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THE FIRST STEKLOV EIGENVALUE ON INFINITE SUBGRAPHS OF INTEGER LATTICES

YASSIN CHEBBI

ABSTRACT. In the context of transient graphs, we study the first Steklov eigenvalue $\sigma_0(\Omega)$ of an infinite subgraph with finite boundary (Ω, \mathcal{B}) of the integer lattice \mathbb{Z}^n . We focus in this paper on finding lower bounds using the technique of discretization of smooth compact Riemannian manifolds with cylindrical boundary. These bounds essentially depend on the discretization of the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with two identical boundaries' isometrics to $\{1\} \times \mathbb{S}^{n-1}$ through quasi-isometries. As a consequence, if $n \geq 4$ and the boundary \mathcal{B} considered as a finite subset of \mathbb{Z}^d where $1 \leq d \leq n - 3$, we show that $\sigma_0(\Omega)|\mathcal{B}|^{\frac{1}{n-1}}$ tends to infinity as the cardinal of \mathcal{B} tends to infinity. Moreover, if $n \geq 3$ and the boundary \mathcal{B} is a sphere, we prove that the first Steklov eigenvalue tends to zero proportionally to $1/|\mathcal{B}|^{\frac{1}{n-1}}$ as the radius tends to infinity and that $\sigma_0(\Omega)|\mathcal{B}|^{\frac{1}{n-1}}$ is bounded.

1. INTRODUCTION

Let M be a smooth Riemannian manifold of dimension $n \geq 2$ with smooth boundary Σ . The Dirichlet-to-Neumann (DtN in short) operator is a first order elliptic pseudo-differential operator, which is known as the voltage-current map in physics, see [9]. The eigenvalues of DtN operator are also known as solutions of the Steklov problem on M , given by

$$\begin{cases} \Delta u = 0 & \text{in } M, \\ \partial_\nu u = \sigma u & \text{on } \Sigma \end{cases}$$

where Δ is the Laplace-Beltrami operator acting on functions on M and ∂_ν is the outward normal derivative along Σ . For $\kappa, r_0 > 0$, let $\mathcal{M}(\kappa, r_0, n)$ be the class of compact n -dimensional manifolds with cylindrical boundary such that it has a Ricci curvature bounded below by $-(n - 1)\kappa$ and an injectivity radius bounded below by r_0 . For a manifold $M \in \mathcal{M}(\kappa, r_0, n)$ then the spectrum of DtN operator is nonnegative, discrete and unbounded see [2] and is given by a sequence of eigenvalues ranked in ascending order as follows:

$$0 = \sigma_0(M) < \sigma_1(M) \leq \sigma_2(M) \leq \dots \nearrow \infty.$$

Let g be a Riemannian metric on M . The Steklov eigenvalues of (M, g) can be characterized by the following variational formula

$$(1.1) \quad \sigma_k(M) := \min_{B \subseteq \mathcal{H}_k(M)} \max_{0 \neq u \in B} \frac{\int_M |\nabla u|^2 dV_M}{\int_\Sigma u^2 dV_\Sigma},$$

where $k \in \mathbb{N} \cup \{0\}$ and $\mathcal{H}_k(M)$ is the set of all k -dimensional subspaces in the Sobolev space $\mathcal{H}^1(M)$ which are orthogonal to constants on Σ .

The Steklov problem was first introduced by Steklov [35] for bounded domains of the plane. This problem has been extensively studied in recent years, and there have been many interesting developments of domains in Euclidean spaces and Riemannian manifolds in literatures, e.g. ([3],[11],[12],[13],[14],[16],[17],[20],[21]). The relationship between the Steklov eigenvalues and the geometry of the underlying space, and also its similarity and difference with the Laplace eigenvalues, have been a main focus of interest and a source of inspiration. Cheeger [8] discovered a close relation between the geometric

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quantity, the isoperimetric constant and the analytic quantity for the first non-trivial eigenvalue of the Laplace-Beltrami operator on closed manifolds.

In the context of finite graphs, the study of spectral geometry associated to the first non-zero eigenvalue of the Steklov problem have been initiated in (Hua, al., [24] and Hassannezhad, al., [27]). The authors define two Cheeger-type constants for the Steklov problem on connected graphs based on the constants by Escobar [21] and Jammes [28] already existing for the continuous case, and give two very interesting lower bound estimates depending on these constants. These give lower bound estimates for the first non-trivial eigenvalues of DtN operators on finite graphs with boundary. In the same context, we refer to a work of Perrin [31], who gives lower bounds for the first non-zero Steklov eigenvalue depending on the extrinsic diameter of the boundary of a finite connected graph. Moreover, Perrin gives an upper bound for the first non-zero Steklov eigenvalue depending on the number of vertices of the graph and of its boundary on a finite graph with boundary included in a Cayley graph associated to a group of polynomial growth [32].

In [26], the authors study the Steklov problem on a finite subgraph of integer lattices and prove a discrete analog for Brock's result (Theorem 3 in [4]). Using the symmetrization approach of Brock's idea which depends on the rotational symmetry of the Euclidean spaces, the authors estimate the Steklov eigenvalues by the geometric quantities of the subset in an integer lattice where they can estimate these quantities by their counterparts in a Euclidean space. This process guarantees to obtain the quantitative estimate of discrete Steklov eigenvalues. Moreover, the authors prove that the multiplicity of trivial eigenvalue 0 of the DtN operator is equal to the number of connected components of the graph.

In [25], the authors introduce a condition approach on infinite subgraphs of graphs centered on the notion of exhaustion. This framework permits to define the Steklov problem on an infinite subgraph with finite or infinite boundary. In this geometric concept, we mention that Anné and Torki-Hamza propose in [1] the notion of χ -completeness on infinite graphs to ensure essential self-adjointness for the Laplace operator. This general condition is characterized by the exhaustion of the graph by a family of cut-off functions satisfying certain properties of boundedness. Recently, Lenz, Schmidt and Wirth proved in [29] that this notion is equivalent to the existence of an intrinsic pseudo metric with finite balls. This context is important to characterize the DtN operator on an infinite boundary of subgraphs.

Moreover, the authors in [25] prove that the first Steklov eigenvalue is always positive on infinite transient graphs with finite boundary. In this context, we focus in the present paper on controlling the first Steklov eigenvalue on infinite subgraphs with finite boundary (Ω, \mathcal{B}) of n -dimensional integer lattices where n greater or equal to 3. Initially, we take the boundary \mathcal{B} as a sphere. Through test functions, we were able to find an upper bound dependent of n and the radius. This leads us to conclude that the first Steklov eigenvalue tends to zero proportionally to $1/|\mathcal{B}|^{\frac{1}{n-1}}$ when the radius tends to infinity. We can extend this result to show that the first Steklov eigenvalue tends to zero for any infinite subgraph with its boundary in \mathbb{Z}^d where $n - 2 \leq d \leq n$.

Let $n \geq 4$, we focus on answering the question that arises naturally: What is the asymptotic behavior of the first Steklov eigenvalue if we take the boundary as a subset of \mathbb{Z}^d , for all $1 \leq d \leq n - 3$? We refer here to [10] where the authors construct a countable sequence of smooth compact surfaces $\{\Omega_N\}_{N \in \mathbb{N}}$ with boundary of perimeter N obtained by expander graphs such that

$$\lim_{N \rightarrow \infty} \sigma_1(\Omega_N) \mathcal{L}(\partial\Omega_N) = \infty,$$

where $\mathcal{L}(\partial\Omega_N)$ the length of boundary. For a closed connected submanifold \mathcal{N} of a closed manifold (M, g) of positive codimension greater or equal 2 equipped with a Riemannian distance d_g , Brisson proves in [Theorem 1.4, [3]] that

$$\lim_{\varepsilon \rightarrow 0} \sigma_1(\Omega_\varepsilon) = \infty,$$

where $\Omega_\varepsilon = M \setminus T_\varepsilon = \{x \in M : d_g(x, \mathcal{N}) \geq \varepsilon\}$, obtained by removing the tubular neighbourhood T_ε .

In [15], the authors prove a uniform spectral comparison inequality between the Steklov eigenvalues of a manifold and those of its discretization. This idea used by Tschanz in [36] to transfer upper bounds of the Steklov spectrum of a bounded domain of M to subgraph of finite polynomial growth Cayley graphs.

This strategy is known as coarse discretization, which is used in the literature to the spectral geometry of the Laplacian on closed Riemannian manifolds. We mention, for example, the work of Brooks [5] using the strategy of discretization to study the first non-zero eigenvalue. Moreover, there is a result in the work of Mantuano [Theorem 3, [30]] to control eigenvalues of the Laplace operator on compact Riemannian manifolds. In [6], Buser constructed compact hyperbolic surfaces with large area and uniformly positive to study the first non-zero eigenvalue. Here, the manifolds were, especially, closely related to Cayley graphs of groups associated to a family of covering spaces. The point of view that will interest us here is the one taken up by Chavel in his book [7], where the question of discretization is very well explained and where he studies in particular the case of isoperimetric inequalities.

Our strategy to find a lower bound of the first Steklov eigenvalue is based on adopting the notion of discretization of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with two identical boundaries isometry to $\{1\} \times \mathbb{S}^{n-1}$ through the quasi-isometry. We begin by solving this problem when we take a subgraph with one vertex on the boundary. Since $\mathbb{Z} \hookrightarrow \mathbb{Z}^n$, we extend this result to get a lower bound for the first Steklov eigenvalue where its boundary as a subset of \mathbb{Z} when n greater or equal to 4. This lower bound is sharp and does not depend on the length of the boundary. This permits to conclude that $\sigma_0(\Omega)|\mathcal{B}|^{\frac{1}{n-1}}$ tends to infinity. More generally, we use the same technique to obtain a lower bound of the first Steklov eigenvalue where the boundary as a subset of \mathbb{Z}^d , for all $1 \leq d \leq n - 3$, see Corollary 2.2.

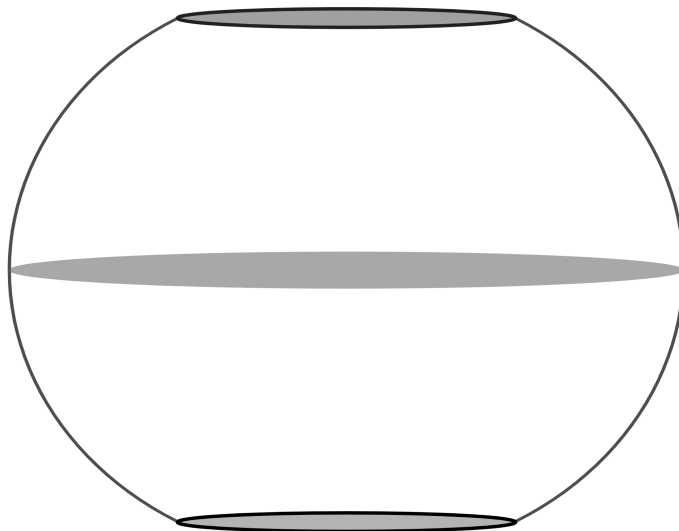


FIGURE 1. A sphere of \mathbb{R}^{n+1} with two identical boundaries.

2. FRAMEWORK AND MAIN RESULTS

The Dirichlet-to-Neumann operator is naturally defined in the discrete setting. We recall some basic definitions on graphs. A graph $G = (\mathcal{V}, \mathcal{E})$ consists of the set of vertices \mathcal{V} and the set of edges \mathcal{E} considered as a subset of $\mathcal{V} \times \mathcal{V}$. Two vertices x, y are called neighbors, denoted by $x \sim y$, if there is an edge e connecting x and y , i.e. $e = (x, y) \in \mathcal{E}$. We denote the set of neighbors of a vertex x by $\mathcal{N}(x)$. The n -dimensional integer lattice graph, still denoted by \mathbb{Z}^n is the graph consisting of the set of vertices $\mathcal{V} = \mathbb{Z}^n$ and the set of edges

$$\mathcal{E} := \left\{ (x, y) \in \mathbb{Z}^n \times \mathbb{Z}^n : \sum_{i=1}^n |x_i - y_i| = 1 \right\}.$$

Let Ω be any subset of \mathbb{Z}^n , we denote by \mathcal{B} the vertex boundary of Ω consisting of vertices in $\mathbb{Z}^n \setminus \Omega$ that are adjacent to some vertices in Ω , i.e.

$$\mathcal{B} := \{x \in \mathbb{Z}^n \setminus \Omega : x \sim y \text{ for some } y \in \Omega\}.$$

In notation, we write $\bar{\Omega} := \Omega \cup \mathcal{B}$ and $\mathcal{E}_{\bar{\Omega}} := \mathcal{E} \cap \bar{\Omega}^2$. This induces the graph structure $(\bar{\Omega}, \mathcal{E}_{\bar{\Omega}})$, still denoted by $\bar{\Omega}$ for simplicity. We denote by $\mathbb{R}^{\bar{\Omega}}$ the set of all real functions on $\bar{\Omega}$, by $l_0^2(\bar{\Omega})$ the subset of functions which are finitely supported and by $l^2(\bar{\Omega})$ the Hilbert space of functions on $\bar{\Omega}$ equipped with the inner product

$$\langle u, v \rangle_{\bar{\Omega}} := \sum_{x \in \bar{\Omega}} u(x)v(x) \quad u, v \in \mathbb{R}^{\bar{\Omega}}.$$

For $u \in l^2(\bar{\Omega})$ the Dirichlet energy of u is defined as

$$(2.2) \quad \mathcal{D}_{\bar{\Omega}}(u) := \sum_{(i,j) \in \mathcal{E}_{\bar{\Omega}}} (u(i) - u(j))^2.$$

Set $i \in \bar{\Omega}$, we define the square of the gradient by

$$|\nabla u|^2(i) := \sum_{j \in \bar{\Omega}, j \sim i} (u(i) - u(j))^2$$

and we mention here that $\mathcal{D}_{\bar{\Omega}}(u) = \|\nabla u\|_{\bar{\Omega}}^2$, for all $u \in l_0^2(\bar{\Omega})$. The Laplace operator $\Delta : \mathbb{R}^{\bar{\Omega}} \rightarrow \mathbb{R}^{\bar{\Omega}}$ on Ω is defined as

$$\Delta u(i) := \sum_{j \in \bar{\Omega}, j \sim i} (u(i) - u(j)), \quad i \in \Omega$$

and the outward normal derivative operator $\partial_\nu : \mathbb{R}^{\bar{\Omega}} \rightarrow \mathbb{R}^{\mathcal{B}}$ on vertex boundary set \mathcal{B} is defined as

$$(2.3) \quad \partial_\nu u(i) := \sum_{j \in \bar{\Omega}, j \sim i} (u(i) - u(j)), \quad i \in \mathcal{B}.$$

An important concept in [25] is that of an exhaustion of an infinite graph by finite subsets to define the Dirichlet-to-Neumann operator in the infinite subgraph setting. A sequence of finite subsets of an infinite graph G satisfying $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \dots \subset \mathcal{O}_k \subset \dots$ and $\mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{O}_k$, denoted by $\mathcal{O} \uparrow \mathcal{V}$, is called an exhaustion of G . For quantities that are monotone in \mathcal{O} , i.e. if $\mathcal{O}_1 \subset \mathcal{O}_2$ (resp. $u(\mathcal{O}_1) \subseteq u(\mathcal{O}_2)$), one has $u(\mathcal{O}_1) \leq u(\mathcal{O}_2)$ (resp. $u(\mathcal{O}_1) \geq u(\mathcal{O}_2)$) and we write $\lim_{\mathcal{O} \uparrow \mathcal{V}} u(\mathcal{O}) := \lim_{k \rightarrow \infty} u(\mathcal{O}_k) = u(\mathcal{V})$. Note that for monotone functions in \mathcal{O} this limit exists, and it does not depend on the choice of the exhaustion. Given \mathcal{B} a finite set of \mathbb{Z}^n . For $k \in \mathbb{N}$, let $\mathcal{O}_k \subset \mathbb{Z}^n$ be a finite subset such that $\mathcal{O}_k \cap \mathcal{B} \neq \emptyset$. In [25], the authors apply the well-known maximum principle, see e.g. [22] that implies to define the Dirichlet-to-Neumann operator in the discrete infinite setting defined as:

$$(2.4) \quad \Lambda : l^2(\mathcal{B}) \rightarrow \mathbb{R}^{\mathcal{B}}, \quad \psi \mapsto \Lambda \psi := \partial_\nu u_\psi$$

where ∂_ν is defined as in (2.3) and $u_\psi \in l^2(\bar{\Omega})$ is the unique harmonic extension of ψ to Ω satisfies

$$u_\psi := \lim_{k \rightarrow \infty} u_\psi^{\mathcal{O}_k},$$

where $u_\psi^{\mathcal{O}_k}$ be the solution of the following Dirichlet problem:

$$\begin{cases} \Delta u_\psi^{\mathcal{O}_k} = 0 & \text{in } \mathcal{O}_k \cap \Omega \\ \partial_\nu u_\psi^{\mathcal{O}_k} = \psi & \text{on } \mathcal{O}_k \cap \mathcal{B} \\ \partial_\nu u_\psi^{\mathcal{O}_k} = 0 & \text{on } \delta \mathcal{O}_k \end{cases}$$

where $\delta \mathcal{O}_k$ is the vertex boundary of \mathcal{O}_k , for any $k \in \mathbb{N}$. This implies to define the Steklov problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \partial_\nu u = \sigma u & \text{on } \mathcal{B} \end{cases}$$

Let σ be an eigenvalue of the Dirichlet-to-Neumann operator Λ and u its associated eigenfunction, then we have that the pair (σ, u) is a solution of the Steklov problem. The solutions of the Steklov problem coincide with the eigenvalues of the discrete DtN operator, which are called the Steklov eigenvalues. The spectrum of the Steklov problem form a discrete sequence ranked in ascending order:

$$0 \leq \sigma_0(\Omega) \leq \sigma_1(\Omega) \leq \dots \leq \sigma_k(\Omega) \leq \dots \leq \sigma_{|\mathcal{B}|-1}(\Omega).$$

where $0 \leq k \leq |\mathcal{B}| - 1$. Due to the variational principle, $\sigma_k(\Omega)$ can be characterized by the Reighley quotient, see [19],

$$(2.5) \quad \sigma_k(\Omega) := \min_{\mathcal{H}} \max_{u \in \mathcal{H}, u \neq 0} \mathcal{R}(u) = \min_{\mathcal{H}} \max_{u \in \mathcal{H}, u \neq 0} \frac{\mathcal{D}_{\overline{\Omega}}(u)}{\|u\|_{\mathcal{B}}^2}$$

where the Dirichlet energy of u , $\mathcal{D}_{\overline{\Omega}}(u)$, is defined as in (2.2), \mathcal{H} is a subset of $l_0^2(\overline{\Omega})$ of dimension $k + 1$ and $\mathcal{R}(u)$ is the Rayleigh quotient associated to the Dirichlet-to-Neumann operator on Ω .

We start with a natural question about the positivity of $\sigma_0(\Omega)$. A random walk is said to be recurrent if it returns to its initial position with probability one, and if it is not recurrent it is called transient. Polya's classic result [34] is the following:

Theorem 1 (Polya's, 1921). *The simple random walk on \mathbb{Z}^n is recurrent in dimensions $n = 1, 2$ and transient in dimension $n \geq 3$.*

Definition 1. *For any finite subset $F \subseteq \overline{\Omega}$, we define the capacity of $\overline{\Omega}$ by*

$$Cap_F(\overline{\Omega}) := \inf_u \|\nabla u\|_{\overline{\Omega}}^2$$

where $u \in l_0^2(\overline{\Omega})$, $0 \leq u \leq 1$ and $u(x) = 1$ for all $x \in F$.

Given an infinite subset Ω of a graph. For the subgraph $(\overline{\Omega}, \mathcal{E}_{\overline{\Omega}})$ to be recurrent there is a sufficient criterion in [37] based on the capacity of $\overline{\Omega}$.

Lemma 2.1. (Woess, 2000) *$\overline{\Omega}$ is recurrent if and only if $Cap_F(\overline{\Omega}) = 0$ for any finite subset $F \subset \overline{\Omega}$.*

In [25], the authors give a necessary and sufficient condition to characterize the positivity of $\sigma_0(\Omega)$ using the above lemma:

Proposition 2.1 (Hua and al., 2018). *Let Ω be an infinite subset of a graph with finite boundary then $\sigma_0(\Omega) = 0$ if and only if $(\overline{\Omega}, \mathcal{E}_{\overline{\Omega}})$ is recurrent.*

Remark 1. *In the context of integer lattices, we combine the previous results to conclude that $\sigma_0(\Omega) > 0$ if and only if $n \geq 3$.*

Let (Ω, \mathcal{B}) be an infinite subgraph with boundary of the integer lattice \mathbb{Z}^n . Given x and y two vertices of $\overline{\Omega}$. A path γ joining x and y is a sequence x_1, \dots, x_ρ in $\overline{\Omega}$ such that $x = x_1$, $y = x_\rho$ and $x_j \sim x_{j+1}$ for all $1 \leq j \leq \rho - 1$. The length of the path γ is the number ρ . The subgraph $\overline{\Omega}$ is connected if any two vertices can be connected by a path. In the sequel, we consider only infinite subgraph with finite boundary of \mathbb{Z}^n and connected. This does not mean that the boundaries are necessarily connected. This allows us to define the combinatorial distance on $\overline{\Omega}$ as follows:

Definition 2 (Distance). *The combinatorial distance between two vertices x and y on $\overline{\Omega}$ is the length of the shortest path on $\overline{\Omega}$ joining x and y , denoted by $d_{\overline{\Omega}}$, i.e*

$$d_{\overline{\Omega}}(x, y) := \min \{|\gamma| : \gamma \text{ joining } x \text{ and } y\}, \text{ for all } x \neq y$$

and $d_{\overline{\Omega}}(x, x) = 0$, for all $x \in \overline{\Omega}$.

The sphere and the ball of \mathbb{Z}^n endowed with an origin \mathcal{O} and a radius R given respectively by

$$\mathcal{S}_R := \{x \in \mathbb{Z}^n : d_{\mathbb{Z}^n}(\mathcal{O}, x) = R\} \text{ and } \mathcal{B}_R := \{x \in \mathbb{Z}^n : d_{\mathbb{Z}^n}(\mathcal{O}, x) < R\}.$$

The main results of this paper are on lower bounds of the first Steklov eigenvalue:

Theorem 2. Set $n \geq 3$ and $R > 0$. Let $(\Omega = \mathbb{Z}^n \setminus \overline{\mathfrak{B}_R}, S_R)$ be an infinity subset of the integer lattice \mathbb{Z}^n with boundary S_R where \mathfrak{B}_R the ball of \mathbb{Z}^n of radius $R > 0$. Then there exists a positive constant $C = C(n) > 0$ such that

$$\sigma_0(\Omega) \geq \frac{C(n-2) \sin^{\frac{n+3}{2}}(1)}{R}.$$

Remark 2. Combining Theorem 2 and Proposition 3.1, we deduce that $\sigma_0(\Omega) \simeq 1/R$ where Ω an infinite subgraph of \mathbb{Z}^n with the boundary \mathcal{B} is a sphere of radius R . This permits to conclude that $\sigma_0(\Omega) |\mathcal{B}|^{\frac{1}{n-1}}$ is bounded.

Corollary 2.1. Set $n \geq 3$ and p an arbitrary vertex of \mathbb{Z}^n . Let $(\Omega, \{p\})$ be an infinite subgraph of the integer lattice \mathbb{Z}^n . Then there exists a positive constant $C = C(n) > 0$ such that

$$\sigma_0(\Omega) \geq C(n-2) \sin^{\frac{n+3}{2}}(1).$$

Proof. Using the same technique in the proof of Theorem 2 with the domain $(1, \delta) \times \mathbb{S}^{n-1}$. \square

Theorem 3. Given $n \geq 4$ and let (Ω, \mathcal{B}) be an infinite subgraph of the integer lattice \mathbb{Z}^n with the boundary \mathcal{B} is considered as a subset of \mathbb{Z} . Then there exists a positive constant $C = C(n) > 0$ such that

$$\sigma_0(\Omega) \geq C(n-3) \sin^{\frac{n+2}{2}}(1).$$

In particular $\sigma_0(\Omega) |\mathcal{B}|^{\frac{1}{n-1}} \rightarrow \infty$ when $|\mathcal{B}|$ tends to infinity.

Remark 3. In the case of \mathbb{Z}^3 , the infinite subgraph in Theorem 3 with its boundary as a subset of \mathbb{Z} of cardinal greater or equal to 2 is more complicated to find a lower bound for the Steklov eigenvalue. We leave this question open. But we show in section 3 that the Steklov eigenvalue of this example tends to zero as the cardinal of the boundary tends to infinity, see Fig. 3.

The lower bound in Theorem 3 is sharp and does not depend on the length of the boundary for all $n \geq 4$. This permits we extend this lower bound to infinite subgraph where its boundary as a finite subset of \mathbb{Z}^d with $1 \leq d \leq n-3$.

Corollary 2.2. Set $n \geq 4$ and let (Ω, \mathcal{B}) be an infinite subgraph of the integer lattice \mathbb{Z}^n . For any $1 \leq d \leq n-3$, we suppose that \mathcal{B} is a finite subset of \mathbb{Z}^d . Then there exists a positive constant $C = C(n) > 0$ such that

$$\sigma_0(\Omega) \geq C(n-d-2) \sin^{\frac{n-d+3}{2}}(1).$$

Remark 4. This yields an interesting consequence that the bottom of the spectrum of (Ω, \mathbb{Z}^d) is greater or equal to $C(n-d-2) \sin^{\frac{n-d+3}{2}}(1)$, for all $n \geq 4$ and $1 \leq d \leq n-3$.

3. UPPER BOUND OF THE FIRST STEKLOV EIGENVALUE

In this section, we would give upper bounds of the first Steklov eigenvalue of some infinite subgraphs of integer lattices with finite boundary. Let us begin with an important example, where we take the boundary considered as a sphere S_R of radius R . The importance of this example appears in the fact that the first Steklov eigenvalue tends to zero proportionally to $1/R \approx 1/|S_R|^{\frac{1}{n-1}}$.

Proposition 3.1. Set $n \geq 3$ and let \mathfrak{B}_R be a ball of the integer lattice \mathbb{Z}^n of radius $R > 0$. Consider $(\Omega = \mathbb{Z}^n \setminus \overline{\mathfrak{B}_R}, S_R)$ an infinity subset of the integer lattice \mathbb{Z}^n . Then there exists a positive constant $C' = C'(n) > 0$ such that

$$\sigma_0(\Omega) \leq \frac{C'}{R}.$$

Proof. Set $a, b \in \mathbb{R}$ and let $u : \mathbb{N} \rightarrow \mathbb{R}$ be a test function given by

$$u(k) := \begin{cases} a + \frac{b}{k^{n-2}} & \text{if } k > R \\ 1 & \text{if } k = R \end{cases}$$

where $u(L) = 0$ for a certain L . We first calculate the Dirichlet energy of u between S_k and S_{k+1} :

$$(u(k) - u(k+1))^2 = b^2 \left(\frac{(k+1)^{n-2} - k^{n-2}}{k^{n-2}(k+1)^{n-2}} \right)^2 \simeq b^2 \left(\frac{k^{n-3}}{k^{n-2}(k+1)^{n-2}} \right)^2 \simeq \frac{b^2}{k^{2(n-1)}}.$$

Since there exist $\alpha, \beta \in (0, \infty)$ such that for all $k \in \mathbb{N}$, $\alpha k^{n-1} \leq |S_k| \leq \beta k^{n-1}$, then we obtain

$$\|\nabla u\|_{S_k}^2 = \sum_{\mathcal{E}_{S_k \times S_{k+1}}} (u(k) - u(k+1))^2 \leq \frac{2n\beta b^2 k^{n-1}}{k^{2(n-1)}} = \frac{2n\beta b^2}{k^{n-1}},$$

where $\mathcal{E}_{S_k \times S_{k+1}} := \{(x, y) \in \mathcal{E} : x \in S_k \text{ and } y \in S_{k+1}\}$. Thus,

$$\|\nabla u\|_{\Omega}^2 = \sum_{k=R}^L \sum_{\mathcal{E}_{S_k \times S_{k+1}}} (u(k) - u(k+1))^2 \leq 2n\beta b^2 \sum_{k=R}^L \frac{1}{k^{n-1}} \simeq \frac{2n\gamma\beta b^2}{R^{n-2}}$$

for a certain positive constant $\gamma > 0$. Moreover, since $u \equiv 1$ on S_R then we have that

$$\|u\|_{S_R}^2 \geq \alpha R^{n-1}.$$

Therefore, the Rayleigh quotient satisfies

$$\mathcal{R}(u) \leq \frac{2n\gamma\beta b^2}{\alpha R^{n-2} R^{n-1}}.$$

It remains to calculate b , we have

$$\begin{cases} u(R) = a + \frac{b}{R^{n-2}} = 1 \\ u(L) = a + \frac{b}{L^{n-2}} = 0 \end{cases}$$

Hence

$$a = -\frac{b}{L^{n-2}} \simeq 0 \text{ and } b = \frac{L^{n-2} R^{n-2}}{L^{n-2} - R^{n-2}} \simeq R^{n-2}, \text{ when } L \rightarrow \infty.$$

Then, we conclude that the Rayleigh quotient

$$\mathcal{R}(u) \leq \frac{C'}{R},$$

where $C' = 2n\gamma\beta/\alpha$. □

Remark 5. As $R \approx |S_R|^{\frac{1}{n-1}}$, this allows us to deduce that the first eigenvalue of Dirichlet-to-Neumann operator $\sigma_0(\Omega)$ tends to zero proportionally to $1/|\mathcal{B}|^{\frac{1}{n-1}}$. In Proposition 3.1, if the function test u take the value 1 on the ball \mathcal{B}_R . We can extend the result to find the same upper bound for the first Steklov eigenvalue of the infinite subgraph $(\Omega = \mathbb{Z}^n \setminus S_R, S_R)$.

We can extend this result to obtain an upper bound of the first Steklov problem on infinite subgraph of integer lattices depending on the diameter of its complementary.

Corollary 3.1. Set $n \geq 3$ and let \mathcal{K} be a connected finite subgraph of the integer lattice \mathbb{Z}^n of diameter \mathcal{D} . Let $\Omega = \mathbb{Z}^n \setminus \overline{\mathcal{K}}$ with its boundary $\delta\mathcal{K}$. Then, there exist a positive constant $C_0 = C_0(n) > 0$ such that

$$\sigma_0(\Omega) \leq \frac{C_0 \mathcal{D}^{n-2}}{|\delta\mathcal{K}|},$$

where $\delta\mathcal{K}$ denotes the vertex boundary of \mathcal{K} and $\overline{\mathcal{K}} = \mathcal{K} \cup \delta\mathcal{K}$.

Proof. We extend the test function u in Proposition 3.1 by 1 on the ball $\mathfrak{B}_{\mathcal{D}/2}$ of radius $\mathcal{D}/2$ and such that $\mathcal{K} \subseteq \mathfrak{B}_{\mathcal{D}/2}$. Hence, it takes the value 1 on the boundary $\delta\mathcal{K}$. Using Proposition 3.1, we obtain that

$$\mathcal{R}(u) \leq \frac{2C' |S_{\mathcal{D}/2}|}{\mathcal{D}|\delta\mathcal{K}|}.$$

Then, its Rayleigh quotient is bounded by $C_0\mathcal{D}^{n-2}/|\delta\mathcal{K}|$ where $C_0 = C'/2^{n-2}$. \square

Remark 6. Corollary 3.1 is interested in the case of infinite subgraphs $\Omega = \mathbb{Z}^n \setminus \overline{\mathcal{K}}$ with its boundary $\delta\mathcal{K}$ where $|\delta\mathcal{K}| \geq \lambda\mathcal{D}^{n-1}$ for all $\lambda > 0$. This permits to conclude that $\sigma_0(\Omega)$ tends to 0 as its diameter tends to infinity. Moreover, we have that $\sigma_0(\Omega)|\delta\mathcal{K}|^{1/n-1}$ is bounded by $C_0/\lambda\mathcal{D}^{\frac{n-2}{n-1}}$, and then it tends to zero.

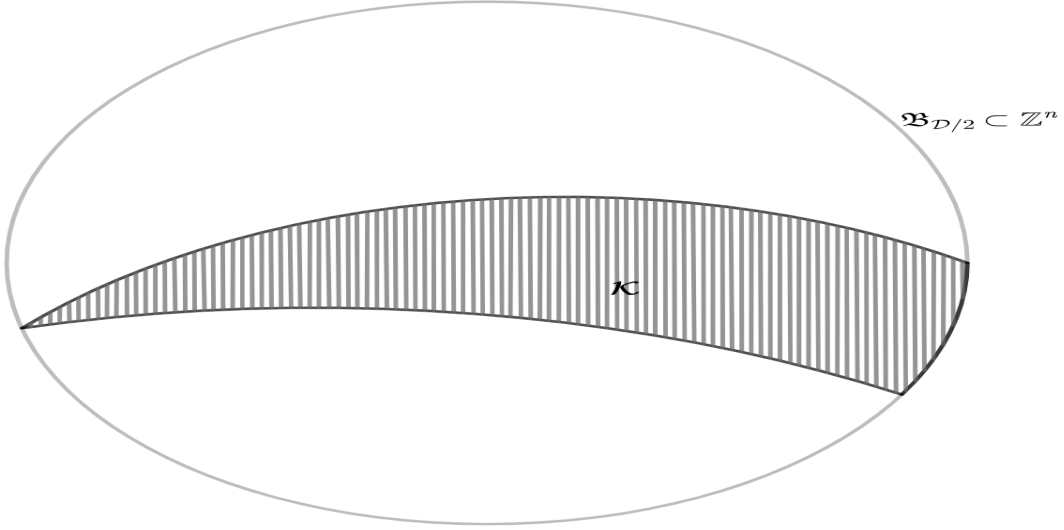


FIGURE 2. An infinite subgraph of \mathbb{Z}^n with its boundary $\delta\mathcal{K}$ of diameter \mathcal{D} in the ball $\mathfrak{B}_{\mathcal{D}/2}$

In the setting of Cayley graphs of groups with polynomial growth, we give a discrete isoperimetric inequality taken from Theorem 1 in [18].

Proposition 3.2 (T. Coulhon and al., 1993). *There exists a constant positive $\gamma > 0$ such that for any subgraph \mathcal{K} of a Cayley graph of groups with polynomial growth of order n , we have that*

$$|\overline{\mathcal{K}}|^{\frac{n-1}{n}} \leq \gamma |\delta\mathcal{K}|,$$

where $\delta\mathcal{K}$ denotes the vertex boundary of \mathcal{K} and $\overline{\mathcal{K}} = \mathcal{K} \cup \delta\mathcal{K}$.

Let $n \geq 4$ and $1 \leq d \leq n-3$. Corollary 2.2 prove that for any infinite subgraph of \mathbb{Z}^n with a boundary considered as a finite subset of \mathbb{Z}^d , we always find that its first Steklov eigenvalue does not tend to 0. The following example shows that if there exists an infinite subgraph with its boundary in \mathbb{Z}^{n-2} , where the first Steklov eigenvalue tends to 0.

Corollary 3.2. *Let \mathfrak{B}_R^{n-1} be a ball of \mathbb{Z}^{n-1} of radius R . Let $\Omega = \mathbb{Z}^n \setminus \overline{\mathfrak{B}_R^{n-1}}$ be an infinite subgraph with the boundary \mathfrak{B} . Then there exists a positive constant $\tau = \tau(n) > 0$ satisfies*

$$\sigma_0(\Omega) \leq \frac{\tau}{R^{\frac{1}{n}}}.$$

Proof. Given \mathfrak{B}_R^{n-1} a ball of \mathbb{Z}^{n-1} of radius R . Let $\Omega = \mathbb{Z}^n \setminus \overline{\mathfrak{B}_R^{n-1}}$ be an infinite subgraph with boundary \mathcal{B} . Then, we have $|\mathcal{B}| \simeq R^{n-2}$. Since $|\mathfrak{B}_R^{n-1}| \simeq R^{n-1}$, then by combining Corollary 3.1 and Proposition 3.2, we get a positive constant $\tau = \tau(n) > 0$ such that $\sigma_0(\Omega) \leq \tau/R^{\frac{1}{n}}$. \square

Remark 7. In the particular case where Ω is an infinite subgraph of \mathbb{Z}^3 with the boundary as a path of length $\rho \geq 2$. We prove that its first Steklov eigenvalue tends to zero as ρ tends to infinity. Indeed, we put the boundary in a ball of \mathbb{Z}^2 radius ρ . We extend the test function $u : \mathbb{N} \cup \{0\} \rightarrow \mathbb{R}$ in Proposition 3.1 by 1 on the ball as shown in the Fig. 3. Using Corollary 3.2, we deduce that

$$\sigma_0(\Omega) \leq \frac{\tau}{\rho^{1/3}}.$$

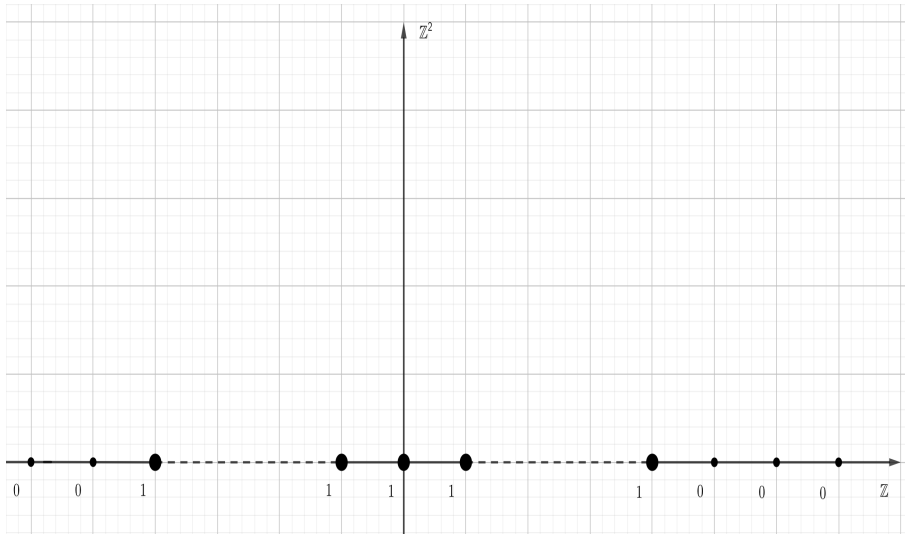


FIGURE 3. An infinite subgraph of \mathbb{Z}^3 with the boundary as a path of \mathbb{Z} where $\sigma_0(\Omega)$ tends to zero.

The following example gives an infinite subgraph Ω of \mathbb{Z}^n with its boundary \mathcal{B} where $\sigma_0(\Omega) |\mathcal{B}|^{\frac{1}{n-1}}$ tends to zero. In this example, we build a boundary of dimension n which consisting of several copies of a path of \mathbb{Z} by making a hole in \mathbb{Z}^n . This allows us to see the impact of the dimension of the boundary on the asymptotic behavior of the first Steklov eigenvalue.

Example 3.1. Set $n \geq 3$. In this example, we make a hole in \mathbb{Z}^n . Let us begin by explaining the hole as shown in the Fig. 4. Let \mathbb{P} be a path of \mathbb{Z} from p to q of length $\rho \geq 2$. For all $s \in \mathbb{N}$, we denote $S_{s,\mathbb{P}} := \{x \in \mathbb{Z}^n : d_{\mathbb{Z}^n}(x, \mathbb{P} \setminus \{p, q\}) = s\}$, the sphere of radius s . Let (Ω, \mathcal{B}) be an infinite subgraph of the integer lattice \mathbb{Z}^n such that $\Omega := \mathbb{Z}^n \setminus \{S_{1,\mathbb{P}} \cup S_{2,\mathbb{P}}\}$, $\mathbb{P} \subseteq \Omega$ and $S_{1,\mathbb{P}} \subset \mathcal{B}$. We give red color to the vertices of \mathcal{B} and black color to the vertices Ω . Hence we obtain that $|\mathcal{B}| \approx 4\rho$. We consider a test function $u : \mathbb{N} \rightarrow \mathbb{R}$ as shown in the Fig. 4 such that $u(p) = u(q) = 0$. Then we have

$$\|\nabla u\|_{\Omega}^2 \simeq 4n \text{ and } \|u\|_{\mathcal{B}}^2 \simeq 2\rho.$$

Therefore, the Rayleigh quotient satisfies

$$\mathcal{R}(u) \simeq \frac{2n}{\rho}.$$

This implies to conclude that

$$\sigma_0(\Omega) |\mathcal{B}|^{\frac{1}{n-1}} \leq \frac{2n}{\rho^{\frac{n-2}{n-1}}} \longrightarrow 0$$

when ρ tends to infinity.

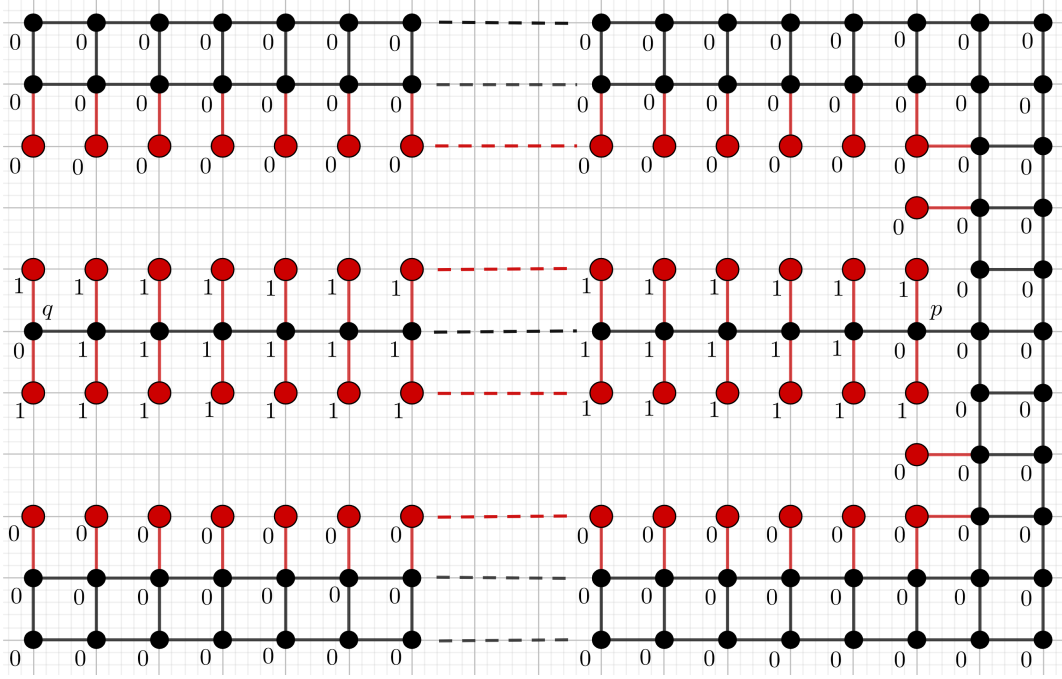


FIGURE 4. An infinite subgraph of \mathbb{Z}^n where $\sigma_0(\Omega) |\mathcal{B}|^{\frac{1}{n-1}}$ tends to zero.

Remark 8. Let $n \geq 4$, we can conjecture that $\sup \left\{ \sigma_0(\Omega) |\mathcal{B}|^{\frac{1}{n-1}} \right\} = \infty$ where Ω an infinite subgraph of \mathbb{Z}^n with the boundary \mathcal{B} considered as a subset of \mathbb{Z}^d , for all $1 \leq d \leq n - 3$, through the example of Brisson in [3] where the boundary of codimension greater or equal 2. This is what we will focus on providing in the sequel of this paper.

4. DISCRETIZATION OF A MANIFOLD

In this section, we would like to mention that this strategy is taken from the work of Colbois and al. in [15]. Let us begin by explaining the strategy of the discretization that leads to a graph with boundary. This discretization of an n -dimensional Riemannian manifold with smooth boundary $M \in \mathcal{M}(\kappa, r_0, n)$, satisfying precise hypothesis:

- † The boundary Σ admits a neighborhood which is isometric to the cylinder $[0, 1] \times \Sigma$, with the boundary corresponding to $\{0\} \times \Sigma$.
- † The Ricci curvature of M is bounded below by $-(n - 1)\kappa$.
- † The Ricci curvature of Σ is bounded below by $-(n - 2)\kappa$.
- † For each point $p \in M$ such that $d(p, \Sigma) > 1$, $\text{inj}_M(p) > r_0$.
- † For each point $p \in \Sigma$, $\text{inj}_\Sigma(p) > r_0$.

Definition 3. Given $\epsilon \in (0, \frac{r_0}{4})$. An ϵ -discretization of a submanifold (\mathcal{N}, Σ) of M is a procedure allowing to associate a graph with boundary $(\Gamma_\epsilon, \mathcal{B}_\Sigma)$ to \mathcal{N} such that (\mathcal{N}, Σ) is roughly isometric to $(\Gamma_\epsilon, \mathcal{B}_\Sigma)$. The graph $\Gamma_\epsilon = (\mathcal{V}_\epsilon, \mathcal{E}_\epsilon)$ obtained is a graph with boundary $(\Gamma_\epsilon, \mathcal{B}_\Sigma)$ declaring \mathcal{B}_Σ as the boundary.

We now explain the procedure of discretization. For $\epsilon \in (0, r_0/4)$, let \mathcal{B}_Σ be a maximal ϵ -separated set in Σ . Let $\mathcal{B}'_\Sigma = \{4\epsilon\} \times \mathcal{B}_\Sigma$ be a copy of \mathcal{B}_Σ in \mathcal{N} located 4ϵ away from Σ . Let $\mathcal{V}_\mathcal{I}$ be a maximal ϵ -separated set in $\mathcal{N} \setminus [0, 4\epsilon) \times \Sigma$ such that $\mathcal{B}'_\Sigma \subset \mathcal{V}_\mathcal{I}$. The set $\mathcal{V}_\epsilon = \mathcal{V}_\mathcal{I} \cup \mathcal{B}_\Sigma$ is endowed with a structure of a graph, declaring

- For all $x, y \in \mathcal{V}_\epsilon$ then $x \sim y \Leftrightarrow d_M(x, y) \leq 3\epsilon$.
- Any $x \in \mathcal{B}_\Sigma$ is adjacent to $x' = (4\epsilon, x) \in \mathcal{B}'_\Sigma$.

Given $1 \leq R < \delta$, let $\Omega := \mathfrak{B}_\delta \setminus \overline{\mathfrak{B}_R}$ be an annular domain of \mathbb{Z}^n . To better explain the relationships between Ω and $(R, \delta) \times \mathbb{S}^{n-1}$ on the mixed Steklov-Dirichlet problems, we present the following diagram:

$$\begin{array}{ccc}
 (\Omega, S_R \cup S_\delta) & \xleftarrow{\text{Roughly isometric}} & (\Gamma_\epsilon, B_\Sigma) \\
 & & \uparrow \epsilon\text{-Discretization} \\
 (\mathcal{N} = (R, \delta) \times \mathbb{S}^{n-1}, g_E) & \xleftarrow{\text{Quasi-isometry}} & (\mathcal{N} = (R, \delta) \times \mathbb{S}^{n-1}, g_S)
 \end{array}$$

where $1 \leq R < \delta$, $g_E := dr^2 + r^2 g_0$ is the Euclidean metric and $g_S := dr^2 + \delta^2 \sin^2(r/\delta) g_0$ correspond to the spherical metric on $(R, \delta) \times \mathbb{S}^{n-1}$.

5. MIXED STEKLOV DIRICHLET PROBLEM

Let (M, g) be a Riemannian manifold with boundary Σ . One way to get upper bounds of the Steklov spectrum on M is to compare it with a mixed Steklov-Dirichlet spectrum on domain $\mathcal{A} \subsetneq M$ such that $\Sigma = \partial\mathcal{A} \cap \Sigma$. We denote by $\partial_{\mathcal{I}}\mathcal{A} := \mathcal{A} \cap M$ the interior boundary of \mathcal{A} such that it is smooth. The following mixed Steklov-Dirichlet problem on \mathcal{A} is:

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{A} \\ u = 0 & \text{on } \Sigma \\ \partial_\nu u = \sigma^{\mathcal{D}} u & \text{on } \partial_{\mathcal{I}}\mathcal{A}. \end{cases}$$

The eigenvalues of this mixed problem form a discrete sequence

$$0 < \sigma_0^{\mathcal{D}}(\mathcal{A}) < \sigma_1^{\mathcal{D}}(\mathcal{A}) \leq \sigma_2^{\mathcal{D}}(\mathcal{A}) \leq \dots \nearrow \infty.$$

and the k -th eigenvalue is given by

$$(5.6) \quad \sigma_k^{\mathcal{D}}(\mathcal{A}) := \min_{B \subset \mathcal{H}_0^1(\mathcal{A})} \max_{0 \neq u \in B} \frac{\int_{\mathcal{A}} |\nabla u|^2 dV_M}{\int_{\Sigma} u^2 dV_\Sigma},$$

where $k \in \mathbb{N} \cup \{0\}$ and $\mathcal{H}_{k,0}(\mathcal{A})$ is the set of all $(k+1)$ -dimensional subspaces in the Sobolev space $\mathcal{H}_0^1(\mathcal{A}) = \{u \in \mathcal{H}^1(\mathcal{A}) : u = 0 \text{ on } \partial_{\mathcal{I}}\mathcal{A}\}$. By comparison between the two variational formulas given in (1.1) and (5.6), we get the following inequality for each $k \in \mathbb{N} \cup \{0\}$:

$$\sigma_k(M) \leq \sigma_k^{\mathcal{D}}(\mathcal{A}).$$

6. LOWER BOUND OF THE FIRST STEKLOV EIGENVALUE

The goal in this section is to prove the main results of Theorems 2 and 3. Our strategy is based on comparison between the mixed Steklov-Dirichlet problems through quasi-isometry.

6.1. Quasi-isometry. One of the essential factors to ensure the spectral comparison inequality between two Riemannian metrics of a compact manifold with smooth boundary is quasi-isometry:

Definition 4 (Quasi-isometry). *Let g_1, g_2 be two Riemannian metrics on a manifold M . We say that g_1 and g_2 are quasi-isometric if there exists a constant $K \geq 1$ such that for every $\xi \in M$ and for every $v \in T_\xi M \setminus \{0\}$, we have*

$$\frac{1}{K} \leq \frac{g_1(v, v)(\xi)}{g_2(v, v)(\xi)} \leq K.$$

The following proposition is borrowed from [15]:

Proposition 6.1 (Colbois and al., 2018). *Let M be a compact manifold of dimension n , with smooth boundary. Let g_1, g_2 be two Riemannian metrics on M which are quasi-isometric with constant $K \geq 1$. The Steklov and Steklov-Dirichlet eigenvalues with respect to g_1 and to g_2 satisfy the following inequality:*

$$\frac{1}{K^{1+\frac{n}{2}}} \leq \frac{\sigma_k(M, g_1)}{\sigma_k(M, g_2)} \leq K^{1+\frac{n}{2}}.$$

In order to compare the Steklov-Dirichlet eigenvalues of the sphere \mathbb{S}^n equipped with Riemannian metric $g_E := dr^2 + r^2 g_0$ and the Steklov-Dirichlet eigenvalues of a subgraph of the integer lattice \mathbb{Z}^n , it was necessary to have a quasi-isometry with the spherical metric $g_S := dr^2 + \delta^2 \sin^2(r/\delta) g_0$ with uniform control on constants.

Proposition 6.2. *Set $1 \leq R \leq r \leq \delta$. The Euclidean metric $g_E(r, \theta) := dr^2 + r^2 g_0(\theta)$ is quasi-isometric to the spherical metric $g_S(r, \theta) := dr^2 + \delta^2 \sin^2(r/\delta) g_0(\theta)$ with ratio $1/\sin^2(1)$ on $(R, \delta) \times \mathbb{S}^{n-1}$. Moreover, this quasi-isometry leads to*

$$\sin^{n+3}(1) \leq \frac{\sigma_k^D((\eta, \delta) \times \mathbb{S}^{n-1}, g_S)}{\sigma_k^D((\eta, \delta) \times \mathbb{S}^{n-1}, g_E)} \leq \frac{1}{\sin^{n+3}(1)}.$$

Proof. For $1 \leq R \leq r \leq \delta$, we set $h_\delta(r) = (\delta/r)^2 \sin^2(r/\delta)$. To show that the Euclidean metric g_E is quasi-isometric to the spherical metric g_S , it suffices to find a uniform control for h . Indeed, h_δ is a decreasing function on $[1, \delta]$. Then, $\sin^2(1) \leq h_\delta(r) \leq \delta^2 \sin^2(1/\delta)$, for all $r \in [1, \delta]$. Since $h_\delta \nearrow 1$ when δ goes to ∞ . Therefore, we obtain $\sin^2(1) \leq h_\delta \leq h_\infty = 1 \leq 1/\sin^2(1)$. Combine Definition 4 and Proposition 6.1 to finish the proof. \square

6.2. Rough isometry. The notion of rough isometry between two metric spaces is an essential factor to study the spectral comparison on the Steklov problem between graphs with boundary.

Definition 5 (Rough isometry). *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ two metric spaces. We say that a map $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$ is a rough isometric if there is $\alpha > 1$ and $\beta \geq 0$ satisfying:*

$$\forall x, y \in \mathcal{X}, \quad \frac{1}{\alpha} d_{\mathcal{X}}(x, y) - \beta \leq d_{\mathcal{Y}}(\Phi(x), \Phi(y)) \leq \alpha d_{\mathcal{X}}(x, y) + \beta$$

and there exists $\tau \geq 0$ such that

$$\bigcup_{x \in \mathcal{X}} \mathcal{B}(\Phi(x), \tau) = \mathcal{Y}.$$

Let's recall a rough isometry between modeled manifold and ϵ -discretization taken from Lemma 12 in [15]:

Lemma 6.1. *For any $\epsilon \in (0, \frac{r_0}{4})$, and any ϵ -discretization $(\Gamma_\epsilon, \mathcal{B}_\Sigma)$ of a manifold M , there exists a rough isometry $\Gamma_\epsilon \hookrightarrow M$ with constants dependent only on ϵ . Indeed, the following estimate holds:*

$$\frac{\epsilon}{4} d_{\Gamma_\epsilon}(x, y) - 10 \leq d_M(x, y) \leq 4\epsilon d_{\Gamma_\epsilon}(x, y).$$

To establish a relation between the mixed Steklov-Dirichlet eigenvalues of annular domains and the Steklov eigenvalues of the discretization of M through coarse discretization sense which is not sensitive to the local geometry and its diameter. The essential condition in choosing ϵ small and bounded by radius r_0 but if $\epsilon \rightarrow 0$, the control of the constants α, β and τ is lost. Therefore, these constants must be better controlled to guarantee the strategy of ϵ -discretization. The following result in [15] gives a control spectral between a domain in the class $\mathcal{M}(\epsilon, r_0, n)$ and its ϵ -discretization:

Theorem 4 (Colbois and al. 2018). *Given $\epsilon \in (0, r_0/4)$, there exist numbers $\alpha, \beta > 0$ depending on κ, r_0, n and ϵ such that any ϵ -discretization $(\mathcal{N}, \mathcal{V}_\Sigma)$ of a manifold $M \in \mathcal{M}(\epsilon, r_0, n)$ the following weaker estimate holds for each $0 \leq k \leq |\mathcal{V}_\Sigma| - 1$:*

$$\frac{\alpha}{k+1} \leq \frac{\sigma_k^D(M)}{\sigma_k^D(\Gamma_\epsilon, \mathcal{V}_\Sigma)} \leq \beta.$$

The following result from [15] gives a spectral comparison through rough isometrics between finite graphs with boundary:

Proposition 6.3 (Colbois and al., 2018). *Given $a \geq 1$ and $b, \tau \geq 0$, there exist constants A, B depending only on a, b, τ and on the maximal degree of vertices, such that any two graphs with boundary $(\Gamma_1, \mathcal{B}_1)$ and $(\Gamma_2, \mathcal{B}_2)$ which are roughly isometric with constants a, b, τ satisfies*

$$A \leq \frac{\sigma_k^{\mathcal{D}}(\Gamma_1, \mathcal{B}_1)}{\sigma_k^{\mathcal{D}}(\Gamma_2, \mathcal{B}_2)} \leq B$$

for each $0 \leq k \leq \min\{|\mathcal{B}_1|, |\mathcal{B}_2|\} - 1$.

We return to find a rough isometry between an annular domain of \mathbb{Z}^n and an ϵ -discretization of a cylinder equipped with the spherical metric g_S with constants depends on only ϵ and n .

Proposition 6.4. *Set $1 \leq R < \delta$ and let $\Omega := \mathfrak{B}_\delta \setminus \overline{\mathfrak{B}_R}$ be an annular domain of \mathbb{Z}^n . Then there exists a rough isometry between $(\mathcal{V}_\epsilon, B_\Sigma)$ and $(\Omega, S_R \cup S_\delta)$ with constants depending on ϵ, n and R where $(\mathcal{V}_\epsilon, B_\Sigma)$ is an ϵ -discretization of $((R, \delta) \times \mathbb{S}^{n-1}, g_S)$. Indeed, the following estimate holds:*

$$\frac{\epsilon}{4} \sin^{\frac{n+3}{2}}(1) d_\Omega(x, y) - \left(\frac{R\pi\epsilon}{2} + 10 \right) \sin^{\frac{n+3}{2}}(1) \leq d_{\Gamma_\epsilon}(x, y) \leq \frac{4\sqrt{n}\epsilon}{\sin^{\frac{n+3}{2}}(1)} d_\Omega(x, y).$$

Proof. Let $1 \leq R < \delta$ and let $\Omega := \mathfrak{B}_\delta \setminus \overline{\mathfrak{B}_R}$ be an annular domain of \mathbb{Z}^n . We consider the cylinder $((R, \delta) \times \mathbb{S}^{n-1}, g_S)$. We denote the ϵ -discretization of $((R, \delta) \times \mathbb{S}^{n-1}, g_S)$ by $(\Gamma_\epsilon, \mathcal{V}_\Sigma)$. To better explain the idea, we give the following diagram:

$$\begin{array}{ccc} (\Omega, S_R \cup S_\delta) & \xrightarrow{\text{Roughly isometric}} & (\Gamma_\epsilon, B_\Sigma) \\ & & \downarrow \text{Roughly isometric} \\ ((R, \delta) \times \mathbb{S}^{n-1}, g_E) & \xrightarrow{\text{Quasi-isometry}} & ((R, \delta) \times \mathbb{S}^{n-1}, g_S) \end{array}$$

We suppose there is a differentiable mapping $\gamma : [0, 1] \rightarrow (\mathcal{N}, \Sigma)$ given by $t \mapsto \gamma(t)$ where $\mathcal{N} = (R, \delta) \times \mathbb{S}^{n-1}$. For all $t \in [0, 1]$, we have $\dot{\gamma}(t) \in T_{\gamma(t)}\mathcal{N}$. By Definition 2.9 in [33], the length of γ is given as follows:

$$l_{g_S}(\gamma) := \int_0^1 \|\dot{\gamma}(t)\|_{g_S} dt = \int_0^1 g_S^{1/2}(\dot{\gamma}(t), \dot{\gamma}(t)) dt.$$

Let $p, q \in (R, \delta) \times \mathbb{S}^{n-1}$, the distance $d_S(p, q)$ is given as follows

$$d_S(p, q) := \inf_\gamma \{l_{g_S}(\gamma) : \gamma : [0, 1] \rightarrow (\mathcal{N}, \Sigma) \text{ a mapping of class } C^1 \text{ such that } \gamma(0) = p \text{ and } \gamma(1) = q\}.$$

Using Proposition 6.2, we obtain

$$\sin^{\frac{n+3}{2}}(1) l_{g_E}(\gamma) \leq l_{g_S}(\gamma) \leq \frac{l_{g_E}(\gamma)}{\sin^{\frac{n+3}{2}}(1)}, \quad \forall \gamma.$$

Since the distance does not depend on the path chosen and only depends on the length then by passing to the infimum

$$(6.7) \quad \sin^{\frac{n+3}{2}}(1) d_E \leq d_S \leq \frac{d_E}{\sin^{\frac{n+3}{2}}(1)}.$$

Given $x, y \in \mathcal{V}_\epsilon$. Lemma 6.1 gives

$$(6.8) \quad \frac{\epsilon}{4} d_{\Gamma_\epsilon}(x, y) - 10 \leq d_S(x, y) \leq 4\epsilon d_{\Gamma_\epsilon}(x, y).$$

where d_{Γ_ϵ} is the combinatorial distance on Γ_ϵ as defined in Definition 2. Combine (6.7) and (6.8), we have that

$$(6.9) \quad \frac{\epsilon}{4} \sin^{\frac{n+3}{2}}(1) d_{\Gamma_\epsilon}(x, y) - 10 \sin^{\frac{n+3}{2}}(1) \leq d_E(x, y) \leq \frac{4\epsilon}{\sin^{\frac{n+3}{2}}(1)} d_{\Gamma_\epsilon}(x, y).$$

Moreover, for any $p, q \in \mathbb{Z}^n$, we have that

$$(6.10) \quad \frac{1}{\sqrt{n}} d_{\mathbb{Z}^n}(p, q) \leq d_E(p, q) \leq d_{\mathbb{Z}^n}(p, q).$$

where $d_{\mathbb{Z}^n}$ is the combinatorial distance on \mathbb{Z}^n as defined in Definition 2. Indeed, if $p, q \in \mathbb{Z}^n$ such that $p \sim q$, we have that $d_E(p, q) = d_{\mathbb{Z}^n}(p, q) = 1$. Let $p, q \in \mathbb{Z}^n$ with $d(p, q) = \kappa$. This means that there exists the shortest path $\{x\}_{1 \leq i \leq \kappa-1}$ between p and q such that $x_1 = p$ and $x_\kappa = q$. The triangle inequality leads to

$$d_E(p, q) \leq \sum_{i=1}^{\kappa-1} d_E(x_i, x_{i+1}) = \sum_{i=1}^{\kappa-1} d_{\mathbb{Z}^n}(x_i, x_{i+1}) = d_{\mathbb{Z}^n}(p, q).$$

On the other hand, for any $p, q \in \mathbb{Z}^n$ we have

$$d_{\mathbb{Z}^n}^2(p, q) \leq \left(\sum_{i=1}^n |p_i - q_i| \right)^2 \leq n \sum_{i=1}^n |p_i - q_i|^2 = n d_E^2(p, q).$$

For $x, y \in \Omega = \mathfrak{B}_\delta \setminus \overline{\mathfrak{B}_R}$, we have that

$$(6.11) \quad d_{\overline{\Omega}}(x, y) - 2R\pi \leq d_{\mathbb{Z}^n}(x, y) \leq d_{\overline{\Omega}}(x, y).$$

Combine (6.9), (6.10) and (6.11), we obtain that

$$(6.12) \quad \frac{\epsilon}{4} \sin^{\frac{n+3}{2}}(1) d_{\overline{\Omega}}(x, y) - \left(\frac{R\pi\epsilon}{2} + 10 \right) \sin^{\frac{n+3}{2}}(1) \leq d_{\Gamma_\epsilon}(x, y) \leq \frac{4\sqrt{n}\epsilon}{\sin^{\frac{n+3}{2}}(1)} d_{\overline{\Omega}}(x, y).$$

for all $x, y \in \Omega$. Moreover, for all $\tau > 0$ we have

$$\bigcup_{x \in \Gamma_\epsilon} \mathfrak{B}(x, \tau) = \Omega,$$

where $\mathfrak{B}(x, \tau)$ denotes the ball of Γ_ϵ which endowed with an origin x and radius τ . Therefore, we finish the proof. \square

The proof of Theorems 2 and 3 is based on comparison between the first Steklov eigenvalues of mixed Steklov-Dirichlet problems on annular domains through a Euclidean metric g_E that is comparable to the spherical metric g_S in the sense of quasi-isometries. The Dirichlet condition makes it possible to restraint the problem to a finite subgraph where we can adapt the recent results for the compact manifolds taken in work of Colbois and al. in [15] to ensure the comparisons while ensuring that the results are extended to infinite domains.

Proof of Theorem 2. Set $n \geq 3$ and $R \geq 1$. Let $\Omega = \mathbb{Z}^n \setminus \overline{\mathfrak{B}_R}$ an infinite subgraph of \mathbb{Z}^n with its boundary S_R . Given $\delta > R$, and we take $\Omega_{R,\delta} = \mathfrak{B}_\delta \setminus \overline{\mathfrak{B}_R}$ as an annular domain of the integer lattice \mathbb{Z}^n with its boundary $S_R \cup S_\delta$. The mixed Steklov-Dirichlet problem on $\Omega_{R,\delta}$ given as follows:

$$(1) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega_{R,\delta} \\ u = 0 & \text{on } S_\delta \\ \partial_\nu u = \sigma^\mathcal{D} u & \text{on } S_R \end{cases}$$

The first Steklov-Dirichlet eigenvalue of the problem (1) is denoted by $\sigma_0^{\mathcal{D}}(\Omega_{R,\delta})$. Let us first denote that $\sigma_0(\Omega) = \lim_{\delta \rightarrow \infty} \sigma_0^{\mathcal{D}}(\Omega_{R,\delta})$. We consider a cylinder equipped to spherical metric g_S and isometry of $(R, \delta) \times \mathbb{S}^{n-1}$ with its boundary $\{R\} \times \mathbb{S}^{n-1} \cup \{\delta\} \times \mathbb{S}^{n-1}$. Then

$$\frac{\sigma_0^{\mathcal{D}}(\Omega_{R,\delta})}{\sigma_0^{\mathcal{D}}((R, \delta) \times \mathbb{S}^{n-1}, g_E)} = \frac{\sigma_0^{\mathcal{D}}(\Omega_{R,\delta})}{\sigma_0^{\mathcal{D}}(\Gamma_\epsilon)} \times \frac{\sigma_0^{\mathcal{D}}(\Gamma_\epsilon)}{\sigma_0^{\mathcal{D}}((R, \delta) \times \mathbb{S}^{n-1}, g_S)} \times \frac{\sigma_0^{\mathcal{D}}((R, \delta) \times \mathbb{S}^{n-1}, g_S)}{\sigma_0^{\mathcal{D}}((R, \delta) \times \mathbb{S}^{n-1}, g_E)}.$$

where Γ_ϵ is an ϵ -discretization of $(R, \delta) \times \mathbb{S}^{n-1}$. We then need to calculate the Steklov-Dirichlet eigenvalues of $(R, \delta) \times \mathbb{S}^{n-1}$ equipped with the Euclidean metric g_E . We will use the following mixed SteklovDirichlet on $(R, \delta) \times \mathbb{S}^{n-1}$:

$$(2) \begin{cases} \Delta u = 0 & \text{in } (R, \delta) \times \mathbb{S}^{n-1} \\ u = 0 & \text{on } R \times \mathbb{S}^{n-1} \\ \partial_\nu u = \sigma^{\mathcal{D}} u & \text{on } \{\eta\} \times \mathbb{S}^{n-1} \end{cases}$$

Let f be an eigenfunction of the first mixed Steklov-Dirichlet eigenvalue then $f : (R, \delta) \times \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ is written as $f(r, p) = u(r)v(p)$ where v is an eigenfunction for the first Laplace eigenvalue of the sphere \mathbb{S}^{n-1} and $u(r) = ar^k + br^{2-k-n}$ is a real valued function defined in (R, δ) such that $u(\delta) = 0$ and $u'(R) = -\sigma_k u(R)$. Hence

$$\begin{cases} (2-n)bR^{1-n} = -\sigma_0^{\mathcal{D}}(a+b)R^{2-n} \\ a + b\delta^{2-n} = 0 \end{cases}$$

On the other hand, let \mathbb{B}_R and \mathbb{B}_δ be the balls in \mathbb{R}^n , endowed with an origin of radius R and δ respectively. Applying Proposition 4 in [17] then the first mixed Steklov-Dirichlet eigenvalue of the annular domain $\mathcal{A}_{R,\delta} := \mathbb{B}_\delta \setminus \overline{\mathbb{B}_R}$ of \mathbb{R}^n , is

$$\sigma_0^{\mathcal{D}}((R, \delta) \times \mathbb{S}^{n-1}, g_E) = \sigma_0^{\mathcal{D}}(\mathcal{A}_{R,\delta}) = \frac{(n-2)R^{1-n}}{R^{2-n} - \delta^{2-n}}.$$

Moreover, Proposition 6.3 shows that there exists A, B depending on only ϵ, n and \mathbb{S}^n such that

$$A \leq \frac{\sigma_0^{\mathcal{D}}(\Omega_{R,\delta})}{\sigma_0^{\mathcal{D}}(\Gamma_\epsilon)} \leq B.$$

Using Theorem 4, we obtain that there exist a constant $\alpha = \alpha(\kappa, r_0, n) > 0$ such that

$$\frac{\sigma_0^{\mathcal{D}}(\Gamma_\epsilon)}{\sigma_0^{\mathcal{D}}((R, \delta) \times \mathbb{S}^{n-1}, g_S)} \geq \alpha.$$

Finally, using Proposition 6.2 we have

$$\frac{\sigma_0^{\mathcal{D}}((R, \delta) \times \mathbb{S}^{n-1}, g_S)}{\sigma_0^{\mathcal{D}}((R, \delta) \times \mathbb{S}^{n-1}, g_E)} \geq \sin^{n+3}(1).$$

Therefore,

$$\sigma_0^{\mathcal{D}}(\Omega_{R,\delta}) \geq \frac{\alpha A (n-2) R^{1-n} \sin^{\frac{n+3}{2}}(1)}{R^{2-n} - \delta^{2-n}}.$$

By passing δ to infinity, we conclude that

$$\sigma_0(\Omega) \geq \frac{\alpha A (n-2) \sin^{\frac{n+3}{2}}(1)}{R}.$$

The proof is finish.

Proof of Theorem 3. Given $\rho \geq 2$ and $n \geq 4$. Without loss of generality, we take Ω an infinite subgraph of \mathbb{Z}^n given by

$$\Omega = \left(\bigcup_{k=1}^{\rho} (k, 0_{\mathbb{Z}^{n-1}}) \right)^c.$$

Let u be an eigenfunction of $\sigma_0(\Omega)$. The Dirichlet energie of u satisfies

$$\begin{aligned} \sum_{i,j \in \mathcal{E}_{\overline{\Omega}}} (u(i) - u(j))^2 &\geq \sum_{k=1}^{\rho} \sum_{i,j \in \mathbb{Z}^{n-1} \times \{k\}, i \sim j} (u(i) - u(j))^2 \\ &\geq \sum_{k=1}^{\rho} \left(\frac{\sum_{i,j \in \mathbb{Z}^{n-1} \times \{k\}, i \sim j} (u(i) - u(j))^2}{u^2(k, 0_{\mathbb{Z}^{n-1}})} \right) u^2(k, 0_{\mathbb{Z}^{n-1}}). \end{aligned}$$

Theorem 2 permits to conclude that there exists a positive constant $C = C(n) > 0$ such that

$$\sum_{i,j \in \mathcal{E}_{\overline{\Omega}}} (u(i) - u(j))^2 \geq C(n-3) \sin^{\frac{n+2}{2}}(1) \sum_{k=1}^{\rho} u^2(k, 0_{\mathbb{Z}^{n-1}}).$$

Since $\|u\|_{\mathcal{B}}^2 = \sum_{k=1}^{\rho} u^2(k, 0_{\mathbb{Z}^{n-1}})$ where $\mathcal{B} := \bigcup_{k=1}^{\rho} (k, 0_{\mathbb{Z}^{n-1}})$ denotes the boundary of Ω , we obtain that

$$\mathcal{R}(u) \geq C(n-3) \sin^{\frac{n+2}{2}}(1).$$

By the variational characterization of the first eigenvalue for the Dirichlet-to-Neumann operator in this setting according to graphs, we obtain that

$$\sigma_0(\Omega) \geq C(n-3) \sin^{\frac{n+2}{2}}(1).$$

Thus, we finish the proof.

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REFERENCES

- [1] C. Anné and N. Torke-Hamza: *The Gauß-Bonnet operator of an infinite graph*, Anal. Math. Phys. **5**, no. 2, 137-159, (2015).
- [2] C. Bandle: *Isoperimetric inequalities and application*, volume 7 of Monographs and Studies in Mathematics. Pitman (Advanced Publishing Program), Boston, Mass. London (1980).
- [3] J. Brisson: *Tubular excision and Steklov eigenvalues*, The Journal of Geometric Analysis volume 32, Article number: 166 (2022).
- [4] F. Brock: *An isoperimetric equality for eigenvalues of the Steklov problem*, Math. Mech., 81(1): 69-71 (2001).
- [5] R. Brooks: *The first eigenvalue in a tower of coverings*. Bull. Amer. Math. Soc. (N.S.), 13(2):137140 (1985).
- [6] P. Buser: *On the bipartition of graphs*. Discrete Appl. Math., 9(1):105109 (1984).
- [7] I. Chavel: *Isoperimetric Inequalities*, Cambridge University Press (2001).
- [8] J. Cheeger: *A lower bound for the smallest eigenvalue of the Laplacian*, In problems in Analysis (Paper dedicated to Salomon Bochner, 1969), Pages 195- 199. Princeton, NJ (1970).
- [9] Alberto P. Caldreón: *On an inverse boundary value problem*, Computational and Applied Mathematics, 25(2), 133-138 (2006).
- [10] B. Colbois, A. Girouard: *The spectral gap of graphs and steklov eigenvalues on surfaces*, Electronic research announcements in mathematical sciences, 21, 19-27, doi:10.3934 (2014).
- [11] B. Colbois, A. Girouard: *Metric upper bounds for Steklov and Laplace eigenvalues*, arXiv:2108.03101v1 (2021).
- [12] B. Colbois and K. Gittins: *Upper bounds for Steklov eigenvalues of submanifolds in Euclidean space via the intersection index*, Differential Geom. Appl., (78) 101777 (2021).
- [13] B. Colbois, A. Girouard, A. Hassannezhad: *The Steklov and Laplacian spectra of Riemannian manifolds with boundary*, J. Funct. Anal., 278 (6), Article 108409 (2020).
- [14] B. Colbois, A. Girouard, K. Gittins: *Steklov eigenvalues of submanifolds with prescribed boundary in Euclidean space*, J. Geom. Anal., 29 (2) pp. 1811-1834 (2019).
- [15] B. Colbois A. Girouard, B. Raveendran: *The Steklov spectrum and coarse discretizations of manifolds with boundary*, Pure Appl. Math. Q., 14(2), 357-392 (2018).

- [16] B. Colbois, A. El Soufi, A. Girouard: *Isoperimetric control of the Steklov spectrum*, J. Funct. Anal., 261 (5) pp. 1384-1399 (2011).
- [17] B. Colbois, S. Verma: *Sharp Steklov upper bound for submanifolds of revolution*, J. Geom. Anal. 31:11214-11225, published online <https://doi.org/10.1007/s12220-021-00678-1> (2021).
- [18] T. Coulhon and L. Saloff-Coste: *Isopérimétrie pour les groupes et les variétés*, Rev. Mat. Iberoam. 9(2), 293-314 (1993).
- [19] E. B. Davies: *Spectral theory and differential operators*, volume 42. Cambridge University Press (1996).
- [20] J. Escobar: *The geometry of the first non-zero Steklov eigenvalue*, Journal of Functional Analysis, 150(2), 544-556 (1997).
- [21] J. Escobar: *An isoperimetric equality and the first Steklov eigenvalue*, J. Funct. Anal, 165(1), 101-116 (1999).
- [22] A. Grigor'yan: *Analysis on graphs*, Lecture Notes, University of Bielefeld (2009).
- [23] A. Girouard and I. Polterovich: *Spectral geometry of the Steklov problem*, Journal of Spectral Theory, 7(2), 321-359 (2017).
- [24] B. Hua, Y. Huang, Z. Wang: *First eigenvalue estimates of Dirichlet-to-Neumann operators on graph*, Calc. Var. 56:178 (2017).
- [25] B. Hua, Y. Huang, Z. Wang: *Cheeger estimates of Dirichlet-to-Neumann operators on infinite subgraphs of graphs*, arXiv:1810.10763v1 (2018).
- [26] W. Han, B. Hua: *Steklov Eigenvalue Problem on Subgraphs of Integer Lattices*, arXiv: 1902.05831v3 (2020).
- [27] A. Hassannezhad and L. Miclo: *Higher order Cheeger inequalities for Steklov eigenvalues*, Ann. Sci. Ecole Norm. Sup, 53(1), 43-88 (2020).
- [28] P. Jammes: *Une inégalité de Cheeger pour le spectre de Steklov*, Institut Fourier, 65(3): 1381-1385 (2015).
- [29] D. Lenz, M. Schmidt and M. Wirth: *Uniqueness of form extensions and domination of semigroups*, J. Funct. Anal. 280, no. 6, 108848, 27 pp (2021).
- [30] T. Mantuano: *Discretization of compact Riemannian manifolds applied to the spectrum of Laplacian*, Ann. Global Anal. Geom., 27(1): 33-46 (2005).
- [31] H. Perrin: *Lower bounds for the first eigenvalue of the Steklov problem on graphs*, Calculus of Variations and Partial Differential Equations volume 58, Article number: 67 (2019).
- [32] H. Perrin: *Isoperimetric Upper Bound for the First Eigenvalue of Discrete Steklov Problems*, The Journal of Geometric Analysis volume 31, pages 8144-8155 (2021).
- [33] M. P. do Carmo: *Riemannian Geometry*, Birkhäuser; DOI 10.1007/978-1-4757-2201-7, 1st ed. 1992, Corr. 14th printing 2013 edition (1 Jan. 1992).
- [34] G. Pólya: *Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Strassenetz*, Math. Ann. 84 149160 (1921).
- [35] V. Stekloff: *Sur les problèmes fondamentaux de la physique mathématique*, Ann. Sci. Ecole Norm. Sup. (3), 19: 455490 (1902).
- [36] L. Tschanz: *Upper bounds for Steklov eigenvalues of subgraphs of polynomial growth Cayley graphs*, Ann Glob Anal Geom, 61(1): 37-55 (2022).
- [37] W. Woess: *Random walks on infinite graphs and groups*, volume 138. Cambridge University Press (2000).

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