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Positive XVAs

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Abstract

Since the 2008 crisis, derivative dealers charge to their clients various add-ons, dubbed XVAs, meant to account for counterparty risk and its capital and funding implications. As banks cannot replicate jump-to-default related cash flows, deals trigger wealth transfers and shareholders need to set capital at risk. We devise an XVA policy, whereby so called contra-liabilities and cost of capital are sourced from bank clients at trade inceptions, on top of the fair valuation of counterparty risk, in order to guarantee to the shareholders a hurdle rate h on their capital at risk. The resulting all-inclusive XVA formula reads $(CVA + FVA + KVA)$, where C sits for credit, F for funding, and where the KVA is a cost of capital risk premium. All these XVA metrics are portfolio-wide, nonnegative and, despite the fact that we include the default of the bank itself in our modeling, they are ultimately unilateral. This makes them naturally in line with the requirement that capital at risk and reserve capital should not decrease simply because the credit risk of the bank has worsened. An application of this approach to a dealer bank reveals, in particular, the XVA implications of the centrally cleared hedging side of the derivative portfolio of the bank.

Keywords: Counterparty risk, market incompleteness, credit valuation adjustment (CVA), funding valuation adjustment (FVA), capital valuation adjustment (KVA), wealth transfer. central counterparties (CCP).

Mathematics Subject Classification: 91B25, 91B26, 91B30, 91G20, 91G40.

JEL Classification: D52, G13, G24, G28, G33, M41.

1 The Sustainable Pricing and Dividends Problem

We devise a pricing and dividend policy for a dealer bank, **sustainable** in the sense of ensuring to its shareholders a constant instantaneous return rate h on their capital at risk, even in the limiting case of a portfolio held on a run-off basis, i.e. without future deals.

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Moreover, the corresponding policy of the bank should satisfy several **regulatory constraints**. Firstly, in order to comply with the Volcker rule that bans proprietary trading for a dealer bank, the market risk of the bank should be hedged as much as possible. As a result, mainly counterparty risk remains. Secondly, reserve capital should be maintained by the bank at the level of its *expected* counterparty credit losses, along two lines: the credit valuation adjustment (CVA) of the bank, meant to cope with the counterparty risk of the bank clients, i.e. with the expected losses of the bank due to client defaults; and the funding valuation adjustment (FVA), meant to cope with the counterparty risk of the bank itself, i.e. with its expected risky funding expenses. Thirdly, capital should be set at risk by the bank to deal with its *exceptional* (above expected) *losses*. The above return rate h is then meant at a hurdle rate for the bank shareholders, i.e. a risk premium for their capital at risk within the bank.

Reserve capital, like capital at risk, should obviously be *nonnegative*. Furthermore, it should not decrease simply because the credit risk of the bank itself has worsened, a property which we refer to as *monotonicity*: See Albanese and Andersen (2014, Section 3.1) for the relevant regulatory wordings.

Further requirements on a solution to the above sustainable pricing and dividend release policy problem are **economic interpretability** and **logical consistency** (for intellectual adhesion by market participants), **numerical feasibility** and **robustness** at the level of a realistic banking portfolio (for practicality), and **minimality** in the sense of being, all things equal, as cheap as possible (for competitiveness).

The design of a pricing and dividend policy satisfying all the above requirements is the main achievement of this article. Although we can not claim for uniqueness, we will see in Section 7.2 that alternative XVA approaches in the literature breach several of the above requirements. For instance, it is not uncommon in the XVA literature to see some *possibly negative FVA metrics, or CVA and KVA metrics that tend to 0 when the default risk of the bank goes to infinity*.

The cost-of-capital XVA approach has been introduced in Albanese, Caenazzo, and Crépey (2016) and developed in various directions in a stream of papers including Albanese, Caenazzo, and Crépey (2017) or Crépey et al. (2020). The intent of this work is to clarify the foundations of this approach, i.e. the cost-of-capital XVA conceptual framework. This is done in an abstract setup, which is then specified to bilateral and centrally cleared trading setups. Related numerics are provided in Albanese, Crépey, Hoskinson, and Saadeddine (2021) and Abbas-Turki, Crépey, and Saadeddine (2021) for the bilateral trading case and by Albanese, Armenti, and Crépey (2020) for the centrally cleared case.

1.1 Solution Setup

The starting point of our solution to the sustainable pricing and dividends problem is an organizational and accounting separation between three kinds of business units within the bank: the *CA desks*, the *trading desks*, and the *management* of the bank. The CA desks are themselves split between the CVA desk and the FVA desk (or Treasury) of the bank, respectively in charge of the default risk triggered by clients and of the risky funding expenses of the bank. The corresponding cash flows are collectively called the contra-assets of the bank. In an informal sense made precise by this paper,

contra-assets (-liabilities)¹ are bank liabilities (assets) that arise as a feedback effect of counterparty credit risk on the counterparty-risk-free assets (liabilities) of the bank.

The CA desks guarantee the trading of the trading desks against client and bank defaults, through an trading margin account, which also funds the trading of the trading desks at the risk-free rate. Thanks to this work accomplished by the CA desks, the trading desks can focus on the market risk of the contracts in their respective business lines, as if there was no counterparty risk (even if some of their positions are liquidated, this will occur at no loss from their perspective). We denote by MtM the amount on the trading margin account of the bank (counted positively when posted by the CA desks) and we write

$$CA = CVA + FVA \tag{1.1}$$

for the overall amount of reserve capital of the bank, which will correspond to the valuation of its contra-assets.

The management of the bank is in charge of its dividend release policy. We consider a level of capital at risk (CR) sufficient to make the bank resilient to a forty-year adverse event, i.e. greater than an economic capital (EC) defined as the expected shortfall of the losses of the bank over one year at the confidence level $\alpha = 97.5\% = 1 - \frac{1}{40}$. The implementation of a sustainable dividend remuneration policy requires a dedicated risk margin account, on which bank profits are initially retained so that they can then be gradually released as dividends at a hurdle rate h on shareholder capital at risk (as opposed to being readily distributed as day-one profit). Counterparty default losses, as also funding payments, are materialities for default if not paid. By contrast, risk margin payments are at the discretion of the bank management, hence they do not represent an actual liability to the bank. As a consequence, the capital valuation adjustment (KVA) amount on the risk margin account is also loss-absorbing, i.e. part of capital at risk (CR). With minimality in view², we thus set

$$CR = \max(EC, KVA). \tag{1.2}$$

All bank accounts are marked-to-model, i.e. continuously and instantaneously readjusted to theoretical target levels, which will be defined in Section 2 in view of yielding a solution to the sustainable pricing and dividends problem. All cash accounts of the bank, as well as all the collateral (assumed all cash for simplicity) posted as a guarantee for the trades, are remunerated at the risk-free rate.

In line with the sustainability requirement edicted in Section 1, the portfolio is supposed to be held on a run-off basis between inception time 0 and its final maturity. At the portfolio inception time 0, the trading desks pay MtM_0 to the clients; the CA desks add an amount MtM_0 on the trading margin account if $MtM_0 > 0$, whereas the trading desks put an amount $(-MtM_0)$ on the trading margin account if $MtM_0 < 0$; the CA desks charge to the clients an amount CA_0 and add it on the reserve capital account; the management of the bank charges the amount KVA_0 to the clients and adds it on the risk margin account. Between time 0 and the bank default time τ (both excluded), mark-to-model readjustments of all bank accounts are on bank shareholders. If the

¹detailed in Table A.1.

²see Section 1, right before Proposition 2.1, and Section 7.4.

bank defaults, any residual amount on the reserve capital and risk margin accounts, as well as any remaining trading cash flows, are transferred to the estate of the defaulted bank, dubbed creditors of the bank hereafter, which is mandated to deal with the liquidation of the bank. These liquidation costs to be born by the creditors are outside the scope of the model, as is also the primary business of the clients of the bank, which motivates their deals with the bank.

See Table 1.1 for a list of the main valuation acronyms used in the paper.

CA	Contra-assets valuation	(1.1) and Table A.1
CL	Contra-liabilities valuation	Table A.1
CR	Capital at risk	Sect. 1 and (1.2)
CVA	Credit valuation adjustment	Sect. 1, (2.12), (B.5), and (6.27)
DVA	Debt valuation adjustment	Table A.1 and (6.31)
EC	Economic capital	Sect. 1 and Definition B.2
FDA	Funding debt adjustment	Table A.1 and (6.32)
FV	Fair valuation of counterparty risk	Table A.1 and Lemma A.1(ii)
FVA	Funding valuation adjustment	Sect. 1, (2.13), (B.6), and (6.28)
KVA	Capital valuation adjustment	Sect. 1.1, (2.17), and (B.7)
MtM	Mark-to-market	Sect. 1.1, (2.11), and (4.5)
SCR	Shareholder capital at risk	Definition 2.3

Table 1.1: Main valuation acronyms and place where they are introduced conceptually and/or concretely specified in the paper.

2 The Cost-of-Capital XVA Equations

2.1 Probabilistic Setup

Implicit to the above-sketched XVA framework is a probabilistic structure, i.e. a measurable space (Ω, \mathcal{A}) endowed with a stochastic basis $(\mathfrak{G}, \mathbb{Q})$, with respect to which all the involved conditional expectations and risk measures are defined. The filtration $\mathfrak{G} = (\mathfrak{G}_t)_{t \in \mathbb{R}_+}$ satisfies the usual conditions. All the processes in the paper are \mathfrak{G} adapted and all the random times of interest are \mathfrak{G} stopping times. The probability measure \mathbb{Q} is used for the linear valuation of cash flows³, using the risk-free asset as our numéraire everywhere⁴.

The XVA matter also crucially entails nontraded assets (or trading constraints). The default risk of most of the clients of the bank is not liquidly priced in the market. Even if a liquid CDS market on the bank is available, this market is not accessible to the bank itself, which, in particular, cannot sell jump-to-default protection on itself.

These hedging limitations of the bank lead us to introduce the financial sub- σ -field $\mathcal{B} \subseteq \mathcal{A}$, on which a risk-neutral measure, equivalent to the restriction to \mathcal{B} of the

³for simplicity, we only consider European derivatives.

⁴this choice of a numéraire simplifies equations by removing all terms related to the (risk-free, see after (1.2)) remuneration of the cash accounts and of the collateral.

physical probability measure (itself defined on \mathcal{A}), is given. We then define \mathbb{Q} to be the uniquely defined probability measure on \mathcal{A} , provided by Artzner, Eisele, and Schmidt (2020, Proposition 2.1), such that (i) \mathbb{Q} coincides with the risk-neutral measure on \mathcal{B} and (ii) \mathbb{Q} and the physical measure coincide conditionally on \mathcal{B} . The probability measure \mathbb{Q} that emerges from this construction is a hybrid of the underlying risk-neutral and physical measures, with respect to which all the XVA computations are then conducted.

Example 2.1 \mathbb{Q} here and \mathbb{P} in Section B should not be confused with the underlying (aforementioned) risk-neutral and physical measures. In the special (unrealistic) case of a bank that would not resort to dynamic hedging, then our probability measure \mathbb{Q} would just be the physical one and we would have $\mathcal{H} = 0$ in the trading loss \mathcal{L} of the bank in (2.9) below. In the opposite (equally unrealistic, cf. Remark A.2) case of a bank that would be perfectly hedged, our probability measure \mathbb{Q} would coincide with the reference risk-neutral probability measure on $\mathcal{B} = \mathcal{A}$ and the trading loss process \mathcal{L} of the bank would vanish (cf. Proposition A.2), as in turn the economic capital and the KVA of the bank. In practice it is mostly jump-to-default risk that cannot be hedged by the bank and should therefore be assessed on a physical ground in our setup.

We denote by T the sum between the final maturity of all claims in the portfolio (e.g. 50 years) and an upper bound $\delta \geq 0$ on the time of liquidating defaulted positions (typically considered in practice to be of the order of one to two weeks). All our cash flow and price processes are modeled as semimartingales, which are all taken in a càdlàg version. All our cash flow processes are cumulative starting from 0 at time 0, stopped at T , and integrable.

For any left-limited process Y , we denote by $Y^{\tau-}$ and ${}^{\tau-}Y$ the respective processes Y stopped before the bank default time τ and starting before τ , i.e.

$$Y^{\tau-} = JY + (1 - J)Y_{\tau-}, \quad {}^{\tau-}Y = Y - Y^{\tau-}, \quad (2.1)$$

where $J = \mathbb{1}_{[0, \tau[}$ is the survival indicator process of the bank.

The \mathbb{Q} expectation and $(\mathfrak{G}_t, \mathbb{Q})$ conditional expectation are denoted by \mathbb{E} and \mathbb{E}_t . Given counterparty risk and its funding and capital consequences, the pricing of a derivative portfolio by the bank will depart from the \mathbb{Q} valuation of its contractually promised cash flows. The following distinction between valuation and shareholder valuation will be important in this regard.

Definition 2.1 Given an optional, integrable process \mathcal{Y} stopped at T (cumulative cash flow stream in the financial interpretation), we call:

(i) value process Z of \mathcal{Y} , the optional projection of $(\mathcal{Y}_T - \mathcal{Y})$, i.e.

$$Z_t = \mathbb{E}_t(\mathcal{Y}_T - \mathcal{Y}_t), \quad t \leq T, \quad (2.2)$$

and Z vanishes on $[T, +\infty)$;

(ii) shareholder value process Y of \mathcal{Y} , any process Y vanishing on $[T, +\infty)$ if $T < \tau$ and such that

$$Y_t = \mathbb{E}_t(\mathcal{Y}_{\tau-} - \mathcal{Y}_t + Y_{\tau-}), \quad t < \tau. \quad (2.3)$$

Note that the shareholder value equation (2.3), for a process Y vanishing on $[T, +\infty)$ if $T < \tau$, is equivalent to

$$Y_t^{\tau-} = \mathbb{E}_t(\mathcal{Y}_{\tau \wedge T}^{\tau-} - \mathcal{Y}_t^{\tau-} + \mathbb{1}_{\{\tau \leq T\}} Y_\tau^{\tau-}), \quad t \leq \tau \wedge T. \quad (2.4)$$

In particular, $(\mathcal{Y} + Y)^{\tau-}$ is then a martingale (stopped before τ).

This makes it apparent that the shareholder value equation (2.3) is actually an equation for $Y^{\tau-}$ and, in fact, a backward stochastic differential equation (BSDE) for $Y^{\tau-}$. This is a nonstandard BSDE, stopped before the bank default time τ . Such a BSDE is tantamount to the notion of recursive valuation of defaultable securities in Collin-Dufresne, Goldstein, and Hugonnier (2004, Section 3.2), in the special case where $R_t(x) = x$ there. This notion is shown to be well posed in their Proposition 2, based on Schönbucher (2004)'s tool of the bank survival pricing measure. We deal with shareholder valuation by the more comprehensive reduction of filtration methodology of Section B, yielding a more complete grasp on the related integrability issues (cf. Crépey and Song (2017, Section 4.2) and Crépey, Sabbagh, and Song (2020, Lemma 5.2)).

2.2 Abstract Trading Cash Flows

In a first stage, to avoid blurring the XVA conceptual picture by the combinatorial complexity of the financial network of the bank, we simply denote by \mathcal{P} , \mathcal{C} and \mathcal{F} the trading cash flows to the trading desks and from the CVA and FVA desks, respectively⁵. These abstract cash flows \mathcal{P} , \mathcal{C} and \mathcal{F} will then be instantiated in the application Sections 4–6.

On top of \mathcal{P} , \mathcal{C} and \mathcal{F} , we also consider the dynamic hedging cash flows (inclusive of the cost of setting up the hedges)

$$\mathcal{H} = \mathcal{H}^{mtm} - \mathcal{H}^{cva} - \mathcal{H}^{fva}, \quad (2.5)$$

where \mathcal{H}^{mtm} is the dynamic hedging loss of the trading desks, whilst \mathcal{H}^{cva} and \mathcal{H}^{fva} are the dynamic hedging gains of the CVA and FVA desks.

Example 2.2 *Assuming a hedge of the trading desks implemented through a repo market on a Black-Scholes stock S with volatility σ , then, supposing no dividends and no repo basis on S , we have until the liquidation of the position that is hedged (with \mathcal{H}^{mtm} and S in units of the risk-free asset that we use as our numéraire):*

$$d\mathcal{H}_t^{mtm} = \zeta_t dS_t = \zeta_t \sigma S_t dW_t. \quad (2.6)$$

Here W is a $(\mathfrak{G}, \mathbb{Q})$ Brownian motion driving S and ζ is the number of stocks shorted by the trading desks as their hedge. Given our choice of the risk-free asset as numéraire, the risk-free cost of funding the hedge is already included in (2.6).

Risky funding for the bank means borrowing at a nonnegative risky spread over the risk-free rate. We assume the funding debt of the bank instantaneously liquidated at the bank default time τ . Accordingly:

⁵unless explicitly specified, an amount paid means effectively paid if positive, received if negative. A similar convention applies to the notions of loss and gain or cost and benefit.

Assumption 2.1 *The process \mathcal{F} is a martingale nondecreasing before τ and stopped at $\tau \wedge T$. Each of the processes $\Xi = \mathcal{H}^{mtm}, \mathcal{H}^{cva}, \mathcal{H}^{fva}$ (hence, the aggregated hedging loss \mathcal{H}) is such that the processes $\Xi^{\tau-}$ and ${}^{\tau-}\Xi$ are martingales.*

Martingale assumptions on the hedging cash flows Ξ and on the risky funding cash flows \mathcal{F} are in line with the definition of \mathbb{Q} provided in Section 2.1. As bank shareholders only perceive bank pre-default cash flows, assuming that the $\Xi^{\tau-}$ processes are also martingales makes it consistent with a bank shareholder centric viewpoint of the different desks of the bank, in line with the fact that the shareholders have the control of the bank as long as it is nondefault (see the end of Section 1.1). A shareholder risk premium will be introduced in a second step, through KVA dividends.

Remark 2.1 *Martingales with martingale $(\cdot)^{\tau-}$ component obviously include all martingales without jump at τ , in particular all continuous martingales. They also include all the \mathfrak{F} (càdlàg) martingales in an immersed reduction of filtration framework $\mathfrak{F} \subseteq \mathfrak{G}$, corresponding to the special case where $\mathbb{P} = \mathbb{Q}$ in the setup of Section B, provided the \mathfrak{F} Azéma supermartingale of τ is continuous and nonincreasing (Crépey, 2015, Lemma 2.1(ii)).*

Remark 2.2 *In the application Sections 4–6, nonnegative losses triggered by client defaults accumulate before the default of the bank itself, so that $\mathcal{C}^{\tau-}$ is nondecreasing, if not for minor corrective terms related to a mismatch of temporality between some cash flows and their hedge during the liquidation periods that separate the defaults from their settlements. In particular, in the limiting case where defaults are instantaneously settled, $\mathcal{C}^{\tau-}$ is nondecreasing (see Remark 6.1).*

2.3 Loss Processes

In our marked-to-model framework (see the end of Section 1.1), the gain process of the trading desks is given, inclusive of the corresponding hedging loss \mathcal{H}^{mtm} , by

$$\mathcal{P} + \text{MtM} - \text{MtM}_0 - \mathcal{H}^{mtm}, \quad (2.7)$$

for some theoretical target MtM to be devised later in view of addressing the requirements of Section 1. Likewise, the loss processes of the CA desks are given, inclusive of their respective hedging gains \mathcal{H}^{cva} (for the CVA desk) and \mathcal{H}^{fva} (for the FVA desk), by

$$\begin{aligned} \mathcal{C} + \text{CVA} - \text{CVA}_0 - \mathcal{H}^{cva} \\ \mathcal{F} + \text{FVA} - \text{FVA}_0 - \mathcal{H}^{fva}, \end{aligned} \quad (2.8)$$

for some to-be-suitably-devised theoretical target CVA and FVA levels. Denoting $\text{CA} = \text{CVA} + \text{FVA}$ (cf. (1.1)), the overall loss of the bank is (recalling (2.5))

$$\mathcal{L} = \mathcal{C} + \mathcal{F} + \text{CA} - \text{CA}_0 - (\mathcal{P} + \text{MtM} - \text{MtM}_0) + \mathcal{H}. \quad (2.9)$$

A dealer bank should not do proprietary trading (cf. Section 1). But the risk of financial loss as a consequence of client default is hard to hedge, because single

name credit default swaps that could in principle be used for that purpose are illiquid. The possibility for the bank of hedging its own jump-to-default is even more questionable. Indeed, for the bank, hedging its default is tantamount to selling jump-to-default protection on itself, which is unfeasible (see Section 2.1). Accordingly, our *reference dynamic hedging case* is when the trading desks are perfectly hedged ($\mathcal{P} + \text{MtM} - \text{MtM}_0 - \mathcal{H}^{mtm} = 0$) whilst the CA desks are not hedged ($\mathcal{H}^{cva} = \mathcal{H}^{fva} = 0$), hence

$$\mathcal{H} = \mathcal{H}^{mtm} = \mathcal{P} + \text{MtM} - \text{MtM}_0 \text{ and } \mathcal{L} = \mathcal{C} + \mathcal{F} + \text{CA} - \text{CA}_0. \quad (2.10)$$

Example 2.3 *Continuing with Example 2.2, the gain process of a trading desk long a delta-hedged option position with payoff $(S_\Theta - K)^+$ (for some fixed maturity Θ) is given, until the liquidation of the position, by $\mathcal{P} + \text{MtM} - \text{MtM}_0 - \int_0^\cdot \zeta_t dS_t$, where $\mathcal{P} = \mathbb{1}_{[\Theta, +\infty)}(S_\Theta - K)^+$ and MtM is the Black-Scholes price process of the option (vanishing from time Θ onward). This trading gain process is therefore 0 if the trader uses the Black-Scholes delta as his hedge, which he should do in order to conform to the Volcker rule.*

Remark 2.3 *Even if (2.10) is our reference dynamic hedging case, we do not assume (2.10) throughout the paper. Indeed, on the one hand, the feasibility of the hedging loss $\mathcal{H}^{mtm} = \mathcal{P} + \text{MtM} - \text{MtM}_0$ by the trading desks is subject to the depth of the dynamic hedging markets, so supposing this hedging loss would be too restrictive. On the other hand, CVA traders do tentatively hedge their market (if not jump-to-default) risk, hence supposing $\mathcal{H}^{cva} = \mathcal{H}^{fva} = 0$ (i.e. $\mathcal{H} = \mathcal{H}^{mtm}$) would also be too restrictive.*

2.4 MtM, CVA, and FVA

Definition 2.2 *MtM, CVA, and FVA are shareholder value processes⁶ of \mathcal{P} , \mathcal{C} , and \mathcal{F} .*

That is, MtM, CVA, and FVA are killed at T on $\{T < \tau\}$ and, for $t < \tau$,

$$\text{MtM}_t = \mathbb{E}_t(\mathcal{P}_{\tau^-} - \mathcal{P}_t + \text{MtM}_{\tau^-}), \quad (2.11)$$

$$\text{CVA}_t = \mathbb{E}_t(\mathcal{C}_{\tau^-} - \mathcal{C}_t + \text{CVA}_{\tau^-}), \quad (2.12)$$

$$\text{FVA}_t = \mathbb{E}_t(\mathcal{F}_{\tau^-} - \mathcal{F}_t + \text{FVA}_{\tau^-}). \quad (2.13)$$

Hence, by Assumption 2.1 and the sentence following (2.4):

Remark 2.4 *The processes MtM and $\text{CA} = \text{CVA} + \text{FVA}$ (cf. (1.1)) are such that each of the trading gains and losses in (2.7)-(2.8), stopped before τ , is a martingale. So is therefore*

$$\mathcal{L}^{\tau^-} = \mathcal{C}^{\tau^-} + \mathcal{F}^{\tau^-} + \text{CA}^{\tau^-} - \text{CA}_0 - (\mathcal{P}^{\tau^-} + \text{MtM}^{\tau^-} - \text{MtM}_0) + \mathcal{H}^{\tau^-} \quad (2.14)$$

(cf. (2.9)), the trading loss of the bank shareholders.

⁶cf. Definition 2.1(ii). Explicit assumptions ensuring existence of all our XVA shareholder value processes, and also their uniqueness within suitable spaces of square integrable solutions, are provided by Lemma B.1, Proposition B.1 and Theorem 6.1.

Given Assumption 2.1 on \mathcal{H} , the reference dynamic hedging case (2.10) is only attainable provided not only $(\mathcal{P} + \text{MtM})^{\tau-}$ (as dictated by (2.11), cf. again the line following (2.4)), but also ${}^{\tau-}(\mathcal{P} + \text{MtM})$, hence $\mathcal{P} + \text{MtM}$ as a whole, is a martingale. Even if we do not assume (2.10) throughout the paper (cf. Remark 2.3), this motivates the following:

Assumption 2.2 *MtM is the value process of \mathcal{P} (in particular, $\text{MtM}_T = 0$).*

Remark 2.5 *In further support of Assumption 2.2, note that, even if the trading desks were able to find clients accepting to deal with the bank on the basis of an MtM process that would be the bank shareholder value of \mathcal{P} but not its value process, the corresponding discrepancy between valuation and shareholder valuation of \mathcal{P} would be an indication of extreme dependence between the mark-to-market risk of the bank and its own default risk, such as the bank trading its own default risk, which should be considered with caution (cf. Remark A.2).*

The processes CVA and FVA are so far unconstrained on $\llbracket \tau, +\infty \llbracket \cap (\{\tau \leq T\} \times \mathbb{R}_+)$. We define them as zero there. As they already vanish on $[T, +\infty)$ if $T < \tau$, either of them, say Y , is in fact killed at $\tau \wedge T$, hence such that

$${}^{\tau-}Y = \mathbf{1}_{\llbracket \tau, +\infty \llbracket (Y - Y_{\tau-}) = -\mathbf{1}_{\llbracket \tau, +\infty \llbracket Y_{\tau-}. \quad (2.15)$$

2.5 Shareholder Capital at Risk and KVA

Since default risk can hardly be hedged, capital needs be set at risk by shareholders, who therefore deserve, in the cost-of-capital pricing approach of this paper, a further KVA add-on as a risk premium.

Economic capital (EC) is the level of capital at risk (CR) that a regulator would like to see on an economic basis. In view of (1.2), where KVA is provided by the clients in the first place (see Section 1.1):

Definition 2.3 *We define the shareholder capital at risk (SCR), to be remunerated at a constant and nonnegative hurdle rate h , as*

$$\text{SCR} = \text{CR} - \text{KVA} = \max(\text{EC}, \text{KVA}) - \text{KVA} = (\text{EC} - \text{KVA})^+, \quad (2.16)$$

where KVA is a shareholder value process of $\int_0^{\cdot} h \text{SCR}_s ds$ killed at $\tau \wedge T$, i.e.

$$\text{KVA}_t = \mathbb{E}_t \left[\int_t^{\tau \wedge T} h (\text{EC}_s - \text{KVA}_s)^+ ds + \text{KVA}_{\tau-} \right], \quad t < \tau, \quad \text{and} \quad (2.17)$$

KVA is killed at $\tau \wedge T$.

Remark 2.6 *The process $\text{KVA}^{\tau-}$ is a supermartingale with drift coefficient $-h \text{SCR} = -h(\text{EC} - \text{KVA})^+$.*

Note the following differential form of (2.17) (cf. (2.4)):

$$\begin{aligned} \text{KVA}_T^{\tau-} &= 0 \text{ on } \{T < \tau\} \text{ and, for } t \leq \tau \wedge T, \\ d\text{KVA}_t^{\tau-} &= -h \text{SCR}_t dt + d\nu_t, \\ &\text{for some martingale } \nu. \end{aligned}$$

This formulation makes it apparent that the KVA corresponds to the amount to be maintained by the bank on its risk margin account in order to be in a position to deliver to its shareholders, dynamically into the future, a hurdle rate h on their capital at risk (SCR). In this sense our KVA addresses the sustainability requirement in Section 1. Moreover the amount on the risk margin account should land off at $\text{KVA}_T = 0$ on $\{T < \tau\}$. Indeed, ending up in the negative would mean an insufficient risk margin for ensuring the hurdle rate h to the shareholders. Conversely, ending up in the positive at $T < \tau$ would mean that the bank is unnecessarily expensive to its clients, which would contradict the minimality requirement in Section 1. The last statement in Proposition B.1(iv) shows that $\text{CR} = \max(\text{EC}, \text{KVA})$ is in fact *the minimal and cheapest* capital at risk process C satisfying the risk admissibility condition $C \geq \text{EC}$ and consistent with the target hurdle rate h on shareholder capital at risk.

Proposition 2.1 *Shareholder dividends*

$$\mathcal{D} = -(\mathcal{L}^{\tau-} + \text{KVA}^{\tau-} - \text{KVA}_0) \tag{2.18}$$

are a submartingale stopped before τ , with drift coefficient $h\text{SCR}$.

Proof. Shareholder trading gains $(-\mathcal{L}^{\tau-})$ and KVA risk margin payments result in a dividend stream (2.18) to shareholders. The stated properties of \mathcal{D} follow from the observations made in Remarks 2.4 and 2.6. ■

3 Cash Flows Allocation Principles

3.1 Financial Network of the Bank

As explained in Section 2.3, banks can hardly hedge jump-to-default risk. Instead they mitigate it, by partitioning their derivative portfolio into netting sets of contracts which are jointly collateralized (guaranteed against default, to some extent) and liquidated following the default of the related counterparty or of the bank itself. The counterparty of the bank in a netting set can be a corporate client, another bank, or a group of financial institutions pooled in the form of a central counterparty (CCP)⁷. See Figure 3.1, where the bank labeled by 0, with financial network emphasized in red, represents the reference bank in the paper. The exchanges on top of Figure 3.1 denote platforms where futures-style positions can be used by the bank for hedging its residual market risk (not already statically offset between different deals) dynamically. Futures-style positions are tantamount to continuously rolled-over instantaneously maturing positions, hence⁸ they are without XVA implications⁹. As will be detailed in Section 5, the trades of the bank with the CCPs, corresponding to the bottom part in Figure 3.1, are fully collateralized back-to-back mark-to-market hedges of client trades. To make it short, the trades of the bank with CCPs and exchanges correspond to its respective static and dynamic hedges.

⁷see Gregory (2014) for an overview on the CCP topic and Menkveld and Vuillemeij (2021) for a recent CCP survey.

⁸XVAs require a positive time horizon to develop.

⁹but they may entail liquidity issues, which are outside the scope of this paper.

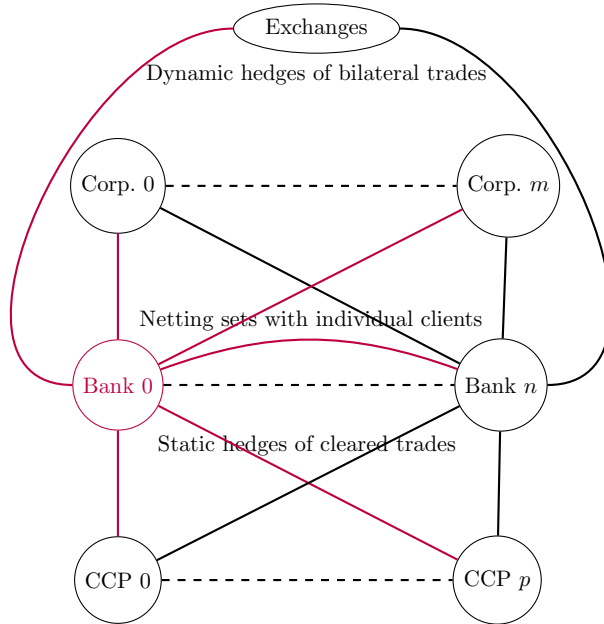


Figure 3.1: The network of corporate clients, banks, CCPs and exchanges. The solid edges represent financial cash flows. Netting sets with individual clients, which can be corporate clients or other banks, correspond to the upper part of the picture (banks and above), and netting sets with CCPs to the lower part (below banks).

3.2 Default Cash Flows

In words to be turned into equations in this paper (see in particular the proofs of Lemmas 4.2 and 5.2), the standing rule regarding the liquidation of a defaulted portfolio is that:

Assumption 3.1 *At the liquidation time of a netting set:*

- *if a counterparty in default is indebted toward the other beyond its posted margin¹⁰, then this debt is only reimbursed at the level of this posted margin plus a fraction (recovery rate of the defaulted party) times the residual debt beyond the margin;*
- *otherwise the debt between the two parties is fully settled.*

Here debt is understood as the sum between, on the one hand, the counterparty-risk-free value of the netting set at liquidation time and, on the other hand, the contractually promised cash flows of the netting set unpaid during the liquidation period that separates the default event from the settlement of the default.

Within the bank, the CVA desk is in charge of these netting set liquidation cash flows.

¹⁰which cannot occur for both jointly, by nonnegativity of initial margins as we will see in Remark 4.4.

3.3 Risky Funding Cash Flows

Regarding now the risky funding cash flows¹¹, let $R \leq \bar{R}$ denote two predictable, $[0, 1]$ valued recovery rates processes, corresponding to two bonds of different seniorities issued by the bank. Assuming the bank default time τ endowed with an intensity $\gamma = J_- \gamma$, we consider the martingales defined by $U_0 = \bar{U}_0 = 1$ and, for $t \geq 0$,

$$\begin{aligned} dU_t &= \lambda_t U_t dt + (1 - R_t) U_{t-} dJ_t = (1 - R_t) U_{t-} (dJ_t + \gamma_t dt) \\ d\bar{U}_t &= \tilde{\lambda}_t \bar{U}_t dt + (1 - \bar{R}_t) \bar{U}_{t-} dJ_t = (1 - \bar{R}_t) \bar{U}_{t-} (dJ_t + \gamma_t dt), \end{aligned}$$

where $dJ_t + \gamma_t dt$ is the compensated jump-to-default martingale of the bank and $\lambda = (1 - R)\gamma$, $\tilde{\lambda} = (1 - \bar{R})\gamma$. We denote by \mathcal{R} and \mathcal{S} two \mathfrak{G} optional processes, with $\mathcal{S} \geq 0$, corresponding to the overall amounts of rehypothecable collateral and segregated collateral posted by the bank Treasury.

Assumption 3.2 \bar{U} and U , where the recovery rate of the bank R is also the recovery rate of the bank toward its clients, are the martingale price processes of two risky funding assets used by the bank for its respective segregated collateral and residual¹² borrowing purposes. As the bank cannot sell default protection on itself, it can only take short positions in U and \bar{U} .

The mechanism through which a recovery rate $\bar{R} \geq R$ may be available on the borrowing debt for segregated collateral is described in Albanese, Armenti, and Crépey (2020, Section 5).

Lemma 3.1 *We have*

$$\begin{aligned} \mathcal{F} &= \int_0^\cdot \lambda_t (\mathcal{R} - \text{CA})_t^+ dt - (1 - R_\tau) (\mathcal{R} - \text{CA})_{\tau-}^+ \mathbf{1}_{\llbracket \tau, \infty \llbracket} \\ &\quad + \int_0^\cdot \tilde{\lambda}_t \mathcal{S}_t dt - (1 - \bar{R}_\tau) \mathcal{S}_{\tau-} \mathbf{1}_{\llbracket \tau, \infty \llbracket} \\ &= \int_0^\cdot ((1 - R_t) (\mathcal{R} - \text{CA})_{t-}^+ + (1 - \bar{R}_t) \mathcal{S}_{t-}) (dJ_t + \gamma_t dt), \end{aligned} \quad (3.1)$$

i.e.

$$\begin{aligned} \mathcal{F}^{\tau-} &= \int_0^\cdot \lambda_t (\mathcal{R} - \text{CA})_t^+ dt + \int_0^\cdot \tilde{\lambda}_t \mathcal{S}_t dt, \\ \tau^-(-\mathcal{F}) &= (1 - R_\tau) (\mathcal{R} - \text{CA})_{\tau-}^+ \mathbf{1}_{\llbracket \tau, \infty \llbracket} + (1 - \bar{R}_\tau) \mathcal{S}_{\tau-} \mathbf{1}_{\llbracket \tau, \infty \llbracket}. \end{aligned} \quad (3.2)$$

Proof. By Assumption 3.2, the risky funding strategy of the bank consists in maintaining short positions (consistent with the condition concluding Assumption 3.2) of $\frac{\mathcal{S}^{\tau-}}{\bar{U}^{\tau-}}$ units of the asset \bar{U} and $\frac{((\mathcal{R} - \text{CA})^+)^{\tau-}}{U^{\tau-}}$ units of U . Given our use of the risk-free asset as numéraire, the self-financing condition on the funding strategy of the FVA desk is then written as $d\mathcal{F}_t = \frac{((\mathcal{R} - \text{CA})^+)^{\tau-}}{U_t^{\tau-}} dU_t + \frac{\mathcal{S}_t^{\tau-}}{\bar{U}_t^{\tau-}} d\bar{U}_t$. This yields (3.1), which splits as (3.2). ■

¹¹beyond the riskless rate that applies to all deposits, as already accounted for by our choice of the riskless asset as the numéraire.

¹²once everything else has already been accounted for, as prescribed by an overall self-financing condition.

4 Bilateral Trading Cash Flows

Table 4.1 gathers the notation pertaining to a netting set ι of financial derivatives between the bank and an individual client (corporate client or other bank, as opposed to a CCP that is considered in the next section). In the case of a netting set between the bank and a client, collateral may come in two layers: rehypothecable variation margins, which are flows designed to track the changes in the (counterparty-risk-free) value of the netting set, possibly complemented by segregated initial margins, meant to cover losses arising during the lapse of time that separates a default from its settlement¹³. All the collateral processes VM^ι , IM^ι and \overline{IM}^ι are assumed to be \mathfrak{G} optional and killed at the final maturity of the bank portfolio. Regarding the first row in Table 4.1, we have by definition (2.2):

$$P_t^\iota = \mathbb{E}_t(\mathcal{P}_T^\iota - \mathcal{P}_t^\iota), t \leq T, \quad (4.1)$$

and P^ι vanishes on $[T, +\infty)$. Also note that, by definition of the various stopping times involved and of T and δ in Section 2.1, we have, for all individual client netting sets ι ,

$$\tau_\iota \leq \tau_\iota^\delta \leq (\tau_\iota + \delta) \wedge T, \text{ and } \tau \leq \tau^\delta \leq \tau + \delta \quad (4.2)$$

holds regarding the bank itself, with liquidation time denoted by τ^δ .

Remark 4.1 *By linearity, (4.1) is the sum over the netting set ι of the analogous quantity pertaining to each individual deal in ι ¹⁴. Hence, P^ι computations reduce to valuations at the individual trade level.*

\mathcal{P}^ι and P^ι	contractually promised cash flows and their value process
τ_ι	$T \wedge$ the default time of the related client
τ_ι^δ	$T \wedge$ the end of the liquidation period of the related client.
R^ι	related predictable, $[0, 1]$ valued recovery rate process
$s_\iota = \tau_\iota \wedge \tau$	$T \wedge$ default time of the netting set ι
$t_\iota = \tau_\iota^\delta \wedge \tau^\delta$	$T \wedge$ liquidation time of the netting set ι
VM^ι	variation margin exchanged between the bank and the client (positive when posted from the client to the bank)
$IM^\iota \geq 0$	initial margin posted by the client to the bank
$\overline{IM}^\iota \geq 0$	initial margin posted by the bank to the client
$\Gamma^\iota = VM^\iota + IM^\iota$	overall collateral amount posted by the client to the bank
$\overline{\Gamma}^\iota = (-VM^\iota) + \overline{IM}^\iota$	overall collateral amount posted by the bank to the client

Table 4.1: Notation pertaining to the individual client netting set ι of contracts of the derivative portfolio of the bank.

¹³we refer the reader to Albanese, Crépey, Hoskinson, and Saadeddine (2021, Section 1.4) for concrete specifications, not needed in this paper, regarding variation and initial margins.

¹⁴assuming all European-style derivatives for simplicity.

In this section we restrict attention to bilateral netting sets and the dynamic hedge of their residual (non already statically offset across different deals) mark-to-market exposure. For each such (non already defaulted) bilateral netting set $\iota \equiv b$, the trading desks and the CVA desk maintain between them an amount P^b of rehypothecable collateral on an trading margin account¹⁵. The process MtM corresponds to the total amount on the trading margin account. At the time s_b where a bilateral netting set b defaults, the corresponding amount on the trading margin account is frozen at its level $P_{s_b-}^b$ in the wait of being transferred (property-wise) from the CVA desk to the trading desks at the liquidation time t_b . During the liquidation period $\llbracket s_b, t_b \rrbracket$, the CVA desk compensates the trading desks for their missed cash flows: contractual trading cash flows $d\mathcal{P}_t^b$ and mark-to-model fluctuations dP_t^b . In addition, at the liquidation time t_b , the liquidation rule of Assumption 3.1 applies to the netting set (and the dynamic hedge of the trading desks related to the netting set b is unwound, i.e. the corresponding term in \mathcal{H} is stopped at t_b).

4.1 Cash Flows to the Trading Desks

In view of the above description, the concrete processes \mathcal{P} (cash flows to the trading desks) and MtM (amount on the trading margin account) corresponding to the present bilateral trading setup are

$$\begin{aligned} \mathcal{P}_0 &= 0, \text{ MtM}_0 = \sum_b P_0^b \text{ and, for } t \in (0, T], \\ d\mathcal{P}_t &= \sum_b \left(\mathbb{1}_{\{t < s_b\}} d\mathcal{P}_t^b + \mathbb{1}_{\{s_b \leq t \leq t_b\}} (d\mathcal{P}_t^b + dP_t^b) + \delta_{t_b}(dt) P_{s_b-}^b \right), \\ d\text{MtM}_t &= \sum_b \left(\mathbb{1}_{\{t < s_b\}} dP_t^b - \delta_{t_b}(dt) P_{s_b-}^b \right). \end{aligned} \quad (4.3)$$

Lemma 4.1 *In a bilateral trading setup, we have*

$$\mathcal{P} = \sum_b \left((\mathcal{P}^b)^{t_b} + \mathbb{1}_{\llbracket s_b, t_b \rrbracket} (P^b - P_{s_b-}^b) + \mathbb{1}_{\llbracket t_b, +\infty \rrbracket} P_{t_b}^b \right), \quad (4.4)$$

$$\text{MtM} = \sum_b \left(\mathbb{1}_{\llbracket 0, s_b \rrbracket} P^b + \mathbb{1}_{\llbracket s_b, t_b \rrbracket} P_{s_b-}^b \right), \text{ and} \quad (4.5)$$

$$\mathcal{P} + \text{MtM} = \sum_b (P^b + P^b)^{t_b}, \quad (4.6)$$

which is a martingale.

Proof. All formulas follow from (4.3) by positivity of the $s_b = \tau_b \wedge \tau$ (having assumed all positive default times). By (4.1), each of the $\mathcal{P}^b + P^b$ is a martingale. Hence so is the sum $(\mathcal{P} + \text{MtM})$. ■

As $\mathcal{P} + \text{MtM}$ is a martingale and $\text{MtM}_T = 0$ (by (4.5) and definition of $s_b \leq t_b \leq T$ with $\iota \equiv b$ in Table 4.1), hence the MtM process in (4.5) is the value process (2.2) of the process \mathcal{P} in (4.4).

¹⁵see Section 1.1.

Remark 4.2 For consistency with the requirement of Definition 2.2 that MtM also be the shareholder value process (2.11) of \mathcal{P} , $\sum_b((\mathcal{P}^b + P^b)^{t_b})^{\tau^-}$ must also be a martingale, which is satisfied if each P^b is not only the value, but also the shareholder value process of \mathcal{P}^b (so that $(\mathcal{P}^b + P^b)^{\tau^-}$ is a martingale). See Remarks 2.1 and 2.5 for a discussion of the ensuing setup.

Remark 4.3 Since $t_b = \tau^\delta \wedge \tau_b^\delta \leq \tau^\delta$, (4.4) and (4.5) imply that \mathcal{P} and MtM are respectively stopped and killed at τ^δ .

4.2 Collateral and Default Cash Flows

Lemma 4.2 In a bilateral trading setup, we have

$$\mathcal{R} = (\text{MtM} - \sum_b \text{VM}^b), \mathcal{S} = \sum_b \overline{\text{IM}}^b \quad (4.7)$$

and the trading cash flows \mathcal{C} from the CVA desk are

$$\begin{aligned} \mathcal{C} = & \sum_{b; \tau_b \leq \tau^\delta} (1 - R_{t_b}^b)(P_{t_b}^b + \mathcal{P}_{t_b}^b - \mathcal{P}_{s_b-}^b - \Gamma_{s_b-}^b)^+ \mathbf{1}_{\llbracket t_b, \infty \llbracket} \\ & - (1 - R_{t_b}) \sum_{b; \tau \leq \tau_b^\delta} (P_{t_b}^b + \mathcal{P}_{t_b}^b - \mathcal{P}_{s_b-}^b + \overline{\Gamma}_{s_b-}^b)^- \mathbf{1}_{\llbracket t_b, \infty \llbracket} \\ & + \sum_b \mathbf{1}_{\llbracket s_b, t_b \llbracket} (\mathcal{P}^b + P^b - (\mathcal{P}^b + P^b)_{s_b-}). \end{aligned} \quad (4.8)$$

Proof. In a bilateral trading setup, the rehypothecable collateral posted by the bank Treasury is the difference between the amount MtM on the trading margin account and the variation margin $\sum_b \text{VM}^b$ posted to the bank by its clients. This yields the formula for \mathcal{R} in (4.7), where the one for \mathcal{S} corresponds to the initial margin posted by the Treasury of the bank.

In view of the description following Table 4.1, during the liquidation period of the bilateral netting set b , the CVA desk loses

$$\begin{aligned} & \int_{s_b-}^{\cdot} \mathbf{1}_{t \leq t_b} (d\mathcal{P}_t^b + dP_t^b) + P_{s_b-}^b \mathbf{1}_{\llbracket t_b, +\infty \llbracket} = \\ & \mathbf{1}_{\llbracket s_b, +\infty \llbracket} ((\mathcal{P}^b)^{t_b} - \mathcal{P}_{s_b-}^b) + \mathbf{1}_{\llbracket s_b, t_b \llbracket} (P^b - P_{s_b-}^b) + \mathbf{1}_{\llbracket t_b, +\infty \llbracket} P_{t_b}^b. \end{aligned} \quad (4.9)$$

Moreover, by application of the liquidation rule of Assumption 3.1 to the bilateral netting set b , the CVA desk receives the following amount from the client at t_b , where $x_b = \mathcal{P}_{t_b}^b - \mathcal{P}_{s_b-}^b + P_{t_b}^b$:

$$\begin{aligned} & \mathbf{1}_{\{\tau_b \leq t_b, x_b > \Gamma_{s_b-}^b\}} (\Gamma_{s_b-}^b + R_{t_b}^b (x_b - \Gamma_{s_b-}^b)) \\ & - \mathbf{1}_{\{\tau \leq t_b, (-x_b) > \overline{\Gamma}_{s_b-}^b\}} (\overline{\Gamma}_{s_b-}^b + R_{t_b} (-x_b - \overline{\Gamma}_{s_b-}^b)) \\ & + \mathbf{1}_{\neg\{(\tau_b \leq t_b, x_b > \Gamma_{s_b-}^b) \vee (\tau \leq t_b, (-x_b) > \overline{\Gamma}_{s_b-}^b)\}} x_b \\ & = x_b - \mathbf{1}_{\{\tau_b \leq t_b\}} (1 - R_{t_b}^b) (x_b - \Gamma_{s_b-}^b)^+ + \mathbf{1}_{\{\tau \leq t_b\}} (1 - R_{t_b}) (x_b + \overline{\Gamma}_{s_b-}^b)^-. \end{aligned} \quad (4.10)$$

Subtracting ($\mathbb{1}_{\llbracket t_b, +\infty \rrbracket} \times (4.10)$), where $x_b = \mathcal{P}_{t_b}^b - \mathcal{P}_{s_b-}^b + P_{t_b}^b$, from (4.9) yields the overall loss process of the CVA desk as

$$\begin{aligned}
& \mathbb{1}_{\{\tau_b \leq t_b\}} (1 - R_{t_b}^b) (x_b - \Gamma_{s_b-}^b)^+ \mathbb{1}_{\llbracket t_b, \infty \rrbracket} - \mathbb{1}_{\{\tau \leq t_b\}} (1 - R_{t_b}) (x_b + \bar{\Gamma}_{s_b-}^b)^- \mathbb{1}_{\llbracket t_b, \infty \rrbracket} \\
& \quad - \mathbb{1}_{\llbracket t_b, +\infty \rrbracket} (\mathcal{P}_{t_b}^b - \mathcal{P}_{s_b-}^b + P_{t_b}^b) \\
& \quad + \mathbb{1}_{\llbracket s_b, +\infty \rrbracket} ((\mathcal{P}^b)^{t_b} - \mathcal{P}_{s_b-}^b) + \mathbb{1}_{\llbracket s_b, t_b \rrbracket} (P^b - P_{s_b-}^b) + \mathbb{1}_{\llbracket t_b, +\infty \rrbracket} P_{t_b}^b \\
& = (1 - R_{t_b}^b) (x_b - \Gamma_{s_b-}^b)^+ \mathbb{1}_{\llbracket t_b, \infty \rrbracket} - \mathbb{1}_{\{\tau \leq t_b\}} (1 - R_{t_b}) (x_b + \bar{\Gamma}_{s_b-}^b)^- \mathbb{1}_{\llbracket t_b, \infty \rrbracket} \\
& \quad + \mathbb{1}_{\llbracket s_b, t_b \rrbracket} (\mathcal{P}^b + P^b - (\mathcal{P}^b + P^b)_{s_b-}),
\end{aligned}$$

where, recalling $t_b = \tau_b^\delta \wedge \tau^\delta$,

$$\begin{aligned}
\{\tau_b \leq t_b\} &= \{\tau_b \leq \tau_b^\delta\} \cap \{\tau_b \leq \tau^\delta\} = \{\tau_b \leq \tau^\delta\} \text{ and} \\
\{\tau \leq t_b\} &= \{\tau \leq \tau_b^\delta\} \cap \{\tau \leq \tau^\delta\} = \{\tau \leq \tau_b^\delta\}.
\end{aligned}$$

By summation over the netting sets b , we obtain the formula (4.8) for \mathcal{C} . ■

Remark 4.4 *Since $\text{IM}^b, \bar{\text{IM}}^b \geq 0$ in $\Gamma^\iota = \text{VM}^\iota + \text{IM}^\iota$ and $\bar{\Gamma}^\iota = (-\text{VM}^\iota) + \bar{\text{IM}}^\iota$ (cf. Table 4.1), we have in (4.10) (cf. the footnote in Assumption 3.1):*

$$\{x_b > \Gamma_{t_b-}^b\} \cap \{(-x_b) > \bar{\Gamma}_{t_b-}^b\} = \{x_b - \text{VM}^b > \text{IM}_{t_b-}^b\} \cap \{(x_b - \text{VM}^b) < -\bar{\text{IM}}_{t_b-}^b\} = \emptyset.$$

5 Centrally Cleared Trading Cash Flows

In this section we consider a financial network of the bank “orthogonal” to the one of Section 4, in the sense that the bank does all its trading with clients in the form of netting sets cleared with a central counterparty (CCP). In the case of a client netting set c of contracts cleared via the bank by a CCP (see Section 3.1), there is a mirroring set of identical deals between the bank and the CCP. The contributions to the trading margin account and the variation margin calls related to the mirroring trades are mirroring the ones on the originating client trade. The bank does not post any initial margin to the related client. In case the client of the netting set c defaults, not only the netting set c between the client and the bank is liquidated, but the mirroring trades between the bank and the CCP are also unwound.

We thus introduce a (single) CCP involving a finite number of clearing members, including the bank itself. The data relative to the different clearing members are indexed by a set of integers i , including $i = 0$ for the bank itself (and disjoint from any other index set in the paper). We still use the notation of Table 4.1 relatively to the individual client netting sets (here, cleared netting sets) of the bank, which we index by $\iota \equiv c$ (for “cleared”), and we use the same notation, but with indices i instead of c , and from the point of view of the CCP (sign-wise), relatively to the portfolios of the clearing members toward the CCP. For instance, \mathcal{P}^c denotes the cash flows promised to the trading desks of the bank related to the cleared netting set c ; \mathcal{P}^i means the cash flows promised by the trading desks of the clearing member i to the CCP, which, by the full collateralization mechanism provided by a counterparty-risk-free CCP as we will see (with an unlimited unfunded default fund), will coincide with the actual cash flows from the trading desks of the clearing member i to the CCP: see Figure 5.1.

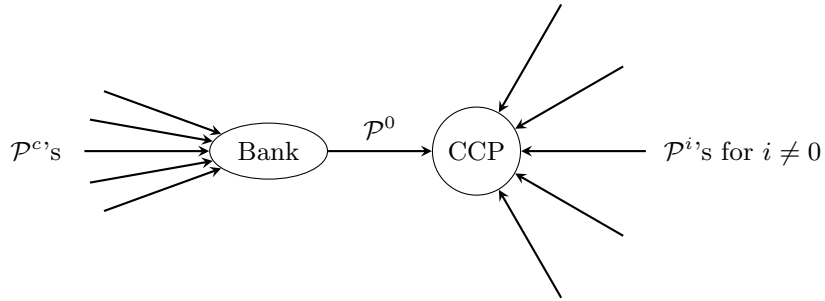


Figure 5.1: Cash flows \mathcal{P}^c from the individual client (here cleared) netting sets c contractually promised to the trading desks of the bank, and cash flows \mathcal{P}^i from the trading desks of the clearing member i to the CCP, in a trading network centrally cleared via a (single) CCP. The clearing members $i \neq 0$ (other than the bank) and their own clients (other than the CCP) are not shown to alleviate the picture.

Remark 5.1 *In practice, the trades of a clearing member with a CCP are partitioned between so-called proprietary trades, which are in effect hedges of its bilateral trades (as proprietary trading as such is forbidden for a dealer bank, which is not allowed to be globally directional in its trading), and mirroring trades of cleared client trades. As discussed around Albanese, Armenti, and Crépey (2020, Figure 1), proprietary CCP trades and the corresponding bilateral trades can typically be rewired in the form of cleared client trades and CCP mirroring trades that are more efficient in XVA terms. Hence we ignore proprietary CCP trades in this paper, for notational simplicity.*

As explained in (Armenti and Crépey, 2017, Remark 3.4), the defaultability of the CCP itself is not an essential issue from an XVA viewpoint. Hence we assume the CCP default-free, i.e. the existence of a default-free clearing member (other than the reference bank).

All the deals between each clearing member i and the CCP are jointly collateralized in terms of variation margin VM^i and initial margin $\overline{\text{IM}}^i \geq 0$ posted by the clearing member to the CCP, along with an additional layer of segregated collateral $\text{DF}^i \geq 0$, depending not only on the CCP portfolio of the clearing member i , but also on the CCP portfolios of the other members, dubbed (funded) default fund contribution of the clearing member i ¹⁶. All the collateral processes VM^i , $\overline{\text{IM}}^i$, DF^i and VM^c are assumed to be \mathfrak{G} optional. As the bank does not post any initial margin to its client in a cleared netting set c , we have

$$\overline{\text{IM}}^c = 0 \text{ (hence } \overline{\Gamma}^c = -\text{VM}^c\text{)}. \quad (5.1)$$

Remark 5.2 *We ignore the skin-in-the-game of the CCP¹⁷, which is negligible from*

¹⁶we refer the reader to Albanese, Crépey, Hoskinson, and Saadeddine (2021, Section 3.2) for concrete specifications, not needed in this paper, regarding the initial margin and default fund contributions posted by a clearing member to a CCP.

¹⁷analog of a default fund contribution that would be posted by the CCP considered as an entity of its own, whereas, for simplicity in our setup, the CCP is nothing more than the collection of its clearing members.

an XVA viewpoint: see Albanese, Armenti, and Crépey (2020, end of Section 3.2).

The following hypothesis completes Assumption 3.1 regarding the liquidation of a netting set in the case where the counterparty of the bank is not an individual client, but a finite collection thereof, pooled in the form of a CCP, so that a rule is required to specify the allocation of the liquidation losses between the (surviving) clearing members.

Assumption 5.1 *In case of a loss triggered by the liquidation of the CCP portfolio of a defaulted clearing member beyond its collateral posted to the CCP, then the loss in excess of its collateral is born by the surviving clearing members, proportionally to \mathfrak{G} optional allocation weights $\mu^i \geq 0$ such that $\mu^i = 0$ on $\llbracket \tau_i, \infty \llbracket$ and $\sum_i \mu^i = 1$.*

The ensuing costs for the clearing members are known as their unfunded default fund contributions (Ghamami, 2015).

5.1 Back-to-back Hedging Cash Flows

Regarding the reference bank, similar to (4.4)-(4.5), but for cleared netting sets c here instead of bilateral netting sets b there, we postulate that the cash flows¹⁸ \mathcal{P}^0 from the trading desks of the bank toward the CCP, are given by

$$\mathcal{P}^0 = \sum_c ((\mathcal{P}^c)^{t_c} + \mathbb{1}_{\llbracket s_c, t_c \llbracket} (P^c - P_{s_c-}^c) + \mathbb{1}_{\llbracket t_c, +\infty \llbracket} P_{t_c}^c). \quad (5.2)$$

Regarding now the globality of the cash flows to the trading desks:

Lemma 5.1 *In a trading setup centrally cleared via a unique CCP, we have $\mathcal{P} = \text{MtM} = 0$.*

Proof. By the analysis having led to (4.3) here applied to the cleared netting sets c , the trading desks get the right-hand side in (5.2) on client deals. By (5.2), they deliver the same amount on CCP deals. Hence $\mathcal{P} = 0$. Besides, the trading margin calls on cleared trades of the bank being mirrored by identical trading margin calls on the mirroring trades between the bank and the CCP, we have $\text{MtM} = 0$. ■

In this sense, in a centrally cleared trading setup, the deals of the bank with the CCP provide a fully collateralized back-to-back hedge to the deals with the clients. The bank is then statically hedged in terms of market risk. But, contrary to a dynamic hedge with futures-style instruments, such a static hedge has XVA implications, which are analysed hereafter.

In view of Lemma 4.1 and Remark 4.3, (5.2) implies that the value process P^0 of \mathcal{P}^0 is worth

$$P^0 = \sum_c (\mathbb{1}_{\llbracket 0, s_c \llbracket} P^c + \mathbb{1}_{\llbracket s_c, t_c \llbracket} P_{s_c-}^c) \quad (\text{in particular, } P_0^0 = \sum_c P_0^c), \quad (5.3)$$

¹⁸both promised and actual cash flows, through the full collateralization mechanism provided by the CCP as we will see.

and that

$$\mathcal{P}^0 + P^0 = \sum_c (\mathcal{P}^c + P^c)^{t_c}, \quad (5.4)$$

$$P^0 \text{ is stopped at } \tau^\delta, P^0 \text{ is killed at } \tau^\delta. \quad (5.5)$$

We postulate a structure of the cash flows \mathcal{P}^i of each clearing member i to the CCP analogous to (5.2) with respect to their own clients (not shown in Figure 5.1 to alleviate the picture) so that, in particular, P^i is killed at τ_i^δ .

5.2 Collateral and Default Cash Flows

For any subset I of the clearing members, let τ_I^δ denote $T \wedge$ the (assumed \mathfrak{G} stopping) time of occurrence of an instantaneous joint default of the clearing members I and only in I , let τ_I^δ denote $T \wedge$ the end of the corresponding liquidation period, and let the breach B^I denote the process

$$\begin{aligned} B^I &= \mathbf{1}_{\llbracket \tau_I^\delta, \infty \llbracket} \sum_{i \in I} (1 - R_{\tau_I^\delta}^i) (P_{\tau_I^\delta}^i + \mathcal{P}_{\tau_I^\delta}^i - \mathcal{P}_{\tau_I^-}^i - \text{VM}_{\tau_I^-}^i - \text{IM}_{\tau_I^-}^i - \text{DF}_{\tau_I^-}^i)^+ \quad (5.6) \\ &= \mathbf{1}_{\llbracket \tau_I^\delta, \infty \llbracket} \sum_{i \in I} (1 - R_{\tau_I^\delta}^i) (\mathcal{P}_{\tau_I^\delta}^i - \mathcal{P}_{\tau_I^-}^i - \text{VM}_{\tau_I^-}^i - \text{IM}_{\tau_I^-}^i - \text{DF}_{\tau_I^-}^i)^+ \end{aligned}$$

(as the $P_{\tau_I^\delta}^i$ for $i \in I$ vanish, cf. the last paragraph of Section 5.1). Hence (cf. (2.1))

$$(B^I)^{\tau^-} = \mathbf{1}_{\{\tau_I^\delta < \tau\}} B^I. \quad (5.7)$$

Lemma 5.2 *In a trading setup centrally cleared via a unique CCP, we have*

$$\mathcal{R} = 0, \mathcal{S} = \overline{\text{IM}}^0 + \text{DF}^0, \quad (5.8)$$

and the trading cash flows \mathcal{C} from the CVA desk are

$$\begin{aligned} \mathcal{C} &= \sum_{c; \tau_c \leq \tau^\delta} (1 - R_{t_c}^c) (P_{t_c}^c + \mathcal{P}_{t_c}^c - \mathcal{P}_{s_{c-}}^c - \Gamma_{s_{c-}}^c)^+ \mathbf{1}_{\llbracket t_c, \infty \llbracket} \quad (5.9) \\ &\quad - \sum_{c; \tau_c \leq \tau_c^\delta} (1 - R_{t_c}^c) (P_{t_c}^c + \mathcal{P}_{t_c}^c - \mathcal{P}_{s_{c-}}^c + \overline{\Gamma}_{s_{c-}}^c)^- \mathbf{1}_{\llbracket t_c, \infty \llbracket} \\ &\quad + \sum_c \mathbf{1}_{\llbracket s_c, t_c \llbracket} (\mathcal{P}^c + P^c - (\mathcal{P}^c + P^c)_{s_{c-}}) \\ &\quad + \sum_I \mu_{\tau_I^\delta}^0 B^I - (1 - R_{\tau^\delta}^0) (\mathcal{P}_{\tau^\delta}^0 - \mathcal{P}_{\tau^-}^0 - \text{VM}_{\tau^-}^0 - \text{IM}_{\tau^-}^0 - \text{DF}_{\tau^-}^0)^+ \mathbf{1}_{\llbracket \tau^\delta, \infty \llbracket} \\ &\quad + \mathbf{1}_{\llbracket \tau, \tau^\delta \llbracket} (\mathcal{P}^0 - \mathcal{P}_{\tau^-}^0). \end{aligned}$$

Proof. The variation margin calls on cleared trades of the bank being mirrored by identical variation margin calls on the mirroring trades between the bank and the CCP. Hence the cleared trades and their mirroring CCP trades have in aggregate no rehypothecable collateral implications for the bank Treasury, which however has the segregated collateral borrowing needs corresponding to the initial margins and funded

default fund contribution to be posted by the bank (recalling from (5.1) that the bank does not post any initial margin to its clients). This yields (5.8).

The cash flows from the CVA desk are, on the one hand, the cash flows from the CVA desk related to the individual client (here cleared) netting sets c , i.e., by the same computation than the one having led to (4.8) (but for the netting set $\iota \equiv c$, here)

$$\begin{aligned} & \sum_{c; \tau_c \leq \tau^\delta} (1 - R_{t_c}^c) (P_{t_c}^c + \mathcal{P}_{t_c}^c - \mathcal{P}_{s_{c-}}^c - \Gamma_{s_{c-}}^c)^+ \mathbb{1}_{\llbracket t_c, \infty \llbracket} \\ & - (1 - R_{t_c}) \sum_{c; \tau \leq \tau_c^\delta} (P_{t_c}^c + \mathcal{P}_{t_c}^c - \mathcal{P}_{s_{c-}}^c + \bar{\Gamma}_{s_{c-}}^c)^- \mathbb{1}_{\llbracket t_c, \infty \llbracket} \\ & + \sum_c \mathbb{1}_{\llbracket s_c, t_c \llbracket} (\mathcal{P}^c + P^c - (\mathcal{P}^c + P^c)_{s_{c-}}). \end{aligned} \quad (5.10)$$

On the other hand, denoting by \mathcal{U}^0 the cumulative process of the unfunded default fund contributions of the bank (see the line following Assumption 5.1), by $x_0 := P_{\tau^\delta}^0 + \mathcal{P}_{\tau^\delta}^0 - \mathcal{P}_{\tau-}^0 = \mathcal{P}_{\tau^\delta}^0 - \mathcal{P}_{\tau-}^0$ (as $P_{\tau^\delta}^0 = 0$), and going by the liquidation rule of Assumption 3.1¹⁹, the cash flows from the CVA desk to the CCP are given by:

$$\begin{aligned} & - \int_0^\cdot \mathbb{1}_{\tau \leq t \leq \tau^\delta} d\mathcal{P}_t^0 + \mathcal{U}^0 + \mathbb{1}_{\llbracket \tau^\delta, \infty \llbracket} \times \\ & \left(\mathbb{1}_{\{x_0 \leq \text{VM}_{\tau-}^0 + \text{IM}_{\tau-}^0 + \text{DF}_{\tau-}^0\}} x_0 + \mathbb{1}_{\{x_0 > \text{VM}_{\tau-}^0 + \text{IM}_{\tau-}^0 + \text{DF}_{\tau-}^0\}} \right. \\ & \quad \left. (\text{VM}_{\tau-}^0 + \text{IM}_{\tau-}^0 + \text{DF}_{\tau-}^0 + R_{\tau^\delta}^0 (x_0 - \text{VM}_{\tau-}^0 - \text{IM}_{\tau-}^0 - \text{DF}_{\tau-}^0)) \right) \\ & = - \int_0^\cdot \mathbb{1}_{\tau \leq t \leq \tau^\delta} d\mathcal{P}_t^0 + \mathcal{U}^0 + x_0 \mathbb{1}_{\llbracket \tau^\delta, \infty \llbracket} \\ & \quad - (1 - R_{\tau^\delta}^0) (x_0 - \text{VM}_{\tau-}^0 - \text{IM}_{\tau-}^0 - \text{DF}_{\tau-}^0)^+ \mathbb{1}_{\llbracket \tau^\delta, \infty \llbracket}, \end{aligned} \quad (5.11)$$

where, as $x_0 = \mathcal{P}_{\tau^\delta}^0 - \mathcal{P}_{\tau-}^0$,

$$\begin{aligned} x_0 \mathbb{1}_{\llbracket \tau^\delta, \infty \llbracket} - \int_0^\cdot \mathbb{1}_{\tau \leq t \leq \tau^\delta} d\mathcal{P}_t^0 & = (\mathcal{P}_{\tau^\delta}^0 - \mathcal{P}_{\tau-}^0) \mathbb{1}_{\llbracket \tau^\delta, \infty \llbracket} - \int_0^\cdot \mathbb{1}_{\tau \leq t \leq \tau^\delta} d\mathcal{P}_t^0 \\ & = \mathbb{1}_{\llbracket \tau, \tau^\delta \llbracket} (\mathcal{P}^0 - \mathcal{P}_{\tau-}^0). \end{aligned} \quad (5.12)$$

Moreover, in view of the last line of (5.11) and by the symmetric cash flow analysis that applies to each clearing member of the CCP (cf. the last sentence of Section 5.1), at each time τ_I^δ , where I ranges over all the possible subsets of the clearing members, the loss to be taken over by the surviving clearing members through their unfunded default fund contributions is $B_{\tau_I^\delta}^I$ (cf. (5.6)). Given the allocation rule of Assumption 5.1 for these losses, we thus have in particular:

$$\mathcal{U}^0 = \sum_I \mu_{\tau_I^\delta}^0 B^I. \quad (5.13)$$

Accounting for (5.12) and (5.13) into (5.11) and adding (5.10) yields (5.9). ■

¹⁹like in the proof of (4.8), but simplified here by the fact that one of the counterparties in the netting set, namely the CCP, is default-free. The term $(-\int_0^\cdot \mathbb{1}_{\tau \leq t \leq \tau^\delta} d\mathcal{P}_t^0)$ is the analog of (4.9).

6 Synthesis

We now consider the realistic situation of a bank involved in an arbitrary combination of bilateral netting sets (as in Section 4) and of cleared netting sets and their mirroring CCP trades (as in Section 5), with possibly multiple CCPs. The index ι now ranges over all the individual client netting sets of the bank (individual as opposed to the netting sets between the bank and the CCPs). We denote by *bil* the collection of all the bilateral netting sets of the bank, indexed by b as per Section 4. The bank can be a clearing member of several CCPs and we index all the CCP related quantities in Section 5 by an additional index *ccp*, ranging in an index set disjoint from any other index set in the paper. For instance, for a given CCP *ccp*, the second index I in *ccp, I* ranges over all the possible subsets of clearing members of the CCP *ccp*; $\sum_{ccp, I}$ is a double sum over all CCPs and, for each of them, over all the corresponding subsets I .

6.1 Cash Flows

Lemma 6.1 *Assuming the reduction of filtration setup of Section B:*

- (i) *For any \mathfrak{G} optional process Y , if $Y^{\tau-}$ is nonnegative, then Y' is nonnegative.*
- (ii) *For any $[0, T]$ valued \mathfrak{G} stopping times θ and η such that $\theta \wedge \tau \leq \eta \wedge \tau$, we have $\theta' \leq \eta'$;*
- (iii) *If, additionally, $\eta \leq \theta + c$ holds for some constant $c \geq 0$, then $\eta' \leq \theta' + c$.*

Proof. We recall from Dellacherie and Meyer (1980, Chapitre VI n^o 17) that the set $\{\mathbf{S} > 0\}$ (where $\mathbf{S} = \mathbb{Q}(\tau > \cdot | \mathfrak{F}_\cdot)$) is a random interval starting from 0 included, which, in the reduction of filtration setup of Section B, is assumed to contain T .

- (i) If $Y^{\tau-}$ is nonnegative, then, for $t \leq T$,

$$Y'_t \mathbf{1}_{\{t < \tau\}} = Y_t^{\tau-} \mathbf{1}_{\{t < \tau\}} \geq 0,$$

hence $\mathbb{E}(Y'_t \mathbf{1}_{\{t < \tau\}} | \mathfrak{F}_t) \geq 0$, i.e. $Y'_t \mathbf{S}_t \geq 0$, where $\mathbf{S}_t > 0$.

- (ii) If $\theta \wedge \tau \leq \eta \wedge \tau$, then $\mathbf{1}_{\llbracket \theta \wedge \tau, +\infty \llbracket} - \mathbf{1}_{\llbracket \eta \wedge \tau, +\infty \llbracket} \geq 0$, hence part (i) implies that $\mathbf{1}_{\llbracket \theta', +\infty \llbracket} - \mathbf{1}_{\llbracket \eta', +\infty \llbracket} \geq 0$, i.e. $\theta' \leq \eta'$.

- (iii) If, additionally, $\eta \leq \theta + c$ holds for some constant $c \geq 0$, then part (i) implies that $\theta' \leq \eta'$ and we have

$$\mathbf{1}_{\llbracket 0, \tau \llbracket} (\eta' - \theta') \mathbf{1}_{\llbracket \eta', +\infty \llbracket} = \mathbf{1}_{\llbracket 0, \tau \llbracket} (\eta - \theta) \mathbf{1}_{\llbracket \eta, +\infty \llbracket} \leq c \mathbf{1}_{\llbracket 0, \tau \llbracket} \mathbf{1}_{\llbracket \eta, +\infty \llbracket}, \quad (6.1)$$

where the left equality means that, for $t \geq 0$,

$$\mathbf{1}_{\{\eta' \leq t < \tau\}} (\eta' - \theta') = \mathbf{1}_{\{\eta \leq t < \tau\}} (\eta - \theta'). \quad (6.2)$$

To justify (6.2), note that the indicators in (6.2) must coincide, otherwise this would imply that either $\eta' \leq t < \tau$ holds but not $\eta \leq t < \tau$, in which case $\eta > t$ and $\eta' \wedge \tau \leq t < \eta \wedge \tau$, or that $\eta \leq t < \tau$ holds but not $\eta' \leq t < \tau$, in which case $\eta' > t$ and $\eta \wedge \tau \leq t < \eta' \wedge \tau$, violating in both cases the identity $\eta \wedge \tau = \eta' \wedge \tau$ (cf. (B.1)). Moreover, as we already know from part (ii) that $\theta' \leq \eta'$ and given the assumption $\theta \wedge \tau \leq \eta \wedge \tau$, if the indicators in (6.2) are both equal to one, then the identities $\eta \wedge \tau = \eta' \wedge \tau$ and $\theta \wedge \tau = \theta' \wedge \tau$ imply that $\theta = \theta'$ and $\eta = \eta'$, so that $\eta' - \theta' = \eta - \theta'$, and the identity

(6.2) holds (which is also obviously the case if the indicators in (6.2) are both equal to zero).

By valuation at time η' of the left and right processes in (6.1) and by taking the $(\mathfrak{F}_{\eta'}, \mathbb{Q})$ conditional expectation of the resulting expressions, we obtain

$$\mathbb{E}((\eta' - \theta')\mathbb{1}_{\{\eta' < \tau\}} | \mathfrak{F}_{\eta'}) \leq c\mathbb{E}(\mathbb{1}_{\{\eta' < \tau\}}\mathbb{1}_{\{\eta \leq \eta'\}} | \mathfrak{F}_{\eta'}) \leq c\mathbb{E}(\mathbb{1}_{\{\eta' < \tau\}} | \mathfrak{F}_{\eta'}),$$

i.e. $(\eta' - \theta')S_{\eta'} \leq cS_{\eta'}$, where $S_{\eta'} > 0$ a.s.. ■

Lemma 6.2 *For each individual client netting set ι , we have*

$$t_\iota - s_\iota \leq \delta \tag{6.3}$$

and, assuming the reduction of filtration setup of Section B,

$$0 \leq \tau'_\iota \leq (\tau_\iota^\delta)' \leq T, (\tau_\iota^\delta)' - \tau'_\iota \leq \delta. \tag{6.4}$$

Proof. We compute

$$\begin{aligned} t_\iota - s_\iota &= \tau_\iota^\delta \wedge \tau^\delta - s_\iota = (\tau_\iota^\delta - s_\iota) \wedge (\tau^\delta - s_\iota) = (\tau_\iota^\delta - \tau_\iota \wedge \tau) \wedge (\tau^\delta - \tau_\iota \wedge \tau) \\ &= ((\tau_\iota^\delta - \tau_\iota) \vee (\tau_\iota^\delta - \tau)) \wedge ((\tau^\delta - \tau_\iota) \vee (\tau^\delta - \tau)) \\ &\leq (\delta \vee (\tau_\iota^\delta - \tau)) \wedge (\delta \vee (\tau^\delta - \tau_\iota)), \end{aligned}$$

by (4.2), where, by (4.2) again: If $\tau \vee \tau_\iota = \tau$, then $\tau_\iota^\delta - \tau \leq \tau_\iota^\delta - \tau_\iota \leq \delta$, otherwise $\tau \vee \tau_\iota = \tau_\iota$ and $\tau^\delta - \tau_\iota \leq \tau^\delta - \tau \leq \delta$. This yields (6.3), from which (6.4) follows, in the reduction of filtration setup of Section B, by application of Lemma 6.1. ■

Writing $M^\iota = \mathcal{P}^\iota + P^\iota$, we denote

$$\begin{aligned} \mathcal{Y}^{ccp} &= \mathbb{1}_{\llbracket \tau, \tau^\delta \rrbracket} (\mathcal{P}^{ccp,0} - \mathcal{P}_{\tau^-}^{ccp,0}) \\ \mathcal{Y}^\iota &= \mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket} (M^\iota - M_{s_\iota-}^\iota) \end{aligned} \tag{6.5}$$

and, in the reduction of filtration setup of Section B,

$$\mathcal{X}^\iota = \mathbb{1}_{\llbracket \tau'_\iota, (\tau_\iota^\delta)' \rrbracket} ((M^\iota)' - (M^\iota)'_{\tau'_\iota-}). \tag{6.6}$$

As we will see these are corrective liquidation period CVA and DVA related cash flows. Note that they all vanish identically in the instantaneous liquidation case where $\delta = 0$ (so that the $\tau_\iota^\delta = \tau_\iota$ and $\tau^\delta = \tau$, by (4.2), and the $(\tau_\iota^\delta)' = \tau'_\iota$, by (6.4)).

Proposition 6.1 *In the realistic situation of a bank involved in an arbitrary combination of bilateral netting sets and of netting sets cleared with one among several available*

CCPs, we have

$$\mathcal{P} = \sum_b ((\mathcal{P}^b)^{t_b} + \mathbf{1}_{\llbracket s_b, t_b \llbracket} (P^b - P_{s_b-}^b) + \mathbf{1}_{\llbracket t_b, +\infty \llbracket} P_{t_b}^b) \quad (6.7)$$

$$\text{MtM} = \sum_b (\mathbf{1}_{\llbracket 0, s_b \llbracket} P^b + \mathbf{1}_{\llbracket s_b, t_b \llbracket} P_{s_b-}^b) \quad (6.8)$$

$$\mathcal{P} + \text{MtM} = \sum_b (P^b + P^b)^{t_b} \quad (6.9)$$

$$\mathcal{R} = \text{MtM} - \sum_b \text{VM}^b, \mathcal{S} = \sum_b \overline{\text{IM}}^b + \sum_{ccp} (\overline{\text{IM}}^{ccp,0} + \text{DF}^{ccp,0}) \quad (6.10)$$

$$\begin{aligned} \mathcal{C} - \sum_l \mathcal{Y}^l - \sum_{ccp} \mathcal{Y}^{ccp} &= \sum_{l; \tau_l \leq \tau^\delta} (1 - R_{t_l}^l) (P_{t_l}^l + \mathcal{P}_{t_l}^l - \mathcal{P}_{s_l-}^l - \Gamma_{s_l-}^l)^+ \mathbf{1}_{\llbracket t_l, \infty \llbracket} \\ &\quad - \sum_{l; \tau_l \leq \tau_l^\delta} (1 - R_{t_l}^l) (P_{t_l}^l + \mathcal{P}_{t_l}^l - \mathcal{P}_{s_l-}^l + \overline{\Gamma}_{s_l-}^l)^- \mathbf{1}_{\llbracket t_l, \infty \llbracket} \\ &\quad + \sum_{ccp, I} \mu_{\tau_I^\delta}^{ccp,0} B^{ccp, I} \\ &\quad - \sum_{ccp} (1 - R_{\tau^\delta}^{ccp,0}) (\mathcal{P}_{\tau^\delta}^{ccp,0} - \mathcal{P}_{\tau-}^{ccp,0} - \text{VM}_{\tau-}^{ccp,0} - \text{IM}_{\tau-}^{ccp,0} - \text{DF}_{\tau-}^{ccp,0})^+ \mathbf{1}_{\llbracket \tau^\delta, \infty \llbracket} \end{aligned} \quad (6.11)$$

$$\begin{aligned} \mathcal{F} &= \int_0^\cdot \lambda_t (\text{MtM} - \sum_b \text{VM}^b - \text{CA})_t^+ dt \\ &\quad - (1 - R_\tau) (\text{MtM} - \sum_b \text{VM}^b - \text{CA})_{\tau-}^+ \mathbf{1}_{\llbracket \tau, \infty \llbracket} \\ &\quad + \int_0^\cdot \tilde{\lambda}_t \left(\sum_b \overline{\text{IM}}^b + \sum_{ccp} (\overline{\text{IM}}^{ccp,0} + \text{DF}^{ccp,0}) \right)_t dt \\ &\quad - (1 - \bar{R}_\tau) \left(\sum_b \overline{\text{IM}}^b + \sum_{ccp} (\overline{\text{IM}}^{ccp,0} + \text{DF}^{ccp,0}) \right)_{\tau-} \mathbf{1}_{\llbracket \tau, \infty \llbracket}, \end{aligned} \quad (6.12)$$

i.e.

$$\begin{aligned} \mathcal{C}^{\tau-} - \sum_l (\mathcal{Y}^l)^{\tau-} &= \sum_{l; \tau_l^\delta < \tau} (1 - R_{t_l}^l) (P_{t_l}^l + \mathcal{P}_{t_l}^l - \mathcal{P}_{s_l-}^l - \Gamma_{s_l-}^l)^+ \mathbf{1}_{\llbracket t_l, \infty \llbracket} \\ &\quad + \sum_{ccp, I} \mu_{\tau_I^\delta}^{ccp,0} B^{ccp, I} \\ \tau-(-\mathcal{C}) + \sum_l \tau-(\mathcal{Y}^l) + \sum_{ccp} \mathcal{Y}^{ccp} &= \\ &\quad \sum_{l; \tau_l \leq \tau_l^\delta} (1 - R_{t_l}^l) (P_{t_l}^l + \mathcal{P}_{t_l}^l - \mathcal{P}_{s_l-}^l + \overline{\Gamma}_{s_l-}^l)^- \mathbf{1}_{\llbracket t_l, \infty \llbracket} \\ &\quad - \sum_{l; \tau_l \leq \tau^\delta, \tau_l^\delta \geq \tau} (1 - R_{t_l}^l) (P_{t_l}^l + \mathcal{P}_{t_l}^l - \mathcal{P}_{s_l-}^l - \Gamma_{s_l-}^l)^+ \mathbf{1}_{\llbracket t_l, \infty \llbracket} \\ &\quad + \sum_{ccp} (1 - R_{\tau^\delta}^{ccp,0}) (\mathcal{P}_{\tau^\delta}^{ccp,0} - \mathcal{P}_{\tau-}^{ccp,0} - \text{VM}_{\tau-}^{ccp,0} - \text{IM}_{\tau-}^{ccp,0} - \text{DF}_{\tau-}^{ccp,0})^+ \mathbf{1}_{\llbracket \tau^\delta, \infty \llbracket} \end{aligned} \quad (6.13)$$

and

$$\begin{aligned}
\mathcal{F}^{\tau^-} &= \int_0^\cdot \lambda_t (\text{MtM} - \sum_b \text{VM}^b - \text{CA})_t^+ dt + \\
&\quad \int_0^\cdot \tilde{\lambda}_t \left(\sum_b \overline{\text{IM}}^b + \sum_{ccp} (\overline{\text{IM}}^{ccp,0} + \text{DF}^{ccp,0}) \right)_t dt \\
\tau^-(-\mathcal{F}) &= (1 - R_\tau) \left(\text{MtM} - \sum_b \text{VM}^b - \text{CA} \right)_{\tau^-}^+ \mathbf{1}_{\llbracket \tau, \infty \llbracket} \\
&\quad + (1 - \bar{R}_\tau) \left(\sum_b \overline{\text{IM}}^b + \sum_{ccp} (\overline{\text{IM}}^{ccp,0} + \text{DF}^{ccp,0}) \right)_{\tau^-} \mathbf{1}_{\llbracket \tau, \infty \llbracket}.
\end{aligned} \tag{6.14}$$

Assuming the reduction of filtration setup of Section B, we have

$$\mathcal{L}' = \mathcal{C}' + \mathcal{F}' + \text{CA}' - \text{CA}_0 - (\mathcal{P}' + \text{MtM}' - \text{MtM}_0) + \mathcal{H}', \tag{6.15}$$

where

$$\text{MtM}' = \sum_b ((P^b)' \mathbf{1}_{\llbracket 0, (\tau_b)'\llbracket} + \mathbf{1}_{\llbracket (\tau_b)', (\tau_b^\delta)'\llbracket} (P^b)'_{\tau_b^-}), \tag{6.16}$$

$$\mathcal{P}' + \text{MtM}' = \sum_b ((\mathcal{P}^b + P^b)' (\tau_b^\delta)') \tag{6.17}$$

$$\begin{aligned}
\mathcal{C}' - \sum_\iota \mathcal{X}^\iota &= \sum_\iota (1 - (R^\iota)'_{(\tau_\iota^\delta)'}) ((P^\iota)'_{(\tau_\iota^\delta)'}) + (\mathcal{P}^\iota)'_{(\tau_\iota^\delta)'}) - (\mathcal{P}^\iota)'_{\tau_\iota^-} - (\Gamma^\iota)'_{\tau_\iota^-})^+ \mathbf{1}_{\llbracket (\tau_\iota^\delta)', \infty \llbracket} \\
&\quad + \sum_{ccp, I} (\mu^{ccp,0})'_{(\tau_I^\delta)'}) (B^{ccp, I})',
\end{aligned} \tag{6.18}$$

$$\begin{aligned}
\mathcal{F}' &= \int_0^\cdot \lambda'_t \left(\text{MtM}' - \sum_b (\text{VM}^b)' - \text{CA}' \right)_t^+ dt + \\
&\quad \int_0^\cdot \tilde{\lambda}'_t \left(\sum_b (\overline{\text{IM}}^b)' + \sum_{ccp} (\overline{\text{IM}}^{ccp,0} + \text{DF}^{ccp,0})' \right)_t dt,
\end{aligned} \tag{6.19}$$

and

$$\begin{aligned}
(B^{ccp, I})' &= \sum_{i \in I} (1 - (R^i)'_{(\tau_I^\delta)'}) \\
&\quad \left(\mathcal{P}_{(\tau_I^\delta)' }^{ccp, i} - \mathcal{P}_{\tau_I^-}^{ccp, i} - \text{VM}_{\tau_I^-}^{ccp, i} - \text{IM}_{\tau_I^-}^{ccp, i} - \text{DF}_{\tau_I^-}^{ccp, i} \right) \mathbf{1}_{\llbracket (\tau_I^\delta)', \infty \llbracket}.
\end{aligned} \tag{6.20}$$

Proof. We partition the portfolio of the bank into the collections of its cleared netting sets ccp, c and their netting set of mirroring deals $ccp, 0$, CCP by CCP, complemented by the collection bil of the bilateral netting sets b of the bank and their dynamic hedge.

We then apply the analysis of Section 5, CCP by CCP, and the analysis of Section 4 to the collection of the bilateral netting sets and their dynamic hedge. As detailed below, this results in respective cash flows $\mathcal{P}^{ccp} = \mathcal{R}^{ccp} = 0, \mathcal{S}^{ccp}, \mathcal{C}^{ccp}$ stemming from the cleared netting sets ccp, c and their back-to-back hedge $ccp, 0$, for each CCP ccp , and

$\mathcal{P}^{bil}, \mathcal{R}^{bil}, \mathcal{S}^{bil}, \mathcal{C}^{bil}$ stemming from the bilateral netting sets and their dynamic hedge, such that (as justified below):

$$\begin{aligned}
\mathcal{S}^{ccp} &= \overline{\text{IM}}^{ccp,0} + \text{DF}^{ccp,0}, \\
\mathcal{C}^{ccp} &= \sum_{c; \tau_c \leq \tau^\delta} (1 - R_{t_c}^{ccp,c}) (P_{t_c}^{ccp,c} + \mathcal{P}_{t_c}^{ccp,c} - \mathcal{P}_{s_{c-}}^{ccp,c} - \Gamma_{s_{c-}}^{ccp,c})^+ \mathbb{1}_{\llbracket t_c, \infty \llbracket} \\
&\quad - \sum_{c; \tau_c \leq \tau_c^\delta} (1 - R_{t_c}) (P_{t_c}^{ccp,c} + \mathcal{P}_{t_c}^{ccp,c} - \mathcal{P}_{s_{c-}}^{ccp,c} + \overline{\Gamma}_{s_{c-}}^{ccp,c})^- \mathbb{1}_{\llbracket t_c, \infty \llbracket} \\
&\quad + \sum_c \mathbb{1}_{\llbracket s_c, t_c \llbracket} (\mathcal{P}^{ccp,c} + P^{ccp,c} - (\mathcal{P}^{ccp,c} + P^{ccp,c})_{s_{c-}}) \\
&\quad + \sum_I \mu_{\tau_I}^{ccp,0} B^{ccp,I} \\
&\quad - (1 - R_{\tau^\delta}^{ccp,0}) (\mathcal{P}_{\tau^\delta}^{ccp,0} - \mathcal{P}_{\tau^-}^{ccp,0} - \text{VM}_{\tau^-}^{ccp,0} - \text{IM}_{\tau^-}^{ccp,0} - \text{DF}_{\tau^-}^{ccp,0})^+ \mathbb{1}_{\llbracket \tau^\delta, \infty \llbracket} \\
&\quad + \mathbb{1}_{\llbracket \tau, \tau^\delta \llbracket} (\mathcal{P}^{ccp,0} - \mathcal{P}_{\tau^-}^{ccp,0}), \\
\mathcal{P}^{bil} &= \sum_b ((\mathcal{P}^b)^{t_b} + \mathbb{1}_{\llbracket s_b, t_b \llbracket} (P^b - P_{s_{b-}}^b) + \mathbb{1}_{\llbracket t_b, +\infty \llbracket} P_{t_b}^b), \\
\text{MtM}^{bil} &= \sum_b (\mathbb{1}_{\llbracket 0, s_b \llbracket} P^b + \mathbb{1}_{\llbracket s_b, t_b \llbracket} P_{s_{b-}}^b), \\
\mathcal{R}^{bil} &= \text{MtM}^{bil} - \sum_b \text{VM}^b, \quad \mathcal{S}^{bil} = \sum_b \overline{\text{IM}}^b, \\
\mathcal{C}^{bil} &= \sum_{b; \tau_b \leq \tau^\delta} (1 - R_{t_b}^b) (P_{t_b}^b + \mathcal{P}_{t_b}^b - \mathcal{P}_{s_{b-}}^b - \Gamma_{s_{b-}}^b)^+ \mathbb{1}_{\llbracket t_b, \infty \llbracket} \\
&\quad - (1 - R_{t_b}) \sum_{b; \tau \leq \tau_b^\delta} (P_{t_b}^b + \mathcal{P}_{t_b}^b - \mathcal{P}_{s_{b-}}^b + \overline{\Gamma}_{s_{b-}}^b)^- \mathbb{1}_{\llbracket t_b, \infty \llbracket} \\
&\quad + \sum_b \mathbb{1}_{\llbracket s_b, t_b \llbracket} (\mathcal{P}^b + P^b - (\mathcal{P}^b + P^b)_{s_{b-}}).
\end{aligned}$$

For each CCP ccp , the formulas for \mathcal{S}^{ccp} and \mathcal{C}^{ccp} in the above follow by application of the respective formulas (5.8) and (5.9) to the collection of the cleared netting sets ccp, c and their netting set of mirroring deals $ccp, 0$. The formulas for \mathcal{P}^{bil} and MtM^{bil} are obtained by application of Lemma 4.1 to the collection of the bilateral netting sets b and their dynamic hedge. The formulas for $\mathcal{R}^{bil}, \mathcal{S}^{bil}$ and \mathcal{C}^{bil} follow by application of the respective formulas (4.7) and (4.8) to the collection of the bilateral netting sets b and their hedge.

The formulas (6.7) through (6.11) then follow from the above ones by summation through the identity $\mathcal{Y} = (\sum_{ccp} \mathcal{Y}^{ccp}) + \mathcal{Y}^{bil}$ that applies for each cash flow $\mathcal{Y} = \mathcal{P}, \mathcal{R}, \mathcal{S}, \mathcal{C}$, reflecting the partition of the portfolio of the bank introduced in the beginning of this proof, noting that the set of all the individual client netting sets ι of the bank is itself partitioned into the subsets of the cleared netting sets ccp, c , for each CCP ccp , complemented by the subset of the bilateral netting sets b .

The formula (6.11) for \mathcal{C} splits as (noting that $(\mathcal{Y}^{ccp})^{\tau^-} = 0$)

$$\begin{aligned}
(\mathcal{C} - \sum_{\iota} \mathcal{Y}^{\iota})^{\tau^-} &= \sum_{\iota; \tau_{\iota} \leq \tau^{\delta}, t_{\iota} < \tau} (1 - R_{t_{\iota}}^{\iota})(P_{t_{\iota}}^{\iota} + \mathcal{P}_{t_{\iota}}^{\iota} - \mathcal{P}_{s_{\iota}-}^{\iota} - \Gamma_{s_{\iota}-}^{\iota})^+ \mathbf{1}_{\llbracket t_{\iota}, \infty \llbracket} \\
&\quad - \sum_{\iota; \tau \leq \tau_{\iota}^{\delta}, t_{\iota} < \tau} (1 - R_{t_{\iota}})(P_{t_{\iota}}^{\iota} + \mathcal{P}_{t_{\iota}}^{\iota} - \mathcal{P}_{s_{\iota}-}^{\iota} + \bar{\Gamma}_{s_{\iota}-}^{\iota})^- \mathbf{1}_{\llbracket t_{\iota}, \infty \llbracket} \\
&\quad + \sum_{ccp, I} \mu_{\tau^{\delta}}^{ccp, 0} B^{ccp, I} \\
\tau^-(-\mathcal{C} + \sum_{\iota} \mathcal{Y}^{\iota}) + \sum_{ccp} \mathcal{Y}^{ccp} &= \sum_{\iota; \tau \leq \tau_{\iota}^{\delta}, t_{\iota} \geq \tau} (1 - R_{t_{\iota}})(P_{t_{\iota}}^{\iota} + \mathcal{P}_{t_{\iota}}^{\iota} - \mathcal{P}_{s_{\iota}-}^{\iota} + \bar{\Gamma}_{s_{\iota}-}^{\iota})^- \mathbf{1}_{\llbracket t_{\iota}, \infty \llbracket} \\
&\quad - \sum_{\iota; \tau_{\iota} \leq \tau^{\delta}, t_{\iota} \geq \tau} (1 - R_{t_{\iota}}^{\iota})(P_{t_{\iota}}^{\iota} + \mathcal{P}_{t_{\iota}}^{\iota} - \mathcal{P}_{s_{\iota}-}^{\iota} - \Gamma_{s_{\iota}-}^{\iota})^+ \mathbf{1}_{\llbracket t_{\iota}, \infty \llbracket} \\
&\quad + \sum_{ccp} (1 - R_{\tau^{\delta}}^{ccp, 0})(\mathcal{P}_{\tau^{\delta}}^{ccp, 0} - \mathcal{P}_{\tau^-}^{ccp, 0} - \text{VM}_{\tau^-}^{ccp, 0} - \text{IM}_{\tau^-}^{ccp, 0} - \text{DF}_{\tau^-}^{ccp, 0})^+ \mathbf{1}_{\llbracket \tau^{\delta}, \infty \llbracket}.
\end{aligned} \tag{6.21}$$

Regarding the index sets over which the sums are taken in (6.21) versus (6.13), note that, as $t_{\iota} = \tau_{\iota}^{\delta} \wedge \tau^{\delta}$: (i) $t_{\iota} < \tau \Leftrightarrow \tau_{\iota}^{\delta} < \tau$, hence $\{\iota; \tau_{\iota} \leq \tau^{\delta}, t_{\iota} < \tau\} = \{\iota; \tau_{\iota}^{\delta} < \tau\}$, (ii) $\{\tau \leq \tau_{\iota}^{\delta}, t_{\iota} < \tau\} = \emptyset$, (iii) $\tau \leq \tau_{\iota}^{\delta} \Rightarrow t_{\iota} \geq \tau$, hence $\{\iota; \tau \leq \tau_{\iota}^{\delta}, t_{\iota} \geq \tau\} = \{\iota; \tau \leq \tau_{\iota}^{\delta}\}$; (iv) $\{c; \tau_{\iota} \leq \tau^{\delta}, t_{\iota} \geq \tau\} = \{c; \tau_{\iota} \leq \tau^{\delta}, \tau_{\iota}^{\delta} \geq \tau\}$. Hence (6.21) reduces to (6.13).

The formulas in (6.12) and (6.14) for \mathcal{F} and its \mathcal{F}^{τ^-} and $\tau^-(-\mathcal{F})$ components are obtained by feeding (3.1)-(3.2) with \mathcal{R} and \mathcal{S} from (6.10).

Assuming the reduction of filtration setup of Section B, (6.15) follows from (2.9) by definition, existence and uniqueness of optional reductions (cf. (B.2)). Moreover we have, for each individual client netting set ι , $s_{\iota} \wedge \tau = (\tau_{\iota} \wedge \tau) \wedge \tau = \tau_{\iota} \wedge \tau$ and $t_{\iota} \wedge \tau = (\tau_{\iota}^{\delta} \wedge \tau^{\delta}) \wedge \tau = \tau_{\iota}^{\delta} \wedge \tau$, hence $s'_{\iota} = \tau'_{\iota}$ and $t'_{\iota} = (\tau_{\iota}^{\delta})'$, by definition, existence and uniqueness of reductions of $[0, T]$ valued stopping times. Then (6.16) and (6.17) follow from the expressions for MtM and $(\mathcal{P} + \text{MtM})$ in (6.7)-(6.9), while (6.18), (6.19) and (6.20) respectively follow from the expressions of $(\mathcal{C})^{\tau^-}$ in (6.13), where $(\mathcal{Y}^{\iota})' = \mathcal{X}^{\iota}$, of $(\mathcal{F})^{\tau^-}$ in (6.14) and of $(B^I)^{\tau^-}$ in (5.6)-(5.7). ■

Corollary 6.1 \mathcal{P} is additive over bilateral trades, \mathcal{C} is additive over individual client netting sets ι and CCPs. The different terms in \mathcal{F} are additive over the corresponding funding sets: bilateral client and CCP netting sets b and $ccp, 0$ for the segregated collateral (\mathcal{S}) related components of \mathcal{F} (two last lines in (6.12)), while the rehypothecable collateral (\mathcal{R}) related component of \mathcal{F} (two first lines in (6.12)) can only be assessed at the level of the overall trading portfolio of the bank.

Proof. The assertion regarding \mathcal{P} follows from (6.7) and by trade additivity of each of the terms \mathcal{P}^b and P^b noted in Remark 4.1. The statements regarding \mathcal{C} and \mathcal{F} follow by inspection of the related processes in (6.11)-(6.12). ■

6.2 Valuation Adjustments

Lemma 6.3 Given stopping times $\alpha \leq \beta$ such that $0 \leq \beta - \alpha \leq \delta$ and a process $Y = \mathbf{1}_{\llbracket \alpha, \beta \llbracket} X$, for a (nonnecessarily adapted) càdlàg process X stopped at $(T - \delta)$ and

such that $X_{\alpha-} = 0$:

- (i) The process $Y = \mathbb{1}_{\llbracket \alpha, \beta \rrbracket} X$ is killed at T ;
- (ii) If Y is adapted, then, for $t \leq T$, $\mathbb{E}_t(Y_T - Y_t) = -Y_t$.

Proof. (i) $T \leq t$ and $\alpha \leq t < \beta \implies \alpha \geq \beta - \delta > t - \delta \geq T - \delta$, so that (X being stopped at $T - \delta$) $X_t = X_{\alpha-} = 0$ and $Y_t = 0$, which also holds in case $T \leq t$ and $\neg(\alpha \leq t < \beta)$.

(ii) As $Y_T = 0$, if Y is adapted, then $\mathbb{E}_t(Y_T - Y_t) = -Y_t$. ■

Let further (cf. (6.5)-(6.6))

$$\begin{aligned} \mathcal{Z}^\iota &= \mathbb{1}_{\llbracket \tau, t_i \rrbracket} \mathbb{1}_{\{\tau_i < \tau \leq t_i\}} (M^\iota - M_{\tau-}^\iota) = \mathbb{1}_{\llbracket \tau, \tau \vee t_i \rrbracket} \mathbb{1}_{\{\tau_i < \tau \leq t_i\}} (M^\iota - M_{\tau-}^\iota) \\ \mathcal{V}^\iota &= -\mathbb{1}_{\llbracket t_i, +\infty \rrbracket} \mathbb{1}_{\{\tau_i < \tau \leq t_i\}} (M_{\tau-}^\iota - M_{s_i-}^\iota). \end{aligned} \quad (6.22)$$

Note that all the processes \mathcal{Y}^{ccp} , \mathcal{Y}^ι , $\mathbb{1}_{\{\tau \leq \tau_i\}} \mathcal{Y}^\iota$, \mathcal{Z}^ι , and \mathcal{V}^ι are \mathfrak{G} adapted and start from 0 at time 0; assuming the reduction of filtration setup of Section B, the process \mathcal{X}^ι is \mathfrak{F} adapted and starts from 0 at time 0.

Corollary 6.2 (i) Each process \mathcal{Y}^{ccp} is killed at T and, for $t \leq T$, the time- t value of \mathcal{Y}^{ccp} is $-\mathcal{Y}_t^{ccp} = -\mathbb{1}_{\tau \leq t < \tau^\delta} (\mathcal{P}_t^{ccp,0} - \mathcal{P}_{\tau-}^{ccp,0})$.

(ii) Each process $\mathbb{1}_{\{\tau \leq \tau_i\}} \mathcal{Y}^\iota$ is killed at T and, for $t \leq T$, the time- t value of $\mathbb{1}_{\{\tau \leq \tau_i\}} \mathcal{Y}^\iota$ is $-\mathbb{1}_{\{\tau \leq \tau_i\}} \mathcal{Y}_t^\iota = -\mathbb{1}_{\{s_i \leq t < t_i\}} \mathbb{1}_{\{\tau \leq \tau_i\}} (M_t^\iota - M_{s_i-}^\iota)$.

(iii) Each process \mathcal{Z}^ι is killed at T and, for $t \leq T$, the time- t value of \mathcal{Z}^ι is $-\mathcal{Z}_t^\iota = -\mathbb{1}_{\{\tau \leq t < t_i\}} \mathbb{1}_{\{\tau_i < \tau \leq t_i\}} (M_t^\iota - M_{\tau-}^\iota)$.

(iv) Assuming the reduction of filtration setup of Section B, each process \mathcal{X}^ι is killed at T and, for $t \leq T$, $\mathbb{E}_t(\mathcal{X}_T^\iota - \mathcal{X}_t^\iota) = -\mathcal{X}_t^\iota = -\mathbb{1}_{\{\tau_i' \leq t < (\tau_i^\delta)'\}} ((M_t^\iota)' - (M_{\tau_i-}^\iota)')$.

Proof. Recall from Section 2.1 that T exceeds by δ the final maturity of all claims in all considered portfolios. Hence all the processes \mathcal{P} , as well as their value processes P , are stopped not only at T , but even at $T - \delta$: in particular (cf. (5.2)),

$$d\mathcal{P}_t^{ccp,0} = \sum_c (\mathbb{1}_{\{t \leq t_c\}} d\mathcal{P}_t^{ccp,c} + \mathbb{1}_{\{s_c \leq t < t_c\}} dP_t^{ccp,c} + P_{t_c}^{ccp,c} \delta_{t_c}(dt)) = \mathbb{1}_{t \leq T - \delta} d\mathcal{P}_t^{ccp,0}.$$

Assuming the reduction of filtration setup of Section B (with optional reductions stopped at T), let us now show that

$$(M^\iota)' = ((M^\iota)^{T-\delta})' = ((M^\iota)')^{T-\delta}, \quad (6.23)$$

so that $(M^\iota)'$ is also stopped at $T - \delta$. To justify (the right equality in) (6.23), note that $((M^\iota)^{T-\delta})'$ and $(M^\iota)'^{T-\delta}$ are two \mathfrak{F} optional processes coinciding with $(M^\iota)^{T-\delta}$ before τ . This holds by definition of an optional reduction, regarding the former. Regarding the latter, we have $(M^\iota)' \mathbb{1}_{\llbracket 0, \tau \rrbracket} = M^\iota \mathbb{1}_{\llbracket 0, \tau \rrbracket}$, which implies by stopping at $T - \delta$ that $((M^\iota)')^{T-\delta} (\mathbb{1}_{\llbracket 0, \tau \rrbracket})^{T-\delta} = (M^\iota)^{T-\delta} (\mathbb{1}_{\llbracket 0, \tau \rrbracket})^{T-\delta}$. But $(\mathbb{1}_{\llbracket 0, \tau \rrbracket})^{T-\delta} = \mathbb{1}_{\{T-\delta \geq \tau\}} \mathbb{1}_{\llbracket 0, \tau \rrbracket} + \mathbb{1}_{\{T-\delta < \tau\}}$, hence $\mathbb{1}_{\llbracket 0, \tau \rrbracket} (\mathbb{1}_{\llbracket 0, \tau \rrbracket})^{T-\delta} = \mathbb{1}_{\llbracket 0, \tau \rrbracket}$, therefore multiplying the above identity by $\mathbb{1}_{\llbracket 0, \tau \rrbracket}$ yields $((M^\iota)')^{T-\delta} \mathbb{1}_{\llbracket 0, \tau \rrbracket} = (M^\iota)^{T-\delta} \mathbb{1}_{\llbracket 0, \tau \rrbracket}$, so that $(M^\iota)'^{T-\delta}$ indeed coincides with $(M^\iota)^{T-\delta}$ before τ , like $((M^\iota)^{T-\delta})'$. By uniqueness (B.2) of optional reductions, the processes $((M^\iota)^{T-\delta})'$ and $(M^\iota)'^{T-\delta}$ therefore coincide.

All results then follow by application of Lemma 6.3, with

- (i) $\alpha = \tau, \beta = \tau^\delta, X = \mathcal{P}^{ccp,0} - \mathcal{P}_{\tau^-}^{ccp,0}$.
- (ii) $\alpha = s_\iota, \beta = t_\iota, X = \mathbb{1}_{\{\tau \leq \tau_\iota\}}(M^\iota - M_{s_\iota}^\iota)$
- (iii) $\alpha = \tau, \beta = \tau \vee t_\iota, X = \mathbb{1}_{\{\tau_\iota < \tau \leq t_\iota\}}(M^\iota - M_{\tau^-}^\iota)$
- (iv) $\alpha = \tau'_\iota, \beta = (\tau_\iota^\delta)', X = (M^\iota)' - (M^\iota)'_{\tau'_\iota}$.

Lemma 6.4 (i) *The process \mathcal{V} is stopped at T .*

(ii) *For $t \leq T$, we have $\mathbb{E}_t \mathcal{V}_T^\iota = \mathcal{V}_t^\iota + W_t^\iota$, where*

$$W_t^\iota = -\mathbb{1}_{\{t < t_\iota\}} \mathbb{E}_t \left[\mathbb{1}_{\{\tau_\iota < \tau \leq t_\iota \leq T\}} (M_{\tau^-}^\iota - M_{\tau_\iota}^\iota) \right].$$

Proof. (i) On $\{T \geq t_\iota\}$, \mathcal{V} is worth $-\mathbb{1}_{\{\tau_\iota < \tau \leq t_\iota\}}(M_{\tau^-}^\iota - M_{s_\iota}^\iota)$ on $[T, +\infty[$. On $\{T < t_\iota\} \setminus \{\tau_\iota < \tau \leq t_\iota\}$, \mathcal{V} vanishes identically, which also holds on $\{T < t_\iota\} \cap \{\tau_\iota < \tau \leq t_\iota\}$, where $\tau > \tau_\iota = s_\iota \geq t_\iota - \delta > T - \delta$ (using (6.3)), so that $M_{\tau^-}^\iota = M_{s_\iota}^\iota$.

(ii) For $t \leq T$,

$$\begin{aligned} \mathbb{E}_t \mathcal{V}_T^\iota &= -\mathbb{E}_t \left[\mathbb{1}_{\{\tau_\iota < \tau \leq t_\iota \leq T\}} (M_{\tau^-}^\iota - M_{s_\iota}^\iota) \right] \\ &= -\mathbb{E}_t \left[\mathbb{1}_{\{t \geq t_\iota\}} \mathbb{1}_{\{\tau_\iota < \tau \leq t_\iota\}} (M_{\tau^-}^\iota - M_{s_\iota}^\iota) \right] \\ &\quad - \mathbb{E}_t \left[\mathbb{1}_{\{t < t_\iota\}} \mathbb{1}_{\{\tau_\iota < \tau \leq t_\iota \leq T\}} (M_{\tau^-}^\iota - M_{s_\iota}^\iota) \right] \\ &= \mathcal{V}_t^\iota - \mathbb{E}_t \left[\mathbb{1}_{\{t < t_\iota\}} \mathbb{1}_{\{\tau_\iota < \tau \leq t_\iota \leq T\}} (M_{\tau^-}^\iota - M_{s_\iota}^\iota) \right] \\ &= \mathcal{V}_t^\iota + W_t^\iota. \blacksquare \end{aligned}$$

Lemma 6.5 (i) *We have*

$$\tau^-(\mathcal{Y}^\iota) = \mathbb{1}_{\{\tau \leq \tau_\iota\}} \mathcal{Y}^\iota + \mathcal{Z}^\iota + \mathcal{V}^\iota. \quad (6.24)$$

(ii) *The time- t value of $\tau^-(\mathcal{Y}^\iota)$ is $-\mathbb{1}_{\{\tau \leq \tau_\iota\}} \mathcal{Y}_t^\iota - \mathcal{Z}_t^\iota + W_t^\iota$.*

Proof. (i) By a repeated use of the definitions (2.1), we have, also using in the third line the identity $(XY)^{\tau^-} = X^{\tau^-} Y^{\tau^-}$ that holds for any left-limited processes (as both sides are stopped before τ and coincide with XY before τ),

$$\begin{aligned} \tau^-(\mathcal{Y}^\iota) &= \tau^- \left(\mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket} (M^\iota - M_{s_\iota}^\iota) \right) \\ &= \mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket} (M^\iota - M_{s_\iota}^\iota) - \left(\mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket} (M^\iota - M_{s_\iota}^\iota) \right)^{\tau^-} \\ &= \mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket} (M^\iota - M_{s_\iota}^\iota) - \left(\pm M^\iota + (M^\iota)^{\tau^-} - M_{s_\iota}^\iota \right) (\mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket})^{\tau^-} \\ &= \tau^- (\mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket}) (M^\iota - M_{s_\iota}^\iota) + \tau^- (M^\iota) (\mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket})^{\tau^-}, \end{aligned}$$

where

$$\begin{aligned} (\mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket})^{\tau^-} &= \mathbb{1}_{\{\tau > t_\iota\}} \mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket} + \mathbb{1}_{\{\tau_\iota < \tau \leq t_\iota\}} \mathbb{1}_{\llbracket s_\iota, +\infty \rrbracket} \\ \tau^- (\mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket}) &= \mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket} - (\mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket})^{\tau^-} = \mathbb{1}_{\{\tau \leq \tau_\iota\}} \mathbb{1}_{\llbracket s_\iota, t_\iota \rrbracket} - \mathbb{1}_{\{\tau_\iota < \tau \leq t_\iota\}} \mathbb{1}_{\llbracket t_\iota, +\infty \rrbracket} \\ \tau^- (M^\iota) &= \mathbb{1}_{\llbracket \tau, +\infty \rrbracket} (M^\iota - M_{\tau^-}^\iota). \end{aligned}$$

Hence

$$\begin{aligned}
\tau^-(\mathcal{Y}^\iota) &= (\mathbb{1}_{\{\tau \leq \tau_i\}} \mathbb{1}_{[s_i, t_i]} - \mathbb{1}_{\{\tau_i < \tau \leq t_i\}} \mathbb{1}_{[t_i, +\infty)}) (M^\iota - M_{s_i-}^\iota) \\
&\quad + (\mathbb{1}_{\{\tau > t_i\}} \mathbb{1}_{[s_i, t_i]} + \mathbb{1}_{\{\tau_i < \tau \leq t_i\}} \mathbb{1}_{[s_i, +\infty)}) \mathbb{1}_{[\tau, +\infty)} (M^\iota - M_{\tau-}^\iota) \\
&= (\mathbb{1}_{\{\tau \leq \tau_i\}} \mathbb{1}_{[s_i, t_i]} (M^\iota - M_{s_i-}^\iota) \\
&\quad + \mathbb{1}_{\{\tau > t_i\}} \mathbb{1}_{[s_i, t_i]} \mathbb{1}_{[\tau, +\infty)} (M^\iota - M_{\tau-}^\iota) \\
&\quad - \mathbb{1}_{\{\tau_i < \tau \leq t_i\}} \mathbb{1}_{[t_i, +\infty)} (M^\iota - M_{s_i-}^\iota) + \mathbb{1}_{\{\tau_i < \tau \leq t_i\}} \mathbb{1}_{[s_i, +\infty)} \mathbb{1}_{[\tau, +\infty)} (M^\iota - M_{\tau-}^\iota),
\end{aligned}$$

where the first two lines respectively equal $\mathbb{1}_{\{\tau \leq \tau_i\}} \mathcal{Y}^\iota$ and 0, whereas the last line equals

$$\begin{aligned}
& - \mathbb{1}_{\{\tau_i < \tau \leq t_i\}} \mathbb{1}_{[t_i, +\infty)} (M^\iota - M_{s_i-}^\iota) + \mathbb{1}_{\{\tau_i < \tau \leq t_i\}} \mathbb{1}_{[t_i, +\infty)} (M^\iota - M_{\tau-}^\iota) \\
&\quad + \mathbb{1}_{\{\tau_i < \tau \leq t_i\}} \mathbb{1}_{[s_i, t_i]} \mathbb{1}_{[\tau, +\infty)} (M^\iota - M_{\tau-}^\iota) \\
&= \mathcal{V}^\iota + \mathbb{1}_{\{\tau_i < \tau \leq t_i\}} \mathbb{1}_{[\tau, t_i]} (M^\iota - M_{\tau-}^\iota) = \mathcal{V}^\iota + \mathcal{Z}^\iota.
\end{aligned}$$

Hence we obtain (6.24).

(ii) follows from part (i), Corollary 6.2(ii)-(iii) and Lemma 6.4(ii). ■

Theorem 6.1 *In the reduction of filtration setup of Section B for a bank involved in an arbitrary combination of bilateral netting sets and of netting sets cleared with one among several available CCPs, if*

$$\begin{aligned}
& \sum_{\iota} (1 - (R^\iota)'_{(\tau_i^\delta)'}) ((P^\iota)'_{(\tau_i^\delta)'} + (\mathcal{P}^\iota)'_{(\tau_i^\delta)'} - (\mathcal{P}^\iota)'_{\tau_i-} - (\Gamma^\iota)'_{\tau_i-})^+ \mathbb{1}_{\{(\tau_i^\delta)' < T\}} \quad (6.25) \\
& \quad + \sum_{ccp, I} (\mu^{ccp, 0})'_{(\tau_i^\delta)'} (B^{ccp, I})'_{(\tau_i^\delta)'} \mathbb{1}_{\{(\tau_i^\delta)' < T\}}
\end{aligned}$$

is \mathbb{P} square integrable, $\sum_{\iota} \mathcal{X}^\iota \in \tilde{\mathbb{S}}'_2$, $\mathbb{E}' \int_0^T \lambda_t^\iota dt < \infty$ and

$$\lambda' (\text{MtM}' - \sum_b (\text{VM}^b)')^+ + \tilde{\lambda}' \left(\sum_{\iota} (\overline{\text{IM}}^\iota)' + \sum_{ccp} (\overline{\text{IM}}^{ccp, 0} + \text{DF}^{ccp, 0})' \right) \text{ and } \lambda' \sum_{\iota} \mathcal{X}^\iota \quad (6.26)$$

are in \mathbb{L}'_2 , then the CVA and FVA equations (2.12)-(2.13) are well posed in \mathbb{S}_2 and we have, for $t \leq T$,

$$\begin{aligned}
\text{CVA}'_t + \sum_{\iota} \mathcal{X}'_t &= \\
& \sum_{\iota} \mathbb{E}'_t \left[(1 - (R^\iota)'_{(\tau_i^\delta)'}) ((P^\iota)'_{(\tau_i^\delta)'} + (\mathcal{P}^\iota)'_{(\tau_i^\delta)'} - (\mathcal{P}^\iota)'_{\tau_i-} - (\Gamma^\iota)'_{\tau_i-})^+ \mathbb{1}_{\{t < (\tau_i^\delta)' \leq T\}} \right] \\
& \quad + \sum_{ccp, I} \mathbb{E}'_t [(\mu^{ccp, 0})'_{(\tau_i^\delta)'} (B^{ccp, I})'_{(\tau_i^\delta)'} \mathbb{1}_{\{t < (\tau_i^\delta)' \leq T\}}],
\end{aligned} \quad (6.27)$$

$$\begin{aligned}
\text{FVA}'_t &= \mathbb{E}'_t \int_t^T \lambda'_s (\text{MtM}' - \sum_b (\text{VM}^b)' - \text{CVA}' - \text{FVA}'_s)^+ ds + \\
& \quad \mathbb{E}'_t \int_t^T \tilde{\lambda}'_s \left(\sum_b (\overline{\text{IM}}^b)' + \sum_{ccp} (\overline{\text{IM}}^{ccp, 0} + \text{DF}^{ccp, 0})' \right)_s ds,
\end{aligned} \quad (6.28)$$

$$\begin{aligned}
d\mathcal{L}'_t &= \sum_{\iota} (1 - (R^\iota)'_{(\tau_t^\delta)'}) ((P^\iota)'_{(\tau_t^\delta)'}) + (\mathcal{P}^\iota)'_{(\tau_t^\delta)'}) - (\mathcal{P}^\iota)'_{\tau_t^-} - (\Gamma^\iota)'_{\tau_t^-})^+ \delta_{(\tau_t^\delta)'}(dt) \\
&+ \sum_{ccp, I} (\mu^{ccp, 0})'_{(\tau_t^\delta)'}) (B^{ccp, I})'_{(\tau_t^\delta)'}) \delta_{(\tau_t^\delta)'}(dt) \\
&+ d(\text{CVA}' + \sum_{\iota} \mathcal{X}^\iota)_t \\
&+ \lambda'_t (\text{MtM}' - \sum_b (\text{VM}^b)' - \text{CVA}' - \text{FVA}')_t^+ dt \\
&+ \tilde{\lambda}'_t (\sum_b (\overline{\text{IM}}^b)' + \sum_{ccp} (\overline{\text{IM}}^{ccp, 0} + \text{DF}^{ccp, 0})'_t) dt \\
&+ d\text{FVA}'_t \\
&- \sum_b \mathbb{1}_{\{t \leq (\tau_b^\delta)'\}} d(\mathcal{P}^b + P^b)'_t + d\mathcal{H}'_t.
\end{aligned} \tag{6.29}$$

Moreover, if $\sum_b ((\mathcal{P}^b + P^b)'_{(\tau_b^\delta)'}) - \mathcal{H}' \in \tilde{\mathbb{S}}_2'^{20}$. then the KVA equation (2.17) is then well posed in \mathbb{S}_2 and we have, for $t \in [0, T]$,

$$\text{KVA}'_t = \mathbb{E}'_t \int_t^T h e^{-h(s-t)} \max(\text{EC}'_s, \text{KVA}'_s) ds. \tag{6.30}$$

Regarding the contra-liabilities defined in Table A.1, we have

$$\begin{aligned}
\text{DVA}_t &+ \sum_{ccp} \mathcal{Y}_t^{ccp} - \sum_{\iota} \mathbb{1}_{\{\tau \leq \tau_t\}} \mathcal{Y}_t^\iota - \sum_{\iota} \mathcal{Z}_t^\iota + \sum_{\iota} W_t^\iota = \\
&\mathbb{E}_t \left[\sum_{\iota; \tau \leq \tau_t^\delta} \mathbb{1}_{\{t < t_i \leq T\}} (1 - R_{t_i}) (P_{t_i}^\iota + \mathcal{P}_{t_i}^\iota - \mathcal{P}_{s_i^-}^\iota + \overline{\Gamma}_{s_i^-}^\iota)^- \right] \\
&- \mathbb{E}_t \left[\sum_{\iota; \tau_i \leq \tau^\delta, \tau_i^\delta \geq \tau} \mathbb{1}_{\{t < t_i \leq T\}} (1 - R_{t_i}) (P_{t_i}^\iota + \mathcal{P}_{t_i}^\iota - \mathcal{P}_{s_i^-}^\iota - \Gamma_{s_i^-}^\iota)^+ \right] \\
&+ \mathbb{E}_t \left[\sum_{ccp} (1 - R_{\tau^\delta}^{ccp, 0}) (\mathcal{P}_{\tau^\delta}^{ccp, 0} - \mathcal{P}_{\tau^-}^{ccp, 0} - \text{VM}_{\tau^-}^{ccp, 0} - \text{IM}_{\tau^-}^{ccp, 0} - \text{DF}_{\tau^-}^{ccp, 0})^+ \right] \\
&+ \mathbb{E}_t \left[\mathbb{1}_{\{t < \tau \leq T\}} \text{CVA}_{\tau^-} \right], \\
\text{FDA}_t &= \mathbb{E}_t \left[\mathbb{1}_{\{t < \tau \leq T\}} (1 - R_\tau) (\text{MtM} - \sum_b \text{VM}^b - \text{CA})_{\tau^-}^+ \right] \\
&+ \mathbb{E}_t \left[\mathbb{1}_{\{t < \tau \leq T\}} (1 - \bar{R}_\tau) \left(\sum_b \overline{\text{IM}}^b + \sum_{ccp} (\overline{\text{IM}}^{ccp, 0} + \text{DF}^{ccp, 0}) \right)_{\tau^-} \right] \\
&+ \mathbb{E}_t \left[\mathbb{1}_{\{t < \tau \leq T\}} \text{FVA}_{\tau^-} \right] = \text{FVA}_t.
\end{aligned} \tag{6.32}$$

Proof. The CVA related statement follows by application of Proposition B.1(i) with $\mathcal{C}' - \sum_{\iota} \mathcal{X}^\iota$ given by (6.18), which is nondecreasing and worth (6.25) at T , also using Corollary 6.2(iv) to obtain the liquidation period corrective term $\sum_{\iota} \mathcal{X}_t^\iota$ in (6.27).

²⁰e.g. in the reference dynamic hedging case (2.10), where $\sum_b ((\mathcal{P}^b + P^b)'_{(\tau_b^\delta)'}) - \text{MtM}_0 - \mathcal{H}' = 0$.

Note that $\text{CVA}' + \sum_{\iota} \mathcal{X}^{\iota} \geq 0$, by (6.27). Hence, also using the 1-Lipschitz property of $x \mapsto (a - x)^+$ in the third line, we have:

$$\begin{aligned}
& (\text{MtM}' - \sum_b (\text{VM}^b)' - \text{CVA}')^+ \\
&= (\text{MtM}' - \sum_b (\text{VM}^b)' - (\text{CVA}' + \sum_{\iota} \mathcal{X}^{\iota}) + \sum_{\iota} \mathcal{X}^{\iota})^+ \\
&\leq (\text{MtM}' - \sum_b (\text{VM}^b)' - (\text{CVA}' + \sum_{\iota} \mathcal{X}^{\iota}))^+ + |\sum_{\iota} \mathcal{X}^{\iota}| \\
&\leq (\text{MtM}' - \sum_b (\text{VM}^b)')^+ + |\sum_{\iota} \mathcal{X}^{\iota}|. \tag{6.33}
\end{aligned}$$

The FVA related statement then follows by application of Proposition B.1(ii), with \mathcal{R} and \mathcal{S} there given by (6.10) so that, by (6.33), $\lambda'(\mathcal{R}' - \text{CVA}')^+ + \tilde{\lambda}'\mathcal{S}'$ is in \mathbb{L}'_2 provided the two terms in (6.26) are in \mathbb{L}'_2 .

The dynamics (6.29) for \mathcal{L}' are obtained by plugging (6.17)–(6.19) into (6.15). This process \mathcal{L}' belongs to $\tilde{\mathbb{S}}'_2$ as the sum (modulo a constant) between $\text{CVA}' + \mathcal{C}'$, the $(\mathfrak{F}, \mathbb{P})$ martingale part of FVA', and $-\sum_b ((\mathcal{P}^b + P^b)'(\tau_b^{\delta})' + \mathcal{H}')$, assumed in $\tilde{\mathbb{S}}'_2$. Hence $\text{EC}' \in \mathbb{L}'_2$, by the Lipschitz property of expected shortfall. By an application of Proposition B.1(iv) (given (6.17)), the KVA equation (2.17) is then well posed in \mathbb{S}_2 and the KVA' formula (6.30) holds.

In view of the definitions of Table A.1, the DVA and FDA formulas respectively follow from the expressions of $\tau^-(-\mathcal{C})$ in (6.13), also using Corollary 6.2(i) and Lemma 6.5(ii), and $\tau^-(-\mathcal{F})$ in (6.14). The last equality in (6.32) is a general consequence, established in Lemma A.1(ii), of the martingale property of \mathcal{F} . ■

Remark 6.1 *The CVA with liquidation period δ is given by two dominant nonnegative terms (the two terms on the right-hand side in (6.27)) and a corrective explicit term $\sum_{\iota} \mathcal{X}^{\iota}$ only nonnull during the liquidation periods (which, therefore, vanishes identically in the $\delta = 0$ case where all liquidations are instantaneous). Similar comments apply to the DVA.*

Remark 6.2 *The industry terminology distinguishes a strict FVA, in the sense of the cost of funding rehypothecable collateral, from an MVA, corresponding to the cost of funding segregated collateral. In this paper, to spare one “VA” notation, we merge the two in an overall FVA meant in the broad sense of the cost of funding the derivative trading of the bank. The strict FVA and the MVA correspond to the first and the second lines on the right-hand side of (6.28).*

7 Discussion

To sum up, the portfolio-wide, all XVA-inclusive, cost-of-capital pricing formula is

$$\text{MtM} - (\text{CVA} + \text{FVA} + \text{KVA}),$$

charged to the clients of the deals at the portfolio inception time 0. The trading margin amount corresponds essentially²¹ to the counterparty-risk-free value of the portfolio. The reserve capital amount $CA = CVA + FVA$ is used by the CVA desk and the bank Treasury for coping with the expected client default losses and risky funding expenses of the bank. The KVA risk margin amount is used by the management of the bank for gradually releasing a dividend risk premium to the shareholders of the bank, at a hurdle rate h on their capital at risk. Figure 7.1 provides a focus on a subset of the (red) financial network of the reference bank labeled by 0 in Figure 3.1, detailing further the respective roles of the different desks within the bank.

In this section we revisit, in the light of the requirements of Section 1, the corresponding cost-of-capital XVA solution to the sustainable pricing and dividend release policy problem.

7.1 Regulatorily Admissible and Sustainable

As stated in Proposition 2.1, shareholder trading gains and KVA risk margin payments result in a $-(\mathcal{L}^{\tau^-} + KVA^{\tau^-} - KVA_0)$ dividend stream, turning the shareholder *equity* process into a submartingale with growth rate h on their capital at risk (yet the shareholder *wealth* process is a martingale and the setup is nonarbitrable in the sense explained after Proposition A.1). This holds even in the case of a portfolio held on a run-off basis, i.e. without the need to enter new deals for generating new profits. This feature addresses the sustainability requirement in Section 1. Moreover, Albanese, Crépey, Hoskinson, and Saadeddine (2021, Section 4.2) shows that the sustainability property of Proposition 2.1 is still valid in the (realistic) case of a trade incremental portfolio, provided the there-defined trade incremental XVA policy is applied at every new deal.

Our setup crucially includes the default time τ of the bank itself, which is the essence of the contra-liabilities wealth transfer issue detailed in Section A. However, accounting for all wealth transfers involved (cf. Proposition A.1), we end up with portfolio-wide nonnegative²² and ultimately unilateral CVA, FVA, and KVA, which price the related (nondecreasing) cash flows until the final horizon T of the XVA problem (cf. (B.5)–(B.7)), as opposed to $\tau \wedge T$. This makes our approach naturally in line with the monotonicity requirement of Section 1, that capital at risk and reserve capital should not decrease simply because the credit risk of the bank has worsened.

7.2 Economically Credible and Logically Consistent

Whereas counterparty jump-to-default risk can fundamentally not be hedged, a large part of the XVA literature relies on a replication paradigm. However, as established in Proposition A.2, in a theoretical, complete counterparty risk market, the all-inclusive XVA formula would simply be $CVA - DVA$, instead of $CVA + FVA + KVA$ when market incompleteness is accounted for.

In particular, the Burgard and Kjaer (2011, 2013, 2017) FVA approach was pioneering, but it breaches several of the requirements stated in Section 1, namely: non-

²¹and exactly so at time 0; see Section 4.1 and (6.8).

²²essentially, cf. Remark 2.2.

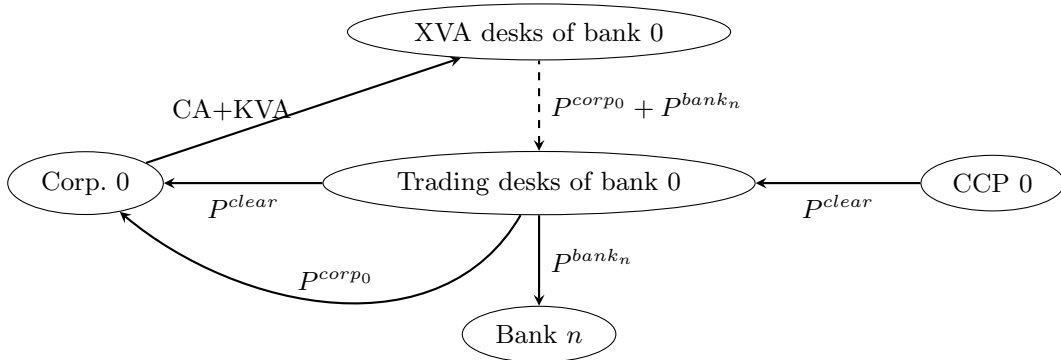


Figure 7.1: Zoom on the reference bank labeled by 0 in Figure 3.1, focusing on its netting sets with Corp. 0, Bank n and CCP 0. The solid arrows represent the direction of deal entry payments between the bank, its clients and the CCP. The reference clearing member bank 0 buys assets from its corporate clients, at a $(CA + KVA)$ -deducted price with respect to their counterparty-risk-free valuation, P^t (cf. Table 4.1), at which transactions between banks are concluded. The dashed arrow refers to the trading margin amount $P^{corp0} + P^{bankn}$ posted by the Treasury to the trading desks of Bank 0, so that these can setup bilateral netting sets valued P^{corp0} and P^{bankn} (on a counterparty-risk-free basis) with Corp. 0 and Bank n , in the way detailed in Section 4. The aggregate $P^{corp0} + P^{bankn}$ is dynamically hedged on exchanges (not shown to alleviate the picture). In addition, the bank has a netting set of cleared deals with Corp. 0, with counterparty-risk-free value P^{clear} , which are statically replicated by identical deals between the bank and CCP 0, in the way detailed in Section 5.

negativity (with an FVA that may become negative in the limiting case of a deeply out-of-the-money portfolio), monotonicity (with CVA and KVA tending to 0 when the default risk of the bank goes to infinity), and economic realism (which is lacking to an “XVA replication paradigm”). Likewise, the Green, Kenyon, and Dennis (2014) KVA approach was pioneering but it breaches monotonicity, economic realism, and (see Section 7.4 below) minimality. The XVA metrics of Bichuch, Capponi, and Sturm (2018) also breach the nonnegativity and monotonicity requirements.

With respect to these XVA replication frameworks, our cost-of-capital XVA approach results in materially different XVA formulas and balance sheet implications. In particular:

- Despite the fact that we include the default of the bank itself in our modeling, our (portfolio-wide) XVA metrics are, ultimately, unilateral (hence do not tend to decrease simply because the credit risk of the bank has worsened), and they are nonnegative (portfolio-wide);
- Our KVA is loss-absorbing: by contrast with the KVA of Green, Kenyon, and Dennis (2014), it does not belong to the loss process \mathcal{L} of the bank (it is not a liability like the CVA and the FVA, it would make no sense to try and replicate it);

- As a consequence of the previous point, our KVA discounts future capital at risk projections at the hurdle rate h (cf. (B.7)-(6.30) and (1.2)), which makes a big difference at the very long time horizon T (such as 50 years) of XVA computations.

In addition, instead of working with economic capital, Green, Kenyon, and Dennis (2014) use approximations in the form of scriptural regulatory capital specifications. This is meant for simplicity but it is less satisfying economically. It is also less self-consistent: under the cost-of-capital, economic capital based, XVA approach, counterparty-risk-free valuations flow into CVA computations, which in turn flow into FVA computations, which all flow into KVA computations. These connections make the counterparty-risk-free valuation, $CA = CVA + FVA$, and KVA equations, thus the derivative pricing problem as a whole, a self-contained and self-consistent problem.

7.3 Numerically Feasible and Robust

These connections can also be exploited for numerical purposes. Albanese, Crépey, Hoskinson, and Saadeddine (2021, Section 5) and Abbas-Turki, Crépey, and Saadeddine (2021) present numerical applications on realistically large bilateral trade portfolios, based on neural net regression computational strategies. See also Albanese, Caenazzo, and Crépey (2017, Section 5) and Abbas-Turki, Diallo, and Crépey (2018) for an alternative, nested Monte Carlo computational strategy. These papers demonstrate the numerical feasibility and scalability of the cost-of-capital XVA approach.

The model risk inherent to XVA computations in general, and to economic capital based KVA computations more specifically, can be addressed by a Bayesian variant of our baseline cost-of-capital XVA approach. This is achieved by combining, in a global simulation, paths of the risk factors obtained in several “good” models, all econometrically realistic and calibrated to the market in counterparty-risk-free valuation terms (cf. Albanese, Crépey, and Iabichino (2021)). Drawing scenarios equally from each makes tails more leptokurtotic and risk measures greater as they are when one picks just a single (even good) model. The difference between the resulting enhanced KVA and a baseline, reference KVA, can be used as a reserve against model risk.

7.4 Minimal

An FVA for rehypothecable collateral computed at the level of a unique funding set, as in the first term of (6.28), avoids the over-conservatism of FVAs for rehypothecable collateral sometimes calculated for simplicity by netting set and aggregated. Indeed, such simplification misses the FVA markdown corresponding to the rehypothecability of (eligible) collateral across the different netting sets of the bank. One should also account for the further FVA markdown due to the possibility for a bank to use its capital at risk as variation margin, which is done in Crépey, Sabbagh, and Song (2020).

Our KVA is minimal in the sense of the last statement in Proposition B.1(iv). An even cheaper KVA as in (Albanese et al., 2017, Proposition 4.2(v)) results from the following variation on our approach in this paper: Upon bank default, notwithstanding the bankruptcy rules recalled in the last paragraph of Section 1.1, the residual risk margin flows back into shareholder equity instead of going to creditors. Likewise, a cheaper FVA as in (Albanese et al., 2017, Proposition 4.2(i)) follows from asserting

that, upon bank default, the residual reserve capital of the FVA desk flows back into shareholder equity instead of going to creditors. However, these violations of the usual bankruptcy rules induce shareholder arbitrage, in the sense of a riskless profit strategy consisting for shareholders in letting the bank default instantaneously at time 0, right after the client portfolio has been setup and the corresponding reserve capital and risk margin amounts have been sourced from the clients. Our approach in this paper, instead, excludes such shareholder arbitrage opportunities (see after Proposition A.1).

Hence, “local departures” from our cost-of-capital XVA solution to the sustainable pricing and dividends problem of Section 1 may be a bit cheaper, but they are less self-consistent. As seen in Section 7.2, more radically different approaches to the problem suffer from severe shortcomings with respect to the requirements of Section 1. In an intuitive formulation, we conclude that the cost of capital XVA solution to the sustainable pricing and dividends problem may not be the only solution, nor is it necessarily “globally minimal”, but it has some “locally minimizing properties, at least in certain directions of the search space”, and we are not aware of any other “distant solution”.

Conclusion: From Replication to Balance Sheet Optimization

The generic²³ KVA formula (6.30), where $\max(EC', KVA') = CR'$ represents the capital at risk (cf. (1.2)), appears as a continuous-time analog of the risk margin formula under the Swiss solvency test cost of capital methodology: See Swiss Federal Office of Private Insurance (2006, Section 6, middle of page 86 and top of page 88). More broadly, as detailed in Table 7.1, the cost-of-capital XVA approach can be seen as an investment banking, genuinely dynamic and continuous-time version of the Solvency II insurance methodology, driven by the same motivation for a sustainable (financial or insurance) system and economy.

This KVA formula (6.30) can be used either in the direct mode, for computing the KVA corresponding to a given target hurdle rate h set by the management of the bank, or in the reverse-engineering mode, like the Black–Scholes model with volatility, for defining the implied hurdle rate associated with the actual amount on the risk margin account of the bank (see after Remark B.1). Cost of capital proxies have always been used to estimate return-on-equity. Whether it is used in the direct or in the implied mode, the KVA is a refinement, dynamic and fine-tuned for derivative portfolios, but the base concept is far older than even the CVA. In the current state of the market, even when they are computed, the KVA and even the MVA (which is included in the FVA in this paper, see Remark 6.2) are not necessarily passed into entry prices. But they are strategically used for collateral and capital optimization purposes. This reflects a switch of paradigm in derivative management, from replication to balance sheet optimization.

²³cf. (B.7).

contra-assets $CA = CVA + FVA$	liabilities best estimate, also called market consistent valuation
priced by conditional expectation of related future cash flows	priced by conditional expectation of related future cash flows
economic capital EC	solvency capital requirement
sized as a conditional expected shortfall of future losses over one year	sized as unconditional expected shortfall of future losses over each successive year
capital valuation adjustment KVA	market value margin or risk margin
sized as a supermartingale with drift coeff. $hSCR$ and zero terminal condition	sized as summed future (deterministic) $hSCR$

Table 7.1: *Left*: Cost-of-capital XVA banking approach; *Right*: Solvency II insurance methodology (with $SCR = (EC - KVA)^+$ for shareholder capital at risk everywhere).

A Wealth Transfer Analysis

This section brings to light the symmetrical companions of the contra-assets, i.e. the contra-liabilities. Put together, contra-assets and contra-liabilities allow analyzing the wealth transfers triggered by the trading of the bank, which occur without giving rise to arbitrage opportunities to shareholders. A view on DVA and FDA as wealth transfers is consistent with the conclusions drawn in a structural default model of the bank by Andersen, Duffie, and Song (2019) (who however do not deal with the KVA).

Using (2.15) that applies to $Y = CVA$ and FVA , Figure A.1 details the split of the overall loss process of the bank, \mathcal{L} in (2.9), as the difference between the pre-bank default loss process, i.e. the shareholder loss process \mathcal{L}^{τ^-} as per (2.14), and the creditor gain process

$$\tau^-(-\mathcal{L}) = \tau^-(-\mathcal{C}) + \tau^-(-\mathcal{F}) + \mathbb{1}_{\llbracket \tau, +\infty \llbracket} CA_{\tau^-} - \tau^-(-\mathcal{P} - MtM) + \tau^-(-\mathcal{H}). \quad (\text{A.1})$$

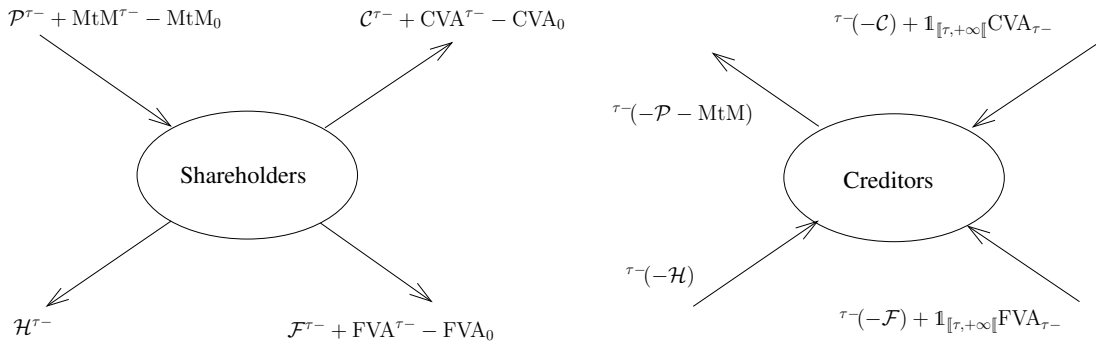


Figure A.1: *Left*: Pre-bank-default trading cash flows \mathcal{L}^{τ^-} . *Right*: Trading cash flows from bank default onward $\tau^-(-\mathcal{L})$.

Table A.1 identifies various cash flows (column 3) and the related value processes²⁴ (columns 1 and 2) involved in the foregoing wealth transfer analysis.

CA	Contra-assets valuation	$\mathcal{C}^{\tau^-} + \mathcal{F}^{\tau^-} + \mathbb{1}_{\llbracket \tau, +\infty \llbracket} \text{CA}_{\tau^-}$
CL	Contra-liabilities valuation	$\tau^-(-\mathcal{L})$
DVA	Debt valuation adjustment	$\tau^-(-\mathcal{C}) + \mathbb{1}_{\llbracket \tau, +\infty \llbracket} \text{CVA}_{\tau^-}$
FDA	Funding debt adjustment	$\tau^-(-\mathcal{F}) + \mathbb{1}_{\llbracket \tau, +\infty \llbracket} \text{FVA}_{\tau^-}$
FV	Fair valuation of counterparty risk	$\mathcal{C} + \mathcal{F}$
KVA^{sh}	Shareholders' KVA	$(\text{KVA}_0 - \text{KVA}^{\tau^-})$
KVA^{cr}	Creditors' KVA	$\mathbb{1}_{\llbracket \tau, +\infty \llbracket} \text{KVA}_{\tau^-}$

Table A.1: Valuation acronym and name (columns 1 and 2) of various cash flows (column 3) involved in the XVA wealth transfer analysis.

Lemma A.1 *We have*

- (i) $\text{CL} = \text{DVA} + \text{FDA}$, which is the value process of both $\tau^-(-\mathcal{L})$ and $(-\mathcal{L})$;
- (ii) $\text{FVA} = \text{FDA}$, $\text{FV} = \text{CA} - \text{CL} = \text{CVA} - \text{DVA}$;
- (iii) $\text{KVA}_t^{sh} = J_t \mathbb{E}_t \int_t^\tau h(\text{EC}_s - \text{KVA}_s)^+ ds$ and $\text{KVA}_t^{cr} = J_t \mathbb{E}_t \text{KVA}_{\tau^-}$, for $t \in \mathbb{R}_+$.

Proof. (i) holds by the definitions of Table A.1, the formula (A.1) for $\tau^-(-\mathcal{L})$, and the martingale (hence, zero-value) properties of $\tau^-(\mathcal{P} + \text{MtM})$, $\tau^- \mathcal{H}$, and \mathcal{L}^{τ^-} (see Assumption 2.1 and Remark 2.4).

(ii) holds by the definitions of Table A.1, by (2.12), (2.13), (1.1), and the fact that \mathcal{F} is a martingale (hence, zero-valued) stopped at $\tau \wedge T$, by Assumption 2.1.

(iii) At $t < \tau$, the value processes of $(\text{KVA}_0 - \text{KVA}^{\tau^-})$ and $\mathbb{1}_{\llbracket \tau, +\infty \llbracket} \text{KVA}_{\tau^-}$ are respectively worth $\mathbb{E}_t(\text{KVA}_t - \text{KVA}_{\tau^-}) = \mathbb{E}_t \int_t^\tau h(\text{EC}_s - \text{KVA}_s)^+ ds$ (by (2.17)) and $\mathbb{E}_t \text{KVA}_{\tau^-}$. From τ onward these value processes vanish, as well as KVA^{sh} and KVA^{cr} . ■

We assume that the shareholders have no other business than their involvement within the bank. The bank creditors, instead, have to face the liquidation costs of the bank, which are outside the scope of our model.

Definition A.1 *We call wealth transferred to the bank shareholders and creditors by the derivative trading of the bank, denoted by \mathcal{W}^{sh} and \mathcal{W}^{cr} , the sum between their respective (received) cash flows and the value process of their cash flows.*

Proposition A.1 *The shareholder and creditor wealth transfer processes are the martingales*

$$\mathcal{W}^{sh} = -(\mathcal{L}^{\tau^-} + \text{KVA}^{\tau^-} - \text{KVA}_0) + \text{KVA}^{sh} \text{ and} \quad (\text{A.2})$$

$$\mathcal{W}^{cr} = \tau^-(-\mathcal{L}) + \text{CL} + \mathbb{1}_{\llbracket \tau, +\infty \llbracket} \text{KVA}_{\tau^-} + \text{KVA}^{cr}, \quad (\text{A.3})$$

which start from the respective values KVA_0^{sh} and $\text{CL}_0 + \text{KVA}_0^{cr}$ at time 0.

²⁴cf. Definition 2.1(i).

Proof. The shareholder cash flows are given by \mathcal{D} in (2.18), where \mathcal{L}^{τ^-} is zero-valued (as a martingale starting from zero) and $\text{KVA}_0 - \text{KVA}^{\tau^-}$ is valued by KVA^{sh} (by definition of the latter in Table A.1). The creditor cash flows are given by ${}^{\tau^-}(-\mathcal{L}) + \mathbb{1}_{\llbracket \tau, +\infty \llbracket} \text{KVA}_{\tau^-}$, where ${}^{\tau^-}(-\mathcal{L})$ is valued by CL, by Lemma A.1(i), and $\mathbb{1}_{\llbracket \tau, +\infty \llbracket} \text{KVA}_{\tau^-}$ by KVA^{cr} (by definition of the latter in Table A.1). ■

Should the shareholders decide to put the bank in default at time 0 right after the portfolio has been set up, they should not make any profit or loss, otherwise this would be an arbitrage for the shareholders. The fact that the shareholder wealth process \mathcal{W}^{sh} starts from $\text{KVA}_0^{sh} > 0$ (positive initial wealth transfer to shareholders, unless the KVA vanishes) suggests that our XVA framework may entail shareholder arbitrage. Yet, given the rules of default settlement stated in Section 1.1, upon bank default, the residual value on the (reserve capital and) risk margin account of the bank goes to creditors. So, even though KVA_0^{sh} is part of the wealth of the shareholders, the shareholders would not monetize KVA_0^{sh} by putting the bank in default at time 0 right after the portfolio has been set up. Hence the positive initial wealth transfer to shareholders does not entail such a shareholder arbitrage.

Likewise, the fact that the creditor wealth transfer martingale \mathcal{W}^{cr} starts from $\text{CL}_0 + \text{KVA}_0^{cr}$ suggests that the derivative trading of the bank may entail a riskless profit to creditors. However, the scope of the model does not include the liquidation costs of the bank. For the creditors to monetize the wealth transfer triggered to them by the derivative portfolio of the bank, the bank has to default and there is a substantial cost associated to that to creditors.

A.1 What-If Analysis

In this section we examine the consequences of an assumption that the bank could hedge out its risks completely, including default risk. As explained in Section 2.3, this assumption is counterfactual, for both practical and legal reasons. However, we endorse it here for the sake of the argument.

We recall from Lemma A.1(i) that the value process of $(-\mathcal{L})$ is CL. To “complete the market”, we now assume that the bank has access to a new deal with a client, insuring the payment of a cash flow stream \mathcal{L} to the bank, along with a time 0 premium CL_0 (initiating the strategy at time 0 where the portfolio is settled). The deal would be fully collateralized, in the sense that the above cash flows would occur independently of the default status of the bank and its client.

Proposition A.2 *Assume that the bank has access to the new deal as defined above, coming on top of the derivative portfolio of the bank and its dynamic hedge considered earlier in the paper. Then*

$$\text{MtM} - \text{FV} = \text{MtM} - (\text{CVA} - \text{DVA}) = \text{MtM} - (\text{CA} - \text{CL})$$

(by Lemma A.1(ii)) is a replication price process for the derivative portfolio of the bank, in the sense that the resulting loss process of the bank (impact of all the hedges included) vanishes, as do in turn the economic capital of the bank, its KVA, and the shareholder and creditor wealth transfer processes.

Proof. Here is the detail of a corresponding replication strategy, starting from time 0 where the portfolio is settled. The trading and CVA desks act exactly as before. The FVA desk passes to the client (at time 0) and to the bank shareholders (on $(0, \tau)$) a diminished add-on FVA – CL, instead of FVA before without the new deal. In addition, a new CL desk puts the upfront premium CL_0 of the new deal on a dedicated cash account, along with a matching liability of CL_0 on the bank balance sheet. This CL account, like all the other ones within the bank, is market-to-model, i.e. reset in continuous time to the value process of the corresponding liability (see Section 1.1), namely to the value CL of the cash flow $(-\mathcal{L})$ due by the bank under the terms of the new deal. Before bank default, the resets to the CL account, which accumulate to $CL_0 - CL$, are passed to bank shareholders, as is the \mathcal{L}^{τ^-} component from the cash flows of the new deal (which thus do not stay on the balance sheet of the bank). Finally, from time τ onward, the $\tau^- \mathcal{L}$ component of the cash flows of the new deal hedges out the $\tau^-(-\mathcal{L})$ cash flows that were previously falling into the hands of the bank creditors.

As a result, the loss of the bank starting before τ (i.e. the process “ τ^- (loss)”) vanishes, whereas before τ the loss of the bank (i.e. the process “(loss) $^{\tau^-}$ ”) is given by

$$\begin{aligned} & \underbrace{\mathcal{C} + \mathcal{F} + \text{FV} - \text{FV}_0 - (\mathcal{P} + \text{MtM} - \text{MtM}_0) + \mathcal{H}}_{\text{modified loss of the CA desks}} \\ & \quad + \underbrace{-\mathcal{L}^{\tau^-} + \text{CL} - \text{CL}_0}_{\text{new hedging loss components passed to shareholders}} \\ & = \mathcal{C} + \mathcal{F} + \text{CA} - \text{CA}_0 - (\mathcal{P} + \text{MtM} - \text{MtM}_0) + \mathcal{H} - \mathcal{L} = 0, \end{aligned}$$

where Lemma A.1(ii) (and $\mathcal{L} = \mathcal{L}^{\tau^-}$ before τ) was used in the first equality and (2.9) in the second one.

Shareholders bear no more trading risk, hence economic capital and the KVA vanish, as do in turn the shareholder and creditor wealth transfer processes. ■

Remark A.1 *Before τ , the amount available free of charge to the bank for its rehypothecable collateral funding purposes is $\text{FV} + \text{CL} = \text{CA}$ as before. Hence the risky funding cash flows for collateral, \mathcal{F} , are not modified by the new deal. The cash flows \mathcal{C} are not affected by the new deal either.*

Due to the new deal, the creditors are left without any resource to address the liquidation costs of the bank. The clients pay $(\text{CA}_0 - \text{CL}_0) - \text{MtM}_0$ instead of $(\text{CA}_0 + \text{KVA}_0) - \text{MtM}_0$ before, hence they are better off by the amount $\text{CL}_0 + \text{KVA}_0$. As before, the bank shareholders are indifferent to the portfolio at the accordingly adjusted price paid by the bank. But this price is now $\text{MtM}_0 - (\text{CA}_0 - \text{CL}_0)$, instead of $\text{MtM}_0 - (\text{CA}_0 + \text{KVA}_0)$ before.

Remark A.2 *The reason why the new deal may alter the XVA picture so deeply is that this deal is different in nature from the contracts eligible to the bank derivative portfolio or its dynamic hedge earlier in the paper. In fact, as \mathcal{L}^{τ^-} is a martingale, the only case where the value process $\text{CL} = \text{DVA} + \text{FVA}$ of $(-\mathcal{L})$ could also be its shareholder value (as required for eligibility to the derivative portfolio or its dynamic hedge earlier in the*

paper, cf. the combined requirements of (2.11) and Assumption 2.2) would be when

$$\text{CL}_t = \mathbb{E}_t(-\mathcal{L}_{\tau-}^{\tau} + \mathcal{L}_t^{\tau-} + \text{CL}_{\tau-}) = \mathbb{E}_t \text{CL}_{\tau-}, \quad t < \tau, \quad (\text{A.4})$$

e.g. if $\text{CL} = \text{DVA} + \text{FVA} = 0$, Hence the new deal cannot be seen as part of the derivative portfolio of the bank or its dynamic hedge earlier in the paper.

Instead, one may interpret the trading loss $-\mathcal{L} + \text{CL} - \text{CL}_0$ triggered by the new deal as an additional line of risky funding cash flows, coming on top of the risky funding cash flows \mathcal{F} earlier in the paper. However, as opposed to \mathcal{F} as per (3.2), this new line of risky funding entails a loss of the bank at its own default, hence a benefit prior to τ , which is how the new deal allows the shareholders to monetize the default of the bank.

In any case, again, the new deal is unfeasible for the bank. In particular, a risky funding benefit of the bank at its own default means that the bank is effectively selling default protection on itself, which is even illegal (cf. the condition concluding Assumption 3.2).

Even if the new deal is unfeasible, the what-if analysis of Proposition A.2 is enlightening on the nature of the XVAs:

Corollary A.1 *The shareholder and creditor wealth transfers (A.2) and (A.3) can be interpreted as the wealth transferred to them by the trading of the bank, due to the inability of the bank to hedge, in particular, jump-to-default risk.*

Proof. Without the new deal, the wealth transfers to shareholders and creditors are given by (A.2) and (A.3), whereas Proposition A.2 shows that these wealth transfers would vanish if the bank had access to the new deal. ■

B Reduction of Filtration Setup

Let there be given a reduced stochastic basis $(\mathfrak{F}, \mathbb{P})$, for a quasi-left continuous filtration $\mathfrak{F} \subseteq \mathfrak{G}$ (also satisfying the usual conditions) and $\mathbb{P} \sim \mathbb{Q}$ on \mathfrak{F}_T , with time- t conditional expectation denoted by \mathbb{E}'_t .

Remark B.1 *In contrast with the setup of Crépey, Sabbagh, and Song (2020) (see Section 1.1 there), where more general anticipated XVA BSDEs are considered, in this paper no specific structure of the $(\mathfrak{F}, \mathbb{P})$ martingales is required. We do not even need to specify the risk drivers in the model. In practice (see e.g. Abbas-Turki, Crépey, and Saadeddine (2021, Section B)), those would typically consist of Brownian motions, driving the market risk factors, and of a continuous-time Markov chain (possibly modulated by the Brownian motions), used for driving the client default events.*

Assumption B.1 *For any \mathfrak{G} predictable, resp. optional process Y , there exists an \mathfrak{F} predictable, resp. optional process Y' , dubbed \mathfrak{F} reduction of Y , that coincides with Y on $\llbracket 0, \tau \rrbracket$, resp. on $\llbracket 0, \tau \llbracket$.*

In particular, any \mathfrak{G} stopping time θ admits an \mathfrak{F} stopping time θ' , dubbed \mathfrak{F} reduction of θ , such that

$$\theta \wedge \tau = \theta' \wedge \tau. \quad (\text{B.1})$$

Remark B.2 As can classically be established by section theorem, for any \mathfrak{G} progressive Lebesgue integrand X such that the \mathfrak{G} predictable projection ${}^{\mathbb{P}}X$ exists,²⁵ the indistinguishable equality $\int_0^\cdot {}^{\mathbb{P}}X_s ds = \int_0^\cdot X_s ds$ holds. As a consequence, one can actually consider the \mathfrak{F} predictable reduction X' of any \mathfrak{G} progressive Lebesgue integrand X (even if this means replacing X by ${}^{\mathbb{P}}X$).

Definition B.1 We call invariance probability measure, a probability measure \mathbb{P} on \mathfrak{F}_T equivalent to the restriction of \mathbb{Q} to \mathfrak{F}_T , such that

- (i) stopping before τ turns $(\mathfrak{F}, \mathbb{P})$ local martingales on $[0, T]$ into $(\mathfrak{G}, \mathbb{Q})$ local martingales on $[[0, \tau \wedge T]]$, and
- (ii) the \mathfrak{F} optional reductions of $(\mathfrak{G}, \mathbb{Q})$ local martingales on $[[0, \tau \wedge T]]$ without jump at τ are $(\mathfrak{F}, \mathbb{P})$ local martingales on $[0, T]$.

We denote by $S = \mathbb{Q}(\tau > \cdot | \mathfrak{F}_\cdot)$ the \mathfrak{F} Azéma supermartingale of τ , i.e. the $(\mathfrak{F}, \mathbb{Q})$ optional projection of $J = \mathbb{1}_{[[0, \tau[}}$.

(Crépey and Song, 2017, Lemma 2.3 and Theorem 3.5) Under Assumption B.1, if $S_T > 0$ a.s., then

$$\text{Two } \mathfrak{F} \text{ optional processes that coincide before } \tau \text{ coincide on } [0, T]. \quad (\text{B.2})$$

For τ also endowed with a $(\mathfrak{G}, \mathbb{Q})$ intensity process $\gamma = \gamma J_-$ such that $e^{\int_0^\tau \gamma_s ds}$ is \mathbb{Q} integrable, then there exists a unique invariance probability measure \mathbb{P} on \mathfrak{F}_T .

Hereafter we endorse the above reduction of filtration setup, so that \mathfrak{F} optional reductions of \mathfrak{G} optional processes are uniquely defined on $[0, T]$, and \mathfrak{F} reductions of $[0, T]$ valued \mathfrak{G} stopping times are uniquely defined. Given our focus on the time interval $[0, T]$ in the paper, we may and do assume that \mathfrak{F} optional reductions are stopped at T (and are thus uniquely defined on \mathbb{R}_+), without loss of generality.

Definition B.2 The economic capital (EC) of the bank is defined as $J \text{ES}$, where, for $t \geq 0$, ES_t is the $(\mathfrak{F}_t, \mathbb{P})$ conditional expected shortfall of the random variable $\mathcal{L}'_{t+1} - \mathcal{L}'_t$ at the confidence level $\alpha = 97.5\%$, in the following $(\mathfrak{F}$ predictable) sense²⁶:

$$\text{ES} = \inf_{\text{rationals } k} \left(k + (1 - \alpha)^{-1} {}^{\mathbb{P}}[(\mathcal{L}'_{+1} - \mathcal{L}' - k)^+] \right),$$

where ${}^{\mathbb{P}}$ denotes the (\mathbb{F}, \mathbb{P}) predictable projection operator²⁷.

Assuming that \mathcal{L}' is an $(\mathfrak{F}, \mathbb{P})$ martingale (as in the setup of Theorem 6.1), EC is non-negative²⁸, finite, and killed at $\tau \wedge T$.

We denote by:

²⁵For which σ integrability of X valued at any stopping time, e.g. X bounded or càdlàg, is enough.

²⁶making $\text{EC}' = \text{ES}$ a suitable input to the BSDE (B.7), see (Crépey et al., 2020, Lemma 2.1), whose proof works in the present setup by the assumed quasi-left continuity of \mathfrak{F} .

²⁷which applies to any raw, non necessarily adapted, càdlàg process, such as $(\mathcal{L}'_{+1} - \mathcal{L}' - k)^+$ for any constant k ; see e.g. He, Wang, and Yan (1992, Theorem V.5.2).

²⁸as expected shortfall of a centered random variable.

- \mathbb{L}_2 , the space of \mathfrak{G} progressive processes X such that

$$\mathbb{E}' \left[\int_0^T (X'_s)^2 ds \right] < +\infty. \quad (\text{B.3})$$

- \mathbb{S}_2 , the space of càdlàg \mathfrak{G} adapted processes Y vanishing on $[\tau \wedge T, +\infty)$ and such that

$$\mathbb{E}' \left[\sup_{t \in [0, T]} (Y'_t)^2 \right] < \infty; \quad (\text{B.4})$$

- \mathbb{L}'_2 , the space of \mathfrak{F} progressive processes X' making the expectation finite in (B.3);
- $\widetilde{\mathbb{S}}'_2$, resp. \mathbb{S}'_2 , the spaces of \mathfrak{F} adapted càdlàg processes Y' making the expectation finite in (B.4), resp. and vanishing on $[T, +\infty)$.

We also consider the following equation for $K = K(C) \in \mathbb{S}_2$, parameterized by $C \in \mathbb{L}_2$:

$$K_t = \mathbb{E}_t \left[\int_t^{\tau \wedge T} h(C_s - K_s) ds + K_{\tau-} \right], \quad t < \tau.$$

The following result can be proven much like Crépey, Sabbagh, and Song (2020, Proposition 6.1 (first part) and Lemma 6.1).

Lemma B.1 *The equations (2.12), (2.13)-(3.1), and (2.17) for processes CVA, FVA, KVA, and K in \mathbb{S}_2 (given $C \in \mathbb{L}_2$) are respectively equivalent, via the correspondence “ $Y = JY'$ ”, to the following equations in \mathbb{S}'_2 : For $t \leq T$,*

$$\text{CVA}'_t = \mathbb{E}'_t(C'_T - C'_t), \quad (\text{B.5})$$

$$\text{FVA}'_t = \mathbb{E}'_t \int_t^T \lambda'_s(\mathcal{R}' - \text{CVA}'_s - \text{FVA}'_s)^+ ds + \mathbb{E}'_t \int_t^T \widetilde{\lambda}'_s S'_s ds, \quad (\text{B.6})$$

$$\text{KVA}'_t = h \mathbb{E}'_t \int_t^T (\text{EC}'_s - \text{KVA}'_s)^+ ds = h \mathbb{E}'_t \int_t^T e^{-h(s-t)} \max(\text{EC}'_s, \text{KVA}'_s) ds, \quad (\text{B.7})$$

$$K'_t = \mathbb{E}'_t \int_t^{\tau \wedge T} h e^{-h(s-t)} C'_s ds = \mathbb{E}'_t \int_t^{\tau \wedge T} h(C'_s - K'_s) ds. \quad (\text{B.8})$$

Proposition B.1 (i) *If $C' \in \widetilde{\mathbb{S}}'_2$, then the formula (B.5) yields a well defined CVA' process in \mathbb{S}'_2 .*

(ii) *Assuming further $\mathbb{E}' \int_0^T \lambda'_t dt < \infty$ and $\lambda'(\mathcal{R}' - \text{CVA}')^+ + \widetilde{\lambda}' S' \in \mathbb{L}'_2$, then the FVA' equation (B.6) is well posed in \mathbb{S}'_2 ;*

(iii) *For any $C' \in \mathbb{L}'_2$, the $K' = K'(C')$ equation (B.8) is well posed in \mathbb{S}'_2 ;*

(iv) *If, additionally, $\text{EC}' \in \mathbb{L}'_2$, then the KVA' equation (B.7) is well posed in \mathbb{S}'_2 , with KVA' nondecreasing with respect to the hurdle rate parameter h ; it holds that $\text{CR} := \max(\text{EC}, \text{KVA}) = \min \text{Adm}$ and $\text{KVA} = \min_{C \in \text{Adm}} K(C)$, where $\text{Adm} := \{C \in \mathbb{L}_2; C \geq \max(\text{EC}, K(C))\}$ and $K = K(C)$.*

Proof. (i) proceeds from Crépey, Sabbagh, and Song (2020, Remark 2.4), whereas (ii)-(iv) follow from Kruse and Popier (2016, Theorem 1 and Proposition 4), where the latter is applied to the BSDE (B.7) for different h in order to prove the monotonicity of the KVA in h . Regarding the last statement in (iv), we have with $\text{CR}' := \max(\text{EC}', \text{KVA}')$:

$$\text{KVA}' = K'(\text{CR}'), \text{ hence } \text{CR}' = \max(\text{EC}', K'(\text{CR}')),$$

by uniqueness of a solution in \mathbb{S}'_2 to (B.7). Therefore $\text{CR}' \in \text{Adm}' := \{C' \in \mathbb{L}'_2; C' \geq \max(\text{EC}', K'(C'))\}$. Moreover, for any $C' \in \text{Adm}'$, we have $h(\text{EC}'_t - K'_t(C'))^+ \leq h(C'_t - K'_t(C'))$. Hence the coefficient of the KVA' BSDE (B.7) never exceeds the coefficient of the linear BSDE (B.8) when both coefficients are evaluated at the solution $K'_t(C')$ of (B.8). Since these are BSDEs with equal (null) terminal condition, the BSDE comparison principle of Proposition 4 in Kruse and Popier (2016)²⁹ applied to the BSDEs (B.7) and (B.8) yields $\text{KVA}' \leq K'(C')$. Consequently, $\text{KVA}' = \min_{C' \in \text{Adm}'} K'(C')$ and, for any $C' \in \text{Adm}'$,

$$C' \geq \max(\text{EC}', K'(C')) \geq \max(\text{EC}', \text{KVA}') = \text{CR}' = \min \text{Adm}'.$$

This shows the last statement in (iv) “with \cdot' everywhere”, from which the statement itself (“without \cdot' ”) readily follows by the equivalences regarding KVA and $K = K(C)$ in Lemma B.1. ■

The fact that KVA is continuous and nondecreasing³⁰ in h as seen in Proposition B.1(iv) allows one to define the *implied hurdle rate* as the value of the target hurdle rate $h \in \mathbb{R}_+$ calibrated, through the KVA formula (B.7) valued at time $t = 0$, to the actual amount on the risk margin account of the bank (if this amount cannot be reached by any $h \in \mathbb{R}_+$, then the implied hurdle rate is deemed infinite). We refer the reader to the concluding paragraph of Albanese, Crépey, Hoskinson, and Saadeddine (2021, Section 3.3) for a simple example in a stylized, one-period XVA model, where the implied hurdle rate can be characterized quite explicitly, shedding light on the impact thereon of the portfolio of the bank, the default intensities for the bank and its counterparty, and the risk aversion of the bank shareholders.

The following immediate corollary to Lemma B.1 and Proposition B.1 echoes the title of the paper:

Corollary B.1 *Assuming all the conditions in Proposition B.1, the FVA and KVA processes that arise by application of Lemma B.1 and Proposition B.1 are nonnegative. So is also CVA provided $C^{\tau-31}$ is nondecreasing³².*

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²⁹Note that jumps are not an issue for comparison in our setup, where the coefficient k “only depends on y^i ”; cf. Kruse and Popier (2016, Assumption (H3')).

³⁰and in fact, unless the processes \mathcal{L}' and EC would vanish, increasing.

³¹hence C' , by B.2.

³²cf. Remark 2.2.

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