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Boundary singular problems for quasilinear equations involving mixed reaction-diffusion

Laurent Véron*

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain, $p > 1$, $1 < q < 2$ and $M > 0$. We present some results concerning the singular boundary behaviour of positive functions satisfying

$$\mathcal{L}_{p,q,M} u := -\Delta u + u^p - M|\nabla u|^q = 0 \quad (1.1)$$

in Ω . The main characteristic of the operator $\mathcal{L}_{p,q,M}$ is that it exhibits a competition between the absorption term u^p and the source reaction term $|\nabla u|^q$, and these terms are not of the same nature. This competition has for consequence the emergence of a rich variety of phenomena. Most of the results presented here have been obtained in collaboration with Bidaut-Véron and Garcia Huidobro [8]. Our study emphasises two directions:

- 1- Existence of solutions with a measure as boundary data.
- 2- Description of the solutions with a boundary isolated singularity.

If $q = \frac{2p}{p+1}$ equation (1.1) is invariant under the scaling transformations T_ℓ , $\ell > 0$ defined by

$$T_\ell[u](x) = \ell^{\frac{2}{p-1}} u(\ell x). \quad (1.2)$$

If $1 < q < \frac{2p}{p+1}$, the absorption term is dominant and the behaviour of singular solutions is modelled by the *Emden-Fowler equation*

$$-\Delta u + u^p = 0. \quad (1.3)$$

If $q > \frac{2p}{p+1}$, the source term is dominant and the behaviour of singular solutions is modelled by an *eikonal equation*

$$u^p - M|\nabla u|^q = 0. \quad (1.4)$$

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Another equation which plays a crucial role is the *Riccatti equation*

$$-\Delta u - M|\nabla u|^q = 0. \quad (1.5)$$

If $q = \frac{2p}{p+1}$ no reaction term is dominant and the value of M becomes fundamental. An important tool for constructing solutions lies in the existence of natural sub and supersolutions which are naturally ordered if they have the same boundary data: the Emden-Fowler equation (resp. the Riccatti equation) is a subsolution (resp. supersolution) for (1.1).

The problems of boundary singularities and measure boundary value problems for related operators have been studied recently, but with different relations between the reaction terms. In the following equation studied in [16] the two reaction effects are cumulative even if they are not of the same nature

$$-\Delta u + u^p + M|\nabla u|^q = 0. \quad (1.6)$$

In that case one term may become dominant but there is no cancelation. The equations with only one absorption term, u^p or $M|\nabla u|^q$ are natural supersolutions.

In the next equation

$$-\Delta u - u^p - M|\nabla u|^q = 0, \quad (1.7)$$

the two reaction effects are source terms. Again there is an addition of effects. The equations with only one source term, u^p or $M|\nabla u|^q$ are natural subsolutions. In the publication [6] it is performed an analysis of the problem which presents some analogy with the one developed here.

The singular boundary value problem for the somewhat similar equation

$$-\Delta u - u^p + M|\nabla u|^q = 0, \quad (1.8)$$

is studied in [10]. Therein, the two reaction terms are also in opposition, a situation which presents some similarities to the one developed here, but the effects of this opposition are very different.

2 Removable boundary singularities

We assume that Ω is a bounded C^2 domain and $0 \in \partial\Omega$. We set $\rho(x) = \text{dist}(x, \partial\Omega)$.

Theorem 2.1 *Let $p \geq \frac{N+1}{N-1}$, $M > 0$ and $u \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{0\})$ is a nonnegative function which satisfies*

$$\mathcal{L}_{p,q,M} u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega \setminus \{0\}. \quad (2.1)$$

Assume that one of the following conditions holds:

(i) either $p = \frac{N+1}{N-1}$ and $1 < q < \frac{N+1}{N}$,

(ii) or $p > \frac{N+1}{N-1}$ and $1 < q \leq \frac{2p}{p+1}$.

Then $u \in L^1(\Omega) \cap L^p_\rho(\Omega)$, $\nabla u \in L^q_\rho(\Omega)$ and

$$\int_{\Omega} (-u\Delta\zeta + (u^p - M|\nabla u|^q)\zeta) dx = 0 \quad \text{for all } \zeta \in \mathbb{X}(\Omega), \quad (2.2)$$

where

$$\mathbb{X}(\Omega) := \{\zeta \in C^1(\overline{\Omega}) : \zeta = 0 \text{ on } \partial\Omega, \Delta\zeta \in L^\infty(\Omega)\}. \quad (2.3)$$

Furthermore, if we assume either (i) that is $p = \frac{N+1}{N-1}$ and $1 < q < \frac{N+1}{N}$, or

(iii) $p > \frac{N+1}{N-1}$ and $1 < q < \frac{2p}{p+1}$, or

(iv) $p > \frac{N+1}{N-1}$, $q = \frac{2p}{p+1}$ and

$$M < m^{**} := (p+1) \left(\frac{(N-1)p - (N+1)}{2p} \right)^{\frac{p}{p+1}}, \quad (2.4)$$

then $u = 0$

Remark. Notice that in the case (i) there exist positive functions satisfying (2.1) with a singularity concentrated at 0. This singularity is not detected in the sense of distributions. The same phenomenon occurs for solutions of

$$-\Delta u = u^p \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega \setminus \{0\}, \quad (2.5)$$

when $\frac{N+1}{N-1} \leq p < \frac{N+1}{N-3}$, see [5].

Proof of Theorem 2.1

Step 1: A priori estimate. If $M \geq 0$, $1 < q < \min\{p, 2\}$ and $u \geq 0$ satisfies (2.1), then for some $c_1 > 0$, we first prove by a modification of the Keller-Osserman method that

$$u(x) \leq c_1 \max \left\{ M^{\frac{1}{p-q}} |x|^{-\frac{q}{p-q}}, |x|^{-\frac{2}{p-1}} \right\} \quad \text{for all } x \in \Omega. \quad (2.6)$$

As a consequence, using the regularity properties of elliptic equations (see e.g. [14]) and the scaling transformation T_ℓ , which is possible as long as $q \leq \frac{2p}{p+1}$, we obtain an estimate on the gradient:

$$|\nabla u(x)| \leq c_2 \max \left\{ |x|^{-\frac{p}{p-q}}, |x|^{-\frac{p+1}{p-1}} \right\} \quad \text{for all } x \in \Omega \cap B_1. \quad (2.7)$$

Step 2: Change of unknown. Set $u = v^b$ with $0 < b \leq 1$, then v satisfies

$$-\Delta v - (b-1) \frac{|\nabla v|^2}{v} + \frac{1}{b} v^{(p-1)b+1} = Mb^{q-1} v^{(b-1)(q-1)} |\nabla v|^q. \quad (2.8)$$

The problem is to get rid of the term on the right-hand side, and this is done as it follows: let $\epsilon > 0$, by Hölder's inequality we have

$$v^{(b-1)(q-1)} |\nabla v|^q \leq \frac{q\epsilon^{\frac{2}{q}}}{2} \frac{|\nabla v|^2}{v} + \frac{2-q}{2\epsilon^{\frac{2}{2-q}}} v^{\frac{(2b-1)q-2(b-1)}{2-q}}.$$

Then (2.8) yields

$$\begin{aligned} -\Delta v + \left(1 - b - M \frac{qb^{q-1}\epsilon^{\frac{2}{q}}}{2}\right) \frac{|\nabla v|^2}{v} \\ + \frac{1}{b} v^{(p-1)b+1} - Mb^{q-1} \frac{2-q}{2\epsilon^{\frac{2}{2-q}}} v^{\frac{(2b-1)q-2(b-1)}{2-q}} \leq 0. \end{aligned} \quad (2.9)$$

Then the question is how to control the exponent of v in order that the absorption becomes dominant for large v , and this necessitates,

$$\frac{(2b-1)q-2(b-1)}{2-q} \leq (p-1)b+1 \iff q \leq \frac{2p}{p+1}.$$

Notice that the condition is independent of b . We fix

$$b = \frac{2}{(N-1)(p-1)} \iff (p-1)b+1 = \frac{N+1}{N-1},$$

which is the removability threshold for boundary isolated singularities of solutions of the Emden-Fowler equation. With this choice the last problem is to control the sign of the coefficient of $\frac{|\nabla v|^2}{v}$

(i) If $p > \frac{N+1}{N-1}$, $q < \frac{2p}{p+1}$, we choose $\epsilon = \left(\frac{2(1-b)}{Mqb^{q-1}}\right)^{\frac{q}{2}}$ and (2.9) is transformed into

$$-\Delta v + \frac{(N-1)(p-1)}{4} v^{\frac{N+1}{N-1}} \leq A. \quad (2.10)$$

By a result of Gmira-Véron [15], v remains bounded and the conclusion follows by a suitable choice of test functions and standard regularity results [14].

(ii) In the case $p > \frac{N+1}{N-1}$, $q = \frac{2p}{p+1}$, we have the same choice of b , but a more refined choice of ϵ in order to take care of M .

(iii) In the case $p = \frac{N+1}{N-1}$, $q < \frac{N+1}{N}$, we use (2.6) to improve the estimate (2.7), and then by iterations to derive the boundedness of u . This is detailed in [8]. \square

Theorem 2.1 can be extended to more general boundary singular sets.

Theorem 2.2 *Assume $p > \frac{N+1}{N-1}$ and $\frac{N+1}{N-1} < r < p$. If one of the following conditions is satisfied:*

(i) either $q = \frac{2p}{p+1}$ and

$$M < m_r^{**} := (p+1) \left(\frac{p-r}{p(r-1)} \right)^{\frac{p}{p+1}}, \quad (2.11)$$

(ii) or $1 < q < \frac{2p}{p+1}$, $r \leq 3$ and M is arbitrary.

Then if $K \subset \partial\Omega$ is a compact set such that $\text{cap}_{\frac{\partial\Omega}{r}, r'}(K) = 0$, any solution u of

$$\mathcal{L}_{p,q,M} u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega \setminus K. \quad (2.12)$$

is identically 0.

Proof. The principle of the proof is somewhat similar: we set $u = v^b$ for some $b \in (0, 1)$ and we reduce (2.12) to an inequality of type

$$-\Delta v + C_1 v^r \leq C_2 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega \setminus K, \quad (2.13)$$

with $C_1, C_2 > 0$. Since $\text{cap}_{\frac{\partial\Omega}{r}, r'}(K) = 0$, it follows by the removability theorem proved in [18] that v is bounded from above and the result is easily obtained by a suitable choice of test functions. \square

3 Measure data problems

The natural space of test functions for analysing boundary value problems is $\mathbb{X}(\Omega)$ already defined in (2.3).

Definition 3.1 Let $\mu \in \mathfrak{M}(\partial\Omega)$ and $p, q \geq 1$. A Borel function u defined in Ω is a weak solution of

$$\begin{aligned} -\Delta u + |u|^{p-1}u - M|\nabla u|^q &= 0 && \text{in } \Omega \\ u &= \mu && \text{in } \partial\Omega, \end{aligned} \quad (3.1)$$

if $u \in L^1(\Omega) \cap L_\rho^p(\Omega)$, $\nabla u \in L_\rho^q(\Omega)$ and

$$\int_{\Omega} (-u\Delta\zeta + (|u|^{p-1}u - M|\nabla u|^q)\zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu \quad \text{for all } \zeta \in \mathbb{X}(\Omega). \quad (3.2)$$

The next two problems in which μ is a Radon measure on $\partial\Omega$ are naturally associated to (3.1).

1- The Emden-Fowler equation

$$\begin{aligned} -\Delta v + |v|^{p-1}v &= 0 && \text{in } \Omega \\ v &= \mu && \text{in } \partial\Omega. \end{aligned} \quad (3.3)$$

2- The Riccatti equation

$$\begin{aligned} -\Delta w - M|\nabla w|^q &= 0 && \text{in } \Omega \\ w &= \mu && \text{in } \partial\Omega. \end{aligned} \quad (3.4)$$

It is proved in [18] that (3.3) admits a solution, necessarily unique, if and only if

$$\text{For any Borel set } E \subset \partial\Omega, \text{cap}_{\frac{2}{p}, p'}^{\partial\Omega}(E) = 0 \implies |\mu|(E) = 0, \quad (3.5)$$

Concerning (3.4) it is proved in [9] that there exists a solution if, for some $C > 0$, μ satisfies

$$\text{For any Borel set } E \subset \partial\Omega, |\mu|(E) \leq C \text{cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(E). \quad (3.6)$$

Combining these two results we prove the following.

Theorem 3.2 *Let $p > 1$, $1 < q < 2$ and μ be a nonnegative Radon measure on $\partial\Omega$ which satisfies*

$$\mu(E) \leq C \min \left\{ \text{cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(E), \text{cap}_{\frac{2}{p}, p'}^{\partial\Omega}(E) \right\} \quad \text{for any Borel set } E \subset \partial\Omega, \quad (3.7)$$

for some $C > 0$. Then there exists $c_0 > 0$ such that for any $0 < c \leq c_0$ there exists a nonnegative weak solution of (3.2) with boundary data $c\mu$. Furthermore the boundary trace of u is the measure $c\mu$.

Remark. No condition involving $\text{cap}_{\frac{2}{p}, p'}^{\partial\Omega}$ (resp. $\text{cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}$) are needed if $1 < p < \frac{N+1}{N-1}$ (resp. $1 < q < \frac{N+1}{N}$) because of the Sobolev-Morrey imbedding theorem.

Abridged proof. Since the positive solution v_μ of (3.3) is a subsolution of $\mathcal{L}_{p,q,M}u = 0$ and is smaller than any solution w_μ of (3.4) which is a supersolution of $\mathcal{L}_{p,q,M}u = 0$, it follows by [11] that there exists a $W_{loc}^{1,2}(\Omega)$ -function u which satisfies $v_\mu \leq u \leq w_\mu$ in Ω and

$$\mathcal{L}_{p,q,M}u = 0 \quad \text{in } \Omega. \quad (3.8)$$

This u is a C^1 function. Consequently, by the sandwich principle,

$$\lim_{\delta \rightarrow 0} \int_{\rho(x)=\delta} w_\mu Z dS(x) = \int_{\partial\Omega} Z d\mu = \lim_{\delta \rightarrow 0} \int_{\rho(x)=\delta} v_\mu Z dS(x) = \lim_{\delta \rightarrow 0} \int_{\rho(x)=\delta} u Z dS(x), \quad (3.9)$$

for all $Z \in C(\overline{\Omega})$, $Z \geq 0$. The restriction $Z \geq 0$ can be dropped and this implies that u admits a boundary trace in the dynamical definition of the boundary trace [19]. Therefore we denote $u = u_\mu$. In order to assert that u_μ is a weak solution in the sense of Definition 3.1, we need some estimates. We denote by $\mathbb{P}_\Omega[\cdot]$ the Poisson operator in Ω .

1- Estimate on solutions: there holds

$$v_\mu \leq \mathbb{P}_\Omega[\mu] \leq w_\mu \leq c\mathbb{P}_\Omega[\mu],$$

see [9], and

$$0 \leq v_\mu \leq u_\mu \leq w_\mu \leq c\mathbb{P}_\Omega[\mu].$$

If μ satisfies the condition (3.6), there exists $C' > 0$ such that for $0 < c \leq C'$ there exists a nonnegative solution $z \in L^1(\Omega) \cap L_\rho^p(\Omega)$ to

$$\begin{aligned} -\Delta z - z^p &= 0 && \text{in } \Omega \\ z &= c\mu && \text{in } \partial\Omega, \end{aligned} \tag{3.10}$$

(see [3]). It clearly satisfies $c\mathbb{P}_\Omega[\mu] \leq z$. Then $w_\mu \in L_\rho^p(\Omega) \implies u_\mu \in L_\rho^p(\Omega)$.

For the gradient, set $\phi = \mathbb{G}_\Omega[u_\mu^p]$, then $\phi \geq 0$ and

$$-\Delta(u_\mu + \phi) = |\nabla u_\mu|^q \geq 0.$$

By Doob's theorem,

$$-\Delta(u_\mu + \phi) \in L_\rho^1(\Omega) \implies |\nabla u_\mu| \in L_\rho^q(\Omega).$$

Since $u_\mu \in L_\rho^p(\Omega)$, $|\nabla u_\mu| \in L_\rho^q(\Omega)$ and u_μ has boundary trace μ , it is straightforward to prove that it is a weak solution. \square

The condition (3.7) can be simplified in most of the case by using classical results on Bessel capacities [1] which endow the general form

$$\text{cap}_{\beta,b}^{\partial\Omega}(E) \leq c \left(\text{cap}_{\alpha,a}^{\partial\Omega}(E) \right)^\theta \quad \text{for all Borel set } E \subset \partial\Omega,$$

under suitable conditions involving $a, p > 1$, $\alpha, \beta > 0$ and $\theta \geq 1$. We prove the two following corollaries,

Corollary 3.3 *Assume $p \geq \frac{N+1}{N-1}$ and $\frac{2p}{p+1} \leq q < 2$. If μ is a nonnegative Radon measure on $\partial\Omega$ which satisfies, for some $C > 0$,*

$$\mu(E) \leq C \text{cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(E) \quad \text{for all Borel set } E \subset \partial\Omega, \tag{3.11}$$

then the conclusions of Theorem 3 hold.

Similarly

Corollary 3.4 *Assume $\frac{N+1}{N} \leq q < \frac{2p}{p+1}$. If μ is a nonnegative Radon measure on $\partial\Omega$ such that for some constant $C > 0$, there holds for any Borel set $E \subset \partial\Omega$,*

$$\mu(E) \leq C \text{cap}_{\frac{2}{p}, p'}^{\partial\Omega}(E), \tag{3.12}$$

then the conclusions of Theorem 3 hold.

Remark. Notice that by Corollary 1, Corollary 2 and the Remark after Theorem 3 we cover the full range $(p, q) \in (1, \infty) \times (1, 2)$ and show that only one Bessel capacity is involved.

4 Separable solutions

Separable solutions of (1.1) are expressed in spherical coordinates $x = (r, \sigma)$ in $\mathbb{R}^N \sim \mathbb{R}_+ \times S^{N-1}$ under the form

$$u(x) = u(r, s) = r^{-\alpha} \omega(s).$$

For equation (1.1), the existence of such solutions in a cone $C_S := (0, \infty) \times S$ generated by a spherical domain $S \subseteq S^{N-1}$, imposes $q = \frac{2p}{p+1}$ and $\alpha = \frac{2}{p-1}$. Then ω satisfies

$$\begin{aligned} \mathcal{S}_{p,M} \omega := & -\Delta' \omega + \alpha(N-2-\alpha)\omega + |\omega|^{p-1} \omega \\ & - M(\alpha^2 \omega^2 + |\nabla' \omega|^2)^{\frac{p}{p+1}} = 0 \quad \text{in } S, \end{aligned} \quad (4.1)$$

where Δ' is the Laplace-Beltrami operator on S^{N-1} .

1- If $S = S^{N-1}$ positive solutions are unique and constant. The problem is completely solvable.

2- If we are dealing with boundary singularities, the model case is $S = S_+^{N-1}$ and the problem of boundary isolated singularities becomes

$$\mathcal{S}_{p,M} \omega = 0 \quad \text{in } S_+^{N-1} \quad \text{and} \quad \omega = 0 \quad \text{on } \partial S_+^{N-1}. \quad (4.2)$$

When $M = 0$ it is proved in [15] that there exists no positive solution if $p \geq \frac{N+1}{N-1}$. Our main result is the following,

Theorem 4.1 *There exists a positive solution ω to problem (4.2) if one of the following conditions is satisfied:*

- (i) either $1 < p < \frac{N+1}{N-1}$ and $M \geq 0$,
- (ii) or $p = \frac{N+1}{N-1}$ and $M > 0$,
- (iii) or $1 < p < 3$ or $p > \frac{N+1}{N-1}$, and $M \geq M_{N,p}$ for some explicit value $M_{N,p} > 0$.

Abridged proof. Existence is obtained by construction of supersolutions (actually large enough constants) and subsolutions under the form $\delta \phi_1$ where ϕ_1 is the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S^{N-1})$ and $\delta > 0$ is small enough. Thus $\mathcal{S}_{p,M}(\delta \phi_1) \leq 0$ and existence follows again by [11]. \square

The existence result (iii) is rather sharp since

Theorem 4.2 *Let $p > \frac{N+1}{N-1}$. If $M \leq m^{**}$, defined by (2.5), there exists no positive solution ω to problem (4.2).*

The proof is delicate and based upon the transformation $\omega = \eta^b$, $b > 0$.

5 Singular solutions

In the subcritical case $1 < p < \frac{N+1}{N-1}$, $0 < q < \frac{N+1}{N}$, then for any $M > 0$ and $k > 0$ there exist *minimal fundamental solutions* that are positive solutions of (1.1) in Ω , vanishing on $\partial\Omega \setminus \{0\}$ and such that

$$\lim_{x \rightarrow 0} \frac{u_k(x)}{\mathbb{P}_\Omega(x)} = k. \quad (5.1)$$

They are solutions of $\mathcal{L}_{p,q,M}u = 0$ in Ω such that $u = k\delta_0$ on $\partial\Omega$.

The correspondence $k \mapsto u_k$ is increasing (between minimal solutions since uniqueness may not hold) and there holds

$$\lim_{x \rightarrow 0} \frac{u_\infty(x)}{\mathbb{P}_\Omega(x)} = \infty. \quad (5.2)$$

Since the functions u_k are uniformly locally bounded from above in $\bar{\Omega} \setminus \{0\}$ by estimate (2.6), there exists $u_\infty = \lim_{k \rightarrow \infty} u_k$.

In order to characterise u_∞ we introduce the following problem,

$$\begin{aligned} -\Delta' \psi + \alpha(N-2-\alpha)\psi + |\psi|^{p-1}\psi &= 0 && \text{in } S_+^{N-1} \\ \psi &= 0 && \text{in } \partial S_+^{N-1}. \end{aligned} \quad (5.3)$$

Existence and uniqueness of a positive solution of (5.3) when $1 < q < \frac{N+1}{N-1}$ has been proved in [15]. In order to describe the singularity at 0, we assume that $\partial\mathbb{R}_+^N \sim \mathbb{R}^{N-1}$ is the tangent hyperplane to $\partial\Omega$ at 0 and the normal vector \mathbf{e}_N is the inner unit normal vector to Ω at 0. We will say that Ω is *in normal situation at 0*. Our main result concerning the behaviour of a positive solution near an isolated singularity on the boundary is the following.

Theorem 5.1 *Let Ω be a smooth C^2 domain with $0 \in \partial\Omega$ in normal situation at 0, $1 < p < \frac{N+1}{N-1}$, $1 < q < \frac{N+1}{N}$ and $M > 0$. Assume that u is a positive function satisfying of (2.1).*

1- *If $1 < q < \frac{2p}{p+1}$, then*

(i) *either*

$$\lim_{r \rightarrow 0} r^\alpha u(r, \cdot) = \psi \quad \text{locally uniformly on } S_+^{N-1}, \quad (5.4)$$

where ψ is the unique positive solution of (5.3),

(ii) *or there exists $k \geq 0$ such that (5.1) holds. If $k = 0$, then $u \equiv 0$ in Ω .*

2- *If $q = \frac{2p}{p+1}$, then*

(i) *either $u \geq u_\infty$ and*

$$\psi \leq \liminf_{r \rightarrow 0} r^\alpha u(r, \cdot) \leq \limsup_{r \rightarrow 0} r^\alpha u(r, \cdot) = \bar{\omega} \quad \text{locally uniformly on } S_+^{N-1}, \quad (5.5)$$

where $\bar{\omega}$ is the maximal positive solution of (4.2),

(ii) *or the statement (ii) in case 1 holds.*

Proof. The proof is lengthy and use the reduction of the problem to a quasi-autonomous second order equation as it was done in [15]. \square

Remark. In case 1-(i) $u = u_\infty$ is the unique positive solution of (1.1) vanishing on $\partial\Omega \setminus \{0\}$ and satisfying (5.2).

If $\frac{2p}{p+1} < q < p$ the blow-up is modelled by the eikonal equation (1.4). The blow-up rate is governed by $r^{-\gamma}$ where the exponent γ is

$$\gamma = \frac{q}{p-q}.$$

Notice that in this range $\gamma > \alpha$. This equation is essentially isotropic, hence it is difficult to construct singular solutions vanishing on the boundary except one point. By a delicate construction with sub and super solutions, we prove:

Theorem 5.2 *Assume $M > 0$, $p > 1$ and $\frac{2p}{p+1} < q < \min\{2, p\}$. Then there exists a positive solution u of (1.1) in \mathbb{R}_+^N , which vanishes on $\partial\mathbb{R}_+^N \setminus \{0\}$ such that*

$$c_3\phi_1(\sigma)r^{-\gamma} \leq u(r, s) \leq c_4 \max\left\{r^{-\alpha}, M^{\frac{1}{p-q}}r^{-\gamma}\right\}, \quad (5.6)$$

for all $(r, s) \in (0, r^*) \times \overline{S_+^{N-1}}$, for some $r^* \in (0, \infty]$ and where $c_3, c_4 > 0$ depend N, p, q . If $Nq > (N-1)p$, then $r^* = \infty$.

The result can be adapted to a solution in a bounded domain Ω with an isolated singularity at the point $0 \in \partial\Omega$.

6 Open problems

Problem 1. It is proved by Bidaut-Véron, Garcia-Huidobro, Véron that if

$$\max\left\{\frac{N}{N-1}, \frac{2p}{p+1}\right\} < q < \min\{2, p\} \quad \text{and} \quad M > 0,$$

there exist infinitely many radial solutions of (1.1) in $B_R \setminus \{0\}$ for small R , which satisfy

$$u(r) = \xi_M r^{-\beta}(1 + o(1)) \quad \text{as } r \rightarrow 0, \quad (6.1)$$

where

$$\beta = \frac{2-q}{q-1} \quad \text{and} \quad \xi_M = \frac{1}{\beta} \left(\frac{(N-1)q - N}{M(p-1)} \right)^{\frac{1}{p-1}}. \quad (6.2)$$

These solutions present the property that their blow-up is smaller than the one of the explicit radial separable solution. It would be interesting to construct similar solutions of (1.1) in \mathbb{R}_+^N (or more likely B_R^+), vanishing on $\partial\mathbb{R}^N \setminus \{0\}$.

Problem 2. Is it possible to define a boundary trace for any positive solution of (1.1) in \mathbb{R}_+^N , noting the fact that such a result holds separately for positive solutions of (1.3) and (1.5) ? Notice that if $u \in L_\rho^p(\Omega)$, then the theory of nonnegative superharmonic, up to a perturbation in $L_\rho^1(\Omega)$ functions [12], applies: thus $\nabla u \in L_\rho^q(\Omega)$ and there exists a nonnegative Radon measure μ such that u solves (3.1). Next, if $\nabla u \in L_\rho^q(\Omega)$ then the theory of boundary trace of positive solutions of Emden-Fowler equation as it is developed in [17] can be easily adapted. In that case there exists a closed set $\mathcal{S} \subset \partial\Omega$ and a nonnegative Radon measure μ in $\mathcal{R} := \partial\Omega \setminus \mathcal{S}$ such that

$$\lim_{\tau \rightarrow 0} \int_{\{x: \rho(x)=\tau\} \cap B_\epsilon(x)} u dS(x) = \infty, \quad (6.3)$$

for all $x \in \mathcal{S}$ and $\epsilon > 0$, and

$$\lim_{\tau \rightarrow 0} \int_{\{x: \rho(x)=\tau\}} \zeta(x) u dS(x) = \int_{\mathcal{R}} \zeta d\mu \quad (6.4)$$

for all $\zeta \in C(\overline{\Omega})$ vanishing in a neighborhood of \mathcal{S} . The difficulty for equation (1.1) comes from the situation where at some boundary points x there holds

$$\int_{\Omega \cap B_\epsilon(x)} u^p \rho dx = \int_{\Omega \cap B_\epsilon(x)} |\nabla u|^q \rho dx = \infty, \quad (6.5)$$

for some $\epsilon > 0$.

Problem 3. Are the weak solutions of the Dirichlet problem with measure boundary data (3.1) unique ? Note that there are a few uniqueness results for solutions with a boundary isolated singularities which can be obtained by using scaling techniques (under geometric restrictions on the domain).

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