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Vincenzo de Risi

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Euclid Upturned

Borelli on the Foundations of Geometry

Vincenzo De Risi

§1. Borelli and the Classical Ideal of Mathematics

The mathematical work of Giovanni Alfonso Borelli (1608-1679) was entirely devoted to the study and interpretation of the Greek classics. His most important mathematical book is the *Euclides restitutus* (“Euclid Restored”) from 1658, in which Borelli offered a complete rewriting of Euclid's *Elements*. In 1663, Borelli's *Euclides* was abridged, simplified and translated into Italian as *Euclide rinnovato* (“Euclid Renewed”) to serve as a student textbook. A third edition also appeared later in Borelli's life (1679), which was essentially a Latin version of the earlier Italian compendium, enriched with a selection of theorems from works by Apollonius and Archimedes.¹ In 1661, Borelli also made a significant contribution to the first Latin edition of Books V, VI and VII of Apollonius' *Conics*, to which he added numerous explanatory notes.²

I heartily thank Mattia Mantovani and Ugo Baldini for their helpful advice on this article.

¹ G.A. BORELLI, *Euclides restitutus, sive prisca geometriae elementa brevius et facilius contexta, in quibus precipue proportionum theoriae nova firmiorique methodo promuntur*, Pisa, F. Onofrio, 1658, in nine books. The Italian edition, entitled *Euclide rinnovato ovvero gli antichi elementi della geometria* (Bologna, Ferroni, 1663) was translated by Domenico Magni in collaboration with Borelli (cf. Borelli's testimony to have worked at the translation in GALLUZZI 1970, p. 284). The 1663 *Euclide* presented, in abbreviated form, only the first five books of the 1658 edition. The second Latin edition (*Euclides restitutus denuo limatus*, Roma, Mascardi, 1679) presented in abbreviated form the first seven books of the first edition, and added a significant preface *ad lectorem geometram* in which Borelli explained the purpose of the volume. The 1679 edition was complemented by a second tome of *Elementa conica Apollonii Pergaei et Archimedis opera* (cf. Borelli's letters on the topic in DERENZINI 1960). Borelli's *Euclides* also had a posthumous fourth edition (Roma, Ercole, 1695) that was similar to the third but added to it (without changes) Book VIII of the 1658 edition. In addition to these works on Euclid, the young Borelli had published a mathematical disputation in Italian: *Discorso del signor Gio. Alfonso Borrelli, academico della Fucina e Professore delle Scienze Matematiche nello Studio della Nobile Città di Messina*, Messina, Brea, 1646.

² On the complex history of this edition, see GIOVANNOZZI 1916, TENCA 1956, GUERRINI 1999. On the importance of this volume by Borelli for the history of science, see DEL CENTINA, FIOCCA 2020. Borelli also contributed to the

In none of these works did Borelli make use of modern mathematics: he does not seem ever to have written one single algebraic formula; not the sum of an infinite series or a study of indivisibles; not a reflection on projections that could lean toward non-metric geometry; not a study on curves that had not already been studied by the Greeks. Borelli remained outside the extraordinary seventeenth-century renewal of mathematics, and it is not surprising that modern histories of science do not ascribe any original theories to him nor are there theorems bearing his name. Today, he is mostly regarded as a successful teacher and a competent interpreter of classical texts.³

Nevertheless, Borelli's *Euclides* is a masterwork in the foundations of mathematics, and arguably the most important treatise on the subject in the seventeenth century. In it, Borelli reformulated most of Euclid's definitions, modified and expanded the axioms, transformed a great many demonstrations in the *Elements*, and made in depth changes to the deductive structure of the classical treatise. Euclid's *Elements* had already been modified many times over the centuries and the text had always remained partially plastic and changeable. Through all these changes, however, the *Elements* had remained a classical text to which editors had added scholia and local amendments. In times closer to Borelli, such modifications had become bolder: Jesuits and Ramists had abridged the text for students, Aristotelians had reshaped some proofs into syllogistic chains, Herigone and Barrow had formalized the *Elements* with new symbolisms.⁴ None of them, however, had changed *the mathematics* of Euclid: the ideas behind the demonstrations and their deductive order had remained the same. By contrast, Borelli's book revolutionized the mathematical core of Euclid's work to such an extent that it could be doubted whether it was still Euclid's.

The starting point of Borelli's attitude towards the *Elements* is probably to be found in Galileo himself. Looking for a mathematics for his new science of mechanics, Galileo did not turn to modern techniques but employed Euclid's classical theory of proportions. Yet, he also felt the need to rewrite the foundations of this theory in the so-called *Fifth Day of Discorsi*.⁵ Galileo's example led to a multitude of variations and rewritings of the theory of proportions by his disciples. The

posthumous edition (Messina, Brea, 1654) of Apollonius' *Conics* edited by Maurolycus (Books I-IV, and a "divination" of Books V and VI). On this endeavor, see BERTOLONI MELI 2001.

³ We know that Borelli taught mathematics in Pisa using his *Euclides* as a textbook, and for instance in 1660 he was asking to send him twelve more copies of the book for his students (Borelli's letter is published in GALLUZZI 1970, p. 273). Some of the books in Borelli's possession are now preserved at the *Biblioteca Nazionale Centrale* in Rome and recognizable by an *ex libris*. In this collection only one edition of Euclid's *Elements* is to be found (edited by Oronce Finé, 1536), although it is crystal clear that Borelli must have known many other editions, and some of them (such as the celebrated one by Clavius, 1574) are also mentioned by him in the *Euclides*. Borelli's sources on elementary geometry, however, remain in many cases entirely conjectural. On Borelli's library, see BALDINI 1996.

⁴ Among Jesuit textbooks, André TACQUET's 1654 edition of the *Elements* is probably the one that dared to make most changes with respect to the classical text. Yet, the deductive structure of Tacquet's book is essentially Euclid's, and it cannot be said to be a complete reworking of the text in any sense comparable to Borelli's. Borelli mentioned Tacquet's work to Magliabechi in 1663, but it is not clear from the context whether he knew it already or not (see GALLUZZI 1970, p. 285). For an example of a Ramist textbook for schoolchildren, which often does not even carry out the demonstrations of theorems, see KECKERMANN 1621, of which Borelli owned a copy (see the previous note). RAMUS' original *Scholae mathematicae* from 1569 were much more extended but did not change the order of Euclid's theorems: they were more a (critical) commentary than a rewriting. A reworking of Euclid's proofs into syllogisms is in the edition by HERLINUS, DASYPIDIUS from 1566. HERIGONE's edition is from 1634, and BARROW's edition from 1655 drew from it. We do not know if Borelli knew this latter work, but he surely owned the works of Archimedes in Barrow's edition, as well as his lectures on geometry and optics.

⁵ GALILEO 1968, vol. 8, pp. 347-62. This essay was circulating among Galileo's disciples and was posthumously published by Viviani in 1674.

result of this approach was a peculiar innovative classicism; faithful and at the same time antagonistic toward Greek authors, which characterizes many Italian works of the time.⁶ At first, Borelli's work fitted perfectly into this cultural framework. As early as 1641 (seventeen years before the publication of the *Euclides*) he, in the wake of Galileo, was working on an alternative theory of proportions that eventually became a part of his book. In 1660, he boasted about a good definition of proportion that

none of the ancients found. But I did.⁷

By this point, however, Borelli had long surpassed Galileo and his contemporaries in both classicism and reformation. He extended the critical approach from the foundations of the theory of proportions to the whole of mathematics, and thus rewrote all of the *Elements*. Moreover, he embraced a hyper-classicist demonstrative ideal that was stricter than that of his contemporaries, and completely rejected modern mathematics and even the innovations made by the Galilean school (such as the geometry of indivisibles).

A clear example of Borelli's upturning of Euclid is given by the deductive structure of Book V of the *Euclides restitutus*. We can list the sequence of theorems proven in this book, in comparison with the matching propositions from the *Elements*:

Elements IV,10 New II,2 II,6 IV,15 IV,16 IV,7 IV,12 IV,9 IV,14 IV,4 IV,8
IV,13 XIII,8 XIII,9 Arch. XII,1 VI,19 XII,2 Arch. Papp. Papp. Ptol. New
I,47 VI,31 New II,4 New II,7 New II,8 New II,5 II,6 II,9 II,10 XIII,4
XIII,6 XIII,3 XIII,1 New II,12 New II,13 New I,48 XIII,12 New XIII,10
New

As can be seen, Book V of Borelli's *Euclides* mixes together Euclid's theorems belonging to the second, fourth and thirteenth books of the *Elements*, adding something from the first, sixth and twelfth books. The order of the propositions is generally quite different from Euclid's sequences of the same books (for *Elements* IV, Borelli proves the theorems in this order: 10, 15, 16, 7, 12, 9, 14, 4, 13). These theorems are also interspersed with results from Archimedes, Ptolemy and Pappus. To hold together such a deductive sequence, Borelli also demonstrates therein ten new theorems. As a bonus, Borelli offered a wholly new proof of the important *Elements* XII,2 (circles are to each other as the squares of the diameters) avoiding the method of exhaustion, which Borelli found hopelessly obscure given its reference to the infinite.⁸

A couple of years after the publication of the *Euclides*, Cosimo Noferi described Borelli's work:

⁶ On the theory of proportions in the Galilean school, see the important GIUSTI 1993.

⁷ "...nessuno degli antichi l'ha trovata; ma l'ho trovata io": Borelli to Alessandro Marchetti, August 21st, 1660, quoted in DERENZINI 1959, p. 228.

⁸ *Euclides*, p. 217. A demonstration of *Elements* XII,2 without exhaustion had already been offered by Maurolycus in the *Praeparatio ad Archimedis opera* that introduced his edition of the works of Archimedes. This edition was not published until 1685 (the theorem is here on p. 6), but we know that Borelli had worked on Maurolycus' manuscripts. In turn, the technique used by Maurolycus for this demonstration seems to have been taken from the pseudo-Archimedean *Liber de curvis superficiebus*, a medieval treatise associated with the name of John of Tynemouth, which can be read today in CLAGETT 1964, pp. 433-557.

...and it seems to me that this book should have for its title *Euclid Mendacious, Transformed, Destroyed, Mutated, Upturned*, and the like, which are more suitable to it than *Euclid Restored*.⁹

These words were echoed by Milliet Dechaies in 1674:

Although the work is excellent and useful, nevertheless the title is wrong, because you will hardly recognize Euclid in *Euclid Restored*.¹⁰

The scandal and the fascination provoked by Borelli's book among his contemporaries was fully justified. For centuries Euclid's *Elements* had been *the paradigm itself* of the deductive method, and the model of any further attempt to present a body of truths as a continuous chain of demonstrations. Innumerable works of logic, mathematics, science, and philosophy had taken the *Elements* as their template and had drawn their rigor and truth by imitating Euclid's book. To revise the deductive sequence of the *Elements*, and to suggest that a better logical arrangement of theorems is possible, was an unprecedented claim in the seventeenth century. It was an attack on mathematics itself.

Borelli felt entitled to justify his complete transformation of Euclid's text with an accurate historical consideration, but one that was highly unusual at the time: namely, that the thirteen books that make up Euclid's *Elements* had not been written by Euclid himself, but were independent essays that the ancient geometer had collected under one title. The polygenesis of the classical text explains, according to Borelli, its structural flaws, and the fact that its theorems can be reorganized according to different and more cohesive thematic groups. This is precisely the purpose of the new structure of the *Euclides restitutus*.¹¹

Borelli employed his abilities as a classical philologist and scholar of ancient mathematical texts in order to *historicize* Euclid's *Elements*. He transformed the eternal paradigm of science into a contingent historical phenomenon. Euclid and his predecessors were historical actors offering a presentation of mathematics according to their own views. Likewise modern geometers legitimately expound mathematics in a different, and possibly better, form.

Borelli's novel presentation of elementary mathematics was guided by an epistemological program. The choice of new definitions and axioms, the new organization of the content, the different proofs, are all grounded on the masterplan of implementing a philosophy of mathematics in actual mathematics. The importance of Borelli's book lays in this philosophically-directed, revisionary approach to the foundations of mathematics.¹²

⁹ C. NOFERI, *Disceptatio pro Euclides*, written in 1658-1663 and never published : "... quare potius mihi videtur Euclides mendax, permutatus, dirutus, pervolutus, inversus, et simili appellatione inscribendus, ipsius volumini magis propria, quam Euclides restitutus" (quoted in GIUSTI 1993, p. 154).

¹⁰ C.F.M. DECHALES, *Cursus*, 1690: "Quamvis autem opus sit bonum, & utile, malè tamen ei titulus est praefixus, cum vix Euclidem in Euclide restituto agnosces" (p. 25).

¹¹ Borelli offers a history of geometry (up to the works of Giovan Battista Benedetti) in the dedication of the *Euclides*. This was not new, and already Clavius had a lengthy historical introduction to the *Elements*, which did not, however, conclude with their imperfection or the need to reform them.

¹² A revisionary approach to Euclid's *Elements* had been followed by Francesco PATRIZI, *Della nuova geometria* (1587), insofar as it reshaped (the first half of the first book of) the *Elements* according to a certain philosophical vision. Yet, the inadequate mathematical skills of Patrizi cannot match with Borelli's real dive into the foundations of geometry. Something similar may be said of Hobbes' own attempts at elementary mathematics. The epistemological importance

Borelli, however, maintained that his own epistemology and methods in geometry were nothing but an implementation of what was already found in Greek authors. He appropriated Greek texts by seeking in them a *classical demonstrative ideal*: a set of methods, techniques, constraints that according to him constituted mathematics as a science. He wanted to extract a philosophy of mathematics from the works of Euclid and Archimedes.

Classical texts do not provide anything similar without a great deal of interpretation, and Borelli was thus *creating* a Greek ideal of demonstration by favoring certain theorems, emphasizing certain definitions, using certain principles, and consciously passing silently over other theorems, definitions and principles. More than this: he *corrected* the Greeks, and Euclid in particular, so that they would fit the bill of Borelli's own classicist epistemology.

Borelli entitled his own masterpiece *Euclid Restored*, because it brought Euclid back to the classical demonstrative ideal for which it was intended. But Borelli also translated the title into Italian with the antiphrastic *Euclid Renewed*, because the restoration of the classical text also involved going *against* the original in the name of its ideal.

In this sense even Borelli's lag behind modern mathematics, his regrettable ignorance of algebra, and his rejection of the so promising analysis of the infinite can be read as conscious epistemological choices. He self-imposed proof constraints tighter than those of Euclid, and much tighter than those of many modern mathematicians who had access to post-classical mathematical tools, because he wanted to prove theorems of elementary mathematics in the *most appropriate* way. Borelli's *Euclides* can thus be seen as the ancestor of a centuries-long tradition enquiring about the standards of rigor, the purity of methods, and the logical tools that can be employed in a mathematical proof. For the first time in history, Borelli showed that it was possible to rewrite a classic of science according to a modern epistemological ideal. Borelli was the first of "Euclid's modern rivals", and the *Euclides restitutus* changed for good the modern approach to the classics of mathematics.¹³

In the following paragraphs, we will see in a little more detail how Borelli's epistemology of mathematics affects the deductive structure and system of principles of the *Euclides*. In §2, we deal with Borelli's epistemology and its theory of principles. In §3, we discuss the axiomatic system of the *Euclides*. In §4 we sketch the new deductive structure of the work and of Book I in particular. In §5 we draw some conclusions on the historical meaning and legacy of the *Euclides restitutus*. We especially concentrate on the first edition of the *Euclides*, which was not abridged for students; and on the geometrical books of it, since Borelli's gave no major contributions to number theory or its foundations.

§2. *The Principles of Mathematics*

Borelli's philosophy of mathematics is set out in the introductory pages of the *Euclides*. Borelli was not a philosopher by profession, and this theory appears to us today as an eclectic variant of Scholastic theories widely held in the seventeenth century. This is all the more remarkable because

of Borelli's work has been recognized by several modern scholars including Koyré. A comprehensive evaluation of Borelli's scholarly work from this perspective can be found in BALDINI 1974.

¹³ The famous expression on the modern rivals titles the 1879 book by Charles DODGSON (Lewis Carroll) on the subject. Borelli is not mentioned in it.

Borelli himself often presented his thinking as entirely anti-Aristotelian and criticized *la ragione muffa e antiquata* ("the moldy and antiquated reason") of the Scholastics.¹⁴

According to Borelli, the objects of geometry are shaped *magnitudes*, or geometrical figures. This is also Aristotle's view: he had defined magnitudes as continuous quantities, such as solids, surfaces and lines.¹⁵ Aristotle's definition of the object of geometry was standard in the seventeenth century. Earlier, at the end of the sixteenth century, some mathematicians had begun to suggest that geometry dealt instead with the properties of *space*, but this idea, which would become dominant between the eighteenth and nineteenth centuries, was still a minority in Borelli's time.¹⁶ Tommaso Campanella, for instance, who is often associated with Borelli, argued that space, not magnitudes, was the proper object of geometry.¹⁷ Borelli himself, however, who is often credited with a theory of "absolute" space in physics, never seems to have connected space and geometry together.¹⁸ By contrast, Borelli believed, like Aristotle, that we arrive at the notions of mathematical magnitudes and figures by abstracting them from physical bodies. Indeed, he criticized Euclid for having begun the *Elements* with the definitions of a point, line, surface and solid, in that order, whereas mathematical abstractionism dictates that one should first define a mathematical solid (which is closer to a physical body), and then move on to define surfaces and lines as further abstractions.¹⁹

The most important part of Borelli's philosophy of mathematics, and the one having consequences on the demonstrative structure of the *Euclides*, is the theory of the principles of demonstration. In a nutshell, it is the following.

Each geometrical figure has (*a*) several essential properties and (*b*) several different procedures by which it can be constructed.²⁰ In each of the two sets, there will be properties and procedures that are "first and most known". We may arbitrarily take any one of these as the *definition* of the figure. All the remaining first and most known essential properties may be taken as *axioms* about the figure, and the remaining first and most known constructive procedures may be taken as *postulates* about the figure. Properties and constructive procedures which are not first and most known, should be proven as propositions.

¹⁴ Borelli to Alessandro Marchetti, February 24th, 1659; in DERENZINI 1959, p. 227. On Borelli's mathematical epistemology, see VASOLI 1969.

¹⁵ See, for example, ARIST. *Rhet.* A 2, 1355^b30-31.

¹⁶ See DE RISI 2015.

¹⁷ Campanella expressed his views on the relationship between space and geometry, which he probably took from Francesco Patrizi, in many of his works. The most complete exposition is in a manuscript first published in ROMANO 1935.

¹⁸ Borelli refers to worldly space as a ground of the science of motion in the *De vi percussiois*, 1667, pp. 3-4.

¹⁹ Vitale Giordano, defending Euclid against Aristotle, protested that Borelli's definitional procedure amounted to *ponere currum ante boves* (putting the cart before the horse). See Giordano to Leibniz, November 11th, 1689; in A II, 2, n. 71, p. 296.

²⁰ Borelli calls the idea of a constructive procedure the *ratio structurae formalis* of a figure, and clearly states that there are several of these for each figure: "ratio structurae formalis alicuius subjecti quanti multiplex esse oportet" (*Euclides*, p. 15). These *rationes structurae* are thus somewhat similar to the mathematical *schemata* of Kant (who may have been familiar with the *Euclides restitutus*, see later in note 55).

An immediate consequence of Borelli's theory of principles is the interchangeability of definitions with axioms and postulates. There are no sharp boundaries among them, and mathematicians have a certain freedom in defining mathematical figures:

...from which follows that any definition may be an axiom, in case the object has already a name. And by contrast any axiom may be a definition, in case the object does not yet have a name.²¹

This point justifies Borelli's upturning of Euclid's system of principles. Definitions, axioms and postulates may be exchanged among them, just as the sequence of theorems of the *Elements* may be rearranged according to different deductive structures.

Borelli advanced a further requirement: for each figure there must be *at least* one constructive procedure. This is because there may be pseudo-definitions of non-existent objects (e.g. "a rectilinear figure with two sides"), and only an explicit constructive procedure may guarantee that the corresponding figure exists.²² A mathematician should offer such a constructive procedure before employing the figure and its properties in proving theorems. Borelli's position on the necessity of constructive definitions was not unique in the seventeenth century, and something similar had been advanced, for example, a few years earlier by Thomas Hobbes—an author that Borelli knew well. There were however some differences among their views. According to Borelli, definitions, axioms and postulates are *ideas*; their linguistic expression is quite contingent. For Hobbes, on the other hand, they are essentially linguistic expressions. As a consequence, for instance, Hobbes admitted that impossible objects may be defined, even if such definitions are useless in science, whereas Borelli denied that such objects have a proper definition. Hobbes dealt with nominal definitions of words, Borelli with real definitions of things.²³

The constructive requirement offers a further constraint, or at least a methodological thread, in selecting the principles of demonstration. Borelli clearly expresses his preference for taking first and most known constructive procedures as definitions, rather than defining objects through essential properties and then show their constructability—even though, according to his own epistemology, this could be done. In fact, he criticizes Euclid and other geometers for behaving differently.

We can give an example illustrating the whole theory. A circumference may be characterized in several ways. It is a conic section, obtained by a particular intersection of a cone with a plane. It is a line, the points of which are equidistant from a single point (the center). It is a line traced by the end of a segment rotating around the other end. The first characterization, according to Borelli, expresses a constructive procedure generating the circumference, but it is not "first and most known". As a consequence, it cannot be employed either as a definition or as a postulate, and

²¹ *Euclides*, p. 16: "Unde colligitur, quod quaelibet definition esse potest Axioma, sin omen iam esse receptum. Et è contra quodlibet axioma esset definitio, si nomen non esset receptum".

²² The example of the figure with two straight sides (*Euclides*, p. 16) became commonplace, and was employed by WOLFF (Latin *Logic*, §533; 1728, p. 404) and later on, famously, by KANT (*KrV*, A220/B268 and A291/B348).

²³ See HOBBS, *De corpore*, I, i, § 5; in *Opera*, vol. 1, pp. 5-6, a work that Borelli owned (see note 3). In the *Euclides*, p. 15, Borelli says that definitions are *cognitiones* and then also *ideae*. He discusses these *primae et notissimae* ideas in *Euclides*, p. 16, where he explains that they are true, maximally evident, and such that further properties of the object may be explained through them. The passage is clearly reminiscent of Aristotle, *An. post.* A 2, 71^b26-30, in which principles of demonstration are said to be *true, explanatory, more familiar and prior*. Borelli's statement on names is in the *Euclides*, p. 15: "nomina sunt signa, ad placitum imposita, ad significandam praecognitionem in animo existentem".

should be proven as a theorem. The second characterization expresses a first and most known (non-constructive) essential property of the circumference, and may be employed either as the definition or as an axiom. The third characterization expresses a first and most known constructive procedure and may be employed either as the definition or as a postulate. Euclid has defined the circumference according to the second characterization, and has assumed the third (even if not explicitly enough) as a postulate. According to Borelli, this may be correct in principle, but wrong in practice. He chooses rather to take the third characterization as the constructive definition of the circumference, and the second one as an axiom expressing an essential property of it.²⁴

Euclides restitutus is full of similar modifications of Euclid's system of principles, which are allowed and encouraged by Borelli's epistemology.

Borelli's choice of principles has also important consequences in the demonstrations in the *Euclides*. He hardly ever makes use of postulates, since the relevant constructive procedures are always included in the definitions. But he also rarely makes use of the definitions themselves. They have the prefatory foundational role of showing the possibility of mathematical objects, but are of little use in actual proofs. As a consequence, in Borelli *axioms* acquire an enormous importance as the only principles that are employed in proving theorems.²⁵

There is more. Borelli's theory of principles would imply the existence of a plethora of axioms: there are a great many (and possibly infinite) statements expressing first and well known essential properties of figures. Nonetheless, Borelli was guided by a principle of economy in the selection of axioms. Although the axioms in the *Euclides* are certainly not all independent of each other – there is no clear concept of deductive independence in seventeenth-century mathematics – they are never redundant and have no significant overlap. Borelli selected them with the aim of maximizing the perspicuity of the principles and at the same time minimizing the number of undemonstrated assumptions. This implies a much deeper reflection than was usual in the early modern age on what are the actual assumptions necessary for proving theorems.

Moreover, Borelli seems to have much tighter epistemic requirements for definitions than for axioms. He states, for instance, that the notion of the infinite is irremediably obscure, and should not pollute the definitions of mathematical objects. Yet, he accepts it in the axioms, provided that they are drawn from constructive definitions. Euclid's definition of a circumference refers to the equality of an infinity of radii, and is therefore unfit as a definition, but it may be employed as an axiom. So in many other cases: Borelli's definition of parallels is local, but his axiom on parallels

²⁴ Here is Borelli's judgment on Euclid's definition of a circumference: "Haec est una ex ijs definitionibus, quae passionem ignotam tradunt. Non enim facilè est videre, an in natura reperiri possit talis figura, quae habeat punctum unicum intra se, à quo omnes rectae, ad eius terminum ductae, sint inter se aequales. Semper enim dubitari potest, an aliqua ex infinitis illis lineis habeat eadem mensuram cum reliquis" (*Euclides*, pp. 9-10). Borelli's idea had a wide following, and for example Christian Wolff, fifty years later, adopted (for the same epistemological reasons) Borelli's definition and axiom of the circumference. See WOLFF 1738, §45 of the *Geometrie*, p. 129 (a different treatment was in the Latin edition of WOLFF 1713, §38 of the *Elementa geometriae*, p. 100).

²⁵ Given the interchangeability between definitions and axioms, Borelli claims that definitions may be used as demonstrative premises of proofs: "definitiones adhibentur in demonstratione ut praemissae" (*Euclides*, p. 15). In actual practice, Borelli did employ sometimes definitions in order to prove theorems, and this is especially true in Book III of the *Euclides*, on proportions, since both Euclid and Galileo had used definitions rather than axioms to ground such a theory. In general, however, Borelli almost exclusively employed axioms in the proof. This is very apparent in elementary geometry, and in the proofs of the entire Book I of the *Euclides* there is only *one* reference to a definition (Proposition 4, p. 22), and a hundred or so references to axioms.

makes use of the infinite; his definitions of proportion avoid recurring to the infinite and are based on commensurable ratios, but his axioms also refer to incommensurable ratios.

Borelli's reliance on axioms rather than definitions in proofs, and his attempt at selecting a small but sufficient number of them in order to prove theorems, distance the *Euclides restitutus* from the other treatises of the seventeenth century and make it similar to a modern textbook. This was the beginning of a long and far-reaching transformation in the logical organization of mathematical books.

§3. *Axiomatization*

Given the importance of the principles of demonstration in Borelli's epistemology, it is worth looking at the axiomatic structure of the *Euclides restitutus*. In the early modern age, Euclid's interpreters had already offered many different systems of axioms for the *Elements*, and as it happens the set of principles was the most widely edited part of the ancient book.²⁶ Since none of these mathematicians had also attempted to significantly reform the theorems and the demonstrations, they did not go much further than *suggesting* that certain statements had to be assumed as principles of demonstration. Borelli's *Euclides* offers one of the most advanced systems of axioms in the seventeenth century, since it is economical and powerful, but most of all it shows when and how these principles need to be actually employed in the proofs.

Euclid's *Elements* were possibly prefaced by 5 postulates and 5 common notions (or axioms).²⁷ By contrast, Borelli's *Euclides restitutus* states a total of 6 postulates and 29 axioms.

In Book I, on plane geometry, Borelli enunciates 4 postulates and 14 axioms. The postulates are Euclid's three postulates on the construction of straight lines and circumferences, plus a fourth principle enunciated by Clavius on the possibility of taking magnitudes larger and smaller than any given magnitude.²⁸ The axioms cover Euclid's standard common notions, and add to them a couple of mereological principles that were found in the tradition (such as that the whole is the sum of the parts). There are two axioms about straight lines, which establish that two straight lines do not enclose a space and cannot have a segment in common (they had already been discussed in late antiquity). These are followed by the above-mentioned axiom on the equality of the radii of the circle; by a new version of the axiom of parallels; and by an axiom on the continuity of figures. Borelli completes the axiomatization of plane geometry in the following books: in Book II, there is a principle stating that equal circles have equal radii, and vice versa; and in Book V, an axiom assuming that a straight line is shorter than the semicircle and any broken line between the same ends. The latter flows from Borelli's definition of a straight line as the shortest line between two points, which was taken from Archimedes but never directly used in proofs.²⁹

²⁶ For a presentation of the different axiomatic systems employed in the *Elements*, see DE RISI 2016.

²⁷ Numbers vary in different manuscripts, and I mention here the numbers appearing in Heiberg's critical edition of the text.

²⁸ CLAVIUS, *Euclidis*, Book I, Postulate 4 (ed. 1574, vol. 1, p. 15').

²⁹ Annotating the *Euclides restitutus*, Leibniz observed that although Borelli defines the straight line as the shortest line, he does not shy away from proving *Elements* I,21 (Book I, proposition 22; *Euclides*, pp. 47-48), which states that one side of the triangle is less than the sum of the other two. But if it made sense to prove this proposition in the *Elements*, it is pleonastic with Borelli's new definition of a straight line. Borelli does not seem bothered by this redundancy, any more than he was by the formulation of the axiom in Book V. Here is Leibniz's annotation, which is still unpublished and found in his copy of Borelli's *Euclides* kept in Hannover: "Non utitur autor hac definitione, cum uti potest, v.g.

One of Borelli's greatest foundational merits is that he attempted the first axiomatization of solid geometry. Neither Euclid's *Elements* nor the following Euclidean tradition had principles for this besides the definitions, and from the point of view of modern axiomatics this is one of the major foundational gaps in ancient, medieval and Renaissance geometry. Book VI of the *Euclides restitutus* offers for the first time some principles, 2 postulates and 5 axioms, for solid geometry. The two postulates are simple extensions of Euclid's ones for plane geometry, and state that it is possible to draw and protract planes in space. Two axioms extend the axiomatic properties of straight line to planes, stating that two planes do not enclose a space and that they cannot have a plane segment in common. The other axioms relate plane geometry to solid geometry by asserting that the intersection of two planes is a line, that if two lines intersect one another they lie in the same plane, and that the line joining two points is in the same plane as them.³⁰ In the years that followed, a great many mathematicians followed Borelli's example, and gradually more complex axiomatizations of solid geometry appeared in the treatises of Arnauld (1667), Roberval (1670s), Lamy (1685), Marchetti (1709), Simpson (1760), and many others, until they became quite standard in the nineteenth century.

In Book III of the *Euclides*, Borelli also offered a new axiomatization of the theory of proportions. In this he was preceded by similar attempts by Maurolycus, Benedetti, Torricelli, and others, who had introduced axioms for a theory that Euclid had left without principles (other than definitions). Borelli's six axioms on proportions partly take up previous attempts, and partly innovate a theory that he was also renewing very deeply at the level of definitions and theorems.³¹ Borelli's most important axiom in the theory is an axiom on the existence of the fourth proportional magnitude after three given magnitudes. This axiom had been already added by Clavius and employed by Galileo, but was by no means uncontroversial, and many mathematicians in the early modern age doubted of the possibility of assuming the non-constructive existence of magnitudes whatsoever.³² Borelli's reliance on this axiom shows that his mathematical constructivism was in fact quite limited. Once again, it only applies to the *definitions*, and not to the axioms. Constructions, for him, are meant to offer a criterion to spell out true definitions from empty nominal expressions without reference (e.g. the rectilinear figure with two sides). But once we know that a certain figure (say, a circle) is possible, we are allowed to assume the existence of any of those figures (i.e. circles

in axioma 14 (quod assumit post prop. 13), utitur potius Clavij definitione seu explicatione, ut axioma confirmet. Et posita hac definitione non erat usu demonstrare propositionem 21 quod duo trianguli latera sint majora tertio".

³⁰ The principle that if two points on a line are in a plane then so is the whole line can be seen as an axiomatic reformulation of *Elements* XI,1, which had often been regarded as an indemonstrable proposition that Euclid had clumsily attempted to prove. Before Borelli, Pierre HERIGONE (1634) had already suggested accepting it simply as an axiom. Borelli's axiom that two intersecting lines lie in a plane is the axiomatic formulation of *Elements* XI,2, which suffered from the same difficulties but had never before been suggested as a principle of the theory. Both principles are still recognizable as axioms I,5 and I,6 in Hilbert's *Grundlagen der Geometrie* (HILBERT 1968, p. 3).

³¹ Borelli's system of axioms for the theory of proportions has been the subject of some modern criticism, which noted that some axioms are pleonastic and provable. See PODETTI 1913 and GIUSTI 1993, who also correctly notes (p. 126, note 23) that the concept of "axiom" in Borelli does not require strict indemonstrability.

³² Borelli's axiom was strongly criticized by SACCHERI in the *Euclides vindicatus* (1733, pp. 107-109; cf. also my discussion of the topic in SACCHERI 2014).

of any magnitude), even if we may not be able to construct each of them in particular. Borelli was not a constructivist according to many modern understandings of the term.³³

There are some conspicuous absences in Borelli's list of axioms. First of all, the *Euclides* completely lacks Euclid's Fourth Postulate, which states that all right angles are equal. This assumption had been criticized since classical antiquity, and was generally regarded as provable; however, it is remarkable that it appears neither as an axiom nor as a theorem in Borelli's work. Second, Borelli never states the famous Axiom of Archimedes ("any magnitude may be multiplied as many times as to exceed any given homogeneous magnitude"), which even various modern authors (including Clavius) had added to the *Elements* as a necessary principle for concluding some of Euclid's demonstrations. Borelli makes use of it throughout the proofs but never makes it explicit in the system of principles.³⁴ Finally, Borelli has no axioms in the arithmetical book of the *Euclides restitutus*, as if the theory of numbers required no principles. Euclid had no axioms for arithmetic either, but modern editions of the *Elements* had formulated several explicit principles. Borelli could not ignore this fact, but left no explanation of his choice of not including them. The epistemology of principles set forth at the beginning of the *Euclides* does not seem to easily apply to the domain of numbers.³⁵

Should we look at the legacy of Borelli's *Euclides*, and the impact it had on modern foundational studies, the single most important axiom in the volume is probably Axiom 13 in Book I. This is one of the first axioms on *continuity* to ever be formulated—a topic that was conspicuously absent from ancient and medieval discussions on geometry and moved to the center of the foundational debate in the modern age. The pioneering role of Borelli in this matter is witnessed by a certain uncertainty surrounding the axiom. It was introduced in the first edition of the *Euclides*, but without comments on its grounds or its usefulness in the book. The axiom is not connected with any definition, and is not constructive. In short, it is not clear how it complies with Borelli's epistemology of principles. The axiom was retained in the 1663 Italian translation of the *Euclides*, but disappeared – again without notice – in the 1679 edition.

The problem at issue is that Euclid draws lines and circles which intersect with one another, but has no principles stating that points of intersection exist. Rather, the existence of points of intersection seems to be read off from the diagram (i.e. the drawing accompanying the text), as if the inspection of a drawing could be sufficient to establish the existence of a mathematical object. In the very first proposition of the ancient text, *Elements* I,1, Euclid draws two circles that overlap

³³ Borelli, for example, had no difficulty with mathematical proofs by *reductio ad absurdum*, whose validity was widely discussed, and sometimes rejected, by seventeenth-century philosophers.

³⁴ This is all the more strange because a large part of the *Euclides* is devoted to demonstrating that so-called angles of contact (e.g., the space bounded by a circumference and a line tangent to it) are not angles at all: see the scholium after Prop. 21 in Book II in the 1658 edition of the *Euclides* (pp. 90-96) and the preface of the 1679 edition. Angles of contact had been discussed since classical antiquity (and, in times closer to Borelli, by Peletier and Clavius) precisely because they do not satisfy the Axiom of Archimedes. Borelli's stance on this issue, therefore, could be read as a further confirmation of the universal validity of this axiom in geometry. On the angles of contact in Borelli, see ITARD 1961. Angelo Marchetti, who generally followed Borelli's choice of axioms, explicitly added the Axiom of Archimedes to the principles of his own *Euclides reformatus* of 1709 (p. 42).

³⁵ Throughout Book VIII on arithmetic, however, Borelli often employs the axioms and results in Book III on the theory of proportions. This was a novelty, since in Euclid the theory of proportions between *magnitudes* set forth in *Elements* V cannot be applied to numbers, and he instead offers an alternative theory of proportions between *numbers* at the beginning of *Elements* VII. Borelli's theory of proportions, on the other hand, by treating commensurable and incommensurable ratios together, claims to be applicable to both natural numbers and continuous magnitudes.

each other, and assumes without further ado that there is a point of intersection between the two circumferences.³⁶

This gap in the axiomatic structure of the *Elements* has played an important role in the modern history of the foundations of geometry, and engendered a lively and important debate. Several continuity axioms were added to Euclid's principles to justify the existence of the points of intersection between figures. Borelli was among the very first mathematicians to conceive of such axioms, but he was not the first. The Renaissance geometer Oronce Finé, in 1532, had suggested adding an axiom about the intersection of circles. Since he did not dare to modify Euclid's text, however, he did not implement this principle in the actual proof of *Elements* I,1. This was only done a century later by Claude Richard (1645), who proved this one proposition thanks to his own axiom on intersections between circles.³⁷

Borelli, who may have known both Finé's and Richard's work,³⁸ formulated a much more general principle:

Axiom 13. If the same straight line is wholly inside two figures, the two figures will have a common part and will intersect one another.³⁹

Borelli's axiom does not refer to circles (like the previous ones) but to figures in general. It is not a local fix of *Elements* I,1, but a new principle of intersections for the whole of geometry: it may be used in several other theorems of the *Elements*, but also in more advanced theories such as the *Conics*. This was an important breakthrough in the foundations of geometry, and similar continuity axioms were later introduced by Pascal, Roberval, Schott, Leibniz, Kästner, and many others. Many of them seem to have been following Borelli's *Euclides* on the topic.⁴⁰

Notwithstanding the generality of Borelli's formulation, the axiom is only explicitly employed by him in the proof of *Elements* I,1— just like Richard's weaker and *ad hoc* axiom. It is not clear why Borelli omitted any reference to it in the proof of the other theorems, nor why he erased the axiom altogether from the last edition of the *Euclides* that appeared in his lifetime. In the 1679 *Euclides*, Borelli added a complex argument within the demonstration of *Elements* I,1 showing – with the help of his other axiom on the equality of the radii of a circle – that the two circles are drawn in such a position that they overlap each other, and thus intersect.⁴¹ According to modern

³⁶ See DE RISI 2021.

³⁷ Oronce FINÉ, *Protomathesis*, Paris, 1532, p. 54^v. In this work Finé does not demonstrate *Elements* I,1, and thus the principle remains unused. Finé did not mention it in the edition of the *Elements* he edited in 1536, probably because he did not want to alter the classical text. Finé's axiom was later mentioned by the Spanish translator of the *Protomathesis*, Pedro Juan de Lastanosa de Monzón, in his Latin edition of the *Elements*. Cf. LASTANOSA, *Elementa*, 1569, p. 45. It is possible that Claude Richard was taking Finé's axiom from here and expounded it in RICHARD, *Euclides*, 1645, pp. 16-17.

³⁸ Borelli owned some of Finé's works, including an edition of Euclid's *Elements* edited by him (see above note 3), but not the *Protomathesis* in which the axiom of continuity appears. Borelli mentions Richard's edition of Apollonius' *Conics* in his letters (see GIOVANNOZZI 1916, p. 5 n. 5).

³⁹ *Euclides*: "XIII. Si eadem recta linea intra duas figuras tota collocata fuerit, habebunt illae figurae partem aliquam commune, & se secabunt" (p. 14).

⁴⁰ See DE RISI 2020.

⁴¹ The model for Borelli's demonstration in the 1679 edition may be Greek, and summarize some considerations found in a demonstration of *Elements* I,22 offered in PROCLUS, *In primum Euclidis* 329-32. Unbeknownst by Borelli, a similar

standards, this was a step back, since a principle of continuity was still needed to draw this last inference. Borelli's 1679 argument is important, however, because it offered a demonstration of something (the mutual position of two circles) that had always been simply assumed to be evident from the diagram as well. Perhaps Borelli wanted to convey a new ideal of proof, according to which geometric demonstrations should be entirely propositional and non-diagrammatic.⁴² If he thought so, he did not make this explicit, and in any case the lack of the continuity axiom was damaging this program. The following generations of mathematicians, however, openly discussed such an ideal of proof, and demonstrations like Borelli's *Elements* I,1 powerfully contributed to such debates.⁴³

Whatever we may think of this inchoative attempt at a gapless proof of *Elements* I,1 and at dealing with the more general issue of continuity in elementary and higher geometry, Borelli's *Euclides* remained for a long time a model for its powerful and yet simple axiomatic system. It was imitated, discussed and corrected by innumerable other axiomatic presentations of mathematics, and several of its specific solutions – on parallels, solid geometry, proportions, continuity – survived as many landmarks in foundational studies.

§4. *The Demonstrative Structure*

The true masterpiece of Borelli's *Euclides* is not to be found in its philosophy or system of axioms, but in the deep reworking of the deductive structure of elementary mathematics. It was also the most difficult part of Borelli's endeavor, and the one which required years of work to be carried out. Revising the deductive order according to a new demonstrative ideal required to write anew many proofs, to invent new theorems, and to rethink the whole logic innervating elementary mathematics. No one had dared so much.

The content of the thirteen books of the *Elements* are reorganized by Borelli into nine thematic books of his *Euclides restitutus*. Thus, Book I of the *Euclides* deals with elementary geometry, and is actually close to the first book of the *Elements*. Book II deals with the properties of the circle, and mixes theorems from *Elements* III and *Elements* X. Book III, from which the whole *Euclides restitutus* had germinated, develops the theory of proportions according to a new method, and thus covers roughly the contents of *Elements* V. Book IV deals with similarity, and thus has *Elements* VI for its basis, but supplements it with all the other theorems on similarity found in Euclid's other books. Book V deals with polygons and their measurement, and draws crosswise from several books of the *Elements*. Book VI deals with solid geometry, and thus covers the main theorems of *Elements* XI and XII. Book VII offers a theory of regular polyhedra, drawing on *Elements* XIII and the

demonstration is also found in some of Richard's personal notes, now in the *Biblioteca de la Real Academia de la Historia* in Madrid (Colección de Cortes, 9/2680-568, f. 1v).

⁴² It cannot be ruled out that Borelli chose *Elements* I,1 as a paradigmatic case, and for this reason he made use of his axiom of continuity (in 1658) and his non-diagrammatic demonstration (in 1679) in this proposition only—assuming that similar axioms and proofs could be extended to the rest of the book. This had happened with the early modern attempts at transforming Euclid's proofs into chains of syllogisms in *Barbara*, which generally only dealt with this one proposition. See the famous proofs by PICCOLOMINI 1547, CLAVIUS 1574 and HERIGONE 1634. Note that all these logical attempts at remolding Euclid's proofs into syllogisms still accepted diagrammatic inferences and unstated assumptions about intersections. In this sense Borelli was doing much better.

⁴³ Entirely non-diagrammatic proofs of *Elements* I,1 are to be found, for instance, in Leibniz' studies on *analysis situs*. See DE RISI 2020.

(spurious) *Elements* XIV. Book VIII, on the other hand, deals with arithmetic, and thus refers to *Elements* VII, VIII and IX. Finally, Book IX deals with incommensurable quantities and is an abbreviated version of *Elements* X. It is remarkable that in this structure arithmetic occupies the penultimate place, and is found after all the geometrical books. In many reworkings of the *Elements* after Borelli, it was instead advanced to the first position, since it is a simpler and more abstract science than geometry. We have seen, however, that for Borelli, more abstract also meant posterior in nature, and therefore to be treated afterwards—just as surfaces and lines are defined after solid bodies.⁴⁴

Borelli did not merely reorganize the large thematic units of the thirteen books, but also altered the deductive structure within each book. In doing so, Borelli simplified and shortened many deductive sequences in the *Elements*, and halved the theorems of the classical text, reducing Euclid's 473 propositions to the 230 propositions of the *Euclides restitutus*.⁴⁵ Something was certainly lost in the editing, but in many cases the richness of the results in the *Elements* was preserved by the elimination of repetitions, by the choice of keeping only the strongest results, and sometimes even by the insertion of new general theorems. In addition, Borelli enriched his own treatment of elementary geometry by adding a number of important theorems that were found in other classical sources—especially in Archimedes and Pappus.

We may examine in greater detail the deductive structure of Book I of the *Euclides restitutus*. The content of this book is still very similar to the first book of the *Elements*, and Borelli has not yet deconstructed Euclid's text to the extent we saw earlier in Book V. The beginnings of geometry are the same for the two authors. Nonetheless, the deductive structure is completely different. Here is the sequence of Borelli's theorems:

Elements I,1 I,2 I,3 I,4 New I,15 I,5 I,8 I,9 I,10 I,11 I,12 I,13 I,14 New
I,29 I,27 I,28 I,31 I,30 I,32 I,16 I,17 I,18 I,6 I,19 I,20 I,21 I,22 I,23
I,26 I,34 I,33 VI,9 Post.5 New I,35 I,30 I,37 I,38 I,41 I,42 I,46

The changes might look negligible at first, but grow into a complete “perversion” of Euclid in the course of the book. Borelli's *Euclides* begins like the *Elements* with the sequence of *Elements* I,1, *Elements* I,2, *Elements* I,3, and *Elements* I,4 dealing with the transportation of segments and the first criterion of congruence of triangles. These demonstrations are similar to those in Euclid. However, then Borelli moves on to consider angle theory, and introduces a new theorem (proven through the technique of superposition) that if two angles are equal then their supplementary angles are also equal. This theorem of Borelli's, which was missing in Euclid, is indeed very deep and has far-reaching consequences. It will be used many times in the modern age, and even Hilbert, in the *Grundlagen der Geometrie*, formulated it as the fundamental theorem of the theory of angles.⁴⁶

⁴⁴ For a treatment of elementary mathematics that puts number theory before geometry, see for example ARNAULD, *Nouveaux Éléments*, 1667; or, in the following century, the textbooks by Christian Wolff. In the times of Borelli, Isaac Barrow had reiterated the conceptual precedence of geometry over arithmetic in his *Lectiones mathematicae*, but this was a rather isolated position (BARROW 1683); these lectures by Barrow are not the same on geometry and optics that had been published earlier and Borelli owned (see note 3).

⁴⁵ This is stated by Borelli in the dedication of the *Euclides*. The number of propositions in the *Elements* varies from edition to edition depending on the manuscripts used. Heiberg's critical edition numbers 460 propositions.

⁴⁶ This is Proposition 5 in Book I of the *Euclides restitutus*, pp. 23-24, proven by Hilbert as Theorem 14 of the *Grundlagen* (HILBERT 1968, p. 17).

Through this new theorem, Borelli is able to immediately prove *Elements* I,15 (about vertically opposite angles), which in Euclid required the sequence of theorems of *Elements* I,5, *Elements* I,7, *Elements* I,8, *Elements* I,11, and *Elements* I,13, which Borelli has not yet proven. Armed with *Elements* I,15, Borelli could demonstrate through it *Elements* I,5 (an isosceles triangle has equal angles at the base). Borelli's proof of this *Elements* I,5 is different from Euclid's, which was very famous and often discussed as too complex and unnatural for such a simple theorem, and was considered difficult for students (the so-called *pons asinorum*). From *Elements* I,5, Borelli proves *Elements* I,8 (the third criterion of congruence of triangles), making use of a demonstration that does not need to assume *Elements* I,7 (which Borelli had not yet proven, and Euclid had employed in the proof). Borelli's demonstration of *Elements* I,8 is found in Proclus attributed to Philo of Byzantium,⁴⁷ who apparently had devised it precisely to show that *Elements* I,8 is deductively independent of *Elements* I,7. Borelli, by inserting Philo's proof into the deductive sequence of the *Euclides restitutus*, shows that such a demonstrative path is indeed globally viable. From here Borelli's book takes up the Euclidean deductive sequence and easily proves the propositions from *Elements* I,9 to *Elements* I,14, concluding the discussion on angles. After these, however, Borelli jumps directly to the theory of parallels, with the group of theorems, in perturbed order, from *Elements* I,27 to I,32. At this point the deductive sequence of the *Euclides* departs even further from that of the *Elements*, and goes on with a sequence of results that has no match in Euclid. The last part of Book I is devoted to a general theory of parallelograms and proves many theorems actually found in Euclid's text and a few more. However, Euclid arguably proved theorems about parallelograms for the sole purpose of arriving at *Elements* I,45, which is a very general theorem of equivalence between polygons. He had so little interest in parallelograms *per se* that he did not even bother to give a definition of them. Borelli, on the other hand, deals with the theory of equivalence of polygons in Book IV, and proves *Elements* I,45 in that book. Consequently, the whole theory of parallelograms (*Elements* I,33 – I,44) is no longer, in Borelli, functional for a further result, but instead becomes an independent research topic. Once again, the deductive sequence within these theorems changes to reflect the shift in function of the propositions and is enriched with results (such as *Elements* VI,9) that were foreign to the subject matter treated by Euclid. Similarly, the last theorem in the first book of the *Elements* is the famous Theorem of Pythagoras, which Borelli proves only in Book V because he believed it is thematically connected with the topics covered there. But this shift disrupts the Euclidean deductive sequence: for example, *Elements* I,46, on the construction of squares, is demonstrated by Borelli at the end of Book I of the *Euclides* as a further complement to the theory of parallelograms rather than as a premise to the Pythagorean Theorem as it was in Euclid.⁴⁸

It is worth taking a closer look at the demonstrative sequence concerning the theory of parallels in Book I, since this is generally considered one of Borelli's masterpieces. Many geometers were dissatisfied with Euclid's original theory and tried to offer better alternatives: a sound theory of parallels was therefore pivotal in determining the success of a radical rewriting of the *Elements*.

Borelli suggested that the difficulties of the classical theory depend on the fact that Euclid had resorted to the notion of the infinite. Euclid, indeed, defines parallel lines as straight lines that never meet, even if extended to infinity. This definition is then supplemented by the Fifth Postulate,

⁴⁷ See PROCLUS, *In primum Euclidis* 266-69. Philo's demonstration was discussed in a scholium by CLAVIUS, *Euclidis*, ed. 1574, pp. 54-55.

⁴⁸ On the deductive structure of the first book of Euclid's *Elements* see NEUENSCHWANDER 1973, MUELLER 1981, VITRAC 1990-2001.

which states that two straight lines that are inclined towards each other (i.e., that form with a transversal interior angles less than two right angles), will eventually meet if extended infinitely (i.e., are not parallel). In both the definition and the postulate appears the Greek term ἄπειρον, the "infinite" or "indefinite," otherwise very rare in Euclid's text. Borelli judges Euclid's definition of parallels to be based on a *passio valdè remota et incomprehensibilis*, and believes that no use can be made of it.⁴⁹

Borelli offers a new definition of parallel lines: parallels are those straight lines which have a perpendicular in common.⁵⁰ This definition has the advantage of not having to consider the two lines in their entirety, that is, infinitely extended. Borelli's definition is *local*. Moreover, although this definition expounds an essential property of parallels rather than their construction, Euclid shows in *Elements* I,11 and 12 how to construct perpendicular straight lines with ruler and compass. Using these propositions, it is thus possible to construct parallel lines according to Borelli's definition. This complies with Borelli's constructive epistemology.⁵¹

Borelli complemented his definition with an explicit axiom: a segment running perpendicular on a straight line traces with its free end another straight line (which is parallel to the first, following Borelli's definition). This statement had appeared as an explicit assumption in the course of Clavius' discussion of the theory of parallels, but no one had assumed it as an axiom before Borelli.⁵² It is a statement equivalent to Euclid's original Fifth Postulate.

From this axiom, Borelli proves a new theorem on parallel lines: if two straight lines have a perpendicular in common, then any other perpendicular to one of them is also perpendicular to the other. In the centuries that followed, Borelli's theorem became a standard formulation of the Fifth Postulate in geometry.⁵³ Through this important theorem, Borelli easily proves that parallel lines (defined according to him) satisfy Euclid's definition of parallels, namely, that they do not meet even if extended infinitely. Further on, he demonstrates Euclid's main propositions requiring

⁴⁹ The expression comes from the scholium after Proposition 16 in Book I (*Euclides*, p. 37).

⁵⁰ There is really an ambiguity in Borelli's definition of parallels, because this definition is offered in 1658 *after* the formulation of the axiom on parallels, and thus it is not clear whether parallel lines are those with a common perpendicular or those that are drawn according to the axiom. However, in the preface to the 1679 edition it is clear that Borelli's definition is the former: "Certum enim est, ex Propositionibus 10. & 11. Primi Elementi quod duae rectae AB, CD in eodem plano existentes possunt perpendiculariter elevari super eandem rectam EH; tunc quidem predictas lineas AB, & CD voco inter se parallelas" (p. iii). It may also be mentioned that the edition of Euclid's *Elements* edited by André Tacquet in 1654 had a definition of parallels as equidistant lines, later enriched by an axiom (pleonastic, for it can be proven from the definition) that two parallels have a common perpendicular (cf. p. 9 and p. 12). It is possible, but not necessary, that Borelli was inspired by this (far less interesting) theory of Tacquet.

⁵¹ It is true, however, that in *Elements* I,12 (*Euclides*, Prop. 11; p. 29) there is a much needed reference to an infinite straight line.

⁵² CLAVIUS, *Euclidis*, ed. 1589, pp. 144-61. The theory was missing from the first edition (1574) of Clavius' work. The construction of parallels through the motion of a perpendicular segment dates to the Arabic Middle Ages (Thābit ibn Qurra, ninth century), and Clavius had attempted to justify it with a quite naïve argument. Borelli's argument does not fare any better, but it largely surpasses Clavius' discussion in making explicit that we are dealing with a new axiom.

⁵³ On Borelli's theorem, see Saccheri's remark: "Ma il punto è dimostrare senza petizione di principio che se una retta è perpendicolare a due rette, qualunque altra che sia perpendicolare ad una di esse, debba essere perpendicolare alla compagna. Io non mi arrendo che il Borelli abbia ciò dimostrato, quando non abbia fatto capo al concetto formale obiettivo di linea retta. Fin qui il Saccheri..." (Ceva to Grandi, August 9th, 1713, in TENCA 1952, p. 38). On the subject, see also: MAIERÙ 1982 and DI STEFANO, FRASCA SPADA 1985.

the Fifth Postulate (*Elements* I,29, and then *Elements* I,32 on the sum of the interior angles of triangles), and finally the Fifth Postulate itself in Euclid's original formulation. In this way he reconstructs, by a different deductive path, the theory of parallels in the *Elements*. It is remarkable that in Borelli's presentation many theorems on parallels that are independent of the validity of the Fifth Postulate (such as *Elements* I,27, *Elements* I,28, *Elements* I,31) are proven *after* the assumption of Borelli's substitute axiom. Even the Exterior Angle Theorem (*Elements* I,16), which Euclid used as an important tool in his own theory of parallels, is not proven by Borelli until after the development of the entire theory. Borelli, in short, never had any interest in separating the results of "absolute geometry" independent of the validity of the Fifth Postulate from those specifically "Euclidean" results that require its validity. His epistemology did not focus on the independence of certain theorems from certain axioms (as is normally the case in today's geometry), but rather on the fact that axioms are immediate consequences of definitions, and therefore they must be formulated, and used, at the beginning of the theory and in place of the definitions themselves.

This brief examination of the deductive structure of Book I of the *Euclides restitutus* shows how complex the task of rewriting the *Elements* was. Borelli had many objectives in doing this: he wanted to prove general and consequence-rich theorems (Borelli's theorem on supplementary angles), simplify the *Elements* for students (the *pons asinorum*), show the logical independence of certain propositions from others (*Elements* I,8 with respect to *Elements* I,7), prove standard axioms and revise classical definitions (the discussion on parallels), thematize new geometrical objects left out by Euclid (parallelograms), eliminate the infinite from proofs, and still others. He accomplished these aims by using alternative proofs found in ancient commentators, rewriting himself some proofs, inventing new theorems, making liberal use of material found in the works of recent mathematicians, and most of all by studying in depth the logical structure of Euclid's treatise.

Borelli showed that the venerable deductive tree planted by Euclid was more changeable than expected, and its roots, trunk and branches could be mixed up as to make new timber for geometry. Following his own ideal of demonstration, he carved his *Euclides restitutus* out of the *Elements*. The material was the same: Borelli never doubted that there is just *one* possible geometry, that had been expounded in Euclid's *Elements*. But the architecture of his *Euclides* was different, since the same *corpus* of mathematical truths may be arranged in different ways.

Borelli's new structure for elementary mathematics offered new insights on geometry itself, by showing connections and logical dependencies that previously had been invisible in Euclid's work. As we have seen, however, something else went lost in the *Euclides*. Some deductive sequences actually present in the ancient text, such as a unified theory of the equivalence of polygonal figures in *Elements* I, 35 – I,45, or the connection between *Elements* I,46 and the Theorem of Pythagoras, or yet the fact that *Elements* I,16 – I,28 are independent from the Fifth Postulate, were missing in Borelli's book. This fact is a straightforward consequence of Borelli's historicization of Euclid: different arrangements of the same mathematical *corpus* may show different mathematical facts. Euclid had his own *interpretation* of mathematics, which stressed and amplified some aspects of it, while leaving others in the shade. Borelli had a different interpretation of mathematics, driven by questions and themes that were foreign to Euclid and had become relevant in the seventeenth century.

In the following centuries, many other mathematicians retained the idea of implementing their own views into alternative deductive systems of elementary geometry. Borelli had made clear that the logical structure of mathematics is not given once and for all, but is a conscious *choice* of the mathematician, who can shape the deductive sequence according to different interests, goals and ideals.

§5. *The Legacy of the Euclides restitutus*

Borelli's *Euclides restitutus* is a fundamental junction in the history of the foundations of mathematics. Borelli's eclecticism and broad interests gathered therein together the foundational reflections of Galileo and his school, the erudition of Clavius, the inventions of Maurolycus and, through him, of some medieval authors, but then also modern innovations of mathematicians from France and Spain such as Finé and Richard, and then again Scholastic and perhaps even Hobbesian theories on definitions and axioms. All these materials were rethought and transformed by Borelli into a new conception of geometry, which he expounded through a radical rereading and transformation of Euclid's *Elements*. His contemporaries realized the importance of Borelli's work in this field, and discussed it at length: it was taken as a model for countless further mathematical treatises, and was harshly criticized in many others. Thus, the threads of the various traditions that Borelli had woven together in the *Euclides* unraveled again in a multitude of new lines; and almost all modern foundational studies, so different from each other and often oblivious of Borelli, owe something to this first attempt at rearranging Euclid.

The most immediate and obvious filiation of the *Euclides restitutus* occurred among Italian scholars.

Vitale Giordano da Bitonto, who was well acquainted with Borelli and his work, published in 1680 (only a few months after Borelli's death) a book that he had probably been preparing for some time but had not yet had the audacity to give to print: a *Euclide restituito* written in Italian. This work offered a reference to Borelli from the title, and Borelli's foundational studies are discussed there in a multitude of places. Giordano's *Euclide restituito*, however, seems to want to oppose, rather than imitate, Borelli's Italian *Euclide rinnovato*. Indeed, Giordano follows Euclid's deductive order, and not Borelli's altered one; he has the axioms and definitions of the Euclidean tradition of Clavius and other Renaissance mathematicians, and not Borelli's; and he presents Greek classical theories, such as that of proportions, rather than Borelli's new ones. Only in the theory of parallels does Giordano try some new foundational paths, but again he takes care to put his own reflections in scholia and notes, without altering the classical text. It would have been clear to all readers that he was *restoring* Euclid's *Elements* to their classical form after they had been so disfigured by Borelli. Giordano was also preserving and expounding, in the margin of the text, some of Borelli's new foundational ideas. It was thus a rejection but also taming of Borelli's work.

In 1709, Angelo Marchetti (son of Alessandro, Borelli's student) published his own *Euclides reformatus*. This work had no philological qualms, and pursued the idea of reforming Euclid through a new deductive structure and new principles—as Borelli had done. Marchetti departed from Borelli, quite explicitly, in terms of specific foundational solutions. In particular, he authored a new theory of proportions, quite different from Euclid's and also from Borelli's, which he opposed to both. Elsewhere, however, Marchetti's different foundational choices were mere variations on Borelli's, and he in fact followed him in almost all important points. It was thus a continuation of the tradition inaugurated by Borelli.

Finally, in 1733, Gerolamo Saccheri gave his *Euclides vindicatus* in print. Again, the title was an explicit reference to the book by Borelli, who is the most discussed author in the volume. In his book, Saccheri presents the reader with two monographic studies on the theory of parallels and the theory of proportions, and in both he sets out to "vindicate" Euclid from the unjust criticisms that modern mathematicians, and Borelli in particular, had made of him on these issues. The hyper-classicism of Saccheri, who like Borelli (but half a century after him) never dealt with modern

mathematics, naturally led him to confront the *Euclides restitutus* and its classicist demonstrative ideal. Saccheri felt however that any deviation from Euclid, like those, very numerous, of Borelli, were as many insults to mathematics and logic. Even more, they were unwarranted attacks on the highest mathematical authority, since Euclid's *Elements* represents for Saccheri an orthodoxy accepting no interpretation that is not a heresy.⁵⁴ The *Euclides vindicatus* was a veritable counter-reformation of Borelli's *Euclides restitutus*. In any case, given the extraordinary importance of Saccheri's theory of parallels for the discovery of non-Euclidean geometries, it can be said that it was this *Euclides* of 1733 that highlighted, albeit critically, the foundational relevance of Borelli's studies in this domain.

The Italian geometrical tradition, however, shows only the most explicit reception of the *Euclides restitutus*. Borelli's work quickly crossed national borders, and was an important source of inspiration for all subsequent treatises on geometry that radically reformed Euclid's *Elements*.

Many of Borelli's mathematical ideas survived in the later geometrical tradition, and in the treatises of the eighteenth and nineteenth centuries it is not difficult to pick up fragments of the *Euclides*: a deductive sequence different from that of the *Elements*, an axiom of continuity that is clearly derived from Borelli, the axiomatization of the geometry of solids, some considerations on the theory of parallels, a rewriting of the Euclidean theorems proven through exhaustion, various considerations on constructivism or the infinite, and much more. Borelli's ideas have influenced many aspects of the modern treatment of elementary mathematics for a very long time.

Borelli's mathematical epistemology even exerted a strong fascination among the philosophical community, which was attracted by the fame and foundational rigor of the *Euclides*. Thus we find discussions of Borelli's work in authors such as Spinoza, who was interested in it for the theory of axioms and definitions; in Leibniz, who commented on the *Euclides* with a multitude of marginal notes and observations of his own; in Wolff, who made use of it in his own philosophy of mathematics, which had such a wide influence in eighteenth-century Germany; and even in Kant, who seems to have regarded Borelli's work as a model and paradigm for mathematical demonstration.⁵⁵

Borelli's legacy in the history of mathematics, however, does not depend only, or so much, on the survival of certain of his particular results in geometry and epistemology. The *Euclides* had shown for the first time the possibility of making use of Greek mathematical texts in a new way, and had thus created a new approach to the classics of this science. After Borelli's book, nothing was as before: dozens and then hundreds of mathematicians wrote treatises on elementary geometry that departed from the structure of the *Elements*. This is an exceptionally important historical

⁵⁴ It is impossible not to see a political and religious contrast between the "revolutionary" Borelli, who participated to the anti-Spanish revolts in Messina, and the conservative Jesuit Saccheri. Among the many studies on Borelli's political views, see for instance BALDINI 1978 and BERTOLONI MELI 1996.

⁵⁵ Spinoza discussed Borelli's philosophy of mathematics in Letter 9 to de Vries (1663). Leibniz was an avid reader of the *Euclides restitutus*, which he annotated with great care, pointing out what he approved ("rectè") and what seemed to him to be inexcusable errors. He had correspondences about Borelli with Vitale Giordano (see *note* 19) and Angelo Marchetti (A III, 6, n. 244, pp. 806-807; A III, 7, n. 196, pp. 779-81). At the end of his life and after writing hundreds of pages of foundational studies in elementary geometry, however, Leibniz was quite dismissive with Borelli: "In Borello multum diligentiae agnosco, ingenio mediocri fuisse videtur" (Leibniz to Johann Bernoulli, October 23rd, 1716; in GM III, p. 971). We have already mentioned in the *notes* 22 and 24 some works in which Wolff quoted Borelli and applied his methodological directions. Kant cites Apollonius' *Conics* in Borelli's edition, but it seems possible that he was also familiar with the *Euclides* and the theory of parallels in particular. On Kant and Borelli see LAYWINE 2014 and HEIS 2020.

phenomenon that powerfully influenced the history of modern mathematics and still marks today's foundational studies. Arnauld's *Elements* of 1667, guided by a Cartesian epistemology; the eighteenth-century textbooks of Simpson and Kästner; Legendre's great *Éléments de géométrie* of 1794; and also the works on elementary geometry by Pasch and Peano, and Hilbert's celebrated *Grundlagen der Geometrie*, which closed the foundational research of the nineteenth century and opened those of today: are as many examples of radical rewriting, inspired by different epistemologies, of the content of Euclid's *Elements*.

All these books and many more, so different from each other and so important for the history of science, belong to the same literary genre. This literary genre was created by Borelli with the *Euclides restitutus*, which was, in many respects, the first modern book of elementary geometry.

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