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# A FOOTNOTE TO A THEOREM OF HALÁSZ

ÉRIC SAĪAS AND KRISTIAN SEIP

ABSTRACT. We study multiplicative functions  $f$  satisfying  $|f(n)| \leq 1$  for all  $n$ , the associated Dirichlet series  $F(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$ , and the summatory function  $S_f(x) := \sum_{n \leq x} f(n)$ . Up to a possible trivial contribution from the numbers  $f(2^k)$ ,  $F(s)$  may have at most one zero or one pole on the one-line, in a sense made precise by Halász. We estimate  $\log F(s)$  away from any such point and show that if  $F(s)$  has a zero on the one-line in the sense of Halász, then  $|S_f(x)| \leq (x/\log x) \exp(c\sqrt{\log \log x})$  for all  $c > 0$  when  $x$  is large enough. This bound is best possible.

Halász obtained in [3, 4] some fundamental results on the mean values of multiplicative functions  $f$  subject to the restriction  $|f(n)| \leq 1$  for all nonnegative integers  $n$ . We denote this class of functions by  $\mathcal{M}$  and set

$$S_f(x) := \sum_{n \leq x} f(n) \quad \text{and} \quad F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

where the latter series converges absolutely for  $\sigma := \operatorname{Re} s > 1$ . Following Montgomery [6], we have the following.

**Halász's theorem.** *Suppose that  $f$  belongs to  $\mathcal{M}$ . Then for every real  $t$  with at most one exception, we have*

$$(1) \quad F(\sigma + it) = o\left(\frac{1}{\sigma - 1}\right), \quad \sigma \searrow 1.$$

*If there exists an exceptional  $t = t_0$  for which (1) does not hold, then*

$$(2) \quad F(\sigma + it_0) \asymp \frac{1}{\sigma - 1}, \quad 1 < \sigma \leq 2.$$

*Moreover, the following three assertions are equivalent:*

- (i)  $S_f(x) = o(x)$ ,  $x \rightarrow \infty$ ;
- (ii) For every real  $t$ ,  $F(\sigma + it) = o(1/(\sigma - 1))$  when  $\sigma \searrow 1$ ;
- (iii) For every real  $t$ , we have

$$\sum_p \frac{1 - \operatorname{Re}(f(p)p^{-it})}{p} = +\infty \quad \text{or} \quad f(2^k) = -2^{ikt} \quad \text{for all } k \geq 1.$$

The three equivalent assertions (i), (ii), (iii) give a more precise statement about the case  $S_f(x) = o(x)$  than what is found in the usual “textbook version” of Halász's theorem; see for example [8, Sect. 4.3]. All the statements above can still be extracted from Satz 1' of [3]. The

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second alternative in item (iii) accounts for a trivial reason for having  $F(\sigma + it) = o(1/(\sigma - 1))$  when  $\sigma \searrow 1$ , namely the existence of  $t$  such that

$$\sum_{k \geq 0} \frac{f(2^k)}{2^{k(\sigma + it)}} = \frac{2^\sigma - 2}{2^\sigma - 1}.$$

In our first theorem, we exclude this possibility by considering the subclass  $\mathcal{M}_2$  of  $\mathcal{M}$  consisting of  $f$  for which  $f(2^k) = 0$  for every  $k \geq 1$ .

We may think of the exceptional case  $t = t_0$  in Halász's theorem as the assertion that  $F(s)$  has a "simple pole" at the point  $s = 1 + it_0$ . Following [7, Thm. 2.1], we find it natural to treat such "poles" on equal terms with possible "zeros" on the line  $\sigma = 1$ . This allows us to incorporate the following consequence of the prime number theorem in the first part of the theorem: if there is such a "zero" or a "pole", there can be no other point of the same kind. This version of Halász's result also comes with a precise estimate:

**Theorem 1.** *Suppose that  $f$  belongs to  $\mathcal{M}_2$ . Then for every real  $t$  with at most one exception,*

$$(3) \quad \lim_{\sigma \searrow 1} \frac{|F(\sigma + it)|^\varepsilon}{\sigma - 1} = +\infty$$

for both  $\varepsilon = -1$  and  $\varepsilon = 1$ . In fact, if there exists a pair  $(\varepsilon, t) = (\varepsilon_0, t_0)$  in  $\{-1, 1\} \times \mathbb{R}$  for which (3) does not hold, then for  $1 < \sigma \leq 3/2$ ,

$$[F(\sigma + it_0)]^{\varepsilon_0} \asymp (\sigma - 1)$$

and

$$(4) \quad \varepsilon_0 \log F(\sigma + it) + \log \zeta(\sigma + it - it_0) = o\left(\sqrt{\log \frac{1}{\sigma - 1}}\right),$$

uniformly for all real  $t$  when  $\sigma \searrow 1$ .

As far as the mean values of  $f$  are concerned, the bound in (4) is of no interest when  $\varepsilon_0 = -1$ . What matters is then only the behavior of  $F(\sigma + it_0)$  when  $\sigma \searrow 1$ , and we will in particular have that  $|S_f(x)|/x$  tends to a positive limit; see [2] for precise information about the relation between  $F(\sigma + it_0)$  and the mean values  $S_f(x)/x$  in the case  $\varepsilon_0 = -1$ . However, when  $\varepsilon_0 = 1$ , the estimate in (4) yields a sharp improvement of the bound in item (i) of Halász's theorem.

**Theorem 2.** *Suppose that  $f$  belongs to  $\mathcal{M}$ . If there exists a real  $t_0$  such that*

$$(5) \quad \sum_p \frac{1 + \operatorname{Re}(f(p)p^{-it_0})}{p} < \infty,$$

then

$$(6) \quad \limsup_{x \rightarrow \infty} \frac{|S_f(x)| \log x}{x \exp(c\sqrt{\log \log x})} = 0$$

for every constant  $c > 0$ . Conversely, if  $\kappa : [3, \infty) \rightarrow \mathbb{R}^+$  satisfies  $\kappa(x) = o(\sqrt{\log \log x})$  when  $x \rightarrow \infty$ , then there exists an  $f$  in  $\mathcal{M}$  such that (5) holds for  $t_0 = 0$  and

$$(7) \quad \limsup_{x \rightarrow \infty} \frac{|S_f(x)| \log x}{x \exp(\kappa(x))} = \infty.$$

We obtain (6) as an immediate consequence of (4) and a celebrated elucidation of item (i) of Halász's theorem, expressed in terms of the size of  $|F(s)|$  close to the 1-line.<sup>1</sup> This result also stems from work of Halász [3, 4]; see Montgomery's paper [5], Tenenbaum's book [8, Sec. III.4.3], or the recent paper [1]. We will therefore give below only the proof of the second part of Theorem 2.

Before proving our two theorems, we establish the following lemma.

**Lemma 1.** *Let  $f(p)$  be a sequence of numbers satisfying  $|f(p)| \leq 1$ . Suppose that there exist  $\varepsilon_0$  in  $\{-1, 1\}$  and a real number  $t_0$  such that*

$$(8) \quad \sum_p \frac{1 + \varepsilon_0 \operatorname{Re}(f(p)p^{-it_0})}{p} < \infty.$$

Then

$$\varepsilon_0 \sum_p \frac{f(p)}{p^s} + \log \zeta(s - it_0) = o\left(\sqrt{\log \frac{1}{\sigma - 1}}\right)$$

uniformly for  $s = \sigma + it$ ,  $\sigma \searrow 1$ , and real  $t$ .

*Proof of Lemma 1.* Our initial assumption is that (8) holds for either  $\varepsilon_0 = -1$  or  $\varepsilon_0 = 1$ . Writing  $\varepsilon_0 f(p)p^{-it_0} =: -|f(p)|e^{i\theta_p}$  with  $-\pi < \theta_p \leq \pi$ , we see that

$$1 + \varepsilon_0 \operatorname{Re}(f(p)p^{-it_0}) = 1 - |f(p)| + |f(p)|(1 - \cos \theta_p) \geq |f(p)|(1 - \cos \theta_p) \geq \frac{|f(p)|}{2\pi} \theta_p^2,$$

so that (8) implies that

$$(9) \quad \sum_p \frac{|f(p)|\theta_p^2}{p} < \infty.$$

We may now write

$$\begin{aligned} \varepsilon_0 \sum_p \frac{f(p)}{p^s} &= \sum_p \frac{\operatorname{Re}(\varepsilon_0 f(p)p^{-it_0})}{p^{s-it_0}} + i \sum_p \frac{\operatorname{Im}(\varepsilon_0 f(p)p^{-it_0})}{p^{s-it_0}} \\ &= - \sum_p \frac{1}{p^{s-it_0}} - i \sum_p \frac{|f(p)| \sin \theta_p}{p^{s-it_0}} + O(1) \\ (10) \quad &= - \log \zeta(s - it_0) - i \sum_p \frac{|f(p)| \sin \theta_p}{p^{s-it_0}} + O(1), \end{aligned}$$

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<sup>1</sup>To this end, we use the classical fact that  $1/\zeta(\sigma + it) \ll \log(|t| + 2)$  holds uniformly for  $\sigma \geq 1$  and real  $t$ .

which holds uniformly for  $\sigma > 1$ . By Mertens's theorem for the sum  $\sum_{p \leq x} 1/p$ , the Cauchy–Schwarz inequality, and (9),

$$\begin{aligned} \sum_p \frac{|f(p) \sin \theta_p|}{p^\sigma} &\leq \log \log \frac{1}{\sigma-1} + O(1) + \sum_{p \geq 1/(\sigma-1)} \frac{|f(p) \sin \theta_p|}{p^\sigma} \\ &\leq \log \log \frac{1}{\sigma-1} + O(1) + \left( \sum_p p^{1-2\sigma} \right)^{1/2} \left( \sum_{p \geq 1/(\sigma-1)} \frac{|f(p)| \theta_p^2}{p} \right)^{1/2} \\ &= o\left( \sqrt{\log \frac{1}{\sigma-1}} \right) \end{aligned}$$

when  $\sigma \searrow 1$ . Plugging this estimate into (10), we obtain the desired bound.  $\square$

*Proof of Theorem 1.* Since  $\zeta(s)$  has a simple pole at  $s = 1$ , is otherwise analytic, and has no zero on  $\sigma = 1$ , the first part of Theorem 1 is an immediate consequence of the second part. To prove the latter assertion, we assume that

$$\frac{|F(\sigma + it_0)|^{\varepsilon_0}}{\sigma - 1}$$

does not tend to  $+\infty$  when  $\sigma \searrow 1$  for some pair  $(\varepsilon_0, t_0)$  in  $\{-1, 1\} \times \mathbb{R}$ . By assumption,  $f$  is in  $\mathcal{M}_2$ , whence

$$F(s) := \prod_{p > 2} \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}.$$

For  $p \geq 3$ , we have  $|\sum_{k=1}^{\infty} f(p^k) p^{-ks}| \leq 1/2$ . We may therefore infer that

$$\log F(s) = \sum_p f(p) p^{-s} + O(1)$$

and hence

$$\log |F(s)| = \sum_p \operatorname{Re}(f(p) p^{-s}) + O(1)$$

when  $\sigma > 1$ . It follows from this and the fact that  $\zeta(s)$  has a simple pole at  $s = 1$  that

$$\frac{|F(s)|^{\varepsilon_0}}{\sigma - 1} \asymp \zeta(\sigma) |F(s)|^{\varepsilon_0} \asymp \exp \left\{ \sum_p \frac{1 + \varepsilon_0 \operatorname{Re}(f(p) p^{-it_0})}{p^\sigma} \right\}$$

when  $1 < \sigma \leq 3/2$ . By monotone convergence,

$$\lim_{\sigma \searrow 1} \sum_p \frac{1 + \varepsilon_0 \operatorname{Re}(f(p) p^{-it_0})}{p^\sigma} = \sum_p \frac{1 + \varepsilon_0 \operatorname{Re}(f(p) p^{-it_0})}{p}.$$

By assumption, this limit is not  $+\infty$ , and hence we may apply Lemma 1 to conclude.  $\square$

*Proof of the second part of Theorem 2.* We will assume that every  $f$  in  $\mathcal{M}$  for which (5) holds with  $t_0 = 0$ , satisfies

$$(11) \quad |S_f(x)| \ll \frac{x}{\log x} \exp(\kappa(x)),$$

and show that this leads to a contradiction.

We may clearly assume that  $\kappa(x)$  is a continuous function. It is also plain that  $\kappa(x)$  may be assumed to be nondecreasing and that  $\kappa(x)/\sqrt{\log\log x}$  may be taken to be a nonincreasing function. Indeed, if  $\kappa(x)$  failed to be nondecreasing, then we could use instead  $\kappa_0(x) := \max_{3 \leq y \leq x} \kappa(y)$ ; should moreover  $\kappa_0(x)/\sqrt{\log\log x}$  fail to be nonincreasing, then we could replace it by

$$\kappa_1(x) := \sqrt{\log\log x} \max_{y \geq x} \frac{\kappa_0(y)}{\sqrt{\log\log y}},$$

which would still be a nondecreasing function being  $o(\sqrt{\log\log x})$  when  $x \rightarrow \infty$ .

By partial summation, we have for every  $1 < \sigma \leq 3/2$  and say  $|t| \leq 1$ ,

$$\begin{aligned} |F(\sigma + it)| &\leq 1 + 2 \int_3^\infty |S_f(y)| y^{-\sigma-1} dy \ll \int_3^\infty \frac{e^{\kappa(y)}}{y^\sigma \log y} dy \\ &\leq \exp\left(\kappa\left(e^{\frac{1}{\sigma-1}}\right)\right) \int_3^{e^{1/(\sigma-1)}} \frac{dy}{y \log y} + \int_{e^{1/(\sigma-1)}}^\infty \frac{e^{-\log\log y + \kappa(y)}}{y^\sigma \log y} dy. \end{aligned}$$

Since  $\kappa(y)/\sqrt{\log\log y}$  is a nonincreasing function, the function  $\log\log y - \kappa(y)$  is eventually increasing, whence the above computation leads to the bound

$$|F(\sigma + it)| \ll \exp\left(\kappa\left(e^{\frac{1}{\sigma-1}}\right)\right) \left(\log \frac{1}{\sigma-1} + 1/e\right).$$

We may write this more succinctly as

$$(12) \quad |F(\sigma + it)| \leq \exp\left(\alpha\left(e^{\frac{1}{\sigma-1}}\right) \sqrt{\log \frac{1}{\sigma-1}}\right),$$

where  $\alpha : [3, \infty) \rightarrow (0, \infty)$  is a nonincreasing function satisfying  $\alpha(x) \rightarrow 0$  when  $x \rightarrow \infty$ .

We now choose a sequence of positive numbers  $x_j$ , growing so rapidly that  $x_j^{\log x_j} < x_{j+1}$  for every  $j \geq 1$  and the sequence

$$a_j := \sqrt{\alpha(x_j^{\log x_j})}$$

is in  $\ell^2$ . We then set

$$\theta_p := \begin{cases} \frac{a_j}{\sqrt{\log\log p}}, & x_j \leq p < x_j^{\log x_j}, \quad -\operatorname{Re}(ip^{-i}) \geq 1/2 \\ 0, & \text{otherwise.} \end{cases}$$

We find that

$$\sum_p \frac{|\theta_p|^2}{p} \leq \sum_{j=1}^\infty \frac{a_j^2}{\log\log x_j} \sum_{\substack{p \leq x_j \\ \log x_j}} \frac{1}{p} \ll \sum_{j=1}^\infty a_j^2,$$

where we in the last step used Mertens's theorem for the sum  $\sum_{p \leq x} 1/p$ . Hence  $\sum_p |\theta_p|^2/p < \infty$  by our choice of the sequence  $a_j$ . Setting  $f(p) := -e^{i\theta_p}$  and using Taylor's theorem to write

$$f(p) = -1 - i\theta_p + O(\theta_p^2),$$

we infer from this that

$$\operatorname{Re} \sum_p f(p) p^{-s} = -\operatorname{Re} \log \zeta(s) - \operatorname{Re} i \sum_p \theta_p p^{-s} + O(1).$$

It does not matter how we define  $f$  for higher prime powers, but for definiteness, let us require that  $f$  be completely multiplicative. Setting  $\sigma = 1 + 1/(\log x_j)^2$  and  $t = 1$ , we then get

$$\begin{aligned} \operatorname{Re} \log F(1 + 1/(\log x_j)^2 + i) &= -\operatorname{Re} i \sum_p \theta_p p^{-1-1/(\log x_j)^2-i} + O(1) \\ &\gg \frac{a_j}{\sqrt{\log \log x_j}} \sum_{x_j \leq p \leq x_j^{\log x_j}} \frac{1}{p} + O(1) \gg a_j \sqrt{\log \log x_j}. \end{aligned}$$

But choosing the same  $\sigma = 1 + 1/(\log x_j)^2$  and  $t = 1$  in (12), we reach the bound

$$\sqrt{\alpha(x_j^{\log x_j})} \gg 1,$$

contradicting that  $\alpha(x) \searrow 0$  when  $x \rightarrow \infty$ , which, as observed above, is a consequence of our assumption that (11) holds for all  $f$  in  $\mathcal{M}$  for which (5) is true.  $\square$

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