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► **To cite this version:**

Benjamin Boutin, Anais Crestetto, Nicolas Crouseilles, Josselin Massot. Modified lawson methods for vlasov equations \*. 2022. hal-03911409

**HAL Id: hal-03911409**

**<https://hal.science/hal-03911409>**

Preprint submitted on 22 Dec 2022

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# MODIFIED LAWSON METHODS FOR VLASOV EQUATIONS\*

BENJAMIN BOUTIN<sup>†</sup>, ANAÏS CRESTETTO<sup>‡</sup>, NICOLAS CROUSEILLES<sup>§</sup>, AND JOSSELIN MASSOT<sup>¶</sup>

**Abstract.** In this work, Lawson type numerical methods are studied to solve Vlasov type equations on a phase space grid. These time integrators are known to satisfy enhanced stability properties in this context since they do not suffer from the stability condition induced from the linear part. We introduce here a class of modified Lawson integrators in which the linear part is approximated in such a way that some geometric properties of the underlying model are preserved, which has important consequences for the analysis of the scheme. Several Vlasov-Maxwell examples are presented to illustrate the good behavior of the approach.

**Key words.** Lawson methods, Vlasov equations, high order.

**MSC codes.** 35L45, 65L06, 65L07, 65M12

**1. Introduction.** In this work, we study high order Eulerian numerical methods to solve Vlasov type equations describing a system of charged particles under the influence of self-consistent electromagnetic fields. In addition to their nonlinear character, the solutions of the Vlasov equations present fine structures and strong gradients which are difficult to capture for the typical time scales used in plasma physics.

Several methods have been proposed in the past to solve efficiently kinetic problems. First, the so-called Particle-In-Cell methods have been introduced and still are extensively used [7, 48, 34, 33]. They are based on a sampling of the distribution function using macro particles which are advanced in time through the characteristics equations, whereas the electromagnetic fields are computed on a spatial grid thanks to grid-projection techniques of the macro-particles. Even if these methods are extremely efficient in high dimensions, they are known to suffer from some numerical noise which prevents from an accurate description of the unknown and which slowly decreases when the number of macro-particles increases. Second, the so-called Eulerian methods which use a grid of the phase space have been considered [10, 32, 25, 44, 2, 42, 43, 23, 49, 13, 9, 3, 15, 6]. These approaches enable the use of high order methods like spectral or finite differences/volumes methods, and as such, capture fine physical phenomena like Landau damping or filamentation. However, the stability condition relating the phase space mesh and the time step might be stringent, making these methods costly both in terms of CPU and memory to reach typical physical time scales. Another approach called the semi-Lagrangian method has been considered to take the best of the two latter approaches ([45, 25, 18, 17]). This method is Eulerian since it requires a grid of the phase space but it is Lagrangian since it follows the characteristics equation to update the unknown from one iteration to the next as such, the method does not suffer from the stability condition. How-

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\*Submitted to the editors January 2023.

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39 ever, to avoid high dimensional costly phase space reconstructions or interpolations,  
 40 they are very often combined with a time splitting. Splitting methods turn out to be  
 41 very powerful for Vlasov problems since the subsystems can often be solved exactly  
 42 in time. However, it is not always the case and high order in time requires a very  
 43 large number of stages, which of course may lead to expensive methods, in particular  
 44 when the splitting involves three or more subsystems (see [12]).

45 Recently (see [16, 15, 12, 39]), exponential time integrators have been investigated  
 46 to numerically solve Vlasov equations as an alternative to above methods. The main  
 47 observation was based on the fact that, in several Vlasov equations, the linear part  
 48 induces the most stringent CFL condition. But since, by essence, the linear part can  
 49 be potentially solved easily, a variation of constant formula can be written to pave the  
 50 way of a new class of methods where the linear part is solved exactly and the nonlinear  
 51 part is solved explicitly in time (to avoid expensive iterative solvers). Thanks to the  
 52 huge bibliography on Runge-Kutta (RK) methods, efficient high order methods can  
 53 then be obtained, whose number of stages (and thus the complexity) is only linear  
 54 with respect to the order of the time integrator. There exists two main families of  
 55 exponential integrators (see [29]) and as investigated in [15], among them, Lawson  
 56 methods [36] are preferred for Vlasov problems.

57 In this work, we investigate and analyse Lawson methods for different Vlasov  
 58 type equations. More specifically, for some complex problems, the exponential of the  
 59 linear part turns out to be difficult (even impossible) to compute, even using dedicated  
 60 softwares and some approximations are required. But due to the specific properties of  
 61 the linear part in Vlasov equations, some standard exponential approximations lead to  
 62 unstable methods. Indeed, when one considers the Vlasov equations, the linear part  
 63 has pure imaginary eigenvalues, which means that the eigenvalues of the exponential  
 64 belong to the unit circle, for any time steps and any space meshes. This important  
 65 geometric property has to be preserved by the approximation for stability reasons.  
 66 We checked that the so-called Padé approximant satisfies this property and the Padé  
 67 strategy turns out to be well adapted in our context. Then, we prove that the order  
 68 of accuracy of the so modified Lawson method (in which the exact exponential is  
 69 replaced by an approximation) becomes  $\mathcal{O}(\Delta t^{\min(m,r)})$  with  $\Delta t$  the time step,  $m$  the  
 70 order of the Lawson method and  $r$  the order of approximation of the exponential  
 71 approximation. As these methods rely on the very popular RK methods, they benefit  
 72 from the huge literature enabling to reach high order efficiently and to be combined  
 73 with adaptive time stepping strategy by using embedded RK methods.

74 In addition, for more simple problems like Vlasov-Ampère or Vlasov-Maxwell,  
 75 for which the exponential of the linear part can be computed exactly, we observe  
 76 that Lawson schemes enable to preserve the charge conservation exactly, for any  
 77 underlying RK method. Indeed, inserting the calculation of the current in the linear  
 78 part enables to solve exactly both the space transport and the Ampère equation, which  
 79 is the key point to ensure that the underlying Poisson equation is satisfied exactly  
 80 (see [44, 33, 19]). Hence, in addition to its efficiency, the versatility of exponential  
 81 integrators turns out to be an interesting aspect in our context.

82 The rest of the paper is organized as follows. First, several aspects of Lawson  
 83 methods are recalled and modified Lawson methods (where the exponential of the  
 84 matrix is approximated) is introduced. Second, some models amenable to Lawson  
 85 integrators are presented and in a third part, the Padé approximants are recalled  
 86 and some important properties in our context are proved. Then, we analyse the  
 87 convergence of the modified Lawson methods for a class of ODEs. Finally, several  
 88 numerical illustrations are discussed.

89 **2. Lawson methods.** In this section, we recall the Lawson methods and intro-  
 90 duce the modified Lawson methods. Lawson methods are a class of time integration  
 91 schemes that are applied to differential equations (which, in our context, come from  
 92 the phase space semi-discretization of Vlasov equations) of the form:

$$93 \quad (2.1) \quad \dot{u}(t) = Lu(t) + g(t, u(t)), \quad u(0) = u_0 \in \mathbb{R}^d,$$

94 where  $L$  is a  $d \times d$  matrix and  $g$  is a, in general nonlinear, function of the unknown  
 95  $u$  and the time  $t \geq 0$  whose values belong to  $\mathbb{R}^d$  (i.e. for  $d \in \mathbb{N}$ , we have  $u : t \in$   
 96  $\mathbb{R}^+ \mapsto u(t) \in \mathbb{R}^d$  and  $g : (t, u) \in \mathbb{R}^+ \times \mathbb{R}^d \mapsto \mathbb{R}^d$ ). Lawson methods are especially  
 97 efficient when applied to problems where  $L$  implies a stringent stability condition if it  
 98 is treated explicitly, which is typically the case of the Vlasov equations (see [15, 12]).

99 **2.1. Standard Lawson methods.** Solving (2.1) with the Lawson method [35]  
 100 relies on the change of variable  $v(t) := \exp(-tL)u(t)$  in (2.1) to get

$$101 \quad (2.2) \quad \dot{v}(t) = \exp(-tL)g(t, \exp(tL)v(t)).$$

Now an explicit Runge-Kutta method is applied to the transformed equation (2.2).  
 We introduce the time discretization  $t^n = n\Delta t$  with  $\Delta t > 0$  the time step,  $n \in \mathbb{N}$  and  
 $u^n$  (resp.  $v^n$ ) denotes the numerical approximation of  $u(t^n)$  (resp.  $v(t^n)$ ). For the  
 sake of simplicity, we first present the method for the explicit Euler scheme. Applying  
 the forward Euler method to (2.2) leads to

$$v(t^n + \Delta t) \approx v^{n+1} = v^n + \Delta t \exp(-t^n L)g(t^n, \exp(t^n L)v^n).$$

Reversing the change of variable yields the following scheme for  $u^n$

$$u(t^n + \Delta t) \approx u^{n+1} = \exp(\Delta t L)u^n + \Delta t \exp(\Delta t L)g(t^n, u^n).$$

102 This is the Lawson-Euler method, also a method of order one. More generally, the  
 103 Lawson method induced by an explicit Runge-Kutta method  $\text{RK}(s, m)$  with  $s$  stages  
 104 of order  $m$  can be written as

$$105 \quad (2.3) \quad \begin{aligned} u_n^{(r)} &= \exp(c_r \Delta t L)u^n + \Delta t \sum_{j=1}^{r-1} a_{r,j} \exp((c_r - c_j)\Delta t L)g(t^n + c_j \Delta t, u_n^{(j)}), \quad r = 1, \dots, s, \\ u^{n+1} &= \exp(\Delta t L)u^n + \Delta t \sum_{j=1}^s b_j \exp((1 - c_j)\Delta t L)g(t^n + c_j \Delta t, u_n^{(j)}), \end{aligned}$$

106 with the underlying explicit Runge-Kutta method defined by its Butcher tableau. The  
 107 method defined by (2.3) will be denoted by  $\text{LRK}(s, m)$ , which is the Lawson method  
 108 induced by the  $\text{RK}(s, m)$  Runge-Kutta method.

109 **2.2. Modified Lawson methods.** For many problems,  $\exp(tL)$  may be diffi-  
 110 cult, costly or even impossible to compute formally for all the involved parameters  
 111 and approximations may be required (see [8, 22, 24, 40]). In the sequel we will de-  
 112 note by  $\text{exp}(tL)$  an approximation of  $\exp(tL)$  and introduce  $\tilde{u}^n$  an approximation of  
 113  $u(t^n)$  for  $n \in \mathbb{N}$  generated by the modified Lawson method. The modified Lawson  
 114 M-LRK( $s, m$ ) method (of order  $m$  with  $s$  stages) is simply obtained from (2.3) where

115 the matrix exponentials  $\exp(tL)$  is approximated by its approximation  $\mathbf{exp}(tL)$  to get  
 (2.4)

$$\begin{aligned} \tilde{u}_n^{(r)} &= \mathbf{exp}(c_r \Delta t L) \tilde{u}^n + \Delta t \sum_{j=1}^{r-1} a_{r,j} \mathbf{exp}((c_r - c_j) \Delta t L) g(t^n + c_j \Delta t, \tilde{u}_n^{(j)}), \quad r = 1, \dots, s, \\ \tilde{u}^{n+1} &= \mathbf{exp}(\Delta t L) \tilde{u}^n + \Delta t \sum_{j=1}^s b_j \mathbf{exp}((1 - c_j) \Delta t L) g(t^n + c_j \Delta t, \tilde{u}_n^{(j)}), \end{aligned}$$

116  
 117 where the coefficients are given by the Butcher tableau.

118 **3. Some Vlasov models amenable to Lawson methods.** In this section,  
 119 we present some Vlasov type models that enter in the framework presented in the  
 120 previous section. We focus here on Eulerian methods which consider a grid of the  
 121 phase space to solve the Vlasov equations. For the different Vlasov models we consider,  
 122 a reformulation into the form (2.1) is proposed so that Lawson or modified Lawson  
 123 methods can be employed. We discuss the eigenvalues of the linear part  $L$ . Indeed,  
 124 in our case  $L$  has a special structure since it has a pure imaginary spectrum which  
 125 has important consequences for the stability of the Lawson schemes studied here.

**3.1. Vlasov-Ampère.** The Vlasov-Ampère equation for the distribution function  $f(t, x, v) \geq 0$  and the electric field  $E(t, x) \in \mathbb{R}$  with  $x \in [0, b]$  and  $v \in \mathbb{R}$  is

$$\frac{\partial f}{\partial t} + v \partial_x f + E \partial_v f = 0, \quad \frac{\partial E}{\partial t} = - \int_{\mathbb{R}} v f dv + \frac{1}{b} \int_0^b \int_{\mathbb{R}} v f dv dx,$$

126 with the initial conditions  $(f_0, E_0)$  such that the Poisson equation is satisfied initially  
 127 (i.e.  $\partial_x E_0 = \int_{\mathbb{R}} f_0 dv - 1$ ) and periodic boundary conditions are imposed in space.

128 Let us detail the numerical strategy in this case. We first semi-discretized the  
 129 phase space domain  $(x, v)$ : we have chosen spectral method in space and high order  
 130 finite differences in velocity. The velocity mesh is  $v_\ell = -v_{\max} + (\ell-1)\Delta v$ ,  $\ell = 1, \dots, N_v$   
 131 where  $v_{\max} > 0$  is the velocity domain truncation and  $\Delta v = (2v_{\max})/(N_v - 1)$  is the  
 132 velocity mesh size. Then, we denote  $\vec{\hat{f}}_k(t) = (\hat{f}_{k,1}, \hat{f}_{k,2}, \dots, \hat{f}_{k,N_v})(t)$  the semi-discrete  
 133 unknown for  $f$  and  $\hat{E}_k(t)$  the semi-discrete unknown for  $E$ ,  $k = 0, \dots, N_x - 1$  ( $N_x$   
 134 being the number of points of the space mesh):

(3.1)

$$135 \quad \hat{f}_{k,\ell}(t) = \frac{1}{b} \int_0^b \exp(-i2\pi kx/b) f(t, x, v_\ell) dx, \quad \hat{E}_k(t) = \frac{1}{b} \int_0^b \exp(-i2\pi kx/b) E(t, x) dx.$$

136 We will introduce the following notations:  $\vec{v} = (v_1, v_2, \dots, v_{N_v}) \in \mathbb{R}^{N_v}$  for the mesh  
 137 in velocity and  $\mathbf{0} = (0, 0, \dots, 0) \in \mathbb{R}^{N_v}$ . We then get

$$138 \quad (3.2) \quad \frac{d}{dt} \begin{pmatrix} \vec{\hat{f}}_k(t) \\ \hat{E}_k(t) \end{pmatrix} + \begin{pmatrix} ik \frac{2\pi}{b} \text{diag}(\vec{v}) & \mathbf{0}^T \\ \Delta v \vec{v} & 0 \end{pmatrix} \begin{pmatrix} \vec{\hat{f}}_k(t) \\ \hat{E}_k(t) \end{pmatrix} + \begin{pmatrix} \widehat{[(ED\vec{f})]}_k \\ 0 \end{pmatrix} = 0,$$

with  $\text{diag}(\vec{v})$  is the diagonal matrix of size  $N_v$  with  $v_1, v_2, \dots, v_{N_v}$  on its diagonal  
 and  $(ED\vec{f}) \in \mathbb{R}^{N_v}$  is a finite difference approximation of  $E \partial_v f$  on the velocity mesh.  
 The (Fourier coefficient of the) current in the Ampère equation is approximated by  
 a standard quadrature  $\sum_{\ell=1}^{N_v} v_\ell \hat{f}_{k,\ell}(t) \Delta v$ . We can then reformulate (3.2) under the  
 following form with  $U_k(t) = (\vec{\hat{f}}_k(t), \hat{E}_k(t)) \in \mathbb{R}^{N_v+1}$

$$\frac{d}{dt} U_k(t) = L U_k(t) + N(U_k(t)),$$

139 which is amenable to Lawson schemes. Moreover,  $L$  is diagonalizable  $L = PDP^{-1}$   
 140 with a pure imaginary spectrum  $\text{Sp}(L) = \{i(2\pi k/b)\vec{v}\}$  and  $\text{cond}(P)$  is uniformly  
 141 bounded with respect to  $N_v$  and  $N_x$ .

142 Let us remark that this approach is different from the one proposed in [15, 16]  
 143 since here the Ampère equation is involved in the linear part whereas the Poisson  
 144 equation was updated at each Runge-Kutta stage in [15, 16]. We believe that the  
 145 current approach is more amenable to the Vlasov-Maxwell equation; moreover, as we  
 146 shall see in Section 6, some properties can be proved using this approach.

**3.2. Linearized hybrid fluid-kinetic model.** The second example considered  
 here is a hybrid model introduced in [46, 47, 30]. In this model, the case of a wave  
 propagation parallel to a uniform magnetic field  $\mathbf{B}_0 = (0, 0, B_0)^T$ ,  $B_0 > 0$  is consid-  
 ered so that the problem becomes one dimensional in space (called  $z$ ) but the three  
 dimensions in velocity are kept:  $\mathbf{v} = (v_x, v_y, v_z) \in \mathbb{R}^3$ . Thus, under some assump-  
 tions (see [30, 12]), a hot/cold decomposition is performed, and after a cold plasma  
 approximation for the cold electron population, the following decomposition of the  
 distribution function is considered:

$$f(t, z, \mathbf{v}) = \rho_c(z)\delta(\mathbf{v} - \mathbf{u}_c(t, z)) + f_h(t, z, \mathbf{v}),$$

with  $\mathbf{j}_c(t, z) = \rho_c(z)\mathbf{u}_c(t, z) = (j_{c,x}, j_{c,y}, 0)(t, z) \in \mathbb{R}^3$  denotes the current,  $\mathbf{u}_c(t, z) \in$   
 $\mathbb{R}^3$  the mean velocity and  $\rho_c(z) \in \mathbb{R}$  the density of the cold particles population.  
 Hence, the linearized hybrid fluid-kinetic model satisfied by

$$U(t, z, \mathbf{v}) = (f_h(t, z, \mathbf{v}), E_x(t, z), E_y(t, z), B_x(t, z), B_y(t, z), j_{c,x}(t, z), j_{c,y}(t, z)),$$

147 can be derived (see [30, 12] for more details). The model is normalized as in [30, 12]  
 148 and reads as ( $\Omega_{pe}$  denotes the ratio between the plasma and cyclotronic frequencies)

$$149 \quad (3.3) \quad \frac{\partial j_{c,x}}{\partial t} = \Omega_{pe}^2 E_x - j_{c,y} B_0, \quad \frac{\partial j_{c,y}}{\partial t} = \Omega_{pe}^2 E_y + j_{c,x} B_0,$$

$$150 \quad (3.4) \quad \frac{\partial B_x}{\partial t} = \partial_z E_y, \quad \frac{\partial B_y}{\partial t} = -\partial_z E_x,$$

$$151 \quad (3.5) \quad \frac{\partial E_x}{\partial t} = -\partial_z B_y - j_{c,x} + \int_{\mathbb{R}^3} v_x f_h \, d\mathbf{v}, \quad \frac{\partial E_y}{\partial t} = \partial_z B_x - j_{c,y} + \int_{\mathbb{R}^3} v_y f_h \, d\mathbf{v},$$

$$152 \quad (3.6) \quad \frac{\partial f_h}{\partial t} + v_z \partial_z f_h - \mathcal{F} \cdot \nabla_{\mathbf{v}} f_h = 0,$$

154 with  $\mathcal{F}(t, z, v_x, v_y, v_z) = (E_x + v_y B_0 - v_z B_y, E_y - v_x B_0 + v_z B_x, v_x B_y - v_y B_x)$ .

155 As in the previous case, we use a Fourier discretization in the space  $z$  direction  
 156 so that the unknown is  $U_k(t) = (\hat{j}_{c,x}, \hat{j}_{c,y}, \hat{B}_x, \hat{B}_y, \hat{E}_x, \hat{E}_y, \hat{f}_h)_k$  and satisfies  $\partial_t U_k =$   
 157  $LU_k + N(t, U_k)$  with

$$158 \quad (3.7) \quad L = \begin{pmatrix} 0 & -B_0 & 0 & 0 & \Omega_{pe}^2 & 0 & 0 \\ B_0 & 0 & 0 & 0 & 0 & \Omega_{pe}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & ik & 0 \\ 0 & 0 & 0 & 0 & -ik & 0 & 0 \\ -1 & 0 & 0 & -ik & 0 & 0 & 0 \\ 0 & -1 & ik & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -v_z ik \end{pmatrix}, \quad N : (t, U) \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \int_{\mathbb{R}^3} v_x (\hat{f}_h)_k \, d\mathbf{v} \\ \int_{\mathbb{R}^3} v_y (\hat{f}_h)_k \, d\mathbf{v} \\ [\mathcal{F} \cdot \nabla_{\mathbf{v}} \hat{f}_h]_k \end{pmatrix}.$$

159 For the velocity space, we use finite differences methods (WENO type). Moreover,  
 160  $L$  is diagonalizable  $L = PDP^{-1}$  with a pure imaginary spectrum and  $\text{cond}(P)$  is  
 161 uniformly bounded with respect to  $N_x, N_v$ .

162 **4. Padé approximant.** In this section we recall the Padé approximant to ap-  
 163 proximate exponential of matrices. Obviously, many other approximations are possi-  
 164 ble (see [41]) but the Padé approximant enjoys some properties that have important  
 165 consequences for the numerical approximation of the Vlasov models presented in the  
 166 previous section in terms of stability of the resulting numerical schemes.

167 The Padé method turns out to be a very popular technique to approximate ex-  
 168ponential of matrices (see for instance [1, 38, 4]). For a given matrix  $L$ ,  $t \in \mathbb{R}$  and  
 169  $(p, q) \in \mathbb{N}^2$ , the Padé approximant is a rational function defined by

$$170 \quad (4.1) \quad P_{p,q}(tL) = h_{p,q}(tL) (k_{p,q}(tL))^{-1},$$

the numerator and denominator being polynomial functions of degree  $p$  and  $q$

$$h_{p,q}(tL) = \sum_{\ell=0}^p \frac{p!}{(p-\ell)!} \frac{(tL)^\ell}{(p+q-\ell)!}, \quad k_{p,q}(tL) = \sum_{\ell=0}^q (-1)^\ell \frac{q!}{(q-\ell)!} \frac{(tL)^\ell}{(p+q-\ell)!}.$$

171 From [1, 41], the error between the exponential of the matrix and the Padé approxi-  
 172 mant satisfies (for a given matrix norm  $\|\cdot\|$ )

$$173 \quad (4.2) \quad \|\exp(tL) - P_{p,q}(tL)\| \leq Ct^{p+q+1}.$$

174 The choice of the Padé approximant is motivated in our case by the following  
 175 proposition.

176 **PROPOSITION 4.1.** *Let  $L$  be a diagonalizable matrix of size  $d \in \mathbb{N}^*$  (there exist  
 177 an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $L = PDP^{-1}$ ) such that  
 178  $\text{Sp}(L) \subset i\mathbb{R}$  and let assume there exists a constant  $C > 0$  such that  $\text{cond}(P) \leq C$ .  
 179 Then, for  $p \in \mathbb{N}$ , the eigenvalues of the Padé approximant  $P_{p,p}(tL)$  of  $\exp(tL)$  belongs  
 180 to the unit circle  $\mathcal{C}(0, 1)$ , i.e.,  $\forall \lambda_j \in \text{Sp}(P_{p,p}(tL))$ , one has  $|\lambda_j| = 1$ .  
 181 Moreover, for any  $t \in \mathbb{R}$  and for any  $n \in \mathbb{N}$*

$$182 \quad (4.3) \quad \|(P_{p,p}(tL))^n\| \leq C.$$

183 *Proof.* Because  $L$  is diagonalizable, there exists an invertible matrix  $P$  such that  
 184  $L = PDP^{-1}$  with  $D$  a diagonal matrix. Since  $\text{Sp}(L) \subset i\mathbb{R}$ , the diagonal terms of  $D$   
 185 can be written as  $D_{j,j} = i\alpha_j$ ,  $\alpha_j \in \mathbb{R}$ ,  $j = 1, \dots, d$ . Computing powers of  $L$  thus  
 186 reduces to look at the power of  $D_{j,j}$ .

When  $p = q$  the Padé approximant  $P_{p,p}(tL)$  can be written as

$$P_{p,p}(tL) = \left( \sum_{\ell=0}^p a_{\ell,p} \frac{(tL)^\ell}{\ell!} \right) \left( \sum_{\ell=0}^p (-1)^\ell a_{\ell,p} \frac{(tL)^\ell}{\ell!} \right)^{-1},$$

with  $a_{\ell,p} = \frac{p!}{(2p-\ell)!}$ . We rewrite the Padé approximant  $P_{p,p}(tL)$  as

$$\begin{aligned} P_{p,p}(tL) &= \left( P \left( \sum_{\ell=0}^p a_{\ell,p} \frac{(tD)^\ell}{\ell!} \right) P^{-1} \right) \left( P \left( \sum_{\ell=0}^p (-1)^\ell a_{\ell,p} \frac{(tD)^\ell}{\ell!} \right) P^{-1} \right)^{-1} \\ &= P \left( \sum_{\ell=0}^p a_{\ell,p} \frac{(tD)^\ell}{\ell!} \right) \left( \sum_{\ell=0}^p (-1)^\ell a_{\ell,p} \frac{(tD)^\ell}{\ell!} \right)^{-1} P^{-1} \equiv P \tilde{D} P^{-1}, \end{aligned}$$

where  $\tilde{D}$  is still a diagonal matrix, as a product of two diagonal matrices. The diagonal terms of  $\tilde{D}_{j,j}$  ( $j = 1, \dots, d$ ) of  $\tilde{D}$  thus writes

$$\begin{aligned} \tilde{D}_{j,j} &= \left( \sum_{\ell=0}^p i^\ell a_{\ell,p} \frac{(t\alpha_j)^\ell}{\ell!} \right) \left( \sum_{\ell=0}^p (-1)^\ell i^\ell a_{\ell,p} \frac{(t\alpha_j)^\ell}{\ell!} \right)^{-1} \\ &= \left( \sum_{\ell=0}^{\lfloor \frac{p}{2} \rfloor} (-1)^\ell a_{\ell,p} \frac{(t\alpha_j)^{2\ell}}{2\ell!} + i \sum_{\ell=0}^{\lfloor \frac{p}{2} \rfloor - 1} (-1)^\ell a_{\ell,p} \frac{(t\alpha_j)^{2\ell+1}}{(2\ell+1)!} \right) \\ &\quad \left( \sum_{\ell=0}^{\lfloor \frac{p}{2} \rfloor} (-1)^\ell a_{\ell,p} \frac{(t\alpha_j)^{2\ell}}{2\ell!} - i \sum_{\ell=0}^{\lfloor \frac{p}{2} \rfloor - 1} (-1)^\ell a_{\ell,p} \frac{(t\alpha_j)^{2\ell+1}}{(2\ell+1)!} \right)^{-1} = \lambda_j^{p,+} / \lambda_j^{p,-}. \end{aligned}$$

187 Observing  $\lambda_j^{p,+} = \overline{\lambda_j^{p,-}}$ ,  $\forall j = 1, \dots, d$  enables to deduce that  $\text{Sp}(P_{p,p}(tL)) \subset \mathcal{C}(0, 1)$ .  
 188 Now let  $t \in \mathbb{R}$  and  $n \in \mathbb{N}$  and observe that we get similarly  $(P_{p,p}(tL))^n = P\tilde{D}^n P^{-1}$ ,  
 189 where  $\tilde{D}^n$  is again a diagonal matrix with diagonal terms that belongs to the unit  
 190 circle, thus a unitary matrix. Therefore  $\|\tilde{D}^n\| = 1$  and the required upper bound (4.3)  
 191 follows with the constant  $C = \text{cond}(P) = \|P\| \|P^{-1}\|$ .  $\square$

192 We end up this section by making some remarks.

*Remark 4.2.* For the case  $p \neq q$ , it is possible to write down the euclidean division of  $h_{p,q} \in \mathbb{R}^p[iX]$  by  $k_{p,q} \in \mathbb{R}^q[iX]$ . There exist  $Q_{p-q} \in \mathbb{R}^{p-q}[iX]$  and  $R \in \mathbb{R}^\ell[iX]$  with  $\ell < q$  such that

$$h_{p,q}(z) = k_{p,q}(z)Q_{p-q}(z) + R(z).$$

Hence, we have

$$P_{p,q}(z) = \frac{h_{p,q}(z)}{k_{p,q}(z)} = Q_{p-q}(z) + \frac{R(z)}{k_{p,q}(z)}.$$

193 In the examples presented in Section 3, the argument  $z$  is a pure imaginary number  
 194 such that  $|z| \rightarrow +\infty$  when  $N_x \rightarrow +\infty$ . Hence, when  $p > q$ , one directly obtains  
 195 that  $|Q_{p-q}(z)| \rightarrow +\infty$  as  $|z| \rightarrow +\infty$  which leads to severe stability issues making this  
 196 case usefulness in practice. A specific example is  $P_{2,1}(z)$  for which we can prove that  
 197  $|P_{2,1}(z)| > 1$  for all  $z \in i\mathbb{R}^*$ . We will present a numerical illustration in Section 7.

198 The case  $p < q$  turns out to be more complicated. Indeed, if  $|p-q|$  is large enough  
 199 ( $|p-q| > 2$  typically), instabilities are observed in the sense that (4.3) is not satisfied.  
 200 Indeed, we investigate numerically the limit when  $n \rightarrow +\infty$  of  $(P_{p,q}(zT/n))^n$  for a  
 201 fixed final time  $T$  and with  $z \in i\mathbb{R}$  such that  $|z|$  is large. First, we observe numerically  
 202 that even if  $|P_{p,q}(z)| \rightarrow 0$  as  $|z| \rightarrow +\infty$ , some values of  $|P_{p,q}(z)|$  are larger than 1,  
 203 thus we observe that  $|(P_{p,q}(zT/n))^n| \rightarrow +\infty$  as  $n \rightarrow \infty$  for a fixed final time  $T = 10$ .  
 204 Finally, still for the case  $p < q$  but when  $1 \leq |p-q| \leq 2$ , we observed that the method  
 205 is stable  $|P_{p,q}(z)| \leq 1, \forall z \in i\mathbb{R}$ .

*Remark 4.3.* Another simple approximation is the Taylor expansion of the exponential, which corresponds to a Padé approximant whose degree denominator  $q$  is zero:

$$\exp(tL) \approx P_{p,0}(tL) = \sum_{\ell=0}^p \frac{(tL)^\ell}{\ell!},$$

206 for which a similar error estimate can be easily derived  $\|\exp(tL) - P_{p,0}(tL)\| \leq Ct^{p+1}$   
 207 but, in that case, as discussed in the previous remark, stability issues are observed.



208 **5. Modified Lawson methods.** In this section, we perform a numerical analy-  
 209 sis of the modified Lawson schemes, in which the exponential of the matrix  $\exp(tL)$   
 210 is approximated by  $\mathbf{exp}(tL)$  verifying for all  $t \in \mathbb{R}$  and all  $n \in \mathbb{N}$ :  $\|(\mathbf{exp}(tL))^n\| \leq C$   
 211 (which is the case for the Padé approximant due to (4.3)). To do so, we consider  
 212 the following system of ODEs (where  $g$  is a smooth function which is assumed to not  
 213 depend on  $t$  for simplicity)

$$214 \quad (5.1) \quad \dot{u}(t) = Lu(t) + g(u(t)), \quad u(0) = u_0 \in \mathbb{R}^d.$$

215 General explicit RK schemes with modified Lawson method are considered latter on,  
 216 but we first give a proof of convergence for the following first order scheme

$$217 \quad (5.2) \quad \tilde{u}^{n+1} = \mathbf{exp}(\Delta t L)\tilde{u}^n + \Delta t \mathbf{exp}(\Delta t L)g(\tilde{u}^n), \quad u^0 = u_0 \in \mathbb{R}^d.$$

218 We perform the following assumptions on the nonlinearity  $g$  and the linear part  $L$ .

219 **ASSUMPTION 5.1. Assumptions on  $g$**

- 220 •  $g$  is a locally Lipschitz continuous function from  $\mathbb{R}^d$  to  $\mathbb{R}^d$ ,
- 221 •  $g'$  is continuous and locally bounded from  $\mathbb{R}^d$  to  $\mathbb{R}^d$ .

222 These assumptions ensure that (5.1) has a unique solution  $u$  defined on some interval  
 223  $[0, T]$ . From the property on  $g$ , we deduce  $\|g(u(t))\| \leq C(\|u(t)\| + 1), \forall t \in [0, T]$   
 224 where the constant  $C > 0$  may depend on  $T$ , or say on a given compact set of  $\mathbb{R}^d$  that  
 225 contains the range  $u([0, T])$  and  $\{0\}$ .

226 *Remark 5.1.* A more tractable assumption would be to consider a global Lipschitz  
 227 continuous function  $g$  so that the above estimate is still available but now at any point  
 228  $u \in \mathbb{R}^d$  with a uniform constant  $C$ :  $\|g(u)\| \leq C(\|u\| + 1), \forall u \in \mathbb{R}^d$ . In order to  
 229 lighten the next proofs, we will use that global inequality and discuss in a subsequent  
 230 remark how to adapt the proofs to the local Lipschitz case only.

231 **ASSUMPTION 5.2. Assumptions on  $L$**

- 232 •  $L$  is conjugated to a skew Hermitian matrix:  $L = PHP^{-1}$  ( $H^* = -H$ ),
- 233 •  $\exists C > 0$  such that  $\text{cond}(P) \equiv \|P\|\|P^{-1}\| \leq C$ .

234 In the proofs, we will use the following version of the Gronwall lemma.

235 **LEMMA 5.2 (Gronwall Lemma).** *Let  $(a_n), (b_n)$  be two nonnegative sequences and*  
 236 *a constant  $M \geq 0$  satisfying  $a_{n+1} \leq M + \sum_{k=0}^n a_k b_k$  for any natural number  $n$ , then*  
 237  *$a_{n+1} \leq M \exp\left(\sum_{k=0}^n b_k\right)$ .*

238 **5.1. First order.** In this subsection, we prove the convergence of the first order  
 239 modified Lawson methods introduced above. For the first order scheme (5.2), we are  
 240 able to prove the following theorem.

**THEOREM 5.3.** *Under the assumptions 5.1 and 5.2 on  $g$  and  $L$  stated above and*  
*with  $\mathbf{exp}(\Delta t L)$  such that  $\|\exp(\Delta t L) - \mathbf{exp}(\Delta t L)\| \leq C\Delta t^{r+1}$ , the first order modified*  
*Lawson method (5.2) satisfies the following error estimate*

$$\|u(t^n) - \tilde{u}^n\| \leq C\Delta t^{\min(1, r)}, \quad \forall n \leq N, \quad \text{with } N \text{ such that } N\Delta t \leq T.$$

241 *Proof.* The first order modified Lawson scheme writes

$$242 \quad (5.3) \quad \tilde{u}^{n+1} = \mathbf{exp}(\Delta t L)\tilde{u}^n + \Delta t \mathbf{exp}(\Delta t L)g(\tilde{u}^n),$$

243 and by induction, it comes

$$244 \quad (5.4) \quad \tilde{u}^n = (\mathbf{exp}(\Delta t L))^n \tilde{u}^0 + \Delta t \sum_{k=0}^{n-1} (\mathbf{exp}(\Delta t L))^{n-k} g(\tilde{u}^k).$$

One thus deduces that  $\tilde{u}^n$  is bounded by considering the norm of (5.4) to get

$$\|\tilde{u}^n\| \leq C\|\tilde{u}^0\| + C\Delta t \sum_{k=0}^{n-1} (\|\tilde{u}^k\| + 1),$$

245 and we conclude thanks to the Gronwall lemma. In comparison with (5.3), the exact  
246 solution writes

$$\begin{aligned} 247 \quad u(t^{n+1}) &= \exp(\Delta t L)u(t^n) + \Delta t \int_{t^n}^{t^{n+1}} e^{(t^{n+1}-t)L} g(u(t)) dt \\ 248 \quad (5.5) \quad &= \exp(\Delta t L)u(t^n) + \Delta t e^{\Delta t L} g(u(t^n)) + \mathbf{r}^n, \end{aligned}$$

249 where  $\mathbf{r}^n$  denotes the local truncation error

$$\begin{aligned} 250 \quad \mathbf{r}^n &= \int_{t^n}^{t^{n+1}} (t - t^n) \int_0^1 \frac{d}{dt} (e^{(t^{n+1}-t)L} g(u(t)))_{t=t^n+\sigma(t-t^n)} d\sigma dt \\ 251 \quad &= \int_{t^n}^{t^{n+1}} (t - t^n) \int_0^1 \exp(\Delta t - \sigma(t - t^n)) \left[ -Lg(u(\cdot)) + g'(u(\cdot)) \frac{du}{dt}(\cdot) \right]_{t^n+\sigma(t-t^n)} d\sigma dt. \end{aligned}$$

252 Defining the local error as  $e^{n+1} = u(t^{n+1}) - \tilde{u}^{n+1}$  and considering the difference of  
253 the two above equations (5.3) and (5.5), one obtains

$$\begin{aligned} 254 \quad e^{n+1} &= \exp(\Delta t L)u(t^n) - \mathbf{exp}(\Delta t L)\tilde{u}^n + \Delta t \left[ \exp(L\Delta t)g(u(t^n)) - \mathbf{exp}(\Delta t L)g(\tilde{u}^n) \right] + \mathbf{r}^n \\ 255 \quad &= \exp(\Delta t L)e^n + \Delta t \exp(\Delta t L) \left[ g(u(t^n)) - g(\tilde{u}^n) \right] \\ 256 \quad &+ (\exp(\Delta t L) - \mathbf{exp}(\Delta t L)) \left[ \tilde{u}^n + \Delta t g(\tilde{u}^n) \right] + \mathbf{r}^n. \end{aligned}$$

By induction, we deduce from there

$$e^n = \exp(n\Delta t L)e^0 + \Delta t \sum_{k=0}^{n-1} \exp((n-k)\Delta t L)\delta_k + \sum_{k=0}^{n-1} \exp((n-1-k)\Delta t L)(\gamma_k + \mathbf{r}^k),$$

257 with  $\delta_k = \left[ g(u(t^k)) - g(\tilde{u}^k) \right]$ ,  $\gamma_k = (\exp(\Delta t L) - \mathbf{exp}(\Delta t L)) \left[ \tilde{u}^k + \Delta t g(\tilde{u}^k) \right]$ . First,  
258 the local truncation error satisfies  $\|\mathbf{r}^n\| \leq C\Delta t^2$ . Second, using the upper bound  
259  $\|\exp(Lt)\| \leq C, \forall t \in [0, T]$  and the Lipschitz bound  $\text{Lip}(g)$  on  $g$ , we deduce  
260  $\|\exp((n-k)\Delta t L)\delta_k\| \leq C\text{Lip}(g)\|e^k\|$ . Finally, using the assumption  $\|\exp(\Delta t L) -$   
261  $\mathbf{exp}(\Delta t L)\| \leq C\Delta t^{r+1}$  together with  $\|\tilde{u}^n\| \leq C, \forall n \leq N$  we get

$$\begin{aligned} 262 \quad \|e^n\| &\leq \|\exp(n\Delta t L)\| \|e^0\| + C\text{Lip}(g)\Delta t \sum_{k=0}^{n-1} \|e^k\| + Cn(\Delta t^2 + \Delta t^{r+1}) \\ 263 \quad &\leq C\text{Lip}(g)\Delta t \sum_{k=0}^{n-1} \|e^k\| + CT(\Delta t + \Delta t^r). \end{aligned}$$

264 We can conclude using again the Gronwall lemma 5.2.

$$265 \quad (5.6) \quad \|e^n\| \leq CT(\Delta t + \Delta t^r) \exp\left(C\text{Lip}(g) \sum_{k=0}^{n-1} \Delta t\right),$$

266 whence finally  $\|e^n\| \leq C\Delta t^{\min(1,r)}$ , provided that  $0 \leq n\Delta t \leq T$ .  $\square$

267 *Remark 5.4* (Local Lipschitz continuity). When considering a only local Lipschitz  
 268 continuous function  $g$ , the previous proof has to be slightly adapted, proving the  
 269 boundedness of the numerical solution within its convergence. For the sake of com-  
 270 pleteness, we give hereafter a short proof in that case for the first order modified  
 271 Lawson method. Actually, the idea of that proof generalizes without difficulty for  
 272 high order modified Lawson methods discussed later on but then without such obser-  
 273 vations.

274 Let us consider a solution  $t \mapsto u(t)$  defined for  $t \in [0, T]$  and introduce the following  
 275 compact set in  $\mathbb{R}^d$ :  $K_T := \bigcup_{t \in [0, T]} \overline{B(u(t), 1)}$  which contains the range of the exact  
 276 solution. Let us denote now  $\text{Lip}(g) > 0$  the Lipschitz constant of  $g$  on  $K_T$ . The above  
 277 analysis is available to get (5.6), provided that any of the terms  $(\tilde{u}_k)_{0 \leq k \leq n}$  belongs  
 278 to  $K_T$ . Now to prove that, we proceed by induction on the integer  $n$  and adjust a  
 279 priori the value  $\Delta t$  to be sufficiently small. To that aim, from the definition of  $K_T$   
 280 the following inequality is sufficient:

$$281 \quad 2CT\Delta t \exp(C\text{Lip}(g)T) < 1.$$

282 **5.2. Arbitrary order and stage.** In this subsection, we extend the convergence  
 283 of the modified Lawson Runge-Kutta method to the general case (order  $m$  and  $s$   
 284 stages). Of course, additional regularity assumptions have to be performed on  $g$  to  
 285 ensure the convergence of the Lawson Runge-Kutta method, *i.e.*

286 **ASSUMPTION 5.3. Assumptions on  $g$**

- 287 •  $g$  is a locally Lipschitz continuous function from  $\mathbb{R}^d$  to  $\mathbb{R}^d$ ,
- 288 •  $g^{(p)}$ ,  $p \leq m + 1$  is continuous and locally bounded from  $\mathbb{R}^d$  to  $\mathbb{R}^d$ .

289 The same assumptions as in the first order case are performed on  $g$  and  $L$ .

290 Thus we have the following theorem.

**THEOREM 5.5.** *Under assumptions 5.3 and 5.2 on  $g$  and  $L$  stated above and with  $\text{exp}(\Delta t L)$  such that  $\|\text{exp}(\Delta t L) - \text{exp}(\Delta t L)\| \leq C\Delta t^{r+1}$ , the modified Lawson method  $M\text{-LRK}(s, m)$  (2.4) of order  $m$  with  $s$  stages satisfies the following error estimate*

$$\|u(t^n) - \tilde{u}^n\| \leq C\Delta t^{\min(m, r)}, \quad \forall n \leq N, \text{ with } N \text{ such that } N\Delta t \leq T.$$

291 *Proof.* In a first step, we prove the boundedness of the solution  $\tilde{u}^n$ . For a given  
 292 integer  $n$ , we first consider the stages iterations for  $1 \leq i \leq s$ , defined from

$$293 \quad \tilde{u}_n^{(i)} = \text{exp}(c_i \Delta t L) \tilde{u}^n + \Delta t \sum_{j=1}^{i-1} a_{ij} \text{exp}((c_i - c_j) \Delta t L) g(\tilde{u}_n^{(j)}).$$

294 From the bound (4.3) and by iteration on  $i$ , we obtain the following estimate

$$295 \quad (5.7) \quad \|\tilde{u}_n^{(i)}\| \leq C(\|\tilde{u}^n\| + \Delta t).$$

296 Now let us introduce  $F_n = \sum_{j=1}^s b_j \text{exp}((1 - c_j) \Delta t L) g(\tilde{u}_n^{(j)})$  so that the global iteration  
 297 over  $u^n$  solves as

$$298 \quad \tilde{u}^n = \text{exp}(\Delta t L)^n \tilde{u}^0 + \Delta t \sum_{k=0}^{n-1} (\text{exp}(\Delta t L))^{n-1-k} F_k.$$

299 Thanks to the assumptions on  $g$  and (5.7), we obtain the following inequality  $\|F_n\| \leq$   
 300  $C(\|\tilde{u}^n\| + 1)$  and thus  $\|\tilde{u}^n\| \leq C(\|\tilde{u}^0\| + T + \Delta t \sum_{k=0}^{n-1} \|\tilde{u}^k\|)$ . Finally from the Gronwall

301 lemma 5.2 we deduce the estimate on  $\tilde{u}^n$

$$302 \quad (5.8) \quad \|\tilde{u}^n\| \leq C(\|\tilde{u}^0\| + T) \exp(CT).$$

303 Next, we consider the error estimate and introduce  $u^n$  the numerical solution  
304 generated by the Lawson method RK( $m, s$ ) (order  $m$  and  $s$  stages) for which the  
305 following error estimates is known to hold (see Theorem 4.7 in [28])

$$306 \quad (5.9) \quad \|u(t^n) - u^n\| \leq C\Delta t^m.$$

Thus, from the following decomposition of the error

$$e^n = u(t^n) - \tilde{u}^n = u(t^n) - u^n + u^n - \tilde{u}^n = u(t^n) - u^n + \mathbf{\epsilon}^n,$$

307 we have to estimate the difference  $\mathbf{\epsilon}^n = u^n - \tilde{u}^n$  between the solution of the classical  
308 Lawson Runge-Kutta scheme  $(u^n)_n$  and the one from the modified Lawson Runge-  
309 Kutta scheme  $(\tilde{u}^n)_n$ . One can write

$$\begin{aligned} 310 \quad \mathbf{\epsilon}^{n+1} &= u^{n+1} - \tilde{u}^{n+1} \\ 311 \quad &= \exp(\Delta t L)u^n - \mathbf{exp}(\Delta t L)\tilde{u}^n \\ 312 \quad &\quad + \Delta t \sum_{j=1}^s b_j \left[ \exp((1 - c_j)\Delta t L)g(u_n^{(j)}) - \mathbf{exp}((1 - c_j)\Delta t L)g(\tilde{u}_n^{(j)}) \right] \\ 313 \quad &= \exp(\Delta t L)\mathbf{\epsilon}^n + \Delta t \sum_{j=1}^s b_j \left[ \exp((1 - c_j)\Delta t L)(g(u_n^{(j)}) - g(\tilde{u}_n^{(j)})) \right] \\ 314 \quad &\quad + \gamma(1)\tilde{u}^n + \Delta t \sum_{j=1}^s b_j \gamma(1 - c_j)g(\tilde{u}_n^{(j)}) \end{aligned}$$

316 where we introduced  $\gamma(z) = \exp(z\Delta t L) - \mathbf{exp}(z\Delta t L)$ . By induction, we get

$$\begin{aligned} 317 \quad \mathbf{\epsilon}^n &= \exp(n\Delta t L)\mathbf{\epsilon}^0 \\ 318 \quad &\quad + \sum_{k=0}^{n-1} \exp(\Delta t L(n - 1 - k)) \left( \Delta t \sum_{j=1}^s b_j \left[ \exp((1 - c_j)\Delta t L)(g(u_k^{(j)}) - g(\tilde{u}_k^{(j)})) \right] \right) \\ 319 \quad (5.10) \quad &\quad + \gamma(1)\tilde{u}^n + \Delta t \sum_{j=1}^s b_j \gamma(1 - c_j)g(\tilde{u}_k^{(j)}). \end{aligned}$$

320 As we can see in (5.10), the error  $\mathbf{\epsilon}_k^{(j)} = u_k^{(j)} - \tilde{u}_k^{(j)}$  ( $j = 1, \dots, s$ ) of the internal stages  
321 of the Lawson and modified Lawson methods has to be estimated. From the definition  
322 of the two Lawson schemes (2.3) and (2.4), it comes

$$\begin{aligned} 323 \quad \mathbf{\epsilon}_k^{(i)} &= u_k^{(i)} - \tilde{u}_k^{(i)} \\ 324 \quad &= \exp(c_i \Delta t L)u^k - \mathbf{exp}(c_i \Delta t L)\tilde{u}^k \\ 325 \quad &\quad + \Delta t \sum_{j=1}^{i-1} a_{i,j} \left[ \exp((c_i - c_j)\Delta t L)g(u_k^{(j)}) - \mathbf{exp}((c_i - c_j)\Delta t L)g(\tilde{u}_k^{(j)}) \right] \\ 326 \quad &= \exp(c_i \Delta t L)\mathbf{\epsilon}^k + \Delta t \sum_{j=1}^{i-1} a_{i,j} \left[ \exp((c_i - c_j)\Delta t L)(g(u_k^{(j)}) - g(\tilde{u}_k^{(j)})) \right] \\ 327 \quad &\quad + \gamma(c_i)\tilde{u}^k + \Delta t \sum_{j=1}^{i-1} a_{i,j} \gamma(c_i - c_j)g(\tilde{u}_k^{(j)}). \end{aligned}$$

As performed at the beginning of the proof, we iterate on  $i$  and then we use the Lipschitz property of  $g$  on the one side, and the error estimate  $\|\exp(\Delta t L) - \mathbf{exp}(\Delta t L)\| \leq C\Delta t^{r+1}$  together with the bound (5.8) on the other side. We thus deduce the following inequality for  $\mathbf{e}_k^{(i)}, i = 1, \dots, s$ :

$$\|\mathbf{e}_k^{(i)}\| \leq C\|\mathbf{e}^k\| + C\Delta t \sum_{j=1}^{i-1} a_{i,j} \|\mathbf{e}_k^{(j)}\| + C\Delta t^{r+1},$$

328 from which it can be proved by induction

$$329 \quad (5.11) \quad \|\mathbf{e}_k^{(i)}\| \leq C\|\mathbf{e}^k\| + C\Delta t^{r+1}, \quad i = 1, \dots, s.$$

330 Now, we come back to (5.10) and use (5.11) and the same lines of arguments as for  
331 the internal stages enable to get

$$332 \quad \|\mathbf{e}^n\| \leq C\|\mathbf{e}^0\| + C\Delta t \sum_{k=0}^{n-1} \left( \sum_{j=1}^s \|\mathbf{e}_k^{(j)}\| + C\Delta t^{r+1} \right) \leq C\Delta t \sum_{k=0}^{n-1} \|\mathbf{e}^k\| + C\Delta t^r,$$

333 and we deduce from the Gronwall lemma 5.2 that  $\|\mathbf{e}^n\| \leq C\Delta t^r$ . Finally, we conclude  
334 using (5.9).  $\square$

335 *Remark 5.6.* As observed for the first order method, the above proof extends to  
336 the local Lipschitz case, with the same arguments.

337 **6. Properties for the Vlasov-Ampère equations.** Regarding the Vlasov-  
338 Ampère case, the (modified) Lawson schemes enjoys some properties that are detailed  
339 in this part.

340 **6.1. Preserving the charge.** An interesting property of the Lawson and modi-  
341 fied Lawson Runge-Kutta methods applied to the Vlasov-Ampère system is the preser-  
342 vation of the charge, *i.e.* the fact that the Poisson equation  $\partial_x E = \int_{\mathbb{R}} f dv - 1$  is pre-  
343 served along time without resolving it. Indeed, let consider the semi-discrete system

$$344 \quad (6.1) \quad \frac{d}{dt} \begin{pmatrix} \vec{f}_k(t) \\ \hat{E}_k(t) \end{pmatrix} = L \begin{pmatrix} \vec{f}_k(t) \\ \hat{E}_k(t) \end{pmatrix} - \begin{pmatrix} [(E\mathcal{D}f)]_k \\ 0 \end{pmatrix},$$

345 where  $(\mathcal{D}f)_\ell$  denotes a consistent finite difference operator applied to the sequence  
346  $(f_\ell)_\ell$ . An example can be  $(\mathcal{D}f)_\ell = (f_{\ell+1} - f_{\ell-1})/(2\Delta v)$  but higher order or uncentered  
347 schemes can be used.

PROPOSITION 6.1. *The schemes (2.3) and (2.4) (where  $\mathbf{exp}$  is chosen as  $P_{p,p}$  the Padé approximant) applied to (6.1) preserve the following discrete Poisson equation*

$$ik \frac{2\pi}{b} \hat{E}_k^n = \Delta v \sum_{\ell=1}^{N_v} \hat{f}_{k,\ell}^n, \quad \forall n \geq 1, \quad \text{for } k \neq 0 \quad \text{and } \hat{E}_{k=0}^n = 0,$$

348 *provided that it is satisfied at the initial time  $n = 0$ .*

*Proof.* First, we write down the exponential of matrices for the Lawson and modified Lawson schemes (let us remark that for simplicity we will denote  $k$  instead of  $k2\pi/b$  with  $[0, b]$  the space interval)

$$\exp(tL) = \begin{pmatrix} \text{diag}(\exp(-ikt\vec{v})) & \mathbf{0}^T \\ \frac{i\Delta v}{k}(\mathbb{1} - \exp(-ikt\vec{v})) & 1 \end{pmatrix}, \quad P_{p,p}(tL) = \begin{pmatrix} \text{diag}(\vec{a}) & \mathbf{0}^T \\ \frac{i\Delta v}{k}(\mathbb{1} - \vec{a}) & 1 \end{pmatrix}$$

349 with  $a_\ell = P_{p,p}(-ikt v_\ell)$ . We can observe that the structure of the matrix  $P_{p,p}(tL)$   
 350 is very close to the one of  $\exp(tL)$ . Thus in the sequel, we will use the notation  $\beta_\ell$   
 351 (the  $\ell$ -th component of  $\vec{\beta} \in \mathbb{R}^{N_v}$ ) which is equal to  $\exp(-ikt v_\ell)$  for the Lawson case  
 352 and which is equal to  $P_{p,p}(-ikt v_\ell)$  in the modified case. We present the proof for the  
 353 first order Lawson case (that is forward Euler) but the proof can be generalized to  
 354 arbitrary explicit Runge-Kutta schemes of order  $m$  and stages  $s$ . Considering the  $N_v$   
 355 first components of (6.1) gives

$$356 \quad (6.2) \quad \hat{f}_{k,\ell}^{n+1} = \beta_\ell \hat{f}_{k,\ell}^n - \Delta t \beta_\ell \left( \widehat{E^n(\mathcal{D}f^n)_\ell} \right)_k, \quad \ell = 1, \dots, N_v, \quad \square$$

357 wheres the last component of (6.1) gives (using (6.2))

$$358 \quad \hat{E}_k^{n+1} = \hat{E}_k^n + \sum_{\ell=1}^{N_v} \frac{i\Delta v}{k} (1 - \beta_\ell) \hat{f}_{k,\ell}^n - \Delta t \sum_{\ell=1}^{N_v} \frac{i\Delta v}{k} (1 - \beta_\ell) \left( \widehat{E^n(\mathcal{D}f^n)_\ell} \right)_k, \\
 359 \quad = \hat{E}_k^n + \sum_{\ell=1}^{N_v} \frac{i\Delta v}{k} (1 - \beta_\ell) \hat{f}_{k,\ell}^n - \Delta t \frac{i\Delta v}{k} \sum_{\ell=1}^{N_v} \left( \widehat{E^n(\mathcal{D}f^n)_\ell} \right)_k - \sum_{\ell=1}^{N_v} \frac{i\Delta v}{k} (\hat{f}_{k,\ell}^{n+1} - \beta_\ell \hat{f}_{k,\ell}^n), \\
 360 \quad = \hat{E}_k^n - \frac{i}{k} \hat{\rho}_k^{n+1} + \frac{i}{k} \hat{\rho}_k^n,$$

361 where we denote  $\hat{\rho}_k^n = \sum_{\ell=1}^{N_v} \hat{f}_{k,\ell}^n \Delta v$  and used  $\sum_{\ell=1}^{N_v} (\mathcal{D}f^n)_\ell = 0$  since the discrete  
 362 operator  $\mathcal{D}$  is chosen to be consistent. Thus, assuming the Poisson equation is satisfied  
 363 initially  $ik\hat{E}_k^0 = \hat{\rho}_k^0$ , this last relation implies by induction  $ik\hat{E}_k^{n+1} = \hat{\rho}_k^{n+1}$  which  
 364 proves the Poisson equation is propagated.

The RK case extends easily. The last stage is for  $f$  (with  $\mathbf{b}_\ell^{(j)} = \exp(-i(1 - c_j)\Delta t k v_\ell)$   
 or  $P_{p,p}(-i(1 - c_j)\Delta t k v_\ell)$ )

$$\hat{f}_{k,\ell}^{n+1} = b_\ell \hat{f}_{k,\ell}^n - \Delta t \sum_{j=1}^s b_j \mathbf{b}_\ell^{(j)} \left( \widehat{E^{(j)}(\mathcal{D}f^{(j)})_\ell} \right)_k, \quad \ell = 1, \dots, N_v,$$

365 and for  $E$ , we have

$$366 \quad \hat{E}_k^{n+1} = \hat{E}_k^n + \sum_{\ell=1}^{N_v} \frac{i\Delta v}{k} (1 - b_\ell) \hat{f}_{k,\ell}^n - \Delta t \sum_{j=1}^s b_j \sum_{\ell=1}^{N_v} \frac{i\Delta v}{k} (1 - \mathbf{b}_\ell^{(j)}) \left( \widehat{E^{(j)}(\mathcal{D}f^{(j)})_\ell} \right)_k,$$

367 which gives the Poisson equation at time  $n + 1$  assuming it is satisfied at time  $n$ .

368 **6.2. Preserving the total energy.** In this part, we propose a way to modify  
 369 the numerical solution to ensure total energy preservation. Usually, preserving the  
 370 total energy may induce the use of implicit methods which are not simple or easy to  
 371 implement and in practice the total energy is quite well preserved when high order  
 372 methods are used in time. The strategy is based on an orthogonal projection technique  
 373 (already employed in [20, 26] to preserve moments in collisional kinetic equations)  
 374 suggested in [27] and detailed in [31] for Hamiltonian PDEs. The strategy proposed  
 375 in this latter reference has two steps: first an explicit high order time integrator  
 376 is used to update the solution of a reformulated system for which the invariant is  
 377 quadratic, and second a projection technique is employed for which the Lagrange  
 378 multiplier can be explicitly obtained. As a result, the method preserves the order of  
 379 the time integrator used in the first step, and is energy-preserving.

380 In our Vlasov context, the energy is quadratic so that we do not have to reformulate the system so as to get a quadratic invariant. But, as explained before, our  
 381 methods preserve the Poisson equation so we modify the strategy proposed in [31] to  
 382 get energy and charge preservations, still retaining the high order accuracy. In the  
 383 sequel, we present the method for the Vlasov-Ampère case.  
 384

385 Let denote by  $(\tilde{f}^{n+1}, \tilde{E}^{n+1})$  the numerical solution obtained from the high order  
 386 Lawson or modified Lawson methods proposed above and denote by  $\tilde{H}^{n+1} =$   
 387  $\sum_{j,\ell} |v_\ell|^2 \tilde{f}_{j,\ell}^{n+1} \Delta x \Delta v + \sum_i \tilde{E}_j^{n+1} \Delta x$  the associated total energy. We used the index  $\ell$   
 388 for the velocity grid and index  $j$  for the grid in space  $x_j = j\Delta x, j = 0, \dots, N_x - 1, \Delta x =$   
 389  $b/N_x$ .

390 We will consider the following correction

$$391 \quad (6.3) \quad f_{j,\ell}^{n+1} = (1 + \lambda_n) \tilde{f}_{j,\ell}^{n+1}, \quad E_i^{n+1} = (1 + \lambda_n) \tilde{E}_j^{n+1},$$

392 where  $\lambda_n$  will be given below and should be of order of the integrator used in the first  
 393 step. Thus the energy at time  $n + 1$  becomes

$$394 \quad H^{n+1} = \sum_{j,\ell} |v_\ell|^2 f_{j,\ell}^{n+1} \Delta x \Delta v + \sum_j (E_j^{n+1})^2 \Delta x$$

$$395 \quad = \tilde{H}^{n+1} + \lambda_n \left[ \sum_{j,\ell} |v_\ell|^2 \tilde{f}_{j,\ell}^{n+1} \Delta x \Delta v + 2 \sum_j (\tilde{E}_j^{n+1})^2 \Delta x \right] + \lambda_n^2 \sum_j (\tilde{E}_j^{n+1})^2 \Delta x$$

$$396 \quad = \tilde{H}^{n+1} - \frac{\delta_n}{D_n} (\gamma_n + 2\alpha_n) + \frac{\delta_n^2 \alpha_n}{D_n^2},$$

397 where we denote  $\lambda_n = -\delta_n/D_n, \gamma_n = \sum_{j,\ell} |v_\ell|^2 \tilde{f}_{j,\ell}^{n+1} \Delta x \Delta v, \alpha_n = \sum_j (\tilde{E}_j^{n+1})^2 \Delta x$  and  
 398  $\delta_n = \tilde{H}^{n+1} - H^n$ . Thus, we look for  $D_n \neq 0$  such that

$$399 \quad (6.4) \quad -\frac{\delta_n}{D_n} (\gamma_n + 2\alpha_n) + \frac{\delta_n^2 \alpha_n}{D_n^2} = -\delta_n,$$

400 so that  $H^{n+1} = \tilde{H}^{n+1} - \delta_n = \tilde{H}^{n+1} - (\tilde{H}^{n+1} - H^n) = H^n$ . Solving (6.4) leads to  
 401  $D_n = \left[ \gamma_n + 2\alpha_n + \sqrt{(\gamma_n + 2\alpha_n)^2 - 4\delta_n \alpha_n} \right] / 2$ .

402 **PROPOSITION 6.2.** *The schemes (2.3) and (2.4) (where `exp` is chosen as  $P_{p,p}$  the*  
 403 *Padé approximant) with the energy correction (6.3) applied to (6.1) preserve the discrete*  
 404 *energy:  $\mathcal{H}^{n+1} = \mathcal{H}^n$  with  $\mathcal{H}^n = \sum_{j,\ell} |v_\ell|^2 f_{j,\ell}^n \Delta v \Delta x + \sum_j (E_j^n)^2 \Delta x$ .*

405 *Proof.* The goal is to ensure that  $\lambda_n = \mathcal{O}(\Delta t^p)$ , which means that the additional  
 406 correction does not deteriorate the order of the scheme. First, since the schemes are  
 407 of order  $p$ , we have  $(\tilde{f}^{n+1}, \tilde{E}^{n+1}) = (\tilde{f}(t^{n+1}), \tilde{E}(t^{n+1})) + \mathcal{O}(\Delta t^{p+1})$  from which we  
 408 deduce  $(\tilde{f}^{n+1}, \tilde{E}^{n+1}) = (f^n, E^n) + \mathcal{O}(\Delta t)$  and thus  $\delta_n = H(t^{n+1}) - H^n + \mathcal{O}(\Delta t^{p+1})$ .  
 409 But since  $H(t^{n+1}) = H(0) = H^0 = H^n$ , we have  $\delta_n = \mathcal{O}(\Delta t^{p+1})$ .  
 410 It remains to prove that  $D^n$  is well defined. Let check that  $(\gamma_n + 2\alpha_n)^2 - 4\delta_n \alpha_n > 0$ ,

$$411 \quad (\gamma_n + 2\alpha_n)^2 - 4\delta_n \alpha_n = \gamma_n^2 + 4\alpha_n^2 + 4\gamma_n \alpha_n - 4\delta_n \alpha_n = \gamma_n^2 + 4\alpha_n^2 + 4\alpha_n(\alpha_n - \delta_n).$$

412 By the definition of  $\alpha_n \equiv \sum_j (\tilde{E}_j^{n+1})^2 \Delta x > 0$  and using  $\delta_n = \mathcal{O}(\Delta t^{p+1})$ , we deduce  
 413 that there exists  $\Delta t_0 > 0$  such that for  $\Delta t < \Delta t_0$ , we have  $\alpha_n - \delta_n > 0$ . Finally, from  
 414 its definition and for  $\Delta t$  small enough,  $D_n$  is of order  $2\mathcal{H}^n$  which is of order 1. We  
 415 conclude  $\lambda_n = \delta_n/D_n = \mathcal{O}(\Delta t^{p+1})$ .  $\square$

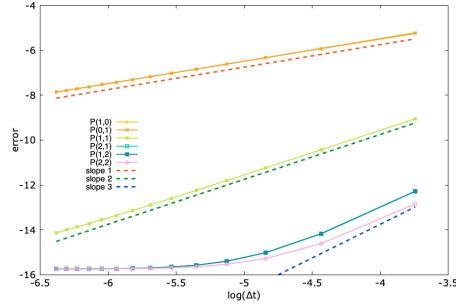


FIG. 1. Order of time accuracy for the two-dimensional linear transport problem approximated by M-LRK(3,3) with the Padé approximant  $P_{p,q}$  for different values of  $p, q$ .

416 *Remark 6.3.* Let us remark that since the same correction is used for  $f$  and  $E$   
 417 and since the above method preserves the Poisson equation between  $(\tilde{f}^{n+1}, \tilde{E}^{n+1})$  the  
 418 corrected solution  $(f^{n+1}, E^{n+1})$  also enjoys the charge preservation.

419 *Remark 6.4.* This correction approach can be extended to Vlasov-Maxwell equa-  
 420 tions in multi-dimension in space and velocity.

421 **7. Numerical results.** In this section, several numerical results are given to  
 422 illustrate the properties of the schemes introduced before. In the sequel,  $LRK(m, s)$   
 423 will denote the (standard) Lawson Runge-Kutta method of order  $m$  and with  $s$  stages  
 424 whereas M-LRK( $m, s$ ) will denote the modified version presented above, *i.e.* where  
 425  $\exp(tL)$  is approximated by  $\text{exp}(tL)$ .

426 **7.1. Explicit case.** First, we investigate a simple problem for which we get an  
 427 exact solution. The goal is to illustrate the error estimate from Theorem 5.5. Thus,  
 428 we consider the following problem satisfied by  $u(t, x, y) \in \mathbb{R}$

429 (7.1) 
$$\partial_t u + a\partial_x u + b\partial_y u = 0, \quad x, y \in \mathbb{R}^2, \quad a, b \in \mathbb{R},$$

430 with the initial condition  $u(0, x, y) = u_0(x, y)$ . We considered the following param-  
 431 eters:  $a = 1, b = 0.75, (x, y) \in [-2, 2]^2$  with periodic boundary conditions. A  
 432 Fourier method is used for the  $y$ -direction whereas finite differences are used for the  
 433  $x$ -direction, and a Lawson RK(3,3) is used for the time discretization. The numerical  
 434 parameters are chosen as follows:  $N_x = N_y = 243$ , and the final time is  $T = 10\sigma\Delta x$ ,  
 435  $\Delta t \in \{\sigma\Delta x/n, n = 1, \dots, 14\}$  with  $\sigma = 1.44$  (which corresponds to the CFL condition  
 436 for WENO5 method coupled with RK(3,3) method).

437 On Figure 1, we plot the order in time of the M-LRK(3,3) methods with different  
 438 order for the Padé approximant  $P_{p,q}$ . It illustrates numerically the order of accuracy  
 439  $\Delta t^{\min(m,r)}$ , ( $m = 3$  here) for different values of  $r = p + q + 1$  that have been proven in  
 440 Theorem 5.5.

441 **7.2. Vlasov-Ampère models .** In this subsection, we focus on Vlasov-Ampère  
 442 models in one dimensional spatial and velocity directions. The case considers the  
 443 classical Vlasov-Ampère model (equivalent to the Vlasov-Poisson one)

We first consider a two stream instability (TSI) test for which the initial condition  
 is

$$f_0(x, v) = \frac{1}{\sqrt{2\pi}} v^2 e^{-v^2/2} (1 + \alpha \cos(kx)),$$



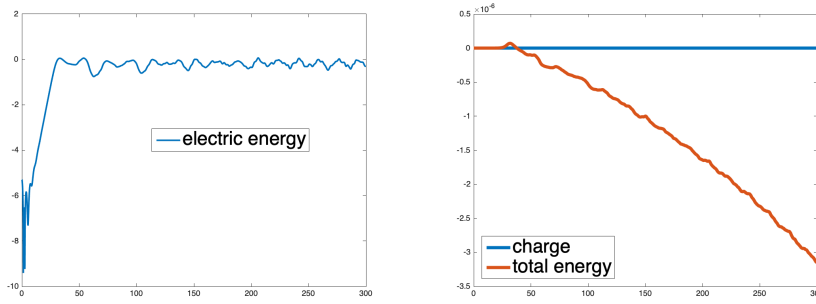


FIG. 2. TSI test: time evolution of the electric energy (left) and of the charge and total energy (right) obtained with LRK(4,4) method with  $\Delta t = 0.1$ ,  $N_x = N_v = 128$  ( $v_{\max} = 14$ ).

444 with  $x \in [0, 2\pi/k]$ ,  $k = 0.5$ ,  $v \in [-9, 9]$  and  $\alpha = 10^{-3}$ . In Figure 2, the time  
 445 evolution of the electric energy (left) and the invariants (total energy and charge)  
 446 are displayed. We used a LRK(4,4) method (fixed time step  $\Delta t = 0.1$ ). The electric  
 447 energy enjoys an exponential behavior and a nonlinear phase, as expected. The total  
 448 energy has a very good behavior since it is preserved up to  $10^{-6}$  and the charge is  
 449 preserved up to machine accuracy. Then, in Figure 3, the same quantities are plotted  
 450 (electric, total energies and charge) but here we use the energy correction explained  
 451 in the previous section. We can observe that now, both charge and total energy  
 452 are preserved, without affecting the time evolution in a sensible way of the electric  
 453 energy. Next, we change the underlying Runge-Kutta method to use adaptive time  
 454 steps to the embedded RK method DP(4,3) from [21] (still combined with the energy  
 455 correction). The tolerance is  $\text{tol} = 10^{-4}$  and the time step is modified if the local  
 456 error  $L_{[p]}^{n+1} = \|f_{[4]}^{n+1} - f_{[3]}^{n+1}\|$  is larger than  $10^{-9}$  (here  $f_{[4]}^{n+1}$  (resp.  $f_{[3]}^{n+1}$ ) denotes the  
 457 numerical solution obtained by the 4-th order (resp. 3-rd order) method). Indeed, if  
 458 the local error is smaller than  $10^{-9}$ , the iteration is accepted and the next time step  
 459 becomes  $\Delta t_{n+1} = \max(0.5, \min(1.5, (\text{tol}/L_{[p]}^{n+1})^{1/3}))\Delta t_n$ , otherwise, the iteration is  
 460 rejected and is performed with a smaller time step  $\Delta t = \max(0.5, (\text{tol}/L_{[p]}^{n+1})^{1/3})\Delta t_n$ .  
 461 Let us remark that for this approach, the Poisson equation is satisfied up to machine  
 462 accuracy  $10^{-14}$ , so as the total mass. In Figure 4 (left), we plot the evolution of the  
 463 time steps together with the electric energy (log-scale). We observe that the method  
 464 automatically computes large time steps (up to  $\Delta t = 3$ ) in the linear phase and then,  
 465 since in the nonlinear phase the amplitude of the electric field is larger, the time steps  
 466 become smaller (mostly between 0.2 and 0.3) to respect the CFL condition induced  
 467 by the nonlinear part. On the right side of Figure 4, we plot the time evolution of the  
 468 charge and total energy which is preserved up to machine accuracy for large time.

We then consider the following initial condition for the (SLD test)

$$f_0(x, v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2} (1 + 0.5 \cos(kx)), \quad x \in [0, 2\pi/k], k = 0.5.$$

469 The same results are displayed in Figures 5 for this Landau case. Similar comments  
 470 can be performed showing the robustness and versatility (change the Runge-Kutta  
 471 method, fixed or adaptive time steps, with or without energy correction) of the ap-  
 472 proach. Let us remark that same comments can be done for M-LRK methods.

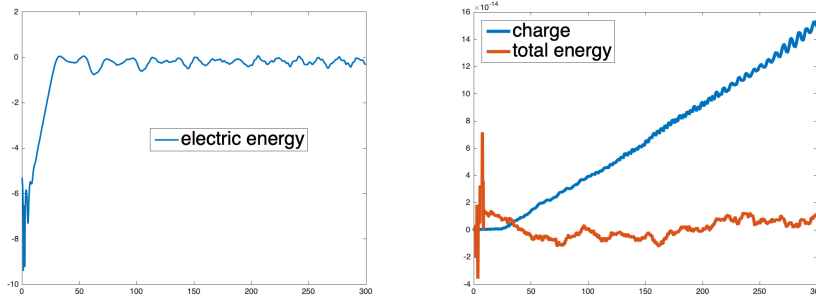


FIG. 3. TSI test: time evolution of the electric energy (left) and of the charge and total energy (right) obtained with LRK(4,4) method with  $\Delta t = 0.1$ ,  $N_x = N_v = 128$ , with energy correction step ( $v_{\max} = 14$ ).

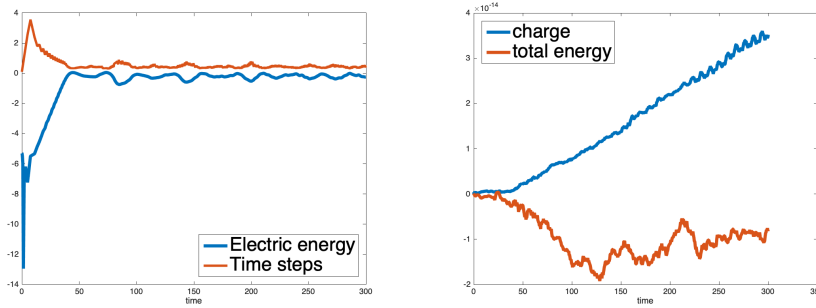


FIG. 4. TSI test: time evolution of the electric energy (left) and of the total energy (right) obtained with Lawson DP(4,3) method combined with an adaptive time step and with energy correction.  $N_x = N_v = 128$  and  $v_{\max} = 14$ .

473 **7.3. Linearized hybrid Vlasov equation 1dx-3dv.** We consider in this part  
 474 the linearized hybrid Vlasov model described in Section 3 and studied in [12, 30]. In  
 475 [12], Lawson method was employed to discretized this model but the linear part was  
 476 split into two parts  $L = L_1 + L_2$  (where  $L_2$  contains the Maxwell part). However since  
 477  $\exp(tL)$  can not be computed for all  $k$  and  $t$  even with scientific software, the  $L_2$  part  
 478 (which corresponds to the Maxwell equations) was put in the nonlinear part so that we  
 479 considered the following form  $\partial_t U = L_1 U + (L_2 U + N(U))$  on which Lawson Runge-  
 480 Kutta integrators were applied since  $\exp(tL_1)$  can be computed easily. In this work,  
 481 we are able to consider modified Lawson integrators on the form  $\partial_t U = LU + N(U)$   
 482 by using an approximation  $\text{exp}(tL)$  of  $\exp(tL)$ . First, we notice that the eigenvalues  
 483 of  $L$  are pure imaginary and the matrix  $L$  can be written as  $L = PHP^{-1}$  with  $P$   
 484 independent from  $k$  and  $H$  is skew hermitian. As such, we are in the framework of  
 485 Proposition 4.1 which suggests the use of Padé approximant to approximate  $\exp(tL)$ .  
 486 This has been implemented within a scientific computing environnement which allows  
 487 for automatic code generation.

We present a test to illustrate the good behavior of the approach in this frame-  
 work. Following [30, 12], the initial condition is given by

$$f_h(t = 0, z, \mathbf{v}) = \frac{\rho_h}{(2\pi)^{3/2} \bar{v}_{\parallel} \bar{v}_{\perp}^2} \exp\left(-\frac{v_z^2}{2\bar{v}_{\parallel}^2} - \frac{(v_x^2 + v_y^2)}{2\bar{v}_{\perp}^2}\right),$$

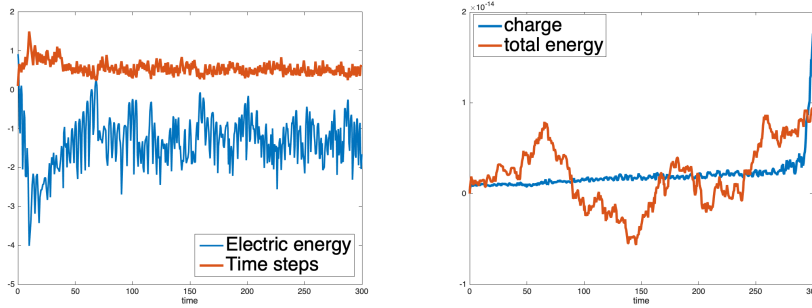


FIG. 5. *SLD test: time evolution of the electric energy (left) and of charge and total energy (right) obtained with Lawson DP(4,3) method combined with an adaptive time step and with energy correction.  $N_x = N_v = 128$  and  $v_{\max} = 14$ .*

488 with  $z \in [0, 2\pi/k]$ ,  $k = 2$ ,  $\bar{v}_{\parallel} = 0.2$ ,  $\bar{v}_{\perp} = 0.6$ ,  $\rho_h = 0.2$  and  $B_x(t = 0, z) = \epsilon \sin(kz)$ ,  
 489 the other unknown ( $E_x, E_y, j_{c,x}, j_{c,y}, B_y$ ) are zero initially. The velocity domain is  
 490 truncated to  $\mathbf{v} = [-3.6, 3.6] \times [-3.6, 3.6] \times [-2.4, 2.4]$  and we consider  $N_x = 27, N_{v_x} =$   
 491  $N_{v_y} = 32, N_{v_z} = 41$  for the phase space discretization.

492 We are interested in the time history of the following energies (magnetic energy,  
 493 electric energy, energy of the cold and hot particles) whose sum is preserved with time

$$494 \quad \mathcal{H}_B(t) = \frac{1}{2} \int (B_x^2(t, z) + B_y^2(t, z)) dz, \quad \mathcal{H}_E(t) = \frac{1}{2} \int (E_x^2(t, z) + E_y^2(t, z) + E_z^2(t, z)) dz,$$

$$495 \quad \mathcal{H}_c(t) = \frac{1}{2\Omega_{pe}^2} \int (j_{c,x}^2(t, z) + j_{c,y}^2(t, z)) dz, \quad \mathcal{H}_h(t) = \frac{1}{2} \iint |\mathbf{v}|^2 f_h(t, z, \mathbf{v}) d\mathbf{v} dz.$$

496 First, we consider the form  $\partial_t U = L_1 U + (L_2 U + N(U))$  and compare the modified  
 497 Lawson method RK(4,4) with Padé  $P_{2,2}$  with the Lawson method (used in [12]). Here  
 498  $\exp(tL_1)$  and  $\mathbf{exp}(tL_1) = P_{2,2}(tL_1)$  are computed, whereas  $L_2$  (which corresponds  
 499 to the Maxwell equations) is considered explicit through the nonlinear term. This  
 500 strategy leads to a stability condition since the Maxwell equations are solved explicitly  
 501 in time. We found out that, for RK(4,4), the condition is  $\Delta t \leq (2\sqrt{2}/\pi)\Delta z \approx 0.11$   
 502 with our numerical parameters (see [12, 33]). In Figure 6, we consider the time  
 503 evolution of the electric energy (with  $\Delta t = 0.05 < 0.11$  to ensure stability) for both  
 504 approaches. The two methods are almost indistinguishable and are able to recover the  
 505 linear phase (for which the growth rate  $\gamma = 0.095$  is in a very good agreement with  
 506 the analytical one). Next, we compare the approach proposed in [12] with the present  
 507 approach where  $\exp(tL)$  is approximated by  $\mathbf{exp}(tL) = P_{2,2}(tL)$ , which does not suffer  
 508 from the stability condition from the Maxwell equations since they are now taken into  
 509 account in the linear part ('Maxwell inside' the linear part). As mentioned before,  
 510 the first approach requires a time step  $\Delta t < 0.11$  to be stable thus we use a slightly  
 511 larger time step  $\Delta t = 0.12$ . In Figure 7, we can observe that the simulation 'Maxwell  
 512 outside' (ie Maxwell equations are outside the linear part as in [12]) is unstable with  
 513  $\Delta t = 0.12$  whereas when using a Padé approximation of the whole  $L$  matrix ('Maxwell  
 514 inside' the linear part), no instability is observed. The same simulation with  $\Delta t = 0.5$   
 515 leads to very similar results which shows the advantages of the current approach.  
 516 In Figure 8, we compare different Padé approximations of  $\exp(tL)$  ( $P_{p,q}(tL)$  with  
 517  $p, q = 1, 2$ ) with  $\Delta t = 0.1$ . As mentioned in Section 4, the choice  $p > q$  leads to  
 518 unstable results which is illustrated in Figure 8 for  $P_{2,1}$  whereas the other choice

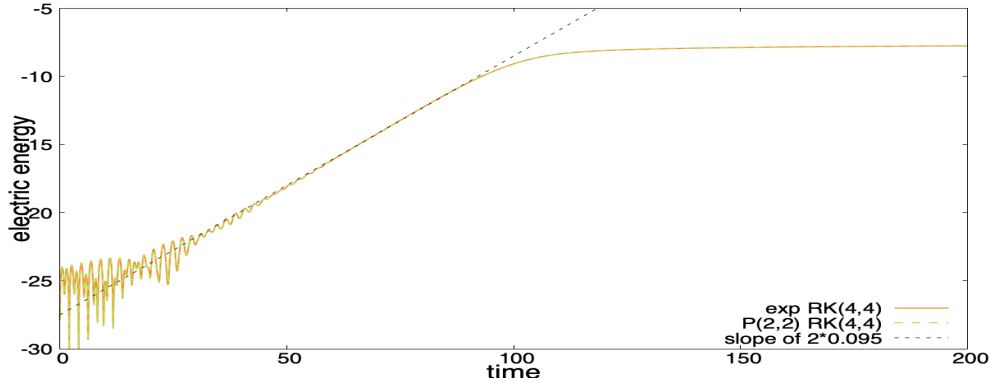


FIG. 6. Time evolution of the electric energy. Comparison between the exact exponential and Padé  $P_{2,2}$  coupled with  $RK(4, 4)$  with  $\Delta t = 0.05$ . Maxwell outside.

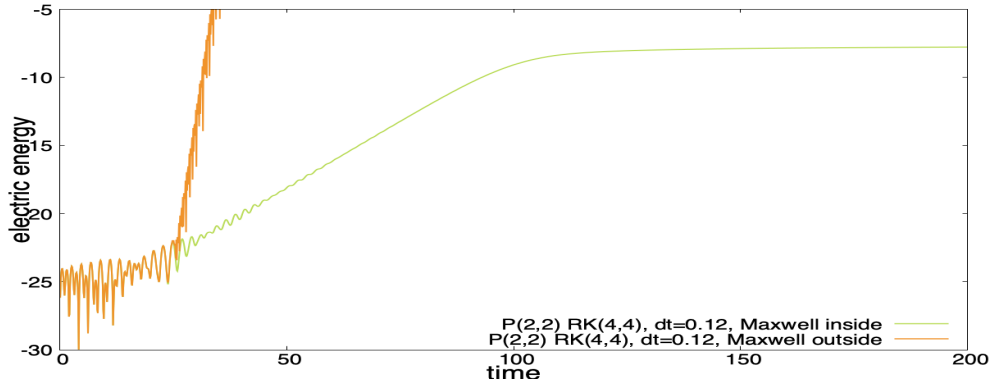


FIG. 7. Time evolution of the electric energy. Comparison between Maxwell inside and Maxwell outside with  $P_{2,2}$  coupled with  $RK(4,4)$  with  $\Delta t = 0.12$ .

519 leads to stable results. Then, we couple our strategy with adaptive time stepping  
 520 using  $DP(4,3)$  time integrator with  $\mathbf{exp}(tL) = P_{2,2}(tL)$ . We plot in Figure 9 the time  
 521 evolution of different energies (electric, cold kinetic and magnetic energies, defined  
 522 in (7.2)) for which the theoretical instability rate is  $\gamma = 0.095$  (see [30, 12]). Since  
 523 energies are displayed, the instability rate is twice the theoretical rate. A very good  
 524 agreement is observed and we can look at the time history of the time steps in Figure  
 525 7.2 (right) which shows that the method is able to take very large time steps (around  
 526  $\Delta t \approx 3$ ) in the linear phase (in which the fields in the nonlinear term are of small  
 527 amplitude) whereas in the nonlinear phase, the time steps are smaller to capture the  
 528 nonlinear effects, in an automatic way.

529 **8. Conclusion.** In this work, we studied Lawson and modified Lawson meth-  
 530 ods for the time integration of Vlasov-type equations. After a suitable phase-space  
 531 discretization, a set of ODEs is obtained in which a linear and nonlinear parts can  
 532 be obtained which is amenable to Lawson methods. The analysis of convergence  
 533 of the modified Lawson methods is carried out and leads to a rate of convergence  
 534  $\mathcal{O}(\Delta t^{\min(m,r)})$  where  $m$  is the order of the underlying Runge-Kutta method and  $r$  is

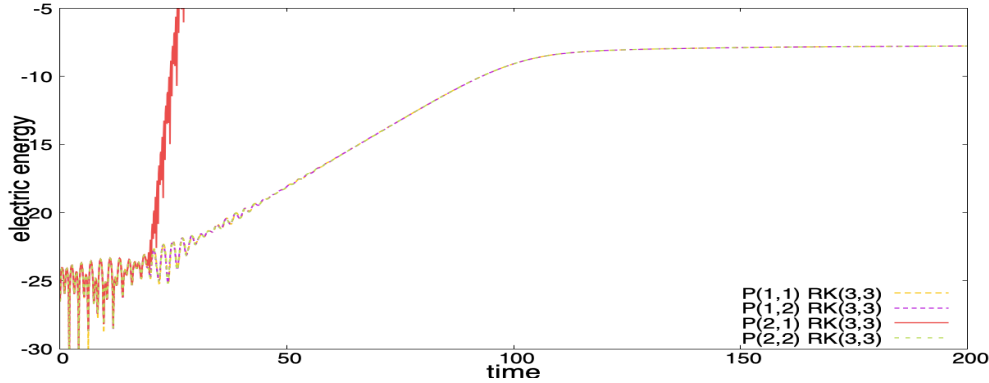


FIG. 8. Time evolution of the electric energy. Comparison of different Padé approximants coupled with RK(3,3) with  $\Delta t = 0.1$ .

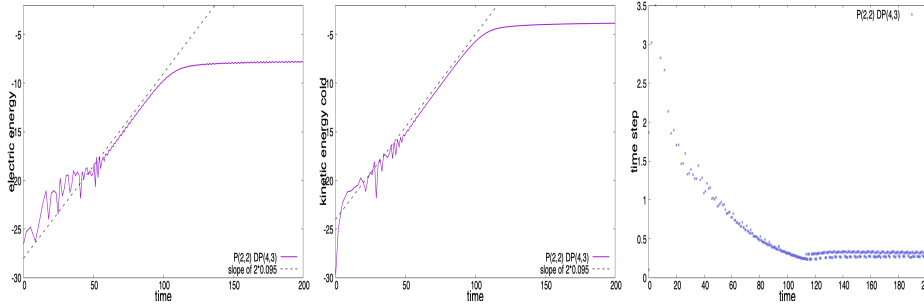


FIG. 9.  $P_{2,2}(tL)$  coupled with  $DP(3,3)$ . Time evolution of the electric energy (left), cold kinetic energy (middle) and time steps (right).

535 the order of approximation of  $\exp(tL)$  by  $\text{exp}(tL)$ . It turns out that Padé approximant  
 536 is a suitable choice regarding its spectral properties leading to stability properties of  
 537 the resulting numerical scheme. Several numerical tests emphasize the good behavior  
 538 of the schemes, in particular they enable to use larger time steps since they are not  
 539 constraint by the stringent CFL condition coming from the linear part. Moreover, in  
 540 the Vlasov-Ampère case, since the linear part is solved exactly, it leads to a scheme  
 541 that preserves the Poisson equation exactly. Let us remark that the construction of  
 542 high order time integrator for multi-dimensional Vlasov-Maxwell equations can be  
 543 performed using different approaches like standard Runge-Kutta or (Hamiltonian)  
 544 splitting techniques. However, these approaches have some drawbacks. On the one  
 545 side, standard RK approaches (like [11, 2, 3]) suffer from stability condition from the  
 546 linear part and charge preservation is not ensured. On the other side, even if Hamil-  
 547 tonian splittings benefit from structure preserving properties when phase-space mesh  
 548 is fine enough (see [14, 9]), for high dimensional problems however, the use of coarse  
 549 meshes destroys the good long time behavior and the construction of high order split-  
 550 ting leads to very costly methods (see [12]). Hence, the proposed method turns out  
 551 to be a viable alternative to design high order time integrators for Vlasov equations.

552 **Acknowledgments.** We would like to acknowledge the assistance of volunteers  
 553 in putting together this example manuscript and supplement.

554

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