbb{E}\left[\sup_{u,q} \left(\sum_{t=1}^{m_n} (g(\mathcal{B}_t/(u x_b))\mathbb{1}(\|\mathcal{B}_t\|_q > (u x_b)) \right. \right. \\ & \quad \left. \left. - g(\mathcal{B}_t^*/(u x_b))\mathbb{1}(\|\mathcal{B}_t^*\|_q > (u x_b)))\right)^2\right]. \end{aligned}$$

We develop the square and obtain a diagonal term

$$\frac{m_n}{k_n} \mathbb{E}\left[\sup_{u,q} (g(\mathcal{B}_1/(u x_b))\mathbb{1}(\|\mathcal{B}_1\|_q > (u x_b)) - g(\mathcal{B}_1^*/(u x_b))\mathbb{1}(\|\mathcal{B}_1^*\|_q > (u x_b)))^2\right].$$

Note that, denoting G the envelop function,

$$(D.94) \quad \begin{aligned} G(\mathcal{B}_1/x_b) & := \sup_{u \in [u_0, s_0], q \in [q_0, q'_0]} g(\mathcal{B}_1/(u x_b))\mathbb{1}(\|\mathcal{B}_1\|_q > (u x_b)) \\ & = g(\mathcal{B}_1/(u_0 x_b))\mathbb{1}(\|\mathcal{B}_1\|_{q_0} > (u_0 x_b)). \end{aligned}$$

Then, by a similar argument as in the proof of Lemma D.1, we can assume the function g is non-negative bounded and thus we can truncate g to g_ε .

This means we now need to control

$$\begin{aligned}
 & \frac{m_n}{k_n} \mathbb{E} \left[\sup_{u,q} (g_\varepsilon(\mathcal{B}_1/(ux_b)) \mathbb{1}(\|\mathcal{B}_1\|_q > (ux_b)) \right. \\
 & \quad \left. - g_\varepsilon(\mathcal{B}_1^*/(ux_b)) \mathbb{1}(\|\mathcal{B}_1^*\|_q > (ux_b)))^2 \right] \\
 &= \frac{m_n}{k_n} \mathbb{E} \left[\sup_{u,q} (g_\varepsilon(\mathcal{B}_{1,\ell}/(ux_b)) \mathbb{1}(\|\mathcal{B}_{1,\ell}\|_q > (ux_b)) \right. \\
 & \quad \left. - g_\varepsilon(\mathcal{B}_{1,\ell}^*/(ux_b)) \mathbb{1}(\|\mathcal{B}_{1,\ell}^*\|_q > (ux_b)))^2 \right] + o(1) \\
 &= \frac{m_n}{k_n} E[G_\varepsilon(\mathcal{B}_{1,\ell}/x_b)^2 \mathbb{1}(\mathcal{B}_{1,\ell} \neq \mathcal{B}_{1,\ell}^*)] + o(1),
 \end{aligned}$$

as $n \rightarrow \infty$, here the second to-last relation holds with the notation $\mathcal{B}_{1,\ell} = \mathbf{X}_{[\ell,b]}$, again relying on the calculation of Lemma D.1. In the last equation, G_ε is again the envelope function in (D.94), but for the function g_ε . The gap of length ℓ allows to use the maximal coupling theorem, notably, Equation (D.93). Thus

$$\begin{aligned}
 & \mathbb{E}[G_\varepsilon(\mathcal{B}_1/x_b)^2 \mathbb{1}(\mathcal{B}_{1,\ell} \neq \mathcal{B}_{1,\ell}^*)] \\
 & \leq \frac{m_n}{k_n} \mathbb{E}[G_\varepsilon(\mathcal{B}_1/x_b)^{2+\delta}]^{2/(2+\delta)} \mathbb{P}(\mathcal{B}_{1,\ell} \neq \mathcal{B}_{1,\ell}^*)^{\delta/(2+\delta)} \\
 & \leq \frac{m_n}{k_n} \mathbb{P}(\|\mathcal{B}_1\| > x_{b_n})^{2/(2+\delta)} \beta_\ell^{\delta/(2+\delta)} \sim (m_n \beta_\ell / k_n)^{\delta/(2+\delta)},
 \end{aligned}$$

as $n \rightarrow \infty$, if G satisfies the assumption **L**.

We now focus on the crossed terms in the development of the square. Note we assume without loss of generality that g is non-negative by considering separately the positive and negative parts. Then note that for $j \neq t$,

$$\begin{aligned}
 & \mathbb{E} \left[\sup_{u,q} \left(g(\mathcal{B}_t/(ux_b)) \mathbb{1}(\|\mathcal{B}_t\|_q > (ux_b)) - g(\mathcal{B}_t^*/(ux_b)) \mathbb{1}(\|\mathcal{B}_t^*\|_q > (ux_b)) \right) \right. \\
 & \quad \left. \times \left(g(\mathcal{B}_j/(ux_b)) \mathbb{1}(\|\mathcal{B}_j\|_q > (ux_b)) - g(\mathcal{B}_j^*/(ux_b)) \mathbb{1}(\|\mathcal{B}_j^*\|_q > (ux_b)) \right) \right] \\
 & \leq \mathbb{E} \left[\sup_{u,q} \left(g(\mathcal{B}_t/(ux_b)) \mathbb{1}(\|\mathcal{B}_t\|_q > (ux_b)) g(\mathcal{B}_j/(ux_b)) \mathbb{1}(\|\mathcal{B}_j\|_q > (ux_b)) \right) \right. \\
 & \quad \left. + \left(g(\mathcal{B}_t^*/(ux_b)) \mathbb{1}(\|\mathcal{B}_t^*\|_q > (ux_b)) g(\mathcal{B}_j^*/(ux_b)) \mathbb{1}(\|\mathcal{B}_j^*\|_q > (ux_b)) \right) \right] \\
 & = \mathbb{E} \left[\sup_{u,q} \left(g(\mathcal{B}_t/(ux_b)) \mathbb{1}(\|\mathcal{B}_t\|_q > (ux_b)) g(\mathcal{B}_j^*/(ux_b)) \mathbb{1}(\|\mathcal{B}_j^*\|_q > (ux_b)) \right) \right],
 \end{aligned}$$

where the last equality follows by the independence of blocks \mathcal{B}_t^* and \mathcal{B}_j^* for $j \neq t$. Finally, the remaining term of the development on the crossed-terms

is of the form

$$\begin{aligned} & \frac{1}{k_n} \sum_{1 \leq j \neq t \leq m} \mathbb{E} \left[\sup_{u,q} g(\mathcal{B}_t/(u x_b)) \mathbb{1}(\|\mathcal{B}_t\|_q > (u x_b)) \right. \\ & \quad \left. \times \sup_{u,q} g(\mathcal{B}_j/(u x_b)) \mathbb{1}(\|\mathcal{B}_j\|_q > (u x_b)) \right] \\ & \leq \frac{m_n}{k_n} \sum_{t=2}^m \mathbb{E}[G(\mathcal{B}_t/x_b)G(\mathcal{B}_1/x_b)]. \end{aligned}$$

This term is bounded by $\sum_{t=1}^{m_n} (m_n \beta_{tb_n}/k_n)^{\delta/(2+\delta)}$ in a similar way as the covariance term is handled in the proof of Proposition D.2. Then, to sum up, provided \mathbf{MX}_β holds, we obtain

$$\mathbb{P} \left(\sup_{u,q} \sqrt{k_n} |\tilde{g}^{\mathbf{Q}}(u, q) - \tilde{g}_{*,*}^{\mathbf{Q}}(u, q)| > \delta \right) \rightarrow 0, \quad n \rightarrow \infty,$$

and this yields the desired result and concludes the proof of Proposition B.1. \square

D.3. Proof of Lemma B.3. An application of Riesz-Thorin Theorem implies that for every fixed $\mathbf{x} \in \tilde{\ell}^{q_0}$, the map

$$(D.95) \quad 1/q \mapsto \log \|\mathbf{x}\|_q, \quad q \in (q_0, q'_0).$$

is convex. Denote $\mathbf{x} \mapsto \psi_q(\mathbf{x})$ the derivative of the function $1/q \mapsto \log \|\mathbf{x}\|_q$ which is defined by

$$\begin{aligned} \psi_q(\mathbf{x}) &= \log(\|\mathbf{x}\|_q^q) - \sum_{t \in \mathbb{Z}} \frac{|\mathbf{x}_t|^q \log(|\mathbf{x}_t|^q)}{\|\mathbf{x}\|_q^q} \\ &= \sum_{t \in \mathbb{Z}} \frac{|\mathbf{x}_t|^q \log(\|\mathbf{x}\|_q^q/|\mathbf{x}_t|^q)}{\|\mathbf{x}\|_q^q}, \end{aligned}$$

which corresponds to the derivative of the $\log \ell^q$ -norm function as in (D.95). It is easy to see that $\psi_q(\mathbf{x}) \geq 0$. Moreover, $1/q \mapsto \psi_q(\mathbf{x})$ is a non-decreasing function. To verify this, it is enough to compute the derivative of the function $1/q \mapsto \psi_q(\mathbf{x})$. Moreover, the convexity of the mapping in (D.95) implies, for all $q_1, q_2 \in (q_0, q'_0)$,

$$\log \|\mathbf{x}\|_{q_1} \geq \log \|\mathbf{x}\|_{q_2} + (q_1^{-1} - q_2^{-1})\psi_{q_2}(\mathbf{x}).$$

Then, rewriting this previous relation yields

$$(D.96) \quad \|\mathbf{x}\|_{q_1} \geq \|\mathbf{x}\|_{q_2} \exp\{(q_1^{-1} - q_2^{-1})\psi_{q_2}(\mathbf{x})\}.$$

It is easy to see that the VC-dimension of our class is larger than two. For example, we can take the point with $x_t^1 = 1$, only if $t = 0$, and the point $x_t^2 = 1/m$, only if $t = 0, 1$, and $2^{1/q_0} < m < 2^{1/q_0'}$. Then, we check readily that $\|\mathbf{x}^1\|_q = 1$ for all $q \in \mathbb{R}$, and $\|\mathbf{x}^2\|_{q_0} > 1 > \|\mathbf{x}^2\|_{q_1}$. Therefore, we conclude that our class of sets separates these two points.

We now show that our class of sets can't shatter three different points. Consider the points \mathbf{x}^1 , \mathbf{x}^2 , and \mathbf{x}^3 with values in $\tilde{\ell}^{q_0}$. Assume that there

exists $q_1, q_2 \in [q_0, q'_0]$ such that $\|\mathbf{x}^1\|_{q_1} > \|\mathbf{x}^2\|_{q_1}$ and $\|\mathbf{x}^2\|_{q_2} > \|\mathbf{x}^1\|_{q_2}$. This means our class picks out the sets $\{\mathbf{x}^1\}$ and $\{\mathbf{x}^2\}$. Assume also, without loss of generality, $q_1 < q_2$. Actually, this means that for all q such that $q_1 < q_2 \leq q$, then $\|\mathbf{x}^2\|_q > \|\mathbf{x}^1\|_q$. Indeed, for $q \geq q_2$ Equation (D.96) implies

$$\begin{aligned} \|\mathbf{x}^1\|_q &\geq \|\mathbf{x}^1\|_{q_1} \exp\{(q^{-1} - q_1^{-1})\psi_{q_1}(\mathbf{x}^1)\} \\ &\geq \|\mathbf{x}^2\|_{q_1} \exp\{(q^{-1} - q_1^{-1})\psi_{q_1}(\mathbf{x}^1)\} \\ &\geq \|\mathbf{x}^2\|_q \exp\{(q^{-1} - q_1^{-1})(\psi_{q_1}(\mathbf{x}^1) - \psi_q(\mathbf{x}^2))\} \\ &\geq \|\mathbf{x}^2\|_q \exp\{(q^{-1} - q_1^{-1})(\psi_{q_1}(\mathbf{x}^1) - \psi_{q_2}(\mathbf{x}^2))\}, \end{aligned}$$

where the last equality holds since $1/q \mapsto \psi_q$ is a non-decreasing function and $q_1 < q_2 \leq q$. In particular, letting $q = q_2$ then the relation $\|\mathbf{x}^2\|_{q_2} > \|\mathbf{x}^1\|_{q_2}$ implies

$$(\psi_{q_1}(\mathbf{x}^1) - \psi_{q_2}(\mathbf{x}^2)) > 0,$$

Moreover, we also have

$$\begin{aligned} \|\mathbf{x}^2\|_q &\geq \|\mathbf{x}^2\|_{q_2} \exp\{(q^{-1} - q_2^{-1})\psi_{q_2}(\mathbf{x}^2)\} \\ &\geq \|\mathbf{x}^1\|_{q_2} \exp\{(q^{-1} - q_2^{-1})\psi_{q_2}(\mathbf{x}^2)\} \\ &\geq \|\mathbf{x}^1\|_q \exp\{(q^{-1} - q_2^{-1})(\psi_{q_2}(\mathbf{x}^2) - \psi_q(\mathbf{x}^1))\} \\ &\geq \|\mathbf{x}^1\|_q \exp\{(q^{-1} - q_2^{-1})(\psi_{q_2}(\mathbf{x}^2) - \psi_{q_1}(\mathbf{x}^1))\}, \end{aligned}$$

but this implies $\|\mathbf{x}^2\|_q > \|\mathbf{x}^1\|_q$, for all q satisfying $q_1 < q_2 \leq q$. Similarly $\|\mathbf{x}^2\|_q < \|\mathbf{x}^1\|_q$, for all q satisfying $q \leq q_1 < q_2$. Assume we can shatter the points $\mathbf{x}^1, \mathbf{x}^2$, and \mathbf{x}^3 . Then there exist $q_i > 0$ satisfying

$$\|\mathbf{x}^i\|_{q_i} > \max\{\|\mathbf{x}^j\|_{q_i}, \|\mathbf{x}^l\|_{q_i}, l, j \neq i\},$$

for every permutation $\{i, l, j\} = \{1, 2, 3\}$. Then, eventually renaming the points, we can suppose without loss of generality $q_1 < q_2 < q_3$. In this case, we claim we can't pick out the set $\{\mathbf{x}^1, \mathbf{x}^3\}$. Indeed, for $q \geq q_2 > q_1$ then $\|\mathbf{x}^1\|_q < \|\mathbf{x}^2\|_q$ and for $q \leq q_2 < q_3$ then $\|\mathbf{x}^2\|_q < \|\mathbf{x}^3\|_q$. \square

APPENDIX E. APPENDIX ON VARIANCE CALCULATIONS

In this Section we compute the Gaussian limit of the Hill and $\ell^{\hat{\alpha}}$ -cluster block estimators under the assumptions of Theorem 3.2. All proofs are deferred to the end of this Section. Recall the Gaussian process

$$\mathbb{G} = (\mathbb{G}_q(g(\cdot/u)),$$

defined in (B.69), and indexed by (u, q, g) such that $u \in [1 - \epsilon, 1 + \epsilon]$, $q \in [\alpha - \epsilon, \alpha + \epsilon]$, and $g \in \mathcal{F}$ as in (B.60). Under the assumptions of Theorem 3.2, and assuming $k/k' \rightarrow \kappa$ and $\kappa > 0$, as $n \rightarrow \infty$, then we have shown in Section B.5 that

$$(E.97) \quad \left\{ \sqrt{k}(\widehat{g}^{\mathbf{Q}}(u, q) - c(q)u^{-\alpha}g^{\mathbf{Q}}(q)) \right\}_{\substack{u \in [1-\epsilon, 1+\epsilon], \\ q \in [\alpha-\epsilon, \alpha+\epsilon], \\ g \in \mathcal{F}}} \xrightarrow{d} \mathbb{G}_\alpha(g(\cdot/u)),$$

uniformly, as $n \rightarrow \infty$. Instead, if $k/k' \rightarrow 0$, as $n \rightarrow \infty$, then the previous result continues to hold with \mathcal{F} replaced by \mathcal{F}_f (see (B.60)) in the previous limit.

The next Lemmas allow us to compute the variance of the Gaussian limit of the Hill and $\ell^{\hat{\alpha}}$ -block estimator.

Lemma E.1 (Asymptotics of the Hill estimator $1/\hat{\alpha}$). *Assume the conditions of Theorem 3.2 are satisfied and $k/k' \rightarrow \kappa$, for $\kappa > 0$. Recall the Hill estimator $1/\hat{\alpha}$ in (2.14), and the functionals h, e , in (2.15) and (2.16). Then,*

$$(E.98) \quad \sqrt{k}(\hat{\alpha} - \alpha) \xrightarrow{d} \alpha \mathbb{G}_\alpha(\alpha h(\cdot/1) - e(\cdot/1)),$$

as $n \rightarrow \infty$.

Lemma E.2 (Asymptotics of the random threshold $\|\mathcal{B}_1\|_{\hat{\alpha},(k)}$). *Assume the conditions of Theorem 3.2 are satisfied and $k/k' \rightarrow \kappa$, for $\kappa \geq 0$.*

$$\sqrt{k}(f_\alpha^{\mathbf{Q}}(\hat{\alpha}) - f_\alpha^{\mathbf{Q}}) \xrightarrow{d} f_\alpha^{\mathbf{Q}} \mathbb{G}_\alpha(f_\alpha(\cdot/1)/f_\alpha^{\mathbf{Q}} - 1(\cdot/1)),$$

as $n \rightarrow \infty$.

E.1. Proof of Lemma E.1. Uniform convergence of the independent-block estimators towards the Gaussian process entails

$$\sqrt{k}((\tilde{h}^{\mathbf{Q}}(u), \tilde{e}^{\mathbf{Q}}(1)) - (u^{-\alpha} h^{\mathbf{Q}}, 1)) \xrightarrow{d} (\mathbb{G}_\alpha(h(\cdot/u)), \mathbb{G}_\alpha(e(\cdot/1))),$$

as $n \rightarrow \infty$. Now notice the identity

$$\sqrt{k}(|\mathbf{X}/x'_{b_n}|_{(\lfloor k'u \rfloor)} - u^{-1/\alpha}) = \sqrt{k}(\tilde{e}^{\mathbf{Q}}(u)^{\leftarrow} - (u^{-\alpha})^{\leftarrow}), \quad u \in [1 - \epsilon, 1 + \epsilon],$$

where $|\mathbf{X}/x'_{b_n}|_{(1)} \geq |\mathbf{X}/x'_{b_n}|_{(2)} \geq \dots \geq |\mathbf{X}/x'_{b_n}|_{(n)}$, are the order statistics of the sample $(|\mathbf{X}_t/x'_{b_n}|)$. Then, by an application of Vervaat's lemma,

$$\sqrt{k}(|\mathbf{X}/x'_{b_n}|_{(\lfloor k' \rfloor)} - 1) \xrightarrow{d} -\alpha^{-1} \mathbb{G}_\alpha(e(\cdot/1)),$$

as $n \rightarrow \infty$, in particular, $|\mathbf{X}/x'_{b_n}|_{(\lfloor k' \rfloor)} \xrightarrow{\mathbb{P}} 1$. Furthermore, denoting $1/\hat{\alpha}$ the Hill estimator in Equation (2.14),

$$\begin{aligned} \sqrt{k} \left(\frac{1}{\hat{\alpha}} - \frac{1}{\alpha} \right) &= \sqrt{k} \alpha^{-1} (\alpha \tilde{h}^{\mathbf{Q}}(|\mathbf{X}|_{\lfloor k' \rfloor} / x'_{b_n}) - 1) \\ &= \sqrt{k} \alpha^{-1} (\alpha \tilde{h}^{\mathbf{Q}}(|\mathbf{X}/x'_{b_n}|_{\lfloor k' \rfloor}) - (|\mathbf{X}/x'_{b_n}|_{\lfloor k' \rfloor})^{-\alpha}) \\ &\quad + \sqrt{k} \alpha^{-1} ((|\mathbf{X}|_{\lfloor k' \rfloor} / x'_{b_n})^{-\alpha} - 1) \\ &\xrightarrow{d} \alpha^{-1} \mathbb{G}_\alpha(\alpha h(\cdot/1) - e(\cdot/1)), \quad n \rightarrow \infty. \end{aligned}$$

Finally, an application of the Delta method yields Equation (E.98) allow us to conclude. \square

E.2. Proof of Lemma E.2. The following Lemma will be useful to proof to complete the proof of Lemma E.2. We defer its proof to the end of this Section.

Lemma E.3 (Asymptotics of $c(\hat{\alpha})$). *Assume the conditions of Theorem 3.2 are satisfied and $k/k' \rightarrow \kappa$, for $\kappa > 0$,*

$$\sqrt{k}(c(\hat{\alpha}) - 1) \xrightarrow{d} -\alpha\psi_\alpha^{\mathbf{Q}} \mathbb{G}_\alpha(h(\cdot/1) - \alpha^{-1}e(\cdot/1)), \quad n \rightarrow \infty,$$

and, for $\kappa = 0$, $\sqrt{k}(c(\hat{\alpha}) - 1) \xrightarrow{\mathbb{P}} 0$, as $n \rightarrow \infty$. Here

$$(E.99) \quad \psi_\alpha^{\mathbf{Q}} = \mathbb{E} \left[\sum_{t \in \mathbb{Z}} |\mathbf{Q}_t|^\alpha \log \left(\frac{\|\mathbf{Q}\|_\alpha}{|\mathbf{Q}_t|} \right) \right],$$

and $\psi_\alpha^{\mathbf{Q}} < \infty$ due to condition **M**.

Under the assumptions of Theorem 3.2 we have $\hat{\alpha} \xrightarrow{\mathbb{P}} \alpha$, as $n \rightarrow \infty$. Moreover, the uniform limit in (E.97) entails

$$\sqrt{k}(\tilde{\Gamma}^{\mathbf{Q}}(u, \hat{\alpha}) - u^{-\alpha}c(\hat{\alpha})) \xrightarrow{d} \mathbb{G}_\alpha(1(\cdot/u)), \quad n \rightarrow \infty.$$

Hence, if $k/k' \rightarrow \kappa$ and $\kappa > 0$, an application of Lemma E.3 implies

$$\begin{aligned} & \sqrt{k}(\tilde{\Gamma}^{\mathbf{Q}}(u, \hat{\alpha}) - u^{-\alpha}) \\ &= \sqrt{k}(\tilde{\Gamma}^{\mathbf{Q}}(u, \hat{\alpha}) - u^{-\alpha}c(\hat{\alpha})) - u^{-\alpha}\sqrt{k}(1 - c(\hat{\alpha})) \\ &\xrightarrow{d} \mathbb{G}_\alpha(1(\cdot/u) - u^{-\alpha}\alpha\psi_\alpha^{\mathbf{Q}}(h(\cdot/1) - \alpha^{-1}e(\cdot/1))), \quad n \rightarrow \infty. \end{aligned}$$

Instead, if $k/k' \rightarrow 0$, we conclude

$$\sqrt{k}(\tilde{\Gamma}^{\mathbf{Q}}(u, \hat{\alpha}) - u^{-\alpha}) \xrightarrow{d} \mathbb{G}_\alpha(1(\cdot/u)), \quad n \rightarrow \infty.$$

In addition, recall the sequence (x_{b_n}) satisfies the assumptions of Proposition 2.2, and (k_n) satisfies $k = k(\alpha) \sim b_n \mathbb{P}(|\mathbf{X}_1| > x_{b_n})$, as $n \rightarrow \infty$. We derive the relation, as $n \rightarrow \infty$,

$$\sqrt{k}(\tilde{\mathbf{Q}}(u, \hat{\alpha})^\leftarrow - (u^{-\alpha})^\leftarrow) = \sqrt{k}(\|\mathcal{B}_1/x_{b_n}\|_{\hat{\alpha}, (\lfloor ku \rfloor)} - u^{-1/\alpha}) + o_{\mathbb{P}}(1),$$

for $u \in [1 - \epsilon, 1 + \epsilon]$. Then, by an application of Vervaat's lemma,

$$\sqrt{k}(\|\mathcal{B}_1/x_{b_n}\|_{\hat{\alpha}, (\lfloor k \rfloor)} - 1) \xrightarrow{d} -\alpha^{-1}\mathbb{G}_\alpha(1(\cdot/1) - \alpha\psi_\alpha^{\mathbf{Q}}(\cdot/1) - \alpha^{-1}e(\cdot/1)),$$

as $n \rightarrow \infty$. These calculations hold in the case $k/k' \rightarrow \kappa$ and $\kappa > 0$. Instead, if $k/k' \rightarrow 0$ a similar argument allow as to conclude

$$\sqrt{k}(\|\mathcal{B}_1/x_{b_n}\|_{\hat{\alpha}, (\lfloor k \rfloor)} - 1) \xrightarrow{d} -\alpha^{-1}\mathbb{G}_\alpha(1(\cdot/1)), \quad n \rightarrow \infty.$$

In particular, $\|\mathcal{B}_1\|_{\hat{\alpha}, (\lfloor k \rfloor)}/x_{b_n} \xrightarrow{\mathbb{P}} 1$ for every $\kappa \geq 0$.

Moreover, since f_α is a continuous non-increasing function, and $c(\hat{\alpha}) \xrightarrow{\mathbb{P}} 1$, similar calculations entail

$$(E.100) \quad \begin{aligned} & \sqrt{k}(f_\alpha^{\mathbf{Q}}(u, \hat{\alpha}) - u^{-\alpha}f_\alpha^{\mathbf{Q}}) \\ &\xrightarrow{d} \mathbb{G}_\alpha(f_\alpha(\cdot/u) - f_\alpha^{\mathbf{Q}}u^{-\alpha}\alpha\psi_\alpha^{\mathbf{Q}}(h(\cdot/1) - \alpha^{-1}e(\cdot/1))). \end{aligned}$$

Overall, we conclude

$$\begin{aligned}
& \sqrt{k} \left(\widehat{f}_\alpha^{\mathbf{Q}}(\widehat{\alpha}) - f_\alpha^{\mathbf{Q}} \right) \\
&= \sqrt{k} \left(\widehat{f}_\alpha^{\mathbf{Q}}(\|\mathcal{B}_1\|_{\widehat{\alpha},(\lfloor k \rfloor)}/x_{b_n}, \widehat{\alpha}) - f_\alpha^{\mathbf{Q}} \mathbf{1} \right) \\
&= \sqrt{k} \left(\widehat{f}_\alpha^{\mathbf{Q}}(\|\mathcal{B}_1\|_{\widehat{\alpha},(\lfloor k \rfloor)}/x_{b_n}, \widehat{\alpha}) - f_\alpha^{\mathbf{Q}}(\|\mathcal{B}_1\|_{\widehat{\alpha},(\lfloor k \rfloor)}/x_{b_n})^{-\alpha} \right) \\
&\quad + \sqrt{k} f_\alpha^{\mathbf{Q}} \left((\|\mathcal{B}_1\|_{\widehat{\alpha},(\lfloor k \rfloor)}/x_{b_n})^{-\alpha} - 1 \right) \\
&\xrightarrow{d} \mathbb{G}_\alpha(f_\alpha(\cdot/1) - f_\alpha^{\mathbf{Q}} \mathbf{1}(\cdot/1)), \quad n \rightarrow \infty,
\end{aligned}$$

and this yields the desired result. \square

E.3. Proof of Lemma E.3. To study the function $q \mapsto c(q)$ in a neighborhood of α we write it as

$$(E.101) \quad c(q) = \mathbb{E}[\|\mathbf{Q}\|_q^\alpha] = (\mathbb{E}[1/\|\mathbf{Q}^{(q)}\|_\alpha^\alpha])^{-1}.$$

Borrowing the Taylor expansions from Remark 4.3 we have

$$\begin{aligned}
& (1/c(q) - 1) \\
&= \mathbb{E}[\|\mathbf{Q}^{(q)}\|_q^q - \|\mathbf{Q}^{(q)}\|_\alpha^\alpha / \|\mathbf{Q}^{(q)}\|_\alpha^\alpha] \\
&= (q - \alpha) \mathbb{E} \left[\sum_{t \in \mathbb{Z}} |\mathbf{Q}_t^{(q)}|^\alpha \log(1/|\mathbf{Q}_t^{(q)}|) / \|\mathbf{Q}^{(q)}\|_\alpha^\alpha \right] + \frac{1}{2}(q - \alpha)^2 \\
&\quad \times \mathbb{E} \left[\frac{1}{\|\mathbf{Q}^{(q)}\|_{q'}^{q'}} \sum_{t \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \frac{|\mathbf{Q}_t^{(q)}|^{q'}}{\|\mathbf{Q}^{(q)}\|_{q'}^{q'}} \frac{|\mathbf{Q}_t^{(q)}|^{q'}}{\|\mathbf{Q}^{(q)}\|_{q'}^{q'}} \log\left(\frac{1}{|\mathbf{Q}_t^{(q)}|}\right) \log\left(\frac{|\mathbf{Q}_t^{(q)}|}{|\mathbf{Q}_j^{(q)}|}\right) \right],
\end{aligned}$$

for $q' \in [q \vee \alpha, q \wedge \alpha]$, $q \in [\alpha - \epsilon, \alpha + \epsilon]$, and $\epsilon > 0$ such that **M** holds. Moreover, an application of the change-of-norms from Equation (2.12) yields

$$\begin{aligned}
& (1/c(q) - 1) \\
&= (q - \alpha) \mathbb{E} \left[\underbrace{\|\mathbf{Q}\|_q^\alpha \sum_{t \in \mathbb{Z}} |\mathbf{Q}_t|^\alpha \log\left(\frac{\|\mathbf{Q}\|_q}{|\mathbf{Q}_t|}\right)}_{:= \psi_{q,\alpha}(\mathbf{Q})} \right] / c(q) + \frac{1}{2}(q - \alpha)^2 \\
&\quad \times \mathbb{E} \left[\underbrace{\|\mathbf{Q}\|_q^\alpha \frac{\|\mathbf{Q}\|_q^{q'}}{\|\mathbf{Q}\|_{q'}^{q'}} \sum_{t \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \frac{|\mathbf{Q}_t|^{q'}}{\|\mathbf{Q}\|_{q'}^{q'}} \frac{|\mathbf{Q}_t|^{q'}}{\|\mathbf{Q}\|_{q'}^{q'}} \log\left(\frac{\|\mathbf{Q}\|_q}{|\mathbf{Q}_t|}\right) \log\left(\frac{|\mathbf{Q}_t|}{|\mathbf{Q}_j|}\right)}_{:= \psi'_{q,q'}(\mathbf{Q})} \right] / c(q) \\
&= (q - \alpha) \mathbb{E}[\psi_{q,\alpha}(\mathbf{Q})] / c(q) + \frac{1}{2}(q - \alpha)^2 \mathbb{E}[\psi'_{q,q'}(\mathbf{Q})] / c(q).
\end{aligned}$$

(E.102)

We start by noting that by an application of Equation (B.77) we can derive the bound

$$0 < \mathbb{E} \left[\sup_{q \in [\alpha - \epsilon, \alpha + \epsilon]} \psi_{q,\alpha}(\mathbf{Q}) \right] \leq \epsilon^{-1} \mathbb{E}[\|\mathbf{Q}\|_{\alpha - \epsilon}^{2\alpha}] < \infty,$$

and this last term is finite due to Assumption **M**.

In what follows we verify that $\mathbb{E}[\sup_{q,q'} |\psi'_{q,q'}|] < \infty$ for q and q' close to α . First note that $|\psi'_{q,q'}|$ is non-increasing in q . Thus it is enough to check that

$$\mathbb{E} \left[\sup_{q \in [\alpha - \varepsilon, \alpha + \varepsilon]} \frac{\|\mathbf{Q}\|_{\alpha - \varepsilon}^{q + \alpha}}{\|\mathbf{Q}\|_q^q} \left| \sum_{t \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \frac{|\mathbf{Q}_t|^q |\mathbf{Q}_j|^q}{\|\mathbf{Q}\|_q^q \|\mathbf{Q}\|_q^q} \log \left(\frac{\|\mathbf{Q}\|_{\alpha - \varepsilon}}{|\mathbf{Q}_t|} \right) \log \left(\frac{|\mathbf{Q}_t|}{|\mathbf{Q}_j|} \right) \right] < \infty,$$

for $\varepsilon > 0$ sufficiently small. Moreover Hölder's inequality provides

$$\begin{aligned} \sum_{t=-\infty}^{\infty} |\mathbf{Q}_t|^{\alpha - \varepsilon} &= \sum_{t=-\infty}^{\infty} |\mathbf{Q}_t|^{(\alpha - \varepsilon)q / (q + \alpha)} |\mathbf{Q}_t|^{(\alpha - \varepsilon)\alpha / (q + \alpha)} \\ &\leq \left(\sum_{t=-\infty}^{\infty} |\mathbf{Q}_t|^q \right)^{(\alpha - \varepsilon) / (q + \alpha)} \left(\sum_{t=-\infty}^{\infty} |\mathbf{Q}_t|^{(\alpha - \varepsilon)\alpha / (q + \varepsilon)} \right)^{(q + \varepsilon) / (q + \alpha)}. \end{aligned}$$

Thus $\|\mathbf{Q}\|_{\alpha - \varepsilon}^{q + \alpha} \leq \|\mathbf{Q}\|_q^q \|\mathbf{Q}\|_{(\alpha - \varepsilon)\alpha / (q + \varepsilon)}^\alpha$. For every chosen $0 < \delta < 1$ we have from Equation (B.77), and assuming that every $|\mathbf{Q}_t| \leq 1$ a.s.,

$$\begin{aligned} \sum_{t \in \mathbb{Z}} \frac{|\mathbf{Q}_t|^q}{\|\mathbf{Q}\|_q^q} |\log(|\mathbf{Q}_t|)|^2 &\leq \sum_{t \in \mathbb{Z}} \frac{|\mathbf{Q}_t|^{q - \delta}}{\|\mathbf{Q}\|_q^q} \\ &\leq \frac{(\sum_{t \in \mathbb{Z}} |\mathbf{Q}_t|^q)^{q / (q - \delta)}}{\|\mathbf{Q}\|_q^q} \\ &\leq \left(\sum_{t \in \mathbb{Z}} |\mathbf{Q}_t|^q \right)^{\delta / (q - \delta)}. \end{aligned}$$

Squaring both sides of the last inequality we obtain an upper bound

$$\begin{aligned} \left(\sum_{t \in \mathbb{Z}} \frac{|\mathbf{Q}_t|^q}{\|\mathbf{Q}\|_q^q} |\log(|\mathbf{Q}_t|)|^2 \right)^2 &\leq \left(\sum_{t \in \mathbb{Z}} |\mathbf{Q}_t|^q \right)^{2\delta / (q - \delta)} \\ &\leq \|\mathbf{Q}\|_q^{q2\delta / (q - \delta)} \\ &\leq \|\mathbf{Q}\|_{\alpha - \delta}^{(\alpha + \varepsilon)2\delta / (\alpha - \varepsilon - \delta)}. \end{aligned}$$

This upper bound is also valid for $|\log(\|\mathbf{Q}\|_q)|$, $q - \varepsilon < q + \varepsilon$, up to constant.

Combining these inequalities we conclude the sufficient condition

$$\mathbb{E} \left[\|\mathbf{Q}\|_{(\alpha - \varepsilon)\alpha / (q + \varepsilon)}^\alpha \|\mathbf{Q}\|_{\alpha - \delta}^{(\alpha + \varepsilon)2\delta / (\alpha - \varepsilon - \delta)} \right] < \infty.$$

Since ε and δ can be taken arbitrarily small this allows to conclude

$$0 < \mathbb{E} \left[\sup_{q, q' \in [\alpha - \varepsilon, \alpha + \varepsilon]} \psi'_{q, q'}(\mathbf{Q}) \right] \leq \mathbb{E} \left[\|\mathbf{Q}\|_{\alpha - \varepsilon}^{\alpha + \varepsilon} \right] < \infty,$$

where the last equality follows again from **M** choosing ε sufficiently small.

Now let k, k' be such that $k/k' \rightarrow \kappa$, for $\kappa > 0$. Recall the asymptotic development of $q \mapsto c(q)$ and the definition of $\psi_{q, \alpha}$ in (E.102). We write

$$\psi_\alpha^{\mathbf{Q}} = \mathbb{E}[\psi_{\alpha, \alpha}(\mathbf{Q})] = \mathbb{E} \left[\sum_{t \in \mathbb{Z}} |\mathbf{Q}_t|^\alpha \log \left(\frac{\|\mathbf{Q}\|_\alpha}{|\mathbf{Q}_t|} \right) \right],$$

which corresponds to the constant in (E.99). Moreover, notice $q \mapsto \mathbb{E}[\psi_{q,\alpha}(\mathbf{Q})]$ is a continuous function at α under condition **M**. Therefore, applying (E.98) and (E.102) we obtain

$$\sqrt{k}(1 - c(\hat{\alpha})) \xrightarrow{d} \alpha \psi_{\alpha}^{\mathbf{Q}} \mathbb{G}_{\alpha}(h(\cdot/1) - \alpha^{-1}e(\cdot/1)),$$

We notice in particular $c(\hat{\alpha}) \xrightarrow{\mathbb{P}} 1$, as $n \rightarrow \infty$.

Moreover, if $k/k' \rightarrow 0$, as $n \rightarrow \infty$, then

$$\sqrt{k'}(1 - c(\hat{\alpha})) \xrightarrow{d} \alpha \psi_{\alpha}^{\mathbf{Q}} \mathcal{N}(0, \mathbb{E}[(h(Y\mathbf{Q}) - \alpha^{-1}e(Y\mathbf{Q}))^2]), \quad n \rightarrow \infty.$$

We omit the details here as this mimics the proof provided in Section B.5, but here we restrict the family \mathcal{F} to \mathcal{F}_h following the notation in (B.61).

This last equation implies $\sqrt{k}(1 - c(\hat{\alpha})) \xrightarrow{\mathbb{P}} 0$, as $n \rightarrow \infty$. Finally, this concludes the proof of Lemma E.3. \square

APPENDIX F. PROOFS OF THE RESULTS OF SECTION 5

F.1. Proof of Proposition 5.2. From the discussion in Section 5.1, we can see that all assumptions in Theorem 3.2 are satisfied. Note that if $p = \alpha$, then $|\mathbf{Q}|$ has a deterministic expression in the shift-invariant space. Moreover, \mathbf{Q} has at most m_0 non-zero coordinates. Thereby, condition **M** is satisfied. Moreover, the index estimators in (4.26), (4.29), and (4.34), with $f_{\alpha} : \mathbf{x} \mapsto \|\mathbf{x}\|_{\infty}^{\alpha} / \|\mathbf{x}\|_{\alpha}^{\alpha}$, $f_{\alpha} : \mathbf{x} \mapsto \|\mathbf{x}\|_1^{\alpha} / \|\mathbf{x}\|_{\alpha}^{\alpha}$, and $f_{\alpha} : \mathbf{x} \mapsto (|\mathbf{x}|_{(j)}^{\alpha} - |\mathbf{x}|_{(j+1)}^{\alpha}) / \|\mathbf{x}\|_{\alpha}^{\alpha}$, respectively, satisfy $\text{Var}(f_{\alpha}(Y\mathbf{Q})) = 0$. In addition, appealing to Remark 4.3, $\partial f_q / \partial q|_{q=\alpha}$ and $\sup_{q \in [\alpha-\epsilon, \alpha+\epsilon]} \partial^2 f_q / \partial q^2|_{q=q'}$ are bounded continuous functions in $\mathcal{G}_+(\ell^{\alpha})$, and this proves **S** holds. Finally, using the change of norms formula in (2.12), we can also show $\text{Var}(f_{\alpha}(Y\mathbf{Q}^{(p)})) = 0$, for any $p \in (0, \infty]$, and this concludes the proof. \square

F.2. Proof of Proposition 5.4. We start by noticing that Equation (5.42) rewrites as: for all $\delta > 0$,

$$(F.103) \quad \lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\sum_{t=1}^n |\sum_{|j|>s} \varphi_j \mathbf{Z}_{t-j} / x_n|^p > \delta)}{n \mathbb{P}(|\mathbf{X}_1| > x)} = 0.$$

Assuming (5.42) holds, **AC** and **CS_p** follow straightforwardly since for all $s > 0$, the series $(\mathbf{X}_t^{(s)})$ is a linear m_0 -dependent sequence with $m_0 = 2s+1$, such that $\mathbf{X}_t^{(s)} = \sum_{|j| \leq s} \varphi_j \mathbf{Z}_{t-j}$. The former satisfies **AC**, **CS_p**, for $p > \alpha/2$, as in Example 5.1.

We now turn to the verification of Equation (F.103). Actually, by monotonicity of the ℓ^p -norms, if (F.103) holds for $\alpha/2 < p < \alpha$, then it also holds for $p \geq \alpha$. In the following we assume $\alpha/2 < p < \alpha$.

For $p \leq 1$, the subadditivity property yields

$$|\sum_{|j|>s} \varphi_j \mathbf{Z}_{t-j}|^p \leq \sum_{|j|>s} |\varphi_j \mathbf{Z}_{t-j}|^p =: I_{1,t}.$$

That the partial sums of $(I_{1,t})$ satisfy (F.103) for $0 < \alpha < 1$ follows from standard arguments, see for instance Section 6.1 of [28]. We provide an alternative prove below, also valid for every $\alpha > 0$.

For $p > 1$, a Taylor decomposition of functional $|\cdot|^p : \mathbb{R} \rightarrow \mathbb{R}$ entails, for all $a, b, \in \mathbb{R}$,

$$|a + b|^p = |a|^p + p \operatorname{sign}(a)|a|^{p-1}b + \frac{p(p-1)}{2}|a|^{p-2}b^2 + \dots + R_{[p]}(a, b),$$

where the remaining term satisfies

$$R_{[p]} = R_{[p]}(a, b) \leq \frac{p(p-1) \cdots (p-[p])}{[p]!} |b - \xi a|^{p-[p]} b^{[p]},$$

for one $\xi \in [0, 1]$. To simplify notation, in the remaining lines of the proof we denote $(|\mathbf{Z}_t|)$ by (Z_t) . Then, the Taylor expression above yields

$$\begin{aligned} & |\sum_{|j|>s} \varphi_j \mathbf{Z}_{t-j} / x_n|^p \\ & \leq |\varphi_s Z_{t-s} / x_n|^p + p |\varphi_s Z_{t-s} / x_n|^{p-1} (\sum_{\substack{|j|\geq s \\ j \neq s}} |\varphi_j Z_{t-j} / x_n|) \\ & \quad + \dots + R_{[p]}. \end{aligned}$$

Moreover, to handle the remaining term $R_{[p]}$, we use subadditivity of the real function $x \mapsto x^{p-[p]}$. Hence,

$$\begin{aligned} & |\sum_{|j|\geq s} \varphi_j \mathbf{Z}_{t-j} / x_n|^p \\ & \leq |\varphi_s Z_{t-s} / x_n|^p + c |\varphi_s Z_{t-s} / x_n|^{p-1} (\sum_{\substack{|j|\geq s \\ j \neq s}} |\varphi_j Z_{t-j} / x_n|) \\ & \quad + \dots + c |\varphi_s Z_{t-s} / x_n|^{p-[p]} (\sum_{\substack{|j|\geq s \\ j \neq s}} |\varphi_j Z_{t-j} / x_n|)^{[p]} \\ & \quad + c \sum_{\substack{|j|\geq s \\ j \neq s}} |\varphi_j Z_{t-j} / x_n|^{p-[p]} (\sum_{\substack{|j|\geq s \\ j \neq s}} |\varphi_j Z_{t-j} / x_n|)^{[p]} \\ & \leq c (I_{0,t} + I_{1,t} + \dots + I_{[p],t} + I_{[p]+1,t}), \end{aligned}$$

where $c > 0$ is a constant of no interest, only depending on p . Relying on the bound above, we require to control the previous $[p]+2$ terms. We argue using a truncation argument. Our goal is to prove that for all $l = 0, \dots, [p] + 1$, for all $\epsilon, \delta > 0$

$$(F.104) \quad \lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\sum_{t=1}^n \overline{I_{l,t}}^\epsilon > \delta)}{n \mathbb{P}(|\mathbf{X}_1| > x_n)} + \frac{\mathbb{P}(\sum_{t=1}^n I_{l,t}^\epsilon > \delta)}{n \mathbb{P}(|\mathbf{X}_1| > x_n)} = 0,$$

where the truncated terms are defined as follows: for $l = 0, \dots, [p]$

$$\begin{aligned}\overline{I}_{l,t}^\epsilon &:= |\overline{\varphi_s Z_{t-s}/x_n}^\epsilon|^{p-l} \left(\sum_{\substack{|j| \geq s \\ j \neq s}} \overline{|\varphi_j Z_{t-j}/x_n}^\epsilon \right)^l, \\ \overline{I}_{[p]+1,t}^\epsilon &:= \sum_{\substack{|j| \geq s \\ j \neq s}} |\overline{\varphi_j Z_{t-j}/x_n}^\epsilon|^{p-[p]} \left(\sum_{\substack{|j| \geq s \\ j \neq s}} \overline{|\varphi_j Z_{t-j}/x_n}^\epsilon \right)^{[p]}, \\ \underline{I}_{l,t_\epsilon} &:= |\underline{\varphi_s Z_{t-s}/x_{n_\epsilon}}|^{p-l} \left(\sum_{\substack{|j| \geq s \\ j \neq s}} \underline{|\varphi_j Z_{t-j}/x_{n_\epsilon}} \right)^l, \\ \underline{I}_{[p]+1,t_\epsilon} &:= \sum_{\substack{|j| \geq s \\ j \neq s}} |\underline{\varphi_j Z_{t-j}/x_{n_\epsilon}}|^{p-[p]} \left(\sum_{\substack{|j| \geq s \\ j \neq s}} \underline{|\varphi_j Z_{t-j}/x_{n_\epsilon}} \right)^{[p]}.\end{aligned}$$

To study each term, we write for $q \in \mathbb{N}$, $\mathcal{J} \subseteq \mathbb{N}$, $(\psi_j) \in \mathbb{R}^{|\mathcal{J}|}$,

$$(F.105) \quad \left(\sum_{j \in \mathcal{J}} \psi_j \right)^q = \sum_{\substack{i_1, \dots, i_q \\ i_j \in \mathcal{J}}} \psi_{i_1} \cdots \psi_{i_q}.$$

We start by analyzing the terms corresponding to the truncation from below. An application of Markov's inequality together with Equation (F.105) yield

$$\begin{aligned}\mathbb{P}(\sum_{t=1}^n \underline{I}_{[p]+1,t_\epsilon} > \delta) &\leq \delta^{-1} n \mathbb{E}[\underline{I}_{[p]+1,t_\epsilon}] \\ &\leq \delta^{-1} n \sum_{\substack{i_1, \dots, i_{[p]+1} \\ |i_j| \geq s}} |\varphi_{i_1} \cdots \varphi_{i_{[p]+1}}| |\varphi_{i_{[p]+1}}|^{p-[p]} \\ &\quad \times \mathbb{E}[|\underline{Z_{-i_1}/x_{n_\epsilon}}| \cdots |\underline{Z_{-i_{[p]}}/x_{n_\epsilon}}| |\underline{Z_{-i_{[p]+1}}/x_{n_\epsilon}}|^{p-[p]}].\end{aligned}$$

Moreover, recall the noise sequence (Z_t) are iid random variables satisfying \mathbf{RV}_α . Therefore, for the expectation we can factor the independent noise terms as the product of at most $[p] + 1$ terms. For each term, the noise random variable Z_{-i_j} will be raised to the power at most p . As $p < \alpha$, we can use Karamata's theorem on each of these terms.

Finally, an application of Karamata's theorem and Potter's bound yield there exists $\kappa > 0$, such that for all $\epsilon, \delta > 0$

$$\frac{\mathbb{P}(\sum_{t=1}^n \underline{I}_{[p]+1,t_\epsilon} > \delta)}{n \mathbb{P}(|Z_1| > x_n)} \leq \frac{\alpha}{\alpha - p} O(\delta^{-1} \epsilon^{-(\alpha - \kappa)} (\sum_{|j| \geq s} |\varphi_j|^{\alpha - \kappa})^p).$$

We conclude that this term is negligible by letting first $n \rightarrow \infty$, and then $s \rightarrow \infty$.

We can follow similar steps as before to study the truncation from below terms $I_{l,t}$, $l = 0, \dots, [p]$. An application of Markov's inequality entails there exists $\kappa > 0$ such that

$$\begin{aligned}\frac{\mathbb{P}(\sum_{t=1}^n \underline{I}_{l,t_\epsilon} > \delta)}{n \mathbb{P}(|Z_1| > x_n)} &\leq \delta^{-1} \frac{\mathbb{E}[|\varphi_s Z_1/x_{n_\epsilon}|^{p-l}]}{\mathbb{P}(|Z_1| > x_n)} \mathbb{E}\left[\left(\sum_{\substack{|j| > s \\ j \neq s}} \underline{|\varphi_j Z_{t-j}/x_{n_\epsilon}}\right)^l\right] \\ &= \frac{\alpha}{\alpha - p + l} O(\delta^{-1} |\varphi_s|^{\alpha - \kappa} \epsilon^{-(\alpha - \kappa)} (\sum_{|j| \geq s} |\varphi_j|^{\alpha - \kappa})^l),\end{aligned}$$

as $n \rightarrow \infty$, where the last relation holds by Karamata's theorem and an application of Potter's bound. Hence, for $l = 0, \dots, [p], [p] + 1$, we conclude letting first $n \rightarrow \infty$, and then $s \rightarrow \infty$. To sum up we have shown that

$$\lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\sum_{t=1}^n \overline{I_{l,t}^\epsilon} > \delta)}{n \mathbb{P}(|\mathbf{X}_1| > x_n)} = 0.$$

We now turn to the terms relative to the truncation from above. In this case, the assumption $n/x_n^p \rightarrow 0$, entails $n \mathbb{E}[\overline{I_{l,t}^\epsilon}] \rightarrow 0$, as $n \rightarrow \infty$, for $l = 0, \dots, [p], [p] + 1$. Therefore, to establish Equation (F.104), it suffices to check the following relation holds:

$$(F.106) \quad \lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\sum_{t=1}^n \overline{I_{l,t}^\epsilon} - \mathbb{E}[\overline{I_{l,t}^\epsilon}] > \delta)}{n \mathbb{P}(|\mathbf{X}_1| > x_n)} = 0.$$

We apply Chebychev's inequality, which together with the stationarity of the series (Z_t) , yields

$$\mathbb{P}(\sum_{t=1}^n \overline{I_{l,t}^\epsilon} - \mathbb{E}[\overline{I_{l,t}^\epsilon}] > \delta) \leq 2 \delta^{-2} n \sum_{t=0}^n \text{Cov}(\overline{I_{l,0}^\epsilon}, \overline{I_{l,t}^\epsilon})$$

As in the arguments for the truncation from above, we start by showing that the term in (F.106) is negligible for $l = [p] + 1$. This reasoning can again be extended for $l = 0, \dots, [p]$. Computation of the covariances then yields

$$\begin{aligned} & \text{Cov}(\overline{I_{[p]+1,0}^\epsilon}, \overline{I_{[p]+1,t}^\epsilon}) \\ &= \sum_{\substack{i_1, \dots, i_{[p]+1} \\ |i_j| \geq s, i_j \neq s}} \sum_{\substack{\ell_1, \dots, \ell_{[p]+1} \\ |\ell_j| \geq s, \ell_j \neq s}} |\varphi_{i_1} \cdots \varphi_{i_{[p]}}| |\varphi_{i_{[p]+1}}|^{p-[p]} |\varphi_{\ell_1} \cdots \varphi_{\ell_{[p]}}| |\varphi_{\ell_{[p]+1}}|^{p-[p]} \\ & \quad \times \text{Cov}(|\overline{Z_{-i_1}/x_n}^{\epsilon \varphi_{i_1}^{-1}} \cdots \overline{Z_{-i_{[p]}}/x_n}^{\epsilon \varphi_{i_{[p]}^{-1}}| |\overline{Z_{-i_{[p]+1}}/x_n}^{\epsilon \varphi_{i_{[p]+1}^{-1}}|^{p-[p]} \\ & \quad |\overline{Z_{t-\ell_1}/x_n}^{\epsilon \varphi_{\ell_1}^{-1}} \cdots \overline{Z_{t-\ell_{[p]}}/x_n}^{\epsilon \varphi_{\ell_{[p]}^{-1}}| |\overline{Z_{t-\ell_{[p]+1}}/x_n}^{\epsilon \varphi_{\ell_{[p]+1}^{-1}}|^{p-[p]}. \end{aligned}$$

Actually, all but a finite number of terms vanish in the previous double sum because the noise sequence (Z_t) are independent random variables. More precisely,

$$\begin{aligned} & \text{Cov}(\overline{I_{[p]+1,0}^\epsilon}, \overline{I_{[p]+1,t}^\epsilon}) \\ &= \sum_{\substack{i_1, \dots, i_{[p]+1} \\ |i_j| \geq s, i_j \neq s}} \sum_{\ell_j \in \{i_1-t, \dots, i_{[p]+1}-t\}} \ell_1, \dots, \ell_{[p]+1} \\ & \quad \times |\varphi_{i_1} \cdots \varphi_{i_{[p]}}| |\varphi_{i_{[p]+1}}|^{p-[p]} |\varphi_{\ell_1} \cdots \varphi_{\ell_{[p]}}| |\varphi_{\ell_{[p]+1}}|^{p-[p]} \\ & \quad \times \text{Cov}(|\overline{Z_{-i_1}/x_n}^{\epsilon \varphi_{i_1}^{-1}} \cdots \overline{Z_{-i_{[p]}}/x_n}^{\epsilon \varphi_{i_{[p]}^{-1}}| |\overline{Z_{-i_{[p]+1}}/x_n}^{\epsilon \varphi_{i_{[p]+1}^{-1}}|^{p-[p]} \\ & \quad |\overline{Z_{t-\ell_1}/x_n}^{\epsilon \varphi_{\ell_1}^{-1}} \cdots \overline{Z_{t-\ell_{[p]}}/x_n}^{\epsilon \varphi_{\ell_{[p]}^{-1}}| |\overline{Z_{t-\ell_{[p]+1}}/x_n}^{\epsilon \varphi_{\ell_{[p]+1}^{-1}}|^{p-[p]}. \end{aligned}$$

Moreover, regarding the last covariance term, we notice that it is sufficient to bound the expectation of the product as

$$\begin{aligned} & \text{Cov}(|\overline{Z_{-i_1}/x_n}{}^{\epsilon\varphi_{i_1}^{-1}} \cdots \overline{Z_{-i_{[p]}}/x_n}{}^{\epsilon\varphi_{i_{[p]}}^{-1}}| |\overline{Z_{-i_{[p]+1}}/x_n}{}^{\epsilon\varphi_{i_{[p]+1}}^{-1}}|^{p-[p]}, \\ & \quad |\overline{Z_{t-\ell_1}/x_n}{}^{\epsilon\varphi_{\ell_1}^{-1}} \cdots \overline{Z_{t-\ell_{[p]}}/x_n}{}^{\epsilon\varphi_{\ell_{[p]}}^{-1}}| |\overline{Z_{t-\ell_{[p]+1}}/x_n}{}^{\epsilon\varphi_{\ell_{[p]+1}}^{-1}}|^{p-[p]}) \\ & \leq \mathbb{E}[|\overline{Z_{-i_1}/x_n}{}^{\epsilon\varphi_{i_1}^{-1}} \cdots \overline{Z_{-i_{[p]}}/x_n}{}^{\epsilon\varphi_{i_{[p]}}^{-1}}| |\overline{Z_{-i_{[p]+1}}/x_n}{}^{\epsilon\varphi_{i_{[p]+1}}^{-1}}|^{p-[p]} \\ & \quad \times |\overline{Z_{t-\ell_1}/x_n}{}^{\epsilon\varphi_{\ell_1}^{-1}} \cdots \overline{Z_{t-\ell_{[p]}}/x_n}{}^{\epsilon\varphi_{\ell_{[p]}}^{-1}}| |\overline{Z_{t-\ell_{[p]+1}}/x_n}{}^{\epsilon\varphi_{\ell_{[p]+1}}^{-1}}|^{p-[p]}]. \end{aligned}$$

Since (Z_t) are iid random variable, the expectation term above can be written as the product of expectations as follows

$$\begin{aligned} & |\varphi_{i_1} \cdots \varphi_{i_{[p]}}| |\varphi_{i_{[p]+1}}|^{p-[p]} |\varphi_{\ell_1} \cdots \varphi_{\ell_{[p]}}| |\varphi_{\ell_{[p]+1}}|^{p-[p]} \\ & \quad \mathbb{E}(|\overline{Z_{-i_1}/x_n}{}^{\epsilon\varphi_{i_1}^{-1}}| \cdots |\overline{Z_{-i_{[p]}}/x_n}{}^{\epsilon\varphi_{i_{[p]}}^{-1}}| |\overline{Z_{-i_{[p]+1}}/x_n}{}^{\epsilon\varphi_{i_{[p]+1}}^{-1}}|^{p-[p]} \\ & \quad \times |\overline{Z_{t-\ell_1}/x_n}{}^{\epsilon\varphi_{\ell_1}^{-1}}| \cdots |\overline{Z_{t-\ell_{[p]}}/x_n}{}^{\epsilon\varphi_{\ell_{[p]}}^{-1}}| |\overline{Z_{t-\ell_{[p]+1}}/x_n}{}^{\epsilon\varphi_{\ell_{[p]+1}}^{-1}}|^{p-[p]}) \\ & = \prod_{\substack{\gamma_1 + \cdots + \gamma_r = p, \\ \gamma'_1 + \cdots + \gamma'_{r'} = p}} |\varphi_{i_{\gamma_j}}|^{\gamma_j} |\varphi_{i_{\gamma_j-t}}|^{\gamma'_j} \mathbb{E}[|\overline{Z_0/x_n}{}^{\epsilon\varphi_{i_{\gamma_j}}^{-1}}|^{\gamma_j} |\overline{Z_0/x_n}{}^{\epsilon\varphi_{i_{\gamma_j-t}}^{-1}}|^{\gamma'_j}], \end{aligned}$$

where $\gamma_j, \gamma'_j \in \{0, 1, \dots, [p]\}$, $\gamma_r, \gamma'_{r'} \in \{0, p - [p], p - [p] + 1, \dots, p\}$ and $1 \leq r, r' \leq [p] + 1$. The product above is a factorization with respect to independent noise terms. We have also used the stationarity of (Z_t) . The new indices $\gamma_1, \dots, \gamma_r$ are defined recursively in terms of the sequence (i_t) . Similarly, we define $\gamma'_1, \dots, \gamma'_{r'}$ from (ℓ_t) . To define γ_1 , first we count the number of times the index i_1 appears in $\mathcal{I} = \{i_1, \dots, i_{[p]}\}$. If $i_1 \neq i_{[p]+1}$, we put γ_1 equal to this count, otherwise, we set γ_1 equal to this count plus $p - [p]$. Then, we look for the next index different than i_1 , say i_j , and set γ_2 as the number of repetitions of i_j in \mathcal{I} plus $p - [p]$ if $i_j \neq i_{[p]+1}$. We continue in this way until the indices i_r and γ_r are defined as previously. We stop as we recognize that all the indices $i_r, i_{r+1}, \dots, i_{[p]+1}$ have already been considered. Therefore, $\gamma_1 + \cdots + \gamma_r = p$. In an identical fashion, we define $\gamma'_1, \dots, \gamma'_{r'}$ from (ℓ_t) . Moreover, notice that for every $\gamma \in \{\gamma_1, \dots, \gamma_r\}$ and $\gamma' \in \{\gamma'_1, \dots, \gamma'_{r'}\}$

$$\begin{aligned} & |\varphi_{i_\gamma}|^\gamma |\varphi_{i_{\gamma-t}}|^{\gamma'} \mathbb{E}[|\overline{Z_0/x_n}{}^{\epsilon\varphi_{i_\gamma}^{-1}}|^\gamma |\overline{Z_0/x_n}{}^{\epsilon\varphi_{i_{\gamma-t}}^{-1}}|^{\gamma'}] \\ & \leq (|\varphi_{i_\gamma}|^{2\gamma} |\varphi_{i_{\gamma-t}}|^{2\gamma'} \mathbb{E}[|\overline{Z_0/x_n}{}^{\epsilon\varphi_{i_\gamma}^{-1}}|^{2\gamma}] \mathbb{E}[|\overline{Z_0/x_n}{}^{\epsilon\varphi_{i_{\gamma-t}}^{-1}}|^{2\gamma'}])^{1/2} \\ & \leq (|\varphi_{i_\gamma}|^{2p} \mathbb{E}[|\overline{Z_0/x_n}{}^{\epsilon\varphi_{i_\gamma}^{-1}}|^{2p}])^{\gamma/2p} (|\varphi_{i_{\gamma-t}}|^{2p} \mathbb{E}[|\overline{Z_0/x_n}{}^{\epsilon\varphi_{i_{\gamma-t}}^{-1}}|^{2p}])^{\gamma'/2p}. \end{aligned}$$

The key property $\gamma_1 + \cdots + \gamma_r + \gamma'_1 + \cdots + \gamma'_{r'} = 2p$ yields

$$\mathbb{P}(Z_0 > x_n) = \prod_{\substack{\gamma_1 + \cdots + \gamma_r = p \\ \gamma'_1 + \cdots + \gamma'_{r'} = p}} (\mathbb{P}(Z_0 > x_n))^{(\gamma + \gamma')/2p}.$$

In this case, we can apply Karamata's Theorem to each one of the expectation terms. Readily,

$$\begin{aligned}
 & \sum_{t=1}^n \text{Cov}(\overline{I_{[p]+1,0}^\epsilon}, \overline{I_{[p]+1,t}^\epsilon}) / \mathbb{P}(Z_0 > x_n) \\
 & \leq \sum_{t=1}^n \sum_{\substack{i_1, \dots, i_{[p]+1} \\ |i_j| \geq s, i_j \neq s}} \sum_{\ell_j \in \{i_1-t, \dots, i_{[p]+1}-t\}} \prod_{\substack{\gamma_1 + \dots + \gamma_r = p \\ \gamma'_1 + \dots + \gamma'_r = p}} \frac{(|\varphi_{i_\gamma}|^{2p} \mathbb{E}[|\overline{Z_0/x_n}^{\epsilon \varphi_{i_\gamma}^{-1}}|^{2p}])^{\gamma/2p} (|\varphi_{i_{\gamma-t}}|^{2p} \mathbb{E}[|\overline{Z_0/x_n}^{\epsilon \varphi_{i_{\gamma-t}}^{-1}}|^{2p}])^{\gamma'/2p}}{(\mathbb{P}(Z_0 > x_n))^{(\gamma+\gamma')/2p}} \\
 & \leq c \sum_{\substack{i_1, \dots, i_{[p]+1} \\ |i_j| \geq s, i_j \neq s}} \sum_{\ell_j \in \{i_1-t, \dots, i_{[p]+1}-t\}} \sum_{t=1}^n \prod_{\substack{\gamma_1 + \dots + \gamma_r = p \\ \gamma'_1 + \dots + \gamma'_r = p}} |\varphi_{i_\gamma}|^{\gamma(\alpha-\kappa)} |\varphi_{i_{\gamma-t}}|^{\gamma'(\alpha-\kappa)} \\
 & = c \sum_{\substack{i_1, \dots, i_{[p]+1} \\ |i_j| \geq s, i_j \neq s}} \sum_{\ell_j \in \{i_1-t, \dots, i_{[p]+1}-t\}} \sum_{t=1}^n |\varphi_{i_j}|^{(\alpha-\kappa)} |\varphi_{\ell_j-t}|^{(\alpha-\kappa)} \\
 & \leq c(p+1) (\sum_{|i| \geq s} |\varphi_i|^{(\alpha-\kappa)})^p (\sum_{j \in \mathbb{Z}} |\varphi_j|^{(\alpha-\kappa)})^p.
 \end{aligned}$$

where $c > 0$, is a constant of no interest. We conclude by letting $s \rightarrow \infty$ that (F.106) holds for $l = [p] + 1$. Similarly, this arguments can be extended for $l = 1, \dots, [p]$. Overall, this shows (F.104) holds, and this concludes the proof. \square

F.3. Proof of Theorem 5.6. For $p > \alpha/2$, we aim to apply Theorem 3.2. First, notice condition (5.44) yields $\|\varphi_t\|_p < \infty$. For any $\kappa' > 0$, consider a sequence (x_n) such that $x_n \in (n^{\kappa'+1/(p \wedge \alpha)}, +\infty)$. Proposition 5.4 implies then that conditions **AC**, **CS_p**, hold, and $n\mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$, as $n \rightarrow \infty$. Fix $\kappa' > 0$, and $x_n = O(n^{\kappa'+1/(p \wedge \alpha)})$, as $n \rightarrow \infty$.

Furthermore, since there exists ϵ such that $\|(\varphi_j)\|_{\alpha-\epsilon} < \infty$, then condition **M** is automatically satisfied by definition of **Q**. In addition, for the $\hat{\alpha}$ -cluster based estimators from Section 4 in (4.26) (4.29), and (4.34) we need to check **S** holds. For this we verify the conditions of Lemma F.1. Actually, Equation (F.117) has already been demonstrated in Proposition 5.4; and it suffices to follow the lines of the proof of Equation (F.106). We can therefore conclude that **S** holds for the $\hat{\alpha}$ -cluster based estimators from Section 4 in (4.26) (4.29), and (4.34).

Finally, to apply Theorem 3.2 it suffices to verify the β -mixing conditions **MX _{β}** holds. Next, we show **MX _{β}** also holds. Choose (k_n) as in (A.51). Then, there exists $\epsilon, \epsilon' > 0$, and a constant $c > 0$, such that

$$\begin{aligned}
 \text{(F.107)} \quad m_n/k_n &= 1/(c(p)b_n \mathbb{P}(|\mathbf{X}_0| > x_{b_n})) \\
 &\leq c x_{b_n}^{(\alpha+\epsilon)} / b_n = c b_n^{-1 + \frac{\alpha}{\alpha \wedge p} + \frac{\epsilon}{\alpha \wedge p} + \kappa'(\alpha+\epsilon)} \\
 &\leq c b_n^{-1 + \frac{\alpha}{p \wedge \alpha} + \epsilon'}.
 \end{aligned}$$

This follows using Potter's bounds. Let $\ell_n = b_n^{(1-\epsilon)}$ such that $\ell_n/b_n \rightarrow 0$. Finally, applying Proposition 5.5, we can find $\epsilon' > 0$ such that the relation below holds

$$m_n \beta_{\ell_n} / k_n = O(b_n^{-\frac{(\rho-1)\alpha}{1+\alpha} + \frac{\alpha}{\alpha \wedge p} + \epsilon'}), \quad n \rightarrow \infty.$$

Then, taking $\rho > 1 + \frac{1+\alpha}{\alpha \wedge p} + \epsilon$ yields $m_n \beta_{\ell_n} / k_n \rightarrow 0$. In this argument, we can choose $\epsilon' > 0$ to be arbitrarily small. Then, assuming (5.44) entails $m_n \beta_{\ell_n} / k_n \rightarrow 0$.

Moreover, let $\delta > 0$ be as in (3.21). Since $\rho > \frac{2}{\delta}(1 + \frac{1}{\alpha}) + 3 + \frac{2}{\alpha}$, equation (5.43) yields $\sum_{t=1}^{\infty} \beta_t^{\delta/(2+\delta)} < \infty$. In this case, there exist $\epsilon > 0$ such that

$$\sum_{t=1}^{m_n} (m_n \beta_{tb_n} / k_n)^{\frac{\delta}{2+\delta}} = O\left(b_n^{\left(-\frac{(\rho-1)\alpha}{1+\alpha} + \frac{\alpha}{\alpha \wedge p} + \epsilon\right) \frac{\delta}{(2+\delta)}}\right).$$

Furthermore, for $p > \alpha/2$, notice $\rho > 3 + 2/\alpha > 1 + (1 + \alpha)/(\alpha \wedge p)$. Similarly as before, notice $\epsilon > 0$ can be made arbitrarily small. Putting everything together, we conclude that (3.24) holds. This completes the proof that \mathbf{MX}_β holds. Since \mathbf{MX}_β holds we can apply Theorem 3.2. Finally, in our setting notice that the sequence (k_n) satisfies

$$k_n \sim c(p)n\mathbb{P}(|\mathbf{X}_0| > x_{b_n}) = o(n b_n^{-\frac{\alpha}{p \wedge \alpha}}), \quad n \rightarrow \infty.$$

This last relation follows using Potter's bounds. This concludes the proof of Theorem 5.6. \square

F.4. Proof of Proposition 5.8. Let (\mathbf{X}_t) be the stationary solution to the SRE (5.45) as in Example 5.7, satisfying \mathbf{RV}_α , for $\alpha > 0$. Then, (\mathbf{X}_t) admits the causal representation in (5.48), where $((\mathbf{A}_t, \mathbf{B}_t))$ is a sequence of iid innovations. Then, backward computations yield

$$(F.108) \quad \mathbf{X}_t = \Pi_t \mathbf{X}_0 + R_t, \quad t \geq 1,$$

where for $1 \leq i \leq t$

$$(F.109) \quad \Pi_{i,t} := \mathbf{A}_i \cdots \mathbf{A}_t, \quad R_t := \sum_{j=1}^t \Pi_{j+1,t} \mathbf{B}_j,$$

with the conventions: $\Pi_{1,t} = \Pi_t$, and $\Pi_{t+1,t} = Id$. Notice that the remaining term R_t is measurable with respect to $\sigma((\mathbf{A}_i, \mathbf{B}_i)_{1 \leq i \leq t})$, and is independent of the sigma-field $\sigma((\mathbf{X}_t)_{t \leq 0})$.

Condition \mathbf{AC} has been shown for Theorem 4.17 in [36]. We focus on showing \mathbf{CS}_p holds for $p \in (\alpha/2, \alpha)$.

To begin, note condition \mathbf{CS}_p was borrowed from Equation (5.2) in [12]. For $p \in (0, \alpha)$, and sequences (x_n) such that $n/x_n^p \rightarrow 0$, as $n \rightarrow \infty$, we have $n\mathbb{E}[|\mathbf{X}_0/x_n|^p] \rightarrow 0$, thus our condition \mathbf{CS}_p and Equation (5.2) in [12]

coincide. More precisely, we show

$$(F.110) \quad \lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} \frac{\mathbb{P}(\|\overline{\mathbf{X}}_{[1,n]}/x_n^\epsilon\|_p^p - \mathbb{E}[\|\overline{\mathbf{X}}_{[1,n]}/x_n^\epsilon\|_p^p] > \delta)}{n\mathbb{P}(|\mathbf{X}_1| > x_n)} = 0.$$

For this reason, we focus on showing (F.110) holds. Actually, we show below that, for (x_n) as in Proposition 5.8, condition \mathbf{CS}_p holds over uniform regions $\Lambda_n = (x_n, \infty)$ in the sense of (F.115). Further, for the purposes of completeness, we show (F.115) holds generally for sequences (x_n) such that $n\mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$, as $n \rightarrow \infty$ in the setting of Example 5.7.

Let $p \in (\alpha/2, \alpha)$, and consider a sequence (x_n) satisfying the assumptions of Proposition 5.8. Consider the region $\Lambda_n = (x_n, \infty)$, and consider $x \in \Lambda_n$. An application of Chebychev's inequality yields

$$(F.111) \quad \mathbb{P}(\|\overline{\mathbf{X}}_{[1,n]}/x^\epsilon\|_p^p - \mathbb{E}[\|\overline{\mathbf{X}}_{[1,n]}/x^\epsilon\|_p^p] > \delta) \leq 2n\delta^{-2} \sum_{t=0}^n I_t,$$

such that we denote $I_t = \text{Cov}(|\overline{\mathbf{X}}_0/x^\epsilon|^p, |\overline{\mathbf{X}}_t/x^\epsilon|^p)$.

Let (Π_t) and (R_t) be as in (F.109) such that (\mathbf{X}_t) satisfies Equation (5.48). We define a new Markov chain $(\mathbf{X}'_t)_{t \geq 0}$ satisfying

$$(F.112) \quad \mathbf{X}'_t := \Pi_t \mathbf{X}'_0 + R_t,$$

with \mathbf{X}'_0 independent of (\mathbf{X}_t) and identically distributed as \mathbf{X}_0 . We can see (\mathbf{X}'_t) as the solution of the SRE (5.45) for the sequence of innovations $((\mathbf{A}'_t, \mathbf{B}'_t))$ where $(\mathbf{A}'_t, \mathbf{B}'_t) = (\mathbf{A}_t, \mathbf{B}_t)$ for $t \leq 0$ and $(\mathbf{A}'_t, \mathbf{B}'_t)_{t \geq 1}$ is an iid sequence independent of $(\mathbf{A}_t, \mathbf{B}_t)$, distributed as the generic element (\mathbf{A}, \mathbf{B}) . Then, following the notation in (F.111), we can rewrite I_t as

$$\begin{aligned} I_t &= \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p |\overline{\mathbf{X}}_t/x^\epsilon|^p] - \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p] \mathbb{E}[|\overline{\mathbf{X}}'_t/x^\epsilon|^p] \\ &\leq \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p (|\overline{\mathbf{X}}_t/x^\epsilon|^p - |\overline{\mathbf{X}}'_t/x^\epsilon|^p)_+] \\ &\leq \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p (|\mathbf{X}_t/x|^p - |\mathbf{X}'_t/x|^p)_+ \mathbf{1}(|\mathbf{X}_t/x| \leq \epsilon) \mathbf{1}(|\mathbf{X}'_t/x| \leq \epsilon)] \\ &\quad + \mathbb{E}[|\overline{\mathbf{X}}_0/x_n^\epsilon|^p |\overline{\mathbf{X}}_t/x^\epsilon|^p \mathbf{1}(|\mathbf{X}'_t/x| > \epsilon)] \\ &= I_{t,1} + I_{t,2}. \end{aligned}$$

We show that $I_{t,1}$ is negligible letting first $n \rightarrow \infty$, and then $\epsilon \downarrow 0$. For this, we consider two cases. First, assume $p > 1$. Then, for the first term $I_{t,1}$, a Taylor decomposition yields

$$\begin{aligned} I_{t,1} &\leq p \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p |\overline{\mathbf{X}}'_t/x^\epsilon - \overline{\mathbf{X}}_t/x^\epsilon|^{2\epsilon} (|\overline{\mathbf{X}}'_t/x^\epsilon| + \xi(|\overline{\mathbf{X}}'_t/x^\epsilon - \overline{\mathbf{X}}_t/x^\epsilon|^{2\epsilon}))^{p-1}] \\ &= p \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p |\Pi_t \mathbf{X}'_0/x - \Pi_t \mathbf{X}_0/x|^{2\epsilon} \\ &\quad \times (|\overline{\mathbf{X}}'_t/x^\epsilon| + \xi(|\Pi_t \mathbf{X}'_0/x - \Pi_t \mathbf{X}_0/x|^{2\epsilon}))^{p-1}], \end{aligned}$$

for some random variable $\xi \in (0, 1)$ a.s. In the last equality, we have used the definition of (\mathbf{X}') in (F.112). Moreover, we can bound $I_{t,1}$ by

$$\begin{aligned} I_{t,1} &\leq p2^{0 \vee (p-1)} \mathbb{E} [|\overline{\mathbf{X}_0/x}^\epsilon|^p |\overline{\Pi_t \mathbf{X}'_0/x - \Pi_t \mathbf{X}_0/x}^{2\epsilon}| |\overline{\mathbf{X}'_t/x}^\epsilon|^{p-1}] \\ &\quad + p2^{0 \vee (p-1)} \mathbb{E} [|\overline{\mathbf{X}_0/x}^\epsilon|^p |\overline{\Pi_t \mathbf{X}'_0/x - \Pi_t \mathbf{X}_0/x}^{2\epsilon}|^p] \end{aligned}$$

Now, an application of Jensen's inequality, Potter's bounds, and Karata's theorem, yield

$$\begin{aligned} I_{t,1} &\leq c \mathbb{E} [|\overline{\mathbf{X}_0/x}^\epsilon|^{2p}]^{1/2} \mathbb{E} [|\overline{\Pi_t \mathbf{X}'_0/x}^{4\epsilon}|^{2p}]^{1/2} \\ &\leq c (\mathbb{E} [|\overline{\mathbf{X}_0/x}^\epsilon|^{2p}] \mathbb{E} [|\Pi_t|_{op}^{\alpha-\delta}] \mathbb{P}(|\mathbf{X}_0| > x))^{1/2}. \\ &= O((\epsilon^{2p-\alpha} \mathbb{E}[|\Pi_t|_{op}^{\alpha-\delta}])^{1/2} \mathbb{P}(|\mathbf{X}_0| > x)), \quad x_n > x_0, \end{aligned}$$

for constants $c > 0$, $x_0 > 0$. Moreover, under the assumptions of Example 5.7 we have $\mathbb{E}[|\Pi_t|_{op}^{\alpha-\delta}] < 1$ for t sufficiently large. Thereby, we conclude

$$(F.113) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \sum_{t=1}^n I_{t,1} / \mathbb{P}(|\mathbf{X}_0| > x) = 0.$$

We now come back to the case where $p < 1$. In this case we can use a subadditivity argument and we conclude by similar steps that relation (F.113) holds for all $p \in (\alpha/2, \alpha)$.

Now, concerning the second term $I_{t,2}$ we have

$$\begin{aligned} I_{t,2} &:= \mathbb{E} [|\overline{\mathbf{X}_0/x}^\epsilon|^p |\overline{\mathbf{X}_t/x}^\epsilon|^p \mathbb{1}(|\mathbf{X}'_t/x| > \epsilon)] \\ &\leq \mathbb{E} [|\overline{\mathbf{X}_0/x}^\epsilon|^p |\overline{\Pi_t \mathbf{X}_0/x}^\epsilon|^p \mathbb{1}(|\mathbf{X}'_t/x| > \epsilon)] \\ &\quad + \mathbb{E} [|\overline{\mathbf{X}_0/x}^\epsilon|^p |\overline{R_t/x}^\epsilon|^p \mathbb{1}(|\mathbf{X}'_t/x| > \epsilon)] \\ &\quad + \mathbb{E} [|\overline{\mathbf{X}_0/x}^\epsilon|^p \mathbb{1}(|\Pi_t \mathbf{X}_0/x| > \epsilon) \mathbb{1}(|\mathbf{X}'_t/x| > \epsilon)] \\ &= O(\mathbb{E} [|\overline{\mathbf{X}_0/x}^\epsilon|^p] \mathbb{E} [|\overline{R_t/x}^\epsilon|^p \mathbb{1}(|\mathbf{X}'_t/x| > \epsilon)]). \end{aligned}$$

Therefore we have,

$$\sum_{t=1}^n I_{t,2} / \mathbb{P}(|\mathbf{X}_0| > x) = O(n \mathbb{E} [|\overline{\mathbf{X}_0/x}^\epsilon|^p]).$$

Assume (x_n) is such that there exists $\kappa' > 0$ satisfying $n/x_n^{p \wedge (\alpha - \kappa')} \rightarrow 0$, as $n \rightarrow \infty$. Then,

$$(F.114) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \frac{\mathbb{P}(\| \overline{\mathbf{X}_{[1,n]}/x}^\epsilon \|_p^p - \mathbb{E}[\| \overline{\mathbf{X}_{[1,n]}/x}^\epsilon \|_p^p] > \delta)}{n \mathbb{P}(|\mathbf{X}_1| > x)} = 0.$$

Moreover, if $n/x_n^p \rightarrow 0$ then $\mathbb{E}[\| \overline{\mathbf{X}_{[1,n]}/x}^\epsilon \|_p^p] \rightarrow 0$ for $p \in (\alpha/2, \alpha)$. In this case, \mathbf{CS}_p holds uniformly over the region Λ_n .

On the other hand, note that we also have $I_t = O(\beta_t)$. Therefore, if we consider a sequence (ℓ_n) such that $\ell_n \rightarrow \infty$, $n \rightarrow \infty$, then we can have

$$\begin{aligned}
 (\text{F.115}) \quad & \mathbb{P}(\|\overline{\mathbf{X}}_{[1,n]}/x_n^\epsilon\|_p^p - \mathbb{E}[\|\overline{\mathbf{X}}_{[1,n]}/x_n^\epsilon\|_p^p] > \delta) / n\mathbb{P}(|\mathbf{X}_0| > x_n) \\
 & \leq 2n\delta^{-2} \left(\sum_{t=0}^{\ell_n} I_t + \sum_{t=\ell_n+1}^n I_t \right) / n\mathbb{P}(|\mathbf{X}_0| > x_n) \\
 & \leq O(\ell_n \mathbb{E}[|\overline{\mathbf{X}}_0/x_n^\epsilon|^p] + \sum_{t=\ell_n+1}^n I_t / \mathbb{P}(|\mathbf{X}_0| > x_n)) \\
 (\text{F.116}) \quad & \leq O(\ell_n \mathbb{E}[|\overline{\mathbf{X}}_0/x_n^\epsilon|^p] + \sum_{t=\ell_n+1}^n \beta_t / \mathbb{P}(|\mathbf{X}_0| > x_n)) = J_1 + J_2.
 \end{aligned}$$

where in the last bound we use the covariance inequality for the (β_t) mixing coefficients. Furthermore, the bound in (F.115) consists of two terms as (F.115) $\leq J_1 + J_2$. If we want J_1 to go to zero as $n \rightarrow \infty$ we can choose $\ell_n := x_n^{p-\delta}$, for some $\delta > 0$. Now, for the second term J_2 , we first note that it is null if $\ell_n > n$ by convention. Otherwise we recall that the mixing-coefficients (β_t) have a geometric decaying rate. Thereby, there exists $\rho \in (0, 1)$ such that we can bound the second term J_2 by

$$\begin{aligned}
 J_2 &= O(\sum_{t=\ell_n+1}^n \rho^t / \mathbb{P}(|\mathbf{X}_0| > x_n)) \\
 &= O(\rho^{\ell_n} / \mathbb{P}(|\mathbf{X}_0| > x_n)) \\
 &\leq O(\rho^{\ell_n} x_n^{(\alpha+\delta)}).
 \end{aligned}$$

Therefore, $J_2 \rightarrow 0$ as $n \rightarrow \infty$ by plugging in the value we set for ℓ_n . Overall, we conclude that for all sequences (x_n) such that $n\mathbb{P}(|\mathbf{X}_n| > x_n) \rightarrow 0$, then $\lim_{n \rightarrow \infty} (\text{F.115}) = 0$ and this shows (F.110). Moreover, we also saw this convergence holds over uniform regions $\Lambda_n = (x_n, \infty)$, in the sense (F.114), if we assume in addition $n/x_n^{p \wedge (\alpha - \kappa')} \rightarrow 0$, as $n \rightarrow \infty$. Finally, this shows that \mathbf{CS}_p holds and this concludes the proof of Proposition 5.8.

F.5. Proof of Theorem 5.10. Our goal it to verify that we can apply Theorem 3.2 as we combine Proposition 5.8 and 5.9. First, notice for any $\kappa' > 0$, if we consider a sequence (x_n) such that $x_n \in (n^{\kappa'+1/(p \wedge \alpha)}, +\infty)$, then conditions \mathbf{AC} , \mathbf{CS}_p hold thanks to Proposition 5.8. Since $c(p) < \infty$ in (2.9), Proposition 2.2 holds and the time series admits a p -cluster process $\mathbf{Q}^{(p)}$. Fix $\kappa' > 0$, and $x_n = O(n^{\kappa'+1/(p \wedge \alpha)})$.

We focus now on the verification of the mixing condition \mathbf{MX}_β in Theorem 3.2. Applying Proposition 5.9, there exists $\rho \in (0, 1)$ such that the coefficients (β_t) satisfy $\beta_t = O(\rho^t)$. Moreover, if we choose (k_n) according to (A.51) as in the linear model case then there exists $\epsilon, \epsilon' > 0$, and a constant $c > 0$, such that

$$m_n/k_n \leq c b_n^{-1 + \frac{\alpha}{p \wedge \alpha} + \epsilon'},$$

and thus $\sum_{t=1}^{m_n} (m_n \beta_{b_n} / k_n)^{\delta/(2+\delta)} = O(\rho^{b_n \delta/(2+\delta)} (b_n)^\epsilon)$, which goes to zero as $n \rightarrow \infty$. Moreover, for all $\eta \in (0, 1)$, choosing $\ell_n = b_n^\eta$, we have $m_n \beta_{\ell_n} / k_n \rightarrow 0$, $b_n / \ell_n \rightarrow 0$, as $n \rightarrow 0$. Therefore, we have verified \mathbf{MX}_β holds. This concludes the proof of Theorem 5.10 since all assumptions of Theorem 3.2 are verified.

It remains to verify that \mathbf{M} and \mathbf{S} hold. To verify \mathbf{M} we check $\mathbb{E}[\|\mathbf{Q}\|_{\alpha-\varepsilon}^{2\alpha}] < \infty$. We use the definition of \mathbf{Q} , a telescoping sum argument and the time-change formula to obtain

$$\begin{aligned} & \mathbb{E}[\|\mathbf{Q}\|_{\alpha-\varepsilon}^{2\alpha}] \\ &= \mathbb{E}\left[\frac{(\sum_{t \in \mathbb{Z}} |\boldsymbol{\Theta}_t|^{\alpha-\varepsilon})^{2\alpha/(\alpha-\varepsilon)}}{\|\boldsymbol{\Theta}\|_\alpha^{2\alpha}}\right] \\ &= \sum_{j \in \mathbb{Z}} \mathbb{E}\left[\frac{(\sum_{t \geq j} |\boldsymbol{\Theta}_t|^{\alpha-\varepsilon})^{2\alpha/(\alpha-\varepsilon)} - (\sum_{t \geq j+1} |\boldsymbol{\Theta}_t|^{\alpha-\varepsilon})^{2\alpha/(\alpha-\varepsilon)}}{\|\boldsymbol{\Theta}\|_\alpha^{2\alpha}}\right] \\ &= \sum_{j \in \mathbb{Z}} \mathbb{E}\left[|\boldsymbol{\Theta}_j|^\alpha \frac{(\sum_{t \geq 0} |\boldsymbol{\Theta}_t|^{\alpha-\varepsilon})^{2\alpha/(\alpha-\varepsilon)} - (\sum_{t \geq 1} |\boldsymbol{\Theta}_t|^{\alpha-\varepsilon})^{2\alpha/(\alpha-\varepsilon)}}{\|\boldsymbol{\Theta}\|_\alpha^{2\alpha}}\right] \\ &= \mathbb{E}\left[\frac{(\sum_{t \geq 0} |\boldsymbol{\Theta}_t|^{\alpha-\varepsilon})^{2\alpha/(\alpha-\varepsilon)} - (\sum_{t \geq 1} |\boldsymbol{\Theta}_t|^{\alpha-\varepsilon})^{2\alpha/(\alpha-\varepsilon)}}{\|\boldsymbol{\Theta}\|_\alpha^{2\alpha}}\right] \end{aligned}$$

By convexity of the function $g(x) = x^{2\alpha/(\alpha-\varepsilon)}$ we obtain

$$\begin{aligned} & \left(1 + \sum_{t \geq 1} |\boldsymbol{\Theta}_t|^{\alpha-\varepsilon}\right)^{2\alpha/(\alpha-\varepsilon)} - \left(\sum_{t \geq 1} |\boldsymbol{\Theta}_t|^{\alpha-\varepsilon}\right)^{2\alpha/(\alpha-\varepsilon)} \\ & \leq \frac{2\alpha}{\alpha-\varepsilon} \left(\sum_{t \geq 1} |\boldsymbol{\Theta}_t|^{\alpha-\varepsilon}\right)^{\alpha+\varepsilon/(\alpha-\varepsilon)} \\ & \leq \frac{2\alpha}{\alpha-\varepsilon} \sum_{t \geq 1} |\boldsymbol{\Theta}_t|^{\alpha+\varepsilon}. \end{aligned}$$

Then,

$$\mathbb{E}[\|\mathbf{Q}\|_{\alpha-\varepsilon}^{2\alpha}] \leq \frac{2\alpha}{\alpha-\varepsilon} \mathbb{E}\left[\sum_{t \geq 1} \frac{|\boldsymbol{\Theta}_t|^{\alpha+\varepsilon}}{\|\boldsymbol{\Theta}\|_\alpha^\alpha}\right] \leq \frac{2\alpha}{\alpha-\varepsilon} \mathbb{E}\left[\sum_{t \geq 1} |\boldsymbol{\Theta}_t|^\varepsilon\right].$$

The latter expression is integrable because $\mathbb{E}[|\mathbf{A}_{op}^{\kappa'}|] < 1$ and $\mathbb{E}[|\boldsymbol{\Theta}_t|^\varepsilon] = \mathbb{E}[|\mathbf{A}_1 \cdots \mathbf{A}_t|_{op}^\varepsilon] \leq \mathbb{E}[|\mathbf{A}_{op}^{\kappa'}|^{t/\kappa'}]$ for every $\varepsilon < \kappa'$. Thus \mathbf{M} holds in Example 5.7.

Finally, to verify \mathbf{S} we rely on Lemma F.1. Actually, Equation (F.117) has already been demonstrated in Proposition 5.8; and it suffices to follow the lines of the proof of Equation (F.111). We can therefore conclude that \mathbf{S} holds for the $\hat{\alpha}$ -cluster based estimators from Section 4 in (4.26) (4.29), and (4.34). \square

Lemma F.1. Consider (\mathbf{X}_t) to be a time series with values in $(\mathbb{R}^d, |\cdot|)$, and consider the $\hat{\alpha}$ -cluster based estimators from Section 4 in (4.26) (4.29), and (4.34). Assume there exists $\eta > 0$ such that $\alpha - \epsilon - \eta > \epsilon/2$ and

$$(F.117) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{E}[(\|\overline{\mathbf{X}}_{[1,n]}/x_n\|_{\alpha-\epsilon}^\delta - \mathbb{E}[\|\overline{\mathbf{X}}_{[1,n]}/x_n\|_{\alpha-\epsilon}^\delta])^2]}{n\mathbb{P}(|\mathbf{X}_1| > x_n)} < \infty.$$

Then, condition **S** holds.

Proof. Proof of Lemma F.1 For this we appeal to Remark 4.3, which shows the functionals $\partial f_q/\partial q|_{q=\alpha}$ and $\sup_{q' \in [\alpha-\epsilon, \alpha+\epsilon]} \partial^2 f_q/\partial q^2|_{q=q'}$ are continuous functions in $\mathcal{G}_+(\ell^\alpha)$. Moreover, we take the example of the extremal index in (4.26) with $f(\mathbf{x}) = \|\mathbf{x}\|_\infty^\alpha/\|\mathbf{x}\|_\alpha^\alpha$, as for (4.29), and (4.34) similar calculations yield the desired result. Then, to verify **S** note that on the event $\{\|\mathbf{X}_{[1,n]}\|_{q'} > x_{b_n}\}$ we have, for $q' > \alpha - \epsilon$,

$$\begin{aligned} 0 &< \frac{\partial f_q(\mathbf{X}_{[1,n]})}{\partial q} \Big|_{q=q'} \\ &= \frac{\|\mathbf{X}_{[1,n]}\|_\infty^q}{\|\mathbf{X}_{[1,n]}\|_{q'}^{q'}} \sum_{t=1}^n \frac{|\mathbf{X}_t|^{q'}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{q'}} \log(\|\mathbf{X}_{[1,n]}\|_\infty/|\mathbf{X}_t|) \\ &\leq \sum_{t=1}^n \frac{|\mathbf{X}_t|^{\alpha-\epsilon}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{\alpha-\epsilon}} \\ &= \frac{\|\mathbf{X}_{[1,n]}/x_n\|_{\alpha-\epsilon}^{\alpha-\epsilon}}{\|\mathbf{X}_{[1,n]}/x_n\|_{q'}^{\alpha-\epsilon}} \leq \|\mathbf{X}_{[1,n]}/x_n\|_{\alpha-\epsilon}^{\alpha-\epsilon}. \end{aligned}$$

Similar calculations yield

$$\begin{aligned} &\left| \frac{\partial^2 f_q(\mathbf{X}_{[1,n]})}{\partial q^2} \Big|_{q=q'} \right| \\ &= \frac{\|\mathbf{X}_{[1,n]}\|_\infty^q}{\|\mathbf{X}_{[1,n]}\|_{q'}^{q'}} \sum_{t=1}^n \sum_{j=1}^n \frac{|\mathbf{X}_t|^{q'}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{q'}} \frac{|\mathbf{X}_j|^{q'}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{q'}} \log(\|\mathbf{X}_{[1,n]}\|_\infty/|\mathbf{X}_j|) \log(|\mathbf{X}_j|/|\mathbf{X}_t|) \\ &\leq \sum_{t=1}^n \sum_{j=1}^n \frac{|\mathbf{X}_t|^{q'}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{q'}} \frac{|\mathbf{X}_j|^{\alpha-\epsilon}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{\alpha-\epsilon}} |\log(|\mathbf{X}_j|/|\mathbf{X}_t|)| \\ &\leq \|\mathbf{X}_{[1,n]}/x_n\|_{\alpha-\epsilon}^{\alpha-\epsilon} \\ &\quad - \sum_{t=1}^n \sum_{j=1}^n \frac{|\mathbf{X}_t|^{q'}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{q'}} \frac{|\mathbf{X}_j|^{\alpha-\epsilon}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{\alpha-\epsilon}} \log(|\mathbf{X}_t|/|\mathbf{X}_j|) \mathbb{1}(|\mathbf{X}_j|/|\mathbf{X}_t| < 1). \\ &= \|\mathbf{X}_{[1,n]}/x_n\|_{\alpha-\epsilon}^{\alpha-\epsilon} + R \end{aligned} \tag{F.118}$$

To study (F.118), we handle separately the two remaining terms. Regarding the remaining term R in (F.118), note,

$$\begin{aligned}
0 &< R \\
&= \sum_{t=1}^n \sum_{j=1}^n \frac{|\mathbf{X}_t|^{q'}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{q'}} \frac{|\mathbf{X}_j|^{\alpha-\epsilon}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{\alpha-\epsilon}} \log(|\mathbf{X}_t|/|\mathbf{X}_j|) \mathbb{1}(|\mathbf{X}_t|/|\mathbf{X}_j| > 1) \\
&\leq \sum_{t=1}^n \sum_{j=1}^n \frac{|\mathbf{X}_t|^{q'}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{q'}} \frac{|\mathbf{X}_j|^{\alpha-\epsilon}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{\alpha-\epsilon}} \left(\frac{|\mathbf{X}_t|}{|\mathbf{X}_j|}\right)^\epsilon \mathbb{1}(|\mathbf{X}_j|/|\mathbf{X}_t| > 1) \\
&= \frac{\|\mathbf{X}_{[1,n]}\|_{q'}^{\alpha-2\epsilon}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{\alpha-\epsilon}} \sum_{t=1}^n \frac{|\mathbf{X}_t|^{q'}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{q'}} |\mathbf{X}_t|^\epsilon.
\end{aligned}$$

Then, by an application of Hölder's inequality, we obtain

$$\begin{aligned}
R &\leq \frac{\|\mathbf{X}_{[1,n]}\|_{q'}^{\alpha-2\epsilon}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{\alpha-\epsilon}} \left(\sum_{j=1}^n \frac{|\mathbf{X}_j|^{pq'}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{pq'}} \right)^{1/p} \left(\sum_{j=1}^n |\mathbf{X}_j|^{p'\epsilon} \right)^{1/p'} \\
&= \frac{\|\mathbf{X}_{[1,n]}\|_{q'}^{\alpha-2\epsilon}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{\alpha-\epsilon}} \frac{\|\mathbf{X}_{[1,n]}\|_{pq'}^{q'}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{q'}} \|\mathbf{X}_{[1,n]}\|_{p'\epsilon}^\epsilon \\
&\leq \frac{\|\mathbf{X}_{[1,n]}\|_{q'}^{\alpha-2\epsilon}}{\|\mathbf{X}_{[1,n]}\|_{q'}^{\alpha-\epsilon}} \|\mathbf{X}_{[1,n]}\|_{p'\epsilon}^\epsilon
\end{aligned}$$

for $p > 1$, and $1/p + 1/p' = 1$. Then, we see it is enough to show that for $\eta < \epsilon < 1$,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[(\|\mathbf{X}_{[1,n]}/x_n\|_{\alpha-2\epsilon}^{\alpha-2\epsilon} \|\mathbf{X}_{[1,n]}/x_n\|_\epsilon)^\epsilon]^{1+\eta} \mathbb{1}(\|\mathbf{X}_{[1,n]}\|_{q'} > x_n)}{\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_{q'} > x_n)} < \infty.$$

Moreover, by an application of Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}[(\|\mathbf{X}_{[1,n]}/x_n\|_{\alpha-2\epsilon}^{\alpha-2\epsilon} \|\mathbf{X}_{[1,n]}/x_n\|_\epsilon)^\epsilon]^{1+\eta} \mathbb{1}(\|\mathbf{X}_{[1,n]}\|_{q'} > x_n) \\
&\leq \mathbb{E}[(\|\mathbf{X}_{[1,n]}/x_n\|_{\alpha-\epsilon}^{\alpha-\epsilon})^{1+\eta}] \\
&\leq \mathbb{E}[\|\mathbf{X}_{[1,n]}/x_n\|_{\alpha-\epsilon+\eta}^{\alpha-\epsilon+\eta}],
\end{aligned}$$

where the last inequality follow by subadditivity when $\eta < 1$. Therefore, it is enough to show there exists $\epsilon, \eta > 0$ such that $\alpha - \epsilon - \eta > \epsilon/2$,

$$\text{(F.119)} \quad \lim_{n \rightarrow \infty} \frac{\mathbb{E}[\|\mathbf{X}_{[1,n]}/x_n\|_{\alpha-\epsilon}^{\alpha-\epsilon} \mathbb{1}(\|\mathbf{X}_{[1,n]}\|_{\alpha-\epsilon-\eta} > x_n)]}{\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_{\alpha-\epsilon} > x_n)} < \infty.$$

We verify that for all $\epsilon, \eta > 0$ such that $\alpha - \epsilon - \eta > \epsilon/2$, and (F.119) holds and an application of Lemma F.1 yields the desired result. Indeed, notice

for all $\delta > 0$,

$$\begin{aligned}
& \frac{\mathbb{E}[\|\mathbf{X}_{[1,n]}/x_n\|_{\alpha-\epsilon}^{\alpha-\epsilon} \mathbf{1}(\|\mathbf{X}_{[1,n]}\|_{\alpha-\epsilon-\eta} > x_n)]}{\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_{\alpha-\epsilon} > x_n)} \\
&= \frac{\mathbb{E}[\overline{\|\mathbf{X}_{[1,n]}/x_n\|_{\alpha-\epsilon}^{\alpha-\epsilon}}^{\delta} \mathbf{1}(\|\mathbf{X}_{[1,n]}\|_{\alpha-\epsilon-\eta} > x_n)]}{\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_{\alpha-\epsilon} > x_n)} \\
&+ \frac{\mathbb{E}[\|\mathbf{X}_{[1,n]}/x_{n_\delta}\|_{\alpha-\epsilon}^{\alpha-\epsilon} \mathbf{1}(\|\mathbf{X}_{[1,n]}\|_{\alpha-\epsilon-\eta} > x_n)]}{\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_{\alpha-\epsilon} > x_n)} \\
\text{(F.120)} \quad &= I + II,
\end{aligned}$$

and here we recall the notation $\overline{\|\mathbf{X}_{[1,n]}/x_n\|_{\alpha-\epsilon}^{\alpha-\epsilon}}^{\delta} = (\overline{\mathbf{X}_1/x_n}^{\delta}, \dots, \overline{\mathbf{X}_n/x_n}^{\delta})$, and $\underline{\|\mathbf{X}_{[1,n]}/x_{n_\delta}\|_{\alpha-\epsilon}^{\alpha-\epsilon}} = (\underline{\mathbf{X}_1/x_{n_\delta}}, \dots, \underline{\mathbf{X}_n/x_{n_\delta}})$. Next, we treat I and II separately. For II it is easy to see

$$I \leq n \frac{\mathbb{E}[\|\mathbf{X}_1/x_{n_\delta}\|_{\alpha-\epsilon}^{\alpha-\epsilon}]}{\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_{\alpha-\epsilon} > x_n)} = \frac{n\mathbb{P}(\|\mathbf{X}_1\| > x_n)}{\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_{\alpha-\epsilon} > x_n)} \frac{\mathbb{E}[\|\mathbf{X}_1/x_{n_\delta}\|_{\alpha-\epsilon}^{\alpha-\epsilon}]}{\mathbb{P}(\|\mathbf{X}_1\| > x_n)}.$$

Then, by an application of Karamata's theorem we see $\lim_{n \rightarrow \infty} II < \infty$. Now to treat I , recall $\mathbb{E}[\overline{\|\mathbf{X}_{[1,n]}/x_{n_\delta}\|_{\alpha-\epsilon}^{\alpha-\epsilon}}] \rightarrow 0$, as $n \rightarrow \infty$ since $\mathbb{E}[\|\mathbf{X}_1\|_{\alpha-\epsilon}^{\alpha-\epsilon}] < \infty$ and $n/x_n^{\alpha-\epsilon} \rightarrow 0$, as $n \rightarrow \infty$. Hence, by Equation (F.117), we conclude $\lim_{n \rightarrow \infty} (I + II) < \infty$, following the notation in (F.120), and this completes the proof. \square

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