Localization, Big-Jump Regime And The Effect Of Disorder For A Class Of Generalized Pinning Models

Giambattista Giacomin, Benjamin Havret

LOCALIZATION, BIG-JUMP REGIME AND THE EFFECT OF DISORDER FOR A CLASS OF GENERALIZED PINNING MODELS

GIAMBATTISTA GIACOMIN AND BENJAMIN HAVRET

Abstract. One dimensional pinning models have been widely studied in the physical and mathematical literature, also in presence of disorder. Roughly speaking, they undergo a transition between a delocalized phase and a localized one. In mathematical terms these models are obtained by modifying the distribution of a discrete renewal process via a Boltzmann factor with an energy that contains only one body potentials. For some more complex models, notably pinning models based on higher dimensional renewals, other phases may be present.

We study a generalization of the one dimensional pinning model in which the energy may depend in a nonlinear way on the contact fraction: this class of models contains the circular DNA case considered for example in [7]. We give a full solution of this generalized pinning model in absence of disorder and show that another transition appears. In fact the systems may display up to three different regimes: delocalization, partial localization and full localization. What happens in the partially localized regime can be explained in terms of the "big-jump" phenomenon for sums of heavy tail random variables under conditioning.

We then show that disorder completely smears this second transition and we are back to the delocalization versus localization scenario. In fact we show that the disorder, even if arbitrarily weak, is incompatible with the presence of a big-jump.

1. Introduction of the model and results

1.1. **Phase transitions, disorder and pinning models.** The pinning model, sometimes called Poland-Scheraga model, comes up in a variety of real world phenomena. For example in the context of DNA denaturation (this is the Poland-Scheraga framework [49]), for polymers in presence of a *defect region* [25, 29, 43], for one dimensional interfaces in two dimensional systems with suitable boundary conditions [52]. But pinning models have also an intrinsic and theoretical interest, due in particular to the following crucial features:

- the model is solvable in its homogeneous version: with this respect we cite in particular [25], but, as pointed out in [36, App. A], the solvability mechanism is in reality just the basics of Renewal Theory developed in mathematics since the 40s with seminal contributions by J. L. Doob, P. Erdös, W. Feller and many others (e.g. [29, App. A] and references therein). Unless we specify otherwise, when we speak of pinning models, like here, we mean *one dimensional* pinning models: there are several higher dimensional generalizations that can and have been considered (e.g. [32] and references therein), and a class is going to be very relevant to us and will soon be mentioned;

(G.G., B.H.) Université de Paris, Laboratoire de Probabilités, Statistique et Modélisation, UMR 8001, F-75205 Paris, France. Corresponding author: G. Giacomin (giacomin@lpsm.paris)

2010 *Mathematics Subject Classification.* 60K35, 60K37, 82B44, 60K10.

Key words and phrases. Pinning Model, Disordered Systems, Critical Behavior, Big-Jump Phenomena, Disorder Relevance.

- the model exhibits a transition between a delocalized and a localized regime which is understood in depth thanks to solvability. Notably, the model depends on a real parameter $\alpha \geq 0$ and the critical phenomenon depends on α is such a way that the (de)localization transition can be of arbitrary order, i.e. from first order (no differentiability) to infinite order (C^∞ transition). In the physical literature the parameter $c = 1 + \alpha$ is typically used, but we are going to stick to α for the natural link with the stable law exponent of the inter-arrival distribution for the underlying renewal process.

Connected to the features we just outlined two research directions are particularly relevant to us:

(1) the effect of disorder on pinning models has been widely studied ([21, 30, 44] and references therein), both because of its importance from the modeling standpoint and because of the whole spectrum of critical phenomena generated by tuning the parameter α. In fact, understanding the effect of disorder on criticality is an important general issue: the stability of criticality under the introduction of disorder is expected to depend on the critical behavior in the homogeneous system. In a nutshell, less singular transitions are expected/predicted to be more stable. This is notably the content of the so called Harris criterion for disorder irrelevance [41] to which we come back below (see in particular Remark 1.10).
(2) (one dimensional) homogeneous pinning models exhibit only one transition, the (de)localization one. But it has been shown that some *generalized* pinning models may exhibit other transitions: this is in particular the case of the generalized Poland-Scheraga (gPS) model that takes into account the fact that the two DNA strands may have different length and that the pairing between the two strands may not be perfect [27, 28, 23]. As pointed out in [31], the gPS model can be seen as a pinning model based on a two dimensional renewal and its solvability nature (first pointed out in [27]) can once again be seen in renewal theory terms. The novel transition exhibited by the gPS is interpreted in the physical literature in analogy with *condensation phenomena* [23]. In renewal theory terms it is a phenomenon for conditioned sums of independent heavy tail random variables that goes under the name of *big-jump* [5, 8, 19]. The big-jump phenomenon has attracted attention in the mathematical community also in connection with other condensation phenomena (see [24] and references therein). We refer to [39, 53], and references therein, for more on big-jump regimes in physical systems.

We consider a generalization of the pinning model which is simpler than the ones just mentioned in (2). This model is based on one dimensional renewals but, in the homogeneous set-up, it exhibits a condensation/big-jump transition, in addition to the (de)localization transition. The circular DNA models studied in [50, 6, 7] have been one of the motivations of our work and appear as a particular case of the family we study. While the big-jump regime may be interpreted as a regime *partial localization*, we will not employ this terminology.

We study the effect of the disorder on this class of models and our main result is that condensation/big-jump transitions do not withstand the introduction of disorder. By this we mean that the transition is completely washed out and there is no condensation/big-jump in presence of disorder, even an arbitrarily weak disorder. In the Harris criterion language, disorder is therefore *relevant*, and in a very drastic way, even if, as we will

explain, big-jump transitions are rather smooth transitions and a direct application of Harris criterion [41] does not suggest disorder relevance.

1.2. **The generalized pinning model.** The model is based on the discrete renewal process $\tau = (\tau_j)_{j=0,1,\dots}$ with $\tau_0 = 0$, that is, if we set $\eta_j := \tau_j - \tau_{j-1}$ we have that $(\eta_j)_{j=1,2,\dots}$ is an IID sequence taking values in $\mathbb{N} := \{1, 2, \dots\}$. By using η for η_j we set $K(n) := \mathbf{P}(\eta = n)$ and assume that

$$K(n) \overset{n\to\infty}{\sim} \frac{C_K}{n^{1+\alpha}}\,, \tag{1.1}$$

for $\alpha > 0$ and a positive constant C_K. While mathematically not really necessary, we assume that $K(n) > 0$ for every $n \in \mathbb{N}$: this does simplify some proofs and it is assumed in part of the literature that we use. We point out that for the (bio-)physical interpretation of the model $K(1) > 0$ is very natural (e.g. [29, Ch. 1]) We stress that $\sum_{n\in\mathbb{N}} K(n) = 1$ and note that $\mathbf{E}[\eta] \in (1, \infty)$ if $\alpha > 1$, while $\mathbf{E}[\eta] = \infty$ if $\alpha \in (0, 1]$. The generalization to regularly varying $K(\cdot)$, see e.g. [29, App. A.4] is possible [42, Ch. 4] and in most cases it is straightforward. But it carries a certain burden of notations and technicalities that cannot be motivated in terms of new phenomena.

We are going to see τ as a random subset of $\mathbb{N} \cup \{0\}$, which almost surely contains 0 and infinitely many other points. In particular $\delta_n := \mathbf{1}_{n\in\tau}$ is the indicator function that there exists j such that $\tau_j = n$ and $|\tau \cap (0, N]| = \sum_{n=1}^{N} \delta_n = \sup\{j = 1, 2, \dots : \tau_j \le N\}$.

The class of models we present is based on a function $(m, N) \mapsto \Psi(m, N)$ defined for $N \in \mathbb{N}$ and $m \in \{1, \dots, N\}$. We give here the conditions we require on Ψ:

Definition 1.1. *We assume that*

$$\Psi(m, N) \;=\; Q(m, N) \exp\left(N H(m/N)\right)\,, \tag{1.2}$$

with

(1) $H : [0, 1] \to \mathbb{R} \cup \{-\infty\}$ concave and real analytic in the interior of its domain; moreover we assume that H is continuous up to the boundary points, including (with abuse of notation) the possibility that $H(0)$ and/or $H(1)$ are equal to $-\infty$.

(2) $Q(m, N) \ge 0$ and

- *for every $b > 0$ there exists $c > 0$ such that for every N and $m \le N$ we have*

$$Q(m, N) \;\le\; c \exp\left(bN\right)\,; \tag{1.3}$$

- *for every $u, v \in (0, 1)$, $u < v$, and every $b > 0$ there exists $c > 0$ such that for every N and m with $m/N \in [u, v]$ we have*

$$Q(m, N) \;\ge\; c \exp\left(-bN\right)\,. \tag{1.4}$$

These conditions readily imply that for $\rho \in (0, 1)$

$$\lim_{\substack{N\to\infty:\\ m/N\to\rho}} \frac{1}{N} \log \Psi(m, N) \;=\; H(\rho)\,. \tag{1.5}$$

Moreover, we will say that H is trivial if $H''(\rho) = 0$ for every $\rho \in (0, 1)$: this is the case of H affine. Whenever H is not trivial, H is strictly convex because it is analytic.

While a full analysis is possible, to keep reasonably concise in the analysis of the (de)localization transition we are going to assume at times that we have

$$H'(x) \overset{x\searrow 0}{=} H'(0) - c_H x + o(x)\,, \tag{1.6}$$

with $c_H > 0$ and $H'(0) := \lim_{x \searrow 0} H'(x)$. In particular, we are going to assume (1.6) only when $H'(0) < \infty$ (we will see that $H'(0) < \infty$ is necessary and sufficient for the existence of a delocalization transition). Note that if $H''(0) < 0$ exists, as limit of $H''(x)$, then (1.6) holds true with $c_H = |H''(0)|$.

We are now ready to define the non disordered model, that is the probability law $\mathbf{P}^{\Psi}_{N,h}$ that depends also on the real parameter h

$$\mathbf{P}^{\Psi}_{N,h}(\{A\}) := \frac{Z^{\Psi}_{N,h}(\{A\})}{Z^{\Psi}_{N,h}}, \tag{1.7}$$

where $A \subset \{0, \dots, N\}$,

$$Z^{\Psi}_{N,h}(\{A\}) := \mathbf{E}\left[\exp\left(h\sum_{j=1}^{N}\delta_j\right)\Psi\left(\sum_{j=1}^{N}\delta_j, N\right)\mathbf{1}_{\tau\cap[0,N]=A}\delta_N\right], \tag{1.8}$$

and $Z^{\Psi}_{N,h} := \sum_A Z^{\Psi}_{N,h}(\{A\})$, that is $Z^{\Psi}_{N,h}$ coincides with the right-hand side of (1.8) without the restriction to $\tau \cap [0, N] = A$. We write $\{A\}$, instead of simply A, because $\{A\}$ is an elementary event and $\mathbf{P}^{\Psi}_{N,h}$ is a probability on the discrete space $\mathcal{P}(\{0, 1, \dots, N\})$, with $\mathcal{P}(\cdot)$ the set of all subsets of $\cdot$. Note that $\mathbf{P}^{\Psi}_{N,h}(\{A\}) = 0$ unless both 0 and N are in A.

For the disordered version of the model we introduce the IID sequence $(\omega_n)_{n\in\mathbb{N}}$ with law $\mathbb{P}$. We assume that $\lambda(s) := \log \mathbb{E}[\exp(s\omega_1)] < \infty$ for every $s \in \mathbb{R}$ and, without loss of generality, we set $\mathbb{E}[\omega_1] = 0$ and $\mathbb{E}[\omega_1^2] = 1$. Moreover the two random sequences τ and ω are independent. For every realization of the disorder sequence, the disordered model has partition function

$$Z^{\Psi}_{N,\omega,\beta,h} := \mathbf{E}\left[\exp\left(\sum_{j=1}^{N}(\beta\omega_j + h)\delta_j\right)\Psi\left(\sum_{j=1}^{N}\delta_j, N\right)\delta_N\right], \tag{1.9}$$

where $\beta \geq 0$. Of course, the definition of $\mathbf{P}^{\Psi}_{N,\omega,\beta,h}$ is immediately inferred by analogy with (1.7).

Observe now that we can write

$$Z^{\Psi}_{N,\omega,\beta,h} = \sum_{m=1}^{N}\exp(mh)\Psi(m, N)\,\mathbf{E}\left[\exp\left(\beta\sum_{j=1}^{N}\omega_j\delta_j\right)\mathbf{1}_{\tau_m=N}\right], \tag{1.10}$$

where we have used that $\{|\tau \cap (0, N]| = m \text{ and } N \in \tau\} = \{\tau_m = N\}$ and if $\beta = 0$, that is in the non disordered case, this expression becomes even more explicit:

$$Z^{\Psi}_{N,h} = \sum_{m=1}^{N}\exp(mh)\Psi(m, N)\,\mathbf{P}(\tau_m = N). \tag{1.11}$$

Remark 1.2. *For the physical motivations and explicit choices of Ψ we refer the reader to Appendix A. A very basic case, that essentially contains the full richness of the model in term of qualitative phenomena, is obtained by choosing $H(\rho) = -\rho^2$ and $Q \equiv 1$, so $\Psi(m, N) = \exp(-m^2/N)$.*

1.3. **Results.** We start with a result that shows that the free energy density exists and that it can be represented in terms of the *free energy density* of a pinning model in which we have fixed the number of contacts.

Theorem 1.3. *For every $\beta \geq 0$ and $\rho \in [0,1]$ we have that $\mathbb{P}(\,\mathrm{d}\omega)$-a.s. the limit*

$$\lim_{\substack{N\to\infty:\\ m/N\to\rho}} \frac{1}{N} \log \mathbf{E}\left[\exp\left(\beta \sum_{j=1}^{N} \omega_j \delta_j \right) \mathbf{1}_{\tau_m = N} \right] =: \mathrm{G}(\beta,\rho)\,, \tag{1.12}$$

exists and $\mathrm{G}(\beta,\rho) \in \mathbb{R}$ is non random. Moreover $\mathrm{G}(\beta,\cdot)$ is concave, $\lim_{\rho\searrow 0} \mathrm{G}(\beta,\rho) = \mathrm{G}(\beta,0) = 0$, $\lim_{\rho\nearrow 1} \mathrm{G}(\beta,\rho) = \mathrm{G}(\beta,1) = \log K(1)$ and $\lim_{\rho\nearrow 1} \partial_\rho \mathrm{G}(\beta,\rho) = -\infty$.

Also the limit

$$\lim_{N\to\infty} \frac{1}{N} \log Z^{\Psi}_{N,\omega,\beta,h} =: \mathrm{F}_H(\beta,h)\,, \tag{1.13}$$

exists for every $\beta \geq 0$ and $h \in \mathbb{R}$, $\mathbb{P}(\,\mathrm{d}\omega)$-a.s. and in $\mathbb{L}^1$. $\mathrm{F}_H(\beta,h) \in \mathbb{R}$ is non random, $\mathrm{F}_H(\beta,\cdot)$ is non decreasing, convex and we have the conjugate variational formulas

$$\mathrm{F}_H(\beta,h) = \sup_{\rho\in[0,1]} \left(h\rho + H(\rho) + \mathrm{G}(\beta,\rho)\right) \quad \text{and} \quad \mathrm{G}(\beta,\rho) = \inf_{h\in\mathbb{R}} \left(\mathrm{F}_H(\beta,h) - \rho h - H(\rho)\right). \tag{1.14}$$

Of course $\mathrm{F}_H(\beta,h)$ is the free energy (density) of the model defined by (1.9). We point out that at this stage that $\partial_\rho \mathrm{G}(\beta,\rho)$ should be interpreted as the limit of the incremental ratio from the left, or from the right: they both exist by concavity. We will see in Proposition 2.2 that $\mathrm{G}(\beta,\cdot)$ is C^∞ if $\beta > 0$ and $\mathrm{G}(0,\cdot)$ is real analytic, except possibly at one point, in which it is in any case at least C^1, see Proposition 3.1.

It is worth pointing out that if $\Psi(m,N) = 1$ for every m and N then the model coincides with the well known disordered pinning model:

$$Z_{N,\omega,\beta,h} := \mathbf{E}\left[\exp\left(\sum_{j=1}^{N} (\beta\omega_j + h)\delta_j \right) \delta_N \right], \tag{1.15}$$

and the corresponding free energy is denoted by $\mathrm{F}(\beta,h)$. Of course also the case $\Psi(m,N) = \exp(am+b)$, a and b real constants, corresponds to trivial modifications of the pinning model. As a matter of fact, Theorem 1.3 is telling us that, whenever H of Definition 1.1 is trivial (i.e., affine), we are dealing with a model with free energy that coincides, up to an additive constant and a shift in h, with the free energy of a pinning model.

In order to better appreciate the results let us consider first the case $\beta = 0$: when, like here, there is no risk of confusion, we drop the dependence on $\beta(=0)$, that is we write $\mathrm{F}_H(h)$ for $\mathrm{F}_H(0,h)$, etc....

The first $\beta = 0$ result says that $h \mapsto \mathrm{F}_H(h)$ may have up to two singularity points if $\alpha > 1$. Otherwise, that is if $\alpha \in (0,1]$, it has at most one singularity. We introduce

$$\rho_c := \frac{1}{\mathbf{E}[\eta]}\,, \tag{1.16}$$

so $\rho_c = 0$ if $\alpha \in (0,1]$ and $\rho_c \in (0,1)$ if $\alpha > 1$. Moreover, given a model (that is given Ψ, hence H), we set

$$h_c^H := -H'(0) \in [-\infty,\infty) \qquad \text{and} \qquad h_b := -H'(\rho_c)\,. \tag{1.17}$$

Of course $h_c^H \le h_b$. Moreover $h_c^H = h_b$ if $\rho_c = 0$, and we can replace *if* with *if and only if* when $H(\cdot)$ is not trivial.

Theorem 1.4. *The function $h \mapsto \mathrm{F}_H(h)$ is real analytic for $h \in \mathbb{R} \setminus \{h_c^H, h_b\}$ and it is not real analytic at h_c^H nor at h_b. Moreover:*

(1) if $h_c^H \in \mathbb{R}$ (and regardless of $h_c^H < h_b$ or $h_c^H = h_b$) $\mathrm{F}_H(h) = H(0) \in \mathbb{R}$ for $h \le h_c^H$ and $\mathrm{F}_H(h) > H(0)$ for $h > h_c^H$ with the $h \searrow h_c^H$ asymptotic behaviors (assuming (1.6)):

- *for $\alpha > 1$ we have $\mathrm{F}_H(\rho) - H(0) \sim (h - h_c^H)^2/(2c_H)$;*
- *for $\alpha \in (0,1]$ we still have $\mathrm{F}_H(\rho) - H(0) \sim (h - h_c^H)^2/(2c_H)$ for $\alpha > 1/2$ and the same is true for $\alpha = 1/2$ but with a prefactor smaller than $1/(2c_H)$. For $\alpha \in (0, 1/2)$ we have $\mathrm{F}_H(\rho) - H(0) \sim c(h - h_c^H)^{1/\alpha}$ for a suitable $c > 0$;*

(2) if $h_c^H < h_b$ (which implies $\alpha > 1$) then the function $h \mapsto \mathrm{F}_H^{reg}(h)$ defined by

$$\mathrm{F}_H^{reg}(h) := \sup_{\rho \in [0,1]} \left(h\rho + H(\rho)\right), \tag{1.18}$$

is real analytic on $(h_c^H, 1)$ and, with $\kappa := \max(\alpha/(\alpha-1), 2)$ and for a suitable constant $c > 0$ (that depends in particular on α), we have (except for the case $\alpha = 2$)

$$\mathrm{F}_H(h) - \mathrm{F}_H^{reg}(h) \overset{h \searrow h_b}{\sim} -c(h - h_b)^\kappa, \tag{1.19}$$

while $\mathrm{F}_H(h) = \mathrm{F}_H^{reg}(h)$ for $h < h_b$. If $\alpha = 2$ the right-hand side in (1.19) is replaced by $-c(h-h_b)^2/\log(1/(h-h_b))$.

Theorem 1.4(2) is established in [6, 7] for specific choices of Ψ, see Appendix A, and for $K(n)$ equal, not simply asymptotically equivalent, to $C_K/n^{1+\alpha}$. The approach in in [6, 7] exploits the expression for the Mellin transform of $K(\cdot)$ in terms of special functions and by doing the asymptotic analysis via identification of singularities in the complex plane. Our analysis is more general and substantially simpler.

A direct consequence of Theorem 1.4 is that $\mathrm{F}_H(\cdot)$ is differentiable. In fact, the two singularity loci are h_c^H and h_b and

- the quadratic behavior at h_c^H, proven assuming (1.6), of course yields differentiability, but we take this occasion to stress that a first order transition, i.e. discontinuous $\mathrm{F}'_H(\cdot)$, happens only if $H(\cdot)$ is trivial: a look at the proof suffices to conclude that the contact fraction is continuous at h_c^H as soon as $H(\cdot)$ is strictly concave, i.e. without assuming (1.6);
- when $h_b > h_c^H$ the critical exponent κ is larger than one, and, again, this yields differentiability.

The transition at h_c^H is a delocalization/localization transition: in fact $\mathrm{F}'_H(h) = 0$ for $h < h_c^H$ and $\mathrm{F}'_H(h) > 0$ for $h > h_c^H$ and $\mathrm{F}'_H(h)$ coincides with $\lim_N \mathbf{E}^\Psi_{N,h}[\sum_{j=1}^N \delta_j]/N$, which is the contact density. Note that for this transition we assume (1.6), so $H(\cdot)$ is non trivial, and the critical exponent coincides with the critical exponent of the standard pinning model only for $\alpha \le 1/2$. For $\alpha > 1/2$ the critical exponent of the $\beta = 0$ pinning model is $\max(1/\alpha, 1)$ [29, Ch. 2].

Remark 1.5. *A more direct view of the delocalization/localization transition can be taken, without assuming $\beta = 0$, by noticing that $\mathrm{F}_H(\beta, h) \ge H(0)$. This fact is straightforward under the stronger condition $\liminf_N (\log Q(1,N))/N \ge 0$ because it suffices to restrict the*

partition function to the event $\tau_1 = N$ (that is, the only contact point is in N) and we obtain for every $b > 0$ and N sufficiently large

$$Z^{\Psi}_{N,\omega,\beta,h} \geq \exp\left(N(H(1/N) - b)\right) \exp(\beta\omega_N + h)\mathbf{P}(\tau_1 = N)\,. \tag{1.20}$$

So $\lim_N (1/N) \log Z^{\Psi}_{N,\omega,\beta,h} \geq H(0)$ a.s. and we are done. For a proof without the additional assumption see Proposition B.2.

The transition at h_b, when $h_b > h^H_c$, corresponds in physical terms to the appearing/disappearing of a *condensation* segment or of a *macroscopic loop*. The underlying phenomenon is well known also in the probability literature and it is called *big-jump* (see [5, 19] and references therein). In order to make this precise we introduce for every N the order statistics of the η sequence *up to* N, that is the order statistics of $\eta_1, \eta_2, \dots, \eta_{|\tau\cap(0,N]|}$, for which we use the notation $\eta_{1,N} \geq \eta_{2,N} \geq \ldots \geq \eta_{|\tau\cap(0,N]|,N}$. Note that this order statistics is empty if $\tau_1 > N$, which never happens because we always work with $N \in \tau$. On the other hand, what may happen is that the sequence contains only one element, that is $\eta_{1,N} = \tau_1 = N$, and in this case we set $\eta_{2,N} = 0$.

Theorem 1.6. *For every $h \neq h^H_c$ we have that in $\mathbf{P}^{\Psi}_{N,h}$-probability*

$$\lim_{N\to\infty} \frac{|\tau \cap (0, N)|}{N} = \rho_h := \mathrm{F}'_H(h)\,, \tag{1.21}$$

and when $\rho_c > 0$ and $h \neq h_b$

$$\lim_{N\to\infty} \frac{\eta_{1,N}}{N} = \left(1 - \frac{\rho_h}{\rho_c}\right)_+, \qquad \lim_{N\to\infty} \frac{\eta_{2,N}}{N} = 0\,. \tag{1.22}$$

If $\rho_c = 0$ and $h > h^H_c = h_b$ we have that $\lim_N \eta_{1,N}/N = 0$.

The only reason to require $h \neq h_b$ is to keep proofs concise: the statement holds without this requirement [42, Ch. 4]. The same is true for $h \neq h^H_c$ and (1.21), if one takes care of excluding the cases in which $\mathrm{F}'_H(h^H_c)$ does not exist: note that, by Theorem 1.4(1), $\mathrm{F}'_H(h^H_c)$ exists and it is equal to zero for $H(\cdot)$ non trivial.

Informally stated, Theorem 1.6 is spelling out the standard fact that $\mathrm{F}'_H(h)$ is the contact fraction and that the largest loop $\eta_{1,N}$ encompasses essentially all the system in the delocalized regime $h < h^H_c$ (but only if $\rho_c > 0$! See Remark 1.7) and it is instead macroscopically negligible (i.e., $\eta_{1,N} = o(N)$) if $h > h_b$. But the key point for us is that when $h^H_c < h_b$, for $h \in (h^H_c, h_b)$ the largest loop $\eta_{1,N}$, normalized by dividing by N, is asymptotically of size $(1 - \rho_h/\rho_c) \in (0, 1)$. Moreover, cf. (1.22), all other loops are macroscopically negligible.

Moreover, as we have seen, when $h^H_c < h_b$ then $\rho_h = \mathrm{F}'_H(h)$ is continuous both at h^H_c and at h_b and this implies that the normalized large loop size behaves continuously at these transitions (in the first case it goes to one, in the second one it vanishes).

Remark 1.7. *The reader may be surprised by the lack of a full path delocalization result for $\rho_c = 0$, i.e. $\alpha \in (0, 1]$, like for $\alpha > 1$. We are convinced that this cannot be obtained with our assumptions on $Q(\cdot, \cdot)$: see the control from below in (1.4) of Def. 1.1. Note in fact that, for example, we can choose $Q(m, N) = \exp(-N^2)$ for $m \leq N/\log N$ and this forces the presence of at least about $N/\log N$ contacts: forcing them to be close to the boundary is very expensive in probability terms.*

Theorem 1.6 can be improved in a number of ways, notably the largest loop for $h > h_b$ is $O(\log N)$, while the second largest loop for $h \in (h_c^H, h_b)$ has a power law scaling and for $h < h_c^H$ is $O(1)$. These issues are developed in [42, Ch. 4], along with a detailed analysis of the critical cases. Our focus is on the effect of the disorder on the system and Theorems 1.4 and 1.6 are sufficient for this purpose.

In fact, the main point of our work is that for $\beta > 0$ the big-jump phenomenon disappears, and this is what we present next, along with an analysis of the effect of the disorder on the (de)localization transition.

Recall Remark 1.5 and set $h_c^H(\beta) := \inf\{h : \mathrm{F}_H(\beta, h) > H(0)\}$. Of course we have $h_c^H(\beta) := \sup\{h : \mathrm{F}_H(\beta, h) = H(0)\}$ and $h_c^H = h_c^H(0)$. If $H(\cdot) \equiv 0$ we use $h_c(\beta)$ for $h_c^H(\beta)$, in parallel with the use of $\mathrm{F}(\beta, h)$ for $\mathrm{F}_H(\beta, h)$. Much work has been done on identifying as precisely as possible $h_c(\beta)$: a through review of the literature is in Section 2. Here we only anticipate that $h_c(\beta) \in [-\lambda(\beta), 0]$ for every $\beta \geq 0$.

Theorem 1.8. *For $\beta > 0$ we have that*

(1) $h \mapsto \mathrm{F}_H(\beta, h)$ is C^∞ for $h \in (h_c^H(\beta), \infty)$ and $h_c^H(\beta) = h_c(\beta) - H'(0)$, so $h_c^H(\beta) > -\infty$ if and only if $H'(0) < \infty$ (like for $\beta = 0$). If $H'(0) < \infty$ (without assuming (1.6)) there exists C_β and $\Delta > 0$ such that if $h - h_c^H(\beta) \in [0, \Delta]$

$$\mathrm{F}_H(\beta, h) \leq H(0) + C_\beta (h - h_c(\beta))^2 , \tag{1.23}$$

and, assuming(1.6), for $\alpha \in (0, 1/2)$ and $\beta \in [0, \beta_\alpha]$, for a suitable choice of $\beta_\alpha > 0$, we have the sharper result

$$\mathrm{F}_H(\beta, h_c^H(\beta) + y) - H(0) \overset{y \searrow 0}{\sim} \mathrm{F}_H(0, h_c^H(0) + y) - H(0) . \tag{1.24}$$

(2) for every h we have that $\mathbb{P}(\,\mathrm{d}\omega)$-a.s. in $\mathbf{P}^\Psi_{N,\omega,\beta,h}$-probability

$$\lim_{N\to\infty} \frac{|\tau \cap (0, N)|}{N} = \partial_h \mathrm{F}_H(\beta, h) , \tag{1.25}$$

and for $h > h_c^H(\beta)$

$$\lim_{N\to\infty} \frac{\eta_{1,N}}{N} = 0 . \tag{1.26}$$

A number of comments are in order:

(1) $h_c^H(\beta)$ may be equal to $-\infty$, but otherwise $\mathrm{F}_H(\beta, \cdot)$ is not analytic at $h_c^H(\beta)$, which is therefore a critical point marking the transition from zero contact density (delocalized regime) to positive contact density (localized regime);
(2) from the proof we see that C_β can be chosen independent of β if we assume (1.6) (see Remark 1.9);
(3) the finite order big-jump transition at h_b has disappeared, but the C^∞ regularity estimate on the free energy leaves open the possibility of an infinite order transition;
(4) nevertheless, (1.26) tells us that the loops in the localized regime do not have macroscopic size, so the large loop phenomenon is washed out by the disorder;
(5) we have decided to leave aside the delicate analysis of the path behavior in the delocalized phase: we certainly expect that results like in [4, 34] can be adapted, but only under stronger conditions on $\Psi(m, N)$ (and the problem is already present for $\beta = 0$, see Remark 1.7).

Two important remarks:

Remark 1.9. *The proof of* (1.23) *exploits the smoothing inequality* [14, 35, 36] *that we recall in* (2.5) *below, but only in part because the result holds as soon as* $H''(\rho)$ *stays bounded away from* 0 *for* ρ *close to zero, and in particular when* (1.6) *holds. And, in view of the* $\beta = 0$ *results in Theorem 1.4(1),* (1.23) *does not establish a smoothing phenomenon. Disorder relevance is certainly expected and it would follow from what is expected to hold for the disordered pinning model (that is, an infinite order transition for* $\alpha \geq 1/2$*, see Remark 2.1). In our model however we can see smoothing for* $\alpha > 1/2$ *if we do not assume* (1.6)*. Notably if we assume for example that* $H'(\rho) - H'(0) \sim -c\rho^\gamma$ *with a* $\gamma > 1$ *and* $c > 0$*:* (1.23) *holds, but Theorem 1.4(1) changes and* $\mathrm{F}_H(0, h_c(0) + \delta)$ *becomes equivalent to* $\delta^{\max(1+1/\gamma, 1/\alpha)}$ *times a positive constant. In this case* (1.23) *does establish a smoothing phenomenon and disorder relevance. Finally,* (1.24) *establishes disorder irrelevance for* $\alpha < 1/2$*.*

Remark 1.10. *The Harris criterion is applied in* [7, Sec. IV] *to the big-jump transition and the claim is that disorder is irrelevant for this transition for* $\alpha \in (1, 2]$*, while for* $\alpha > 2$ *disorder is "marginal", i.e. at the boundary between irrelevance and relevance. This is in contrast with Theorem 1.8 which proves relevance of the disorder for every* $\alpha > 1$*. It would be of course very interesting to understand what is happening in the Harris' perspective. We take this occasion to point out that the "instability" of the big-jump transition under the effect of disorder has been observed also in* [40, 47]*. In* [40, 47] *the disorder is introduced in such a way that the renewal structure is preserved and explicit computations can be performed. In our case there is no such structure and our results follow from the smoothing inequality bound for the standard pinning case* [14, 35, 36]*. While we believe that our disorder relevance result for big-jump transitions should hold in greater generality, our approach does not generalize in an evident way, notably not to the tightly related gPS model mentioned in Section 1.1. The contribution* [9] *deals with the disorder (ir)relevance issue in the gPS model, but only for the localization transition.*

Organization of the paper.

- In Section 2 we present the main ideas on how we deal with the disorder and we provide a proof of Theorem 1.8(1), relying on the variational formulas of Theorem 1.3 and on the uniform strict convexity bound of Theorem B.1.
- In Section 3 we provide a full analysis of $\mathrm{F}_H(0, h)$. In particular, this section contains the proof of Theorem 1.4.
- In Section 4 we analyse the trajectories of the process for $\beta = 0$ (proof of Theorem 1.6).
- In Section 5 we complete the proof of Theorem 1.8, by proving part (2) that concerns the trajectories: no big-jump for $\beta > 0$.
- In Section 6 we take care of the free energy existence issues and of the variational formulas (proof of Theorem 1.3).
- In Appendix A we explain how the circular DNA case [6, 7] fits in our framework and in Appendix B we prove that $\partial_h^2 \mathrm{F}(\beta, h) > 0$ for every $h > h_c(\beta)$ and we complete Remark 1.5.

2. Exploiting the Legendre transform and the key role of the standard pinning model

Let us start by pointing out the direct consequence of (1.14)

$$\mathrm{F}(\beta, h) = \sup_{\rho \in [0,1]} \left(h\rho + \mathrm{G}(\beta, \rho) \right) \quad \text{and} \quad \mathrm{G}(\beta, \rho) = \inf_{h \in \mathbb{R}} \left(\mathrm{F}(\beta, h) - \rho h \right) . \tag{2.1}$$

The strategy we employ is to obtain information on $\mathrm{G}(\beta,\rho)$, defined in (1.12), via the second formula in (2.1) and what we know on $\mathrm{F}(\beta,h)$. We start therefore by collecting here the relevant known results on $\mathrm{F}(\beta,h)$: some of these results are straightforward, but most of them are the outcome of the work of several contributors.

(**P1**) Basic convexity and monotonicity properties, together with some relatively standard bounds, show that $\mathrm{F}(\beta,\cdot)$ is convex, it is equal to 0 for $h \le h_c(\beta)$ and it is positive and increasing for $h > h_c(\beta)$: $h_c(\beta) \le 0$ and for more on its value see (**P4**).

(**P2**) $h \mapsto \mathrm{F}(\beta,h)$ is C^∞ for $h > h_c(\beta)$ [37, th. 2.1] (in [37] a concentration condition is required on the law of ω_1, but this is not used in the proof of Theorem 2.1) and it is analytical if $\beta = 0$ for $h > h_c(0)$, see e.g. either [29, Ch. 2] or [30, Ch. 2]. Of course it is also analytical for $h < h_c(\beta)$ and $h_c(\beta)$ is a non analyticity point. We add that for every $\beta \ge 0$ it is straightforward to show that $\lim_{h\to\infty} \partial \mathrm{F}_h(\beta,h) = 1$.

(**P3**) For $\beta = 0$ the model is solvable [25, 29, 30]: we have already pointed out that $h_c(0) = 0$, but the sharp behavior of the free energy and its derivatives at criticality is available too. That is, for $\alpha \in (0,1)$ there exists $c_\alpha > 0$ such that

$$\mathrm{F}(0,h) \overset{h\searrow 0}{\sim} c_\alpha h^{1/\alpha}\,, \tag{2.2}$$

and (2.2) holds also if we differentiate $k \in \mathbb{N}$ times both sides. If $\alpha > 1$ instead $\mathrm{F}(0,h) \sim h/\mathbf{E}[\eta]$ and this statement can be differentiated once. If $\alpha = 1$ instead $\mathrm{F}(0,h) \sim C_K h/\log(1/h)$ (one differentiation allowed). These results imply the rougher statement

$$\frac{\log \mathrm{F}(0,h)}{\log h} \overset{h\searrow 0}{\sim} \max\left(1, \frac{1}{\alpha}\right)\,. \tag{2.3}$$

Notably, the transition is of first order if $\alpha > 1$ and it is of higher order if $\alpha \in (0,1]$:

$$\lim_{h\searrow 0} \partial_h \mathrm{F}(0,h) \;=\; \frac{1}{\mathbf{E}[\eta]} \begin{cases} > 0 & \text{if } \alpha > 1\,, \\ = 0 & \text{if } \alpha \in (0,1]\,. \end{cases} \tag{2.4}$$

(**P4**) For $\beta > 0$ we have $-\lambda(\beta) \le h_c(\beta) < h_c(0) = 0$ (see [30, Ch. 3] and [29, Ch. 4]; see [2] and [29, Section. 5.2] for the strict inequality). Moreover $h_c(\beta) > -\lambda(\beta)$ for $\alpha \ge 1/2$ [3, 11, 15, 20], but $h_c(\beta) = -\lambda(\beta)$ for $\alpha \in (0,1/2)$ and $\beta \le \beta_\alpha$, for a suitable $\beta_\alpha > 0$ [1, 46, 51].

(**P5**) For $\beta > 0$ there exists $c_\beta > 0$ and $\Delta_0 > 0$ such that for every $\Delta \in (0,\Delta_0]$

$$0 \;<\; \mathrm{F}(\beta, h_c(\beta)+\Delta) - \mathrm{F}(\beta, h_c(\beta)) \;\le\; c_\beta \Delta^2\,. \tag{2.5}$$

The lower bound in (2.5) is trivial, the upper bound is the smoothing inequality [14, 35, 36] . We stress that (2.5) directly implies that, regardless of the value of α, for $\beta > 0$ we have

$$\lim_{h\searrow h_c(\beta)} \partial_h \mathrm{F}(\beta,h) \;=\; 0\,. \tag{2.6}$$

(**P6**) If $\alpha \in (0,1/2)$ (and $\beta \le \beta_\alpha$, see (**P4**)) we have [1, 38, 46, 51]

$$\mathrm{F}(\beta, h_c(\beta)+y) \overset{y\searrow 0}{\sim} \mathrm{F}(0,y)\,. \tag{2.7}$$

Remark 2.1. *The truly open problem for the disordered pinning model, and, as a matter of fact, for every disorder relevant model, is what is the precise critical behavior when disorder is relevant, see* [30, Ch. 5] *for a discussion and references. For the pinning model in the relevant disorder regime it is now expected that the transition becomes of infinite order. One of the main reason is that the model is expected to be in the strong or infinite disorder universality class* [17, 26, 44]*, see also the more recent contribution* [22]*. In this line there have been also some mathematical progress* [10, 16]*, but they do not impact directly the pinning model.*

2.1. **Legendre transform viewpoint on homogeneous pinning.** Recalling (1.11) and (1.12) we see that $\mathrm{G}(\rho) = \mathrm{G}(0,\rho)$ has a very simple expression:

$$\mathrm{G}(\rho) \;=\; \lim_{\substack{N\to\infty:\\ m/N\to\rho}} \frac{1}{N} \log \mathbf{P}\left(\tau_m = N\right) \,, \tag{2.8}$$

and arbitrarily precise estimates on $\mathrm{G}(\cdot)$ can be obtained, see in particular Proposition 3.1 that is resumed in part in Fig. 1 and in Fig. 2, and their captions. The behavior of $\mathrm{F}(\cdot) = \mathrm{F}(0,\cdot)$ can then be extracted from $\mathrm{G}(\cdot)$ via (2.1). In particular, the non analytic behavior of $\mathrm{G}(\cdot)$ at $\rho_c > 0$ yields a jump of size ρ_c in $\mathrm{F}'(\cdot)$ at $h_c(0) = 0$. The jump disappears if $\rho_c = 0$. Moreover, these implications can be reversed, and the behavior of $\mathrm{G}(\cdot)$ can be inferred from the one of $\mathrm{F}(\cdot)$.

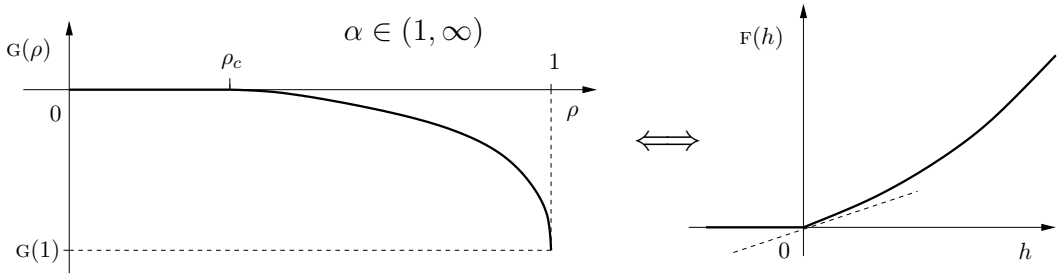

FIGURE 1. The figure illustrates the (Legendre transform) link between $\mathrm{G}(\cdot)$ and $\mathrm{F}(\cdot)$ for $\alpha > 1$ (and $\beta = 0$). We stress here that the flat portion of $\mathrm{G}(\cdot)$, that is $\mathrm{G}(\rho) = 0$ for $\rho \in [0, \rho_c]$ with $\rho_c = 1/\sum_n nK(n)$, has a direct counterpart the first order localization transition for the pinning model: quantitatively, $\lim_{h\searrow 0} \mathrm{F}'(h) = \rho_c$. Other features stressed in the graph are (1) the fact that $\mathrm{G}(1) = -\log K(1)$ is finite but $\lim_{\rho\nearrow 1} \mathrm{G}'(\rho) = -\infty$ and (2) that $\lim_{\rho\to\rho_c} \mathrm{G}'(\rho) = 0$. These two features are going to be central for our generalized pinning model: feature (1) forces the optimizing contact density to be smaller than one and feature (2) makes the big-jump transition of order two or larger.

The non analyticity at $\rho = \rho_c$ can be viewed as a phase transition: in fact, $\mathrm{G}(h)$ capture the exponential asymptotic behavior of $\mathbf{P}\left(\tau_m = N\right)$, for $m/N \sim \rho$, and we can view $\mathbf{P}\left(\tau_m = N\right) = \mathbf{E}[\mathbf{1}_{\tau_m = N}]$ as the partition function of the model which is just the renewal conditioned to $\tau_m = N$ (in Section 4 this probability will be denoted by $\mathbf{Q}_{N,m}$). The trajectories $\mathbf{Q}_{N,m}$ are substantially different when $m/N \sim \rho$ is below or above ρ_c and the phenomenon is known in probability as the *big-jump* phenomenon: if $\rho < \rho_c$ a single large excursion takes care of the anomalously low contact density, in fact the *typical* contact density for the renewal is ρ_c (this is very well known Renewal Theorem, see [30, App. A] and references therein). Instead the system constrained to a contact density $\rho > \rho_c$ globally modifies itself to accomodate more excursions. See Fig. 3 for a visual explanation: Proposition 4.1 is a mathematical presentation of the big-jump transition.

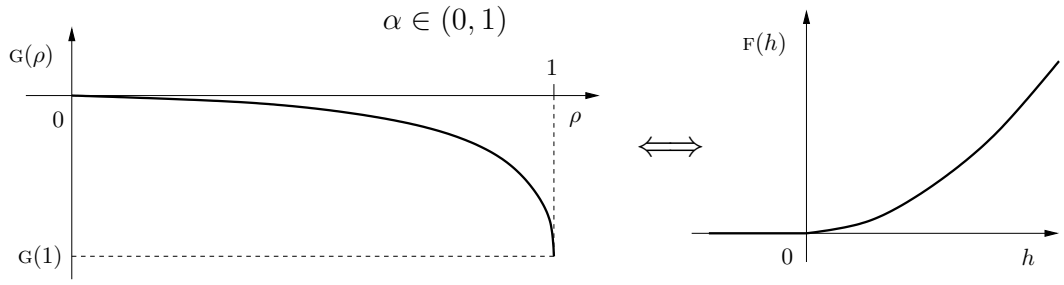

FIGURE 2. The figure illustrates the (Legendre transform) link between $\mathrm{G}(\cdot)$ and $\mathrm{F}(\cdot)$ for $\alpha \in (0,1)$ (and $\beta = 0$). This time $\mathrm{G}(\cdot)$ is analytic over all the domain. The corresponding behavior of $\mathrm{F}(\cdot)$ is on the right and the difference with the case $\alpha > 1$ is that $\mathrm{F}'(\cdot)$ exists also at the origin. In fact, the smaller α is, the more $\mathrm{F}(\cdot)$ is regular at the origin. But the most prominent fact is that $\rho_c = 0$ is equivalent $\mathrm{F}'(\cdot)$ being C^1 in 0.

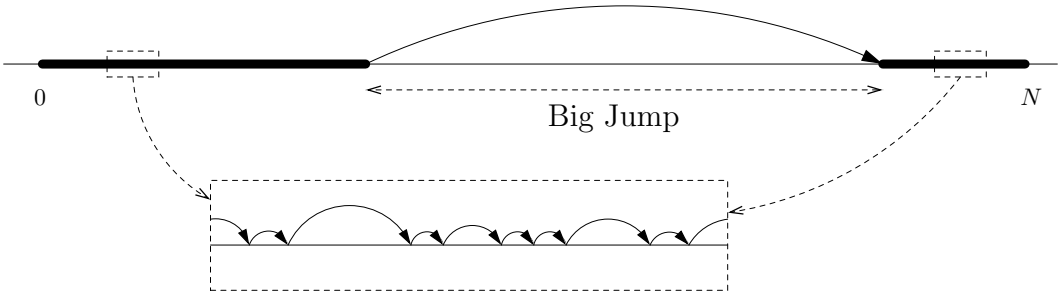

FIGURE 3. The big-jump phenomenon that happens when we condition the renewal to have m contacts between before N, with $m/N \sim \rho$ smaller than the typical value ρ_c. The system behaves typically, so with contact density $\rho_c = 1/\mathbf{E}[\eta]$, and compensates for the low global contact density by making a *big-jump* of length $\sim (1-\rho/\rho_c)N$, randomly (uniformly) placed in the interval.

2.2. **Legendre transform viewpoint on disordered pinning.** The key point here is simply that (2.6) is telling us that there are no longer two scenarios, but only one: qualitatively, the one of $\beta = 0$ and $\alpha \in (0,1]$. Here is a central statement for our analysis:

Proposition 2.2. *For $\beta > 0$ we have that $\rho \mapsto \mathrm{G}(\beta,\rho)$ is C^∞ in the interior of its domain of definition, that is for $\rho \in (0,1)$. Moreover $\lim_{\rho \searrow 0} \mathrm{G}(\beta,\rho) = 0$ and $\lim_{\rho \searrow 0} \partial_h \mathrm{G}(\beta,\rho) = -h_c(\rho)$. Finally, $\partial_h^2 \mathrm{G}(\beta,\rho) < 0$ for every $\rho \in (0,1)$ and there exists $c > 0$ (depending on β) such that for every ρ*

$$\mathrm{G}(\beta,\rho) \,\le\, -h_c(\beta)\rho - c\rho^2\,. \tag{2.9}$$

*For $\alpha \in (0,1/2)$ and $\beta \in [0,\beta_\alpha]$ (β_α given in (**P4**)) we have that*

$$\mathrm{G}(\beta,\rho) + h_c(\beta)\rho \overset{\rho \searrow 0}{\sim} \mathrm{G}(0,\rho)\,. \tag{2.10}$$

Proof. Fix $\beta > 0$. By the second identity in (2.1) for every $\rho \in (0,1)$

$$\mathrm{G}(\beta,\rho) \,=\, \mathrm{F}(\beta,h_\rho) - \rho h_\rho\,, \tag{2.11}$$

with $h_\rho = h$ unique solution to $\rho = \partial_h \mathrm{F}(\beta,h)$: note that, by (2.6), by the large h remark at the end of (**P2**) and the strict convexity of $\mathrm{F}(\beta,\cdot)$ (see Theorem B.1), $\partial_h \mathrm{F}(\beta,\cdot)$ is a

bijection from $(h_c(\beta),\infty)$ to $(0,1)$. Fully exploiting Theorem B.1, i.e. using $\partial_h^2 \mathrm{F}(\beta,h)>0$ for $h>h_c(\beta)$, by the Implicit Function Theorem we see that $\rho \mapsto h_\rho$ is C^∞, so $\mathrm{G}(\beta,\cdot)$ is C^∞ too.

By differentiating once (2.11) we obtain $\partial_\rho \mathrm{G}(\beta,\rho) = -h_\rho$ which tends to $-h_c(\beta)$ for $\rho \searrow 0$. By differentiating once more we obtain $-\partial_\rho^2 \mathrm{G}(\beta,\rho) = 1/\partial_h^2 \mathrm{F}(\beta,h_\rho) \in (0,\infty)$ and we have all the claimed estimates except (2.9), that we consider next. Since $h_\rho \searrow h_c(\beta)$ when $\rho \searrow 0$, for every $\Delta_0>0$ we have

$$\mathrm{G}(\beta,\rho) \;=\; -h_c(\beta)\rho + \inf_{h\in(h_c(\beta),h_c(\beta)+\Delta)} \left(\mathrm{F}(\beta,h) - \rho(h-h_c(\beta))\right)\,, \tag{2.12}$$

provided that ρ is smaller than constant that depends on Δ_0. Therefore by (**P5**) we obtain $\mathrm{G}(\beta,\rho) \le -h_c(\beta)\rho - \rho^2/(4c_\beta)$ and (2.9) follows.

Finally, (2.10) follows directly from (2.7) of (**P6**): this analysis coincides with the $\beta=0$ analysis, developed in greater generality in Section 3. □

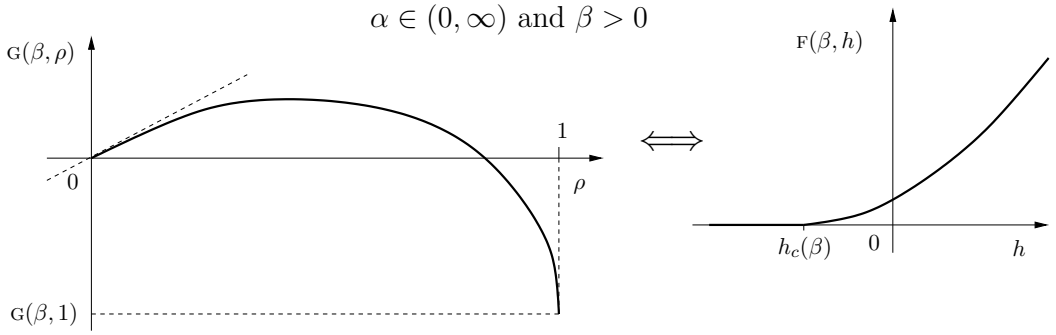

FIGURE 4. The figure illustrates the (Legendre transform) link between $\mathrm{G}(\cdot)$ and $\mathrm{F}(\cdot)$ for $\alpha>0$ and $\beta>0$. $\mathrm{F}(\beta,\cdot)$ has a non analyticity point at a critical value $h_c(\beta)<0$: to the left of this critical value the free energy is zero and to the right it is positive, C^∞ and strictly convex. From this we can extract $\mathrm{G}(\beta,\cdot)$ is C^∞ and strictly concave. The positive slope of $\mathrm{G}(\beta,\cdot)$ at the origin, more precisely $\lim_{\rho\searrow 0}\partial_\rho \mathrm{G}(\beta,\rho) = -h_c(\beta)$, is a direct consequence $h_c(\beta)<0$ (see Proposition 2.2). The fact that $\mathrm{G}(\beta,\cdot)$ is strictly concave strongly hints to the similarity with the case of Fig. 2.

2.3. **The generalized pinning model: free energy and transitions.** The free energy $\mathrm{F}_H(\beta,h)$ is just given by (1.14) via elementary considerations given the properties and features of $\mathrm{G}(\beta,\rho)$. These features are richer for $\beta=0$ (Fig. 1 and Fig. 2) and they reduce to Fig. 4 for $\beta>0$. In particular

- $\mathrm{G}(0,\rho)$ has a singularity at $\rho=\rho_c>0$ that directly reflects on a singularity of $\mathrm{F}_H(0,h)$ at $h_b=-H'(\rho_c)$, and corresponds to the big-jump transition: the proof is in Section 4, but the result can be readily understood because the variational formula suggests that the system will behave like a renewal constrained to a contact density $\rho=\rho_h$, where ρ_h is the optimal density;
- when $\beta>0$ instead this singularity disappears and, modulo the shift of the critical point $h_c(0)$ to $h_c(\beta)$, that generates the positive slope at the origin, Fig. 4 is analogous to Fig. 2. Therefore the transition at h_b disappears: the proof that the trajectories of the process have no big-jump transition is given in Section 5.

Here we provide the proof that Proposition 2.2 yields, via Legendre transform, the properties of $\mathrm{F}_H(\beta,h)$, for $\beta>0$, given in Theorem 1.8, see also Fig. 5.

Proof of Theorem 1.8(1). The result is already known if $H(\cdot)$ is trivial, but the argument we give applies in general. By (1.14) and Proposition 2.2 we have that

$$\mathrm{F}_H(\beta,h) \,=\, h\rho_h + H(\rho_h) + \mathrm{G}(\beta,\rho_h)\,, \tag{2.13}$$

with $\rho=\rho_h$ unique solution of $h=-H'(\rho)-\partial_\rho \mathrm{G}(\beta,\rho)$. Note that $\partial^2_\rho \mathrm{G}(\beta,\rho)<0$ for $\rho\in(0,1)$ yields that $h\mapsto\rho_h$ is a C^∞ bijection between $(h_c(\beta)-H'(0),\infty)$ and $(0,1)$. In particular $\mathrm{F}_H(h)>H(0)$ for $h>h_c(\beta)-H'(0)$ and, by continuity, $\mathrm{F}_H(h_c(\beta)-H'(0))=H(0)$. On the other hand, $\rho_h=0$ for $h<h_c(\beta)-H'(0)$. So $h_c^H(\beta)=h_c(\beta)-H'(0)$. The fact that $\mathrm{F}_H(\beta,\cdot)\in C^\infty$ on $\mathbb{R}\setminus\{h_c^H(\beta)\}$ is also a direct consequence of (2.13) above $h_c^H(\beta)$, and of the triviality of the free energy below $h_c^H(\beta)$.

Let us turn to (1.23). We claim that for every $\beta>0$ there exists a constant $c>0$ such that $\rho_h\le c(h-h_c^H(\beta))$ for h sufficiently close to $h_c^H(\beta)$. This suffices to show (1.23) because from the variational formula (1.14) and $H(\rho)\le H(0)+\rho H'(0)$, i.e. concavity, and $\mathrm{G}(\beta,\rho)\le -h_c(\beta)\rho$ (Proposition 2.2) directly yield

$$\mathrm{F}_H(\beta,h) \,\le\, H(0)+(h-h_c^H(\beta))\rho_h\,, \tag{2.14}$$

which is (1.23) if we use the claim. To prove the claim we use the implicit characterization of ρ_h for $h>h_c(\beta)$ that we write as

$$h-h_c^H(\beta) \,=\, g_\beta(\rho_h) \;\text{ with }\; g_\beta(\rho) \,:=\, -(H'(\rho)-H'(0))-(\partial_\rho \mathrm{G}(\beta,\rho)+h_c(\beta))\,. \tag{2.15}$$

Note that $g_\beta(\cdot)$ is smooth and increasing and it satisfies $g_\beta(\rho)\ge C\rho$ for a positive constant C: this is obvious if we assume (1.6) (and in this case C does not depend on β), but it is true in general because (2.9) implies $-\partial_\rho \mathrm{G}(\beta,\rho)-h_c(\beta)\ge c\rho$ (this is simply because if $f(\cdot)$ is convex, $f'(\rho)\ge f(\rho)/\rho$). Therefore $g_\beta^{-1}:[0,\infty)\longrightarrow[0,1)$ satisfies $g_\beta^{-1}(x)\le x/C$, so

$$\rho_h \,=\, g_\beta^{-1}\left(h-h_c(\beta)\right) \,\le\, \frac{h-h_c(\beta)}{C}\,, \tag{2.16}$$

and the claim is proven.

The proof of (1.24) is analogous to the one for $\beta=0$ (once again: the $\beta=0$ analysis is developed in detail in Section 3), because of the sharp estimate (2.10), which, by convexity, holds also if we formally differentiate both sides of the asymptotic equivalence. □

3. Free energy in the non disordered case: proof of Theorem 1.4

Recall that $\rho_c=1/\mathbf{E}[\eta]\in[0,1)$ and that $\rho_c=0$ if $\alpha\in(0,1)$ and $\rho_c>0$ if $\alpha>1$. In the next statement c is a positive constant for which we have an explicit expression in terms of α, moments of η and C_K: we are going to specify on what c depends, except for C_K (see (1.1)) that is omitted, because c depends on C_K in all the cases, either directly or via $\mathbf{E}[\eta]$ and $\mathbf{E}[\eta^2]$.

Proposition 3.1 (Basic properties of G)**.** *For every $\rho\in[0,1]$ the limit in* (1.12) *with $\beta=0$ exists and*

$$\mathrm{G}(\rho) \,=\, \inf_{x\ge 0}\left(x+\rho\log\mathbf{E}\left[\exp(-x\eta)\right]\right)\,, \tag{3.1}$$

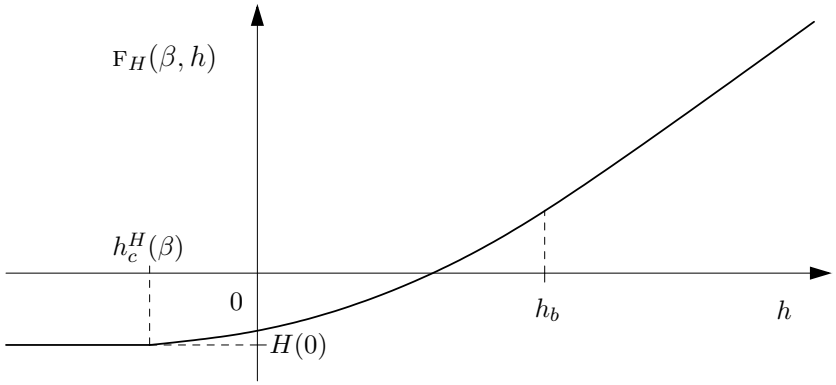

FIGURE 5. This is the graph of $\mathrm{F}_H(\beta,h)$ for $H(\cdot)$ non trivial and $H'(0)<\infty$, hence $H(0)>-\infty$ too. We see the (de)localization transition at $h_c^H(\beta)=-H'(0)+h_c(\beta)$ and we remark that the contact density is continuous at this transition. The transition at h_b is instead present only if $\beta=0$ (and $\alpha>1$), but it is difficult to appreciate it in this image because the contact density is also in this case continuous, or more regular, at the transition and the free energy is non trivial, i.e. non affine, both on the left and on the right of h_b.

from which the concavity of $\mathrm{G}(\cdot)$ *is evident and we have also the uniform estimate:*

$$\lim_{\varepsilon\searrow 0}\limsup_{N\to\infty}\sup_{\substack{m\in\{1,\dots,N:\}\\ |m/N-\rho|\le\varepsilon}}\left|\mathrm{G}(\rho)-\frac{1}{N}\log\mathbf{P}\left(\tau_m=N\right)\right| = 0\,. \tag{3.2}$$

Moreover

(1) *if* $\alpha\in(0,1]$ *(see Fig. 2) then* $\mathrm{G}(\cdot)$ *is analytic and negative on* $(0,1)$ *and, if we exclude* $\alpha=1$*, for* $\rho\searrow 0$

$$\mathrm{G}(\rho)\sim -c\rho^{1/(1-\alpha)} \quad \textit{and} \quad \mathrm{G}'(\rho)\sim -\frac{c}{1-\alpha}\rho^{\alpha/(1-\alpha)}\,, \tag{3.3}$$

and c *depends on* α*. For* $\alpha=1$ *instead* $-\mathrm{G}'(\rho)=\exp(-(1+o(1))/(C_K\rho))$ *and* $-\mathrm{G}(\rho)=o(-\mathrm{G}'(\rho))$*.*

(2) *if* $\alpha>1$ *(see Fig. 1) then* $\mathrm{G}(\rho)=0$ *for* $\rho\in[0,\rho_c]$ *and* $\mathrm{G}(\cdot)$ *is analytic and negative on* $(\rho_c,1)$*. Moreover with* $\delta:=\rho-\rho_c$ *and* $\kappa:=\max(\alpha/(\alpha-1),2)$ *in the limit* $\delta\searrow 0$ *we have for* $\alpha\neq 2$

$$\mathrm{G}(\rho_c+\delta)\sim -c\,\delta^{\kappa} \quad \textit{and} \quad \mathrm{G}'(\rho_c+\delta)\sim -\kappa c\,\delta^{\kappa-1}\,, \tag{3.4}$$

where c *depends on* α *and* $\mathbf{E}[\eta]$ *when* $\alpha\in(1,2)$ *and it depends on* $\mathbf{E}[\eta]$ *and* $\mathbf{E}[\eta^2]$ *if* $\alpha>2$*. When* $\alpha=2$ *we have instead* $\mathrm{G}(\rho)\sim c\delta^2/\log(1/\delta)$ *and* $\mathrm{G}'(\rho)\sim 2c\delta/\log(1/\delta)$ *with* c *that depends on* $\mathbf{E}[\eta]$*.*

(3) $\lim_{\rho\nearrow 1}\mathrm{G}(\rho)=\mathrm{G}(1)=\log K(1)\in(-\infty,0)$ *and* $\lim_{\rho\nearrow 1}\mathrm{G}'(\rho)=-\infty$*.*

Proof. Existence of $\mathrm{G}(\rho)$ and (3.1) can be established at the same time by standard arguments: for the upper bound it suffices to apply the Markov inequality to $\mathbf{P}\left(\tau_m\le N\right)=\mathbf{P}\left(\exp(-x\tau_m)\ge\exp(-xN)\right)$, for $x>0$; for the lower bound the standard exponential tilt argument covers the case $\rho>\rho_c$, while for $\rho<\rho_c$ the lower bound is easily achieved by selecting trajectories that make a suitable big-jump (no exponential cost), so that in the rest of the system the contact density is ρ_c and this matches with the typical behavior

of the renewal (again, no exponential cost): if $\rho = \rho_c$ the argument is the same, but the big-jump is empty. Details of the proof can be found in [42, § 4.3]; in a more general context these estimates can be found for example in [12, 13].

Now we set $g_\rho(x) := x + \rho \log \mathbf{E}[\exp(-x\eta)]$ and remark that for every $\rho \in (0,1)$ the function $g_\rho(\cdot)$ is strictly convex and $\lim_{x\to\infty} g_\rho(x) = \infty$ because in this limit $\mathbf{E}[\exp(-x\eta)] \sim K(1)\exp(-x)$. Therefore the infimum in the right-hand side of (3.1) is reached at a unique point $x_\rho \geq 0$. Since for $x > 0$

$$g'_\rho(x) \,=\, 1 - \rho \frac{\mathbf{E}[\eta \exp(-x\eta)]}{\mathbf{E}[\exp(-x\eta)]} \,=:\, 1 - \rho \mathbf{E}_x[\eta]\,, \tag{3.5}$$

we readily see that if $\rho \leq \rho_c$ then $g'_\rho(x) > 0$ for every $x > 0$ so $x_\rho = 0$ and $\mathrm{G}(\rho) = g_\rho(0) = 0$. If $\rho > \rho_c$ instead $x_\rho > 0$, because in this case $g'_\rho(x) = 0$, that is $\rho \mathbf{E}_x[\eta] = 1$ can be solved with $x \in (0,1)$. Since $x \mapsto \log \mathbf{E}[\exp(-x\eta)]$ is real analytic on $(0,\infty)$ from the analytic implicit function theorem one readily obtains that $\rho \mapsto x_\rho$ is analytic on $(\rho_c, 1)$ and this property passes directly to $\mathrm{G}(\cdot)$, because $\mathrm{G}(\rho) = g_\rho(x_\rho)$.

The rest of the proof is concerned with the asymptotic behaviors for $\rho \searrow \rho_c$ and $\rho \nearrow 1$. Key formulas for this are $\mathbf{E}_{x_\rho}[\eta] = 1/\rho$ (cf. (3.5)) and

$$\mathrm{G}(\rho) \,=\, x_\rho + \rho \log \mathbf{E}\left[\exp(-x_\rho \eta)\right] \quad \text{and} \quad \mathrm{G}'(\rho) \,=\, \log \mathbf{E}\left[\exp(-x_\rho \eta)\right]\,. \tag{3.6}$$

For the case $\rho \nearrow 1$ we observe that, for $x \nearrow \infty$, both $\mathbf{E}[\exp(-x\eta)]$ and $\mathbf{E}[\eta\exp(-x\eta)]$ are equal to $K(1)\exp(-x) + O(\exp(-2x))$. Therefore $\mathbb{E}_x[\eta] = 1 + O(\exp(-x))$ and this implies $x_\rho \nearrow \infty$ as $\rho \nearrow 1$ with $1 - \rho = O(\exp(-x_\rho))$. Therefore

$$\mathrm{G}(\rho) \,=\, x_\rho + \rho \log\left(K(1) e^{-x_\rho} + O\left(e^{-2x_\rho}\right)\right) \,=\, x_\rho(1-\rho) + \rho \log K(1) + O\left(e^{-x_\rho}\right)\,, \tag{3.7}$$

so $\lim_{\rho\nearrow 1} \mathrm{G}(\rho) = \log K(1)$. By using the second expression in (3.6), we get $\mathrm{G}'(\rho) \sim -x_\rho$, in particular $\mathrm{G}'(1) = -\infty$.

We are left with $\rho \searrow \rho_c$ that we separate into $\rho_c = 0$ and $\rho_c > 0$. Let us first remark that in both cases $\lim_{\rho\searrow\rho_c} \mathrm{G}(\rho) = \mathrm{G}(\rho_c) = 0$. This is obvious by concavity when $\rho_c > 0$. If $\rho_c = 0$ it suffices to use the first expression in (3.6) and the fact that $\mathbf{E}_{x_\rho}[\eta] = 1/\rho \nearrow \infty$ when $\rho \searrow 0$, so $x_\rho \to 0$ in this limit. This remark simplifies the analysis because the asymptotic analysis of $\mathrm{G}(\cdot)$ near ρ_c follows from by integrating the corresponding estimate on $\mathrm{G}'(\cdot)$.

For $\rho_c = 0$, i.e. $\alpha \in (0,1]$, by Riemann sum approximation we readily find that $\mathbf{E}_x[\eta] \sim \mathbf{E}[\eta\exp(-x\eta)]$ for $x \searrow 0$ and

$$\mathbf{E}[\eta \exp(-x\eta)] \overset{x\searrow 0}{\sim} C_K \times \begin{cases} \left(\int_0^\infty y^{-\alpha} e^{-y}\,\mathrm{d}y\right) x^{\alpha-1} & \text{if } \alpha \in (0,1)\,, \\ \log(1/x) \text{ if } \alpha = 1\,. \end{cases} \tag{3.8}$$

Of course $\int_0^\infty y^{-\alpha} e^{-y}\,\mathrm{d}y = \Gamma(1-\alpha)$, but we will not keep track of the precise value of the constants and we content ourselves with remarking that we have obtained for $\alpha \in (0,1)$ that $x_\rho \sim c_\alpha \rho^{1/(1-\alpha)}$. Now we can insert this result into the second identity in (3.6) that in this limit becomes $\mathrm{G}'(\rho) \sim -\mathbf{E}[1 - \exp(-x_\rho\eta)]$: the sharp asymptotic behavior of the right-hand side is again a matter of Riemann sum approximation for x_ρ that tends to zero. So for for $\alpha \in (0,1)$ we obtain $\mathrm{G}'(\rho) \sim -C_K(\Gamma(1-\alpha)/\alpha) x_\rho^\alpha \sim -c\rho^{\alpha/(1-\alpha)}$. For $\alpha = 1$, going back to (3.8) we see that $x_\rho = \exp(-(1+o(1))/(C_K\rho))$, so by using $\mathbf{E}[1 - \exp(-x\eta)] \sim C_K x \log(1/x)$ from which, using (3.6), we obtain $\mathrm{G}'(\rho) \sim -C_K x_\rho \log(1/x_\rho)$, which implies $\mathrm{G}'(\rho) = \exp(-(1+o(1))/(C_K\rho))$ and, by convexity of $-\mathrm{G}(\cdot)$, we see that $0 \leq \mathrm{G}(\rho)/\mathrm{G}'(\rho) \leq \rho$.

For $\rho_c > 0$, i.e. $\alpha > 1$, the analysis is different according to whether $\mathbf{E}[\eta^2] < \infty$ or not:

(1) if $\mathbf{E}[\eta^2] < \infty$, that is $\alpha > 2$, and $x \searrow 0$ we have

$$\mathbf{E}_x[\eta] \,=\, \frac{\mathbf{E}[\eta \exp(-x\eta)]}{\mathbf{E}[\exp(-x\eta)]} \,=\, \frac{\mathbf{E}[\eta] - \mathbf{E}[\eta^2]x(1+o(1))}{1 - \mathbf{E}[\eta]x(1+o(1))} \,=\, \mathbf{E}[\eta] - \mathrm{var}(\eta)x + o(x)\,. \tag{3.9}$$

Therefore $x_{\rho_c+\delta} \sim (\mathbf{E}[\eta]^2/\mathrm{var}(\eta))\delta$, so $\mathrm{G}'(\rho) \sim -\mathbf{E}[\eta]x_\rho$ directly yields the result for $\mathrm{G}'(\rho_c+\delta)$.

(2) if $\mathbf{E}[\eta^2] = \infty$ we consider separately $\alpha \in (1,2)$ and $\alpha = 2$. In the first case we use

$$\mathbf{E}_x[\eta] \,=\, \frac{\mathbf{E}[\eta \exp(-\eta x)]}{1+O(x)} \,=\, \mathbf{E}[\eta] - C_K \left(\int_0^\infty \frac{1-e^{-y}}{y^\alpha}\, \mathrm{d}y \right) x^{\alpha-1} + o\left(x^{\alpha-1}\right)\,, \tag{3.10}$$

so $x_{\rho_c+\delta} \sim c_\alpha \delta^{1/(\alpha-1)}$, $\mathrm{G}'(\rho_c+\delta) \sim -\mathbf{E}[\eta]x_{\rho_c+\delta} \sim -\mathbf{E}[\eta]c_\alpha\delta^{1/(\alpha-1)}$. For $\alpha = 2$ we have $\mathbf{E}_x[\eta] = \mathbf{E}[\eta] - C_K(1+o(1))x\log(1/x)$ that entails $\delta \sim \rho_c^2 C_K x_{\rho_c+\delta}\log x_{\rho_c+\delta}$ so $x_{\rho_c+\delta} \sim \delta/(\rho_c^2 C_K \log(1/\delta))$ and with $\mathrm{G}'(\rho) \sim -\mathbf{E}[\eta]x_\rho$ we conclude.

□

Proof of Theorem 1.4. Of course we are going to use intensively

$$\mathrm{F}_H(h) \,=\, \sup_{\rho\in[0,1]} \left(h\rho + H(\rho) + \mathrm{G}(\rho)\right)\,,\quad \mathrm{F}'_H(h) \,=\, \rho_h \;\text{ and }\; H'(\rho_h) + \mathrm{G}'(\rho_h) \,=\, -h\,, \tag{3.11}$$

where the second equation identifies the unique optimizer ρ_h, as long as $\rho_h > 0$. So the second and third identity are written for $\rho_h > 0$. Note also that (3.3) implies that $\mathrm{G}(\cdot)$ is C^1 also at ρ_c.

We start with the case $h_c^H = -H'(0) > -\infty$: one readily sees that $\rho_h = 0$ for $h \le h_c^H$ and, by the first identity in (3.11), $\mathrm{F}_H(h) = H(0)$ for these values of h. On the other hand, for $h > h_c^H$, the second identity in (3.11) can be written as $-(H'(\rho_h) - H'(0)) - \mathrm{G}'(\rho_h) = (h - h_c^H)$ and we see that it has a strictly positive solution $\rho_h (= \mathrm{F}'_H(h))$ because the left-hand side is an increasing function of ρ_h, and this directly yields $\mathrm{F}_H(h) > H(0)$. Therefore h_c^H is a critical (i.e., non analyticity) point.

We now recall that we assume (1.6). Here is the $h \searrow h_c^H$ analysis:

- if $\rho_c > 0$ then $h - h_c^H = -H'(\rho_h) + H'(0) \sim c_H\rho_h$, so $\mathrm{F}'_H(\rho) = \rho_h \sim (h-h_c^H)/c_H$ and $\mathrm{F}_H(\rho) - H(0) \sim (h-h_c^H)^2/(2c_H)$;
- if $\rho_c = 0$ then $h - h_c^H = -H'(\rho_h) + H'(0) - \mathrm{G}'(\rho_h)$ and, by Proposition 3.1(1), we have (with $c_\alpha = c/(1-\alpha)$)

$$h - h_c^H \,\sim\, \begin{cases} c_\alpha \rho_h^{\alpha/(1-\alpha)} & \text{if } \alpha \in (0,1/2)\,, \\ (c_{1/2} + c_H)\rho_h & \text{if } \alpha = 1/2\,, \\ c_H\rho_h & \text{if } \alpha \in (1/2,1]\,, \end{cases} \tag{3.12}$$

and, like above, from $\mathrm{F}'_H(\rho) = \rho_h$ we extract the claimed asymptotic behaviors.

We turn now to h_b, of course when $h_b > h_c^H$ (so $\alpha > 1$ and $H(\cdot)$ is non trivial) otherwise we are in the case we just considered. The origin of the h_b singularity is simply the fact that $\mathrm{F}_H(h)$ is determined by different variational problems according to whether $h < h_b$ and $h > h_b$. In fact $h \le h_b$ means $\rho_h \le \rho_c$, i.e. $\mathrm{G}(\rho) = 0$, and the variational problem in this case reduces to $\mathrm{F}_H(h) = \mathrm{F}_H^{\mathrm{reg}}(h) = \sup_{\rho\in[0,1]}(h\rho + H(\rho))$. For $h > h_b$ instead $\rho_h > \rho_c$, i.e. $\mathrm{G}(\rho) > 0$, and one has to use the full expression for $\mathrm{F}_H(\cdot)$.

Let us analyse the singularity at h_b. We start by remarking that we are just need to do a local analysis at $h_b = -H'(\rho_c)$: by introducing $J(y) := H(\rho_c + y) - H(\rho_c) - H'(\rho_c)y$ and $G(y) = \mathrm{G}(\rho_c + y)$, we can work with

$$F(x) := \sup_y \left(xy + J(y) + G(y)\right) \quad \text{and} \quad F^{\mathrm{reg}}(x) := \sup_y \left(xy + J(y)\right) . \tag{3.13}$$

The maximizer y_x for $F(\cdot)$ is the (unique) solution of $U'(y_x) = -x$ with $U = J + G$. We remark from the start that $F(x) \le F^{\mathrm{reg}}(x)$ and the inequality is strict for $x > 0$.

Consider first the case of κ which is not an integer and set $\mathtt{k} = \lfloor \kappa \rfloor$, so $\mathtt{k} = 2, 3, \ldots$: this means that we are considering $\alpha \in (1, 2)$. For $y \searrow 0$ we have

$$U(y) \;=\; -a_2 y^2 - a_3 y^3 - \ldots - a_{\mathtt{k}} y^{\mathtt{k}} - b_\kappa y^\kappa + o(y^\kappa) \,, \tag{3.14}$$

where $b_\kappa > 0$ is the constant c in (3.4). Moreover, still by (3.4) , we have

$$U'(y) \;=\; -2a_2 y - 3a_3 y^2 - \ldots - \mathtt{k} a_{\mathtt{k}} y^{\mathtt{k}-1} - \kappa b_\kappa y^{\kappa-1} + o\left(y^{\kappa-1}\right) . \tag{3.15}$$

Note that $a_2 = |J''(0)| > 0$, but the other $a_\cdot$ coefficients are just real numbers. From (3.15) we extract that as $x \searrow 0$

$$F'(x) \;=\; y_x \;=\; c_1 x + c_2 x^2 + \ldots + c_{\mathtt{k}-1} x^{\mathtt{k}-1} - c_\kappa x^{\kappa-1} + o\left(x^{\kappa-1}\right) , \tag{3.16}$$

where $c_1 = 1/(2a_2)$ and $c_\kappa = \kappa b_\kappa/(2a_2)^\kappa$. so

$$\begin{aligned} F(x) \;&=\; \frac{1}{2}c_1 x^2 + \frac{1}{3}c_2 x^3 + \ldots + \frac{1}{\mathtt{k}}c_{\mathtt{k}-1} x^{\mathtt{k}} - \frac{1}{\kappa}c_\kappa x^{\kappa-1} + o\left(x^{\kappa-1}\right) \\ &=:\; P_{\mathtt{k}}(x) - \frac{1}{\kappa}c_\kappa x^{\kappa-1} + o\left(x^{\kappa-1}\right) , \end{aligned} \tag{3.17}$$

where the last line defines $P_{\mathtt{k}}(x)$, a polynomial of degree $\mathtt{k}$.

Since the analysis we have developed can be repeated for $F^{\mathrm{reg}}(x)$, that is with $U = J$, in an essentially identical (in fact, simpler) way, and considering $x \to 0$ (not simply $x \searrow 0$) we readily see that $F^{\mathrm{reg}}(x) = P_{\mathtt{k}}(x) + O(x^{\mathtt{k}+1})$. This completes the case of κ non integer.

Let us consider now the cases $\kappa = \mathtt{k} = 2, 3, \ldots$.

When $\kappa = 2$, that is $\alpha = 2$ (and $\mathbf{E}[\eta^2] = \infty$), for $x \searrow 0$ we have $U'(y) = -2a_2 y - 2b_2 y/\log(1/y)$+h.o., of course h.o. means higher order. Therefore $y_x = x/(2a_2) - (b_2/(2a_2^2))x/\log(1/x)$+h.o., which yields $F(x) = x^2/(4a_2) + (b_2/(4a_2^2))x^2/\log(1/x)$+h.o.. For $x \nearrow 0$ it suffices to repeat the same analysis, but this time there is no logarithmic terms and $(b_2/(4a_2^2))x^2/\log(1/x)$ becomes simply $O(x^3)$.

For $\kappa = \mathtt{k} = 3, 4, \ldots$ the analysis changes slightly because the last two explicit terms in the right-hand side of (3.14) and (3.15) have the same behavior. Therefore the coefficient appearing in front of the term $y^{\mathtt{k}}$ in the development for $U(y)$ is $a_{\mathtt{k}} + b_{\mathtt{k}}$, respectively $a_{\mathtt{k}}$, when $y \searrow 0$, respectively $y \nearrow 0$. This mismatch directly reflects on a mismatch in the $x^{\mathtt{k}}$ term of the two developments for $F(x)$. □

4. Path properties in the non disordered case: proof of Theorem 1.6

The basic step is observing that the probability $\mathbf{P}^{\Psi}_{N,h}$ introduced in (1.7) is a superposition of probabilities in which the number of contacts is fixed:

$$\mathbf{P}^{\Psi}_{N,h}(\cdot) := \frac{\sum_{m=1}^N \exp(hm)\Psi(m,N)\mathbf{P}(\tau_m = N)\mathbf{Q}_{N,m}(\cdot)}{\sum_{m=1}^N \exp(hm)\Psi(m,N)\mathbf{P}(\tau_m = N)} , \tag{4.1}$$

where $\mathbf{Q}_{N,m}$ is the law of $\tau \cap [0, N]$ conditioned to $\tau_m = N$. So $\mathbf{Q}_{N,m}$ is the law of the renewal conditioned to visiting N in precisely m steps and can of course be viewed as a non

disordered pinning model conditioned to having m contacts (one of which is at N). But this process is very relevant well beyond pinning models and in fact it has been studied in depth: we collect here the results we will use (that are only a minimal part of what is available in the literature).

Recall the notations introduced for Theorem 1.6:

Proposition 4.1. *In two parts, the first applies only to $\alpha > 1$, the second one is general:*

(1) for every $\rho \in [0, \rho_c)$ and every $\varepsilon \in (0, (\rho_c - \rho)/2)$ we have

$$\lim_{N\to\infty} \inf_{m:\,|m/N-\rho|\le\varepsilon} \mathbf{Q}_{N,m}\left(\left|\frac{\eta_{1,N}}{N} - \left(1-\frac{\rho}{\rho_c}\right)\right| + \frac{\eta_{2,N}}{N} \le \frac{2\varepsilon}{\rho_c}\right) = 1\,. \tag{4.2}$$

(2) for every $\rho \in (\rho_c, 1)$ and every $\varepsilon \in (0, \min(\rho-\rho_c, 1-\rho)/2)$

$$\lim_{N\to\infty} \inf_{m:\,|(m/N)-\rho|\le\varepsilon} \mathbf{Q}_{N,m}\left(\left|\frac{\eta_{1,N}}{N}\right| \le \varepsilon\right) = 1\,. \tag{4.3}$$

Proof. The first part is a result in the big-jump domain and one can directly apply the (much sharper and much more general) result in [5, Th. 1] (see also [19]) that implies that for $\rho < \rho_c$ and every $\widetilde{\varepsilon} > 0$

$$\lim_{\substack{N\to\infty:\\ m/N\sim\rho}} \mathbf{Q}_{N,m}\left(\left|\frac{\eta_{1,N}}{N} - \left(1-\frac{\rho}{\rho_c}\right)\right| + \frac{\eta_{2,N}}{N} > \widetilde{\varepsilon}\right) = 0\,. \tag{4.4}$$

Therefore if we set

$$p_{N,m} := \mathbf{Q}_{N,m}\left(\left|\frac{\eta_{1,N}}{N} - \left(1-\frac{\rho}{\rho_c}\right)\right| + \frac{\eta_{2,N}}{N} > \frac{2\varepsilon}{\rho_c}\right), \tag{4.5}$$

then $\lim_{N\to\infty} p_{N,m_N} = 0$ if $m_N/N \to \rho < \rho_c$. This implies that, with ρ and ε like in the statement $\lim_{N\to\infty} \sup_{m:|m/N-\rho|\le\varepsilon} p_{N,m_N} = 0$ because otherwise there exists $p > 0$ and a subsequence $(N_j)_{j\in\mathbb{N}}$ such that $\lim_j m_j/N_j \in [\rho-\varepsilon, \rho+\varepsilon]$ and $p_{N_j,m_j} \ge p$ for every j, in contradiction with $\lim_N p_{N,m_N} = 0$.

The second part is in the Large Deviation regime and we can perform the standard *tilting procedure* in a direct way because of the constraint that there are exactly m contacts and the last one is in N. Explicitly:

$$\mathbf{Q}_{N,m}(A) = \mathbf{P}\left(\tau^{(q)} \in A,\, \tau^{(q)}_m = N\right) \Big/ \mathbf{P}\left(\tau^{(q)}_m = N\right), \tag{4.6}$$

with $\tau^{(q)}$ the renewal process with inter-arrival probability distribution given by $K_q(n) \propto K(n)\exp(-qn)$, and $q = q(m/N)$ is the unique solution of $\mu_q := \sum_n nK_q(n) = N/m$. We can now apply the Local Central Limit Theorem for triangular arrays (see e.g. [18, Th. 1.2]) that gives

$$\lim_{m\to\infty} \sup_{M\in\mathbb{N}} \left| \sqrt{\sigma_q^2 m}\, \mathbf{P}\left(\tau^{(q)}_m = M\right) - f_{\mathcal{N}}\left(\frac{M - m\mu_q}{\sqrt{\sigma_q^2 m}}\right)\right| = 0\,, \tag{4.7}$$

with μ_q and σ_q^2 respectively sum and variance of $\tau_1^{(q)}$, and $f_{\mathcal{N}}(\cdot)$ is the density of a standard Gaussian variable. From (4.7) we readily extract that with ρ and ε as required in the

statement there exists $c > 0$ and $m_0 > 0$ such that, uniformly in $|m/N - \rho| \le \varepsilon$ we have that $\mathbf{P}\left(\tau_m^{(q)} = N\right) \ge 1/(c\sqrt{m})$ for every $m \ge m_0$. Therefore

$$\mathbf{Q}_{N,m}\left(\left|\frac{\eta_{1,N}}{N}\right| > \varepsilon\right) \le c\sqrt{m}\,\mathbf{P}\left(\sup_{j=1,\dots,m} \tau_j^{(q)} - \tau_{j-1}^{(q)} > \varepsilon N\right) \le cm^{3/2}\mathbf{P}\left(\tau_1^{(q)} > \varepsilon N\right), \tag{4.8}$$

and the right-most term vanishes, with an exponential rate, when $m \to \infty$. □

Now we observe that, in analogy with (4.1) we have

$$Z_{N,h}^{\Psi} = \sum_{m=1}^{N} \exp(hm)\Psi(m,N)\mathbf{P}(\tau_m = N), \tag{4.9}$$

Recall that we use $\rho_h = \mathrm{F}'_H(h)$ for $h \ne h_c^H$, cf. (1.21), so ρ_h is the unique point that maximizes $\rho \mapsto \rho h + H(\rho) + \mathrm{G}(\rho)$.

The central estimate for the proof of Theorem 1.6 is:

Lemma 4.2. *For every $\varepsilon > 0$ and every $h \ne h_c^H$*

$$Z_{N,h}^{\Psi} \overset{N\to\infty}{\sim} Z_{N,h}^{\Psi,\varepsilon} := \sum_{\substack{m\in\{1,\dots,N\}:\\ |m/N-\rho_h|\le\varepsilon}} \exp(hm)\Psi(m,N)\mathbf{P}(\tau_m = N). \tag{4.10}$$

Proof. We have

$$\begin{aligned} 0 \le Z_{N,h}^{\Psi} - Z_{N,h}^{\Psi,\varepsilon} &\le c \sum_{\substack{m\in\{1,\dots,N\}:\\ |m/N-\rho_h|>\varepsilon}} \exp\left((h+b)m + NH(m/N)\right)\mathbf{P}(\tau_m = N) \\ &\le c^2 \sum_{\substack{m\in\{1,\dots,N\}:\\ |m/N-\rho_h|>\varepsilon}} \exp\left((h+2b)m + NH(m/N) + N\mathrm{G}\left(m/N\right)\right) \\ &\le c^2 N \exp\left(2bN + N \sup_{\rho:\,|\rho-\rho_h|>\varepsilon} (\rho h + H(\rho) + \mathrm{G}(\rho))\right), \end{aligned} \tag{4.11}$$

where in the first step we have used the hypothesis (1.3), so $b > 0$ can be chosen arbitrarily small and $c = c(b)$ just needs to be chosen sufficiently large. In the second step instead we used (3.2) of Proposition 3.1. Now it suffices to remark that

$$\sup_{\rho:\,|\rho-\rho_h|>\varepsilon} (\rho h + H(\rho) + \mathrm{G}(\rho)) = \mathrm{F}_H(h) - q_\varepsilon, \tag{4.12}$$

with $q_\varepsilon > 0$ (here we use $h \ne h_c^H$, but we stress that this is needed only if $H(\cdot)$ is trivial) and therefore, by choosing $b = q_\varepsilon/5$ for N sufficiently large, we have

$$0 \le Z_{N,h}^{\Psi} - Z_{N,h}^{\Psi,\varepsilon} \le \exp\left(\mathrm{F}_H(h) - q_\varepsilon/2\right). \tag{4.13}$$

Since $\log Z_{N,h}^{\Psi} \sim N\mathrm{F}_H(h)$ we are done. □

Proof of Theorem 1.6. (1.21) follows because

$$\mathbf{P}_{N,h}^{\Psi}\left(A_N^0\right) = \frac{Z_{N,h}^{\Psi,\varepsilon}}{Z_{N,h}^{\Psi}} \overset{N\to\infty}{\longrightarrow} 1 \quad \text{with} \quad A_N^0 := \left\{\tau : \left|\frac{|\tau \cap (0,N]|}{N} - \rho_h\right| \le \varepsilon\right\}, \tag{4.14}$$

by Lemma 4.2. For $\rho_c > 0$ ($\alpha > 1$ and we assume $h \neq h_b$) we consider the event

$$A_N^1 := \left\{ \tau : \left| \frac{\eta_{1,N}}{N} - \left(1 - \frac{\rho_h}{\rho_c}\right)_+ \right| + \frac{\eta_{2,N}}{N} \leq \frac{2\varepsilon}{\rho_c} \right\} . \tag{4.15}$$

By Lemma 4.2 we have

$$\mathbf{P}_{N,h}^{\Psi}\left(A_N^0 \cap A_N^1\right) \overset{N\to\infty}{\sim} \frac{1}{Z_{N,h}^{\Psi,\varepsilon}} \sum_{m:\,|m/N-\rho_h|\leq\varepsilon} \exp(hm)\Psi(m,N)\mathbf{P}(\tau_m = N)\mathbf{Q}_{N,m}\left(A_N^1\right) . \tag{4.16}$$

Recall now that we assume $h \neq h_b$, so $\rho_h \neq \rho_c$. By Proposition 4.1 we have that for ε sufficiently small $\mathbf{Q}_{N,m}\left(A_N^1\right)$ tends to one as $N \to \infty$, with the constraint we have on m, and this readily entails that numerator and denominator in the right-hand side of (4.16) are asymptotically equivalent, so (1.22) is established. In the case $\rho_c = 0$ we change the event A_N^1, but the argument is the same. □

5. Path properties in the disordered case: proof of Theorem 1.8(2)

We start with an estimate on the disordered pinning model that is in the spirit of the sharper, but also, to a certain extent, different (see Remark 5.2), result in [37, Theorem 2.5].

Lemma 5.1. *Consider the $\Psi \equiv 1$ model for h such that $\mathrm{F}(\beta,h) > 0$. Then for every $\gamma \in (0, \mathrm{F}(\beta,h)]$ there exists an a.s. finite random variable $N_0(\gamma,\omega)$ such that for $N \geq N_0(\gamma,\omega)$*

$$\mathbf{P}_{N,\omega,\beta,h}\left(\eta_{1,N} > \gamma N\right) \leq \exp\left(-\gamma N \mathrm{F}(\beta,h)/2\right) . \tag{5.1}$$

Proof. Set $\mathbf{P}_{N,\omega} = \mathbf{P}_{N,\omega,\beta,h}$. We also choose $\gamma \in (0, \mathrm{F}(\beta,h)/2]$. The key estimate is

$$\begin{aligned} \mathbf{P}_{N,\omega}\left(\eta_{1,N} > \gamma N\right) &= \sum_{\substack{n_1,n_2\in\{1,\dots,N\}:\\ n_2-n_1>\gamma N}} \frac{Z_{n_1,\omega}K(n_2-n_1)e^{\beta\omega_{n_2}}Z_{n_2,\theta^{n_2}\omega}}{Z_{N,\omega}} \\ &\leq \sum_{\substack{n_1,n_2\in\{1,\dots,N\}:\\ n_2-n_1>\gamma N}} \frac{Z_{n_1,\omega}K(n_2-n_1)e^{\beta\omega_{n_2}}Z_{n_2,\theta^{n_2}\omega}}{Z_{n_2,\omega}Z_{n_2,\theta^{n_2}\omega}} \leq \sum_{\substack{n_1,n_2\in\{1,\dots,N\}:\\ n_2-n_1>\gamma N}} \frac{Z_{n_1,\omega}}{Z_{n_2,\omega}}e^{\beta\omega_{n_2}} . \end{aligned} \tag{5.2}$$

Now we observe that $Z_{n_1,\omega} \leq C(\omega)\exp(n_1\mathrm{F}(\beta,h) + \gamma/6)$ for every $n_1 \in \mathbb{N}$ and $Z_{n_2,\omega} \geq \exp(n_2\mathrm{F}(\beta,h) - \gamma/6)$ for every $n_2 \geq \gamma N$ and for N larger than a random threshold, possibly dependent also on γ. Therefore

$$\mathbf{P}_{N,\omega}\left(\eta_{1,N} > \gamma N\right) \leq C(\omega)N^2\exp(-(\mathrm{F}(\beta,h) - \gamma/3)\gamma N) \leq \exp(-\gamma N\mathrm{F}(\beta,h)/2) , \tag{5.3}$$

always for N larger than a random threshold. □

Remark 5.2. *Lemma 5.1 is a rough version of the sharp statement (in probability) for the size of the largest excursion in the localized phase* [37, Th. 2.5]. *In* [37, Th. 2.5] *the size of the largest excursion in the localized phase is shown to be equal, in $\mathbb{P}(\,\mathrm{d}\omega)$ probability and to leading order, to $c(\beta,h)\log N$, for a suitable choice of $c(\beta,h) > 0$. Lemma 5.1 however, with respect to* [37, Th. 2.5], *gives a $\mathbb{P}(\,\mathrm{d}\omega)$-almost sure estimate and, above all, an exponential decay rate in the quenched probability proportional to N.*

Let us consider now the constrained case. Recalling (1.12) we set

$$\mathcal{Z}_{N,m,\omega,\beta} := \mathbf{E}\left[\exp\left(\beta\sum_{j=1}^{N}\omega_j\delta_j\right)\mathbf{1}_{\tau_m=N}\right], \tag{5.4}$$

Proposition 5.3. *Choose any $\rho\in(0,1)$. Then for every $\gamma>0$ there exists ε_0 such that for $\varepsilon\in(0,\varepsilon_0)$*

$$\lim_{N\to\infty}\sup_{m:\,|m/N-\rho|\le\varepsilon}\mathbf{Q}_{N,m,\omega,\beta}\left(\eta_{1,N}>\gamma N\right) = 0\,. \tag{5.5}$$

Proof. It suffices to prove the result for γ small. Set $A_N=\{\eta_{1,N}>\gamma N\}$. We have

$$\mathbf{P}_{N,\omega,\beta,h}(A_N) = \frac{1}{Z_{N,\omega,\beta,h}}\sum_{m=1}^{N}e^{hm}\mathcal{Z}_{N,m,\omega,\beta}\mathbf{Q}_{N,m,\omega,\beta}(A_N)\,, \tag{5.6}$$

Therefore for every $\rho\in(0,1)$, every $\varepsilon>0$ and every h we have

$$\sup_{m:\,|m/N-\rho|\le\varepsilon}\mathbf{Q}_{N,m,\omega,\beta}(A_N) \le \frac{Z_{N,\omega,\beta,h}}{\inf_{m:\,|m/N-\rho|\le\varepsilon}e^{hm}\mathcal{Z}_{N,m,\omega,\beta}}\mathbf{P}_{N,\omega,\beta,h}(A_N)\,. \tag{5.7}$$

Choose h such that $\partial_h\mathrm{F}(\beta,h)=\rho$, so that $\mathbb{P}(\,\mathrm{d}\omega)$-a.s.

$$\lim_{N\to\infty}\frac{1}{N}\log Z_{N,\omega,\beta,h} = \mathrm{G}(\beta,h)+h\rho\,. \tag{5.8}$$

Therefore we can use Proposition 6.2 to bound the ratio of partition functions in (5.7) thus obtaining that there exists $c_\rho>0$ such that for N larger than a random threshold

$$\sup_{m:\,|m/N-\rho|\le\varepsilon}\mathbf{Q}_{N,m,\omega,\beta}(A_N) \le \exp\left(c_\rho\varepsilon N\right)\mathbf{P}_{N,\omega,\beta,h}(A_N)\,. \tag{5.9}$$

By combining this last estimate with (5.1) we see that for $\varepsilon<\gamma\mathrm{F}(\beta,h)/(2c_\rho)$ – and satisfying also the other smallness requirements in Proposition 6.2 – we have that a.s.

$$\lim_{N\to\infty}\sup_{m:\,|m/N-\rho|\le\varepsilon}\mathbf{Q}_{N,m,\omega,\beta}(A_N) = 0\,. \tag{5.10}$$

□

Proof of Theorem 1.8(2). We proceed like in the $\beta=0$ case, see (4.11) of Lemma 4.2, replacing $\mathbf{P}(\tau_m=N)$ with $\mathcal{Z}_{N,m,\omega,\beta}$ and by using Proposition 6.2 instead of Proposition 3.1. We use ρ_h the optimizer of the first variational problem in (1.14) and we exploit the hypothesis (1.3). This way we see that there exists $N_0(\omega)\overset{\text{a.s}}{<}\infty$ such that for $N\ge N_0(\omega)$

$$\begin{aligned}0 \le Z^{\Psi}_{N,\omega,\beta,h}-Z^{\Psi,\varepsilon}_{N,\omega,\beta,h} &\le c\sum_{\substack{m\in\{1,\dots,N\}:\\ |m/N-\rho_h|>\varepsilon}}\exp\left((h+b)m+NH(m/N)\right)\mathbf{P}(\tau_m=N)\mathcal{Z}_{N,m,\omega,\beta}\\ &\le c^2N\exp\left(2bN+N\sup_{\rho:\,|\rho-\rho_h|>\varepsilon}\left(\rho h+H(\rho)+\mathrm{G}(\beta,\rho)\right)\right),\end{aligned} \tag{5.11}$$

where $Z^{\Psi,\varepsilon}_{N,\omega,\beta,h}$ is the direct generalization of the analogous quantity in the $\beta=0$ case, see Lemma 4.2. Since b can be chosen arbitrarily small and by (strict) concavity of

$H(\cdot) + \mathrm{G}(\beta, \cdot)$ (note that, since $\beta > 0$, $\mathrm{G}(\beta, \cdot)$ is strictly concave) we obtain also in this case that for every $\varepsilon > 0$ there exists $p_\varepsilon > 0$ such that

$$0 \le Z^{\Psi}_{N,\omega,\beta,h} - Z^{\Psi,\varepsilon}_{N,\omega,\beta,h} \le \exp\left(N(\mathrm{F}_H(\beta,h) - p_\varepsilon)\right), \tag{5.12}$$

for N larger than an a.s . finite random quantity. So the fact that the ratio of $Z^{\Psi,\varepsilon}_{N,\omega,\beta,h}$ and $Z^{\Psi}_{N,\omega,\beta,h}$ tends a.s. to one takes care of (1.25) because $\rho_h = \partial_h \mathrm{F}_H(\beta,h)$ for every h.

For (1.26) we use the $\beta > 0$ version of (4.1), that is

$$\mathbf{P}^{\Psi}_{N,\omega,\beta,h}(\cdot) = \frac{\sum_{m=1}^N \exp(hm)\Psi(m,N)\mathcal{Z}_{N,m,\omega,\beta}\mathbf{Q}_{N,m}(\cdot)}{\sum_{m=1}^N \exp(hm)\Psi(m,N)\mathcal{Z}_{N,m,\omega,\beta}}. \tag{5.13}$$

Now we fix any $h > h^H_c(\beta)$, so $\rho_h > 0$, and we observe that, by (1.25), we have that for every $\varepsilon > 0$ and a.s.

$$\mathbf{P}^{\Psi}_{N,\omega,\beta,h}\left(\eta_{1,N} > \gamma N\right) \overset{N\to\infty}{\sim} \mathbf{P}^{\Psi}_{N,\omega,\beta,h}\left(\left|\sum_{n=1}^N \delta_j - \rho_h N\right| \le \varepsilon N,\, \eta_{1,N} > \gamma N\right). \tag{5.14}$$

We can now insert this event into (5.13) and, by using Proposition 5.3, we readily see that for ε small the right-hand side of (5.14) vanishes a.s. when $N \to \infty$. Since $\gamma > 0$ can be chosen arbitrarily small, we are done. □

6. On free energies and variational formulas: proof of Theorem 1.3

6.1. **On $\mathrm{G}(\beta,\rho)$.** Recall that the definition (5.4) of $\mathcal{Z}_{N,m,\omega,\beta}$. For $\rho \in (0,1]$ we set

$$\widetilde{\mathcal{Z}}_{N,\omega}(\rho) := \min_{m \in \{\lfloor \rho N \rfloor, \lceil \rho N \rceil\}} \mathcal{Z}_{N,m,\omega,\beta}, \tag{6.1}$$

where the set over which the minimum is taken reduces to a singleton if ρN is integer. Note that $\mathcal{Z}_{N,m,\omega,\beta}$ is zero if $m = 0$, so $\widetilde{\mathcal{Z}}_{N,\omega}(\rho) = 0$ whenever $\rho < 1/N$.

Lemma 6.1. *$(\log \widetilde{\mathcal{Z}}_{N,\omega}(\rho))_{N=1,2,\dots}$ is super-additive, namely: for every $N_1, N_2 \in \mathbb{N}$ we have*

$$\log \widetilde{\mathcal{Z}}_{N_1+N_2,\omega}(\rho) \ge \log \widetilde{\mathcal{Z}}_{N_1,\omega}(\rho) + \log \widetilde{\mathcal{Z}}_{N_2,\theta^{N_1}\omega}(\rho), \tag{6.2}$$

where $(\theta\omega)_n = \omega_{n+1}$.

Proof. First of all remark that for every $b, c \ge 0$

$$\lfloor b \rfloor + \lfloor c \rfloor \le \lfloor b + c \rfloor \le \lceil b + c \rceil \le \lceil b \rceil + \lceil c \rceil, \tag{6.3}$$

which implies that both the lower and upper integer part of $c+b$ coincide with the sum of a suitably chosen combination of upper and/or lower integer parts of b and c. For example if neither b nor c is an integer, then either $\lceil b+c \rceil = \lceil b \rceil + \lceil c \rceil$ or $\lceil b+c \rceil = \lceil b \rceil + \lfloor c \rfloor = \lfloor b \rfloor + \lceil c \rceil$. On the other hand, if b is an integer and c is not $\lceil b+c \rceil = b + \lceil c \rceil = \lfloor b \rfloor + \lceil c \rceil = \lceil b \rceil + \lceil c \rceil =$. The case in which b and c are both integers is of course trivial.

Then remark also that

$$\log \mathcal{Z}_{N_1+N_2,m_1+m_2,\omega,\beta} \ge \log \mathcal{Z}_{N_1,m_1,\omega,\beta} + \log \mathcal{Z}_{N_2,m_2,\theta^{N_1}\omega,\beta}, \tag{6.4}$$

which follows by restricting the expectation in the definition of $\mathcal{Z}_{N_1+N_2,m_1+m_2,\omega,\beta}$ to the event $\tau_{m_1} = N_1$ and by using the independence of the increments of τ.

Since $\lceil \rho(N_1+N_2) \rceil$ is one among $\lceil \rho N_1 \rceil + \lceil \rho N_2 \rceil$, $\lceil \rho N_1 \rceil + \lfloor \rho N_2 \rfloor$ and $\lfloor \rho N_1 \rfloor + \lceil \rho N_2 \rceil$ and since exactly the same holds true is we switch upper integer parts with lower integer parts, we readily see that (6.2) holds and the proof is complete. □

Proposition 6.2. *For every $\rho \in [0,1]$ and every $\beta \geq 0$ the limit*

$$\lim_{\substack{N\to\infty \\ m/N\to\rho}} \frac{1}{N}\mathbb{E}\log \mathcal{Z}_{N,m,\omega,\beta} =: \mathrm{G}(\beta,\rho)\,, \tag{6.5}$$

exists and the convergence holds $\mathbb{P}(\,\mathrm{d}\omega)$-a.s, with the same (deterministic) limit, without averaging with respect to $\mathbb{P}$. Moreover, if we set

$$D_{N,\varepsilon,\rho}(\omega) := \sup_{\substack{m\in\{1,\dots,N\}: \\ |m/N-\rho|\leq\varepsilon}} \left| \frac{1}{N}\log \mathcal{Z}_{N,m,\omega,\beta} - \mathrm{G}(\beta,\rho)\right|, \tag{6.6}$$

for $\rho \in (0,1)$ and $\varepsilon \in (0, \min(\rho/2,(1-\rho)/2)$ we can exhibit a constant $c_\rho > 0$, with $\sup_{\rho\in[b,1-b]} < \infty$ for every $b \in (0,1/2)$, such that $D_{N,\varepsilon,\rho} \leq c_\rho\varepsilon$ for every ω and $N \geq N_0(\varepsilon,\rho,\omega)$, with $\mathbb{P}(N_0(\varepsilon,\rho,\omega) < \infty) = 1$.

For $\rho = 0$ we have $\mathrm{G}(\beta,0) = 0$ and there exists $c_0 > 0$ such that $D_{N,\varepsilon,\rho} \leq c_0\varepsilon$ for every $\varepsilon \in (0,1/2)$ and for N larger than an a.s. finite random variable, like above. For $\rho = 1$ instead $\mathrm{G}(\beta,1) = \log K(1)$ and there exists $c_1 > 0$ such that $D_{N,\varepsilon,1} \leq c_1\varepsilon + \mathrm{G}(1-2\varepsilon) - \mathrm{G}(1)$ for N larger than a suitable a.s. finite random variable.

Proof. For this proof we we fix $\beta \geq 0$ and drop the dependence on β from $\mathcal{Z}_{N,m,\omega,\beta}$ We treat first the case $\rho \in (0,1)$. In this case we apply Kingman Sub-additive Ergodic Theorem [45], but one has to take care of the fact that $\mathcal{Z}_{N,m,\omega} = 0$ for $\rho N < 1$. We deal with this by considering $N_0 = N_0(\rho) = \lceil 1/\rho \rceil$ and by focusing on $(\log \widetilde{\mathcal{Z}}_{nN_0,\omega}(\rho))_{n=1,2,\dots}$. By Lemma 6.1 and by Kingman Sub-additive Ergodic Theorem we have that

$$\lim_{n\to\infty} \frac{1}{nN_0}\log \widetilde{\mathcal{Z}}_{nN_0,\omega}(\rho) = \lim_{n\to\infty} \frac{1}{nN_0}\mathbb{E}\log \widetilde{\mathcal{Z}}_{nN_0,\omega}(\rho) =: \mathrm{G}(\beta,\rho)\,, \tag{6.7}$$

where the first limit is $\mathbb{P}(\,\mathrm{d}\omega)$-a.s.. We now proceed to a *surgical procedure* to compare the partition function $\mathcal{Z}_{N,m,\omega}$ of the systems that satisfy $|m/N - \rho| \leq \varepsilon$ with $\widetilde{\mathcal{Z}}_{nN_0,\omega}(\rho)$, n suitably chosen: we are therefore going to establish (6.6), from which (6.5) follows. By the same trick used in the proof of Lemma 6.1 we have that

$$\log \mathcal{Z}_{N,m,\omega} \geq \log \widetilde{\mathcal{Z}}_{N-\ell_N,m',\omega}(\rho) + \log \mathcal{Z}_{\ell_N,m-m',\theta^{N-\ell_N}\omega} =: T_1 + T_2\,, \tag{6.8}$$

where $N-\ell_N$ is a multiple of N_0 and $m' := \lfloor \rho(N-\ell_N)\rfloor$. Recall that we have $|m/N-\rho| \leq \varepsilon$ and that we have the constraint that

$$1 \leq m - m' \leq \ell_N\,, \tag{6.9}$$

which simply means that the *second portion of the system* contains at least one contact and no more than its length. These requirements are satisfied if

$$\ell_N \geq \max\left(\frac{2+\varepsilon N}{\rho}, \frac{1+\varepsilon N}{1-\rho}\right), \tag{6.10}$$

and we can therefore assume that in addition

$$\limsup_N \frac{\ell_N}{N} \leq \varepsilon \max\left(\frac{1}{\rho}, \frac{1}{1-\rho}\right). \tag{6.11}$$

Note also that these definition require N sufficiently large, more precisely N is larger than a multiple of N_0 and the proportionality constant depends on ρ. Everything we claim

below is for these values of N. Let us remark now that for the term T_1 in (6.8) we have

$$\begin{aligned}\frac{1}{N}\log\widetilde{\mathcal{Z}}_{N-\ell_N,\omega}(\rho) &\geq \left(1-\frac{\ell_N}{N}\right)\frac{1}{N-\ell_N}\log\widetilde{\mathcal{Z}}_{N-\ell_N,\omega}(\rho)\\ &\geq \left(1-\frac{\ell_N}{N}\right)\mathrm{G}(\beta,\rho)-\left|\frac{1}{N-\ell_N}\log\widetilde{\mathcal{Z}}_{N-\ell_N,\omega}(\rho)-\mathrm{G}(\beta,\rho)\right| \qquad (6.12)\\ &\geq (1-c\varepsilon)\,\mathrm{G}(\beta,\rho)-\left|\frac{1}{N-\ell_N}\log\widetilde{\mathcal{Z}}_{N-\ell_N,\omega}(\rho)-\mathrm{G}(\beta,\rho)\right|,\end{aligned}$$

with $c = 2\max(1/\rho, 1/(1-\rho))$, cf. (6.11) (this requires N sufficiently large), and the rightmost term that vanishes a.s. as $N\to\infty$.

We have now to bound T_2 from below. We proceed via a rough bound that consists in selecting only one renewal trajectory (simply the one for which the $m-m'$ contacts are at the right end of the system):

$$\begin{aligned}\log\mathcal{Z}_{\ell_N,m-m',\theta^{N-\ell_N}\omega} &\geq \log\mathcal{Z}_{\ell_N-(m-m'-1),1,\theta^{N-\ell_N}\omega}+\sum_{j=1}^{m-m'-1}\log\mathcal{Z}_{1,1,\theta^{N-j}\omega}\\ &\geq \log K(\ell_N-(m-m'-1))+(m-m'-1)\log K(1)+\beta\sum_{j=0}^{m-m'-1}\omega_{N-j}\\ &\geq -c\varepsilon N+\beta\sum_{j=0}^{m-m'-1}\omega_{N-j} \geq -c\varepsilon N-\beta\sum_{j=0}^{\ell_N-1}|\omega_{N-j}|, \qquad (6.13)\end{aligned}$$

where in the step before the last $c>0$ depends on ρ and we have simply used that $m-m'\leq \ell_N = O(\varepsilon N)$, see (6.10) and (6.11). Note that $\mathbb{E}\sum_{j=0}^{\ell_N-1}|\omega_{N-j}| = \ell_N\mathbb{E}[|\omega_1|] = O(\varepsilon N)$ and that, by an elementary Large Deviation bound via exponential Markov inequality, we see that $\mathbb{P}(\sum_{j=0}^{\ell_N-1}|\omega_{N-j}|\geq 2\ell_N\mathbb{E}[|\omega_1|])\leq \exp(-c\ell_N)\leq\exp(-c'\varepsilon N)$ so that, by Borel-Cantelli, $\sum_{j=0}^{\ell_N-1}|\omega_{N-j}|\leq 2\ell_N = O(\varepsilon N)$ for N sufficiently large, how large may depend on ω.

The upper bound is obtained exploiting the same idea: the first step is to observe that

$$\log\mathcal{Z}_{N,m,\omega} \leq \log\widetilde{\mathcal{Z}}_{N+\ell_N,\omega}(\rho)-\min_{m'\in\{m_+,m_-\}}\log\mathcal{Z}_{\ell_N,m'-m,\theta^N\omega}, \qquad (6.14)$$

where $m_- = \lfloor\rho(N+\ell_N)\rfloor$ and $m_+ = \lceil\rho(N+\ell_N)\rceil$. Once again, the first term on the right-hand side is controlled using (6.7) and we need a lower bound on $\log\mathcal{Z}_{\ell_N,m'-m,\theta^N\omega}$, like before. With the same procedure we obtain

$$\min_{m'\in\{m_+,m_-\}}\log\mathcal{Z}_{\ell_N,m'-m,\theta^N\omega} \geq -c\varepsilon N-\beta\sum_{j=0}^{\ell_N-1}|\omega_{N-j}|. \qquad (6.15)$$

This term can be bounded a.s. precisely like for the lower bound, and, by putting upper and lower bound together we obtain the bound for $\rho\in(0,1)$ on $D_{N,\varepsilon,\rho}(\omega)$, cf. (6.6), claimed in Proposition 6.2. Note that this bound directly implies (6.5).

For the case $\rho = 0$ we can use the same trick as in (6.13) to get the lower bound

$$\log\mathcal{Z}_{N,m,\omega} \geq \log K(N-m+1)+(m-1)\log K(1)-\beta\sum_{j=N-m+1}^{N}|\omega_j|, \qquad (6.16)$$

and it is straightforward to see that $-\log \mathcal{Z}_{N,m,\omega}$ is bounded above by a constant time ε plus a random contribution that is also $O(\varepsilon N)$ both in $\mathbb{L}^1$ and a.s.. So $(\log \mathcal{Z}_{N,m,\omega})_-$ is under control and it suffices to remark that $\mathbb{E}\mathcal{Z}_{N,m,\omega} = \exp(\lambda(\beta)m)\mathbb{P}(\tau_m = N) \le \exp(\lambda(\beta)m)$ which is in turn bounded by $\exp(\lambda(\beta)\varepsilon N)$. Therefore $\mathbb{E}\log \mathcal{Z}_{N,m,\omega} \le 2\lambda(\beta)N\varepsilon$ and Borel-Cantelli yields also that $\log \mathcal{Z}_{N,m,\omega} \le 2\lambda(\beta)m \le \lambda(\beta)N\varepsilon$ for N larger than a constant that may depend on ω. This completes the proof for $\rho = 0$.

For the case $\rho = 1$ we write

$$\log \mathcal{Z}_{N,m,\omega} \,=\, \log \mathbf{P}\left(\tau_m = N\right) + \log \mathbf{E}_{N,m}\left[\exp\left(-\beta \sum_{j=1}^{N} \omega_j(1-\delta_j)\right)\right] + \beta \sum_{j=1}^{N} \omega_j\,, \quad (6.17)$$

where $\mathbf{Q}_{N,m}(\cdot) = \mathbf{P}(\cdot|\tau_m = N)$, like in Section 4. By Proposition 3.1, notably (3.2), we have $|\log \mathbf{P}\left(\tau_m = N\right) - \textsc{g}(1)| \le \textsc{g}(1-2\varepsilon)$ for N sufficiently large. The last term is also easily disposed of since by standard estimates for IID sequence of centered variables in $\mathbb{L}^p$ for every p we have that this term is a.s. $O(N^c)$, any $c > 1/2$.

We are therefore left with controlling the second term in (6.17). By Jensen inequality have the lower bound

$$\log \mathbf{E}_{N,m}\left[\exp\left(-\beta \sum_{j=1}^{N} \omega_j(1-\delta_j)\right)\right] \,\ge -\beta \sum_{j=1}^{N} \omega_j u_{N,m}(j)\,, \quad (6.18)$$

with $u_{N,m}(j) = (1-\mathbf{E}_{N,m}[\delta_j])$. The bound in $\mathbb{L}^1$ is obvious because $\sum_j u_{N,m}(j) = N-m \le \varepsilon N$. For the a.s. bound the Markov inequality yields

$$\begin{aligned}\mathbb{P}\left(\sum_{j=1}^{N} \omega_j u_{N,m}(j) \,\ge\, \varepsilon N\right) \,&\le\, \exp\left(-t\varepsilon N + \sum_j \lambda(u_{N,m}(j)t)\right)\\ &\le\, \exp\left(-t\varepsilon N + \sum_j (u_{N,m}(j))^2 t^2\right) \,\le\, \exp\left(-t\varepsilon N + \varepsilon N t^2\right) \,\le\, \exp\left(-c\varepsilon N\right)\,, \quad (6.19)\end{aligned}$$

where we have used that $\lambda(u) \sim u^2/2$ for u small, so $\lambda(u) \le u^2$ for $|u| \le u_0$, and $u_{N,m}(j) \le 1$ as well as $\sum_j u_{N,m}(j) = N - m \le \varepsilon N$. In the last step c is the maximum of $t - t^2$ for $t \in [0, u_0]$. The Borel-Cantelli Lemma warrants that the quantity in (6.18) is bounded below by $-2\beta\varepsilon N$ for N larger than a random threshold.

For the upper bound it suffices to remark that

$$\mathbb{E}\mathbf{E}_{N,m}\left[\exp\left(\beta \sum_{j=1}^{N} \omega_j(1-\delta_j)\right)\right] \,=\, \exp((N-m)\lambda(\beta)) \,\le\, \exp(\lambda(\beta)\varepsilon N)\,, \quad (6.20)$$

which, by using again the Markov inequality and Borel-Cantelli, yields the a.s. bound we are looking for. □

Proposition 6.3. $\textsc{g}(\beta,\cdot)$ *is concave on* $[0,1]$ *and it is continuous up to the boundary. Moreover*

$$\lim_{\rho \nearrow 1} \frac{\textsc{g}(\beta,1) - \textsc{g}(\beta,\rho)}{1-\rho} \,=\, -\infty\,. \quad (6.21)$$

Proof. Choose $\lambda \in (0,1)$ and $\rho_1, \rho_2 \in [0,1]$ with $\rho_1 < \rho_2$. We have

$$\frac{1}{N}\mathbb{E}\log \mathcal{Z}_{N,\lceil \lambda\rho_1+(1-\lambda)\rho_2 N\rceil,\omega} \geq \frac{1}{N}\mathbb{E}\log \mathcal{Z}_{\lceil \lambda N\rceil,\lceil \lambda\rho_1 N\rceil,\omega} + \frac{1}{N}\mathbb{E}\log \mathcal{Z}_{N-\lceil \lambda N\rceil,\lceil \lambda\rho_1+(1-\lambda)\rho_2 N\rceil-\lceil \lambda\rho_1 N\rceil,\omega}\,. \quad (6.22)$$

By Proposition 6.2 we can pass to the limit $N \to \infty$ and we obtain

$$\mathrm{G}\left(\beta, \lambda\rho_1 + (1-\lambda)\rho_2\right) \geq \lambda \mathrm{G}(\beta,\rho_1) + (1-\lambda)\mathrm{G}(\beta,\rho_2)\,, \quad (6.23)$$

so $\mathrm{G}(\cdot)$ is concave, hence continuous because it is bounded. Both the continuity at 0 and 1, with $\mathrm{G}(\beta,0) = 0$ and $\mathrm{G}(\beta,1) = \log K(1)$, follow directly from the estimates in Proposition 6.2 (we observe that the continuity in $(0,1)$ can be extracted directly from Proposition 6.2 as well).

For what concerns (6.21) we need an adequate lower bound on $\mathrm{G}(\beta,\rho)$. This follows by taking the expectation of both sides of (6.17) and (6.18). This way we obtain

$$\mathbb{E}\log \mathcal{Z}_{N,m,\omega} \geq \log \mathbf{P}\left(\tau_m = N\right)\,, \quad (6.24)$$

and the general bound $\mathrm{G}(\beta,\rho) \geq \mathrm{G}(0,\rho)$. This inequality becomes an equality at $\rho = 1$ (this follows once again from (6.17)) and therefore $\mathrm{G}(\beta,1) - \mathrm{G}(\beta,\rho) \leq \mathrm{G}(0,1) - \mathrm{G}(0,\rho)$, so that the claim follows from the analogous claim for the case $\beta = 0$. □

6.2. **Proof Theorem 1.3.**

Proof. Let us make the preliminary remark that it suffices to show a.s. convergence because $(\log Z^{\Psi}_{N,\omega,\beta,h}/N)_{N=1,2,\dots}$ is uniformly integrable and therefore we have also convergence in $\mathbb{L}^1$. Uniform integrability can be established by making upper and lower bounds on $Z^{\Psi}_{N,\omega,\beta,h}$ in the spirit of the repeated estimates we used in the proof of Proposition 6.2 (but what suffices here is substantially rougher), so one readily sees that there exists $C > 0$ (that depends on Ψ, on $K(\cdot)$ and on h) such that

$$\frac{1}{N}\left|\log Z^{\Psi}_{N,\omega,\beta,h}\right| \leq \frac{\beta}{N}\sum_{j=1}^{N}|\omega_j| + C\,. \quad (6.25)$$

Since the expectation of the square of the right-hand side is bounded uniformly in N, uniform integrability is proven.

We now proceed with proving a.s. convergence by suitable lower and upper bounds on the sequence. Remark that the expected limit $\sup_{\rho\in[0,1]}\left(h\rho + H(\rho) + \mathrm{G}(\beta,\rho)\right)$ is in fact a maximum which is uniquely achieved at $\rho_h \in [0,1)$ by the assumptions on $H(\cdot)$ and by what we have proven on $\mathrm{G}(\beta,\cdot)$ (that is, Proposition 3.1 and Proposition 6.3).

For the lower bound it suffices to remark that for every ρ (1.10) (with the notation (5.4)) yields thanks to (1.4) of Definition 1.1

$$Z^{\Psi}_{N,\omega,\beta,h} \geq c(b)\exp\left(mh + NH(m/N) - bN\right)\mathcal{Z}_{N,m,\omega,\beta}\,, \quad (6.26)$$

With b that can be chosen arbitrarily small. If $\rho_h \in (0,1)$ it suffices to choose $\rho = \rho_h$ and pass to the limit: by Proposition 6.2 we obtain that a.s.

$$\liminf_{\substack{N\to\infty:\\ m/N\sim\rho_h}} \frac{1}{N}Z^{\Psi}_{N,\omega,\beta,h} \geq \rho_h h + H(\rho_h) + \mathrm{G}(\beta,\rho_h) - b\,. \quad (6.27)$$

Since $b > 0$ is arbitrary, we are done for the case $\rho_h \in (0,1)$. If $\rho_h = 0$ we can repeat the same argument for $\rho = \rho_j$, with $\rho_j \searrow 0$, and the lower bound analysis is complete.

For the upper bound we use (1.3) of Definition 1.1

$$Z^{\Psi}_{N,\omega,\beta,h} \,\le\, c(b)\sum_{m=1}^{N}\exp\left(mh+NH(m/N)+bN\right)\mathcal{Z}_{N,m,\omega,\beta}\,, \tag{6.28}$$

so

$$\limsup_N \frac{1}{N}\log Z^{\Psi}_{N,\omega,\beta,h} \,\le\, \limsup_N \max_{m=1,\dots,N}\left(\frac{m}{N}h+H\left(\frac{m}{N}\right)+\frac{1}{N}\log\mathcal{Z}_{N,m,\omega,\beta}+b\right). \tag{6.29}$$

To deal with the maximum we fix a small value positive value of $\widetilde{\rho}$ and the grid of densities $\rho_j := \widetilde{\rho}+j(1-2\widetilde{\rho})/M$, M a positive integer, for $j=0,1,\dots,M$: we can therefore group the maximum into $M+3$ blocks. We can now apply Proposition 6.2 in taking the limit $N\to\infty$, with $\varepsilon = 1/M$ for for the blocks corresponding to the densities $\rho_0=\widetilde{\rho},\rho_1,\dots,\rho_M=1-\widetilde{\rho}$, and with $\varepsilon=\widetilde{\rho}$ fro the two boundary blocks. It is then a matter of sending first M to ∞ and $\widetilde{\rho}$ to zero. Since also b can be chosen arbirarily small, we conclude that a.s.

$$\limsup_N \frac{1}{N}\log Z^{\Psi}_{N,\omega,\beta,h} \,\le\, \sup_{\rho\in[0,1]}\left(h\rho+H(\rho)+\textsc{g}(\beta,\rho)\right). \tag{6.30}$$

This completes the proof of Theorem 1.3. □

Appendix A. Circular DNA models

In [6, 7] (to which we refer also for a more complete literature) the problem of modeling circular DNA is considered: circular DNA corresponds notably to the genetic structures called *plasmids* that are present in cells. Plasmids are also used for genetic manipulations. If a doubled stranded DNA has a circular structure, that is if the strands are not free at their ends but form an ring, then the separation of the two strands – even just locally, global separation may not be possible – generates a conflict with the double helix structure and the physical properties of the DNA polymer. In fact, in a double stranded DNA with free ends, the local separation of the two strands just induces a rotation in the chain. But in the circular case (Fig. 6) this rotation cannot take place: the ring of the two strands has a winding number that can change if the backbone of DNA polymer can absorb, typically with an energetic cost, this *(over)twist*. Another way of absorbing the winding number is by forming nontrivial spatial structures called *supercoils* at locations where the two strands are attached (this induces a strain on the base pairs involved, so the relative contact energy changes): everybody has experienced the formation of supercoils when trying to disentangle two ropes, or even just one rope.

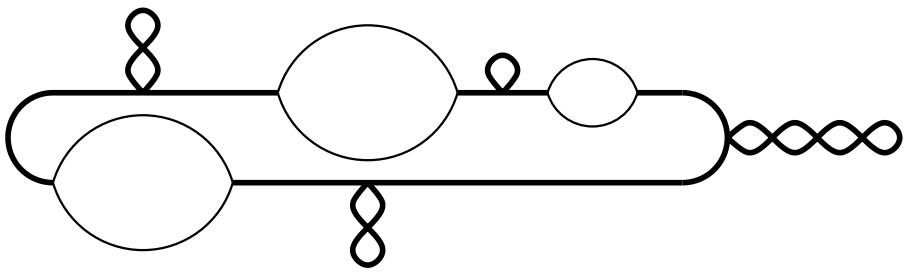

Figure 6. A schematic image of a circular DNA with three loops and four supercoil sections. The thick line represents the segments of DNA on which the two stands are in contact. The thin line represents a single strand portion, and this happens only in loops.

The physical phenomena we just described are very complex: the free end case is already of great complexity! Nonetheless the (simple!) Poland-Scheraga model turns out to be a very relevant model for DNA denaturation study in the free end case (see references in [7, 29]). The circular case is tackled in [6, 7]: a model is built from the Poland-Scheraga and the partition function of the model, with our notations, is (1.9). The function $\log \Psi(m, N)$, cf. Def. 1.1, that enters the definition can be seen as a nonlocal energy. Note that the functional dependence in Ψ is only on the length of the polymer and on the number of contacts m. Moreover, with m contacts the total length of the loops is $N - m$.

Let us take a closer look at the models considered in [6, 7]:

(1) *Model with overtwist.* This first model is particularly simple:

$$\Psi(m, N) \,=\, \exp\left(-\chi(N-m)^2/m\right)\,, \tag{A.1}$$

with $\chi > 0$. In this case $Q(m, N) \equiv 1$ and $H(\rho) = -\chi(1-\rho)^2/\rho$. We refer to [6, 7] for explanations about the choice of the precise shape of this energy term. Simply put, the smaller the number m of contacts is, the less likely the configuration is. In other words, the opening of loops is penalized. The rationale is that if the loop length increases, then overtwist is produced in the backbone, which is costly from the energetic viewpoint.

(2) *Model with overtwist and supercoils.* This is less straightforward and it involves choosing n supercoils among the m contacts. This is done by fair coin flipping: there is no loss of generality in this choice because a bias corresponds simply to an energetic change for supercoil contacts and in the model there is a real parameter w that accounts for that. Here is the choice in [7]:

$$\Psi(m, N) \,=\, \sum_{n=0}^{m} \frac{1}{2^m} \binom{m}{n} \exp\left(nw - \chi \frac{(N-m-n)^2}{m}\right)\,, \tag{A.2}$$

where $\chi > 0$ and w are constants. The fair coin structure and the energetic term for supercoils are clear: added to that there is a penalization term that favors $N - n \approx m$. Recall that in the simple overtwist case m close to N is favored. This is simply because n supercoils are formed and the twist that remains has to be absorbed by the portion of DNA, of length $N - n$, which is not in supercoil form.

The choice (A.2) is in the framework of Definition 1.1 with

$$H(\rho) = \sup_{\zeta \in [0,\rho]} \psi(\zeta, \rho) = \psi(\zeta_0(\rho), \rho), \tag{A.3}$$

where

$$\psi(\zeta, \rho) := \zeta w - \rho \log 2 - \chi \frac{(1-\rho-\zeta)^2}{\rho} + \rho \log \rho - \zeta \log(\zeta) - (\rho - \zeta) \log(\rho - \zeta). \tag{A.4}$$

$Q(m, N)$ is therefore (implicitly) defined and one can derive the asymptotic behavior for $N \to \infty$ and m/N asymptotically constant:

$$Q(m, N) \sim q\left(\frac{m}{N}\right), \quad \text{with} \quad q(\rho) := \sqrt{\frac{\rho}{\zeta_0(\rho)(\rho - \zeta_0(\rho))|\partial_\zeta^2 \psi(\zeta_0(\rho), \rho)|}}. \tag{A.5}$$

The function ψ is concave on the convex domain $\{0 \le \zeta \le \rho \le 1\}$. Thus, H is also concave. Moreover H is analytic on $(0, 1)$.

(3) *Model with supercoils.* This is the $\chi = \infty$ limit of (A.2). In this limit $N - n = m$ and, since $n \le m$, we have that $m \ge N/2$ and that the opening of a loop must be compensated by at least as many supercoils. Explicitly we obtain

$$\Psi(m, N) \;=\; \frac{1}{2^m}\binom{m}{N-m} e^{Nw - mw} \mathbf{1}_{[1/2,1]}(m/N)\,. \tag{A.6}$$

Since $\Psi(m, N) = 0$ for $m < N/2$, this limit case does not fall into the framework of Definition 1.1. A wider set-up that encompasses all models in in [6, 7] can be found in [42, Ch. 4].

.

We remark that in both examples (1) and (2) $H(0) = -\infty$ and (of course) $H'(0) = \infty$. So, with the convention we have chosen to consider *localized* both the partly and fully localized cases, the circular DNA models are localized for all values of the parameters and they display a big-jump transition if $\beta = 0$ and $\alpha > 1$.

Appendix B. On the strict convexity of the disordered pinning free energy

Theorem B.1. *Consider the $\Psi \equiv 1$ model, that is the disordered pinning model. For every $\beta \ge 0$ and every $h > h_c(\beta)$ we have $\partial_h^2 \mathrm{F}(\beta, h) > 0$.*

Proof. Let us remark that for $\beta = 0$ the result can be established by explicit computations, but the proof that we give here works for the $\beta = 0$ case as well. In this proof $\mathbf{P}_{N,\omega} = \mathbf{P}_{N,\omega,\beta,h}$ and $\mathrm{Var}_{N,\omega}$ is the variance with respect to $\mathbf{P}_{N,\omega}$. We know from [37, Proof of Theorem 2.1] that for $h > h_c(\beta)$

$$\partial_h^2 \mathrm{F}(\beta, h) \;=\; \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\, \mathrm{Var}_{N,\omega}\left(\sum_{j=1}^{N-1} \delta_j\right)\,. \tag{B.1}$$

We are going to condition on even sites, so let us replace N with $2N$ and let us denote by $\mathcal{F}_e$ the σ-algebra generated by δ_j with j even: $\mathrm{Var}_{N,\omega,e}(\cdot)$ is going to denote the variance with respect to $\mathbf{P}_{N,\omega}(\,\cdot\,|\mathcal{F}_e)$. By Jensen's inequality

$$\mathrm{Var}_{2N,\omega}\left(\sum_{j=1}^{2N-1} \delta_j\right) \;\ge\; \mathbf{E}_{2N,\omega}\left[\mathrm{Var}_{2N,\omega,e}\left(\sum_{j=1}^{N} \delta_{2j-1}\right)\right]\,. \tag{B.2}$$

We know consider the conditional variance on the set $E_\sigma := \{\tau : \delta_{2j} = \sigma_j \text{ for } j = 1, 2, \dots, N-1\}$ for every $\sigma \in \{0,1\}^{\{1,\dots,N-1\}}$. We set $n(\sigma) := \sum_{j=1}^{N-1} \sigma_j$ and $\ell_0 := 0$ and we define iteratively $\ell_{j+1} = \min\{\ell > \ell_j : \sigma_\ell = 1\}$ for $j \le n(\sigma) - 1$. We then redifine ℓ_j to be $2\ell_j$ and set also $\ell_{n(\sigma)+1} := 2N$. Therefore $\ell_0, \ell_1, \dots, \ell_{n(\sigma)+1}$ are the $n(\sigma)$ pinned even sites, plus 0 and $2N$ that are pinned from the start. Note that

$$\sum_{j=1}^{N} \delta_{2j-1} \;=\; \sum_{k=1}^{n(\sigma)+1} \sum_{j=1+\ell_{k-1}/2}^{\ell_k/2} \delta_{2j-1}\,, \tag{B.3}$$

and remark that, under $\mathbf{P}_{N,\omega}(\,\cdot\,|\mathcal{F}_e)(\tau)$ with $\tau \in E_\sigma$, the random variables

$$\left(\sum_{j=1+\ell_{k-1}/2}^{\ell_k/2} \delta_{2j-1}\right)_{k=1,\dots,n(\sigma)+1}\,, \tag{B.4}$$

are independent. Therefore on E_σ

$$\mathrm{Var}_{2N,\omega,e}\left(\sum_{j=1}^{N}\delta_{2j-1}\right) = \sum_{k=1}^{n(\sigma)+1}\mathrm{Var}_{2N,\omega,e}\left(\sum_{j=1+\ell_{k-1}/2}^{\ell_k/2}\delta_{2j-1}\right) \geq \sum_{\substack{k=1,\dots,n(\sigma)+1\\ \ell_k-\ell_{k-1}=2}}\mathrm{Var}_{2N,\omega,e}\left(\delta_{\ell_k-1}\right). \quad \text{(B.5)}$$

Since δ_{ℓ_k-1}, under the conditional measure we are considering, is just a Bernoulli random variable with parameter (we use the short-cut notation $\omega=\omega_{\ell_k-1}$)

$$p(\omega) := \frac{K(1)^2\exp(h+\beta\omega)}{K(1)^2\exp(h+\beta\omega)+K(2)}, \quad \text{(B.6)}$$

we see that for k such that $\ell_k-\ell_{k-1}=2$

$$\mathrm{Var}_{2N,\omega,e}\left(\delta_{\ell_k-1}\right) = p\left(\omega_{\ell_k-1}\right)\left(1-p\left(\omega_{\ell_k-1}\right)\right) =: \sigma^2\left(\omega_{\ell_k-1}\right), \quad \text{(B.7)}$$

and therefore

$$\mathrm{Var}_{2N,\omega,e}\left(\sum_{j=1}^{N}\delta_{2j-1}\right) \geq \mathbf{E}_{2N,\omega}\left[\sum_{k=0}^{N-1}\delta_{2k}\delta_{2k+2}\sigma^2\left(\omega_{\ell_k-1}\right)\right]. \quad \text{(B.8)}$$

Now we set $\sigma^2_\star(L) := \inf\{\sigma^2(\omega) : |\omega|\leq L\} > 0$. We remark that $\sigma^2_\star(L)>0$ for every $L>0$, but in what follows we are forced to work with L such that $\mathbb{P}(|\omega|<L)>0$, that is for L above a threshold. With this notation

$$\begin{aligned}\mathbb{E}\mathrm{Var}_{2N,\omega,e}\left(\sum_{j=1}^{N}\delta_{2j-1}\right) &\geq \sigma^2_\star(L)\mathbb{E}\mathbf{E}_{2N,\omega}\left[\sum_{k=0}^{N-1}\delta_{2k}\delta_{2k+2}\mathbf{1}_{|\omega_{2k+1}|\leq L}\right]\\ &\geq \sigma^2_\star(L)\left(\mathbb{E}\mathbf{E}_{2N,\omega}\left[\sum_{k=0}^{N-1}\delta_{2k}\delta_{2k+2}\right]-N\mathbb{P}\left(|\omega|>L\right)\right),\end{aligned} \quad \text{(B.9)}$$

and we are left with showing that

$$q := \liminf_N \frac{1}{N}\mathbb{E}\mathbf{E}_{2N,\omega}\left[\sum_{k=0}^{N-1}\delta_{2k}\delta_{2k+2}\right] > 0, \quad \text{(B.10)}$$

because it suffices to choose L so that $\mathbb{P}\left(|\omega|>L\right)\leq q/2$ to obtain, see (B.1)-(B.2), that $\partial^2_h \mathrm{F}(\beta,h)\geq \sigma^2_\star(L)q/4>0$.

In order to establish (B.10) we want to show that the quantity under analysis is bounded below by $\lim_N N^{-1}\mathbb{E}\mathbf{E}_{2N,\omega}[\sum_{j=1}^{2N}\delta_j]$, which is equal to $2\partial\mathrm{F}(\beta,h)>0$, times a positive constant. This can be done by explicit estimates, but for sake of conciseness we use that, for any choice of a sequence $(b_N)_{N\in\mathbb{N}}$ of positive integer numbers satisfying $\lim_N b_N=\infty$ and $\lim_N b_N/N=0$, by [37, Theorem 2.2] we have uniformly on $k\in[b_N,N-b_n]\cap\mathbb{N}$

$$\lim_{N\to\infty}\mathbb{E}\mathbf{E}_{2N,\omega}[\delta_k] = \partial_h\mathrm{F}(\beta,h) \quad \text{and} \quad \lim_{N\to\infty}\mathbb{E}\mathbf{E}_{2N,\omega}[\delta_k\delta_{k+2}] =: \mathrm{L}(\beta,h), \quad \text{(B.11)}$$

where the second statement is just the existence of the limit and (B.10) follows once $\mathrm{L}(\beta,h)>0$ is shown. For this we write $\mathbf{E}_{2N,\omega}[\delta_k\delta_{k+2}]=\mathbf{E}_{2N,\omega}[\delta_k]\mathbf{E}_{2N,\omega}[\delta_{k+2}|\delta_k=1]$ and

$$\mathbf{E}_{2N,\omega}[\delta_{k+2}|\delta_k=1] \,=\, \frac{Z_{2,\theta^k\omega,\beta,h}Z_{N-k-2,\theta^{k+2}\omega,\beta,h}}{Z_{N-k,\theta^k\omega,\beta,h}} \,\geq\, C\exp\left(-\beta\left(|\omega_{k+1}|+|\omega_{k+2}|\right)\right)\,, \tag{B.12}$$

where the constant $C>0$ depends on h and on $K(\cdot)$: this estimate is a standard surgery procedure ([29, Ch. 2], full details can be found in [42, Sec. 5.5]) for which one uses notably the regularly varying character of $K(\cdot)$. Therefore

$$\mathbf{E}_{2N,\omega}[\delta_k\delta_{k+2}] \,\geq Ce^{-2\beta L}\mathbf{E}_{2N,\omega}[\delta_k]\left(1-\mathbf{1}_{|\omega_{k+1}|+|\omega_{k+2}|>L}\right)\,, \tag{B.13}$$

and, in turn, we have

$$\mathbb{E}\mathbf{E}_{2N,\omega}[\delta_k\delta_{k+2}] \,\geq\, Ce^{-2\beta L}\left(\mathbb{E}\mathbf{E}_{2N,\omega}[\delta_k]-\mathbb{P}\left(|\omega_1|>L\right)\right)\,. \tag{B.14}$$

It suffices now to choose L so that $\mathbb{P}\left(|\omega_1|>L\right)\leq\partial_h\mathrm{F}(\beta,h)/2$ to obtain that, uniformly in k like in (B.11), we have

$$\liminf_N \mathbb{E}\mathbf{E}_{2N,\omega}\left[\delta_k\delta_{k+2}\right] \,\geq\, \frac{1}{2}Ce^{-2\beta L}\partial_h\mathrm{F}(\beta,h) \,>\, 0\,, \tag{B.15}$$

and we are done. □

We include here the result proved under restrictive conditions in Remark 1.5.

Proposition B.2. *For every $\beta\geq 0$ and every h*

$$\mathrm{F}_H(\beta,h) \,\geq\, H(0)\,. \tag{B.16}$$

Proof. We can assume $H(0)>-\infty$ and, with $b>0$ and u, v and $c(b)$ like in (1.4) of Definition 1.1, we obtain that

$$\begin{aligned} Z^{\Psi}_{N,\omega\beta,h} \,\geq\, & c(b)\exp\left(N\min_{\rho\in(0,b]}H(\rho)-bN\right) \\ & \mathbf{E}\left[\exp\left(\beta\sum_{j=1}^N(\beta\omega_j+h)\delta_j\right)\,;\,\tau_{\lfloor bN\rfloor}=N,\,\frac{\tau}{N}\cap\left((0,b)\cup(1-b,1)\right)=\emptyset\right] \end{aligned}\,, \tag{B.17}$$

a bound that is obtained simply by restricting the partition function to the renewals with $\lfloor bN\rfloor$ contacts and all at distance at least bN from the boundary. Therefore

$$\liminf_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z^{\Psi}_{N,\omega\beta,h} \,\geq\, H(0)+\liminf_{N\to\infty}\frac{1}{N}\mathbb{E}\log\mathbf{E}\left[\exp\left(\sum_{j=1}^N(\beta\omega_j+h)\delta_j\right)\,;\,E_{N,b}\right]\,, \tag{B.18}$$

with $E_{N,b}:=\{\tau_{\lfloor bN\rfloor}=N,\,(\tau/N)\cap((0,b)\cup(1-b,1))=\emptyset\}$. Using $\mathbf{P}'(\cdot):=\mathbf{P}(\cdot|E_{N,b})$ we see that by Jensen's inequality the quantity of which we take inferior limit in the right-hand side of the last expression is bounded below by

$$\frac{1}{N}\mathbb{E}\left[\sum_{j=1}^N(\beta\omega_j+h)\mathbf{E}'[\delta_j]\right]+\frac{1}{N}\log\mathbf{P}\left(E_{N,b}\right) \,=\, h\frac{\lfloor bN\rfloor}{N}+\frac{1}{N}\log\mathbf{P}\left(E_{N,b}\right)\,. \tag{B.19}$$

Of course the limit of the first term is hb, which can be made arbitrarily small by choosing b small. The remaining term is bounded below, for $N \to \infty$, by a (negative) quantity that vanishes as $b \searrow 0$ because $\mathbf{P}(E_{N,b})$ is bounded below by $K(\lceil bN \rceil)^2$ times $\mathbf{P}(\tau_{\lfloor bN \rfloor - 2} = N - 2\lceil bN \rceil)$, so, by Proposition 3.1, $\liminf_N (1/N) \log \mathbf{P}(E_{N,b}) = 0$ for $\alpha > 1$ and it vanishes as $b \searrow 0$ for $\alpha \in (0, 1]$. □

Acknowledgements

We are grateful to Quentin Berger and Hubert Lacoin for several exchanges. We thank in particular Hubert Lacoin for suggesting the argument of proof of Theorem B.1. G.G. also acknowledges support from grant ANR-15-CE40-0020.

References

[1] K. S. Alexander, *The effect of disorder on polymer depinning transitions*, Commun. Math. Phys. **279** (2008), 117-146.
[2] K. S. Alexander and V. Sidoravicius, *Pinning of polymers and interfaces by random potentials*, Ann. Appl. Probab. **16** (2006), 636-669.
[3] K. S. Alexander and N. Zygouras, *Quenched and annealed critical points in polymer pinning models*, Comm. Math. Phys. **291** (2009), 659-689.
[4] K. S. Alexander and N. Zygouras, *Path properties of the disordered pinning model in the delocalized regime*, Ann. Appl. Prob. **24** (2014), 599-615.
[5] I. Armendáriz and M. Loulakis, *Conditional distribution of heavy tailed random variables on large deviations of their sum*, Stoch. Proc. Appl. **121** (2011), 1138-1147.
[6] A. Bar, A. Kabakçıoğlu and D. Mukamel, *Denaturation of circular DNA: Supercoil mechanism*, Phys. Rev. E **84** (2011), 041935.
[7] A. Bar, A. Kabakçıoğlu and D. Mukamel, *Denaturation of circular DNA: Supercoils and overtwist*, Phys. Rev. E **86** (2012), 061904.
[8] Q. Berger, G. Giacomin and M. Khatib, *DNA melting structures in the generalized Poland-Scheraga model*, ALEA, Lat. Am. J. Probab. Math. Stat. **15** (2018), 993 -1025.
[9] Q. Berger, G. Giacomin and M. Khatib, *Disorder and denaturation transition in the generalized Poland-Scheraga model*, Ann. H. Lebesgue **3** (2020), 299-339.
[10] Q. Berger, G. Giacomin and H. Lacoin, *Disorder and critical phenomena: the $\alpha = 0$ copolymer model*, Probab. Theory Rel. Fields, **174** (2019), 787–819.
[11] Q. Berger and H. Lacoin, *Pinning on a defect line: characterization of marginal disorder relevance and sharp asymptotics for the critical point shift*, J. Inst. Math. Jussieu, **17** Issue 2 (2018), 305-346.
[12] A.A. Borovkov, K.A. Borovkov, *On probabilities of large deviations for random walks. I. Regularly varying distribution tails*, Theory Probab. Appl. **46** (2000), 193-213.
[13] A. A. Borovkov and A. A. Mogul'skiĭ, *On large deviations of sums of independent random vectors on the boundary and outside of the cramér zone I*, Theory Probab.Appl. **53** (2009), No. 2, 301-311.
[14] F. Caravenna and F. den Hollander , *A general smoothing inequality for disordered polymers*, Electron. Commun. Probab. **18** (2013), 1-15 .
[15] F. Caravenna, F. L. Toninelli and N. Torri , *Universality for the pinning model in the weak coupling regime*, Ann. Probab. **45**, (2017), 2154-2209.
[16] X. Chen, V. Dagard, B. Derrida, Y. Hu, M. Lifshits and Z. Shi, *The Derrida–Retaux conjecture on recursive models*, arXiv:1907.01601
[17] C. Dasgupta and S.-K. Ma, *Low-temperature properties of the random Heisenberg anti-ferromagnetic chain*, Phys. Rev. B **22** (1980), 1305-1319.
[18] B. Davis and D. McDonald, *An elementary proof of the local central limit theorem*, J. Theoret. Probab. **8** (1995), 693-701.
[19] D. Denisov, A. B. Dieker and V. Shneer, *Large deviations for random walks under subexponentiality: the big-jump domain*, Ann. Probab. **36** (5), pp. 1946-199, 2008.
[20] B. Derrida, G. Giacomin, H. Lacoin and F. L. Toninelli, *Fractional moment bounds and disorder relevance for pinning models*, Commun. Math. Phys. **287** (2009), 867-887.
[21] B. Derrida, V. Hakim and J. Vannimenus, *Effect of disorder on two-dimensional wetting*, J. Statist. Phys. **66** (1992), 1189–1213.

[22] B. Derrida and M. Retaux, *The depinning transition in presence of disorder: a toy model*, J. Stat. Phys. **156** (2014), 268-290.
[23] T. R. Einert, H. Orland and R. R. Netz, *Secondary structure formation of homopolymeric single-stranded nucleic acids including force and loop entropy: implications for DNA hybridization*, Eur. Phys. J. E **34** (2011), 55 (15 pages).
[24] P. A. Ferrari, C. Landim and V. V. Sisko, *Condensation for a fixed number of independent random variables*, J. Stat. Phys. **128** (2007), 1153-1158.
[25] M. E. Fisher, *Walks, walls, wetting, and melting*, J. Statist. Phys. **34** (1984), 667-729.
[26] D. S. Fisher, *Critical behavior of random transverse-field Ising spin chains*, Phys. Rev. B **51** (1995), 6411-6461.
[27] T. Garel and H. Orland, *On the role of mismatches in DNA denaturation*, arXiv:cond-mat/0304080
[28] T. Garel and H. Orland, *Generalized Poland-Scheraga model for DNA hybridization*, Biopolymers **75** (2004), 453-467.
[29] G. Giacomin, *Random polymer models*, Imperial College Press, World Scientific, 2007.
[30] G. Giacomin, *Disorder and critical phenomena through basic probability models*, École d'été de probablités de Saint-Flour XL-2010, Lecture Notes in Mathematics **2025**, Springer, 2011.
[31] G. Giacomin and M. Khatib, *Generalized Poland Sheraga denaturation model and two dimensional renewal processes*, Stoch. Proc. Appl. **127** (2017), 526-573.
[32] G. Giacomin and H. Lacoin, *The disordered lattice free field pinning model approaching criticality*, arXiv:1912.10538
[33] G. Giacomin, H. Lacoin and F. L. Toninelli, *Marginal relevance of disorder for pinning models*, Comm. Pure Appl. Math. **63** (2010), 233-265.
[34] G. Giacomin and F. L. Toninelli, *Estimates on path delocalization for copolymers at selective interfaces*, Probab. Theory Rel. Fields **133** (2005), 464-482.
[35] G. Giacomin and F. L. Toninelli, *Smoothing of depinning transitions for directed polymers with quenched disorder*, Phys. Rev. Lett. **96** (2006), 070602.
[36] G. Giacomin and F. L. Toninelli, *Smoothing effect of quenched disorder on polymer depinning transitions*, Commun. Math. Phys. **266** (2006), 1-16.
[37] G. Giacomin and F. L. Toninelli, *The localized phase of disordered copolymers with adsorption*, ALEA, Lat. Am. J. Probab. Math. Stat. **1** (2006), 149-180.
[38] G. Giacomin and F. L. Toninelli, *On the irrelevant disorder regime of pinning models*, Ann. Probab. **37** (2009), 1841-1875.
[39] C. Godrèche, *Condensation for random variables conditioned by the value of their sum*, J. Stat. Mech. Theory Exp. **6** (2019), 063207, 34 pp.
[40] S. Grosskinsky, P. Chleboun and G. M. Schütz, *Instability of condensation in the zero-range process with random interaction*, Phys. Rev. E **78** (2008), 030101.
[41] A. B. Harris, *Effect of random defects on the critical behaviour of Ising models* J. Phys. C **7** (1974), 1671-1692.
[42] B. Havret, *On the Lyapunov exponent of random transfer matrices and on pinning models with constraints*, PhD thesis, Université de Paris (2019), https://tel.archives-ouvertes.fr/tel-02478078
[43] F. den Hollander, *Random polymers*, Lectures from the 37th Probability Summer School held in Saint-Flour, 2007. Lecture Notes in Mathematics **1974**, Springer-Verlag, 2009.
[44] F. Iglói and C. Monthus, *Strong disorder RG approach of random systems*, Phys. Rep. **412** (2005), 277-431.
[45] J. F. C. Kingman, *Subadditive Ergodic Theory*, Ann. Probab. **1** (1973), 882-909.
[46] H. Lacoin, *The martingale approach to disorder irrelevance for pinning models*, Electron. Commun. Probab. **15** (2010), 418-427.
[47] L. C. G. del Molino, P. Chleboun and S. Grosskinsky, *Condensation in randomly perturbed zero-range processes*, J. Phys. A **45** (2012), 205001.
[48] R. A. Neher and U. Gerland, *Intermediate phase in DNA melting*, Phys. Rev. E **73** (2006), 030902R.
[49] D. Poland and H. A. Scheraga, *Theory of helix-coil transitions in biopolymers;: Statistical mechanical theory of order-disorder transitions in biological macromolecules*, Academic Press, 1970.
[50] J. Rudnick and R. Bruinsma, *Effects of torsional strain on thermal denaturation of DNA*, Phys. Rev. E **65** (2002), 030902(R).
[51] F. L. Toninelli, *A replica-coupling approach to disordered pinning models*, Commun. Math. Phys. **280** (2008), 389-401.

[52] Y. Velenik, *Localization and delocalization of random interfaces*, Probab. Surv. **3** (2006), 112-169.
[53] A. Vezzani, E. Barkai and R. Burioni, *Single-big-jump principle in physical modeling*, Phys. Rev. E **100** (2019), 012108