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December 23, 2022

## ON THE ASYMPTOTICS OF EXTREMAL $\ell^p$ -BLOCKS CLUSTER INFERENCE

GLORIA BURITICÁ AND OLIVIER WINTENBERGER

ABSTRACT. Extremes occur in stationary regularly varying time series as short periods with several large observations, known as extremal blocks. We study cluster statistics summarizing the behavior of functions acting on these extremal blocks. Examples of cluster statistics are the extremal index, cluster size probabilities, and other cluster indices. The purpose of our work is twofold. First, we state the asymptotic normality of block estimators for cluster inference based on consecutive observations with large  $\ell^p$ -norms, for  $p > 0$ . Second, we verify the conditions we require on classic models such as linear models and solutions of stochastic recurrence equations. Regarding linear models, we prove that the asymptotic variance of classical index cluster-based estimators is null as first conjectured in [26]. We illustrate our findings on simulations.

### 1. INTRODUCTION

We study stationary heavy-tailed time series with regularly varying distributions; cf. [5]. In this framework, extremal observations cluster: an extreme value triggers a short period with numerous large observations. This behavior is known to perturb classical inference procedures tailored for independent observations like high quantile inference; see [20]. This clustering effect can be summarized with the *extremal index*, initially introduced in [34] and [35]. We can interpret it as the inverse of the mean number of consecutive exceedances above a high threshold in a short period of time. In this article, we aim to infer such properties of the clustering effect by letting functionals act on consecutive observations with extremal behavior. For example, we can recover the extremal index from this setting and other important indices of the extremes of the series.

We consider cluster statistics of regularly varying time series  $(\mathbf{X}_t)$  with values in  $(\mathbb{R}^d, |\cdot|)$ , and tail index  $\alpha > 0$ ; a formal definition is conferred to Section 2.2. For cluster inference, we consider a sample  $\mathbf{X}_{[1,n]}$  together with the sequence  $(b_n)$ , and we define the sample of disjoint blocks  $(\mathcal{B}_j)_{j=1,\dots,m_n}$  as blocks of consecutive observations with

$$(1.1) \quad \mathcal{B}_j := (\mathbf{X}_{(j-1)b_n+1}, \dots, \mathbf{X}_{jb_n}) = \mathbf{X}_{(j-1)b_n+[1:b_n]},$$

such that  $b_n \rightarrow \infty$ ,  $m_n = n/b_n \rightarrow \infty$ , as  $n \rightarrow \infty$ . We follow the  $p$ -clusters theory developed in [12] for a fixed  $p \in (0, \infty]$ . The extremal behavior of

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regularly varying time series is modeled by the conditional behavior of a block  $\mathcal{B}_j$  given that its  $\ell^p$ -norm is large:

$$(1.2) \quad \mathbb{P}(\mathcal{B}_1/x_{b_n} \in A \mid \|\mathcal{B}_1\|_p > x_{b_n}) \xrightarrow{w} \mathbb{P}(Y\mathbf{Q}^{(p)} \in A), \quad n \rightarrow \infty,$$

such that  $Y$  is independent of  $\mathbf{Q}^{(p)} \in \ell^p$ ,  $\mathbb{P}(Y > y) = y^{-\alpha}$ , for  $y > 1$ ,  $\|\mathbf{Q}^{(p)}\|_p = 1$  a.s. The weak convergence holds for a family of shift-invariant continuity sets  $A \subset \ell^p$ , and  $(x_n)$  is a suitable sequence satisfying  $\mathbb{P}(\|\mathcal{B}_1\|_p > x_{b_n}) \rightarrow 0$ , as  $n \rightarrow \infty$ . The spectral  $p$ -cluster process  $\mathbf{Q}^{(p)}$  models the short period's behavior under the rare event that its  $\ell^p$ -norm is large. We study  $p$ -cluster statistics of the form

$$(1.3) \quad f_p^{\mathbf{Q}} = \mathbb{E}[f_p(Y\mathbf{Q}^{(p)})],$$

for suitable  $\ell^p$ -continuity functions  $f_p : \ell^p \rightarrow \mathbb{R}$ , invariant to the shift operator. To infer the cluster statistic (1.3), we use the disjoint blocks estimators proposed in [12], and defined as

$$(1.4) \quad \widehat{f}_p^{\mathbf{Q}} := \frac{1}{k} \sum_{t=1}^{m_n} f_p(\mathcal{B}_t / \|\mathcal{B}\|_{p,(k+1)}) \mathbf{1}(\|\mathcal{B}_t\|_p > \|\mathcal{B}\|_{p,(k+1)}),$$

where  $\|\mathcal{B}\|_{p,(1)} \geq \|\mathcal{B}\|_{p,(2)} \geq \dots \geq \|\mathcal{B}\|_{p,(m_n)}$ , denotes the sequence of order statistics of the  $\ell^p$ -norms of blocks defined in (1.1).

The main goal of this article is to establish the asymptotic normality of the block estimators in Equation (1.4), tailored for cluster inference. We state moment, mixing and bias assumptions yielding the existence of a sequence  $(k_n)$ , satisfying  $k = k_n \rightarrow \infty$ ,  $m_n/k_n \rightarrow \infty$  such that

$$(1.5) \quad \sqrt{k}(\widehat{f}_p^{\mathbf{Q}} - f_p^{\mathbf{Q}}) \xrightarrow{d} \mathcal{N}(0, \text{Var}(f_p(Y\mathbf{Q}^{(p)}))), \quad n \rightarrow \infty,$$

and the limit is a centered Gaussian distribution. As a result, we see that the asymptotic variance of the blocks estimator can be computed in terms of the  $p$ -cluster  $\mathbf{Q}^{(p)}$ . In general, it is possible to obtain numerous representations of one cluster statistics in (1.3) combining the choice of  $p$  with  $f_p$ . This follows by the change-of-norms equation given in [12] that we recall in (2.12). For example, the extremal index admits distinct  $p$ -clusters representations in the form of (1.3) if we use  $p = \infty$  or  $p = \alpha$ , the tail-index of the series; cf. [13]. This strategy points to new inference methodologies to estimate the same statistic. The asymptotic result in (1.5) allows us to compare the variances of these inference procedures tuned with different  $p$ .

We show that introducing  $\ell^p$ -norm block order statistics in (1.4), instead of order statistics of the sample  $(|\mathbf{X}_t|)$  as in [18, 14], can lead to a better asymptotic variance for cluster inference. We give examples of variance reduction in the case of linear models with short-range dependence, for inference of classical indices. In our examples, the asymptotic variance  $\text{Var}(f_p(Y\mathbf{Q}^{(p)}))$  is null because of the deterministic properties of the spectral  $p$ -cluster process of linear models. For linear models, the advantage of replacing thresholds with block maxima records was previously investigated

in [26]. Existing works [17, 18, 14, 33] following [26] focus on cluster of exceedances inference such that  $p = \infty$ . Our asymptotic result comforts and extends the heuristics presented in [26] for  $p = \infty$  and linear models to the case  $p < \infty$  and general models. To prove the asymptotic normality of block estimators, we rely on the asymptotics of disjoint block estimators studied in Theorem 2.10. in [17], and the theory of empirical processes therein. We also follow the modern overview in [33]. To handle the asymptotics of extremal  $\ell^p$ -blocks, we build on the large deviation principles studied in [12], and appeal to the  $p$ -cluster processes theory therein.

The blocks estimator in Equation (1.4) compares favourably to state-of-the-art methodologies for cluster inference in terms of asymptotic variance. Considering inference through extremal  $\ell^p$ -blocks with  $p < \infty$  has already proven in [12] to be advantageous in terms of bias. Choosing  $p = \alpha$  can be useful in practice to make tuning the parameters in the blocks estimator less susceptible to time dependencies; see [12]. We illustrate on simulations that the  $\ell^\alpha$ -block estimator is competitive both in terms of bias and variance for finite sample sizes. This approach performs well even when the tail index  $\alpha > 0$  has to be estimated with a Hill-type procedure.

The article is organized as follows. Preliminaries on mixing coefficients, regular variation, and the  $p$ -clusters theory of stationary time series are compiled in Section 2. In Section 3 we present our main result in Theorem 3.1, stating the asymptotic normality of the block estimators introduced in Equation (1.4). We work under mixing, moment, and bias conditions on the series that we also present in Section 3. Section 4 studies examples of extremal cluster inference such as estimation of the extremal index, the cluster size probabilities, and the cluster index for sums. We conclude by verifying our conditions on classical models such as linear processes and stochastic recurrence equations in Section 5. In the case of linear models with short-range dependence, Theorem 5.6 states that the  $\ell^p$ -block estimators of all the aforementioned quantities have null-asymptotic variance. Thereby, they are super-efficient for cluster inference of important indices as conjectured by [26] for  $p = \infty$ . We illustrate the finite-sample performances of our estimators in Section 6. All proofs are deferred to Section A.

**1.1. Notation.** We consider stationary time series  $(\mathbf{X}_t)$  taking values in  $\mathbb{R}^d$ , that we endow with a norm  $|\cdot|$ . Let  $p > 0$ , and  $(\mathbf{x}_t) \in (\mathbb{R}^d)^{\mathbb{Z}}$ . Define the  $p$ -modulus function  $\|\cdot\|_p : (\mathbb{R}^d)^{\mathbb{Z}} \rightarrow [0, +\infty]$  as

$$\|(\mathbf{x}_t)\|_p^p := \sum_{t \in \mathbb{Z}} |\mathbf{x}_t|^p,$$

and define the sequential space  $\ell^p$  as

$$\ell^p := \{(\mathbf{x}_t) \in (\mathbb{R}^d)^{\mathbb{Z}} : \|(\mathbf{x}_t)\|_p^p < +\infty\},$$

with the convention that, for  $p = \infty$ , the space  $\ell^\infty$  refers to sequences with finite supremum norm. For any  $p \in (0, +\infty]$ , the  $p$ -modulus functions induce a distance  $d_p$  in  $\ell^p$ , and for  $p \in [1, +\infty)$ , it defines a norm. Abusing notation, we call them  $\ell^p$ -norms for  $p \in (0, +\infty]$ . Let  $\tilde{\ell}^p = \ell^p / \sim$  be the shift-invariant

quotient space where:  $(\mathbf{x}_t) \sim (\mathbf{y}_t)$  if and only if there exists  $k \in \mathbb{Z}$  such that  $\mathbf{x}_{t-k} = \mathbf{y}_t$ ,  $t \in \mathbb{Z}$ . We also consider the metric space  $(\tilde{\ell}^p, \tilde{d}_p)$  such that for  $[\mathbf{x}], [\mathbf{y}] \in \tilde{\ell}^p$ ,

$$\tilde{d}_p([\mathbf{x}], [\mathbf{y}]) = \inf_{k \in \mathbb{Z}} \{d_p(\mathbf{x}_{t-k}, \mathbf{y}_t), (\mathbf{x}_t) \in [\mathbf{x}], (\mathbf{y}_t) \in [\mathbf{y}]\},$$

and without loss of generality, we write an element  $[\mathbf{x}]$  in  $\tilde{\ell}^p$  also as  $(\mathbf{x}_t)$ . Further details on the shift-invariant spaces are deferred to [12, 4].

The operator norm for  $d \times d$  matrices,  $\mathbf{A} \in \mathbb{R}^{d \times d}$ , is defined as  $|\mathbf{A}|_{op} := \sup_{|\mathbf{x}|=1} |\mathbf{A}\mathbf{x}|$ . The truncation operations of  $(\mathbf{x}_t)$  at the level  $\epsilon$ , for  $\epsilon > 0$ , are defined by

$$(\bar{\mathbf{x}}_t^\epsilon) = (\mathbf{x}_t \mathbb{1}_{|\mathbf{x}_t| \leq \epsilon}), \quad (\underline{\mathbf{x}}_{t\epsilon}) = (\mathbf{x}_t \mathbb{1}_{|\mathbf{x}_t| > \epsilon}).$$

The notation  $a \wedge b$  denotes the minimum between two constants  $a, b \in \mathbb{Z}$ , and  $a \vee b$  denotes its maximum. We write  $\log^+(x) = \log(x) \vee 0$ , for  $x \in (0, \infty)$ . We sometimes write  $\mathbf{x}$  for the sequence  $\mathbf{x} := (\mathbf{x}_t) \in (\mathbb{R}^d)^\mathbb{Z}$ . Furthermore, for  $a, b \in \mathbb{Z}$ , and  $a \leq b$ , we write as  $\mathbf{x}_{[a,b]}$  the vector  $(\mathbf{x}_t)_{t=a, \dots, b}$  taking values in  $(\mathbb{R}^d)^{b-a+1}$ . We sometimes write  $\mathbf{x}_{[a,b]} \in \tilde{\ell}^p$ , which means we take the natural embedding of  $\mathbf{x}_{[a,b]}$  in  $\tilde{\ell}^p$  defined by assigning zeros to undefined coefficients. It will be convenient to write  $\mathcal{G}_+(\tilde{\ell}^p)$  for the continuous non-negative functions on  $(\tilde{\ell}^p, \tilde{d}_p)$  which vanish in a neighborhood of the origin.

## 2. PRELIMINARIES

**2.1. Mixing coefficients.** Let  $(\mathbf{X}_t)$  be an  $\mathbb{R}^d$ -valued strictly stationary time series defined over a probability space  $((\mathbb{R}^d)^\mathbb{Z}, \mathcal{A}, \mathbb{P})$ . The properties of stationary sequences are usually studied through mixing coefficients. Denote the past and future  $\sigma$ -algebras by

$$\mathcal{F}_{t \leq 0} := \sigma((\mathbf{X}_t)_{t \leq 0}), \quad \mathcal{F}_{t \geq h} := \sigma((\mathbf{X}_t)_{t \geq h}), \quad h \geq 1,$$

respectively. We recall the definition of mixing coefficients  $(\rho_h), (\beta_h)$ , below

$$\begin{aligned} \rho_h &= \sup_{f \in L^2(\mathcal{F}_{t \leq 0}), g \in L^2(\mathcal{F}_{t \geq h})} |\text{Corr}(f, g)|, \\ \beta_h &= d_{TV}(\mathbb{P}_{\mathcal{F}_{t \leq 0} \otimes \mathcal{F}_{t \geq h}}, \mathbb{P}_{\mathcal{F}_{t \leq 0}} \otimes \mathbb{P}_{\mathcal{F}_{t \geq h}}), \end{aligned}$$

where  $d_{TV}(\cdot, \cdot)$  is the total variation distance between two probability measures:  $((\mathbb{R}^d)^\mathbb{Z}, \mathcal{A}, \mathbb{P}_1)$ ,  $((\mathbb{R}^d)^\mathbb{Z}, \mathcal{A}, \mathbb{P}_2)$ , and  $\mathbb{P}_1 \otimes \mathbb{P}_2(A \times B) = \mathbb{P}_1(A)\mathbb{P}_2(B)$ , for  $A, B \in \mathcal{A}$ . For a summary on mixing conditions see [8, 16, 42].

**Remark 2.1.** The  $\rho$ -mixing coefficients  $(\rho_t)_{t \geq 0}$  were introduced in [32], and popularized due to the Ibragimov central limit theorem for dependent stationary sequences in [28]. This theorem states that sufficient conditions for the central limit of stationary sequences to hold are the mixing condition:  $\sum_{t=1}^{\infty} \rho_{2^t} < \infty$ , together with a moment assumption of order  $\kappa > 2$ . The aforementioned mixing condition was studied in detail in [7, 40, 46, 48]; see [8] for a review. However, aside from the  $m_0$ -dependent case and the

Gaussian case (where  $\rho_t = 0$  for  $t > m_0$  and  $\rho_t \leq \pi\beta_t$ , respectively) there is no general recipe for computing  $\rho$ -mixing rates.

**Remark 2.2.** A detailed interpretation of the  $\beta$ -mixing coefficients ( $\beta_h$ ) in terms of the total variation distance can be found in Chapter 1.2 in [16]. These mixing coefficients are well adapted while working with Markov processes. Indeed, a strictly stationary Harris recurrent Markov chain  $(\mathbf{X}_t)$ , satisfies  $\beta_t \rightarrow 0$  as  $t \rightarrow \infty$ ; see Theorem 3.5 in [8].

**2.2. Regular variation.** We consider stationary time series  $(\mathbf{X}_t)$  taking values in  $(\mathbb{R}^d, |\cdot|)$  and that it is regularly varying with tail index  $\alpha > 0$ : all its finite-dimensional vectors are multivariate regularly varying of the same index. In this case we write  $(\mathbf{X}_t)$  satisfies  $\mathbf{RV}_\alpha$ . Borrowing the ideas in [5],  $(\mathbf{X}_t)$  satisfies  $\mathbf{RV}_\alpha$  if and only if, for all  $h \geq 0$ , there exists a vector  $(\Theta_t)_{|t| \leq h}$ , taking values in  $(\mathbb{R}^d)^{2h+1}$  such that

$$(2.6) \quad \mathbb{P}(x^{-1}(\mathbf{X}_t)_{|t| \leq h} \in \cdot \mid |\mathbf{X}_0| > x) \xrightarrow{d} \mathbb{P}(Y(\Theta_t)_{|t| \leq h} \in \cdot), \quad x \rightarrow +\infty,$$

where  $Y$  is independent of  $(\Theta_t)_{|t| \leq h}$  and  $\mathbb{P}(Y > y) = y^{-\alpha}, y > 1$ . We call the sequence  $(\Theta_t)$ , taking values in  $(\mathbb{R}^d)^{\mathbb{Z}}$ , the spectral tail process.

The time series  $(\Theta_t)$  does not inherit the stationarity property of the series. Instead, the time-change formula of [5] holds: for any  $s, t \in \mathbb{Z}, s \leq 0 \leq t$  and for any measurable bounded function  $f : (\mathbb{R}^d)^{t-s+1} \rightarrow \mathbb{R}$ ,

$$(2.7) \quad \mathbb{E}[f(\Theta_{s-i}, \dots, \Theta_{t-i}) \mathbb{1}(|\Theta_{-i}| \neq 0)] = \mathbb{E}[|\Theta_i|^\alpha f(\Theta_s/|\Theta_i|, \dots, \Theta_t/|\Theta_i|)].$$

**2.3.  $p$ -cluster processes.** Let  $(\mathbf{X}_t)$  be a stationary time series satisfying  $\mathbf{RV}_\alpha$ . For  $p > 0$ , we say the series admits a  $p$ -cluster process  $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$  if there exists a well-chosen sequence  $(x_n)$ , satisfying

$$(2.8) \quad \mathbb{P}(\|\mathbf{X}_{[1,n]}\|_p > x_n) \sim n c(p) \mathbb{P}(|\mathbf{X}_1| > x_n), \quad n \rightarrow \infty$$

with  $c(p) \in (0, \infty)$ ,  $n\mathbb{P}(|\mathbf{X}_1| > x_n) \rightarrow 0$ , and

$$(2.9) \quad \mathbb{P}(\mathbf{X}_{[1,n]}/x_n \in \cdot \mid \|\mathbf{X}_{[1,n]}\|_p > x_n) \xrightarrow{w} \mathbb{P}(Y\mathbf{Q}^{(p)} \in \cdot), \quad n \rightarrow \infty,$$

where  $Y$  is independent of  $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$ ,  $\mathbb{P}(Y > y) = y^{-\alpha}$ , for  $y > 1$ ,  $\|\mathbf{Q}^{(p)}\|_p = 1$  a.s., and the limit in (2.9) holds in  $(\tilde{\ell}^p, \tilde{d}_p)$ . We study below the anti-clustering and vanishing-small values conditions noted  $\mathbf{AC}$ ,  $\mathbf{CS}_p$ , respectively, which guarantee the existence of  $p$ -clusters. We rephrase next the Theorem 2.1. of [12].

**Proposition 2.3.** *Let  $(\mathbf{X}_t)$  be a stationary time series satisfying  $\mathbf{RV}_\alpha$ . Let  $(x_n)$  be a sequence such that  $n\mathbb{P}(|\mathbf{X}_1| > x_n) \rightarrow 0$ , as  $n \rightarrow \infty$  and  $p > 0$ . For all  $\delta, \epsilon > 0$ , assume*

$$\mathbf{AC}: \lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(\|\mathbf{X}_{[s,n]}\|_\infty > \epsilon x_n \mid |\mathbf{X}_1| > \epsilon x_n),$$

$$\mathbf{CS}_p: \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\|\overline{\mathbf{X}_{[1,n]}/x_n}^\epsilon\|_p > \delta)}{n\mathbb{P}(|\mathbf{X}_1| > x_n)} = 0.$$

Then, if  $p \geq \alpha$ , Equation (2.8) holds with  $c(\infty) \leq c(p) \leq c(\alpha) = 1$ , and  $(\mathbf{X}_t)$  admits a  $p$ -cluster process  $\mathbf{Q}^{(p)}$  in the sense of (2.9). If  $p < \alpha$ , existence of the  $p$ -cluster process holds if  $\mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_p^\alpha] < \infty$ . In this case, Equation (2.8) holds with  $c(p) = \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_p^\alpha]$ .

From Proposition 2.3 we see that assuming **AC**, and **CS** $_\alpha$ , the time series  $(\mathbf{X}_t)$  admits an  $\alpha$ -cluster  $\mathbf{Q}^{(\alpha)}$ , where  $\alpha > 0$ , denotes the tail index. In this case, appealing to Proposition 3.1. in [12], we have

$$(2.10) \quad \mathbf{Q}^{(\alpha)} \stackrel{d}{=} \boldsymbol{\Theta} / \|\boldsymbol{\Theta}\|_\alpha, \quad \in \tilde{\ell}^\alpha,$$

where  $(\boldsymbol{\Theta}_t)$  is the spectral tail process from Equation (2.6). Moreover, if **CS** $_p$ , **CS** $_{p'}$ , and  $\mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_p^\alpha] + \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_{p'}^\alpha] < \infty$  also hold, then the  $p, p'$ -clusters exist and are related by the change-of-norms formula below

$$(2.11) \quad \begin{aligned} & \mathbb{P}(\mathbf{Q}^{(p)} \in \cdot) \\ &= c(p)^{-1} \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_p^\alpha \mathbb{1}(\mathbf{Q}^{(\alpha)} / \|\mathbf{Q}^{(\alpha)}\|_p \in \cdot)] \end{aligned}$$

$$(2.12) \quad = \frac{c(p')}{c(p)} \mathbb{E}[\|\mathbf{Q}^{(p')}\|_p^\alpha \mathbb{1}(\mathbf{Q}^{(p')} / \|\mathbf{Q}^{(p')}\|_p \in \cdot)].$$

Since  $\|\mathbf{Q}^{(p)}\|_p = 1$  a.s. for any  $p > 0$ , then  $c(\alpha) = 1$ , and  $\mathbb{E}[\|\mathbf{Q}^{(p')}\|_p^\alpha] = c(p)/c(p')$ , where  $c(p), c(p')$ , are as in Equation (2.8).

**Remark 2.4.** We can check readily that if  $\sum_{t=1}^\infty \rho_t < \infty$ , then **AC** holds for all sequences of levels  $(x_n)$  satisfying  $n\mathbb{P}(|\mathbf{X}_1| > x_n) \rightarrow 0$ , as  $n \rightarrow \infty$ .

**Remark 2.5.** Using the monotonicity of norms, we see straightforwardly that **CS** $_p$  implies the condition **CS** $_{p'}$ , for  $p' > p > 0$ . If  $p > \alpha$ , where  $\alpha$  is the tail index, the condition **CS** $_p$  is always satisfied for sequences  $(x_n)$  such that  $n\mathbb{P}(|\mathbf{X}_1| > x_n) \rightarrow 0$ , as  $n \rightarrow \infty$ . In the case  $\alpha/2 < p \leq \alpha$ , if  $\sum_{t=0}^\infty \rho_t < \infty$ , then condition **CS** $_p$  holds for sequences  $(x_n)$  such that there exists  $\kappa > 0$ , satisfying  $n/x_n^{p \wedge (\alpha - \kappa)} \rightarrow 0$ , as  $n \rightarrow \infty$ . This follows by Remarks 5.1. and 5.2. in [12].

### 3. ASYMPTOTIC NORMALITY

**3.1. Main result.** Let  $(\mathbf{X}_t)$  be an  $\mathbb{R}^d$ -valued stationary time series satisfying **RV** $_\alpha$ . Assume the conditions of Proposition 2.3 hold for  $p > 0$ , the series admits a  $p$ -cluster process  $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$ , and (2.9) holds for a sequence of high levels  $(x_n)$  satisfying  $\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_p > x_n) \rightarrow 0$ . Recall the block estimator in (1.4) is tuned with the block lengths  $(b_n)$ , and the number  $(k_n)$  of extremal blocks. The total number of disjoint blocks in a sample is denoted  $(m_n)$  with  $m_n = \lfloor n/b_n \rfloor$ . We assume the relation between  $(k_n)$  and  $(b_n)$

$$(3.13) \quad \begin{aligned} k & := k_n = \lfloor m_n \mathbb{P}(\|\mathbf{X}_{[1,n]}\|_p > x_{b_n}) \rfloor \\ & \sim n c(p) \mathbb{P}(|\mathbf{X}_1| > x_{b_n}), \quad n \rightarrow \infty, \end{aligned}$$

holds, where  $c(p) \in (0, \infty)$  are as in (2.8). In what follows, if the sequences  $(x_n)$ ,  $(b_n)$ ,  $(k_n)$ ,  $(m_n)$ , appear then they coincide with the ones mentioned here.

Our main result is presented in this section in Theorem 3.1. It establishes the asymptotic normality of the block estimator in (1.4) under the moment, bias, and mixing assumptions that we introduce below.

**L:** Let  $f_p : \tilde{\ell}^p \rightarrow \mathbb{R}$ , be such that  $f_p \in \mathcal{G}_+(\tilde{\ell}^p)$ . Moreover assume  $u \mapsto f_p(\mathbf{x}_t)/u$  is a non-increasing function, and assume there exists  $\delta > 0$  such that for all  $u > 0$ ,

$$(3.14) \quad \frac{m_n}{k_n} \mathbb{E}[f_p(\mathcal{B}_1/u x_{b_n})^{2+\delta}] = O(1),$$

as  $n \rightarrow \infty$ , and  $\mathbb{E}[f_p(Y\mathbf{Q}^{(p)})^2] < \infty$ .

**B:** Let  $f_p : \tilde{\ell}^p \rightarrow \mathbb{R}$  satisfy **L**. Fix  $\epsilon > 0$ , and assume the bias conditions

$$(3.15) \quad 0 = \lim_{n \rightarrow \infty} \sqrt{k} \sup_{u \in [1-\epsilon, 1+\epsilon]} \left| \frac{\mathbb{E}[f_p(\mathcal{B}_1/u x_{b_n}) \mathbb{1}(\|\mathcal{B}_1/x_{b_n}\|_p > u)]}{\mathbb{P}(\|\mathcal{B}_1\|_p > x_{b_n})} - u^{-\alpha} f_p^{\mathbf{Q}} \right|,$$

$$(3.16) \quad 0 = \lim_{n \rightarrow \infty} \sqrt{k} \sup_{u \in [1-\epsilon, 1+\epsilon]} \left| \frac{\mathbb{P}(\|\mathcal{B}_1/x_{b_n}\|_p > u)}{\mathbb{P}(\|\mathcal{B}_1\|_p > x_{b_n})} - u^{-\alpha} \right|,$$

where  $f_p^{\mathbf{Q}}$  is as in (1.3).

**MX:** Assume the mixing coefficients  $(\beta_t)$  satisfy the condition

$$(3.17) \quad \lim_{n \rightarrow \infty} m_n \beta_{b_n} = 0.$$

**MX $_{\beta}$ :** Let  $f_p : \tilde{\ell}^p \rightarrow \mathbb{R}$  satisfy **L**, and let  $\delta > 0$  be such that (3.14) hold. Assume there exists a sequences  $(\ell_n)$ , satisfying  $\ell_n \rightarrow \infty$ , and the mixing coefficients  $(\beta_t)$  satisfy  $m_n \beta_{\ell_n}/k_n \rightarrow 0$ ,  $\ell_n/b_n \rightarrow 0$ , as  $n \rightarrow \infty$ , and

$$(3.18) \quad \lim_{n \rightarrow \infty} \sum_{t=1}^{m_n} (m_n \beta_{tb_n}/k_n)^{\frac{\delta}{2+\delta}} = 0.$$

If  $f_p$  is bounded, we assume  $\sum_{t=1}^{m_n} m_n \beta_{tb_n}/k_n \rightarrow 0$  instead of (3.18).

**MX $_{\rho}$ :** Assume the correlation coefficients  $(\rho_t)$  satisfy  $\sum_{t=1}^{m_n} \rho_{tb_n} \rightarrow 0$  and

$$(3.19) \quad \lim_{s \rightarrow \infty} \lim_{b_n \rightarrow \infty} \sum_{t=s}^{b_n} \mathbb{P}(|\mathbf{X}_t| > \epsilon x_{b_n} \mid |\mathbf{X}_0| > \epsilon x_{b_n}) = 0.$$

We state in Theorem 3.1 below our main result on the asymptotic normality of the blocks estimator. We defer its proof to Section A.

**Theorem 3.1.** *Let  $(\mathbf{X}_t)$  be a stationary time series satisfying  $\mathbf{RV}_{\alpha}$ . Assume the conditions of Proposition 2.3 hold, such that the series admits a  $p$ -cluster*



process  $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$ . Consider  $f_p : \tilde{\ell}^p \rightarrow \mathbb{R}$  such that  $\mathbf{L}$  and  $\mathbf{B}$  hold. Assume also  $\mathbf{MX}$  holds, and assume either  $\mathbf{MX}_\beta$  or  $\mathbf{MX}_\rho$  hold. Then, there exists  $(k_n)$  such that  $k = k_n \rightarrow \infty$ ,  $m_n/k_n \rightarrow \infty$ , and

$$(3.20) \quad \sqrt{k}(\widehat{f_p^{\mathbf{Q}}} - f_p^{\mathbf{Q}}) \xrightarrow{d} \mathcal{N}(0, \text{Var}(f_p(Y^{\mathbf{Q}^{(p)}}))), \quad n \rightarrow \infty.$$

Here we follow the notation from equations (1.3) and (1.4).

**Remark 3.2.** The proof of Theorem 3.1 follows the functional central limit theorem stated in Theorem C.4.5 in [33]. Condition  $\mathbf{L}$  restricts the family of functions we can consider for inference. Equation (3.14) entails a Lindeberg-type condition holds.

#### 4. CLUSTER STATISTICS

In view of Theorem 3.1, we derive asymptotic normality of classical cluster index estimators in extreme value theory.

**4.1. The extremal index.** Let  $(\mathbf{X}_t)$  be a stationary time series in  $(\mathbb{R}^d, |\cdot|)$  satisfying  $\mathbf{RV}_\alpha$ . The extremal index  $\theta_{|\mathbf{X}|}$  of the series  $(|\mathbf{X}_t|)$  is a measure of serial clustering introduced in [34] and [35]. We recall the extremal index estimator proposed in [13], based on extremal  $\ell^\alpha$ -blocks.

**Corollary 4.1.** Consider  $f_p : \tilde{\ell}^p \rightarrow \mathbb{R}$  to be the function  $\mathbf{x} \mapsto \|\mathbf{x}\|_\infty^\alpha / \|\mathbf{x}\|_\alpha^\alpha$ . Assume the conditions of Theorem 3.1 hold for  $p = \alpha$ . Let  $\theta_{|\mathbf{X}|} = \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_\infty^\alpha]$ , hence we deduce an estimator

$$(4.21) \quad \widehat{\theta}_{|\mathbf{X}|} = \frac{1}{k} \sum_{t=1}^m \frac{\|\mathcal{B}_t\|_\infty^\alpha}{\|\mathcal{B}_t\|_\alpha^\alpha} \mathbb{1}(\|\mathcal{B}_t\|_\alpha > \|\mathcal{B}\|_{\alpha, (k+1)}),$$

such that

$$\sqrt{k}(\widehat{\theta}_{|\mathbf{X}|} - \theta_{|\mathbf{X}|}) \xrightarrow{d} \mathcal{N}(0, \text{Var}(\|\mathbf{Q}^{(\alpha)}\|_\infty^\alpha)), \quad n \rightarrow \infty.$$

*Proof.* The proof of Corollary 4.1 follows directly as  $f_p \in \mathcal{G}_+(\tilde{\ell}^p)$  is a bounded continuous function satisfying  $\mathbf{L}$ .  $\square$

For comparison we review the blocks estimator based on extremal  $\ell^\infty$ -blocks proposed in [25]:

$$(4.22) \quad \widehat{\theta}_{|\mathbf{X}|}^B = \frac{1}{k_n b_n} \sum_{t=1}^{m_n} \mathbb{1}(\|\mathcal{B}_t\|_\infty > |\mathbf{X}|_{(k+1)}).$$

Direct computations from Example 10.4.2 in [33] yield

$$\sqrt{k}(\widehat{\theta}_{|\mathbf{X}|}^B - \theta_{|\mathbf{X}|}) \xrightarrow{d} \mathcal{N}(0, \sigma_\theta^2), \quad n \rightarrow \infty,$$

where  $\sigma_\theta^2 \in [0, +\infty)$ , and

$$\begin{aligned}
 \sigma_\theta^2 &= \theta_{|\mathbf{x}|}^2 \sum_{j \in \mathbb{Z}} \mathbb{E}[|\Theta_j|^\alpha \wedge 1] - \theta_{|\mathbf{x}|} \\
 (4.23) \quad &= \theta_{|\mathbf{x}|}^2 \sum_{j \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \mathbb{E}[|\mathbf{Q}_{j+t}^{(\alpha)}|^\alpha \wedge |\mathbf{Q}_t^{(\alpha)}|^\alpha] - \theta_{|\mathbf{x}|}.
 \end{aligned}$$

The last equality follows appealing to the time-change formula in (2.7) and Equation (2.10). As a result, we can compare the asymptotic variances of  $\hat{\theta}_{|\mathbf{x}|}$  and  $\hat{\theta}_{|\mathbf{x}|}^B$  in the cases where  $\mathbf{Q}^{(\alpha)}$  is known. This is the topic of Section 5.

**4.2. The cluster index for sums.** Let  $(\mathbf{X}_t)$  be a stationary time series with values in  $(\mathbb{R}^d, |\cdot|)$  satisfying  $\mathbf{RV}_\alpha$ . We recall [39] coined the constant  $c(1)$  in (2.8) as the cluster index for sums. We review a cluster-based estimator of it, introduced in [12], based on extremal  $\ell^\alpha$ -blocks.

**Corollary 4.2.** *Consider  $f_p : \tilde{\ell}^p \rightarrow \mathbb{R}$  to be the function  $\mathbf{x} \mapsto \|\mathbf{x}\|_1^\alpha / \|\mathbf{x}\|_\alpha^\alpha$ . Assume the conditions of Theorem 3.1 hold for  $p = \alpha \wedge 1$ , and  $\alpha < 2$ . Let  $c(1) = \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_1^\alpha] < \infty$ , hence one deduces an estimator*

$$(4.24) \quad \hat{c}(1) = \frac{1}{k} \sum_{t=1}^m \frac{\|\mathcal{B}_t\|_1^\alpha}{\|\mathcal{B}_t\|_\alpha^\alpha} \mathbb{1}(\|\mathcal{B}_t\|_\alpha > \|\mathcal{B}\|_{\alpha, (k+1)}),$$

such that

$$\sqrt{k}(\hat{c}(1) - c(1)) \xrightarrow{d} \mathcal{N}(0, \text{Var}(\|\mathbf{Q}^{(\alpha)}\|_1^\alpha)), \quad n \rightarrow \infty,$$

and  $c(1)$  is as in (2.8) with  $p = 1$ .

*Proof.* The proof of Corollary 4.2 follows directly from Theorem 3.1 as  $f_p \in \mathcal{G}_+(\ell^p)$  is a bounded continuous function satisfying  $\mathbf{L}$ .  $\square$

Another sums index cluster-based estimator we can consider is the one proposed in [33] based on extremal  $\ell^\infty$ -blocks:

$$(4.25) \quad \hat{c}^B(1) = \frac{1}{kb_n} \sum_{t=1}^{m_n} \mathbb{1}(\|\mathcal{B}_t\|_1 > |\mathbf{X}|_{(k+1)}).$$

Then, relying on Example 10.4.2 in [33],

$$\sqrt{k}(\hat{c}^B - c(1)) \xrightarrow{d} \mathcal{N}(0, \sigma_{c(1)}^2), \quad n \rightarrow \infty.$$

for a constant  $\sigma_{c(1)}^2 \in [0, +\infty)$  defined by

$$(4.26) \quad \sigma_{c(1)}^2 = c(1)^2 \sum_{j \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \mathbb{E}[|\mathbf{Q}_{j+t}^{(\alpha)}|^\alpha \wedge |\mathbf{Q}_t^{(\alpha)}|^\alpha] - c(1).$$

Similarly as in Example 4.1, whenever  $\mathbf{Q}^{(\alpha)}$  is known, we can compare directly the asymptotic variances relative to the estimators  $\hat{c}(1)$  and  $\hat{c}^B(1)$ . Section 5 covers this topic for classical models where the cluster process is known.

**4.3. The cluster sizes.** In general, a classical approach to model serial exceedances is using point processes as in [35] and [26]. For the levels  $(a_n)$ , satisfying  $n\mathbb{P}(|X_1| > a_n) \rightarrow 1$ , as  $n \rightarrow \infty$ , and for every fixed  $x > 0$  consider the point process of exceedances with state space  $(0, 1]$ :

$$\eta_{n,x}(\cdot) := \bar{N}_n(\{\mathbf{y} : |\mathbf{y}| > x\} \times \cdot) = \sum_{i=1}^n \varepsilon_{i/n}(\cdot) \mathbb{1}(|\mathbf{X}_i| > x a_n).$$

Under mixing and anti-clustering conditions, for fixed  $x > 0$ , we can express the limiting point process in [26] such as

$$\begin{aligned} \eta_{n,x}(\cdot) &\xrightarrow{d} \eta_x(\cdot) := \bar{N}(\{\mathbf{y} : |\mathbf{y}| > x\} \times \cdot) \\ &= \sum_{i=1}^{\infty} \sum_{j \in \mathbb{Z}} \mathbb{1}(\Gamma_i^{-1/\alpha} |\mathbf{Q}_{ji}^{(\alpha)}| > x) \varepsilon_{U_i}(\cdot), \end{aligned}$$

where the points  $(U_i)$  are iid uniformly distributed on  $(0, 1)$ ,  $(\Gamma_i)$  are the points of a standard homogeneous Poisson process, and  $(\mathbf{Q}_{ji}^{(\alpha)})$  are iid copies of the cluster process  $\mathbf{Q}^{(\alpha)}$ . Using the independence among these three processes, one can easily rewrite the limit as

$$(4.27) \quad \eta_x(t) = \sum_{i=1}^{N_x(t)} \xi_i, \quad 0 < t \leq 1,$$

where

- $N_x$  is a homogeneous Poisson process on  $(0, 1]$  with intensity  $x^{-\alpha}$ ,
- for an iid sequence  $(Y_i)$  of Pareto $(\alpha)$ -distributed random variables which is also independent of  $(\mathbf{Q}_i^{(\alpha)})$ ,

$$\xi_i = \sum_{j \in \mathbb{Z}} \mathbb{1}(Y_i |\mathbf{Q}_{ij}^{(\alpha)}| > 1),$$

- $N_x, (\xi_i)$  are independent.

Relying on the point process of exceedances representation in (4.27), the random variables  $(\xi_i)$  can be interpreted as counts of serial exceedances from one cluster. Furthermore, we deduce the relation  $\mathbb{P}(\xi_1 > 0) = \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_{\infty}^{\alpha}] = \theta_{|\mathbf{X}|}$ , and also get an expression for the cluster size probabilities

$$(4.28) \quad \begin{aligned} \mathbb{P}(\xi_1 = j) &= \mathbb{E}[|\mathbf{Q}^{(\alpha)}|_{(j)}^{\alpha} - |\mathbf{Q}^{(\alpha)}|_{(j+1)}^{\alpha}] \\ &= \mathbb{E}[\pi_j^{\mathbf{Q}}(\mathbf{Q}^{(\alpha)})] = \pi_j, \quad j \geq 1. \end{aligned}$$

The statistic  $\pi_j$  can be understood as the probability of recording a cluster of length  $j$ . The blocks estimator provide natural estimators

$$(4.29) \quad \hat{\pi}_j = \frac{1}{k} \sum_{t=1}^m \frac{|\mathcal{B}_t|_{(j)}^{\alpha} - |\mathcal{B}_t|_{(j+1)}^{\alpha}}{\|\mathcal{B}_t\|_{\alpha}^{\alpha}} \mathbb{1}(\|\mathcal{B}_t\|_{\alpha} > \|\mathcal{B}\|_{\alpha, (k+1)}),$$

$|\mathcal{B}_t|_{(1)} \geq |\mathcal{B}_t|_{(2)} \geq \dots \geq |\mathcal{B}_t|_{(b)}$  are the order statistics of  $\mathcal{B}_t$ , the  $t$ -th block.

**Corollary 4.3.** Consider the function  $\pi_j^{\mathbf{Q}} : \tilde{\ell}^p \rightarrow \mathbb{R}$  defined by  $\pi_j^{\mathbf{Q}}(\mathbf{x}) := (|\mathbf{x}|_{(j)}^\alpha - |\mathbf{x}|_{(j+1)}^\alpha) / \|\mathbf{x}\|_\alpha^\alpha$ , where  $|\mathbf{x}|_{(1)} \geq |\mathbf{x}|_{(2)} \geq \dots$ . Assume the conditions of Theorem 3.1 hold for  $p = \alpha$ . Then, for all  $j \geq 1$  we have

$$(4.30) \quad \sqrt{k}(\hat{\pi}_j - \pi_j) \xrightarrow{d} \mathcal{N}(0, \text{Var}(\pi_j^{\mathbf{Q}}(\mathbf{Q}^{(\alpha)}))), \quad n \rightarrow \infty.$$

Corollary 4.3 provides a novel procedure for estimating cluster size probabilities based on extremal  $\ell^\alpha$ -blocks. As in the previous examples, the asymptotic variance can be computed as long as  $\mathbf{Q}^{(\alpha)}$  is known. This allows for comparison with the other cluster-based inference procedures provided in [25, 21, 45]. One advantage of our methodology is that we can straightforwardly infer the asymptotic variances of cluster sizes since we express them as cluster statistics in (4.30). Moreover, inference through extremal  $\ell^\alpha$ -blocks has already proven to be useful in [12] for fine-tuning the hyperparameters of the estimators, see also the discussion in Section 6.

## 5. MODELS

**5.1. Linear  $m_0$ -dependent sequences.** We consider  $(\mathbf{X}_t)$  to be a  $m_0$ -dependent time series with values in  $(\mathbb{R}^d, |\cdot|)$  satisfying  $\mathbf{RV}_\alpha$ .

**Example 5.1.** The time series  $(\mathbf{X}_t)$  is a linear moving average of order  $m_0 \geq 1$  if it satisfies

$$(5.31) \quad \mathbf{X}_t := \mathbf{Z}_t + \varphi_1 \mathbf{Z}_{t-1} + \dots + \varphi_{m_0} \mathbf{Z}_{t-m_0}, \quad t \in \mathbb{Z},$$

with  $\mathbb{R}^d$ -variate iid innovations  $(\mathbf{Z}_t)$  satisfying  $\mathbf{RV}_\alpha$ , and  $(\varphi_j) \in \mathbb{R}^{m_0}$ .

Alternatively, the max-moving average of order  $m_0 \geq 1$  satisfies

$$(5.32) \quad X_t := \max\{Z_t, \varphi_1 Z_{t-1}, \dots, \varphi_{m_0} Z_{t-m_0}\}, \quad t \in \mathbb{Z},$$

with  $\mathbb{R}_+$ -variate iid innovations  $(Z_t)$  satisfying  $\mathbf{RV}_\alpha$ , and  $(\varphi_j) \in \mathbb{R}_+^{m_0}$ .

Then both moving averages satisfy  $\mathbf{RV}_\alpha$  with  $|\mathbf{Q}^{(\alpha)}|$  admitting the same deterministic expression  $(|\varphi_t| / \|(\varphi_j)\|_\alpha)$  in  $\tilde{\ell}^\alpha$ , see for instance Proposition 3.1. in [12] and Chapter 5 of [33].

Let  $p > \alpha/2$ . For all  $\kappa > 0$ , a sequence satisfying  $x_n = O(b_n^{\kappa+1/(p \wedge \alpha)})$  verifies  $\mathbf{AC}$  and  $\mathbf{CS}_p$ . This is a consequence of Remark 2.5. Choosing  $(x_n)$  in this way implies there exist  $\kappa' > 0$ , and  $(k_n)$  satisfying

$$(5.33) \quad k_n = O(n b_n^{-\kappa' - \alpha/(p \wedge \alpha)}),$$

such that Equation (3.13) holds from an application of Potter's bound. Since  $\kappa$  can be chosen arbitrarily small,  $\kappa'$  can also be chosen arbitrarily close to zero. Keeping this in mind, we can state the Proposition below. The proof is postponed to Section B.

**Proposition 5.2.** Consider  $(\mathbf{X}_t)$  to be a  $m_0$ -dependent time series with values in  $(\mathbb{R}^d, |\cdot|)$ . Consider  $p > \alpha/2$ , and a sequence  $(k_n)$  satisfying (5.33), such that  $k = k_n \rightarrow \infty$ ,  $m_n/k_n \rightarrow \infty$ . Consider  $f_p : \tilde{\ell}^p \rightarrow \mathbb{R}$ , and assume  $\mathbf{L}$

and **B** hold. Then,

$$\sqrt{k}(f_p^{\widehat{\mathbf{Q}}} - f_p^{\mathbf{Q}}) \xrightarrow{d} \mathcal{N}(0, \text{Var}(f_p(Y\mathbf{Q}^{(p)}))), \quad n \rightarrow \infty.$$

In particular, the  $\alpha$ -cluster based estimators from Section 4 in (4.21) (4.24), and (4.29), are asymptotically normally distributed. Their asymptotic variances are null in the case of the moving averages of Example 5.1.

**5.2. Linear processes.** In this section we consider stationary linear processes  $(\mathbf{X}_t)$  with values in  $(\mathbb{R}^d, |\cdot|)$  satisfying  $\mathbf{RV}_\alpha$ .

**Example 5.3.** Consider  $(\mathbf{X}_t)$  to be an  $\mathbb{R}^d$ -variate sequence satisfying

$$(5.34) \quad \mathbf{X}_t = \sum_{j \in \mathbb{Z}} \varphi_j \mathbf{Z}_{t-j}, \quad t \in \mathbb{Z},$$

for a sequence of iid innovations  $(\mathbf{Z}_t)$  satisfying  $\mathbf{RV}_\alpha$ , and a sequence  $(\varphi_j)$  in  $\mathbb{R}^{\mathbb{Z}}$ . Moreover, assume there exists  $\kappa > 0$  such that  $\|(\varphi_j)\|_{(\alpha-\kappa) \wedge 2} < \infty$ .

In the setting of Example 5.3, a stationary solution  $(\mathbf{X}_t)$  exists and satisfies  $\mathbf{RV}_\alpha$  (c.f. [15, 37]). Proposition 5.4 below demonstrates conditions **AC**, **CS<sub>p</sub>** hold for  $p > \alpha/2$ , and a suitable sequence  $(x_n)$  such that  $n\mathbb{P}(|\mathbf{X}_1| > x_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore, the time series  $(\mathbf{X}_t)$  admits an  $\alpha$ -cluster process  $\mathbf{Q}^{(\alpha)}$ , which we can compute in terms of the filter  $(\varphi_j)$ , and the spectral measure of the random variable  $\mathbf{Z}_0$ , denoted by  $\boldsymbol{\Theta}_0^{\mathbf{Z}}$ , with  $|\boldsymbol{\Theta}_0^{\mathbf{Z}}| = 1$  a.s. We obtain the expression, cf. Chapter 5 of [33],

$$(5.35) \quad \mathbf{Q}^{(\alpha)} \stackrel{d}{=} (\varphi_t / \|(\varphi_j)\|_\alpha) \boldsymbol{\Theta}_0^{\mathbf{Z}}, \quad \in \tilde{\ell}^\alpha.$$

Note again that the norm of the  $\alpha$ -cluster process, i.e.,  $|\mathbf{Q}^{(\alpha)}|$ , is deterministic in  $\tilde{\ell}^\alpha$ . Assuming  $\|(\varphi_j)\|_p < \infty$ , we can compute the indices  $c(p)$  in (2.8) by

$$(5.36) \quad c(p) = \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_p^\alpha] = \|(\varphi_j)\|_p^\alpha / \|(\varphi_j)\|_\alpha^\alpha < \infty.$$

Classic examples of these heavy-tailed linear models are auto-regressive moving averages, i.e., ARMA processes, with iid regularly varying noise; cf. [10].

The proposition below guarantees that the assumptions of Proposition 2.3 hold. We defer its proof to Section B.1.

**Proposition 5.4.** Consider  $(\mathbf{X}_t)$  to be a linear process with values in  $(\mathbb{R}^d, |\cdot|)$ , as in Example 5.3. Consider  $p > \alpha/2$ , and a sequence  $(x_n)$  such that  $n/x_n^{p \wedge (\alpha-\kappa)} \rightarrow 0$ ,  $n \rightarrow \infty$ , for some  $\kappa > 0$ . Then it holds for all  $\delta > 0$

$$(5.37) \quad \lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}_{[1,n]}/x_n - \mathbf{X}_{[1,n]}^{(s)}/x_n\|_p^p > \delta)}{n\mathbb{P}(|\mathbf{X}_1| > x_n)} = 0,$$

where  $\mathbf{X}_t^{(s)} := \sum_{|j| \leq s} \varphi_j \mathbf{Z}_{t-j}$ . Thus **AC** and **CS<sub>p</sub>** are satisfied.

We now review the mixing properties of a linear process. We recall below the statement in Theorem 2.1. in [41] (see Lemma 15.3.1. in [33]).

**Proposition 5.5.** *Consider  $(\mathbf{X}_t)$  to be a causal linear process with values in  $(\mathbb{R}^d, |\cdot|)$ , as in Example 5.3 with  $\varphi_j = 0$ , for  $j < 0$ . Assume the distribution of  $\mathbf{Z}_0$  is absolutely continuous with respect to the Lebesgue measure in  $\mathbb{R}^d$ , and has a density  $g_Z$  satisfying*

- i)  $\int |g(\mathbf{x} - \mathbf{y}) - g(\mathbf{x})| d\mathbf{x} = O(|\mathbf{y}|)$ , for all  $\mathbf{y} \in \mathbb{R}^d$ ,
- ii)  $\varphi_t = O(t^{-\rho})$ , for  $t \geq 0$ , and  $\rho > 2 + 1/\alpha$ ,
- iii)  $\sum_{j=0}^{\infty} \varphi_j \mathbf{x}^j \neq 0$ , for all  $\mathbf{x} \in \mathbb{R}^d$  with  $|\mathbf{x}| < 1$ ,

Then, for all  $0 < \kappa < \alpha$ , the mixing coefficients  $(\beta_t)$  satisfy

$$(5.38) \quad \beta_t = O\left(t^{1 - \frac{(\rho-1)(\alpha-\kappa)}{1+\alpha-\kappa}}\right).$$

Combining Propositions 5.4 and 5.5, we state below the asymptotic normality of the  $p$ -cluster based estimators for linear processes in Theorem 5.6. We defer its proof to Section B.2.

**Theorem 5.6.** *Consider  $(\mathbf{X}_t)$  to be a causal linear process with values in  $(\mathbb{R}^d, |\cdot|)$ , as in Example 5.3. Let  $\rho > 0$ , and assume the conditions of Proposition 5.5 hold with  $\varphi_t = O(t^{-\rho})$ , for  $t > 0$ . Consider  $p > \alpha/2$ , and a sequence  $(k_n)$  satisfying (5.33), such that  $k = k_n \rightarrow \infty$ ,  $m_n/k_n \rightarrow \infty$ . Consider  $f_p: \ell^p \rightarrow \mathbb{R}$ , and assume **L** and **B** hold. Furthermore, assume*

- i) for  $\delta > 0$  as in (3.14),

$$\rho > 3 + \frac{2}{\alpha} + \frac{2}{\delta}\left(1 + \frac{1}{\alpha}\right),$$

- ii) for all  $\kappa > 0$ ,  $nb_n \frac{(\rho-1)(\alpha-\kappa)}{1+\alpha-\kappa} \rightarrow 0$ , as  $n \rightarrow \infty$ .

If  $f_p$  is bounded, condition i) can be replaced by  $\rho > 3 + 2/\alpha$ . Then,

$$\sqrt{k}(f_p^{\widehat{\mathbf{Q}}} - f_p^{\mathbf{Q}}) \xrightarrow{d} \mathcal{N}\left(0, \text{Var}(f_p(Y^{\mathbf{Q}(p)}))\right), \quad n \rightarrow \infty.$$

In particular, the  $\alpha$ -cluster based estimators from Section 4 in (4.21) (4.24), and (4.29), are asymptotically normally distributed and their asymptotic variances are null.

Regarding cluster inference in the case of linear models, the  $\alpha$ -cluster approach has an optimal asymptotic variance for shift-invariant functionals since we use the  $\ell^\alpha$ -norm order statistics. For this reason, it compares favourably with state-of-the-art blocks estimator. For example, for the extremal index, the super-efficient estimator in (4.21) has a lower asymptotic variance than the blocks estimator in (4.22). Indeed the asymptotic variance  $\sigma_\theta^2$  of the latter, computed in (4.23), is not necessarily null. For example, for the autoregressive process of order one AR(1) one has  $\sigma_\theta^2 = 1 - \theta_{|\mathbf{x}|} > 0$ .

The main drawback of the  $\alpha$ -cluster estimator is that we must infer  $\alpha$ . We propose to use a consistent and unbiased Hill estimator of  $1/\alpha$ ; see [25, 44]. Numerical experiments in Section 6 show this works fine in practice.

**5.3. Affine stochastic recurrence equation solution under Kesten's conditions.** In this section we focus on the causal solution to the affine

stochastic recurrence equation SRE under Kesten's conditions. To guarantee the existence of a solution  $(\mathbf{X}_t)$ , with values in  $(\mathbb{R}^d, |\cdot|)$  as in (5.39) satisfying  $\mathbf{RV}_\alpha$ , we rely on Theorem 2.1. and Theorem 2.4 in [2]. For an overview, we refer to [11]. In what follows, we study time series  $(\mathbf{X}_t)$  as in the Example 5.7 below.

**Example 5.7.** Consider  $(\mathbf{X}_t)$  to be a sequence with values in  $\mathbb{R}^d$  satisfying

$$(5.39) \quad \mathbf{X}_t = \mathbf{A}_t \mathbf{X}_{t-1} + \mathbf{B}_t, \quad t \in \mathbb{Z},$$

where  $((\mathbf{A}_t, \mathbf{B}_t))$  is an iid sequence of non-negative random  $d \times d$  matrices with generic element  $\mathbf{A}$ , and non-negative random vectors with generic element  $\mathbf{B}$  taking values in  $\mathbb{R}^d$ . For the existence of a causal stationary solution, we assume

- i)  $\mathbb{E}[\log^+ |\mathbf{A}|_{op}] + \mathbb{E}[\log^+ |\mathbf{B}|] < \infty$ ,
- ii) under i), assume the Lyapunov exponent of  $(\mathbf{A}_t)$ , denoted  $\gamma$ , satisfies

$$\gamma := \lim_{n \rightarrow \infty} n^{-1} \log |\mathbf{A}_t \cdots \mathbf{A}_1|_{op} < 0, \quad \text{a.s.}$$

To guarantee the heavy-tailedness condition  $\mathbf{RV}_\alpha$ , we also assume

- iii)  $\mathbf{B} \neq \mathbf{0}$  a.s., and  $\mathbf{A}$  has no zero rows a.s.
- iv) there exists  $\kappa > 0$  such that  $\mathbb{E}[|\mathbf{A}|_{op}^\kappa] < 1$ ,
- v) the set  $\Gamma$  from Equation (5.40) generates a dense group on  $\mathbb{R}$ ,

$$(5.40) \quad \begin{aligned} \Gamma &= \{ \log |\mathbf{a}_n \cdots \mathbf{a}_1|_{op} : n \geq 1, \mathbf{a}_n \cdots \mathbf{a}_1 > 0, \\ &\quad \mathbf{a}_n, \dots, \mathbf{a}_1 \text{ are in the support of } \mathbf{A}'\text{'s distribution} \}, \end{aligned}$$

- vi) there exists  $\kappa_1 > 0$  such that  $\mathbb{E}[(\min_{i=1, \dots, d} \sum_{j=1}^d A_{ij})^{\kappa_1}] \geq d^{\kappa_1/2}$ , and  $\mathbb{E}[|\mathbf{A}|_{op}^{\kappa_1} \log^+ |\mathbf{A}|_{op}] < \infty$ .
- vii) under i) – vi), there exists a unique  $\alpha > 0$  such that

$$(5.41) \quad \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{E}[|\mathbf{A}_n \cdots \mathbf{A}_1|_{op}^\alpha] = 0,$$

and  $\mathbb{E}[|\mathbf{B}|^\alpha] < \infty$ . If  $d > 1$  assume  $\alpha$  is not an even integer.

The  $\mathbb{R}^d$ -variate series  $(\mathbf{X}_t)$ , satisfying (5.39) and i) – vii), admits a causal stationary solution and satisfies  $\mathbf{RV}_\alpha$ , with  $\alpha > 0$  as in Equation (5.41).

The previous example is motivated by the seminal Kesten's paper [31]. We follow Theorem 2.1. in [2] to state conditions i) – ii) of Example 5.7. Under the conditions i) – ii), the unique solution  $(\mathbf{X}_t)$  of (5.39) has the a.s. causal representation

$$(5.42) \quad \mathbf{X}_t = \sum_{i \geq 0} \mathbf{A}_{t-i+1} \cdots \mathbf{A}_t \mathbf{B}_{t-i}, \quad t \in \mathbb{Z},$$

where the first summand is  $\mathbf{B}_t$  for  $i = 0$  by convention; for an overview see [11].

One of the main reasons why the solutions to SRE as in Example 5.7 have received strong interest, is because  $(\mathbf{X}_t)$  satisfies  $\mathbf{RV}_\alpha$  even when the innovations  $((\mathbf{A}_t, \mathbf{B}_t))$  are light-tailed. This feature was first noticed in [31] where the original Kesten's assumptions were introduced. In Kesten's framework,

a causal stationary solution to the SRE exists as in (5.42), and the extremes of the series occur due to the sums of infinitely many terms of growing length products appearing in (5.42); see [6] for a review. Further, the community adopted the simplified Kesten's conditions stated by Goldie in [22] for univariate SRE. These conditions also aim to capture the heavy-tailed feature under lighter-tailed innovations. In Example 5.7, we borrow the conditions *iii) – vii)* established for the multivariate setting from Theorem 2.4 and Corollary 2.7. in [2]. Then, a solution  $(\mathbf{X}_t)$  as in Example 5.7 satisfies  $\mathbf{RV}_\alpha$ , for  $\alpha > 0$ , and the index of regular variation  $\alpha$  is the unique solution to the Equation (5.41). We are also interested in Example 5.7 because it models classic econometric time series such as the squared ARCH( $p$ ), and the volatility of GARCH( $p, q$ ) processes; see [11].

Concerning the extremes of  $(\mathbf{X}_t)$  in Example 5.7, the forward spectral tail process satisfies the relation

$$\Theta_t = \mathbf{A}_t \cdots \mathbf{A}_1 \Theta_0, \quad t \geq 0,$$

where  $(\mathbf{A}_t)$  is an iid sequence distributed as  $\mathbf{A}$ ; see [30]. The backward spectral tail process has a cumbersome representation that we omit here; c.f. [30]. We state in Proposition 5.8 sufficient conditions on  $(\mathbf{A}, \mathbf{B})$  yielding assumptions  $\mathbf{AC}$ ,  $\mathbf{CS}_p$  hold for  $p > \alpha/2$ , and a suitable sequence  $(x_n)$  such that  $n\mathbb{P}(|\mathbf{X}_1| > x_n) \rightarrow 0$  as  $n \rightarrow \infty$ . In this case the time series  $(\mathbf{X}_t)$  admits an  $\alpha$ -cluster process  $\mathbf{Q}^{(\alpha)}$ . We recall the identity from Equation (8.6) of [12]:  $c(p) = \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_p^\alpha] = \mathbb{E}[\|(\Theta_t)_{t \geq 0}\|_p^\alpha - \|(\Theta_t)_{t \geq 1}\|_p^\alpha]$ , for  $c(p)$  as in (2.8). Then, letting  $p = \alpha/2$ , a straightforward computation yields

$$\begin{aligned} c(p) &\leq 2 \mathbb{E}[\|(\Theta_t)_{t \geq 0}\|_p^{\alpha-p}] = 2 \mathbb{E}[\sum_{t \geq 0} |\mathbf{A}_t \cdots \mathbf{A}_1 \Theta_0|^p] \\ &\leq 2s \sum_{t \geq 0} (\mathbb{E}[|\mathbf{A}_s \cdots \mathbf{A}_1|_{op}^p])^t, \end{aligned}$$

and  $\mathbb{E}[|\mathbf{A}_s \cdots \mathbf{A}_1|_{op}^p] < 1$ , for  $p < \alpha$  and  $s \geq 1$  fixed sufficiently large in the setting of Example 5.7. Hence, for  $p \in (\alpha/2, \alpha)$ ,  $c(p) < \infty$  in (2.8), and then the series admits a  $p$ -cluster process  $\mathbf{Q}^{(p)}$ .

We state now Proposition 5.8 which verifies conditions  $\mathbf{AC}$ ,  $\mathbf{CS}_p$  for the SRE equation. The proof is postponed to Section B.3.

**Proposition 5.8.** *Let  $(\mathbf{X}_t)$  be a stationary time series with values in  $(\mathbb{R}^d, |\cdot|)$ , as in Example 5.7. Let  $p > \alpha/2$ , and consider  $(x_n)$  such that there exists  $\kappa > 0$  satisfying  $n/x_n^{p \wedge (\alpha - \kappa)} \rightarrow 0$ , as  $n \rightarrow \infty$ . Then,  $(x_n)$  satisfies conditions  $\mathbf{AC}$  and  $\mathbf{CS}_p$ .*

In the setting of SRE equations, condition  $\mathbf{AC}$  has been shown in Theorem 4.17 in [38]. In [38], the authors already considered a condition similar to  $\mathbf{CS}_p$ . Parallel to their setting, we propose a proof of Proposition 5.8 which shows  $\mathbf{CS}_p$  holds over uniform regions  $\Lambda_n = (x_n, \infty)$  such that  $n/x_n^p \rightarrow 0$ , as  $n \rightarrow \infty$ , in the sense of (2.69). Thereby, our proof extends Theorem 4.17 in [38] to uniform regions  $\Lambda_n$  not having an upper bound.

Concerning the mixing properties of  $(\mathbf{X}_t)_{t \geq 0}$  as in Example 5.7, we use that it is a Markov chain and that  $\mathbf{X}_0$  has the stationary distribution. As



mentioned in Remark 2.2, we can then use Markov chain's theory to compute its mixing coefficients; cf. [36]. We review Theorem 2.8. in [2], yielding an exponential decay of the mixing-coefficients  $(\beta_t)$  of the series. For a general treatment see Chapter 4.2 in [11].

**Proposition 5.9.** *Consider a time series  $(\mathbf{X}_t)$  with values in  $(\mathbb{R}^d, |\cdot|)$ , as in Example 5.7. Assume there exists a Borel measure  $\mu$  on  $(\mathbb{R}^d, |\cdot|)$ , such that the Markov chain  $(\mathbf{X}_t)_{t \geq 0}$  is  $\mu$ -irreducible, i.e., for all  $C \subset \mathbb{R}^d$  with  $\mu(C) > 0$ ,*

$$(5.43) \quad \sum_{t=0}^{\infty} \mathbb{P}(\mathbf{X}_t \in C \mid \mathbf{X}_0 = \mathbf{x}) > 0, \quad \mathbf{x} \in \mathbb{R}.$$

*Then  $(\mathbf{X}_t)$  has mixing coefficients  $(\beta_t)$  satisfying  $\beta_t = O(\rho^t)$  for some  $\rho \in (0, 1)$ , and we say it is strongly mixing with geometric rate. Moreover,  $(\mathbf{X}_t)_{t \geq 0}$  is irreducible with respect to the Lebesgue measure if  $(\mathbf{A}, \mathbf{B})$  admits a density.*

We can now state the asymptotic normality of cluster-based estimator for SRE solutions in Theorem 5.10 below. The proof is postponed to Section 2.4.

**Theorem 5.10.** *Consider  $(\mathbf{X}_t)$  to be the causal solution to the SRE in (5.39) with values in  $(\mathbb{R}^d, |\cdot|)$ , as in Example 5.7. Assume the conditions of Proposition 5.9 hold. Consider  $p > \alpha/2$ , and a sequence  $(k_n)$  satisfying (5.33), such that  $k = k_n \rightarrow \infty$ ,  $m_n/k_n \rightarrow \infty$ . Consider  $f_p : \ell^p \rightarrow \mathbb{R}$ , and assume  $\mathbf{L}$  and  $\mathbf{B}$  hold. Assume  $\log(n)/b_n \rightarrow 0$ , as  $n \rightarrow \infty$ . Then,*

$$\sqrt{k}(\widehat{f_p^{\mathbf{Q}}} - f_p^{\mathbf{Q}}) \xrightarrow{d} \mathcal{N}(0, \text{Var}(f(Y^{\mathbf{Q}^{(p)}}))), \quad n \rightarrow \infty.$$

*In particular, the  $\alpha$ -cluster based estimators from Section 4 in (4.21) (4.24), and (4.29), are asymptotically normally distributed.*

**Remark 5.11.** In this example, the asymptotic variances of the  $\alpha$ -cluster based estimators from Section 4 in (4.21) (4.24), and (4.29) are non-null. The limiting variances in Theorem 5.10 are difficult to compare with the existing ones in the literature because of the complexity of the distribution of  $\mathbf{Q}^{(p)}$ . However, we provide simple  $\ell^\alpha$ -block estimators of the asymptotic variances in Section 6.

## 6. NUMERICAL EXPERIMENTS

This section aims to illustrate the finite-sample performance of the  $\alpha$ -cluster estimators on time series  $(\mathbf{X}_t)$  with tail-index  $\alpha > 0$ . In all the models we consider in Section 5, we work under the assumption that the tuning parameters of the  $\alpha$ -cluster satisfy (5.33). We take  $\kappa' = 1$  in (5.33) which yields  $b = \sqrt{n/k}$ . In this case, the implementation of our estimators can be written solely as a function of  $k$ . Recall  $k = k_n$  must satisfy  $k \rightarrow \infty$  and  $m/k \rightarrow \infty$  with  $m = \lfloor n/b \rfloor$ . Numerical comparisons of our  $\alpha$ -cluster based approach with other existing estimators for the extremal index and the cluster index are at the advantage of our approach; see [13] and [12].

The code of all numerical experiments is available at: [https://github.com/GBuritica/cluster\\_functionals.git](https://github.com/GBuritica/cluster_functionals.git).

**6.1. Cluster size probabilities.** We reviewed in Section 4.3 how cluster sizes play a key role to model the serial behavior of exceedances. In this section, we implement the cluster size probabilities estimation procedure from Equation (4.29) in an example of a solution to the SRE under Kesten's conditions.

**Example 6.1.** Consider the non-negative univariate random variables  $A$ ,  $B$ , defined by  $\log A = N - 0.5$ , where  $N$  denotes a standard Gaussian random variable, and  $B$  is uniformly distributed in  $[0, 1]$ . Let  $(X_t)$  be the solution to the SRE in (5.39). Then,  $(X_t)$  satisfies  $\mathbf{RV}_\alpha$  with  $\alpha = 1$ . If  $(A_j)$  is a sequence of iid random variables with generic element  $A$ , then

$$Q_t^{(\alpha)} \stackrel{d}{=} \Pi_t / \|(\Pi_j)\|_\alpha, \quad t \in \mathbb{Z},$$

with

$$\Pi_t \stackrel{d}{=} \begin{cases} A_t \cdots A_1 & \text{if } t \geq 1, \\ A_t \cdots A_{-1} & \text{if } t \leq -1, \\ 1 & \text{if } t = 0. \end{cases}$$

This follows by Example 6.1 in [30], and Proposition 3.1 in [12]. Then, for  $p > \alpha/2$ , the  $p$ -cluster based estimators (4.21) (4.24), and (4.29) are asymptotically normally distributed.

Recall the cluster sizes  $\pi_1, \pi_2, \dots$ , defined in (4.28). We infer the cluster sizes of Example 6.1 using  $\alpha$ -cluster estimates. To illustrate Theorem 5.10, we run a Monte-Carlo simulation experiment based on 1 000 samples  $(X_t)_{t=1, \dots, n}$  of length  $n = 12\,000$  from Example 6.1. For each sampled trajectory, we obtain estimates  $\hat{\pi}_1, \hat{\pi}_2, \dots$ , letting  $k = 8$  and  $b = 38$  in (4.29). We use the known value of the tail-index  $\alpha = 1$ . We also estimate the extremal index  $\theta_X$  of the series from Equation (4.21). Theorem 5.10 yields, for  $j \geq 0$ ,

$$(6.44) \quad \text{Var}(\sqrt{k}(\hat{\pi}_j - \pi_j^{\mathbf{Q}})) \rightarrow \text{Var}(\pi_j^{\mathbf{Q}}(\mathbf{Q}^{(\alpha)})), \quad n \rightarrow \infty,$$

where  $\pi_j^{\mathbf{Q}}$  are the cluster functional yielding the cluster sizes  $\pi_j$  with the notation in (4.30). Notice that the asymptotic variance of our cluster sizes estimate is again a cluster statistic that we can infer. We compute an estimate of the asymptotic variance in (6.44) using cluster-based estimates, and compare this estimate with the empirical variance obtained from the Monte-Carlo simulation study. Figure 1 plots the profile of the limit Gaussian distribution where the asymptotic variance is computed in these two ways. As expected from Equation (6.44), the curves overlap, even if  $k$  is small.

In the case of SRE equations, the cluster sizes were studied in detail in [24]. The authors proposed a method to approximate the true values when

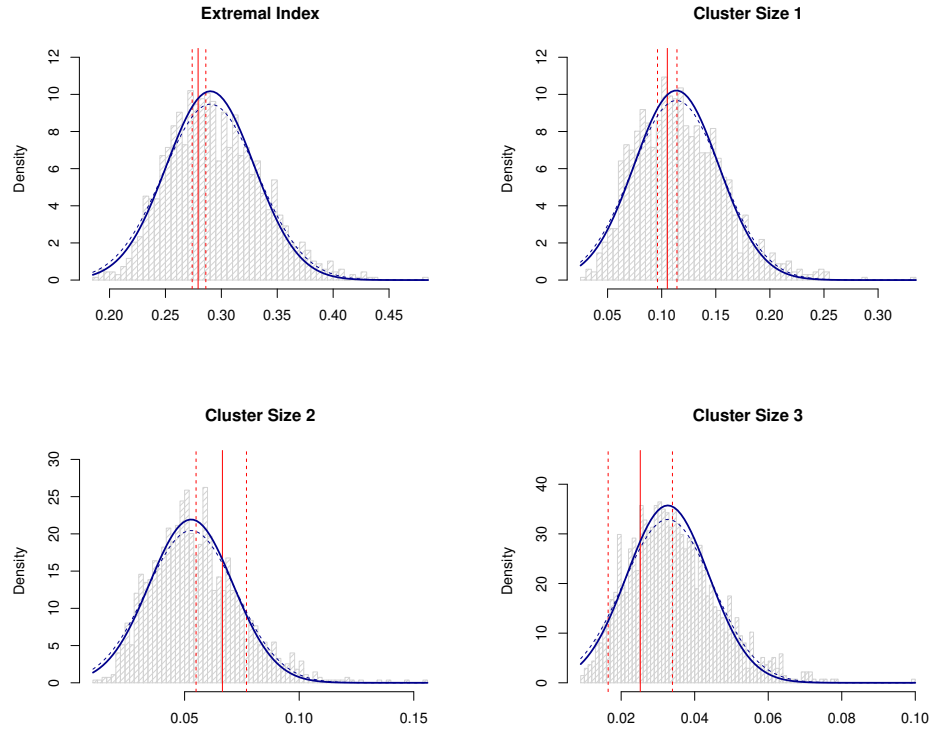


FIGURE 1. Histogram of estimates  $\hat{\theta}_X$  of the extremal index using (4.21), and the cluster size probability  $\hat{\pi}_1$ ,  $\hat{\pi}_2$ ,  $\hat{\pi}_3$ , using (4.28). We simulate 1 000 samples  $(X_t)_{t=1,\dots,n}$  of Example 6.1 with  $n = 12\,000$ . The dotted curve is the estimated Gaussian density from the Monte-Carlo simulation study. The solid curve is the estimated Gaussian density using a cluster-based estimate of the asymptotic variance defined in (6.44). The red lines point to the Monte-Carlo approximation of the real values with standard deviation. These were computed using Equation 3.5 in [24], and a simulation study with 10 000 samples of length 500 000.

the tail-index  $\alpha$ , and the random variable  $A$  are known. We approximate true values using Equation 3.5 in [24], and a Monte-Carlo study with 10 000 samples of length 500 000. The obtained values are pointed out in red in Figure 1. We see that this choice of  $k$  yields estimates centered around the true value.

**6.2. Replacing  $\alpha$  by  $\hat{\alpha}$ .** So far we have used known values of  $\alpha$ . In this section, we illustrate that the  $\alpha$ -cluster-based estimators perform well in simulation when we replace  $\alpha$  with a Hill-type estimator  $\hat{\alpha}$ . We use the

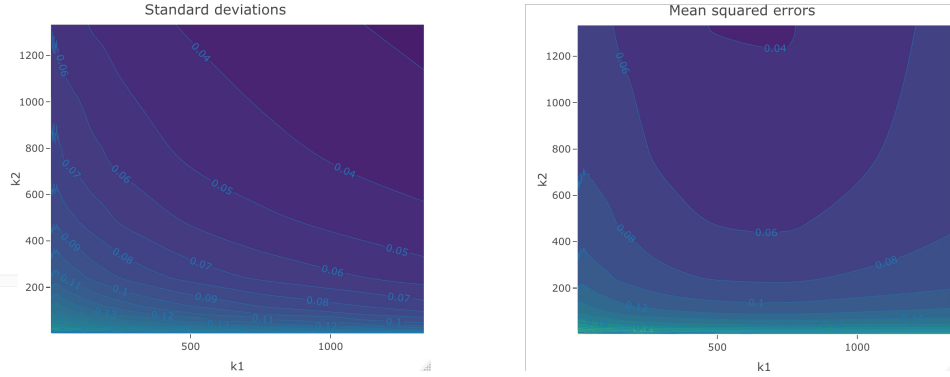


FIGURE 2. Heatmap with contour curves of standard deviations and mean squared errors for estimates of the extremal index  $k1 = k \mapsto \hat{\theta}_X(k)$  in (4.21), where we estimate the tail-index using a Hill-type estimator  $k2 = k \mapsto \hat{\alpha}(k)$ . We simulate 500 samples  $(X_t)_{t=1,\dots,n}$  of an  $\text{AR}(\varphi)$  model with absolute value student( $\alpha$ ) noise for  $n = 12\,000$ ,  $\varphi = 0.5$ ,  $\alpha = 1$ , such that  $\theta_X = 0.5$ .

bias-correction procedure in [23] and write an estimator of the tail-index as  $k' \mapsto \hat{\alpha}(k')$  where the tuning parameter  $k'$  is the number of higher order statistics of  $(|\mathbf{X}_t|)$  that we use for inference. For consistency of the tail-index estimator, we must take  $k' = k'_n$  satisfying  $k' \rightarrow \infty$  and  $n/k' \rightarrow \infty$ . Therefore, we recommend choosing the tuning sequence of the tail-index and of the cluster estimators as  $(k'_n)$ ,  $(k_n)$ , respectively, such that  $k'/k \rightarrow \infty$ . Roughly speaking, the cluster statistics capture the block extremal behavior whereas the tail-index points to an extremal property of margins. As a consequence, the variance of the Hill procedure should not affect the variance of the cluster estimates asymptotically. To illustrate this point, we simulate 500 samples  $(X_t)_{t=1,\dots,n}$  of an  $\text{AR}(\varphi)$  model with absolute value student( $\alpha$ ) noise for  $n = 12\,000$ ,  $\alpha = 1$  and  $\varphi \in \{0.5, 0.7\}$ , and for samples of Example 5.7. We estimate the extremal index  $\hat{\theta}_X(k)$  as in (4.21) where we replace  $\alpha$  by  $\hat{\alpha}(k')$ . Recall that for an  $\text{AR}(\varphi)$  model the asymptotic variances of the extremal index estimator are null. We see in Figures 2, 3 and 4 that in practice we have to choose  $k$  small to reduce the bias of the estimator. Moreover, the estimation procedure is robust with respect to  $k'$  therefore we recommend taking  $k'$  large to reduce variance. Similar results were found for  $n = 3\,000$ ,  $n = 5\,000$ , and  $n = 8\,000$  and these are available upon request. To conclude, we see in Figures 2, 3 and 4 that standard deviations are small, and thus the error of cluster inference is mainly due to bias. We recommend choosing  $k$  small and  $k'$  larger in all settings.

**6.3. Conclusion.** Our main theoretical result in Theorem 3.1 states asymptotic normality of  $\alpha$ -cluster-based disjoint blocks estimators  $\hat{f}_\alpha^{\mathcal{Q}}(k)$ , based on

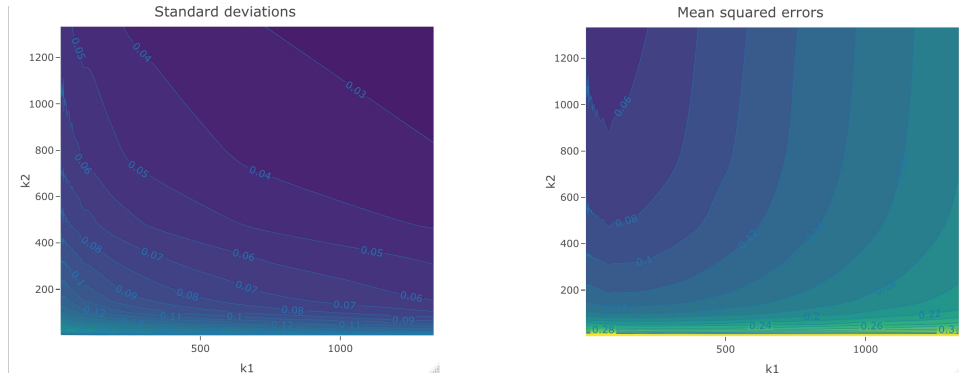


FIGURE 3. Heatmap with contour curves as in Figure 2. Here we simulate 500 samples  $(X_t)_{t=1,\dots,n}$  of an  $\text{AR}(\varphi)$  model with absolute value student( $\alpha$ ) noise for  $n = 12\,000$ ,  $\varphi = 0.7$ ,  $\alpha = 1$ , such that  $\theta_X = 0.3$ .

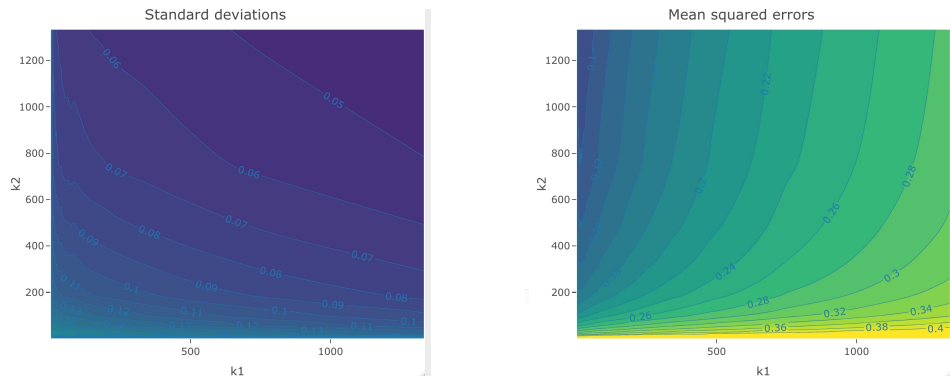


FIGURE 4. Heatmap with contour curves as in Figure 2. Here we simulate 500 samples  $(X_t)_{t=1,\dots,n}$  of Example (6.1) for  $n = 12\,000$  such that  $\theta_X \approx 0.2792$ .

$k$  extremal  $\ell^\alpha$ -blocks, where  $\alpha$  is the tail index of the series. The advantage of  $\alpha$ -cluster-based methods is that the choice of  $k$  is robust to time dependencies; see [12] who have shown already consistency of  $\hat{f}_\alpha^{\mathbf{Q}}$  estimators. Equation (3.20) characterizes their asymptotic variance in terms of a cluster statistic that we can infer. We further show in Section 4 that many important indices in extremes can be written in terms of an  $\alpha$ -cluster statistic, e.g., the extremal index and cluster sizes. Section 5 verifies that our assumptions hold for numerous models like causal linear models and SRE solutions under Kesten's conditions. For linear models, we obtain super-efficient estimators with null asymptotic variance for classical indices. In the examples we considered, our estimators have a small variance that can be easily estimated. To illustrate the performance of our  $\alpha$ -cluster inference

methodology, we run finite-sample simulations in Section 6. In practice, the main drawback of our estimator is that  $\alpha$  needs to be estimated. Our simulations support that replacing  $\alpha$  by  $\hat{\alpha}(k')$  as in Section 6.2 does not have a big impact on the asymptotic variance. This is because  $k$  needs to be chosen small to obtain unbiased estimates, whereas  $k'$  can be chosen larger. Then, even if we choose  $k$  small, the uncertainty of our procedure is well quantified by plugging an estimate of the asymptotic variance in the Gaussian limit. In the cases where the limit degenerates, further analysis needs to be investigated to assess the uncertainty.

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## APPENDIX A. PROOF OF THE MAIN RESULT

In the following proofs, we assume the conditions of Proposition 2.3 hold. In this setting, the time series  $(\mathbf{X}_t)$  admits a  $p$ -cluster  $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$  and (2.8), (2.9), hold for  $(x_n)$ . For inference purposes we fix a sequence of block lengths  $(b_n)$ , and we write  $m_n = \lfloor n/b_n \rfloor$ , such that  $b_n \rightarrow \infty, m_n \rightarrow \infty$ . We assume that the relation

$$(A.45) \quad k := k_n = \lfloor m_n \mathbb{P}(\|\mathcal{B}_1\|_p > x_{b_n}) \rfloor$$

$$(A.46) \quad \sim n c(p) \mathbb{P}(|\mathbf{X}_0| > x_{b_n}), \quad n \rightarrow \infty,$$

holds, where  $c(p) \in (0, \infty)$  are as in (2.8), and we can verify  $m_n/k_n \rightarrow \infty$ .

To state asymptotic normality of the estimator in Equation (1.4), with thresholds chosen as order statistics of the  $\ell^p$ -norms, we study the functional deterministic threshold estimator defined by

$$(A.47) \quad \widetilde{f}_p^{\mathbf{Q}}(u) := \frac{1}{k_n} \sum_{t=1}^{m_n} f(\mathcal{B}_t/u x_{b_n}) \mathbb{1}(\|\mathcal{B}_t\|_p > u x_{b_n}), \quad u > 0.$$

In what follows, the sequences  $(x_n)$ ,  $(b_n)$ ,  $(m_n)$ ,  $(k_n)$ , that we consider, defining the estimator in (A.47), are the ones fixed above.

With this notation, Section A.1 states mixing rates for consistency of the blocks estimators in (1.4). Section A.2 studies the covariance structure of the deterministic threshold estimators in (A.47). Finally, the proof of Theorem 3.1 is deferred to Section A.3.

**A.1. Consistency of  $p$ -cluster based blocks estimators.** The following Lemma is proven in Section A.5.

**Lemma A.1.** *Let  $(\mathbf{X}_t)$  be an  $\mathbb{R}^d$ -valued stationary time series satisfying  $\mathbf{RV}_\alpha$ . Let  $p > 0$ , and assume the conditions of Proposition 2.3 such that  $(\mathbf{X}_t)$  admits a  $p$ -cluster process  $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$ . Let  $f \in \mathcal{G}_+(\tilde{\ell}^p)$  be a bounded Lipschitz continuous function. If either (1) or (2) below hold, then the sequences  $(x_{b_n})$*



and  $(b_n)$  satisfy the relation

$$(A.48) \quad \left| \mathbb{E} \left[ e^{\left\{ -\frac{1}{k} \sum_{t=1}^m f(x_b^{-1} \mathcal{B}_t) \right\}} \right] - \mathbb{E} \left[ e^{\left\{ -\frac{1}{k} \sum_{t=1}^{\lfloor m/k \rfloor} f(x_b^{-1} \mathcal{B}_t) \right\}} \right]^k \right| \rightarrow 0, \\ n \rightarrow \infty,$$

where  $(k_n)$  is chosen as in (A.46).

(1) The  $\rho$ -mixing coefficients satisfy

$$\lim_{n \rightarrow +\infty} \sum_{t=0}^{\lfloor \log_2(n) \rfloor} \rho_{b_n 2^t / k_n} = 0.$$

(2) There exists a sequence  $(\ell_n)$ , satisfying  $\ell_n \rightarrow \infty$ , as  $n \rightarrow \infty$ , such that the  $\beta$ -mixing coefficients satisfy

$$\lim_{n \rightarrow +\infty} m_n \beta_{\ell_n} / k_n = \lim_{n \rightarrow +\infty} \ell_n / b_n = 0.$$

**Corollary A.2.** Consider the assumptions in Lemma A.1, and assume either condition (1) or (2) therein hold. Then for any bounded function  $f \in \mathcal{G}_+(\tilde{\ell}^p)$ , the  $p$ -cluster based blocks estimator in (1.4) satisfies

$$\widehat{f}_p^{\mathbf{Q}} \xrightarrow{\mathbb{P}} f_p^{\mathbf{Q}}, \quad n \rightarrow \infty.$$

*Proof of Corollary A.2.* We applying Theorem 4.1 in [12]. The mixing assumption therein is verified in Lemma A.1 stated above. Therefore, we deduce the consistency of the blocks estimators and this concludes the proof.  $\square$

**A.2. Covariance  $p$ -cluster based blocks estimators.** We state below Proposition A.3. This is an intermediate result in the purpose of proving Theorem 3.1. First, we introduce the next condition on the covariance structure of the estimators in Equation (A.47).

(C'): Let  $\mathcal{F} \subseteq \mathcal{G}_+(\tilde{\ell}^p)$  be the set with two functions:  $(\mathbf{x}_t) \mapsto f(\mathbf{x}_t)$  and  $(\mathbf{x}_t) \mapsto 1$ . Consider the family of deterministic threshold estimators as in (A.47) defined, for all  $\epsilon > 0$ , by

$$(A.49) \quad \mathcal{T} = \{ \widetilde{g}_p^{\mathbf{Q}}(u) \}_{\{g(\cdot/u): u \in [1-\epsilon, 1+\epsilon], g \in \mathcal{F}\}}.$$

Assume the asymptotic normality of the finite-dimensional parts of, i.e.,

$$(A.50) \quad \sqrt{k} (\widetilde{g}_p^{\mathbf{Q}}(u) - u^{-\alpha} g_p^{\mathbf{Q}}) \xrightarrow{f.i.d.i.} \mathbb{G}(g(\cdot/u)), \quad \widetilde{g}_p^{\mathbf{Q}}(u) \in \mathcal{T},$$

as  $n \rightarrow +\infty$ , where  $g_p^{\mathbf{Q}}$  is a cluster statistic as in (1.3), and  $\mathbb{G}$  is a centered Gaussian process satisfying

$$(A.51) \quad \begin{aligned} & \text{Cov}(\mathbb{G}(g(\cdot/u)), \mathbb{G}(h(\cdot/v))) \\ &= \int_{u \vee v}^{\infty} \mathbb{E}[g(y \mathbf{Q}^{(p)}/u) h(y \mathbf{Q}^{(p)}/v)] d(-y^{-\alpha}), \end{aligned}$$

for  $u, v \in [1 - \epsilon, 1 + \epsilon]$ , and  $g, h \in \mathcal{F}$ , such that  $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$  is the  $p$ -cluster process of the series.

The following Proposition is shown in Section A.4.

**Proposition A.3.** *Let  $(\mathbf{X}_t)$  be a stationary time series satisfying  $\mathbf{RV}_\alpha$ . Let  $p > 0$ , and assume the conditions of Proposition 2.3 such that  $(\mathbf{X}_t)$  admits a  $p$ -cluster process  $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$ . Consider  $f_p : \tilde{\ell}^p \rightarrow \mathbb{R}$  such that  $\mathbf{L}$  holds. Assume also the conditions  $\mathbf{MX}, \mathbf{B}, \mathbf{C}'$ , hold. Then,*

$$(A.52) \quad \sqrt{k}(\widehat{f}_p^{\mathbf{Q}} - f_p^{\mathbf{Q}}) \xrightarrow{d} \mathcal{N}(0, \text{Var}(f_p(Y\mathbf{Q}^{(p)}))), \quad n \rightarrow +\infty.$$

where  $Y$  is independent of  $\mathbf{Q}^{(p)}$ , and  $\mathbb{P}(Y > y) = y^{-\alpha}$ , for  $y > 1$ . Moreover, uniform asymptotic normality of the family  $\mathcal{T}$  holds.

We list a couple of remarks of Proposition A.3 below.

**Remark A.4.** *Concerning condition  $\mathbf{C}'$ , Proposition A.3 requires that for  $f_p \in \mathcal{F}$  satisfies that  $u \mapsto f_p(\mathbf{x}_t)/u$  is a non-increasing function, for all  $(\mathbf{x}_t) \in \tilde{\ell}^p$ . Then  $(f_p(\cdot/u))_{u \in [1-\epsilon, 1+\epsilon]}$  is a linearly ordered family and thus it is a VC-class; see Remark 2.11 in [17]. This monotonicity condition is always satisfied by the indicator functions  $u \mapsto \mathbb{1}(\|\mathbf{x}_t\|_p > u)$ , for  $p > 0$ . It is also the case for functions projecting into the  $\tilde{\ell}^p$  sphere as the cluster indices studied in [12] (see Examples 4.1, 4.2).*

**Remark A.5.** *We interpret the bias condition  $\mathbf{B}$  in Theorem A.3 as follows. First, recall from (A.46),  $k \sim n c(p) \mathbb{P}(|\mathbf{X}_0| > x_b)$ , as  $n \rightarrow \infty$ . Notice that if we fix  $n$ , and let first  $b \rightarrow \infty$ , then the desired relations in (3.16), (3.15), hold. Therefore, we require block lengths  $b_n$  sufficiently large to guarantee  $\mathbf{B}$  holds.*

**A.3. Proof of Theorem 3.1.** In order to prove Theorem 3.1 it is enough to show that condition  $\mathbf{C}'$  in Proposition A.3 holds under the mixing condition  $\mathbf{MX}_\beta$  or  $\mathbf{MX}_\rho$ . This is the purpose of the next lemma whose proof is postponed to Section A.6.

**Lemma A.6.** *Let  $(\mathbf{X}_t)$  be an  $\mathbb{R}^d$ -valued stationary time series satisfying  $\mathbf{RV}_\alpha$ . Let  $p > 0$ , and assume the conditions of Proposition 2.3 hold such that  $(\mathbf{X}_t)$  admits a  $p$ -cluster process  $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$ . Consider  $g, h : \tilde{\ell}^p \rightarrow \mathbb{R}$  such that  $\mathbf{L}$  holds. Assume also the conditions  $\mathbf{MX}, \mathbf{B}$ , and either condition  $\mathbf{MX}_\rho$ , or  $\mathbf{MX}_\beta$  hold, then*

$$k \text{Cov}(g_p^{\mathbf{Q}}(u), h_p^{\mathbf{Q}}(v)) \rightarrow \text{Cov}(\mathbb{G}(g(\cdot/u)), \mathbb{G}(h(\cdot/v))), \\ n \rightarrow +\infty,$$

and  $\mathbb{G}$  is a centered Gaussian process with covariance structure as in (A.51). Recall the deterministic threshold estimators are defined in (A.47).

Relying on Lemma A.6 and Proposition A.3 we can conclude the proof of Theorem 3.1.

*Proof of Theorem 3.1.* Consider the assumptions in Lemma A.6, and assume either condition  $\mathbf{MX}_\rho, \mathbf{MX}_\beta$  hold. Consider a function  $f_p : \tilde{\ell}^p \rightarrow \mathbb{R}$  satisfying  $(\mathbf{L})$ . Recall the family  $\mathcal{T}$  in (A.49). Asymptotic normality of the finite-dimensional parts of  $\mathcal{T}$  hold by Lemma A.6. Indeed, by the Wold

device it is enough to check that every linear combination of deterministic threshold estimators in (A.47) is asymptotically normal with Gaussian limit  $\mathbb{G}$  and covariance structure as in (A.51). This holds since linear combinations of estimators in (A.47) are again a deterministic threshold estimator as in (A.47). Finally, this shows that  $\mathbf{C}'$  holds. Thus, we can apply Proposition A.3 and this concludes the proof.  $\square$

**Remark A.7.** *Concerning condition  $\mathbf{MX}$ , we require it in Theorem 3.1 to show equicontinuity of the family  $\mathcal{T}$  in (A.49). Furthermore, to check condition  $\mathbf{C}'$ , i.e., to show (A.50) holds with limit Gaussian process  $\mathbb{G}$  and covariance (A.51), we require additional mixing conditions:  $\mathbf{MX}_\rho$  or  $\mathbf{MX}_\beta$ .*

#### A.4. Proof of Proposition A.3.

*Proof.* Let  $f \in \mathcal{G}_+(\tilde{\ell}^p)$  be a fixed bounded function. Recall the set  $\mathcal{F}$  containing two functions:  $(\mathbf{x}_t) \mapsto f(\mathbf{x}_t)$  and  $(\mathbf{x}_t) \mapsto 1$ , and the family  $\mathcal{T}$  in (A.53) defined by

$$(A.53) \quad \mathcal{T} = \{\widetilde{g}_p^{\mathbf{Q}}(u)\}_{\{g(\cdot/u): u \in [1-\epsilon, 1+\epsilon], g \in \mathcal{F}\}}$$

for  $\epsilon > 0$  as in (3.16).

We separate the proof in two steps. We start by assuming that the uniform asymptotic normality of estimators indexed by  $\mathcal{T}$  holds. We show in this case that (A.52) holds. In the second part of the proof we will show the uniformity of the limit Gaussian process indexed by  $\mathcal{T}$ .

For the first step of the proof, assume that

$$\sqrt{k}(\widetilde{g}_p^{\mathbf{Q}}(u) - u^{-\alpha} g_p^{\mathbf{Q}}) \xrightarrow{d} \mathbb{G}(g(\cdot/u)), \quad \widetilde{g}_p^{\mathbf{Q}} \in \mathcal{T},$$

as  $n \rightarrow \infty$  holds uniformly, and  $\mathbb{G}$  is a Gaussian process with structure as in (A.51), that we recall below:

$$\begin{aligned} \text{Cov}(\mathbb{G}(g(\cdot/u)), \mathbb{G}(h(\cdot/v))) &= \int_{u \vee v}^{\infty} \mathbb{E}[g(y\mathbf{Q}^{(p)}/u)h(y\mathbf{Q}^{(p)}/v)]d(-y^{-\alpha}) \\ &= c(g(\cdot/u), h(\cdot/v)), \end{aligned}$$

for  $g, h \in \mathcal{F}$ , and  $u, v \in [1 - \epsilon, 1 + \epsilon]$ . We also write  $c(g(\cdot/u), g(\cdot/u)) = c(g(\cdot/u))$  in what follows.

Then, for  $g \in \mathcal{F}$ ,  $u \in [1 - \epsilon, 1 + \epsilon]$ , we have

$$\sqrt{k}(\widetilde{g}_p^{\mathbf{Q}}(u)/g_p^{\mathbf{Q}} - u^{-\alpha}) \xrightarrow{d} \mathbb{G}(g(\cdot/u))/g_p^{\mathbf{Q}}, \quad n \rightarrow \infty,$$

Taking  $1 \in \mathcal{F}$  to be the constant function one:  $1(\mathbf{x}_t) = 1$ , yields  $1_p^{\mathbf{Q}} = 1$ . Moreover,

$$\begin{aligned} &\sqrt{k}(1_p^{\mathbf{Q}}(u)^{\leftarrow} - (u^{-\alpha})^{\leftarrow}) \\ &= \sqrt{k}(\|\mathcal{B}_1/x_{b_n}\|_{p,(\lfloor ku \rfloor)} - u^{-1/\alpha}), \quad u \in [1 - \epsilon, 1 + \epsilon]. \end{aligned}$$

Then, by an application of Vervaat's lemma,

$$\sqrt{k}(\|\mathcal{B}_1/x_{b_n}\|_{p,(\lfloor k \rfloor)} - 1) \xrightarrow{d} -\alpha^{-1}\mathbb{G}(1(\cdot/1)),$$

as  $n \rightarrow \infty$ . In particular,  $\|\mathcal{B}_1\|_{p,(\lfloor k \rfloor)}/x_{b_n} \xrightarrow{\mathbb{P}} 1$ , and the following joint convergence holds

$$\sqrt{k}((\widetilde{f}_p^{\mathbf{Q}}(u), \|\mathcal{B}_1\|_{p,(k)}/x_b) - (f_p^{\mathbf{Q}}(u), 1)) \xrightarrow{d} (\mathbb{G}(f(\cdot/u)), -\alpha^{-1}\mathbb{G}(1(\cdot/1))),$$

uniformly for  $u \in [1 - \epsilon, 1 + \epsilon]$ , as  $n \rightarrow \infty$ . Furthermore,

$$\begin{aligned} \sqrt{k}(\widehat{f}_p^{\mathbf{Q}} - f_p^{\mathbf{Q}}) &= \sqrt{k}f_p^{\mathbf{Q}}(\widetilde{f}_p^{\mathbf{Q}}(\|\mathcal{B}_1\|_{p,(k)}/x_{b_n})/f_p^{\mathbf{Q}} - 1) \\ &= \sqrt{k}f_p^{\mathbf{Q}}(\widetilde{f}_p^{\mathbf{Q}}(\|\mathcal{B}_1\|_{p,(k)}/x_{b_n})/f_p^{\mathbf{Q}} - (\|\mathcal{B}_1\|_{p,(\lfloor k \rfloor)}/x_{b_n})^{-\alpha}) \\ &\quad + \sqrt{k}f_p^{\mathbf{Q}}((\|\mathcal{B}_1\|_{p,(k)}/x_{b_n})^{-\alpha} - 1) \\ &\xrightarrow{d} f_p^{\mathbf{Q}}\mathbb{G}(f(\cdot/1)/f_p^{\mathbf{Q}} - 1(\cdot/1)) \end{aligned}$$

To sum up,  $\sqrt{k}(\widehat{f}_p^{\mathbf{Q}} - f_p^{\mathbf{Q}}) \xrightarrow{d} \mathcal{N}(0, \text{Var}(f(Y\mathbf{Q}^{(p)})))$ , and we recognize that the variance term in (A.52).

We now turn to the second step to conclude the proof. We must show

$$\sqrt{k}(\widetilde{g}_p^{\mathbf{Q}}(u) - g_p^{\mathbf{Q}}(u)) \xrightarrow{d} \mathbb{G}, \quad \widetilde{g}_p^{\mathbf{Q}}(u) \in \mathcal{T},$$

uniformly as  $n \rightarrow \infty$ . We can replace the expected value of  $\widetilde{g}_p^{\mathbf{Q}}(u)$  directly by the limit  $g_p^{\mathbf{Q}}(u)$  due to the bias assumption (B). We have assumed convergence of the finite-dimensional parts of  $\mathcal{T}$  holds as in (A.50). Then, it remains to check the asymptotic equicontinuity of the family

$$\{\sqrt{k}(\widetilde{g}_p^{\mathbf{Q}}(u) - g_p^{\mathbf{Q}}(u)) : \widetilde{g}_p^{\mathbf{Q}}(u) \in \mathcal{T}\}$$

as this will yield the uniformity of the Gaussian limit.

Actually, it is enough to check separately equicontinuity on  $\{\sqrt{k}(\widetilde{g}_p^{\mathbf{Q}}(u) - g_p^{\mathbf{Q}}(u))\}_{u \in [u_0, s_0]}$ , for each  $g \in \mathcal{F}$ , with  $u_0 = 1 - \epsilon < + < 1 + \epsilon = s_0$ . This holds since the family  $\mathcal{F}$  contains only a finite number of functions. Let's fix  $g \in \mathcal{F}$  and define the semi-metric  $d(\cdot, \cdot)$  by

$$d(g(\cdot/u), g(\cdot/s)) := |u^{-\alpha} - s^{-\alpha}| \mathbb{E}[g(Y\mathbf{Q}^{(p)})^2], \quad s, u > 0.$$

Then  $(\widetilde{g}_p^{\mathbf{Q}}(u))_{u \in [u_0, s_0]}$  is a totally bounded family for this semi-metric. The remaining of the proof consists of two steps. First, we show the sequence  $(\mathcal{B}_t)_{t=1, \dots, m_n}$  can be replaced by a sequence  $(\mathcal{B}_t^*)_{t=1, \dots, m_n}$ , containing iid blocks distributed as  $\mathcal{B}_1$ . Second, we show that the conditions (i), (ii), (iii), of Theorem C.4.5 in [33] hold; see also Theorem 2.3. in [18].

For the first part, for replacing the sequence  $(\mathcal{B}_t)$  by  $(\mathcal{B}_t^*)$  we argue as in Section 10.6. in [33]. For any  $\delta > 0$ ,

$$\begin{aligned} & \mathbb{P}\left(\sup_{\substack{u,s \in [u_0, s_0] \\ d(g(u^{-1}), g(s^{-1})) < \delta}} \sqrt{k} |g_p^{\widetilde{\mathbf{Q}}}(u) - g_p^{\widetilde{\mathbf{Q}}}(s)| > \delta\right) \\ & \leq 2\mathbb{P}\left(\sup_{\substack{u,s \in [u_0, s_0] \\ d(g(u^{-1}), g(s^{-1})) < \delta}} \sqrt{k} |g_{p,*}^{\widetilde{\mathbf{Q}}}(u) - g_{p,*}^{\widetilde{\mathbf{Q}}}(s)| > \delta/2\right) + 4m_n\beta_{b_n}. \end{aligned}$$

where  $g_{p,*}^{\widetilde{\mathbf{Q}}}(u) = \frac{1}{k} \sum_{t=1, \dots, m_n, t \text{ even}} g(\mathcal{B}_t^*/u x_b) \mathbb{1}(\|\mathcal{B}_t^*\|_p > u x_b)$ , and the last bound follows by Lemma 2 in [19].

We are now in the framework of Theorem C.4.5. in [33], and it is enough to check conditions (i), (ii), (iii) therein. To verify the Lindeberg condition (i) it suffices to verify that for every  $\eta > 0$

$$\begin{aligned} I &= \frac{m_n}{k_n} \mathbb{E}[g(\mathcal{B}_1/u x_{b_n})^2 \mathbb{1}(g(\mathcal{B}_1/u x_{b_n}) > \sqrt{\eta k_n})] \\ &\rightarrow 0. \end{aligned}$$

Indeed, we have

$$\begin{aligned} I &\leq \frac{m_n}{k_n} \mathbb{E}[g(\mathcal{B}_1/u x_{b_n})^{2+\delta}]^{\frac{2}{2+\delta}} \mathbb{P}(g(\mathcal{B}_1/u x_{b_n}) > \sqrt{\eta k_n})^{\frac{\delta}{2+\delta}} \\ &\leq (\eta k_n)^{\frac{-(2+\delta)}{2}} \frac{m_n}{k_n} \mathbb{E}[g(\mathcal{B}_1/u x_{b_n})^{2+\delta}], \end{aligned}$$

where  $\delta > 0$  is as in (3.14). Then, by **L** we deduce that  $I \rightarrow 0$ .

We now verify (ii) (this is also condition (D1) in [18]). The convergence of the finite-dimensional distributions of  $\mathcal{T}$  yields, for  $s > u$ ,

$$\begin{aligned} & k \mathbb{E}[(g_{p,*}^{\widetilde{\mathbf{Q}}}(u) - g_{p,*}^{\widetilde{\mathbf{Q}}}(s))^2] \\ & \rightarrow c(g(\cdot/u)) + c(g(\cdot/s)) - 2c(g(\cdot/u), g(\cdot/s)) \\ & = \int_1^{+\infty} (u^{-\alpha} + s^{-\alpha}) g(y \mathbf{Q}^{(p)})^2 - 2s^{-\alpha} g(y \mathbf{Q}^{(p)}) g((s/u) y \mathbf{Q}^{(p)}) d(-y^{-\alpha}). \end{aligned}$$

Since  $v \mapsto g(\cdot/v)$  is a non-increasing function, then we have

$$\begin{aligned} \lim_{k \rightarrow +\infty} k \mathbb{E}[(g_{p,*}^{\widetilde{\mathbf{Q}}}(u) - g_{p,*}^{\widetilde{\mathbf{Q}}}(s))^2] &\leq |u^{-\alpha} - s^{-\alpha}| \mathbb{E}[g(Y \mathbf{Q}^{(p)})^2] \\ &= d(g(\cdot/u), g(\cdot/s)). \end{aligned}$$

To sum up we have shown that under the bias conditions the following limit holds.

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow +\infty} \sup_{\substack{u,s \in [u_0, s_0] \\ d(g(\cdot/t), g(\cdot/s)) < \delta}} k \mathbb{E}[(g_{p,*}^{\widetilde{\mathbf{Q}}}(u) - g_{p,*}^{\widetilde{\mathbf{Q}}}(s))^2] = 0.$$

Thus, from this we conclude that (ii) holds. Finally, the entropy condition in (iii) holds with respect to the random metric  $(d_n, (g(\cdot/u))_{u \in [u_0, s_0]})$  defined

by

$$\begin{aligned} & (d_n(g(\cdot/u), g(\cdot/s)))^2 \\ &= \frac{1}{k} \sum_{t=1}^{m_n} (g(\mathcal{B}_t^*/(u x_b)) \mathbb{1}(\|\mathcal{B}_t^*\|_p > u x_b) - g(\mathcal{B}_t^*/(s x_b)) \mathbb{1}(\|\mathcal{B}_t^*\|_p > s x_b))^2. \end{aligned}$$

Indeed, the family of functions  $(g_p^{\widetilde{\mathbf{Q}}}(t))_{t \in [t_0, s_0]}$  is linearly ordered thus it forms a VC(2)-class (cf. [17], Remark 2.11 and the discussion on condition (D3) in [18]). We conclude the uniform asymptotic normality of the estimators indexed by  $\mathcal{T}$  and Theorem A.52 follows.  $\square$

### A.5. Proof of Lemma A.1.

Condition (1)  $\implies$  (A.48). We start by denoting disjoint blocks as

$$(A.54) \quad \mathcal{B}_t := \mathbf{X}_{(t-1)b+[1,b]}, \quad \mathcal{B}_t^* := \mathbf{X}_{(t-1)b+[1,b]}^*,$$

$t = 1, \dots, m$ , such that  $(\mathcal{B}_t^*)_{t=1, \dots, m}$  is a sequence of iid blocks, distributed as  $\mathcal{B}_1$ , independent of  $(\mathcal{B}_t)_{t=1, \dots, m}$ . Then, the mean value theorem entails  $|e^{-x} - e^{-y}| \leq |x - y|$ , thus

$$\begin{aligned} & \left| \mathbb{E} \left[ \exp \left\{ -\frac{1}{k} \sum_{t=1}^m f(x_b^{-1} \mathcal{B}_t) \right\} \right] - \mathbb{E} \left[ \exp \left\{ -\frac{1}{k} \sum_{t=1}^m f(x_b^{-1} \mathcal{B}_t^*) \right\} \right] \right|^2 \\ & \leq \mathbb{E} \left[ \left( \frac{1}{k} \sum_{t=1}^m f(x_b^{-1} \mathcal{B}_t) - \frac{1}{k} \sum_{t=1}^m f(x_b^{-1} \mathcal{B}_t^*) \right)^2 \right] \\ & = I. \end{aligned}$$

The term  $I$  can be controlled using the correlation coefficients  $(\rho_h)$ , defined in Section 2.1. It follows from stationarity and an application of Theorem 1 in [47] that there exists a constant  $c > 0$  such that

$$I \leq 2c \frac{m}{k^2} \mathbb{E} [f(x_b^{-1} \mathcal{B}_1)^2] \exp \left\{ c \sum_{t=0}^{\lfloor \log_2(n) \rfloor} \rho_{b 2^t} \right\}$$

Moreover, by (2.9), for any function  $g \in \mathcal{G}_+(\tilde{\ell}_p)$ ,

$$\left| \frac{m}{k} \mathbb{E} [g(x_b^{-1} \mathcal{B}_1)] - \int_0^\infty g(y \mathbf{Q}^{(p)}) d(-y^{-\alpha}) \right| \rightarrow 0, \quad n \rightarrow \infty.$$

Hence, if  $\exp \left\{ \sum_{t=0}^{\lfloor \log_2(n) \rfloor} \rho_{b 2^t} \right\} / k \rightarrow 0$  as  $n \rightarrow \infty$ , (A.48) holds. Thus, since  $k \rightarrow +\infty$  we deduce that condition (1) in Lemma A.1 implies (A.48) and this concludes the proof.

(2)  $\implies$  (A.48). Consider a sequence  $\ell := \ell_n \rightarrow \infty$ . We denote disjoint blocks as

$$\mathcal{B}_{t,\ell} := \mathbf{X}_{(t-1)b+[1,b-\ell]}, \quad t = 1, \dots, m.$$

such that for  $\ell = 0$  we keep the notation in (A.54). Notice that for all  $\delta > 0, \epsilon > 0$ , and for every bounded Lipschitz-continuous function  $f \in \mathcal{G}_+(\tilde{\ell}^p)$  it

holds

$$\begin{aligned}
& \left| \mathbb{E} \left[ \exp \left\{ -\frac{1}{k} \sum_{t=1}^m f(x_b^{-1} \mathcal{B}_t) \right\} \right] - \mathbb{E} \left[ \exp \left\{ -\frac{1}{k} \sum_{t=1}^m f(x_b^{-1} \mathcal{B}_{t_\epsilon}) \right\} \right] \right| \\
& \leq \mathbb{E} \left[ \left| \frac{1}{k} \sum_{t=1}^m f(x_b^{-1} \mathcal{B}_t) - \frac{1}{k} \sum_{t=1}^m f(x_b^{-1} \mathcal{B}_{t_\epsilon}) \right| \right] \\
& \leq \mathbb{E} \left[ \frac{1}{k} \sum_{t=1}^m |f(x_b^{-1} \mathcal{B}_t) - f(x_b^{-1} \mathcal{B}_{t_\epsilon})| \right] \\
& = o(m \mathbb{P}(\|\overline{\mathcal{B}_1/x_b}^\epsilon\|_p > \delta)/k)
\end{aligned}$$

This term vanishes by condition  $\mathbf{CS}_p$ . Moreover, define

$$I = \left| \mathbb{E} \left[ \exp \left\{ -\frac{1}{k} \sum_{t=1}^m f_\epsilon(x_b^{-1} \mathcal{B}_t) \right\} \right] - \mathbb{E} \left[ \exp \left\{ -\frac{1}{k} \sum_{t=1}^m f_\epsilon(x_b^{-1} \mathcal{B}_{t,\ell}) \right\} \right] \right|,$$

where  $f_\epsilon(\mathbf{x}_t) := f(\mathbf{x}_{t_\epsilon})$ . Then, there exists a constant  $c > 0$  such that

$$\begin{aligned}
I & \leq c \frac{1}{k} \mathbb{P} \left( \max_{1 \leq j \leq m} \max_{1 \leq i \leq \ell} |\mathbf{X}_{(j-1)b-i+1}| > \epsilon x_b \right) \\
& \leq c \frac{m}{k} \mathbb{P}(\|\mathcal{B}_{1,\ell}\|_\infty > \epsilon x_b) \\
& \leq c \frac{m\ell}{k} \mathbb{P}(|\mathbf{X}_0| > \epsilon x_b) \sim \ell/b (c\epsilon^{-\alpha}/c(p)) = O(\ell/b).
\end{aligned}$$

Thus, we conclude that  $\lim_{n \rightarrow +\infty} \ell_n/b_n = 0$  is a sufficient condition yielding  $I \rightarrow 0$  as  $n \rightarrow \infty$ . Furthermore, recall the definition of the mixing coefficients  $(\beta_h)$  in Section 2.1. Then,

$$\begin{aligned}
& \left| \mathbb{E} \left[ \exp \left\{ -\frac{1}{k} \sum_{t=1}^m f_\epsilon(x_b^{-1} \mathcal{B}_{t,\ell}) \right\} \right] - \mathbb{E} \left[ \exp \left\{ -\frac{1}{k} \sum_{t=1}^m f_\epsilon(x_b^{-1} \mathcal{B}_t^*) \right\} \right] \right| \\
& \leq \frac{m}{k} \|f\|_\infty 2 d_{TV} \left( \mathcal{L}(\mathcal{B}_{t,\ell}) \otimes \underbrace{\mathcal{L}(\mathbf{X}_1) \otimes \cdots \otimes \mathcal{L}(\mathbf{X}_1)}_{\ell \text{ times}}, \mathcal{L}(\mathcal{B}_t) \right) \\
& \leq \frac{m}{k} \|f\|_\infty 2\beta_{\ell_n} \rightarrow 0, \quad n \rightarrow \infty.
\end{aligned}$$

We use first the definition of the total variation distance, and second a reformulation of the distance in terms of the mixing coefficients  $(\beta_h)$ . Hence we deduce that (2) in Lemma A.1 implies (A.48).

**A.6. Proof of Lemma A.6.** To prove Lemma A.6 we will use the following Lemma.

**Lemma A.8.** *Let  $(\mathbf{X}_t)$  be an  $\mathbb{R}^d$ -valued stationary time series satisfying  $\mathbf{RV}_\alpha$ . For  $p > 0$ , assume the conditions of Proposition 2.3 such that  $(\mathbf{X}_t)$  admits a  $p$ -cluster process  $\mathbf{Q}^{(p)} \in \tilde{\ell}^p$ . Consider functions  $g, h : \tilde{\ell}^p \rightarrow \mathbb{R}$  satisfying  $\mathbf{L}$ . Assume that one of the two conditions below holds:*

- (1) *The coefficients  $(\rho_t)$  satisfy  $\rho_t \rightarrow 0$ , as  $t \rightarrow \infty$ , and (3.19) holds.*
- (2) *There exists a sequences  $(\ell_n)$ , satisfying  $\ell_n \rightarrow \infty$ , and*

$$\lim_{n \rightarrow +\infty} m_n \beta_{\ell_n} / k_n = \lim_{n \rightarrow +\infty} \ell_n / b_n = 0.$$

*Then, for all  $t = 2, 3, \dots$ , the relation below holds*

$$(A.55) \quad \lim_{n \rightarrow \infty} \frac{m}{k} \text{Cov}(g(x_b^{-1} \mathcal{B}_1) h(x_b^{-1} \mathcal{B}_t)) = 0.$$

*Proof of Lemma A.8.* We assume with no loss of generality that the functions  $g, h : \tilde{\ell}^p \rightarrow \mathbb{R}$  satisfying  $\mathbf{L}$  take non-negative values. Notice that for all  $t > 0$

$$(A.56) \frac{m}{k} \text{Cov}(g(x_b^{-1} \mathcal{B}_1) h(x_b^{-1} \mathcal{B}_t)) = \frac{m}{k} \mathbb{E}[g(x_b^{-1} \mathcal{B}_1) h(x_b^{-1} \mathcal{B}_t)] + o(1),$$

since  $m \mathbb{E}[h^2(x_b^{-1} \mathcal{B}_1)]/k \rightarrow \mathbb{E}[h^2(Y \mathbf{Q}^{(p)})]$ , as  $n \rightarrow \infty$ , by the moment assumption in (3.14). Moreover,

$$\begin{aligned} I &= \frac{m}{k} \mathbb{E}[g(x_b^{-1} \mathcal{B}_1) \mathbb{1}(g(x_b^{-1} \mathcal{B}_1) > \eta) h(x_b^{-1} \mathcal{B}_t)] \\ &\leq \frac{m}{k} \mathbb{E}[g(x_b^{-1} \mathcal{B}_1)^2 \mathbb{1}(g(x_b^{-1} \mathcal{B}_1) > \eta)]^{1/2} \mathbb{E}[h(x_b^{-1} \mathcal{B}_1)^2]^{1/2} \\ &\leq \frac{m}{k} (\eta)^{-\delta/2} \mathbb{E}[g(x_b^{-1} \mathcal{B}_1)^{2+\delta}]^{\frac{1}{2+\delta}} \mathbb{E}[g(x_b^{-1} \mathcal{B}_1)^{2+\delta}]^{\frac{\delta}{2(2+\delta)}} \mathbb{E}[h(x_b^{-1} \mathcal{B}_1)^2]^{1/2}. \end{aligned}$$

We deduce from Equation (3.14) that  $I = O(\eta^{-\delta/2})$ , thus this term is negligible letting  $n \rightarrow \infty$ , and then  $\eta \rightarrow \infty$ . Therefore,

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{m}{k} \mathbb{E}[g(x_b^{-1} \mathcal{B}_1) h(x_b^{-1} \mathcal{B}_t)] \\ &= \lim_{\eta \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{m}{k} \mathbb{E}[(g \wedge \eta)(x_b^{-1} \mathcal{B}_1) (h \wedge \eta)(x_b^{-1} \mathcal{B}_t)]. \end{aligned}$$

We conclude that it suffices to establish (A.55) for continuous bounded functions. We consider Lipschitz-continuous bounded functions  $f, f' : \tilde{\ell}^p \rightarrow \mathbb{R}$  in  $\mathcal{G}_+(\tilde{\ell}^p)$ . The extension to continuous bounded functions then holds following a Portmanteau argument. Now notice

$$\begin{aligned} &\frac{m}{k} \mathbb{E}[f(x_b^{-1} \mathcal{B}_1) f'(x_b^{-1} \mathcal{B}_t)] \\ &= \frac{m}{k} \mathbb{E}[f(\underline{x_b^{-1} \mathcal{B}_1}_\epsilon) f'(x_b^{-1} \mathcal{B}_t)] + \frac{m}{k} \mathbb{E}[(f(x_b^{-1} \mathcal{B}_1) - f(\underline{x_b^{-1} \mathcal{B}_1}_\epsilon)) f'(x_b^{-1} \mathcal{B}_t)] \end{aligned}$$

For the second term, we rely on condition  $\mathbf{CS}_p$  since  $f, f'$  are bounded Lipschitz-continuous functions. In this case, the second term is negligible letting first  $n \rightarrow \infty$  and then  $\epsilon \downarrow 0$ . Similarly, we deduce from condition  $\mathbf{CS}_p$  that for  $t = 2, 3, \dots$ , and for all  $\epsilon > 0$ ,

$$(A.57) \quad \begin{aligned} &\frac{m}{k} \mathbb{E}[f(x_b^{-1} \mathcal{B}_1) f'(x_b^{-1} \mathcal{B}_t)] \\ &\sim \frac{m}{k} \mathbb{E}[f(\underline{x_b^{-1} \mathcal{B}_1}_\epsilon) f'(\underline{x_b^{-1} \mathcal{B}_t}_\epsilon)], \end{aligned}$$

if we let  $n \rightarrow \infty$  and then  $\epsilon \downarrow 0$ . We now consider the cases (1)  $\implies$  (A.55), and (2)  $\implies$  (A.55) separately.



(1)  $\implies$  (A.55). If  $t = 2$ , we use a telescopic sum decomposition to rewrite the term in (A.57). Hence for any  $s > 0$ ,

$$\begin{aligned} & \frac{m}{k} \mathbb{E}[f(\underline{x}_b^{-1} \mathcal{B}_1) f'(\underline{x}_b^{-1} \mathcal{B}_t)] \\ & \sim \frac{\mathbb{E}[\frac{1}{b} \sum_{i=s}^b (f(\underline{x}_b^{-1} \mathbf{X}_{[0,i]_\epsilon}) - f(\underline{x}_b^{-1} \mathbf{X}_{[1,i]_\epsilon})) f'(\underline{x}_b^{-1} \mathbf{X}_{i+(t-1)b+[1,b]_\epsilon})]}{c(p) \mathbb{P}(|\mathbf{X}_0| > x_b)} \\ & = I. \end{aligned}$$

Notice the finite terms in  $s$  are negligible as  $n \rightarrow \infty$ . Then,

$$|I| \leq \|f\|_\infty \frac{\mathbb{E}[\mathbb{1}(|\mathbf{X}_0| > \epsilon x_b) \frac{1}{b} \sum_{i=s}^b f'_\epsilon(\underline{x}_b^{-1} \mathbf{X}_{[t-1]b+i+[1,b]})]}{c(p) \mathbb{P}(|\mathbf{X}_0| > x_b)}.$$

Furthermore, the term  $I$  can be bounded in absolute value by decomposing the function  $f'$  as a telescopic sum again as follows. We write  $f_\epsilon(\mathbf{x}_t) := f(\underline{x}_{t_\epsilon})$  and  $f'_\epsilon(\mathbf{x}_t) := f'(\underline{x}_{t_\epsilon})$  to simplify notation. Hence

$$\begin{aligned} |I| & \leq \|f\|_\infty \mathbb{E}[\mathbb{1}(|\mathbf{X}_0| > \epsilon x_b) \\ & \quad \times \frac{\frac{1}{b} \sum_{i=s}^b \sum_{j=1}^{b-1} (f'_\epsilon(\underline{x}_b^{-1} \mathbf{X}_{(t-1)b+i+[1,j]}) - f'_\epsilon(\underline{x}_b^{-1} \mathbf{X}_{(t-1)b+i+[1,j+1]}))]}{c(p) \mathbb{P}(|\mathbf{X}_0| > x_b)} \\ & \quad + \|f\|_\infty \|f'\|_\infty \frac{\mathbb{E}[\mathbb{1}(|\mathbf{X}_0| > \epsilon x_b) \times \frac{1}{b} \sum_{i=s}^b \mathbb{1}(|\mathbf{X}_{(t-1)b+i+1}| > \epsilon x_b)]}{c(p) \mathbb{P}(|\mathbf{X}_0| > x_b)} \\ & \leq \|f\|_\infty \|f'\|_\infty \frac{\mathbb{E}[\mathbb{1}(|\mathbf{X}_0| > \epsilon x_b) \frac{1}{b} \sum_{i=s}^b \sum_{j=1}^{b-1} \mathbb{1}(|\mathbf{X}_{(t-1)b+i+j}| > \epsilon x_b)]}{c(p) \mathbb{P}(|\mathbf{X}_0| > x_b)} \\ & \quad + o(1) \\ & = O(\sum_{i=(t-2)b+s}^{tb} \mathbb{P}(|\mathbf{X}_i| > \epsilon x_b \mid |\mathbf{X}_0| > \epsilon x_b)) \end{aligned}$$

For  $t = 2$ , we can control the last expression using condition (3.19). We let first  $n \rightarrow \infty$  and then  $s \rightarrow \infty$  which yields  $|I| \rightarrow 0$ . This we conclude (A.55) holds for Lipschitz-continuous functions, and this shows the case  $t = 2$ .

On the other hand, notice that for  $t = 3, 4, \dots$ ,

$$\begin{aligned} \frac{m}{k} \text{Cov}(g(\underline{x}_b^{-1} \mathcal{B}_1) h(\underline{x}_b^{-1} \mathcal{B}_t)) & \leq \rho_{(t-2)b} \frac{m}{k} (\mathbb{E}[g(\underline{x}_b^{-1} \mathcal{B}_1)^2] \mathbb{E}[h(\underline{x}_b^{-1} \mathcal{B}_1)^2])^{1/2} \\ & = O(\rho_{(t-2)b}). \end{aligned}$$

Hence, assuming  $\rho_{b_n} \rightarrow 0$ , as  $n \rightarrow \infty$ , we deduce (A.55) holds for continuous functions  $g, h : \tilde{\ell}^p \rightarrow \mathbb{R}$  in  $\mathcal{G}_+(\tilde{\ell}^p)$  satisfying the moment assumption in (3.14), and this concludes the proof of (1)  $\implies$  (A.55).

(2)  $\implies$  (A.55). In this case, and arguing as in the first steps of the proof, it suffices to show that for all  $t = 2, 3, \dots$ ,

$$\lim_{n \rightarrow \infty} \frac{m}{k} \text{Cov}(f_\epsilon(\underline{x}_b^{-1} \mathcal{B}_1) f'_\epsilon(\underline{x}_b^{-1} \mathcal{B}_t)) = 0.$$

Consider the sequence  $(\ell_n)$  satisfying condition (2), and recall the notation  $\mathcal{B}_{t,\ell} = \mathbf{X}_{(t-1)b+[1,b-\ell]}$ . We can use similar steps as in the proof of Lemma A.1

and replace  $\mathcal{B}_1$  by  $\mathcal{B}_{1,\ell}$  inside the covariance term. For this step we require  $\ell_n/b_n \rightarrow 0$  as  $n \rightarrow \infty$ . Moreover,

$$\frac{m}{k} \text{Cov}(f_\epsilon(x_b^{-1}\mathcal{B}_{1,\ell})f'_\epsilon(x_b^{-1}\mathcal{B}_t)) \leq 2\|f\|_\infty\|f'\|_\infty\frac{m}{k}\beta_{\ell+(t-2)b}.$$

Finally, this concludes the proof of (2)  $\implies$  (A.55) since  $m_n\beta_{\ell_n}/k_n \rightarrow 0$ , as  $n \rightarrow \infty$ .  $\square$

*Proof of Lemma A.6.* Let  $g, h : \tilde{\ell}^p \rightarrow \mathbb{R}$ , be two functions verifying **(L)**. Then,

$$\begin{aligned} & \text{Cov}\left(\frac{1}{k}\sum_{t=1}^m g(x_b^{-1}\mathcal{B}_t), \frac{1}{k}\sum_{t=1}^m h(x_b^{-1}\mathcal{B}_t)\right) \\ &= \frac{m}{k^2} \text{Var}[g(x_b^{-1}\mathcal{B}_1)h(x_b^{-1}\mathcal{B}_1)] + \frac{2}{k^2} \sum_{1 \leq t < j \leq m} \text{Cov}(g(x_b^{-1}\mathcal{B}_t), h(x_b^{-1}\mathcal{B}_j)) \\ &= \frac{(I + II)}{k} \end{aligned}$$

The second term  $II$  can be rewritten in the following way by stationarity:

$$\begin{aligned} II &= \frac{m}{k} \frac{1}{m} \sum_{1 \leq t < j \leq m} \text{Cov}(g(x_b^{-1}\mathcal{B}_t)h(x_b^{-1}\mathcal{B}_j)) \\ &= \frac{1}{m} \sum_{1 \leq t \leq m} \sum_{1 < j-t \leq m-t} \frac{m}{k} \text{Cov}(g(x_b^{-1}\mathcal{B}_1)h(x_b^{-1}\mathcal{B}_{j-t})) \\ &= \frac{1}{m} \sum_{1 \leq t \leq m} \sum_{1 < j \leq m-t} \frac{m}{k} \mathbb{E}[g(x_b^{-1}\mathcal{B}_1)h(x_b^{-1}\mathcal{B}_j)] \\ &= \frac{1}{m} \sum_{1 < j \leq m-1} \sum_{1 \leq t \leq m-j} \frac{m}{k} \text{Cov}[g(x_b^{-1}\mathcal{B}_1)h(x_b^{-1}\mathcal{B}_j)] \\ &= \sum_{1 < j \leq m-1} \left(1 - \frac{j}{m}\right) \frac{m}{k} \text{Cov}(g(x_b^{-1}\mathcal{B}_1)h(x_b^{-1}\mathcal{B}_j)) \end{aligned}$$

Then, following the lines of Lemma A.8, we conclude that assuming  $\sum_{t=1}^{m_n} \rho_{tb_n} \rightarrow 0$ , we deduce  $II \rightarrow 0$ .

Moreover, A direct corollary of Theorem 1.1. in [42], directly stated in equation (1.12b) therein, yields for  $j = 3, 4, \dots$ , and  $\delta > 0$ ,

$$\begin{aligned} & \frac{m}{k} \text{Cov}(g(x_b^{-1}\mathcal{B}_1)h(x_b^{-1}\mathcal{B}_j)) \\ & \leq 2\frac{m}{k} \beta_{\frac{\delta}{(j-2)b_n}}^{\frac{\delta}{(2+\delta)}} \mathbb{E}[g(x_b^{-1}\mathcal{B}_1)^{2+\delta}]^{\frac{1}{2+\delta}} \mathbb{E}[h(x_b^{-1}\mathcal{B}_1)^{2+\delta}]^{\frac{1}{2+\delta}}, \\ & = O\left(\left(\frac{m}{k} \beta_{(j-2)b_n}\right)^{\frac{\delta}{(2+\delta)}}\right). \end{aligned}$$

The last equality holds by condition (3.14). Actually, the above inequality can be extended to bounded random variables letting  $\delta \uparrow \infty$ . Finally, for  $j = 2$  we use the result in Lemma A.8. To sum up, we have also shown that if  $\sum_{t=1}^{m_n} (m_n\beta_{tb_n}/k_n)^{\delta/(2+\delta)} \rightarrow 0$  together with **(MX)** hold, then  $II \rightarrow 0$  as  $n \rightarrow \infty$ .

To conclude, this shows

$$\begin{aligned} I &= \frac{m}{k} \text{Var}(g(x_b^{-1}\mathcal{B}_1)h(x_b^{-1}\mathcal{B}_1)) \\ & \rightarrow c(g, h) := \int_0^\infty g(y\mathbf{Q}^{(p)})h(y\mathbf{Q}^{(p)})d(-y^{-\alpha}), \end{aligned}$$

and this concludes the proof.  $\square$

## APPENDIX B. PROOFS OF THE RESULTS OF SECTION 5

*Proof of Proposition 5.2.* From the discussion in Section 5.1, we can see that all assumptions in Theorem 3.1 are satisfied. Notice that if  $p = \alpha$ , then  $|\mathbf{Q}^{(\alpha)}|$  has a deterministic expression in the shift-invariant space. Then, the index estimators in (4.21), (4.24), and (4.29), with  $f_p : \mathbf{x} \mapsto \|\mathbf{x}\|_\infty^\alpha / \|\mathbf{x}\|_\alpha^\alpha$ ,  $f_p : \mathbf{x} \mapsto \|\mathbf{x}\|_1^\alpha / \|\mathbf{x}\|_\alpha^\alpha$ , and  $f_p : \mathbf{x} \mapsto (|\mathbf{x}|_{(j)}^\alpha - |\mathbf{x}|_{(j+1)}^\alpha) / \|\mathbf{x}\|_\alpha^\alpha$ , respectively, satisfy  $\text{Var}(f_p(Y\mathbf{Q}^{(\alpha)})) = 0$ . Finally, using the change of norms formula in (2.11), we can also show  $\text{Var}(f_p(Y\mathbf{Q}^{(p)})) = 0$ , for any  $p > 0$ , and this concludes the proof.  $\square$

**B.1. Proof of Proposition 5.4.** We start by noticing that Equation (5.37) rewrites as: for all  $\delta > 0$ ,

$$(B.58) \quad \lim_{s \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \frac{\mathbb{P}(\sum_{t=1}^n |\sum_{|j|>s} \varphi_j \mathbf{Z}_{t-j} / x_n|^p > \delta)}{n\mathbb{P}(|\mathbf{X}_1| > x)} = 0.$$

Assuming (5.37) holds, **AC** and **CS<sub>p</sub>** follow straightforwardly since for all  $s > 0$ , the series  $(\mathbf{X}_t^{(s)})$  is a linear  $m_0$ -dependent sequence with  $m_0 = 2s + 1$ , such that  $\mathbf{X}_t^{(s)} = \sum_{|j| \leq s} \varphi_j \mathbf{Z}_{t-j}$ . The former satisfies **AC**, **CS<sub>p</sub>**, for  $p > \alpha/2$ , as in Example 5.1.

We now turn to the verification of Equation (B.58). Actually, by monotonicity of the  $\ell^p$ -norms, if (B.58) holds for  $\alpha/2 < p < \alpha$ , then it also holds for  $p \geq \alpha$ . In the following we assume  $\alpha/2 < p < \alpha$ .

For  $p \leq 1$ , the subadditivity property yields

$$|\sum_{|j|>s} \varphi_j \mathbf{Z}_{t-j}|^p \leq \sum_{|j|>s} |\varphi_j \mathbf{Z}_{t-j}|^p =: I_{1,t}.$$

For  $p > 1$ , a Taylor decomposition of functional  $|\cdot|^p : \mathbb{R} \rightarrow \mathbb{R}$  entails, for all  $a, b \in \mathbb{R}$ ,

$$|a + b|^p = |a|^p + p \text{sign}(a) |a|^{p-1} b + \frac{p(p-1)}{2} |a|^{p-2} b^2 + \dots + R_{[p]}(a, b),$$

where the remaining term satisfies

$$R_{[p]} = R_{[p]}(a, b) \leq \frac{p(p-1) \dots (p-[p])}{[p]!} |b - \xi a|^{p-[p]} b^{[p]},$$

for one  $\xi \in [0, 1]$ . To simplify notation, in the remaining lines of the proof we denote  $(|\mathbf{Z}_t|)$  by  $(Z_t)$ . Then, the Taylor expression above yields

$$\begin{aligned} & |\sum_{|j|>s} \varphi_j \mathbf{Z}_{t-j} / x_n|^p \\ & \leq |\varphi_s Z_{t-s} / x_n|^p + p |\varphi_s Z_{t-s} / x_n|^{p-1} (\sum_{\substack{|j| \geq s \\ j \neq s}} |\varphi_j Z_{t-j} / x_n|) \\ & \quad + \dots + R_{[p]}. \end{aligned}$$

Moreover, to handle the remaining term  $R_{[p]}$ , we use subadditivity of the real function  $x \mapsto x^{p-[p]}$ . Hence,

$$\begin{aligned}
 & \left| \sum_{|j| \geq s} \varphi_j \mathbf{Z}_{t-j} / x_n \right|^p \\
 & \leq |\varphi_s \mathbf{Z}_{t-s} / x_n|^p + c |\varphi_s \mathbf{Z}_{t-s} / x_n|^{p-1} \left( \sum_{\substack{|j| \geq s \\ j \neq s}} |\varphi_j \mathbf{Z}_{t-j} / x_n| \right) \\
 & \quad + \cdots + c |\varphi_s \mathbf{Z}_{t-s} / x_n|^{p-[p]} \left( \sum_{\substack{|j| \geq s \\ j \neq s}} |\varphi_j \mathbf{Z}_{t-j} / x_n| \right)^{[p]} \\
 & \quad + c \sum_{\substack{|j| \geq s \\ j \neq s}} |\varphi_j \mathbf{Z}_{t-j} / x_n|^{p-[p]} \left( \sum_{\substack{|j| \geq s \\ j \neq s}} |\varphi_j \mathbf{Z}_{t-j} / x_n| \right)^{[p]} \\
 & \leq c (I_{0,t} + I_{1,t} + \cdots + I_{[p],t} + I_{[p]+1,t}),
 \end{aligned}$$

where  $c > 0$  is a constant of no interest, only depending on  $p$ . Relying on the bound above, we require to control the previous  $[p]+2$  terms. We argue using a truncation argument. Our goal is to prove that for all  $l = 0, \dots, [p]+1$ , for all  $\epsilon, \delta > 0$

$$(B.59) \quad \lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\sum_{t=1}^n \overline{I}_{l,t}^\epsilon > \delta)}{n \mathbb{P}(|\mathbf{X}_1| > x_n)} + \frac{\mathbb{P}(\sum_{t=1}^n \underline{I}_{l,t}^\epsilon > \delta)}{n \mathbb{P}(|\mathbf{X}_1| > x_n)} = 0,$$

where the truncated terms are defined as follows: for  $l = 0, \dots, [p]$

$$\begin{aligned}
 \overline{I}_{l,t}^\epsilon & := \overline{|\varphi_s \mathbf{Z}_{t-s} / x_n|^{p-l} \left( \sum_{\substack{|j| \geq s \\ j \neq s}} |\varphi_j \mathbf{Z}_{t-j} / x_n|^\epsilon \right)^l}, \\
 \overline{I}_{[p]+1,t}^\epsilon & := \sum_{\substack{|j| \geq s \\ j \neq s}} \overline{|\varphi_j \mathbf{Z}_{t-j} / x_n|^{p-[p]} \left( \sum_{\substack{|j| \geq s \\ j \neq s}} |\varphi_j \mathbf{Z}_{t-j} / x_n|^\epsilon \right)^{[p]}}, \\
 \underline{I}_{l,t}^\epsilon & := \underline{|\varphi_s \mathbf{Z}_{t-s} / x_n|^{p-l} \left( \sum_{\substack{|j| \geq s \\ j \neq s}} |\varphi_j \mathbf{Z}_{t-j} / x_n|^\epsilon \right)^l}, \\
 \underline{I}_{[p]+1,t}^\epsilon & := \sum_{\substack{|j| \geq s \\ j \neq s}} \underline{|\varphi_j \mathbf{Z}_{t-j} / x_n|^{p-[p]} \left( \sum_{\substack{|j| \geq s \\ j \neq s}} |\varphi_j \mathbf{Z}_{t-j} / x_n|^\epsilon \right)^{[p]}}.
 \end{aligned}$$

To study each term, we write for  $q \in \mathbb{N}$ ,  $\mathcal{J} \subseteq \mathbb{N}$ ,  $(\psi_j) \in \mathbb{R}^{|\mathcal{J}|}$ ,

$$(B.60) \quad \left( \sum_{j \in \mathcal{J}} \psi_j \right)^q = \sum_{\substack{i_1, \dots, i_q \\ i_j \in \mathcal{J}}} \psi_{i_1} \cdots \psi_{i_q}.$$

We start by analyzing the terms corresponding to the truncation from below. An application of Markov's inequality together with Equation (B.60) yield

$$\begin{aligned}
 & \mathbb{P}(\sum_{t=1}^n \underline{I}_{[p]+1,t}^\epsilon > \delta) \\
 & \leq \delta^{-1} n \mathbb{E}[\underline{I}_{[p]+1,t}^\epsilon] \\
 & \leq \delta^{-1} n \sum_{\substack{i_1, \dots, i_{[p]+1} \\ |i_j| \geq s}} |\varphi_{i_1} \cdots \varphi_{i_{[p]}}| |\varphi_{i_{[p]+1}}|^{p-[p]} \\
 & \quad \times \mathbb{E} \left[ \left| \frac{Z_{-i_1}}{x_n} \right|_{\epsilon \varphi_{i_1}^{-1}} \cdots \left| \frac{Z_{-i_{[p]}}}{x_n} \right|_{\epsilon \varphi_{i_{[p]}}^{-1}} \left| \frac{Z_{-i_{[p]+1}}}{x_n} \right|_{\epsilon \varphi_{i_{[p]+1}}^{-1}} \right]^{p-[p]}.
 \end{aligned}$$

Moreover, recall the noise sequence  $(Z_t)$  are iid random variables satisfying  $\mathbf{RV}_\alpha$ . Therefore, for the expectation we can factor the independent noise terms as the product of at most  $[p] + 1$  terms. For each term, the noise random variable  $Z_{-i_j}$  will be raised to the power at most  $p$ . As  $p < \alpha$ , we can use Karamata's theorem on each of these terms.

Finally, an application of Karamata's theorem and Potter's bound yield there exists  $\kappa > 0$ , such that for all  $\epsilon, \delta > 0$

$$\frac{\mathbb{P}(\sum_{t=1}^n \overline{I_{[p]+1,t}^\epsilon} > \delta)}{n\mathbb{P}(|Z_1| > x_n)} \leq \frac{\alpha}{\alpha - p} O(\delta^{-1} \epsilon^{-(\alpha-\kappa)} (\sum_{|j| \geq s} |\varphi_j|^{\alpha-\kappa})^p).$$

We conclude that this term is negligible by letting first  $n \rightarrow \infty$ , and then  $s \rightarrow \infty$ .

We can follow similar steps as before to study the truncation from below terms  $I_{l,t}$ ,  $l = 0, \dots, [p]$ . An application of Markov's inequality entails there exists  $\kappa > 0$  such that

$$\begin{aligned} \frac{\mathbb{P}(\sum_{t=1}^n \overline{I_{l,t}^\epsilon} > \delta)}{n\mathbb{P}(|Z_1| > x_n)} &\leq \delta^{-1} \frac{\mathbb{E}[|\varphi_s Z_1/x_{n_\epsilon}|^{p-l}]}{\mathbb{P}(|Z_1| > x_n)} \mathbb{E}[(\sum_{\substack{|j| > s \\ j \neq s}} |\varphi_j| Z_{t-j}/x_{n_\epsilon})^l] \\ &= \frac{\alpha}{\alpha - p + l} O(\delta^{-1} |\varphi_s|^{\alpha-\kappa} \epsilon^{-(\alpha-\kappa)} (\sum_{|j| \geq s} |\varphi_j|^{\alpha-\kappa})^l), \end{aligned}$$

where the last relation holds by Karamata's theorem and an application of Potter's bound. Hence, for  $l = 0, \dots, [p], [p] + 1$ , we conclude letting first  $n \rightarrow \infty$ , and then  $s \rightarrow \infty$ . To sum up we have shown that

$$\lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\sum_{t=1}^n \overline{I_{l,t}^\epsilon} > \delta)}{n\mathbb{P}(|\mathbf{X}_1| > x_n)} = 0.$$

We now turn to the terms relative to the truncation from above. In this case, the assumption  $n/x_n^p \rightarrow 0$ , entails  $n\mathbb{E}[\overline{I_{l,t}^\epsilon}] \rightarrow 0$ , as  $n \rightarrow \infty$ , for  $l = 0, \dots, [p], [p] + 1$ . Therefore, to establish Equation (B.59), it suffices to check the following relation holds:

$$(B.61) \quad \lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\sum_{t=1}^n \overline{I_{l,t}^\epsilon} - \mathbb{E}[\overline{I_{l,t}^\epsilon}] > \delta)}{n\mathbb{P}(|\mathbf{X}_1| > x_n)} = 0.$$

We apply Chebychev's inequality, which together with the stationarity of the series  $(Z_t)$ , yields

$$\mathbb{P}(\sum_{t=1}^n \overline{I_{l,t}^\epsilon} - \mathbb{E}[\overline{I_{l,t}^\epsilon}] > \delta) \leq 2\delta^{-2} n \sum_{t=0}^n \text{Cov}(\overline{I_{l,0}^\epsilon}, \overline{I_{l,t}^\epsilon})$$

As in the arguments for the truncation from above, we start by showing that the term in (B.61) is negligible for  $l = [p] + 1$ . This reasoning can again be

extended for  $l = 0, \dots, [p]$ . Computation of the covariances then yields

$$\begin{aligned} & \text{Cov}(\overline{I_{[p]+1,0}}^\epsilon, \overline{I_{[p]+1,t}}^\epsilon) \\ &= \sum_{\substack{i_1, \dots, i_{[p]+1} \\ |i_j| \geq s, i_j \neq s}} \sum_{\substack{\ell_1, \dots, \ell_{[p]+1} \\ |\ell_j| \geq s, \ell_j \neq s}} |\varphi_{i_1} \cdots \varphi_{i_{[p]}}| |\varphi_{i_{[p]+1}}|^{p-[p]} |\varphi_{\ell_1} \cdots \varphi_{\ell_{[p]}}| |\varphi_{\ell_{[p]+1}}|^{p-[p]} \\ & \quad \times \text{Cov}(|\overline{Z_{-i_1}/x_n}^{\epsilon\varphi_{i_1}^{-1}} \cdots \overline{Z_{-i_{[p]}}/x_n}^{\epsilon\varphi_{i_{[p]}^{-1}}| |\overline{Z_{-i_{[p]+1}}/x_n}^{\epsilon\varphi_{i_{[p]+1}^{-1}}|^{p-[p]}, \\ & \quad |\overline{Z_{t-\ell_1}/x_n}^{\epsilon\varphi_{\ell_1}^{-1}} \cdots \overline{Z_{t-\ell_{[p]}}/x_n}^{\epsilon\varphi_{\ell_{[p]}^{-1}}| |\overline{Z_{t-\ell_{[p]+1}}/x_n}^{\epsilon\varphi_{\ell_{[p]+1}^{-1}}|^{p-[p]}). \end{aligned}$$

Actually, all but a finite number of terms vanish in the previous double sum because the noise sequence  $(Z_t)$  are independent random variables. More precisely,

$$\begin{aligned} & \text{Cov}(\overline{I_{[p]+1,0}}^\epsilon, \overline{I_{[p]+1,t}}^\epsilon) \\ &= \sum_{\substack{i_1, \dots, i_{[p]+1} \\ |i_j| \geq s, i_j \neq s}} \sum_{\ell_j \in \{i_1-t, \dots, i_{[p]+1}-t\}} \ell_1, \dots, \ell_{[p]+1} \\ & \quad \times |\varphi_{i_1} \cdots \varphi_{i_{[p]}}| |\varphi_{i_{[p]+1}}|^{p-[p]} |\varphi_{\ell_1} \cdots \varphi_{\ell_{[p]}}| |\varphi_{\ell_{[p]+1}}|^{p-[p]} \\ & \quad \times \text{Cov}(|\overline{Z_{-i_1}/x_n}^{\epsilon\varphi_{i_1}^{-1}} \cdots \overline{Z_{-i_{[p]}}/x_n}^{\epsilon\varphi_{i_{[p]}^{-1}}| |\overline{Z_{-i_{[p]+1}}/x_n}^{\epsilon\varphi_{i_{[p]+1}^{-1}}|^{p-[p]}, \\ & \quad |\overline{Z_{t-\ell_1}/x_n}^{\epsilon\varphi_{\ell_1}^{-1}} \cdots \overline{Z_{t-\ell_{[p]}}/x_n}^{\epsilon\varphi_{\ell_{[p]}^{-1}}| |\overline{Z_{t-\ell_{[p]+1}}/x_n}^{\epsilon\varphi_{\ell_{[p]+1}^{-1}}|^{p-[p]}). \end{aligned}$$

Moreover, regarding the last covariance term, we notice that it is sufficient to bound the expectation of the product as

$$\begin{aligned} & \text{Cov}(|\overline{Z_{-i_1}/x_n}^{\epsilon\varphi_{i_1}^{-1}} \cdots \overline{Z_{-i_{[p]}}/x_n}^{\epsilon\varphi_{i_{[p]}^{-1}}| |\overline{Z_{-i_{[p]+1}}/x_n}^{\epsilon\varphi_{i_{[p]+1}^{-1}}|^{p-[p]}, \\ & \quad |\overline{Z_{t-\ell_1}/x_n}^{\epsilon\varphi_{\ell_1}^{-1}} \cdots \overline{Z_{t-\ell_{[p]}}/x_n}^{\epsilon\varphi_{\ell_{[p]}^{-1}}| |\overline{Z_{t-\ell_{[p]+1}}/x_n}^{\epsilon\varphi_{\ell_{[p]+1}^{-1}}|^{p-[p]}). \\ & \leq \mathbb{E}[|\overline{Z_{-i_1}/x_n}^{\epsilon\varphi_{i_1}^{-1}} \cdots \overline{Z_{-i_{[p]}}/x_n}^{\epsilon\varphi_{i_{[p]}^{-1}}| |\overline{Z_{-i_{[p]+1}}/x_n}^{\epsilon\varphi_{i_{[p]+1}^{-1}}|^{p-[p]} \\ & \quad \times |\overline{Z_{t-\ell_1}/x_n}^{\epsilon\varphi_{\ell_1}^{-1}} \cdots \overline{Z_{t-\ell_{[p]}}/x_n}^{\epsilon\varphi_{\ell_{[p]}^{-1}}| |\overline{Z_{t-\ell_{[p]+1}}/x_n}^{\epsilon\varphi_{\ell_{[p]+1}^{-1}}|^{p-[p]}]. \end{aligned}$$

Since  $(Z_t)$  are iid random variable, the expectation term above can be written as the product of expectations as follows

$$\begin{aligned} & |\varphi_{i_1} \cdots \varphi_{i_{[p]}}| |\varphi_{i_{[p]+1}}|^{p-[p]} |\varphi_{\ell_1} \cdots \varphi_{\ell_{[p]}}| |\varphi_{\ell_{[p]+1}}|^{p-[p]} \\ & \quad \mathbb{E}(|\overline{Z_{-i_1}/x_n}^{\epsilon\varphi_{i_1}^{-1}} \cdots \overline{Z_{-i_{[p]}}/x_n}^{\epsilon\varphi_{i_{[p]}^{-1}}| |\overline{Z_{-i_{[p]+1}}/x_n}^{\epsilon\varphi_{i_{[p]+1}^{-1}}|^{p-[p]} \\ & \quad \times |\overline{Z_{t-\ell_1}/x_n}^{\epsilon\varphi_{\ell_1}^{-1}} \cdots \overline{Z_{t-\ell_{[p]}}/x_n}^{\epsilon\varphi_{\ell_{[p]}^{-1}}| |\overline{Z_{t-\ell_{[p]+1}}/x_n}^{\epsilon\varphi_{\ell_{[p]+1}^{-1}}|^{p-[p]}). \\ &= \prod_{\substack{\gamma_1 + \dots + \gamma_r = p, \\ \gamma'_1 + \dots + \gamma'_r = p}} |\varphi_{i_{\gamma_j}}|^{\gamma_j} |\varphi_{i_{\gamma_j} - t}|^{\gamma'_j} \mathbb{E}[|\overline{Z_0/x_n}^{\epsilon\varphi_{i_{\gamma_j}}^{-1}}|^{\gamma_j} |\overline{Z_0/x_n}^{\epsilon\varphi_{i_{\gamma_j} - t}^{-1}}|^{\gamma'_j}], \end{aligned}$$

where  $\gamma_j, \gamma'_j \in \{0, 1, \dots, [p]\}$ ,  $\gamma_r, \gamma'_{r'} \in \{0, p - [p], p - [p] + 1, \dots, p\}$  and  $1 \leq r, r' \leq [p] + 1$ . The product above is a factorization with respect to independent noise terms. We have also used the stationarity of  $(Z_t)$ . The new indices  $\gamma_1, \dots, \gamma_r$  are defined recursively in terms of the sequence  $(i_t)$ . Similarly, we define  $\gamma'_1, \dots, \gamma'_{r'}$  from  $(\ell_t)$ . To define  $\gamma_1$ , first we count the number of times the index  $i_1$  appears in  $\mathcal{I} = \{i_1, \dots, i_{[p]}\}$ . If  $i_1 \neq i_{[p]+1}$ , we put  $\gamma_1$  equal to this count, otherwise, we set  $\gamma_1$  equal to this count plus  $p - [p]$ . Then, we look for the next index different than  $i_1$ , say  $i_j$ , and set  $\gamma_2$  as the number of repetitions of  $i_j$  in  $\mathcal{I}$  plus  $p - [p]$  if  $i_j \neq i_{[p]+1}$ . We continue in this way until the indices  $i_r$  and  $\gamma_r$  are defined as previously. We stop as we recognize that all the indices  $i_r, i_{r+1}, \dots, i_{[p]+1}$  have already been considered. Therefore,  $\gamma_1 + \dots + \gamma_r = p$ . In an identical fashion, we define  $\gamma'_1, \dots, \gamma'_{r'}$  from  $(\ell_t)$ . Moreover, notice that for every  $\gamma \in \{\gamma_1, \dots, \gamma_r\}$  and  $\gamma' \in \{\gamma'_1, \dots, \gamma'_{r'}\}$

$$\begin{aligned} & |\varphi_{i_\gamma}|^\gamma |\varphi_{i_{\gamma-t}}|^{\gamma'} \mathbb{E}[|\overline{Z_0/x_n}^{\epsilon\varphi_{i_\gamma}^{-1}}|^\gamma |\overline{Z_0/x_n}^{\epsilon\varphi_{i_{\gamma-t}}^{-1}}|^{\gamma'}] \\ & \leq (|\varphi_{i_\gamma}|^{2\gamma} |\varphi_{i_{\gamma-t}}|^{2\gamma'} \mathbb{E}[|\overline{Z_0/x_n}^{\epsilon\varphi_{i_\gamma}^{-1}}|^{2\gamma}] \mathbb{E}[|\overline{Z_0/x_n}^{\epsilon\varphi_{i_{\gamma-t}}^{-1}}|^{2\gamma'}])^{1/2}. \\ & \leq (|\varphi_{i_\gamma}|^{2p} \mathbb{E}[|\overline{Z_0/x_n}^{\epsilon\varphi_{i_\gamma}^{-1}}|^{2p}])^{\gamma/2p} (|\varphi_{i_{\gamma-t}}|^{2p} \mathbb{E}[|\overline{Z_0/x_n}^{\epsilon\varphi_{i_{\gamma-t}}^{-1}}|^{2p}])^{\gamma'/2p}. \end{aligned}$$

The key property  $\gamma_1 + \dots + \gamma_r + \gamma'_1 + \dots + \gamma'_{r'} = 2p$  yields

$$\mathbb{P}(Z_0 > x_n) = \prod_{\substack{\gamma_1 + \dots + \gamma_r = p \\ \gamma'_1 + \dots + \gamma'_{r'} = p}} (\mathbb{P}(Z_0 > x_n))^{(\gamma + \gamma')/2p}.$$

In this case, we can apply Karamata's Theorem to each one of the expectation terms. Readily,

$$\begin{aligned} & \sum_{t=1}^n \text{Cov}(\overline{I_{[p]+1,0}}^\epsilon, \overline{I_{[p]+1,t}}^\epsilon) / \mathbb{P}(Z_0 > x_n) \\ & \leq \sum_{t=1}^n \sum_{\substack{i_1, \dots, i_{[p]+1} \\ |i_j| \geq s, i_j \neq s}} \sum_{\ell_j \in \{i_1-t, \dots, i_{[p]+1}-t\}} \prod_{\substack{\gamma_1 + \dots + \gamma_r = p \\ \gamma'_1 + \dots + \gamma'_{r'} = p}} \\ & \quad \frac{(|\varphi_{i_\gamma}|^{2p} \mathbb{E}[|\overline{Z_0/x_n}^{\epsilon\varphi_{i_\gamma}^{-1}}|^{2p}])^{\gamma/2p} (|\varphi_{i_{\gamma-t}}|^{2p} \mathbb{E}[|\overline{Z_0/x_n}^{\epsilon\varphi_{i_{\gamma-t}}^{-1}}|^{2p}])^{\gamma'/2p}}{(\mathbb{P}(Z_0 > x_n))^{(\gamma + \gamma')/2p}} \\ & \leq c \sum_{\substack{i_1, \dots, i_{[p]+1} \\ |i_j| \geq s, i_j \neq s}} \sum_{\ell_j \in \{i_1-t, \dots, i_{[p]+1}-t\}} \sum_{t=1}^n \\ & \quad \prod_{\substack{\gamma_1 + \dots + \gamma_r = p \\ \gamma'_1 + \dots + \gamma'_{r'} = p}} |\varphi_{i_\gamma}|^{\gamma(\alpha-\kappa)} |\varphi_{i_{\gamma-t}}|^{\gamma'(\alpha-\kappa)}. \\ & = c \sum_{\substack{i_1, \dots, i_{[p]+1} \\ |i_j| \geq s, i_j \neq s}} \sum_{\ell_j \in \{i_1-t, \dots, i_{[p]+1}-t\}} \sum_{t=1}^n |\varphi_{i_j}|^{(\alpha-\kappa)} |\varphi_{\ell_j-t}|^{(\alpha-\kappa)}. \\ & \leq c(p+1) (\sum_{|i| \geq s} |\varphi_i|^{(\alpha-\kappa)})^p (\sum_{j \in \mathbb{Z}} |\varphi_j|^{(\alpha-\kappa)})^p. \end{aligned}$$

where  $c > 0$ , is a constant of no interest. We conclude by letting  $s \rightarrow \infty$  that (B.61) holds for  $l = [p] + 1$ . Similarly, this arguments can be extended for  $l = 1, \dots, [p]$ . Overall, this shows (B.59) holds, and this concludes the proof.  $\square$

**B.2. Proof of Theorem 5.6.** For  $p > \alpha/2$ , we aim to apply Theorem 3.1. First, notice condition *i*) yields  $\|\varphi_t\|_p < \infty$ . For all  $\kappa > 0$ , consider a sequence  $(x_n)$  such that  $x_n \in (n^{\kappa+1/(p \wedge \alpha)}, +\infty)$ . Proposition 5.4 implies then that conditions **AC**, **CS<sub>p</sub>**, hold, and  $n\mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$ , as  $n \rightarrow \infty$ . Fix  $\kappa > 0$ , and  $x_n = O(n^{\kappa+1/(p \wedge \alpha)})$ .

To apply Theorem 3.1 it suffices to verify the  $\beta$ -mixing conditions **MX** and **MX <sub>$\beta$</sub>** . As for (3.17), an application of Proposition 5.5 yields

$$\begin{aligned} m_n \beta_{b_n} &= O(m_n b_n^{1 - \frac{(\rho-1)(\alpha-\kappa)}{1+\alpha-\kappa}}) \\ &= O(n b_n^{-\frac{(\rho-1)(\alpha-\kappa)}{1+\alpha-\kappa}}), \end{aligned}$$

then *ii*) yields  $m_n \beta_{b_n} \rightarrow 0$ , as  $n \rightarrow \infty$ . This shows **MX** holds.

Next, we show **MX <sub>$\beta$</sub>**  also holds. Choose  $(k_n)$  as in (A.45). Then, there exists  $\epsilon, \epsilon' > 0$ , and a constant  $c > 0$ , such that

$$\begin{aligned} \text{(B.62)} \quad m_n/k_n &= 1/(c(p)b_n \mathbb{P}(|\mathbf{X}_0| > x_{b_n})) \\ &\leq c x_{b_n}^{(\alpha+\epsilon)}/b_n = c b_n^{-1 + \frac{\alpha}{\alpha \wedge p} + \frac{\epsilon}{\alpha \wedge p} + \kappa(\alpha+\epsilon)} \\ &\leq c b_n^{-1 + \frac{\alpha}{p \wedge \alpha} + \epsilon'}. \end{aligned}$$

This follows using Potter's bounds. Let  $\ell_n = b_n^{(1-\epsilon)}$  such that  $\ell_n/b_n \rightarrow 0$ . Finally, applying Proposition 5.5, we can find  $\epsilon' > 0$  such that the relation below holds

$$m_n \beta_{\ell_n}/k_n = O(b_n^{-\frac{(\rho-1)\alpha}{1+\alpha} + \frac{\alpha}{\alpha \wedge p} + \epsilon'}).$$

Then, taking  $\rho > 1 + \frac{1+\alpha}{\alpha \wedge p} + \epsilon$  yields  $m_n \beta_{\ell_n}/k_n \rightarrow 0$ . In this argument, we can choose  $\epsilon' > 0$  to be arbitrarily small. Then, assuming *i*) entails  $m_n \beta_{\ell_n}/k_n \rightarrow 0$ .

Moreover, let  $\delta > 0$  be as in (3.14). Since  $\rho > \frac{2}{\delta}(1 + \frac{1}{\alpha}) + 3 + \frac{2}{\alpha}$ , equation (5.38) yields  $\sum_{t=1}^{\infty} \beta_t^{\delta/(2+\delta)} < \infty$ . In this case, there exist  $\epsilon > 0$  such that

$$\sum_{t=1}^{m_n} (m_n \beta_{tb_n}/k_n)^{\frac{\delta}{2+\delta}} = O(b_n^{(-\frac{(\rho-1)\alpha}{1+\alpha} + \frac{\alpha}{\alpha \wedge p} + \epsilon)\frac{\delta}{(2+\delta)}}).$$

Furthermore, for  $p > \alpha/2$ , notice  $\rho > 3 + 2/\alpha > 1 + (1 + \alpha)/(\alpha \wedge p)$ . Similarly as before, notice  $\epsilon > 0$  can be made arbitrarily small. Putting everything together, we conclude that (3.18) holds. This completes the proof that **MX <sub>$\beta$</sub>**  holds. Since both **MX** and **MX <sub>$\beta$</sub>**  hold we can apply Theorem 3.1. Finally, in our setting notice that the sequence  $(k_n)$  satisfies

$$k_n \sim c(p)n\mathbb{P}(|\mathbf{X}_0| > x_{b_n}) = o(n b_n^{-\frac{\alpha}{p \wedge \alpha}}).$$

This follows using Potter's bounds. This concludes the proof of Theorem 5.6.  $\square$



**B.3. Proof of Proposition 5.8.** Let  $(\mathbf{X}_t)$  be the stationary solution to the SRE (5.39) as in Example 5.7, satisfying  $\mathbf{RV}_\alpha$ , for  $\alpha > 0$ . Then,  $(\mathbf{X}_t)$  admits the causal representation in (5.42), where  $((\mathbf{A}_t, \mathbf{B}_t))$  is a sequence of iid innovations. Then, backward computations yield

$$(B.63) \quad \mathbf{X}_t = \Pi_t \mathbf{X}_0 + R_t, \quad t \geq 1,$$

where for  $1 \leq i \leq t$

$$(B.64) \quad \Pi_{i,t} := \mathbf{A}_i \cdots \mathbf{A}_t, \quad R_t := \sum_{j=1}^t \Pi_{j+1,t} \mathbf{B}_j,$$

with the conventions:  $\Pi_{1,t} = \Pi_t$ , and  $\Pi_{t+1,t} = Id$ . Notice that the remaining term  $R_t$  is measurable with respect to  $\sigma((\mathbf{A}_i, \mathbf{B}_i)_{1 \leq i \leq t})$ , and is independent of the sigma-field  $\sigma((\mathbf{X}_t)_{t \leq 0})$ .

Condition **AC** has been shown for Theorem 4.17 in [38]. We focus on showing  $\mathbf{CS}_p$  holds for  $p \in (\alpha/2, \alpha)$ .

To begin, note condition  $\mathbf{CS}_p$  was borrowed from Equation (5.2) in [12]. For  $p \in (0, \alpha)$ , and sequences  $(x_n)$  such that  $n/x_n^p \rightarrow 0$ , as  $n \rightarrow \infty$ , we have  $n\mathbb{E}[|\mathbf{X}_0/x_n|^p] \rightarrow 0$ , thus our condition  $\mathbf{CS}_p$  and Equation (5.2) in [12] coincide. More precisely, we show

$$(2.65) \quad \lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} \frac{\mathbb{P}(\|\overline{\mathbf{X}}_{[1,n]}/x_n^\epsilon\|_p^p - \mathbb{E}[\|\overline{\mathbf{X}}_{[1,n]}/x_n^\epsilon\|_p^p] > \delta)}{n\mathbb{P}(|\mathbf{X}_1| > x_n)} = 0.$$

For this reason, we focus on showing (2.65) holds. Actually, we show below that, for  $(x_n)$  as in Proposition 5.8, condition  $\mathbf{CS}_p$  holds over uniform regions  $\Lambda_n = (x_n, \infty)$  in the sense of (2.70). Further, for the purposes of completeness, we show (2.70) holds generally for sequences  $(x_n)$  such that  $n\mathbb{P}(|\mathbf{X}_0| > x_n) \rightarrow 0$ , as  $n \rightarrow \infty$  in the setting of Example 5.7.

Let  $p \in (\alpha/2, \alpha)$ , and consider a sequence  $(x_n)$  satisfying the assumptions of Proposition 5.8. Consider the region  $\Lambda_n = (x_n, \infty)$ , and consider  $x \in \Lambda_n$ . An application of Chebychev's inequality yields

$$(2.66) \quad \mathbb{P}(\|\overline{\mathbf{X}}_{[1,n]}/x^\epsilon\|_p^p - \mathbb{E}[\|\overline{\mathbf{X}}_{[1,n]}/x^\epsilon\|_p^p] > \delta) \leq 2n\delta^{-2} \sum_{t=0}^n I_t,$$

such that we denote  $I_t = \text{Cov}(|\overline{\mathbf{X}}_0/x^\epsilon|^p, |\overline{\mathbf{X}}_t/x^\epsilon|^p)$ .

Let  $(\Pi_t)$  and  $(R_t)$  be as in (B.64) such that  $(\mathbf{X}_t)$  satisfies Equation (5.42). We define a new Markov chain  $(\mathbf{X}'_t)_{t \geq 0}$  satisfying

$$(2.67) \quad \mathbf{X}'_t := \Pi_t \mathbf{X}'_0 + R_t,$$

with  $\mathbf{X}'_0$  independent of  $(\mathbf{X}_t)$  and identically distributed as  $\mathbf{X}_0$ . We can see  $(\mathbf{X}'_t)$  as the solution of the SRE (5.39) for the sequence of innovations  $((\mathbf{A}'_t, \mathbf{B}'_t))$  where  $(\mathbf{A}'_t, \mathbf{B}'_t) = (\mathbf{A}_t, \mathbf{B}_t)$  for  $t \leq 0$  and  $(\mathbf{A}'_t, \mathbf{B}'_t)_{t \geq 1}$  is an iid sequence independent of  $(\mathbf{A}_t, \mathbf{B}_t)$ , distributed as the generic element  $(\mathbf{A}, \mathbf{B})$ .

Then, following the notation in (2.66), we can rewrite  $I_t$  as

$$\begin{aligned}
 I_t &= \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p |\overline{\mathbf{X}}_t/x^\epsilon|^p] - \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p |\overline{\mathbf{X}}'_t/x^\epsilon|^p] \\
 &\leq \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p (|\overline{\mathbf{X}}_t/x^\epsilon|^p - |\overline{\mathbf{X}}'_t/x^\epsilon|^p)_+] \\
 &\leq \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p (|\mathbf{X}_t/x|^p - |\mathbf{X}'_t/x|^p)_+ \mathbb{1}(|\mathbf{X}_t/x| \leq \epsilon) \mathbb{1}(|\mathbf{X}'_t/x| \leq \epsilon)] \\
 &\quad + \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p |\overline{\mathbf{X}}_t/x^\epsilon|^p \mathbb{1}(|\mathbf{X}'_t/x| > \epsilon)] \\
 &= I_{t,1} + I_{t,2}.
 \end{aligned}$$

We show that  $I_{t,1}$  is negligible letting first  $n \rightarrow \infty$ , and then  $\epsilon \downarrow 0$ . For this, we consider two cases. First, assume  $p > 1$ . Then, for the first term  $I_{t,1}$ , a Taylor decomposition yields

$$\begin{aligned}
 I_{t,1} &\leq p \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p |\overline{\mathbf{X}}'_t/x - \mathbf{X}_t/x^{2\epsilon}| |\overline{\mathbf{X}}'_t/x^\epsilon + \xi(\overline{\mathbf{X}}'_t/x - \mathbf{X}_t/x^{2\epsilon})|^{p-1}] \\
 &= p \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p |\overline{\Pi}_t \mathbf{X}'_0/x - \Pi_t \mathbf{X}_0/x^{2\epsilon}| \\
 &\quad \times |\overline{\mathbf{X}}'_t/x^\epsilon + \xi(\overline{\Pi}_t \mathbf{X}'_0/x - \Pi_t \mathbf{X}_0/x^{2\epsilon})|^{p-1}],
 \end{aligned}$$

for some random variable  $\xi \in (0, 1)$  a.s. In the last equality, we have used the definition of  $(\mathbf{X}')$  in (2.67). Moreover, we can bound  $I_{t,1}$  by

$$\begin{aligned}
 I_{t,1} &\leq p 2^{0 \vee (p-1)} \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p |\overline{\Pi}_t \mathbf{X}'_0/x - \Pi_t \mathbf{X}_0/x^{2\epsilon}| |\overline{\mathbf{X}}'_t/x^\epsilon|^{p-1}] \\
 &\quad + p 2^{0 \vee (p-1)} \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^p |\overline{\Pi}_t \mathbf{X}'_0/x - \Pi_t \mathbf{X}_0/x^{2\epsilon}|^p]
 \end{aligned}$$

Now, an application of Jensen's inequality, Potter's bounds, and Karamata's theorem, yield

$$\begin{aligned}
 I_{t,1} &\leq c \mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^{2p}]^{1/2} \mathbb{E}[|\overline{\Pi}_t \mathbf{X}'_0/x^{4\epsilon}|^{2p}]^{1/2} \\
 &\leq c (\mathbb{E}[|\overline{\mathbf{X}}_0/x^\epsilon|^{2p}] \mathbb{E}[|\Pi_t|_{op}^{\alpha-\delta}] \mathbb{P}(|\mathbf{X}_0| > x))^{1/2}. \\
 &= O((\epsilon^{2p-\alpha} \mathbb{E}[|\Pi_t|_{op}^{\alpha-\delta}])^{1/2} \mathbb{P}(|\mathbf{X}_0| > x)), \quad x_n > x_0,
 \end{aligned}$$

for constants  $c > 0$ ,  $x_0 > 0$ . Moreover, under the assumptions of Example 5.7 we have  $\mathbb{E}[|\Pi_t|_{op}^{\alpha-\delta}] < 1$  for  $t$  sufficiently large. Thereby, we conclude

$$(2.68) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \sum_{t=1}^n I_{t,1} / \mathbb{P}(|\mathbf{X}_0| > x) = 0.$$

We now come back to the case where  $p < 1$ . In this case we can use a subadditivity argument and we conclude by similar steps that relation (2.68) holds for all  $p \in (\alpha/2, \alpha)$ .

Now, concerning the second term  $I_{t,2}$  we have

$$\begin{aligned}
I_{t,2} &:= \mathbb{E}[\overline{\mathbf{X}_0/x}^\epsilon |^p \overline{\mathbf{X}_t/x}^\epsilon |^p \mathbb{1}(|\mathbf{X}'_t/x| > \epsilon)] \\
&\leq \mathbb{E}[\overline{\mathbf{X}_0/x}^\epsilon |^p |\Pi_t \mathbf{X}_0/x|^\epsilon |^p \mathbb{1}(|\mathbf{X}'_t/x| > \epsilon)] \\
&\quad + \mathbb{E}[\overline{\mathbf{X}_0/x}^\epsilon |^p \overline{R_t/x}^\epsilon |^p \mathbb{1}(|\mathbf{X}'_t/x| > \epsilon)] \\
&\quad + \mathbb{E}[\overline{\mathbf{X}_0/x}^\epsilon |^p \mathbb{1}(|\Pi_t \mathbf{X}_0/x| > \epsilon) \mathbb{1}(|\mathbf{X}'_t/x| > \epsilon)] \\
&= O(\mathbb{E}[\overline{\mathbf{X}_0/x}^\epsilon |^p] \mathbb{E}[\overline{R_t/x}^\epsilon |^p \mathbb{1}(|\mathbf{X}'_t/x| > \epsilon)]).
\end{aligned}$$

Therefore we have,

$$\sum_{t=1}^n I_{t,2}/\mathbb{P}(|\mathbf{X}_0| > x) = O(n\mathbb{E}[\overline{\mathbf{X}_0/x}^\epsilon |^p]).$$

Assume  $(x_n)$  is such that there exists  $\kappa > 0$  satisfying  $n/x_n^{p \wedge (\alpha - \kappa)} \rightarrow 0$ , as  $n \rightarrow \infty$ . Then,

$$(2.69) \quad \lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \frac{\mathbb{P}(\|\overline{\mathbf{X}_{[1,n]}/x}^\epsilon\|_p^p - \mathbb{E}[\|\overline{\mathbf{X}_{[1,n]}/x}^\epsilon\|_p^p] > \delta)}{n\mathbb{P}(|\mathbf{X}_1| > x)} = 0.$$

Moreover, if  $n/x_n^p \rightarrow 0$  then  $\mathbb{E}[\|\overline{\mathbf{X}_{[1,n]}/x}^\epsilon\|_p^p] \rightarrow 0$  for  $p \in (\alpha/2, \alpha)$ . In this case,  $\mathbf{CS}_p$  holds uniformly over the region  $\Lambda_n$ .

On the other hand, note that we also have  $I_t = O(\beta_t)$ . Therefore, if we consider a sequence  $(\ell_n)$  such that  $\ell_n \rightarrow \infty$ ,  $n \rightarrow \infty$ , then we can have

$$\begin{aligned}
(2.70) \quad &\mathbb{P}(\|\overline{\mathbf{X}_{[1,n]}/x_n}^\epsilon\|_p^p - \mathbb{E}[\|\overline{\mathbf{X}_{[1,n]}/x_n}^\epsilon\|_p^p] > \delta)/n\mathbb{P}(|\mathbf{X}_0| > x_n) \\
&\leq 2n\delta^{-2} \left( \sum_{t=0}^{\ell_n} I_t + \sum_{t=\ell_n+1}^n I_t \right) / n\mathbb{P}(|\mathbf{X}_0| > x_n). \\
&\leq O(\ell_n \mathbb{E}[\overline{\mathbf{X}_0/x_n}^\epsilon |^p] + \sum_{t=\ell_n+1}^n I_t / \mathbb{P}(|\mathbf{X}_0| > x_n)) \\
(2.71) \quad &\leq O(\ell_n \mathbb{E}[\overline{\mathbf{X}_0/x_n}^\epsilon |^p] + \sum_{t=\ell_n+1}^n \beta_t / \mathbb{P}(|\mathbf{X}_0| > x_n)) = J_1 + J_2.
\end{aligned}$$

where in the last bound we use the covariance inequality for the  $(\beta_t)$  mixing coefficients. Furthermore, the bound in (2.70) consists of two terms as (2.70)  $\leq J_1 + J_2$ . If we want  $J_1$  to go to zero as  $n \rightarrow \infty$  we can choose  $\ell_n := x_n^{p-\delta}$ , for some  $\delta > 0$ . Now, for the second term  $J_2$ , we first note that it is null if  $\ell_n > n$  by convention. Otherwise we recall that the mixing-coefficients  $(\beta_t)$  have a geometric decaying rate. Thereby, there exists  $\rho \in (0, 1)$  such that we can bound the second term  $J_2$  by

$$\begin{aligned}
J_2 &= O(\sum_{t=\ell_n+1}^n \rho^t / \mathbb{P}(|\mathbf{X}_0| > x_n)) \\
&= O(\rho^{\ell_n} / \mathbb{P}(|\mathbf{X}_0| > x_n)) \\
&\leq O(\rho^{\ell_n} x_n^{(\alpha+\delta)}).
\end{aligned}$$

Therefore,  $J_2 \rightarrow 0$  as  $n \rightarrow \infty$  by plugging in the value we set for  $\ell_n$ . Overall, we conclude that for all sequences  $(x_n)$  such that  $n\mathbb{P}(|\mathbf{X}_n| > x_n) \rightarrow 0$ , then  $\lim_{n \rightarrow \infty} (2.70) = 0$  and this shows (2.65). Moreover, we also saw this convergence holds over uniform regions  $\Lambda_n = (x_n, \infty)$ , in the sense (2.69), if we assume in addition  $n/x_n^{p \wedge (\alpha - \kappa)} \rightarrow 0$ , as  $n \rightarrow \infty$ . Finally, this shows that  $\mathbf{CS}_p$  holds and this concludes the proof of Proposition 5.8.

**2.4. Proof of Theorem 5.10.** Our goal is to verify that we can apply Theorem 3.1 as we combine Proposition 5.8 and 5.9. First, notice for all  $\kappa > 0$ , if we consider a sequence  $(x_n)$  such that  $x_n \in (n^{\kappa+1/(p \wedge \alpha)}, +\infty)$ , then conditions  $\mathbf{AC}$ ,  $\mathbf{CS}_p$  hold thanks to Proposition 5.8. Since  $c(p) < \infty$  in (2.8), Proposition 2.3 holds and the time series admits a  $p$ -cluster process  $\mathbf{Q}^{(p)}$ . Fix  $\kappa > 0$ , and  $x_n = O(n^{\kappa+1/(p \wedge \alpha)})$ .

We focus now on the verification of the mixing condition  $\mathbf{MX}$  and  $\mathbf{MX}_\beta$  in Theorem 3.1. Applying Proposition 5.9, there exists  $\rho \in (0, 1)$  such that the coefficients  $(\beta_t)$  satisfy  $\beta_t = O(\rho^t)$ . Therefore,

$$m_n \beta_{b_n} = O(m_n \rho^{b_n}) = O(n \rho^{b_n} / b_n),$$

and an application of assumption *i*) yields  $\mathbf{MX}$ . Moreover, if we choose  $(k_n)$  according to (A.45) as in the linear model case then there exists  $\epsilon, \epsilon' > 0$ , and a constant  $c > 0$ , such that

$$m_n / k_n \leq c b_n^{-1 + \frac{\alpha}{p \wedge \alpha} + \epsilon'},$$

and thus  $\sum_{t=1}^{m_n} (m_n \beta_{b_n} / k_n)^{\delta/(2+\delta)} = O(\rho^{b_n \delta/(2+\delta)} (b_n)^\epsilon)$ , which goes to zero as  $n \rightarrow \infty$ . Moreover, for all  $\eta \in (0, 1)$ , choosing  $\ell_n = b_n^\eta$ , we have  $m_n \beta_{\ell_n} / k_n \rightarrow 0$ ,  $b_n / \ell_n \rightarrow 0$ , as  $n \rightarrow \infty$ . Therefore, we have verified  $\mathbf{MX}_\beta$  holds. This concludes the proof of Theorem 5.10 since all assumptions of Theorem 3.1 are verified.  $\square$

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