ASYMPTOTIC BEHAVIOR OF THE LINEARIZED COMPRESSIBLE BAROTROPIC NAVIER-STOKES SYSTEM WITH A TIME VARYING DELAY TERM IN THE BOUNDARY OR INTERNAL FEEDBACK

Subrata Majumdar

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SUBRATA MAJUMDAR*

Abstract. In this paper, we consider the linearized compressible barotropic Navier-Stokes system in a bounded interval $(0, L)$ with a time-varying delay term acting in the Dirichlet boundary or internal feedback of the hyperbolic component. Assuming some suitable conditions on the time-dependent delay term and the coefficients of feedback (delayed or not), we study the exponential stability of the concerned hyperbolic-parabolic system. Due to the presence of the time-varying delay term, the corresponding spatial operator is also time-dependent. Using classical semigroup theory with Kato's variable norm approach, we first show the existence and uniqueness of the Navier-Stokes system with time delay, acting in the boundary or interior. Next, we prove the two stabilization results by means of interior delay feedback and boundary delay. In both cases, we establish the exponential stability results by introducing some suitable functional energy and using the Lyapunov function approach.

1. Introduction

1.1. **Setting of the problem.** Control and stability of *fluid flow* have been a significant topic of study and have numerous useful applications. Many researchers have been interested in the subject of the controllability of fluid flows, more so for *incompressible flow* (see [8], [32], [35], [61], [62], [64]) than for *compressible flow* (see [25], [26]). The stability analysis of the linearized compressible Navier-Stokes system is of interest to us in this research.

The Navier-Stokes equations in $\Omega \subset \mathbb{R}^N$ for a *compressible isothermal barotropic fluid* consists of *continuity equation* and the *momentum equation*:

$$\begin{cases} \partial_t \zeta(t,x) + \operatorname{div}[\zeta(t,x)\,\mathbf{v}(t,x)] = 0, \\ \zeta[\partial_t \mathbf{v}(t,x) + (\mathbf{v}(t,x).\nabla)\mathbf{v}(t,x)] = -\nabla p(t,x) + \nu\Delta\mathbf{v}(t,x) + (\lambda+\nu)\nabla\left(\operatorname{div}\mathbf{v}(t,x)\right), \end{cases}$$

$\zeta(t,x)$ and $\mathbf{v}(t,x) = (v_1(t,x), v_2(t,x), ..., v_N(t,x))$ denote *fluid density* and *velocity vector* in $\mathbb{R}^N$, p denotes the pressure and we assume that it satisfies the constitutive law

$$p(t,x) = a\zeta^\gamma(t,x) \quad t > 0,\ x \in \Omega \ \text{ for } a > 0, \gamma \geq 1.$$

The viscosity coefficients ν, λ are the constants satisfying the *thermodynamic restrictions*, $\nu > 0$, $\lambda + \nu \geq 0$ (see [29] for more details). The second equation can be written component wise as follows

$$\zeta\left(\partial_t v_i + \mathbf{v}.\nabla v_i\right) = -\partial_{x_i} p + \nu\Delta v_i + (\lambda+\nu)\partial_{x_i}[\operatorname{div}\,\mathbf{v}].$$

We write the Navier-Stokes equations for a viscous compressible barotropic fluid in a bounded subset $\Omega = (0, L), L < \infty$ of the real line as

$$\begin{gathered} \partial_t \zeta(t,x) + (\zeta v)_x(t,x) = 0, \\ \zeta(t,x)[\partial_t v(t,x) + v(t,x)v_x(t,x)] + (p(\zeta))_x(t,x) - \nu v_{xx}(t,x) = 0, \\ p(t,x) = a\zeta^\gamma(t,x) \quad t > 0,\ x \in \Omega, \end{gathered} \tag{1.1}$$

here ζ, v, p, ν are fluid density, velocity, pressure and viscosity like above. To get the linearized system around $(Q_0, V_0), Q_0 > 0, V_0 \geq 0$, we perform a change of variable:

$$\sigma(t,x) = \zeta(t,x) - Q_0,\ u(t,x) = v(t,x) - V_0,$$

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*Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai- 400076;
e-mail: sm18rs016@iiserkol.ac.in.

* Corresponding author: Subrata Majumdar.

and collect the system including first order terms of σ, u and this leads to the linearized compressible Navier-Stokes system around (Q_0, V_0) as follows

$$\begin{cases} \sigma_t + V_0\sigma_x + Q_0 u_x = 0, & (t,x) \in (0,\infty)\times(0,L), \\ u_t - \frac{\nu}{Q_0}u_{xx} + V_0 u_x + a\gamma Q_0^{\gamma-2}\sigma_x = 0, & (t,x) \in (0,\infty)\times(0,L). \end{cases} \tag{1.2}$$

From the perspective of controllability and stabilization, the compressible Navier-stokes equations (1.2) is one of the fascinating topics of research. Lots of work have been done in this direction, and many results have yet to be explored, for example, the stabilizability of Navier-Stokes system (1.2) with Dirichlet boundary data

$$\sigma(t,0) = 0, u(t,0) = u(t,L) = 0. \tag{1.3}$$

We refer to the works [23], [22] [46], where controllability and stabilization of the hyperbolic-parabolic coupled system (1.2) by means of interior control with periodic boundary data have been studied. In [10] and [42], these null controllability results have been extended to more general transport-parabolic $(d \times d)$ systems with constant coefficients in one dimensional torus $\mathbb{T}$. Null and approximate controllability of (1.2) with Dirichlet boundary data can be found in [11] and [19], respectively. More precisely, boundary null controllability result has been explored in [11] with the control acting in the hyperbolic component $(\sigma(t,0) = q(t), u(t,0) = u(t,L) = 0)$, whereas approximate controllability has been studied with localized interior control in [19]. We must mention that in the work [46], the authors studied the rapid feedback stabilization (exponential stabilization with arbitrary prescribed decay rate) of (1.2) with an interior control acting everywhere in the parabolic equation in periodic set-up. Boundary stabilizability result of (1.2) with $V_0 = 0$ with certain decay ω_0 has been studied in [7], [20] and [24] in Dirichlet set up $(u(t,0) = u(t,1) = 0)$ by spectral analysis and the method of backstepping, respectively.

Let us write the system (1.2) in the abstract formation for Dirichlet boundary conditions (1.3):

$$\begin{cases} \dot{\mathbf{U}}(t) = \mathcal{A}\mathbf{U}(t),\ t > 0, \\ \mathbf{U}(0) = (\sigma_0, u_0)^T, \end{cases}$$

where the operator $\mathcal{A}$ is defined as following:

$$\begin{cases} \mathcal{D}(\mathcal{A}) = \{(\sigma,u) \in H^1(0,L)\times H^2(0,L)\cap H_0^1(0,L) : \sigma(0) = 0\}, \\ \mathcal{A}(\sigma,u)^T = \left(-V_0\sigma_x - Q_0 u_x, \frac{\nu}{Q_0}u_{xx} - V_0 u_x - a\gamma Q_0^{\gamma-2}\sigma_x\right)^T. \end{cases}$$

The operator $\mathcal{A}$ is maximal dissipative in $L^2(0,L)\times L^2(0,L)$. Thus it generates a contraction semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ of continuous operator. Moreover, the semigroup is exponentially stable, that is for any $\mathbf{U}_0 = (\sigma_0, u_0)^T \in L^2(0,L)\times L^2(0,L)$, there exist positive constants C, ω_0 such that

$$\|\mathcal{S}(t)\mathbf{U}_0\|_{L^2(0,L)\times L^2(0,L)} \leq Ce^{-\omega_0 t}\|\mathbf{U}_0\|_{L^2(0,L)\times L^2(0,L)}, t \geq 0,$$

see [36] for more details. In this paper, we show that despite the time-varying delay acting in the boundary or interior of the hyperbolic component, the concerned system retains its asymptotic behaviour with the presence of an additional damping term acting in the same component. We explore the time-delayed phenomenon in the asymptotic analysis for the Navier-Stokes system (1.4). More precisely, we study the well-posedness and stability analysis of the linearized compressible barotropic Navier-Stokes system with the effect of delay depending on time acting in the boundary term or as interior feedback. In the next section, we discuss the stability problems which are the main concerns of this article.

1.2. **Brief description about the problem.** Let us take $V_0 > 0$ in (1.2). Introducing the change of variables

$$\sigma(t,x) \mapsto \frac{1}{\varepsilon}\sigma\left(\frac{t}{\upsilon}, \frac{x}{\delta}\right), \quad u(t,x) \mapsto u\left(\frac{t}{\upsilon}, \frac{x}{\delta}\right),$$

with the following choices of $\epsilon, \upsilon, \delta > 0$,

$$\varepsilon := \left(a\gamma Q_0^{\gamma-3}\right)^{-1/2}, \quad \upsilon := \frac{Q_0 V_0^2}{\nu}, \quad \delta := \frac{Q_0 V_0}{\nu},$$

one can recast the system (1.2) into the following simplified version:

$$\begin{cases} \sigma_t + \sigma_x + bu_x = 0, & (t,x) \in (0,\infty)\times(0,\delta L), \\ u_t - u_{xx} + u_x + b\sigma_x = 0 & (t,x) \in (0,\infty)\times(0,\delta L), \end{cases} \tag{1.4}$$

with $b = \frac{Q_0}{V_0}\left(a\gamma Q_0^{\gamma-3}\right)^{1/2}$. We utilize the advantage of this reduction of the number of system parameters in our computation by considering the system (1.4) in our analysis. To describe the introductory study about the problems, let us first consider the following system:

$$\begin{cases} \sigma_t + \sigma_x + bu_x = 0, & (t,x) \in (0,\infty)\times(0,L), \\ u_t - u_{xx} + u_x + b\sigma_x = 0, & (t,x) \in (0,\infty)\times(0,L), \\ \sigma(t,0) = \alpha\sigma(t,L) + \beta\sigma(t-\tau(t),L),\ u(t,0) = u(t,L) = 0, & t \in (0,\infty), \\ \sigma(0,x) = \sigma_0(x),\ u(0,x) = u_0(x), & x \in (0,L), \\ \sigma(t-\tau(0),L) = z_0(t-\tau(0)), & 0 < t < \tau(0), \end{cases} \tag{1.5}$$

in which we assume that the time varying delay $\tau(t)$ satisfies the following properties

$$\tau \in W^{2,\infty}[0,T], \text{ for all } T > 0, \tag{1.6}$$

$$0 < \tau_0 \le \tau(t) \le M, \text{ for all } t \ge 0, \tag{1.7}$$

$$\dot{\tau}(t) \le m < 1, \text{ for all } t \ge 0, \text{ where } 0 \le m < 1. \tag{1.8}$$

Under the following condition on the damping parameters α, β:

$$|\alpha| + |\beta| + m < 1, \tag{1.9}$$

we prove that the functional energy associated to the system (1.5)

$$E(t) = \frac{1}{2}\int_0^L \sigma^2(t,x)dx + \frac{1}{2}\int_0^L u^2(t,x)dx + \frac{|\beta|\tau(t)}{2}\int_0^1 \sigma^2(t-\tau(t)\rho,L)d\rho,\ t \ge 0 \tag{1.10}$$

decays exponentially to zero. That is, there exist some positive constants $C > 0$, $\mu > 0$ such that the following is the infinite time behaviour of the energy of the solution of (1.5):

$$E(t) \le CE(0)e^{-\mu t},\ \forall t > 0.$$

Next, we study the same issue for the linearized compressible Navier-Stokes equation with internal time varying delay

$$\begin{cases} \sigma_t + \sigma_x + bu_x + a(x)\sigma(t,x) + c(x)\sigma(t-\tau(t),x) = 0, & (t,x) \in (0,\infty)\times(0,L), \\ u_t - u_{xx} + u_x + b\sigma_x = 0, & (t,x) \in (0,\infty)\times(0,L), \\ u(t,0) = u(t,L) = \sigma(t,0) = 0, & t \in (0,\infty), \\ \sigma(0,x) = \sigma_0(x),\ u(0,x) = u_0(x), & x \in (0,L), \\ \sigma(t-\tau(0),x) = z_0(t-\tau(0),x), & x \in (0,L), 0 < t < \tau(0), \end{cases} \tag{1.11}$$

where $a, c \in L^\infty(0,L)$ are two non-negative functions with $\operatorname{supp} c = \omega \subset (0,L)$ and

$$c(x) \ge c_0 \text{ a.e. in } \omega, \text{ where } c_0 \text{ is a positive number}, \tag{1.12}$$

$$a(x) \ge k_0 + \frac{2-m}{2(1-m)}c(x) \text{ a.e in } \omega, \text{ where } k_0 > 0. \tag{1.13}$$

Under the above assumptions on the damping functions a, c, we prove the exponential stability of the system by using the energy

$$E(t) = \frac{1}{2}\int_0^L \sigma^2(t,x)dx + \frac{1}{2}\int_0^L u^2(t,x)dx + \frac{\tau(t)}{2}\int_\omega\int_0^1 \xi(x)\sigma^2(t-\tau(t)\rho,x)\,dx\,d\rho,\ t \ge 0,$$

where $\xi \in L^\infty(0,L)$, $\operatorname{supp}\xi = \omega$, a non negative function to be chosen later.

1.3. **Bibliographical comments and motivation.** Delay effects arise in many applications and practical problems and may destabilize an otherwise exponentially stable system, see [44], [45], [27]. However, particular choices of delays may provide the exponential stability. Thus, it is important to explore the impact of the delay in the asymptotic behaviour of dynamical systems. Throughout years, exponential stability analysis of a dynamical systems governed by partial differential equations with time delay has gained immense interest among researchers. In [28], R. F. Datko et al. have shown that an arbitrarily small time delay in the feedback control may destabilize a distributed system described by the wave equation. Thereafter, numerous number of works have been performed on the stabilization of wave equations with time-delayed feedback controls [49], [50], [5], [33], [37], [52], [18], [4], [3]. Internal delay of an abstract wave equation with Kelvin-Voigt damping has been studied in [6]. For semilinear wave type equation see [57], nonlinear wave equation with switching delay see [33]. In the constant time delay case, we refer to the other related works: [54] for Schrodinger equation, [9] for KdV equation with boundary time-delay, [63] for KdV equation with interior delay feedback, [58] KdV equation with star shaped network, [12],

[13], and [17] for Kawahara equation with boundary and interior time delay feedback, respectively, [38] for KdV-Burger equation, Kuramoto-Sivashinsky equation with the time delay in the nonlinear term [65], Benjamin-Bona-Mahony equation [31], microbeam equation [30], other evolution equation with time delay feedback see [55].

On the other hand, there is a rich literature regarding the stability of the dynamical system with time-varying delay feedback. In [56], S. Nicaise et al. have studied the time-varying delay phenomenon for a parabolic and a hyperbolic model. More precisely, exponential stability has been studied for heat and wave equations in one dimension with boundary time-varying delays. This work motivates us to consider the time-dependent delay effect on the coupled hyperbolic-parabolic model, namely the linearized compressible Navier-Stokes system. To the best of the author's knowledge, there is no such result where the stability analysis of the linearized compressible Navier-Stokes system with time-dependent boundary or interior delay has been taken into account. This is indeed a motivation of this manuscript. Due to the presence of hyperbolic-parabolic coupling of two different orders of PDEs (transport-heat), the existence theory is not straightforward like heat or wave equations cases. We need to adopt a vanishing viscosity technique to establish the well-posedness of the time-dependent semigroup. Moreover, it is worth mentioning that, for the constant time delay case, there are some works related to the exponential decay analysis for coupled hyperbolic-parabolic systems with same orders spatial operators (heat-wave). More precisely, in the works [47] and [48], the authors have explored the exponential stabilization issue with internal delay acting in the first and second equation, respectively, for the system of linear thermoelasticity.

For the sake of completeness of the bibliographical study, let us mention some other related works for the time delayed system. In [53], S. Nicaise et al. have extended their previous result [56] for multi-dimensional wave system. Internal feedback stabilization with boundary delay for wave equation has been established [51]. Recently, in [59], Parada et al. have utilized the above mentioned works to produce stability analysis of KdV equation with interior and boundary time varying delay. Time-dependent delay phenomenon for second order evolution equation can be found in [34]. It is worth mentioning that, in the works [14], [16], [15], the authors have considered incompressible Navier-Stokes equations and studied its asymptotic behaviour with the distributed constant delay term (delay acting in the interior). Boundary time delay effect for the incompressible Navier-Stokes equations and higher dimensional compressible Navier-Stokes equations can be the objects of future research.

1.4. **Organization.** In the Section 2, we establish the existence and uniqueness of the solution of the linearized Navier-Stokes equation with boundary delay feedback. Exponential stability of the concerned system has been shown here, see Theorem 2.5. We study the well-posedness and stability analysis (Theorem 3.3) of (1.11), that is, the linearized Navier-Stokes system with internal delay in Section 3.

2. Exponential stability of Navier-Stokes with boundary delay

This section is devoted to the well-posedness and stability analysis of the linearized compressible Navier-Stokes system with a boundary delay feedback. At first, we show the existence and uniqueness of the Navier-Stokes system (1.5) with the presence of boundary delay term. We start by showing the well-posedness of the system (1.5) using semigroup theory and Kato's variable norm technique [41].

2.1. **Well-posedness.** We will mainly follow the work [56] to prove the well-posedness of the time varying system (1.5). Let us introduce a new variable to tackle the effect of the delay term

$$z(t,\rho) = \sigma(t - \tau(t)\rho, L), \rho \in (0,1) \text{ and } t > 0.$$

It can be checked that z satisfies the following set of equations

$$\begin{cases} \tau(t) z_t + (1 - \dot{\tau}(t)\rho) z_\rho = 0, & t \in (0,\infty), \\ z(t,0) = \sigma(t,L), & t \in (0,\infty), \\ z(0,\rho) = z_0(-\tau(0)\rho), & \rho \in (0,1). \end{cases} \tag{2.1}$$

Thus combining (1.5) and (2.1), we write the full system as follows:

$$\begin{cases} \sigma_t + \sigma_x + bu_x = 0, & (t,x) \in (0,\infty)\times(0,L), \\ u_t - u_{xx} + u_x + b\sigma_x = 0, & (t,x) \in (0,\infty)\times(0,L), \\ \tau(t)z_t + (1-\dot{\tau}(t)\rho)z_\rho = 0, & (t,\rho) \in (0,\infty)\times(0,1), \\ u(t,0) = u(t,L) = 0, & t \in (0,\infty), \\ \sigma(t,0) = \alpha\sigma(t,L) + \beta\sigma(t-\tau(t),L), & t \in (0,\infty), \\ z(t,0) = \sigma(t,L), & t \in (0,\infty), \\ \sigma(0,x) = \sigma_0(x),\ u(0,x) = u_0(x), & x \in (0,L), \\ z(0,\rho) = z_0(-\tau(0)\rho), & \rho \in (0,1), \\ \sigma(t-\tau(0),L) = z_0(t-\tau(0)), & 0 < t < \tau(0). \end{cases} \tag{2.2}$$

Now we will show the existence and uniqueness of the system (2.2) in a semigroup theory framework. Let us first write this system in infinite dimensional ODE set up. Denote

$$\mathbf{U} = (\sigma, u, z)^T, \mathbf{U}_0 = (\sigma_0, u_0, z_0(-\tau(0)\cdot))^T.$$

Thereafter, one can recast the above system as the following abstract ODE:

$$\begin{cases} \dot{\mathbf{U}}(t) = \mathcal{A}(t)\mathbf{U}(t), & t > 0, \\ \mathbf{U}(0) = \mathbf{U}_0, \end{cases} \tag{2.3}$$

where the time dependent operator $\mathcal{A}(t)$ can be written as

$$\mathcal{A}(t)\mathbf{U} = \begin{pmatrix} -\sigma_x - bu_x \\ u_{xx} - u_x - b\sigma_x \\ -\frac{1-\dot{\tau}(t)\rho}{\tau(t)} z_\rho \end{pmatrix}, \tag{2.4}$$

with the domain

$$\mathcal{D}(\mathcal{A}(t)) = \big\{(\sigma,u,z) \in H^1(0,L)\times H^2(0,L)\cap H_0^1(0,L)\times H^1(0,1) : z(0) = \sigma(L), \sigma(0) = \alpha\sigma(L) + \beta z(1)\big\}. \tag{2.5}$$

Clearly

$$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)),\ \forall\, t > 0.$$

Let us introduce the Hilbert space

$$\mathcal{H} = L^2(0,L)\times L^2(0,L)\times L^2(0,1),$$

equipped with the following inner product

$$\left\langle \begin{pmatrix} \sigma_1 \\ u_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} \sigma_2 \\ u_2 \\ z_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^L \sigma_1\sigma_2 dx + \int_0^L u_1u_2 dx + \int_0^1 z_1z_2 d\rho. \tag{2.6}$$

Now, using the variable norm theory of Kato [40], [39], [41], we will show the existence and uniqueness of the above abstract system with time dependent operator. Similar types of analysis has been done in many works, see [56], [53], [43], [59] for instances. The main idea of the existence-uniqueness theory is to show that the triplet $\{\mathcal{A}, \mathcal{H}, \mathcal{D}(\mathcal{A}(0)\}$ with $\mathcal{A} = \{\mathcal{A}(t) : t \in [0,T]\}$, for some fixed $T > 0$ forms a constant domain system. More precisely, the following theorem of Kato (see [41, Theorem 1.9]) is enough to serve our purpose:

Theorem 2.1. *Let us assume that*

- $\mathcal{D}(\mathcal{A}(0))$ *is dense in* $\mathcal{H}$,
- $\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)), \forall t > 0$,
- $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* $\mathcal{H}$ *for all* $t \in [0,T]$ *and the family* $\mathcal{A} = \{\mathcal{A}(t) : t \in [0,T]\}$ *is stable with stability constant* C *and* ω_0 *independent of* t *(i.e. the semigroup* $\{S_t(s)\}_{s\geq 0}$ *generated by* $\mathcal{A}(t)$*) satisfies* $\|S_t(s)u\|_{\mathcal{H}} \leq Ce^{\omega_0 s}\|u\|_{\mathcal{H}}$, *for all* $u \in \mathcal{H}$ *and* $s \geq 0$*),*
- $\partial_t\mathcal{A}(t)$ *belongs to* $L^\infty_*([0,T]; B(\mathcal{D}(\mathcal{A}(0)), \mathcal{H}))$, *the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0,T]$ *into the set* $B(\mathcal{D}(\mathcal{A}(0)), \mathcal{H}))$ *of bounded operators from* $\mathcal{D}(\mathcal{A}(0))$ *into* $\mathcal{H}$.

Then, the system (2.3) has a unique solution $\mathbf{U} \in \mathcal{C}([0,T]; \mathcal{D}(\mathcal{A}(0))) \cap \mathcal{C}^1([0,T]; \mathcal{H})$, *when the initial data lies in* $\mathcal{D}(\mathcal{A}(0))$.

Henceforth, we need to verify all the assumptions of the above Theorem 2.1 with the operator $(\mathcal{A}(t), \mathcal{D}(\mathcal{A}(0)))$ defined in (2.4)-(2.5) and this will ensure that the system (2.3) has a unique solution.

First, we define the following time dependent inner product on $\mathcal{H}$:

$$\left\langle \begin{pmatrix} \sigma_1 \\ u_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} \sigma_2 \\ u_2 \\ z_2 \end{pmatrix} \right\rangle_t = \int_0^L \sigma_1\sigma_2 dx + \int_0^L u_1u_2 dx + \tau(t)|\beta| \int_0^1 z_1z_2 d\rho.$$

It can be shown that the norm $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_t$ are equivalent. Indeed,

$$\min(1, |\beta|\tau_0) \|(\sigma, u, z)\|_{\mathcal{H}} \leq \|(\sigma, u, z)\|_t \leq (1 + |\beta|M) \|(\sigma, u, z)\|_{\mathcal{H}}. \tag{2.7}$$

Now, we are in the position of proving the well-posedness of our system (2.3):

Theorem 2.2. *For any $\mathbf{U}_0 = (\sigma_0, u_0, z_0(-\tau(0)\cdot))^T \in \mathcal{H}$, there exists a unique solution $\mathbf{U} \in \mathcal{C}([0,\infty); \mathcal{H})$ of the Navier-Stokes system (2.3). Moreover, if $\mathbf{U}_0 \in \mathcal{D}(\mathcal{A}(0))$, then the solution $\mathbf{U} \in \mathcal{C}([0,\infty); \mathcal{D}(\mathcal{A}(0))) \cap \mathcal{C}^1([0,\infty); \mathcal{H})$.*

Proof. We prove this result by showing all the assumptions of the Theorem 2.1 hold for the system (2.3).

- Straightforward analysis will show that $\mathcal{D}(\mathcal{A}(0))$ is dense in $\mathcal{H}$. Indeed, let $(f, g, w)^T \in \mathcal{H}$ such that $(f, g, w)^T \in \mathcal{D}(\mathcal{A}(0))^\perp$. Thus by definition, we have

  $$\int_0^L f(x)\sigma(x)dx + \int_0^L g(x)u(x)dx + \int_0^1 z(\rho)w(\rho)d\rho = 0, \text{ for all } (\sigma, u, z) \in \mathcal{D}(\mathcal{A}(0)).$$

  Let us take $\sigma = 0, u = 0$ and $z \in C_c^\infty(0,1)$. As $(0,0,z)^T \in \mathcal{D}(\mathcal{A}(0))$, we have

  $$\int_0^1 z(\rho)w(\rho)d\rho = 0.$$

  By density argument, we obtain $w = 0$. In a similar manner taking $z = 0, \sigma = 0, u \in C_c^\infty(0,L)$ or $z = 0, u = 0, \sigma \in C_c^\infty(0,L)$, we can prove that $f = g = 0$. Therefore we deduce that $\mathcal{D}(\mathcal{A}(0))^\perp = \{\mathbf{0}\}$. Thus we have $\mathcal{D}(\mathcal{A}(0))$ is dense in $\mathcal{H}$.

- By definition, the second conditions of Theorem 2.1 holds.

- Hereinafter, we show that the family $\mathcal{A} = \{\mathcal{A}(t), t \in [0,T]\}$ is stable. Let us denote $\Phi = (\sigma, u, z)^T$. It can be shown that

  $$\frac{\|\Phi\|_t}{\|\Phi\|_s} \leq e^{\frac{\hat{c}}{2\tau_0}|t-s|}, \forall\, t, s \in [0,T], \tag{2.8}$$

  where $\hat{c}$ is a positive constant. Indeed for all $t, s \in [0,T]$, we have

$$\|\Phi\|_t^2 - \|\Phi\|_s^2 e^{\frac{\hat{c}}{\tau_0}|t-s|} = (1 - e^{\frac{\hat{c}}{\tau_0}|t-s|}) \left(\int_0^L u^2(x)dx + \int_0^L \sigma^2(x)dx \right) + \left(\tau(t) - e^{\frac{\hat{c}}{\tau_0}|t-s|}\tau(s) \right) |\beta| \int_0^1 z^2(\rho)d\rho.$$

  Note that $(1 - e^{\frac{\hat{c}}{\tau_0}|t-s|}) \leq 0$. Furthermore, $\left(\tau(t) - e^{\frac{\hat{c}}{\tau_0}|t-s|}\tau(s) \right)$ is also non positive for some constant $\hat{c} > 0$. Indeed, as $\tau \in W^{2,\infty}[0,T], \forall\, T > 0$, by mean value theorem

  $$\tau(t) = \tau(s) + (t-s)\tau'(a),\ a \in (s,t),$$

  and therefore

  $$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{\hat{c}}{\tau_0}|t-s| \leq e^{\frac{\hat{c}}{\tau_0}|t-s|}.$$

  Thus we get the estimate (2.8).

- Next, we will show that $\mathcal{A}(t)$ is an infinitesimal generator of a contraction semigroup. To prove this, we first establish that the operator $\mathcal{A}(t)$ is maximal dissipative up to some bounded perturbation. First we evaluate $\langle \mathcal{A}(t)\mathbf{U}, \mathbf{U} \rangle_t$ for some fixed t. Let us take $\mathbf{U} = (\sigma, u, z)^T \in \mathcal{D}(\mathcal{A}(t))$. Then we have

  $$\langle \mathcal{A}(t)\mathbf{U}, \mathbf{U} \rangle_t = \left\langle \begin{pmatrix} -\sigma_x - bu_x \\ u_{xx} - u_x - b\sigma_x \\ -\frac{1-\dot{\tau}(t)\rho}{\tau(t)} z_\rho \end{pmatrix}, \begin{pmatrix} \sigma \\ u \\ z \end{pmatrix} \right\rangle_t.$$

Performing integration by parts and using the boundary conditions we obtain

$$\begin{aligned}\langle \mathcal{A}(t)\mathbf{U},\mathbf{U}\rangle_t =& -\int_0^L \sigma\sigma_x dx - b\int_0^L \sigma u_x dx - \int_0^L uu_x dx + \int_0^L uu_{xx}dx - b\int_0^L u\sigma_x dx + |\beta|\tau(t)\int_0^1 \frac{\dot\tau(t)\rho-1}{\tau(t)} zz_\rho\, d\rho\\ =&\frac{1}{2}\left(-\sigma^2(L)+\sigma^2(0)\right) - \frac{|\beta|\dot\tau(t)}{2}\int_0^1 z^2 d\rho + \frac{|\beta|(\dot\tau(t)-1)}{2}z^2(1) + \frac{|\beta|}{2}z^2(0) - \int_0^L u_x^2 dx\\ =&\frac{1}{2}\left(\alpha\sigma(L)+\beta z(1)\right)^2 - \frac{1}{2}\sigma^2(L) - \frac{|\beta|\dot\tau(t)}{2}\int_0^1 z^2 d\rho + \frac{|\beta|(\dot\tau(t)-1)}{2}z^2(1) + \frac{|\beta|}{2}\sigma^2(L) - \int_0^L u_x^2 dx.\end{aligned}$$

Let us define the following function

$$k(t) = \frac{(\dot\tau(t)^2+1)^{1/2}}{2\tau(t)},\ t\in[0,T]. \tag{2.9}$$

Thus we further have,

$$\begin{aligned}\langle \mathcal{A}(t)\mathbf{U},\mathbf{U}\rangle_t - k(t)\langle \mathbf{U},\mathbf{U}\rangle_t =& \frac{1}{2}\left(\alpha^2\sigma^2(L)+2\alpha\beta\sigma(L)z(1)+\beta^2 z^2(1)\right) + \frac{1}{2}(|\beta|-1)\sigma^2(L) - \int_0^L u_x^2\, dx\\ &- \left(\frac{|\beta|\dot\tau(t)}{2} + k(t)\frac{|\beta|\tau(t)}{2}\right)\int_0^1 z^2 d\rho - k(t)\int_0^L \sigma^2 dx - k(t)\int_0^L u^2 dx + \frac{|\beta|(\dot\tau(t)-1)}{2}z^2(1)\\ \le& \frac{1}{2}\left(\alpha^2\sigma^2(L)+2\alpha\beta\sigma(L)z(1)+\beta^2 z^2(1)\right) + \frac{1}{2}(|\beta|-1)\sigma^2(L) + \frac{|\beta|(m-1)}{2}z^2(1)\\ =&\frac{1}{2}\langle \mathbf{B}\Psi,\Psi\rangle_{\mathbb{R}^2},\end{aligned}$$

where $\Psi = (\sigma(L), z(1))^T$ and

$$\mathbf{B} = \begin{pmatrix} \alpha^2 - 1 + |\beta| & \alpha\beta \\ \alpha\beta & \beta^2 + |\beta|(m-1)\end{pmatrix}. \tag{2.10}$$

To prove

$$\langle \mathcal{A}(t)\mathbf{U},\mathbf{U}\rangle_t - k(t)\langle \mathbf{U},\mathbf{U}\rangle_t \le 0,$$

it is enough to show that $\mathbf{B}$ is negative definite matrix. Indeed, the following lemma will prove it. This implies that $\widetilde{\mathcal{A}}(t) = \mathcal{A}(t) - k(t)\mathcal{I}$ is a dissipative operator with the inner product $\langle\cdot,\cdot\rangle_t$.

Lemma 2.3. *Under the assumption (1.9), the matrix $\mathbf{B}$ defined in (2.10) is negative definite.*

Proof. Indeed, thanks to (1.9), we compute:

$$\begin{aligned}trace(\mathbf{B}) =& \alpha^2 - 1 + |\beta| + \beta^2 + |\beta|(m-1)\\ <& |\alpha| + |\beta| - 1 + |\beta|m < 0.\end{aligned}$$

In particular $\alpha^2 - 1 + \beta < 0$. Also,

$$\begin{aligned}det(\mathbf{B}) =& \alpha^2|\beta|(m-1) - \beta^2 - |\beta|(m-1) + |\beta|^3 + \beta^2(m-1)\\ =& |\beta|\left(\alpha^2(m-1) - m + 1 - 2|\beta| + \beta^2 + |\beta|m\right)\\ =& |\beta|\left(\alpha^2 m - \alpha^2 + (1-|\beta|)(1-|\beta|-m)\right) > 0.\end{aligned}$$

As $\mathbf{B}$ is a symmetric matrix, we write $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22}\end{pmatrix}$. Here $b_{11} < 0$, $b_{11}b_{22} - b_{12}^2 > 0$. Taking $\mathbf{x} = (x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$, we compute

$$\begin{aligned}\langle \mathbf{Bx},\mathbf{x}\rangle_{\mathbb{R}^2} =& b_{11}x^2 + 2b_{12}xy + b_{22}y^2\\ =& b_{11}\left(x + \frac{b_{12}}{b_{11}}y\right)^2 + \left(\frac{b_{22}b_{11} - b_{12}^2}{b_{11}}\right)y^2 < 0.\end{aligned}$$

Therefore $\mathbf{B}$ is negative definite. □

Next, we prove that for all $t\in[0,T]$, $\mathcal{A}(t)$ is maximal. Thus we need to prove that for some $\lambda > 0$,

$$\mathcal{R}(\lambda\mathcal{I} - \mathcal{A}(t)) = \mathcal{H}.$$

Let us first assume that $(f,g,h)^T \in \mathcal{H}$. We need to find $\mathbf{U} = (\sigma,u,z)^T \in \mathcal{D}(\mathcal{A}(t))$ such that $(\lambda\mathcal{I} - \mathcal{A}(t))\mathbf{U} = (f,g,h)^T$, which is equivalent to find $\mathbf{U} = (\sigma,u,z)^T \in \mathcal{D}(\mathcal{A}(t))$ such that

$$\begin{cases} \lambda\sigma + \sigma_x + bu_x = f, \\ \lambda u - u_{xx} + u_x + b\sigma_x = g, \\ \lambda z + \left(\frac{1-\dot\tau(t)\rho}{\tau(t)}\right) z_\rho = h, \\ u(0) = u(L) = 0, \\ \sigma(0) = \alpha\sigma(L) + \beta z(1), \\ z(0) = \sigma(L). \end{cases} \tag{2.11}$$

We mainly follow the work [56] to solve the above problem. If we are able to find u and σ with proper regularity then the third equation of the system (2.11) will give the solution z as follows:

$$z(\rho) = \begin{cases} \sigma(L)e^{-\lambda\tau(t)\rho} + \tau(t)e^{-\lambda\tau(t)\rho}\int_0^\rho e^{\lambda\tau(t)s}h(s)\,ds, & \text{if } \dot\tau(t) = 0, \\ \sigma(L)e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\rho)} + e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\rho)}\int_0^\rho e^{-\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)s)}\frac{h(s)\tau(t)}{(1-\dot\tau(t)s)}\,ds, & \text{if } \dot\tau(t) \neq 0. \end{cases}$$

Putting $\rho = 1$, we write $z(1) = \sigma(L)\mathcal{S}_0 + \mathcal{S}_h$, where

$$\mathcal{S}_0 = \begin{cases} e^{-\lambda\tau(t)}, & \text{if } \dot\tau(t) = 0, \\ e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))}, & \text{if } \dot\tau(t) \neq 0, \end{cases}$$

and

$$\mathcal{S}_h = \begin{cases} \tau(t)e^{-\lambda\tau(t)}\int_0^1 e^{\lambda\tau(t)s}h(s)\,ds, & \text{if } \dot\tau(t) = 0, \\ e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))}\int_0^1 e^{-\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)s)}\frac{h(s)\tau(t)}{(1-\dot\tau(t)s)}\,ds & \text{if } \dot\tau(t) \neq 0. \end{cases}$$

Thus, (σ,u) satisfies the following system

$$\begin{cases} \lambda\sigma + \sigma_x + bu_x = f, \\ \lambda u - u_{xx} + u_x + b\sigma_x = g, \\ u(0) = u(L) = 0, \\ \sigma(0) = (\alpha + \beta\mathcal{S}_0)\sigma(L) + \beta\mathcal{S}_h. \end{cases} \tag{2.12}$$

Let us consider a constant function ψ_0 defined by (with fixed t)

$$\psi_0(x) = \frac{\beta\mathcal{S}_h(t)}{(1-\alpha-\beta\mathcal{S}_0)},$$

and we assume that $\tilde\alpha = \alpha + \beta\mathcal{S}_0(t)$. Let us denote $\eta = \sigma - \psi_0$. Then, we can see that (η,u) satisfy the following equations

$$\begin{cases} \lambda\eta + \eta_x + bu_x = \tilde f, \\ \lambda u - u_{xx} + u_x + b\eta_x = g, \\ u(0) = u(L) = 0, \\ \eta(0) = \tilde\alpha\eta(L), \end{cases} \tag{2.13}$$

where $\tilde f = f - \lambda\psi_0 \in L^2(0,L)$. Thus the problem (2.13) boils down to showing that there exists $(\eta,u) \in \mathcal{D} = \left\{H^1(0,L)\times H^2(0,L)\cap H_0^1(0,L) : \eta(0) = \tilde\alpha\eta(L)\right\}$ satisfying (2.13).
We solve this problem by using vanishing viscosity approach. Let $\epsilon > 0$. Instead of solving directly the above problem (2.13), we first deal with the following regularized problem

$$\begin{aligned} \lambda\eta - \epsilon\eta_{xx} + \eta_x + bu_x &= \tilde f, \\ \lambda u - u_{xx} + b\eta_x + u_x &= g, \end{aligned} \tag{2.14}$$

with the following boundary conditions

$$\eta(0) = \tilde\alpha\eta(L), \quad \tilde\alpha\eta_x(0) = \eta_x(L), \quad u(0) = 0, \quad u(L) = 0.$$

Let us proceed through the following steps.

Step 1. We consider the space $\mathcal{V}$, given by

$$\mathcal{V} = \left\{(\eta,u) \in H^1(0,L)\times H^1(0,L) \ : \ \eta(0) = \tilde\alpha\eta(L), \quad u(0) = 0, \quad u(L) = 0\right\}.$$

Using Lax-Milgram theorem, we first prove that the system (2.14) has a unique solution in $\mathcal{V}$. Let us define the operator $\mathcal{B}:\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ by

$$\mathcal{B}\left(\begin{pmatrix}\eta\\u\end{pmatrix},\begin{pmatrix}\psi\\\varphi\end{pmatrix}\right)=\lambda\int_0^L\eta\psi dx+\epsilon\int_0^L\eta_x\psi_x dx+b\int_0^L u_x\psi dx+\int_0^L\eta_x\psi dx$$
$$+\lambda\int_0^L u\varphi dx+\int_0^L u_x\varphi_x dx+b\int_0^L\eta_x\varphi dx+\int_0^L u_x\varphi dx,$$

for all $\begin{pmatrix}\eta\\u\end{pmatrix},\begin{pmatrix}\psi\\\varphi\end{pmatrix}\in\mathcal{V}$. Then, one can show that B is continuous and coercive. Thus, by Lax-Milgram theorem, for every $\epsilon>0$, there exists a unique solution $(\eta^\epsilon,u^\epsilon)\in\mathcal{V}$ such that

$$\mathcal{B}\left(\begin{pmatrix}\eta^\epsilon\\u^\epsilon\end{pmatrix},\begin{pmatrix}\psi\\\varphi\end{pmatrix}\right)=\mathcal{G}\left(\begin{pmatrix}\psi\\\varphi\end{pmatrix}\right),\quad\forall\begin{pmatrix}\psi\\\varphi\end{pmatrix}\in\mathcal{V},\tag{2.15}$$

where $\mathcal{G}:\mathcal{V}\to\mathbb{R}$ is the linear functional given by

$$\mathcal{G}\left(\begin{pmatrix}\psi\\\varphi\end{pmatrix}\right):=\int_0^L\tilde{f}\psi dx+\int_0^L g\varphi dx.$$

Let us take $\begin{pmatrix}\psi\\0\end{pmatrix}\in\mathcal{V}$ in (2.15), so that we obtain

$$\lambda\int_0^L\eta^\epsilon\psi dx+\epsilon\int_0^L\eta^\epsilon_x\psi_x dx+b\int_0^L u^\epsilon_x\psi dx+\int_0^L\eta^\epsilon_x\psi dx=\int_0^L\tilde{f}\psi dx.\tag{2.16}$$

Similarly, by taking $\begin{pmatrix}0\\\varphi\end{pmatrix}\in\mathcal{V}$, we get

$$\lambda\int_0^L u^\epsilon\varphi dx+\int_0^L u^\epsilon_x\varphi_x dx+b\int_0^L u^\epsilon_x\varphi dx+b\int_0^L\eta^\epsilon_x\varphi dx=\int_0^L g\varphi dx.\tag{2.17}$$

These shows that the equations (2.14) are satisfied in the sense of distribution. Consequently it gives the regularity of η^ϵ_{xx} and u^ϵ_{xx} in $L^2(0,L)$. Thus the equation (2.14) is satisfied in the strong sense. Now multiplying the first equation of (2.14) by $\psi\in H^1(0,L)$ with $\psi(0)=\tilde{\alpha}\psi(L)$, and using integration by parts we obtain

$$\lambda\int_0^L\eta^\epsilon\psi dx+\epsilon\int_0^L\eta^\epsilon_x\psi_x dx-\epsilon\left(\eta^\epsilon_x(L)\psi(L)-\eta^\epsilon_x(0)\psi(0)\right)+b\int_0^L u^\epsilon_x\psi dx+\int_0^L\eta^\epsilon_x\psi dx=\int_0^L\tilde{f}\psi dx.\tag{2.18}$$

Comparing the above equation with (2.16), we obtain $\eta^\epsilon_x(L)=\tilde{\alpha}\eta^\epsilon_x(0)$.

Step 2. Now, from (2.15), observe that

$$\mathcal{B}\left(\begin{pmatrix}\eta^\epsilon\\u^\epsilon\end{pmatrix},\begin{pmatrix}\eta^\epsilon\\u^\epsilon\end{pmatrix}\right)=\mathcal{G}\left(\begin{pmatrix}\eta^\epsilon\\u^\epsilon\end{pmatrix}\right).$$

which yields, applying integration by parts and Young's inequality

$$\lambda\int_0^L|\eta^\epsilon|^2dx+\epsilon\int_0^L|\eta^\epsilon_x|^2dx+\frac{1}{2}(1-\tilde{\alpha}^2)|\eta^\epsilon(L)|^2+\lambda\int_0^L|u^\epsilon|^2dx+\int_0^L|u^\epsilon_x|^2dx$$
$$\leq\frac{\lambda}{2}\int_0^L|\eta^\epsilon|^2dx+\frac{\lambda}{2}\int_0^L|u^\epsilon|^2dx+C\left(\int_0^L\tilde{f}^2\,dx+\int_0^L g^2\,dx\right)$$

Note that $|\tilde{\alpha}|<1$, by (1.9) and the definition of $\tilde{\alpha}$. This shows that $(u^\epsilon)_{\epsilon\geq0}$ is bounded in $H^1(0,L)$, $(\eta^\epsilon)_{\epsilon\geq0}$ is bounded in $L^2(0,L)$ and $(\sqrt{\epsilon}\eta^\epsilon_x)_{\epsilon\geq0}$ is bounded in $L^2(0,L)$. Since the spaces $H^1(0,L)$ and $L^2(0,L)$ are reflexive, there exist subsequences, still denoted by $(u^\epsilon)_{\epsilon\geq0}$, $(\eta^\epsilon)_{\epsilon\geq0}$, and functions $\eta\in L^2(0,L)$ and $u\in H^1(0,L)$, $l\in\mathbb{R}$ such that

$$u^\epsilon\rightharpoonup u\text{ in }H^1(0,L),\text{ and }\eta^\epsilon\rightharpoonup\eta\text{ in }L^2(0,L),\ \eta^\epsilon(L)\to l.\tag{2.19}$$

Furthermore, we have

$$\int_0^L|\epsilon\eta^\epsilon_x|^2dx=\epsilon\int_0^L|\sqrt{\epsilon}\eta^\epsilon_x|^2dx\to0,\text{ as }\epsilon\to0.\tag{2.20}$$

Multiplying the first equation of (2.14) by $\psi \in C_c^\infty(0,L)$ and applying integration by parts, we get,

$$\lambda \int_0^L \eta^\epsilon \psi dx + \epsilon \int_0^L \eta_x^\epsilon \psi_x dx + b\int_0^L u_x^\epsilon \psi dx - \int_0^L \eta^\epsilon \psi_x = \int_0^L \tilde{f}\psi dx,\ \forall \psi \in C_c^\infty(0,L).$$

Then, passing to the limit $\epsilon \to 0$, we obtain

$$\lambda \int_0^L \eta\psi dx + b\int_0^L u_x \psi dx - \int_0^L \eta\psi_x dx = \int_0^L \tilde{f}\psi dx,$$

and the above relation is true $\forall \psi \in \mathcal{C}_c^\infty(0,L)$. As a consequence,

$$\eta_x + \lambda\eta + bu_x = \tilde{f}, \tag{2.21}$$

in the sense of distribution and therefore $\eta_x = f - bu_x - \lambda\eta \in L^2(0,L)$; in other words, $\eta \in H^1(0,L)$.

Step 3. We now show $u(0) = u(L) = 0$. Since the inclusion map $i : H^1(0,L) \hookrightarrow \mathcal{C}^0[0,L]$ is compact and $u^\epsilon \rightharpoonup u$ in $H^1(0,L)$, we obtain

$$u^\epsilon \to u \quad \text{in } \mathcal{C}^0[0,L].$$

Thus, $(u^\epsilon(0), u^\epsilon(L)) \to (u(0), u(L))$. Since $u^\epsilon(0) = u^\epsilon(L) = 0$ for all $\epsilon > 0$, we have

$$u(0) = u(L) = 0.$$

Similarly from the second equation of (2.14), one can deduce that

$$\lambda u - u_{xx} + u_x + b\eta_x = g, \tag{2.22}$$

in the sense of distribution and therefore $u_{xx} \in L^2(0,L)$, that is $u \in H^2(0,L)$.
Now replacing ψ from the equation of (2.16) by $\varphi \in C_c^\infty(0,L)$ and using (2.19), (2.20) and (2.21), we have

$$\lim_{\epsilon\to 0} \int_0^L \eta_x^\epsilon \varphi = \int_0^L \eta_x^\epsilon \varphi. \tag{2.23}$$

By density argument, we can say that $\eta_x^\epsilon \rightharpoonup \eta_x$ in $L^2(0,L)$. Similarly using the equations (2.17) and (2.22) we have $u^\epsilon \rightharpoonup u$ in $H^2(0,L)$.
We now show $\eta(0) = \tilde{\alpha}\eta(L)$. Let us first denote the space

$$\mathcal{W} = \left\{(\eta, u) \in H^1(0,L) \times H^2(0,L) \ : \ u(0) = 0, \quad u(L) = 0\right\}. \tag{2.24}$$

$\mathcal{W}$ is a Hilbert space with graph norm and $\mathcal{D}$ is a closed subspace of $\mathcal{W}$. Hence the weak closure coincides with the strong closure of $\mathcal{D}$ in $\mathcal{W}$. In our analysis, we have actually shown that (η, u) lies in the weak closure of $\mathcal{D}$ as $(\eta^\epsilon, u^\epsilon) \in \mathcal{D}$. Thus it follows that $(\eta, u) \in \mathcal{D}$. In particular, we have $\eta(0) = \tilde{\alpha}\eta(L)$.

Therefore we have proved that $\mathcal{A}(t)$ is maximal operator for any fixed $t \in [0,T]$. As $k(t) > 0$ defined in (2.9), it can be shown that the time dependent operator $\lambda\mathcal{I} - \widetilde{\mathcal{A}} = (\lambda + k(t))\mathcal{I} - \mathcal{A}$ is also surjective for some $\lambda > 0$. This shows that $\widetilde{\mathcal{A}}$ is maximal. Hence $\widetilde{\mathcal{A}}(t)$ generates a strongly continuous semigroup of contraction on $\mathcal{H}$. Also by (2.8), $\widetilde{\mathcal{A}} = \{\widetilde{\mathcal{A}}(t) : t \in [0,T]\}$ is a stable family of generators in $\mathcal{H}$ with suitably constants independent of t.

- We need to establish the last assumption of Theorem 2.1, that is, $\partial_t\mathcal{A}(t) \in L_*^\infty([0,T]; B(\mathcal{D}(\mathcal{A}(0)), \mathcal{H}))$. First we compute

$$\partial_t\mathcal{A}(t)\mathbf{U} = \begin{pmatrix} 0 \\ 0 \\ \frac{\tau(t)\ddot{\tau}(t)\rho - \dot{\tau}(t)(\dot{\tau}(t)\rho - 1)}{\tau^2(t)} z_\rho \end{pmatrix}. \tag{2.25}$$

It can be shown that $\frac{\tau(t)\ddot{\tau}(t)\rho - \dot{\tau}(t)(\dot{\tau}(t)\rho-1)}{\tau^2(t)} z_\rho$ is bounded on $[0,T]$ by the properties of τ (see (1.6), (1.7), (1.8)). Next from (2.9), we have

$$\dot{k}(t) = \frac{\ddot{\tau}(t)\dot{\tau}(t)}{2\tau(t)\left(1 + \dot{\tau}^2(t)\right)^{1/2}} - \frac{\dot{\tau}(t)\left(1 + \dot{\tau}^2(t)\right)^{1/2}}{2\tau^2(t)} \tag{2.26}$$

and it is bounded in $[0,T]$ for any $T > 0$. Thus combining (2.25) and (2.26), we finally have: $\partial_t\widetilde{\mathcal{A}}(t) \in L_*^\infty([0,T]; B(\mathcal{D}(\mathcal{A}(0)), \mathcal{H}))$.

Thanks to the Theorem 2.1, there exists unique solution $\widetilde{\mathbf{U}} \in \mathcal{C}([0,\infty);\mathcal{D}(\mathcal{A}(0))) \cap \mathcal{C}^1([0,\infty);\mathcal{H})$ of the following abstract system

$$\begin{cases} \partial_t \widetilde{\mathbf{U}}(t) = \widetilde{\mathcal{A}}(t)\widetilde{\mathbf{U}}(t), \\ \widetilde{\mathbf{U}}(0) = \mathbf{U}_0. \end{cases}$$

Let us denote

$$\mathbf{U}(t) = e^{\int_0^t k(s)ds}\widetilde{\mathbf{U}}(t).$$

It can be shown that this $\mathbf{U}(t)$ is the unique solution of (2.3). Because, on differentiation we have

$$\begin{aligned} \partial_t \mathbf{U}(t) =& k(t)e^{\int_0^t k(s)ds}\widetilde{\mathbf{U}}(t) + e^{\int_0^t k(s)ds}\partial_t\widetilde{\mathbf{U}}(t) \\ =& e^{\int_0^t k(s)ds}\left(k(t)\widetilde{\mathbf{U}}(t) + \widetilde{\mathcal{A}}(t)\widetilde{\mathbf{U}}(t)\right) \\ =& e^{\int_0^t k(s)ds}\mathcal{A}(t)\widetilde{\mathbf{U}}(t) \\ =& \mathcal{A}(t)\mathbf{U}(t). \end{aligned}$$

Thus the proof is completed. □

2.2. **Boundary stability result.** In this section, we prove one of the main results regarding the boundary stability of the system (1.5). It relies on the choice of suitable Lyapunov function and crucial energy estimate.

2.2.1. ***Energy decay.*** We recall the energy functional defined in (1.10)

$$E(t) = \frac{1}{2}\int_0^L \sigma^2(t,x)dx + \frac{1}{2}\int_0^L u^2(t,x)dx + \frac{|\beta|\tau(t)}{2}\int_0^1 \sigma^2(t-\tau(t)\rho, L)d\rho, \ t \geq 0. \tag{2.27}$$

The choice of this energy functional is classical when we deal with the time-varying delay effect through the boundary, see [56]. Below we prove that this energy (2.27) is non-increasing.

Proposition 2.4. *Let us assume that (1.6), (1.7), (1.8) and (1.9) hold. Then for all regular solution of the system (1.5), the energy E in (2.27) is non-increasing and satisfies:*

$$\dot{E}(t) \leq \frac{1}{2}\left\langle \mathbf{B}\begin{pmatrix}\sigma(t,L)\\ \sigma\left(t-\tau(t),L\right)\end{pmatrix}, \begin{pmatrix}\sigma(t,L)\\ \sigma\left(t-\tau(t),L\right)\end{pmatrix}\right\rangle_{\mathbb{R}^2} - \int_0^L u_x^2 \leq 0.$$

Proof. Indeed on differentiation (2.27) with respect to t, we have:

$$\dot{E}(t) = \int_0^L \sigma\sigma_t dx + \int_0^L uu_t dx + \frac{|\beta|\dot{\tau}(t)}{2}\int_0^1 \sigma^2(t-\tau(t)\rho,L)d\rho + |\beta|\tau(t)\int_0^1 \sigma(t-\tau(t)\rho,L)\partial_t\sigma(t-\tau(t)\rho,L)d\rho. \tag{2.28}$$

Plugging the expressions for σ_t, u_t from (1.5) and applying integration by parts, we have:

$$\begin{aligned} \int_0^L \sigma\sigma_t dx + \int_0^L uu_t dx =& \frac{1}{2}\left(\alpha^2\sigma^2(t,L) + 2\alpha\beta\sigma(t,L)\sigma\left(t-\tau(t),L\right) + \beta^2\sigma^2\left(t-\tau(t),L\right)\right) \\ & - \frac{1}{2}\sigma^2(t,L) - \int_0^1 u_x^2\, dx. \end{aligned} \tag{2.29}$$

Thanks to the identity

$$-\tau(t)\partial_t\sigma(t-\tau(t)\rho,L) = (1-\dot{\tau}(t)\rho)\partial_\rho\sigma(t-\tau(t)\rho,L)$$

and then applying integration by parts, we have

$$\begin{aligned} & |\beta|\tau(t)\int_0^1 \sigma(t-\tau(t)\rho,L)\partial_t\sigma(t-\tau(t)\rho,L)d\rho \\ &= -|\beta|\int_0^1 (1-\dot{\tau}(t)\rho)\sigma(t-\tau(t)\rho,L)\partial_\rho\sigma(t-\tau(t)\rho,L)d\rho \\ &= -\frac{|\beta|\dot{\tau}(t)}{2}\int_0^1 \sigma^2(t-\tau(t)\rho,L)d\rho + \frac{1}{2}|\beta|\sigma^2(t,L) + \frac{|\beta|(\dot{\tau}(t)-1)}{2}\sigma^2\left(t-\tau(t),L\right). \end{aligned} \tag{2.30}$$

Thus combining (2.29) and (2.30), we obtain from (2.28),

$$\begin{aligned} \dot{E}(t) =& \frac{1}{2}\left(\alpha^2\sigma^2(t,L) + 2\alpha\beta\sigma(t,L)\sigma\left(t-\tau(t),L\right) + \beta^2\sigma^2\left(t-\tau(t),L\right)\right) + \frac{1}{2}(|\beta|-1)\sigma^2(t,L) \\ & + \frac{|\beta|(\dot{\tau}(t)-1)}{2}\sigma^2\left(t-\tau(t),L\right) - \int_0^1 u_x^2\, dx. \end{aligned} \tag{2.31}$$

Recall the negative definite matrix $\mathbf{B}$ in (2.10)

$$\mathbf{B} = \begin{pmatrix} \alpha^2 - 1 + |\beta| & \alpha\beta \\ \alpha\beta & \beta^2 + |\beta|(m-1) \end{pmatrix}.$$

From (2.31), we have further,

$$\dot{E}(t) \leq \frac{1}{2}\left\langle \mathbf{B}\begin{pmatrix} \sigma(t,L) \\ \sigma\left(t-\tau(t),L\right)\end{pmatrix}, \begin{pmatrix} \sigma(t,L) \\ \sigma\left(t-\tau(t),L\right)\end{pmatrix}\right\rangle_{\mathbb{R}^2} - \int_0^1 u_x^2 \leq 0.$$

□

This result regarding the energy E is not sufficient to address the exponential stability issues of the studied system. In the next section we answer this question by choosing appropriate Lyapunov function.

2.2.2. ***Exponential stability***. Let us take $\gamma_1, \gamma_2 \in (0,1)$ are positive constants, small enough, to be chosen later. Consider the Lyapunov functional

$$\mathcal{E}(t) = E(t) + \gamma_1\mathcal{E}_1(t) + \gamma_2\mathcal{E}_2(t), t > 0, \gamma_1, \gamma_2 \in (0,1), \tag{2.32}$$

where

$$\mathcal{E}_1(t) = \frac{1}{2}\int_0^L e^{-\lambda x}\sigma^2(t,x)dx, \text{ for some } \lambda > 0. \tag{2.33}$$

$$\mathcal{E}_2(t) = \frac{\tau(t)}{2}\int_0^1 (1-\rho)\sigma^2(t-\tau(t)\rho, L)d\rho. \tag{2.34}$$

It is easy to note that $E(t) \leq \mathcal{E}(t)$, for all $t \geq 0$. On the other hand, it follows that

$$\begin{aligned}\gamma_1\mathcal{E}_1(t) + \gamma_2\mathcal{E}_2(t) &\leq \frac{\gamma_1}{2}\int_0^L \sigma^2(t,x)dx + \frac{\gamma_2\tau(t)}{2}\int_0^1 \sigma^2(t-\tau(t)\rho, L)d\rho \\ &\leq \frac{\gamma_1}{2}\int_0^L \sigma^2(t,x)dx + \frac{\gamma_2|\beta|\tau(t)}{2|\beta|}\int_0^1 \sigma^2(t-\tau(t)\rho, L)d\rho \\ &\leq \max\left\{\gamma_1, \frac{\gamma_2}{|\beta|}\right\}E(t),\end{aligned}$$

that is,

$$E(t) \leq \mathcal{E}(t) \leq \left(1 + \max\left\{\gamma_1, \frac{\gamma_2}{|\beta|}\right\}\right)E(t), \text{ for all } t \geq 0. \tag{2.35}$$

This ensures that the Lyapunov $\mathcal{E}$ is equivalent to the energy E.

Theorem 2.5. *Let us assume that (1.6), (1.7), (1.8) and (1.9) hold. Let us also assume that* $\mathbf{U}_0 = (\sigma_0, u_0, z_0(-\tau(0)\cdot))^T \in \mathcal{H}$. *Then the energy E defined in (2.27) decays exponentially, that is there exist positive constants C, μ such that the following happens*

$$E(t) \leq CE(0)e^{-\mu t}, \forall t > 0,$$

where for $\gamma_1, \gamma_2 \in (0,1)$ small enough,

$$C \leq \left(1 + \max\left\{\gamma_1, \frac{\gamma_2}{|\beta|}\right\}\right)$$

and

$$\mu \leq \min\left\{\frac{\pi^2}{L^2}\left(1 - \frac{b\gamma_1}{2}\right), \frac{\gamma_2(1-m)}{2M(\gamma_2 + |\beta|)}, \frac{\gamma_1(\lambda - b)}{2(\gamma_1 + e^{\lambda L})}\right\}, \lambda > b.$$

Proof. First we assume that the solution of the system is sufficiently smooth by taking $\mathbf{U}_0 = (\sigma_0, u_0, z_0(-\tau(0)\cdot))^T \in \mathcal{D}(\mathcal{A}(0))$. We perform a differentiation on (2.33) and write

$$\begin{aligned}\dot{\mathcal{E}}_1(t) &= \int_0^L e^{-\lambda x}\sigma\sigma_t\, dx = -\int_0^L e^{-\lambda x}\sigma\sigma_x\, dx - b\int_0^L e^{-\lambda x}\sigma u_x\, dx \\ &= -\frac{\lambda}{2}\int_0^L e^{-\lambda x}\sigma^2\, dx - \frac{e^{-\lambda L}}{2}\sigma^2(t,L) + \frac{1}{2}\sigma^2(t,0) - b\int_0^L e^{-\lambda x}\sigma u_x\, dx.\end{aligned} \tag{2.36}$$

Differentiating both sides of (2.34),we have

$$\begin{aligned}\dot{\mathcal{E}}_2(t) =&\frac{\dot\tau(t)}{2}\int_0^1(1-\rho)\sigma^2(t-\tau(t)\rho,L)d\rho+\tau(t)\int_0^1(1-\rho)\sigma(t-\tau(t)\rho,L)\partial_t\sigma(t-\tau(t)\rho,L)d\rho\\ =&\frac{\dot\tau(t)}{2}\int_0^1(1-\rho)\sigma^2(t-\tau(t)\rho,L)d\rho\\ &-\int_0^1(1-\rho)(1-\dot\tau(t)\rho)\sigma(t-\tau(t)\rho,L)\partial_\rho\sigma(t-\tau(t)\rho,L)d\rho.\end{aligned}\tag{2.37}$$

In the last term of (2.37), we have used the identity $-\tau(t)\partial_t\sigma(t-\tau(t)\rho,L)=(1-\dot\tau(t)\rho)\partial_\rho\sigma(t-\tau(t)\rho,L)$. Performing integration by parts for the last term of the above identity we have

$$\begin{aligned}\dot{\mathcal{E}}_2(t) =&\frac{\dot\tau(t)}{2}\int_0^1(1-\rho)\sigma^2(t-\tau(t)\rho,L)d\rho-\frac{\dot\tau(t)}{2}\int_0^1(1-\rho)\sigma^2(t-\tau(t)\rho,L)d\rho\\ &-\frac12\int_0^1(1-\dot\tau(t)\rho)\sigma^2(t-\tau(t)\rho,L)\,d\rho+\frac12\sigma^2(t,L).\end{aligned}$$

Thus we have finally

$$\dot{\mathcal{E}}_2(t)=-\frac12\int_0^1(1-\dot\tau(t)\rho)\sigma^2(t-\tau(t)\rho,L)+\frac12\sigma^2(t,L).\tag{2.38}$$

Using (2.31), (2.36) and (2.38), for some $\mu>0$ we have

$$\begin{aligned}\dot{\mathcal{E}}(t)+2\mu\mathcal{E}(t)\le&\frac12\left\langle\mathbf{B}\begin{pmatrix}\sigma(t,L)\\ \sigma\left(t-\tau(t),L\right)\end{pmatrix},\begin{pmatrix}\sigma(t,L)\\ \sigma\left(t-\tau(t),L\right)\end{pmatrix}\right\rangle_{\mathbb{R}^2}-\frac{\lambda\gamma_1}{2}\int_0^L e^{-\lambda x}\sigma^2\,dx-\frac{\gamma_1e^{-\lambda L}}{2}\sigma^2(t,L)\\ &+\frac{\gamma_1}{2}\sigma^2(t,0)-b\gamma_1\int_0^L e^{-\lambda x}\sigma u_x\,dx-\int_0^L u_x^2\,dx-\frac{\gamma_2}{2}\int_0^1(1-\dot\tau(t)\rho)\sigma^2(t-\tau(t)\rho,L)\,d\rho+\frac{\gamma_2}{2}\sigma^2(t,L)\\ &+\mu\int_0^L\sigma^2\,dx+\mu\int_0^L u^2\,dx+\mu|\beta|\tau(t)\int_0^1\sigma^2(t-\tau(t)\rho,L)d\rho+\mu\gamma_1\int_0^L e^{-\lambda x}\sigma^2(t,x)\,dx\\ &+\gamma_2\mu\tau(t)\int_0^1(1-\rho)\sigma^2(t-\tau(t)\rho,L)d\rho\\ \le&\frac12\left\langle\mathbf{B}\begin{pmatrix}\sigma(t,L)\\ \sigma\left(t-\tau(t),L\right)\end{pmatrix},\begin{pmatrix}\sigma(t,L)\\ \sigma\left(t-\tau(t),L\right)\end{pmatrix}\right\rangle+\frac{\gamma_1}{2}\sigma^2(t,0)+\frac{\gamma_2}{2}\sigma^2(t,L)-\frac{\lambda\gamma_1}{2}\int_0^L e^{-\lambda x}\sigma^2\,dx\\ &-b\gamma_1\int_0^L e^{-\lambda x}\sigma u_x\,dx-\int_0^L u_x^2\,dx+\mu\int_0^L\sigma^2\,dx+\mu\int_0^L u^2\,dx+\mu\gamma_1\int_0^L e^{-\lambda x}\sigma^2(t,x)\,dx\\ &+\left(\mu|\beta|M+\gamma_2\mu M-\frac{\gamma_2}{2}(1-m)\right)\int_0^1\sigma^2(t-\tau(t)\rho,L)d\rho.\end{aligned}\tag{2.39}$$

Let us simplify the first three terms of the right hand side of (2.39). Now,

$$\begin{aligned}&\frac12\left\langle\mathbf{B}\begin{pmatrix}\sigma(t,L)\\ \sigma\left(t-\tau(t),L\right)\end{pmatrix},\begin{pmatrix}\sigma(t,L)\\ \sigma\left(t-\tau(t),L\right)\end{pmatrix}\right\rangle_{\mathbb{R}^2}+\frac{\gamma_1}{2}\sigma^2(t,0)+\frac{\gamma_2}{2}\sigma^2(t,L)\\ &=\frac12\left\langle\mathbf{B}\begin{pmatrix}\sigma(t,L)\\ \sigma\left(t-\tau(t),L\right)\end{pmatrix},\begin{pmatrix}\sigma(t,L)\\ \sigma\left(t-\tau(t),L\right)\end{pmatrix}\right\rangle_{\mathbb{R}^2}+\frac{\gamma_1}{2}\left(\alpha\sigma(t,L)+\beta z(t,1)\right)^2+\frac{\gamma_2}{2}\sigma^2(t,L)\\ &=\frac12\left\langle(\mathbf{B}+\mathbf{B}_{\gamma_2}^{\gamma_1})\begin{pmatrix}\sigma(t,L)\\ \sigma\left(t-\tau(t),L\right)\end{pmatrix},\begin{pmatrix}\sigma(t,L)\\ \sigma\left(t-\tau(t),L\right)\end{pmatrix}\right\rangle_{\mathbb{R}^2},\end{aligned}$$

where

$$\mathbf{B}_{\gamma_2}^{\gamma_1}=\begin{pmatrix}\alpha^2\gamma_1+\gamma_2 & \alpha\beta\gamma_1\\ \alpha\beta\gamma_1 & \beta^2\gamma_1\end{pmatrix}=\gamma_1\begin{pmatrix}\alpha^2 & \alpha\beta\\ \alpha\beta & \beta^2\end{pmatrix}+\gamma_2\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

Since $\mathbf{B}$ is negative definite and trace and determinant map is continuous, we can find γ_1,γ_2 small enough such that $\mathbf{B}+\mathbf{B}_{\gamma_2}^{\gamma_1}$ is negative definite.
Thanks to the Poincaré inequality with best constant, we have

$$\mu\int_0^L u^2(x)\,dx\le\frac{\mu L^2}{\pi^2}\int_0^L u_x^2(x)\,dx,\ \text{ as } u\in H_0^1(0,1).\tag{2.40}$$

Using Young's inequality we estimate the following:

$$b\gamma_1\int_0^L e^{-\lambda x}\sigma u_x\,dx \le \frac{b\gamma_1}{2}\int_0^L e^{-\lambda x}\sigma^2\,dx + \frac{b\gamma_1}{2}\int_0^L e^{-\lambda x}u_x^2\,dx. \tag{2.41}$$

Thus combining (2.39), (2.40) and (2.41), we have :

$$\dot{\mathcal{E}}(t)+2\mu\mathcal{E}(t) \le \underbrace{\left(\frac{\mu L^2}{\pi^2}-1+\frac{b\gamma_1}{2}\right)}_{\mathcal{J}_1}\int_0^L u_x^2(t,x)dx + \underbrace{\left(\mu|\beta|M+\gamma_2\mu M-\frac{\gamma_2}{2}(1-m)\right)}_{\mathcal{J}_2}\int_0^1\sigma^2(t-\tau(t)\rho,L)d\rho$$
$$+\underbrace{\left(-\frac{\lambda\gamma_1}{2}+\frac{b\gamma_1}{2}+\mu e^{\lambda L}+\mu\gamma_1\right)}_{\mathcal{J}_3}\int_0^L e^{-\lambda x}\sigma^2(t,x)dx.$$

Now we are at the position of choosing our parameters involved in the Lyapunov function to achieve the required estimate:

$$\dot{\mathcal{E}}(t)+2\mu\mathcal{E}(t)\le 0. \tag{2.42}$$

- To have $\mathcal{J}_1 \le 0$ we need to take $\mu \le \frac{\pi^2}{L^2}(1-\frac{b\gamma_1}{2})$ and also there is a restriction on the small parameter γ_1 used in the Lyapunov expression as $\gamma_1 \le \frac{2}{b}$.
- $\mathcal{J}_2 \le 0$ iff $\mu \le \frac{\gamma_2(1-m)}{2M(\gamma_2+|\beta|)}$.
- If we take $\lambda > b$ and $\mu < \frac{\gamma_1(\lambda-b)}{2(\gamma_1+e^{\lambda L})}$, then we have $\mathcal{J}_3 \le 0$.

By the conditions of the Theorem 2.5 we have (2.42). Therefore using the inequality (2.35) we have

$$E(t) \le \left(1+\max\left\{\gamma_1,\frac{\gamma_2}{|\beta|}\right\}\right)E(0)e^{-2\mu t},\ \forall\, t>0.$$

A standard density argument allows to extend the desired result to $\mathbf{U}_0 \in \mathcal{H}$. □

3. Exponential Stability of NSE with time varying internal delay

In this section, we study the stability analysis of the Navier-Stokes equations (1.11) with internal time-delay feedback. As before, we first study the well-posedness of the internal time varying delay system. Then using Lyapunov approach, we prove the exponential stability of the system.

3.1. Well-posedness of the problem.

This section is devoted to the well-posedness of the above system (1.11). As the boundary delay case, let us introduce the variable

$$z(t,x,\rho)=\sigma(t-\tau(t)\rho,x),\ x\in\omega,\ \rho\in(0,1),t>0.$$

Then the system (1.11) reduces to the following:

$$\begin{cases}
\sigma_t+\sigma_x+bu_x+a(x)\sigma(t,x)+c(x)\sigma(t-\tau(t),x)=0, & (t,x)\in(0,\infty)\times(0,L),\\
u_t-u_{xx}+u_x+b\sigma_x=0, & (t,x)\in(0,\infty)\times(0,L),\\
\tau(t)z_t+(1-\dot\tau(t)\rho)z_\rho=0, & x\in\omega,\ (t,\rho)\in(0,\infty)\times(0,1),\\
u(t,0)=u(t,L)=\sigma(t,0)=0, & t\in(0,\infty),\\
z(t,x,0)=\sigma(t,x), & x\in\omega,\ t\in(0,\infty),\\
z(0,x,\rho)=z_0(-\tau(0)\rho,x), & x\in\omega,\rho\in(0,1),\\
\sigma(0,x)=\sigma_0(x),\ u(0,x)=u_0(x), & x\in(0,L),\\
\sigma(t-\tau(0),x)=z_0(t-\tau(0),x), & x\in(0,L),0<t<\tau(0).
\end{cases} \tag{3.1}$$

Let us set

$$\mathbf{U}=(\sigma,u,z)^T,\mathbf{U}_0=(\sigma_0,u_0,z_0(-\tau(0)\cdot,\cdot))^T.$$

Thus we write the above system as:

$$\begin{cases}\dot{\mathbf{U}}(t)=\mathcal{A}(t)\mathbf{U}(t), & t>0,\\ \mathbf{U}(0)=\mathbf{U}_0,\end{cases} \tag{3.2}$$

where the time dependent operator $\mathcal{A}(t)$ can be written as

$$\mathcal{A}(t)\mathbf{U}=\begin{pmatrix}-\sigma_x-bu_x-a\sigma-cz_e(\cdot,1)\\ u_{xx}-u_x-b\sigma_x\\ -\frac{1-\dot\tau(t)\rho}{\tau(t)}z_\rho\end{pmatrix},$$

where z_e is the zero extension of z outside ω, with the domain

$$\mathcal{D}(\mathcal{A}(t)) = \big\{(\sigma,u,z) \in H^1(0,L)\times H^2(0,L)\cap H_0^1(0,L)\times L^2(\omega,H^1(0,1)) : z(x,0)=\sigma|_\omega(x), \sigma(0)=0\big\}.$$

Clearly $\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)), \forall t>0$. Let us introduce the Hilbert space $\widetilde{\mathcal{H}} = L^2(0,L)\times L^2(0,L)\times L^2(\omega\times(0,1))$ induced with the following inner product depending on time

$$\left\langle \begin{pmatrix}\sigma_1\\u_1\\z_1\end{pmatrix}, \begin{pmatrix}\sigma_2\\u_2\\z_2\end{pmatrix} \right\rangle_t = \int_0^L \sigma_1\sigma_2 dx + \int_0^L u_1u_2 dx + \tau(t)\int_\omega\int_0^1 \xi(x) z_1 z_2 \, dx\, d\rho, \tag{3.3}$$

where ξ is a non-negative function in $L^\infty(0,L)$ such that supp ξ = supp $c = \omega$ and it satisfies following condition

$$\frac{1}{1-m}c(x) + k_0 \le \xi(x) \le 2a(x) - c(x) - k_0 \text{ a.e. in } \omega. \tag{3.4}$$

By the definition of the inner product (3.3) we further have:

$$\min(1,\tau_0 c_0)\,\|(\sigma,u,z)\|^2_{\widetilde{\mathcal{H}}} \le \|(\sigma,u,z)\|_t^2 \le \Big(1+2M\,\|a\|_{L^\infty(0,L)}\Big)\,\|(\sigma,u,z)\|^2_{\widetilde{\mathcal{H}}}, \tag{3.5}$$

where $\|.\|_{\widetilde{\mathcal{H}}}$ is the usual norm on $\widetilde{\mathcal{H}}$. Now we are ready to prove the well-posedness result for the linear system (1.11) using time dependent semigroup theory.

Theorem 3.1. *Let $a, c \in L^\infty(0,L)$ be two non-negative functions with (1.12), (1.13) and we also assume (1.6), (1.7), (1.8) and (3.4). For any $\mathbf{U}_0 = (\sigma_0, u_0, z_0(-\tau(0)\cdot,\cdot))^T \in \widetilde{\mathcal{H}}$, there exists a unique solution $\mathbf{U}\in\mathcal{C}([0,\infty);\widetilde{\mathcal{H}})$ of (3.2). Moreover, if $\mathbf{U}_0\in\mathcal{D}(\mathcal{A}(0))$, then $\mathbf{U}\in\mathcal{C}([0,\infty);\mathcal{D}(\mathcal{A}(0)))\cap\mathcal{C}^1([0,\infty);\widetilde{\mathcal{H}})$.*

Proof. To prove the existence-uniqueness of the solution of (3.2), we follow similar kind of argument as the boundary time delay case. Since the analysis is standard, we only give a sketch of the proof.

At first, it is elementary to check that $\mathcal{D}(\mathcal{A}(0))$ is dense in $\widetilde{\mathcal{H}}$ and $\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)), \forall t>0$. Let us take $\mathbf{U} = (\sigma,u,z)^T \in \mathcal{D}(\mathcal{A}(t))$. We compute

$$\begin{aligned}
\langle \mathcal{A}(t)\mathbf{U},\mathbf{U}\rangle_t =& -\int_0^L \sigma\sigma_x\,dx - b\int_0^L \sigma u_x\,dx + \int_0^L u u_{xx}\,dx - \int_0^L u u_x\,dx - b\int_0^L \sigma u_x\,dx - \int_0^L a(x)\sigma^2\,dx\\
&- \int_\omega c(x)\sigma(x) z(x,1)dx + \tau(t)\int_\omega\int_0^1 \xi(x)\frac{\dot\tau(t)\rho-1}{\tau(t)} z z_\rho\, d\rho dx\\
=& \frac{1}{2}\left(-\sigma^2(L)+\sigma^2(0)\right) - \int_0^L u_x^2\,dx - \int_0^L a(x)\sigma^2\,dx - \int_\omega c(x)\sigma(x) z_e(x,1)\,dx\\
&+ \frac{1}{2}\int_\omega \xi(x)(-1+\dot\tau(t)) z^2(x,1)\,dx + \frac{1}{2}\int_\omega \xi(x) z^2(x,0)\,dx - \frac{\dot\tau(t)}{2}\int_\omega\int_0^1 \xi(x) z^2\,d\rho dx.
\end{aligned}$$

Applying the inequality $ab \le \frac{a^2}{2} + \frac{b^2}{2}$ for the term $\int_\omega c(x)\sigma(x)z(x,1)$, we further obtain

$$\begin{aligned}
\langle \mathcal{A}(t)\mathbf{U},\mathbf{U}\rangle_t \le& \int_\omega \left(-a(x) + \frac{c(x)}{2} + \frac{\xi(x)}{2}\right)\sigma^2(x)dx + \frac{1}{2}\int_\omega \left(c(x)+\xi(x)(\dot\tau(t)-1)\right) z^2(x,1)\,dx\\
&- \frac{\dot\tau(t)}{2}\int_\omega\int_0^1 \xi(x) z^2(x,\rho)\,d\rho dx.
\end{aligned}$$

Thanks to (3.4), we have $\left(-a(x) + \frac{c(x)}{2} + \frac{\xi(x)}{2}\right) < 0$ and $(c(x)+\xi(x)(\dot\tau(t)-1)) < c(x)+\xi(x)(m-1) < 0$, a.e. in ω.

Let us recall the function $k(t) = \frac{(1+\dot\tau^2(t))^{1/2}}{2\tau(t)}$. Therefore one can easily show that the time dependent operator $\widetilde{\mathcal{A}}(t) = \mathcal{A}(t) - k(t)\mathcal{I}$ is dissipative.

Next, we will compute $\langle \mathcal{A}(t)^*\mathbf{U},\mathbf{U}\rangle_t$, where $\mathcal{A}(t)^*$, the adjoint of the operator $\mathcal{A}(t)^*$ is defined by:

$$\mathcal{A}(t)^*\mathbf{U} = \begin{pmatrix} \sigma_x + bu_x - a\sigma + \xi z_e(\cdot,0)\\ u_{xx} + u_x + b\sigma_x\\ \frac{1-\dot\tau(t)\rho}{\tau(t)} z_\rho - \frac{\dot\tau(t)}{\tau(t)} z\end{pmatrix},$$

with the domain:

$$\begin{aligned}
\mathcal{D}(\mathcal{A}(t)^*) =& \big\{(\sigma,u,z)\in H^1(0,L)\times H^2(0,L)\cap H_0^1(0,L)\times L^2(\omega,H^1(0,1))|\\
&\sigma(L)=0, z(x,1) = \frac{-c(x)}{\xi(x)(1-\dot\tau(t))}\sigma|_\omega(x)\big\}.
\end{aligned}$$

Let us consider $\mathbf{U} = (\sigma, u, z)^T \in \mathcal{D}(\mathcal{A}(t)^*)$. Thus applying integration by parts, we obtain

$$\begin{aligned}\langle \mathcal{A}(t)^*\mathbf{U}, \mathbf{U}\rangle_t =& \int_0^L \sigma\sigma_x\,dx + b\int_0^L \sigma u_x\,dx + \int_0^L uu_{xx}\,dx + \int_0^L uu_x\,dx + b\int_0^L \sigma u_x\,dx - \int_0^L a(x)\sigma^2\,dx\\ &- \int_\omega c(x)\sigma(x)z(x,0)\,dx + \tau(t)\int_\omega\int_0^1 \xi(x)\frac{-\dot\tau(t)\rho+1}{\tau(t)} zz_\rho\,dx\,d\rho - \tau(t)\int_\omega\int_0^1 \xi(x)\frac{\dot\tau(t)}{\tau(t)}z^2\,(x,\rho)dx\,d\rho\\ =&\frac{1}{2}(\sigma^2(L)-\sigma^2(0)) - \int_0^L u_x^2\,dx - \int_\omega a(x)\sigma^2\,dx - \int_\omega\int_0^1 c(x)\sigma(x)z(x,0)\,dx + \frac{1}{2}\int_\omega\int_0^1 \dot\tau(t)\xi(x)z^2(x,\rho)\,d\rho\,dx\\ &+\frac{1}{2}\int_\omega \xi(x)(-1+\dot\tau(t))z^2(x,1)\,dx - \frac{1}{2}\int_\omega \xi(x)z^2(x,0)\,dx - \dot\tau(t)\int_\omega\int_0^1 \xi(x)z^2(x,\rho)d\rho dx.\end{aligned}$$

Utilizing the boundary data, we derive

$$\begin{aligned}\langle \mathcal{A}(t)^*\mathbf{U}, \mathbf{U}\rangle_t =& -\frac{1}{2}\sigma^2(0) - \int_0^L u_x^2\,dx - \int_\omega a(x)\sigma^2\,dx - \int_\omega\int_0^1 c(x)\sigma(x)z(x,0)\,dx - \frac{1}{2}\int_\omega\int_0^1 \dot\tau(t)\xi(x)z^2(x,\rho)\,dx\\ &+\frac{1}{2}\int_\omega \frac{c^2(x)}{\xi(x)(1-\dot\tau(t))}\sigma^2(x)\,dx - \frac{1}{2}\int_\omega \xi(x)z^2(x,0)\,dx\\ \le&\frac{1}{2}\int_\omega\left(-2a(x)+\xi(x)+\frac{c^2(x)}{\xi(x)(1-\dot\tau(t))}\right)\sigma^2(x) - \frac{1}{2}\int_\omega\int_0^1 \dot\tau(t)\xi(x)z^2(x,\rho)dxd\rho.\end{aligned}$$

Thanks to the inequality (3.4) and the upper bound condition (1.8) of $\dot\tau(t)$, we have

$$\frac{c^2(x)}{\xi(x)(1-\dot\tau(t))} \le \frac{c(x)}{\xi(x)}\frac{c(x)}{(1-m)} \le c(x), \text{ a.e. in } \omega.$$

Again by (3.4), we further obtain $\left(-2a(x)+\xi(x)+\frac{c^2(x)}{\xi(x)(1-\dot\tau(t))}\right) \le -k_0 < 0$, a.e. in ω.

Therefore as in the previous case, straightforward computations show that the time dependent operator $\widetilde{\mathcal{A}}(t)^* = \mathcal{A}(t)^* - k(t)\mathcal{I}$ is dissipative. It can be shown that $\widetilde{\mathcal{A}}(t)$ is a densely defined closed linear operator. Henceforth $\widetilde{\mathcal{A}}(t)$ generates a C_0 semigroup of contraction on $\widetilde{\mathcal{H}}$ (see [60]).

Proceeding with similar argument as boundary delay feedback cases, one can prove (2.8). And thus the family $\widetilde{\mathcal{A}} = \{\widetilde{\mathcal{A}}(t) : t \in [0,T]\}$ is stable with stability constants C and ω_0 independent of t. Hence, the first three conditions of Theorem 2.1 hold. Finally, one can also show that $\partial_t\widetilde{\mathcal{A}}(t) \in L^\infty_*([0,T]; B(\mathcal{D}(\mathcal{A}(0)), \widetilde{\mathcal{H}}))$. Thanks to Theorem 2.1, the following ODE

$$\begin{cases}\dot{\widetilde{\mathbf{U}}}(t) = \widetilde{\mathcal{A}}(t)\widetilde{\mathbf{U}}(t), \quad t>0,\\ \widetilde{\mathbf{U}}(0) = \mathbf{U}_0,\end{cases} \tag{3.6}$$

has a unique solution $\widetilde{\mathbf{U}} \in \mathcal{C}([0,\infty); \mathcal{D}(\mathcal{A}(0))) \cap \mathcal{C}^1([0,\infty); \widetilde{\mathcal{H}})$, provided $\mathbf{U}_0 \in \mathcal{D}(\mathcal{A}(0))$. Thus $\mathbf{U}(t) = e^{\int_0^t k(s)ds}\widetilde{\mathbf{U}}(t)$ solves the ODE (3.2). □

3.2. **Stability analysis.** In this section, we prove the exponential stability result for the system (1.11). Let us first consider the energy for the system (1.11)

$$E(t) = \frac{1}{2}\int_0^L \sigma^2(t,x)dx + \frac{1}{2}\int_0^L u^2(t,x)dx + \frac{\tau(t)}{2}\int_\omega\int_0^1 \xi(x)\sigma^2(t-\tau(t)\rho,x)d\rho,\ t\ge 0. \tag{3.7}$$

Our first goal is to show that this energy is a decreasing function of time.

Proposition 3.2. *Let $a, c \in L^\infty(0,L)$ be two non-negative functions with (1.12), (1.13) and we also assume that (1.6), (1.7), (1.8) and (3.4) hold. Then for all regular solutions of the system (1.11), the energy E in (3.7) is non-increasing and satisfies:*

$$\dot E(t) < 0.$$

Proof. Differentiating E with respect to t and using the equation (1.11) and applying integration by parts successively, we get,

$$\begin{aligned}\dot{E}(t) &= \int_0^L \sigma\sigma_t dx + \int_0^L uu_t dx + \frac{\dot{\tau}(t)}{2}\int_\omega\int_0^1 \xi(x)\sigma^2(t-\tau(t)\rho,x)d\rho \\ &\qquad + \tau(t)\int_\omega\int_0^1 \xi(x)\sigma(t-\tau(t)\rho,x)\partial_t\sigma(t-\tau(t)\rho,x)d\rho\,dx \\ &= -\frac{1}{2}\sigma^2(L) - \int_0^L u_x^2\,dx - \int_0^1 a(x)\sigma^2\,dx - \int_\omega c(x)\sigma(x)\sigma(t-\tau(t),x)\,dx \\ &\qquad - \frac{1}{2}\int_\omega \xi(x)(1-\dot{\tau}(t))\sigma^2(t-\tau(t),x)dx + \frac{1}{2}\int_\omega \xi(x)\sigma^2(t,x)\,dx\end{aligned}$$

Here, we have inserted the identity $-\tau(t)\partial_t\sigma(t-\tau(t)\rho,x) = (1-\dot{\tau}(t)\rho)\partial_\rho\sigma(t-\tau(t)\rho,x)$, $x\in\omega$ in the term

$$\tau(t)\int_\omega\int_0^1 \xi(x)\sigma(t-\tau(t)\rho,x)\partial_t\sigma(t-\tau(t)\rho,x)d\rho\,dx$$

and performing integration by parts we derived

$$\begin{aligned}\tau(t)\int_\omega\int_0^1 \xi(x)\sigma(t-\tau(t)\rho,x)\partial_t\sigma(t-\tau(t)\rho,x)d\rho dx &= -\frac{\dot{\tau}(t)}{2}\int_\omega\int_0^1 \xi(x)\sigma^2(t-\tau(t)\rho,x)d\rho dx \\ &\quad - \frac{1}{2}\int_\omega \xi(x)(1-\dot{\tau}(t))\sigma^2(t-\tau(t),x)dx + \frac{1}{2}\int_\omega \xi(x)\sigma^2(t,x)dx\end{aligned}$$

On simplification, we obtain

$$\dot{E}(t) \le \frac{1}{2}\int_\omega \left(-2a(x)+\xi(x)+c(x)\right)\sigma^2(t,x)dx + \frac{1}{2}\int_\omega \left(c(x)-(1-m)\xi(x)\right)\sigma^2(t-\tau(t),x)dx - \int_0^L u_x^2 dx.$$

Thanks to the inequality (3.4), we obtain $\dot{E}(t) < 0$. □

Now, we will establish that the energy E decays exponentially towards the origin. Let us consider the Lyapunov functional

$$\mathcal{E}(t) = E(t) + \gamma_1\mathcal{E}_1(t) + \gamma_2\mathcal{E}_2(t), t>0, \gamma_1,\gamma_2\in(0,1),$$

where

$$\mathcal{E}_1(t) = \frac{1}{2}\int_0^L e^{-\lambda x}\sigma^2(t,x)dx, \text{ for some } \lambda>0. \tag{3.8}$$

$$\mathcal{E}_2(t) = \frac{\tau(t)}{2}\int_\omega\int_0^1 (1-\rho)\sigma^2(t-\tau(t)\rho,x)dxd\rho. \tag{3.9}$$

It is easy to note that $E(t)\le\mathcal{E}(t)$, for all $t\ge 0$. On the other hand, it follows that

$$\begin{aligned}\gamma_1\mathcal{E}_1(t)+\gamma_2\mathcal{E}_2(t) &\le \frac{\gamma_1}{2}\int_0^L \sigma^2(t,x)dx + \frac{\gamma_2\tau(t)}{2}\int_\omega\int_0^1 \sigma^2(t-\tau(t)\rho,x)dx\,d\rho \\ &\le \frac{\gamma_1}{2}\int_0^L \sigma^2(t,x)dx + \frac{\gamma_2\tau(t)}{2c_0}\int_\omega\int_0^1 \xi(x)\sigma^2(t-\tau(t)\rho,x)\,dx\,d\rho \\ &\le \max\left\{\gamma_1,\frac{\gamma_2}{c_0}\right\}E(t),\end{aligned}$$

that is,

$$E(t)\le\mathcal{E}(t)\le\left(1+\max\left\{\gamma_1,\frac{\gamma_2}{c_0}\right\}\right)E(t), \text{ for all } t\ge 0. \tag{3.10}$$

This ensures that the Lyapunov $\mathcal{E}$ is equivalent to the energy E. Now, we prove our exponential stability result for interior delay case.

Theorem 3.3. *Let $a,c\in L^\infty(0,L)$ be two non-negative functions with (1.12), (1.13) and we also assume that (1.6), (1.7), (1.8), (1.13) and (3.4) hold. Let us also assume that $\mathbf{U}_0 = (\sigma_0,u_0,z_0(-\tau(0)\cdot,\cdot))^T\in\widetilde{\mathcal{H}}$. Then the energy E defined in (3.7) decays exponentially, that is there exist positive constants C,μ such that the following happens*

$$E(t)\le CE(0)e^{-\mu t},\ \forall t>0,$$

where for $\gamma_1, \gamma_2 \in (0,1)$ *small enough,*

$$C \leq \left(1 + \max\left\{\gamma_1, \frac{\gamma_2}{c_0}\right\}\right)$$

and

$$\mu \leq \min\left\{\frac{\pi^2}{L^2}\left(1 - \frac{b\gamma_1}{2}\right), \frac{\gamma_2(1-m)}{2M(\gamma_2 + \|\xi\|_{L^\infty(0,L)})}, \frac{\gamma_1(\lambda - b)}{2(\gamma_1 + e^{\lambda L})}\right\},\ \lambda > b.$$

Proof. Analogous to boundary feedback case, here we first assume that the solution of the system (1.11) is sufficiently smooth by taking $\mathbf{U}_0 = (\sigma_0, u_0, z_0(-\tau(0)\cdot, \cdot))^T \in \mathcal{D}(\mathcal{A}(0))$. A standard density argument allows to extend the desired result to $\mathbf{U}_0 \in \widetilde{\mathcal{H}}$.

Differentiating (3.8) with respect to t, we have

$$\begin{aligned}
\dot{\mathcal{E}}_1(t) &= \int_0^L e^{-\lambda x}\sigma\sigma_t\, dx \\
&= -\int_0^L e^{-\lambda x}\sigma\sigma_x\, dx - b\int_0^L e^{-\lambda x}\sigma u_x\, dx - \int_0^L a(x)e^{-\lambda x}\sigma^2\, dx - \int_\omega e^{-\lambda x}c(x)\sigma(t-\tau,x)\sigma(t,x)\, dx \\
&= -\frac{\lambda}{2}\int_0^L e^{-\lambda x}\sigma^2\, dx - \frac{e^{-\lambda L}}{2}\sigma^2(t,L) + \frac{1}{2}\sigma^2(t,0) - b\int_0^L e^{-\lambda x}\sigma u_x\, dx - \int_0^L a(x)e^{-\lambda x}\sigma^2\, dx \\
&\qquad - \int_\omega e^{-\lambda x}c(x)\sigma(t-\tau,x)\sigma(t,x)\, dx \\
&\leq -\frac{\lambda}{2}\int_0^L e^{-\lambda x}\sigma^2 dx - b\int_0^L e^{-\lambda x}\sigma u_x\, dx - \int_0^L a(x)e^{-\lambda x}\sigma^2\, dx - \int_\omega e^{-\lambda x}c(x)\sigma(t-\tau,x)\sigma(t,x)\, dx.
\end{aligned} \tag{3.11}$$

Differentiating (3.9) with respect to t we have

$$\dot{\mathcal{E}}_2(t) = \frac{\dot\tau(t)}{2}\int_\omega\int_0^1 (1-\rho)\sigma^2(t-\tau(t)\rho,x)dxd\rho + \tau(t)\int_\omega\int_0^1 (1-\rho)\sigma(t-\tau(t)\rho,x)\partial_t\sigma(t-\tau(t)\rho,x)dxd\rho \tag{3.12}$$

Thanks to the identity $-\tau(t)\partial_t\sigma(t-\tau(t)\rho,x) = (1-\dot\tau(t)\rho)\partial_\rho\sigma(t-\tau(t)\rho,x)$, $x\in\omega$, we write the last term of the above identity as follows:

$$\begin{aligned}
&\tau(t)\int_\omega\int_0^1 (1-\rho)\sigma(t-\tau(t)\rho,x)\partial_t\sigma(t-\tau(t)\rho,x)dxd\rho \\
&\quad = -\int_\omega\int_0^1 (1-\rho)(1-\dot\tau(t)\rho)\sigma(t-\tau(t)\rho,x)\partial_\rho\sigma(t-\tau(t)\rho,x)dxd\rho \\
&\quad = -\frac{1}{2}\int_\omega\int_0^1 (1-\dot\tau(t)\rho)\sigma^2(t-\tau(t)\rho,x)dxd\rho - \frac{1}{2}\dot\tau(t)\int_\omega\int_0^1 (1-\rho)\sigma^2(t-\tau(t)\rho,x)dxd\rho \\
&\qquad + \frac{1}{2}\int_\omega \sigma^2(t,x)dx.
\end{aligned}$$

The above identity along with (3.12) provides

$$\dot{\mathcal{E}}_2(t) = -\frac{1}{2}\int_\omega\int_0^1 (1-\dot\tau(t)\rho)\sigma^2(t-\tau(t)\rho,x)dx + \int_\omega \sigma^2(t,x)dx. \tag{3.13}$$

Combining (3.11) and (3.13) and using Young's inequality, we proceed

$$\begin{aligned}\dot{\mathcal{E}}(t)+2\mu\mathcal{E}(t)\leq&\frac{1}{2}\int_\omega\underbrace{(-2a(x)+\xi(x)+c(x)+\gamma_1c(x)+\gamma_2)}_{\mathcal{I}_1}\sigma^2(t,x)\,dx\\&+\frac{1}{2}\int_\omega\underbrace{(c(x)-(1-m)\xi(x)+\gamma_1c(x))}_{I_2}\sigma^2(t-\tau(t),x)dx-\int_{(0,L)\setminus\omega}a(x)\sigma^2dx\\&+\frac{1}{2}\int_\omega\int_0^1\underbrace{(2\mu M\xi(x)-(1-m)\gamma_2+M2\mu\gamma_2)}_{\mathcal{I}_3}\sigma^2(t-\tau(t)\rho,x)dx\\&+\frac{1}{2}\int_0^L\underbrace{(-\gamma_1\lambda-\gamma_1a(x)+2\mu e^{\lambda L}+2\gamma_1\mu+b\gamma_1)}_{\mathcal{I}_4}e^{-\lambda x}\sigma^2dx+\underbrace{\left(-1+\frac{\mu L^2}{\pi^2}+\frac{b\gamma_1}{2}\right)}_{\mathcal{I}_5}\int_0^L u_x^2dx.\end{aligned}$$

Thus to prove $\dot{\mathcal{E}}(t)+2\mu\mathcal{E}(t)<0$, we need to choose the Lyapunov parameters γ_1,γ_2 and the decay μ in such a way that $\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3,\mathcal{I}_4,\mathcal{I}_5$ are negative.

- As $(-2a(x)+\xi(x)+c(x))<0, c(x)-(1-m)\xi(x)<0$ a.e. on ω, we can choose γ_1,γ_2 small enough to get $\mathcal{I}_1,\mathcal{I}_2<0$. In fact, the inequality (3.4) ensures that the following choices serve our purpose:
$$\gamma_1\leq\inf_\omega\left\{\frac{2a(x)-\xi(x)-c(x)}{c(x)},\ \frac{(1-m)\xi(x)-c(x)}{c(x)}\right\},$$
$$\gamma_2\leq\inf_\omega\Big\{2a(x)-\xi(x)-c(x)-\gamma_1c(x)\Big\}.$$
- Clearly, if we consider $\mu\leq\frac{(1-m)\gamma_2}{2M\left(\|\xi\|_{L^\infty(0,L)}+\gamma_2\right)}$, then $\mathcal{I}_3\leq0$.
- To prove that $\mathcal{I}_4<0$, that is, $\mu<\frac{\gamma_1(\lambda+a(x)-b)}{2(e^{\lambda L}+\gamma_1)}$, it is enough to consider that
$$\mu<\frac{\gamma_1\,(\lambda-b)}{2(e^{\lambda L}+\gamma_1)},\lambda>b.$$
- To have $\mathcal{I}_5<0$ we need to take $\mu<\frac{\pi^2}{L^2}(1-\frac{b\gamma_1}{2})$ and also there is a restriction on the small parameter γ_1 used in the Lyapunov expression as $\gamma_1<\frac{2}{b}$. This completes the Theorem 3.3.

□

4. Conclusion

In this article, we have investigated the time-dependent delay effect in the asymptotic behaviour of the linearized compressible Navier-Stokes equations. We established two well-posedness results of the concerned system with the time delay term acting in the boundary and in the interior of the hyperbolic component. We have explored the exponential stability results of the Navier-Stokes system under some conditions on the damping parameters and delayed terms as well. In this work, we have assumed that the delay function is strictly positive; see (1.7). An interesting question arises that, can we demonstrate our stability analysis for the degenerate delay case, like [53]? We will address this question in the near future. It is reasonable to examine the impact of the time dependent delay acting in the parabolic component of the linearized Navier-Stokes system. Also, it will be interesting to explore these stability results in other related fluid models, such as linearized compressible Navier-Stokes equations in the case of Creeping flow [21] or linearized compressible Navier-Stokes system with Maxwell's law [2], and [1].

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Conflict of Interest

The author declare that he has no conflict of interest.

ORCID https://orcid.org/0000-0001-6724-6943

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Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai- 400076
Email address: sm18rs016@iiserkol.ac.in, subratamajumdar634@gmail.com