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# Modeling Under Location Uncertainty: A Convergent Large-Scale Representation of the Navier-Stokes Equations



Arnaud Debussche, Berenger Hug, and Etienne Mémin

**Abstract** We construct martingale solutions for the stochastic Navier-Stokes equations in the framework of the modelling under location uncertainty (LU). These solutions are pathwise and unique when the spatial dimension is 2D. We then prove that if the noise intensity goes to zero, these solutions converge, up to a subsequence in dimension 3, to a solution of the deterministic Navier-Stokes equation. This warrants that the LU Navier-Stokes equations can be interpreted as a large-scale model of the deterministic Navier-Stokes equation.

## 1 Introduction

For several years there has been a burst of activity to devise stochastic representations of fluid flow dynamics. These models are strongly motivated in particular by climate and weather forecasting issues and the need to provide accurate ensemble of large-scale flow realisations [2]. Yet, elaborating such stochastic dynamics on *ad hoc* grounds can be highly detrimental to the system of interest [4]. A minimal mathematical requirement for satisfactory large-scale flow dynamics representation is that a weak solution of the Large Eddy Simulation (LES) scheme converges toward a weak solution of the fine-scale deterministic Navier-Stokes equations in 3D and toward the unique solution for the 2D Navier-Stokes equations. The convergence of some classical LES models toward the true fine scale dynamics is well known in the deterministic case [3, 7]. However, the question of convergence of stochastic parametrization toward solutions of the deterministic equations at the limit of vanishing noise is not always clear.

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In this study we show that stochastic Navier-Stokes models defined within the modelling under location uncertainty principle (LU) [9] have martingale solutions in 3D and a unique strong solution—in the probabilistic sense—in 2D. Moreover, in 3D in the limit of vanishing noise there exists a subsequence converging in law toward a weak solution of the deterministic Navier-Stokes equations and in 2D the whole sequence converges toward the unique solution. As such these results enable to consider the LU representation as a valid large-scale stochastic representation of flow dynamics that is more amenable to ensemble forecasting and data assimilation than deterministic model due to an improved variability.

## 2 Modelling Under Location Uncertainty

The LU formulation relies mainly on the following time-scale separation assumption of the flow:

$$dX_t = u(X_t, t) dt + \sigma(X_t, t) dW_t, \quad (1)$$

where  $X : \mathbb{R}^+ \times \Omega \rightarrow \mathcal{S}$  is the Lagrangian displacement defined within the bounded domain  $\mathcal{S} \subset \mathbb{R}^d$  ( $d = 2$  or  $3$ ) with smooth boundary, and  $u : \mathbb{R}^+ \times \mathcal{S} \times \Omega \rightarrow \mathcal{S}$  denotes the large-scale velocity that is both spatially and temporally correlated, while  $\sigma dW$  is a highly oscillating unresolved component (also called noise term) that is only correlated in space.

More precisely, we consider a cylindrical Wiener process  $W$  on  $L^2(\mathcal{S}, \mathbb{R}^d)$ , the space of square integrable functions on  $\mathcal{S}$  with values in  $\mathbb{R}^d$ ,

$$W = \sum_{i \in \mathbb{N}} \hat{\beta}^i e_i,$$

where  $(e_i)_{i \in \mathbb{N}}$  is a Hilbertian orthonormal basis of  $L^2(\mathcal{S}, \mathbb{R}^d)$  and  $(\hat{\beta}_i)_{i \in \mathbb{N}}$  is a sequence of independent standard brownian motions on a stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  ([11]). The above does not converge in  $L^2(\mathcal{S}, \mathbb{R}^d)$  but in any larger Hilbert space  $U$  such that the embedding of  $L^2(\mathcal{S}, \mathbb{R}^d)$  into  $U$  is Hilbert-Schmidt, for instance  $U$  can be the  $L^2(\mathcal{S})$  based Sobolev space  $H^{-\alpha}(\mathcal{S})$  for some  $\alpha > d/2$ .

The spatial structure of the noise is specified through a time dependent deterministic integral covariance operator  $\sigma_t$  defined from a bounded and symmetric kernel  $\hat{\sigma}$ :

$$\sigma_t f(x) := \int_{\mathcal{S}} \hat{\sigma}(x, y, t) f(y) dy, \quad f \in L^2(\mathcal{S}, \mathbb{R}^d).$$

For each  $(x, y, t)$ ,  $\hat{\sigma}(x, y, t)$  is a  $d \times d$  symmetric tensor. Since  $\hat{\sigma}$  is bounded in  $x$ ;  $y$  and  $t$ ,  $\sigma(x, t)$  maps  $L^2(\mathcal{S}, \mathbb{R}^d)$  into itself and is Hilbert-Schmidt. Then, the noise can be written as the Wiener process:

$$\sigma_t W_t = \sum_{i \in \mathbb{N}} \hat{\beta}_t^i \sigma_t e_i,$$

where the series converges in  $L^2(\mathcal{S}, \mathbb{R}^d)$  almost surely and in  $L^p(\Omega)$  for all  $p \in \mathbb{N}$  and Eq. (1) should be understood in the Itô sense. We may further write the dependance of the Wiener process in terms of the other variables:

$$\sigma_t W_t(x, \omega) = \sum_{i \in \mathbb{N}} \hat{\beta}_t^i(\omega) \sigma_t e_i(x),$$

We consider a divergence free noise:

$$\nabla_x \cdot \hat{\sigma}(x, y, t) = 0, \quad x, y \in \mathcal{S}, \quad t \geq 0.$$

Also, for each  $t \in \mathbb{R}^+$ , there exists  $(\phi_n(t))_n$  a complete orthogonal system composed by eigenfunctions of the covariance operator at each time  $t \in \mathbb{R}$  and another sequence of independent standard brownian motions, on the same stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ , such that we have the representation:

$$\sigma_t W_t = \sum_{k=0}^{\infty} \phi_k(t) \beta_t^k.$$

This Gaussian random field is associated to the two-times, two-points covariance tensor given by

$$Q(x, y, t, t') = \mathbb{E} \left( \sigma_t dW_t(x) [\sigma_{t'} dW_{t'}]^T(y) \right) = \int_{\mathcal{S}} \hat{\sigma}(x, z, t) \hat{\sigma}(y, z, t') dy \delta(t-t'),$$

with the diagonal part (i.e one time auto-correlation), referred to in the following as the variance tensor, and denoted by

$$a(x, t) = \int_{\mathcal{S}} \hat{\sigma}(x, y, t) \hat{\sigma}(x, y, t) dy = \sum_{k=0}^{\infty} \phi_k(x, t) \phi_k^T(x, t). \quad (2)$$

In a way similar to the classical derivation of Navier-Stokes equations, the LU setting is based on a stochastic representation of the Reynolds transport theorem (SRTT) [9], describing the rate of change of a random scalar  $q$  within a volume  $V(t)$  transported by the stochastic flow (1). For incompressible unresolved flows, (i.e.  $\nabla \cdot \sigma = 0$ ), the SRTT reads

$$d \left( \int_{V(t)} q(x, t) dx \right) = \int_{V(t)} (\mathbb{D}_t q + q \nabla \cdot (u - u_s) dt) dx, \quad (3a)$$

$$\mathbb{D}_t q = d_t q + (u - u_s) \cdot \nabla q dt + \sigma dW_t \cdot \nabla q - \frac{1}{2} \nabla \cdot (a \nabla q) dt, \quad (3b)$$

where  $d_t q(x, t) = q(x, t + dt) - q(x, t)$  stands for the forward time-increment of  $q$  at a fixed point  $x$ ,  $\mathbb{D}_t$  is introduced as the stochastic transport operator in [9, 12] and plays the role of the material derivative. Recall that  $u$  is the large-scale velocity used in (1) and  $a$  is defined in (2). Note also that we omit to mention the dependance of  $\sigma$  on time.

This operator is derived from the Itô-Wentzell formula [8] to express the differentiation of a stochastic process transported by the flow [9]. The drift  $u_s = \frac{1}{2} \nabla \cdot a$ , coined as the Itô-Stokes drift (ISD) in [1], represents through the divergence of the variance tensor, the effects of the small-scale inhomogeneity on the large-scale flow component. This term can be understood as a generalization of the Stokes drift associated to the waves orbital motion. In addition to this modified advection, the stochastic transport operator involves an inhomogeneous diffusion driven by the variance tensor, which can be interpreted as a subgrid diffusion term attached to the mixing operated by the small scales. It can be noticed that this term would only be implicitly represented in Stratonovich integral form. However, the ISD would remain [1]. The remaining term corresponds to the advection by the random term. It can be observed by a direct application of Itô on the norm of the scalar that the positive energy brought by this (backscattering) term is exactly compensated by the energy loss by the diffusion [12]. Due to that, for a transported quantity, its energy is conserved pathwise, or in other words: for any realization of the flow.

The above SRTT (3a) and Newton's second principle (in a distributional sense) allow us to derive the following stochastic equations of motions (see Sect. 5 of [9] or Sect. 2.2–2.3 of [10]), which for any noise scaling  $\varepsilon > 0$  parameter and for all points of  $\mathcal{S}$  reads, using  $\sigma$ ,  $u_s$ ,  $a$  introduced above:

$$\begin{aligned} d_t u + (u - \varepsilon^2 u_s) \cdot \nabla u \, dt + \varepsilon \sigma dW_t \cdot \nabla u - \frac{1}{2} \varepsilon^2 \nabla \cdot (a \nabla u) \, dt \\ = -\frac{1}{\rho} \nabla(p \, dt + dp_t^\sigma) + \frac{1}{R_e} \Delta(u \, dt + \varepsilon \sigma dW_t), \end{aligned} \quad (4)$$

with the incompressibility conditions

$$\nabla \cdot (u - \varepsilon^2 u_s) = 0 \quad , \quad \nabla \cdot \sigma = 0 \quad , \quad (5)$$

and associated with Dirichlet boundary condition  $u(t, x) = 0$  and  $\widehat{\sigma}(x, y, t) = 0$  for all  $x \in \partial\mathcal{S}$  and  $t > 0$ . The initial condition is denoted by  $u(0, x) = u_0(x)$  for all  $x \in \mathcal{S}$ . As usual,  $u(t, x) = (u_1(t, x), \dots, u_d(t, x))$  and  $p(t, x)$  stands for the velocity and the pressure of the fluid, respectively. The term  $dp_t^\sigma$  corresponds to the Brownian (martingale) part of the pressure. The Ito-Stokes drift  $u_s$  is defined as  $u_s := \frac{1}{2} \nabla \cdot a$  and  $\rho$  stands for the fluid density. The dimensioning constant  $R_e = UL/\nu$  denotes the Reynolds number, sets from the ratio of the product of characteristic length and velocity scales,  $UL$ , with the kinematics viscosity  $\nu$ . As for the noise scaling parameter,  $\varepsilon$ , it encodes a scale of the unresolved energy and should converge to zero when all the flow components are resolved. Meaning thus there is no noise and the system corresponds trivially to the deterministic Navier-Stokes system.

Although the system corresponds to the Navier-Stokes for zero noise, the convergence toward weak (strong) solutions of the 3D (2D) deterministic Navier-Stokes, respectively, at the limit of vanishing noise needs to be assessed. This is the results we aim to prove in this paper.

First of all, in order to work with a pressure-free system through a divergence-free Leray projection, we proceed to the change of variable  $v := u - \varepsilon^2 u_s$  in (4) to rewrite the system with a classical incompressibility condition on  $v$ :

$$\begin{aligned} d_t v + v \cdot \nabla v \, dt - \frac{1}{R_e} \Delta v \, dt + \varepsilon^2 (v \cdot \nabla) u_s \, dt - \frac{\varepsilon^2}{2} \nabla \cdot (a \nabla v) \, dt \\ - \frac{\varepsilon^4}{2} \nabla \cdot (a \nabla u_s) \, dt - \frac{\varepsilon^2}{R_e} \Delta u_s \, dt + \varepsilon^2 \partial_t u_s \, dt = - \frac{1}{\rho} \nabla (p \, dt + d p_t^\sigma) - \\ (\varepsilon \sigma \, dW_t \cdot \nabla) v - (\varepsilon^3 \sigma \, dW_t \cdot \nabla) u_s + \frac{\varepsilon}{R_e} \Delta (\sigma \, dW_t), \end{aligned} \quad (6)$$

with the incompressibility conditions

$$\nabla \cdot v = 0 \quad \nabla \cdot \sigma = 0, \quad (7)$$

for all points in  $\mathcal{S}$  together with Dirichlet boundary conditions  $v(t, x) = 0$ ,  $\widehat{\sigma}(x, y, t) = 0$  for all  $x \in \partial \mathcal{S}$  and  $t > 0$  and the initial condition  $v(0, x) = v_0(x) := u_0(x) - \varepsilon^2 u_s(0, x)$  for all  $x \in \mathcal{S}$ . In the following section we specify the spaces on which this system is defined, rewrite it in an equivalent abstract form and state our main result.

### 3 Notations and Main Result

Let  $\mathcal{V}$  be the space of infinitely differentiable  $d$ -dimensional vector fields  $u$  on  $\mathcal{S}$ , with compact support strictly contained in  $\mathcal{S}$ , and satisfying  $\nabla \cdot u = 0$ . We denote by  $H$  the closure of  $\mathcal{V}$  in  $L^2(\mathcal{S}, \mathbb{R}^d)$  and  $V$  the closure of  $\mathcal{V}$  in the Sobolev space  $H^1(\mathcal{S}, \mathbb{R}^d)$ . The space  $H$  is endowed with the  $L^2(\mathcal{S}, \mathbb{R}^d)$  inner product. This inner product and its induced norm are noted:

$$(u, v)_H := (u, v)_{L^2(\mathcal{S})} \quad \text{and} \quad |u|_H := \|u\|_{L^2(\mathcal{S})}.$$

As for space  $V$ , thanks to Poincaré inequality, it is endowed with the  $H_0^1(\mathcal{S}, \mathbb{R}^d)$  inner product and its associated norm, denoted respectively as

$$((u, v))_V := (\nabla u, \nabla v)_{L^2(\mathcal{S})} \quad \text{and} \quad \|u\|_V := \|\nabla u\|_{L^2(\mathcal{S})}.$$

We may define then the Gelfand triple  $V \subset H \subset V'$  where  $V'$  is the dual space of  $V$  relative to  $H$ . We denote by  $\langle \cdot, \cdot \rangle_{V' \times V}$  the duality pairing between  $V'$  and  $V$ . The space of Hilbert-Schmidt operators from  $H$  to  $H$  is denoted by  $\mathcal{L}_2(H)$  and  $\| \cdot \|_{\mathcal{L}_2}$  is its norm.

System (4) may be rewritten in an equivalent simplified pressure-free formulation by using the Leray projection  $P : L^2(\mathcal{S}, \mathbb{R}^d) \rightarrow H$  of  $L^2(\mathcal{S}, \mathbb{R}^d)$  onto the space  $H$  of divergence-free vectorial functions. Applying Leray's projector to (6), we obtain

$$\begin{aligned} d_t v - \frac{1}{R_e} P(\Delta v dt) + P(v \cdot \nabla v dt) \\ + P \left( \varepsilon^2 (v \cdot \nabla) u_s dt - \frac{\varepsilon^2}{2} \nabla \cdot (a \nabla v) dt - \frac{\varepsilon^4}{2} \nabla \cdot (a \nabla u_s) dt - \frac{\varepsilon^2}{R_e} \Delta u_s dt + \varepsilon^2 \partial_t u_s dt \right) \\ = P \left( \frac{\varepsilon}{R_e} \Delta(\sigma dW_t) - (\varepsilon \sigma dW_t \cdot \nabla) v - (\varepsilon^3 \sigma dW_t \cdot \nabla) u_s \right). \end{aligned} \quad (8)$$

This system can finally be rewritten in the following simplified abstract form

$$\begin{cases} d_t v(t) + Av(t) dt + Bv(t) dt + F_\varepsilon v(t) dt = G_\varepsilon v(t) dW_t, \\ v(0) = v_0. \end{cases} \quad (9)$$

The deterministic terms  $A$ ,  $B$ ,  $F_\varepsilon$  and the stochastic term  $G_\varepsilon$  are described below.

Several kinds of solutions can be defined for stochastic partial differential equations. As for deterministic PDEs, these can be strong, weak or mild (semi-group) solutions. When the solutions are constructed for a fixed Wiener process  $W$  on a given stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ , they are strong in the probabilistic sense. As usual in 3D, due to the lack of uniqueness, we work with weaker solutions, called martingale solutions, that consists in looking for solutions defined as a triplet composed of a stochastic basis, a Wiener process and an adapted process.

More precisely, we say that there is a martingale solution of system (9) if there exists a stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ , a cylindrical Wiener process  $W$  on  $L^2(\mathcal{S}; \mathbb{R}^d)$  and a progressively measurable process  $v : [0, T] \times \Omega \rightarrow H$ , with

$$v \in L^2(\Omega \times [0, T]; V) \cap L^2(\Omega, C^0([0, T]; H)),$$

such that  $\mathbb{P} - a.e.$ ,  $v$  satisfies for all time  $t \in [0, T]$

$$v(t) + \int_0^t Av(s) ds + \int_0^t Bv(s) ds + \int_0^t Fv(s) ds = v_0 + \int_0^t G(v(s)) dW_s, \quad (10)$$

where the equality must be understood in the weak sense. We will show, for all  $\varepsilon > 0$ , the existence in 3D of a martingale solution for the LU representation of the Navier-Stokes equations for noises associated with a smooth enough diffusion

tensor kernel  $\widehat{\sigma}$  in space and time. In 2D, this solution is unique and strong in the probabilistic sense. This result is summarized in the following theorem.

**Theorem 1** *Let  $d = 2$  or  $3$  and assume that the noise is smooth enough in the sense that its variance tensor and Ito-Stokes drift are such that*

$$\sup_{t \in [0, T]} \sum_{k=0}^{\infty} \|\phi_k(t)\|_{H^3(\mathcal{S})}^2 < \infty, \quad (11)$$

$$u_s \in L^\infty(0, T; H^3(\mathcal{S}, \mathbb{R}^d)); \quad \partial_t u_s \in L^\infty(0, T; H) \text{ and } a \nabla u_s \in L^\infty(0, T; V). \quad (12)$$

Then, for all  $\varepsilon > 0$ , Eq. (10) admits a martingale solution. Moreover, for  $d = 2$ , any solution of (10) is strong in the probabilistic sense and unique.

Moreover, when  $\varepsilon \rightarrow 0$ , for  $d = 3$ , there exists a subsequence of  $(u_\varepsilon)_{\varepsilon > 0}$  which converges in law to a solution of the deterministic Navier-Stokes equation. For  $d = 2$ , the whole sequence converges to the unique solution of the Navier-Stokes equation.

The condition of Theorem 1 simplifies when the covariance operator does not depend on time or if the ISD is divergence free. In both cases the condition on the temporal derivative of the ISD are not necessary. We note also, that for a spatially homogeneous noise, the variance tensor is constant and the ISD cancels. However this may happen only on a periodic domain or on the full space. The assumptions on the noise are anyway non optimal but it is not the purpose of this paper to consider non spatially smooth noise since in practice it is smooth.

Note that condition (11) is satisfied for instance if we choose  $\sigma$  independent on  $t$  and equal to  $A^{-r}$  with  $r$  large enough where  $A$  is the Stokes operator defined below. Indeed, in this case  $\phi_k = \lambda_k^{-r} e_k$  where  $(e_k)_k$  is an orthonormal complete system of eigenvectors of  $A$  associated to the eigenvalues  $(\lambda_k)_k$  and  $\|\phi_k(t)\|_{H^3(\mathcal{S})}^2 = \lambda_k^{3-2r}$ .

The behavior of the eigenvalues:  $\lambda_k \sim k^{2/d}$  allows to conclude that (11) follows. Since  $u_s = \frac{1}{2} \nabla \cdot a$  and  $a$  is defined by (2), (12) holds also for  $r$  large enough since  $\|u_s\|_{H^3(\mathcal{S})} \leq \sum_{k=0}^{\infty} \|\phi_k(t)\|_{H^4(\mathcal{S})}^2$ . Finally, since  $A^{-r}$  is self-adjoint and Hilbert-Schmidt for  $r > d/4$ , it is associated to a symmetric kernel  $\widehat{\sigma}$  which is bounded for  $r$  large enough.

These convergence results open new interesting possibilities for the study of turbulence or for the proposition of new large-scale representations of fluid dynamics. From the theoretical point of view, it might be interesting to explore multiscale versions of the LU representation based on spatial filtering together with nested noise models. This would generalize classical large eddy models in which the noise would depend on the spatial filtering applied. The coarser the filtering the larger the noise. Energy transfer between scales would then be very interesting to study in this probabilistic setting. Stochastic Karman-Howarth-Monin equations for energy exchanges across scales could be obtained by this way. From a practical point of view, these convergence results justify the setting of such stochastic models to represent large-scale solutions of the Navier-Stokes equations.

## 4 Proofs of the Main Result

We introduce the Stokes operator:  $Av := -\frac{1}{R_e} P(\Delta v)$  on the domain  $\mathcal{D}(A) := V \cap H^2(\mathcal{S}, \mathbb{R}^d)$ . Let  $b$  be the trilinear form and  $B$  the bilinear operator defined for all  $u, v$  and  $w \in V$  by

$$b(u, v, w) = \int_{\mathcal{S}} w(x) [u(x) \cdot \nabla] v(x) dx = (B(u, v), w)_H.$$

Recall that for all  $u, v$  and  $w \in V$ :  $b(u, v, w) = -b(u, w, v)$ . As usual, we set  $B(u) = B(u, u)$ . We then define  $F$  by:

$$\begin{aligned} F(v) &= \varepsilon^2 B(v, u_s) - \frac{\varepsilon^2}{2} P \nabla \cdot (a \nabla v) - \frac{\varepsilon^4}{2} P \nabla \cdot (a \nabla u_s) - \varepsilon^2 A u_s \\ &\quad + \varepsilon^2 \partial_t u_s, \quad v \in V. \end{aligned} \quad (13)$$

It can be seen that  $F(v) \in V'$ . We next write the noise term as

$$G(v) dW_t = \sum_{k=0}^{\infty} \left( -\varepsilon A \phi_k - \varepsilon B(\phi_k, v) - \varepsilon^3 B(\phi_k, u_s) \right) d\beta_{t,k},$$

where, as for  $\sigma$ , we omit to write dependance of  $\phi_k$  on  $t$ . With these notations, (8) may indeed be rewritten as (9).

Let  $(e_i)_{i \geq 0}$  be the Hilbertian basis of  $H$  consisting of eigenvectors of  $A$ . We use the finite dimensional orthogonal projector  $P_n, n \in \mathbb{N}$ , onto  $Span(e_0, \dots, e_n)$  and the projected operators:

$$B^n := P_n B \quad F^n = P_n F \quad G^n = P_n G.$$

The Galerkin approximation of (9) is given by:

$$\begin{cases} d_t v_n(t) + A v_n(t) dt + B^n[v_n(t)] dt + F^n[v_n(t)] dt = G^n[v_n(t)] dW_t, \\ v_n(0) = P_n(v_0). \end{cases} \quad (14)$$

This is a finite dimensional system of a stochastic differential equation with smooth coefficients. It has a unique local solution, by the estimate (17) below it is global.

Apply Itô formula to  $F(x) = |x|_H^p$  for  $p \geq 2$ :

$$\begin{aligned} d_t |v_n(t)|_H^p &= p |v_n(t)|_H^{p-2} (v_n(t), G^n(v_n(t)) dW_t)_H \\ &\quad - p |v_n(t)|_H^{p-2} (v_n(t), A v_n(t) + B^n v_n(t) + F^n v_n(t))_H dt \\ &\quad + \frac{p(p-2)}{2} (G^n v_n(t), v_n(t))_H^2 |v_n(t)|_H^{p-4} dt + \frac{p}{2} \|G^n v_n(t)\|_{\mathcal{L}_2(H)}^2 |v_n(t)|_H^{p-2} dt. \end{aligned} \quad (15)$$

We have  $(v_n(t), Av_n(t))_H = \frac{1}{R_e} \|v_n(t)\|_V^2$ ,  $(v_n(t), B^n v_n(t))_H = 0$  and

$$\begin{aligned} (v_n(t), F^n v_n(t))_H &= \varepsilon^2 ([v_n(t) \cdot \nabla] u_s, v_n(t))_H - \frac{\varepsilon^2}{2} (v_n(t), \nabla \cdot (a \nabla v_n(t)))_H \\ &\quad - \frac{\varepsilon^4}{2} (v_n(t), \nabla \cdot (a \nabla u_s))_H + \varepsilon^2 (A u_s, v_n(t))_H + \varepsilon^2 (\partial_t u_s, v_n(t))_H \\ &:= F_1^n + F_2^n + F_3^n + F_4^n + F_5^n. \end{aligned}$$

Under the assumption (12) in Theorem 1, we have the estimate:

$$|F_1^n + F_3^n + F_4^n + F_5^n| \leq C(\varepsilon^2 + \varepsilon^4) |v_n(t)|_H^2 + C(\varepsilon^2 + \varepsilon^4)$$

with  $C > 0$  a finite constant. And by the definition of  $a$ , we have

$$F_2^n = \frac{\varepsilon^2}{2} \sum_{k=0}^{\infty} |(\phi_k \cdot \nabla) v_n(t)|_{L^2(\mathcal{S})}^2.$$

Furthermore, using (11),

$$\frac{1}{2} \|G^n v_n(t)\|_{\mathcal{L}_2^2(l^2(H))}^2 \leq \frac{\varepsilon^2}{2} \sum_{k=0}^{\infty} |(\phi_k \cdot \nabla) v_n(t)|_{L^2(\mathcal{S})}^2 + C\varepsilon^2 + 2\varepsilon^2 |v_n(t)|_H^2$$

and the first term corresponds exactly to  $F_2^n$ . Finally, using again (11),

$$(G^n v_n(t), v_n(t))_H^2 \leq 2C(\varepsilon^2 + \varepsilon^6) |v_n(t)|_H^2.$$

Hence

$$\begin{aligned} d_t |v_n(t)|_H^p + \frac{p}{R_e} |v_n(t)|_H^{p-2} \|v_n(t)\|_V^2 &\leq p |v_n(t)|_H^{p-2} (v_n(t), G^n(v_n(t)) dW_t)_H \\ &\quad + C(\varepsilon^2 + \varepsilon^4) |v_n(t)|_H^p + C[(\varepsilon^2 + \varepsilon^6)^\alpha + (\varepsilon^2 + \varepsilon^4)] \end{aligned} \quad (16)$$

with  $C > 0$  depending on  $p$  (and not on  $\varepsilon$  and  $n$ ). We then use classical arguments based in particular on Burkholder-Davis-Gundy inequality to deduce:

$$\frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |v_n(t)|_H^p + \int_0^T |v_n(t)|_H^{p-2} \|v_n(t)\|_V^2 \right] \leq \mathbb{E} [|v_0|_H^p] + C\varepsilon^2. \quad (17)$$

Arguing as in [6], we prove that the laws  $(\mathcal{L}(v_n))_n$  are tight in  $L^2([0, T]; H)$  and in  $C^0([0, T]; \mathcal{D}(A^{-3/2}))$ .

By the Skorohod's embedding theorem, there exists a stochastic basis  $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_t, \overline{\mathbb{P}})$  with  $L^2([0, T]; H) \cap C^0([0, T]; \mathcal{D}(A^{-3/2}))$ -valued random variables  $\overline{v}_n$  for  $n \geq 1$  and  $\overline{v}$  such that  $\overline{v}_n$  has the same law as  $v_n$  on  $L^2([0, T]; H) \cap C^0([0, T]; \mathcal{D}(A^{-3/2}))$  and  $C^0([0, T], U_0)$  cylindrical Wiener processes  $\overline{W}^n$  for  $n \geq 1$  together with  $\overline{W}$  such that (by thinning the sequences)

$$\overline{v}_n \rightarrow \overline{v} \text{ in } L^2([0, T]; H) \cap C^0([0, T]; \mathcal{D}(A^{-3/2})) \quad \overline{\mathbb{P}} \text{ a.s.} \quad (18)$$

$$\overline{W}^n \rightarrow \overline{W} \text{ in } C^0([0, T], U_0) \quad \overline{\mathbb{P}} \text{ a.s.} \quad (19)$$

For all integers  $n$ ,  $\overline{v}_n$  verifies

$$\overline{v}_n(t) - P_n(v_0) + \int_0^t [A\overline{v}_n(r) + B^n\overline{v}_n(r) + F^n\overline{v}_n(r)] dr = \int_0^t G^n(\overline{v}_n(r)) d\overline{W}_r^n. \quad (20)$$

We may let  $n \rightarrow \infty$  in this equation and prove that  $\overline{v}$  verifies for almost surely  $(t, \omega) \in [0, T] \times \overline{\Omega}$

$$\overline{v}(t) - v_0 + \int_0^t (A\overline{v}(r) + B\overline{v}(r) + F\overline{v}(r)) dr = \int_0^t G(\overline{v}(r)) d\overline{W}_r \quad (21)$$

in the weak sense. For instance, let  $w$  be a smooth test function, then:

$$\int_0^t (B^n(\overline{v}_n(r), w))_H dr = \int_0^t b((\overline{v}_n(r), (\overline{v}_n(r), w)) dr = - \int_0^t b((\overline{v}_n(r), w, (\overline{v}_n(r)) dr$$

and by the almost sure strong convergence in  $L^2(0, T, H)$  this converges to  $-\int_0^t b((\overline{v}(r), w, (\overline{v}(r)) dr$  when  $n \rightarrow \infty$ .

It can be shown that (17) holds for  $\overline{v}_n$  and letting  $n \rightarrow \infty$  we obtain a bound on  $\overline{v}$ . In particular,  $\overline{v} \in L^2(\overline{\Omega}; L^2([0, T], V)) \cap L^2(\overline{\Omega}; L^\infty([0, T], H))$ . We then use the mild form of this equation to prove that  $\overline{v} \in C^0([0, T], H)$  almost surely.

For  $d = 2$ , we consider  $v_1$  and  $v_2$  two solutions of (9) on the same probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$  and, using Ito formula and classical estimates, prove that

$$\mathbb{E} \left[ \sup_{0 \leq r \leq T} e(r) |(v_1 - v_2)(r)|_H^2 \right] = 0,$$

where  $e(t) := \exp\left(-\alpha \int_0^t \|v_2(r)\|_V^2 dr\right)$  for a well chosen  $\alpha$ . As  $\mathbb{E} \int_0^T \|v_2(r)\|_V^2 dr < \infty$ , we deduce  $\mathbb{P}$  a.s.  $v_1 = v_2$  for all  $t \in [0, T]$ . We have proved that pathwise uniqueness holds for  $d = 2$ . Then, using an argument due to Gyongy and Krylov (see for instance [5], Sect. 5), we conclude that the whole sequence  $(v_n)_n$  converges to a unique solution of (21).

Let  $v_0 \in H$ . For all  $\varepsilon > 0$ , we have proved that the abstract problem (8) admits martingale solutions  $(v_\varepsilon)_{\varepsilon > 0}$ . We then study if  $(v_\varepsilon)_{\varepsilon > 0}$  converges when  $[\varepsilon \rightarrow 0^+]$  to a solution  $v$  of the following deterministic Navier-Stokes equation

$$\begin{cases} d_t v(t) + Av(t) dt + Bv(t) dt = 0 \\ v(0) = v_0 . \end{cases} \quad (22)$$

When  $d = 2$ , the solution  $v_\varepsilon$  is strong and unique. The deterministic Eq. (22) admits also a unique weak solution  $v$ . By classical estimate, we prove:

$$\mathbb{E}_\varepsilon \left[ \sup_{0 \leq t \leq T} e(t) |v_\varepsilon(t) - v(t)|_H^2 \right] \xrightarrow{\varepsilon \rightarrow 0^+} 0,$$

where  $e(t) := \exp\left(-\alpha \int_0^t \|v(r)\|_V^2 dr\right)$  for some  $\alpha > 0$ .

When  $d = 3$ , inequality (17) shows that  $(\mathcal{L}(v_{\varepsilon_n}))_n$  are tight in  $L^2([0, T]; H) \cap C^0([0, T]; \mathcal{D}(A^{-3/2}))$ . Using Skorohod embedding theorem, we show that a subsequence converges to the law a weak solution of (22).

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