

Online Appendix to “Pecuniary Externalities in Competitive Economies with Limited Pledgeability”

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Abstract

This is the online appendix to [Martins-da-Rocha et al. \(2022\)](#) (henceforth MPV). In this document, we present the proofs of the results and provide details regarding technical arguments. Additional issues are also discussed.

A Proofs of the Results in Sections 2 and 3

A.1 Proof of Proposition 2.1

Let $(q, (c^i, a^i, D^i)_{i \in I})$ be an equilibrium with limited pledgeability. Since pledgeable income is nonnegligible, we must have

$$\sum_{i \in I} \text{PV}(y^i | s^0) \leq \frac{1}{\varepsilon} \sum_{i \in I} \text{PV}(\ell^i | s^0).$$

By the decomposition property property (2.4), we have that $\text{PV}(\ell^i | s^0) < \infty$ for each agent i , so we deduce that the aggregate wealth of the economy $\sum_{i \in I} \text{PV}(y^i | s^0)$ must be finite. Since consumption markets clear, we obtain that the present value of optimal consumption is finite for all agents. In addition, due to the Inada’s condition, the optimal consumption is

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strictly positive.¹ Lemma A.1 in [Martins-da-Rocha and Vailakis \(2017\)](#) then implies that the following market transversality condition holds true:²

$$\lim_{t \rightarrow \infty} \sum_{s^t \in S^t} p(s^t)[a^i(s^t) + D^i(s^t)] = 0. \quad (\text{A.1})$$

The decomposition property property (2.4) implies that, for each i , there exists a nonnegative discounted martingale process M^i such that $D^i = \text{PV}(\ell^i) + M^i$. Condition A.1 can then be rewritten as follows:

$$\lim_{t \rightarrow \infty} \sum_{s^t \in S^t} p(s^t)a^i(s^t) = -p(s^0)M^i(s^0).$$

Since bond markets clear, we deduce that $\sum_{i \in I} M^i(s^0) = 0$, proving the desired result: $M^i = 0$ for each i .

A.2 Proof of Theorem 3.1

The proof of Theorem 3.1 in MPV exploits two intermediate results. The first and crucial observation, that has no analogue in the absence of output contraction, is to show that the present value of foregone endowment imposes a lower bound on not-too-tight debt limits. A direct implication of this property is that the process $\text{PV}(\ell^i)$ is finite. This is summarized in the following lemma.

Lemma A.1. *Not-too-tight debt limits are at least as large as the present value of endowment losses, i.e., for each agent i , $D^i(s^t) \geq \text{PV}(\ell^i|s^t)$ at any event s^t .*

A natural approach to prove this result is to show that $D^i(s^t) \geq \ell^i(s^t) + \tilde{D}^i(s^t)$, where $\tilde{D}^i(s^t) := \sum_{s^{t+1} \succ_{s^t} s^t} q(s^{t+1})D^i(s^{t+1})$ is the present value of next period's debt limits, and then use a standard iteration argument. Because, in equilibrium, debt limits are not too tight, this is equivalent to proving that agent i does not have an incentive to default when her net asset position is $\ell^i(s^t) + \tilde{D}^i(s^t)$, i.e.,

$$V^i(D^i, -\ell^i(s^t) - \tilde{D}^i(s^t)|s^t) \geq V_{\ell^i}^i(0, 0|s^t). \quad (\text{A.2})$$

¹See the supplemental material of [Martins-da-Rocha and Santos \(2019\)](#) for a detailed proof.

²The market transversality condition differs from the individual transversality condition. Indeed, due to the lack of commitment, agent i 's debt limits may bind, in which case we do not necessarily have that $p(s^t) = \beta^t \pi(s^t)u'(c^i(s^t))/u'(c^i(s^0))$.

By definition, the value function $V_{\ell^i}^i$ satisfies:

$$V_{\ell^i}^i(0, 0|s^t) \geq u(y^i(s^t) - \ell^i(s^t)) + \beta \sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) V_{\ell^i}^i(0, 0|s^{t+1}). \quad (\text{A.3})$$

If we had an equality in (A.3), then inequality (A.2) would be straightforward. Indeed, consuming $y^i(s^t) - \ell^i(s^t)$ and borrowing up to each debt limit $D^i(s^{t+1})$ at event s^t leads to the right-hand side continuation utility in (A.3) and satisfies the solvency constraint at event s^t in the budget set defining the left-hand side of (A.2). Unfortunately, in our environment where agents can save upon default condition (A.3) may not hold as an equality.³ Overcoming this problem is the technical challenge in the proof of Lemma A.1. The formal argument is presented below.

The second observation is that the process $\text{PV}(\ell^i)$ of present values of endowment losses, when it is finite, is itself not too tight. The following lemma provides the formal statement. The proof follows from a simple translation invariance of the flow budget constraints.

Lemma A.2. *If $\text{PV}(\ell^i|s^0)$ is finite, then the process $\text{PV}(\ell^i)$ is not too tight, i.e.,*

$$V^i(\text{PV}(\ell^i), -\text{PV}(\ell^i|s^t)|s^t) = V_{\ell^i}^i(0, 0|s^t), \quad \forall s^t \succeq s^0.$$

Equipped with Lemma A.1 and Lemma A.2, we can now provide a simple proof of Theorem 3.1 in MPV.

Proof of Theorem 3.1. Fix a process D^i of not-too-tight debt limits. Lemma A.1 implies that $\text{PV}(\ell^i|s^0)$ is finite. From Lemma A.2 we also deduce that the process $\underline{D}^i := \text{PV}(\ell^i)$ is not too tight. [Martins-da-Rocha and Santos \(2019\)](#) show that the difference between two processes of not-too-tight debt limits must be an exact rollover process. Therefore, there exists a process M^i satisfying the exact rollover property such that $D^i = \underline{D}^i + M^i$. By Lemma A.1, $D^i \geq \underline{D}^i$, in which case the process M^i must be nonnegative. \square

A.2.1 Proof of Lemma A.1

Since we are exclusively concerned with the single-agent problem, we simplify notation by dropping the superscript i . Let D be a process of not-too-tight limits. We first show that

³In the simpler environment where, upon default, saving is not possible (as it is the case in [Alvarez and Jermann 2000](#)) condition (A.3) always hold as an equality.

there exists a nonnegative process \underline{D} satisfying

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D(s^{t+1}), \underline{D}(s^{t+1})\}, \quad \text{for all } s^t \succeq s^0. \quad (\text{A.4})$$

Indeed, let Φ be the mapping $B \in \mathbb{R}^\Sigma \mapsto \Phi B \in \mathbb{R}^\Sigma$ defined by

$$(\Phi B)(s^t) := \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D(s^{t+1}), B(s^{t+1})\}, \quad \text{for all } s^t \succeq s^0.$$

Denote by $[0, \bar{D}]$ the set of all processes $B \in \mathbb{R}^\Sigma$ satisfying $0 \leq B \leq \bar{D}$ where

$$\bar{D}(s^t) := \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D(s^{t+1}), \quad \text{for all } s^t \succeq s^0.$$

The mapping Φ is continuous (for the product topology), and we have $\Phi[0, \bar{D}] \subseteq [0, \bar{D}]$. Since $[0, \bar{D}]$ is convex and compact (for the product topology), it follows that Φ admits a fixed point \underline{D} in $[0, \bar{D}]$.

Claim A.1. *The process \underline{D} is tighter than the process D , i.e., $\underline{D} \leq D$.*

Proof of Claim A.1. Fix a node s^t . Since $V_\ell(0, 0|s^t) = V(D, -D(s^t)|s^t)$ and $V(D, \cdot|s^t)$ is strictly increasing, it is sufficient to show that $V(D, -\underline{D}(s^t)|s^t) \geq V_\ell(0, 0|s^t)$. Denote by (c, \tilde{a}) the optimal consumption and bond holdings in the budget set $B_\ell(0, 0|s^t)$ for some arbitrary event s^t .⁴ We let \hat{D} be the process defined by $\hat{D}(s^t) := \min\{D(s^t), \underline{D}(s^t)\}$ for all s^t . Observe that

$$\begin{aligned} y(s^t) - \underline{D}(s^t) &= y(s^t) - \ell(s^t) - \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \hat{D}(s^{t+1}) \\ &= c(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) [\tilde{a}(s^{t+1}) - \hat{D}(s^{t+1})] \\ &= c(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a(s^{t+1}) \end{aligned}$$

where $a(s^{t+1}) := \tilde{a}(s^{t+1}) - \hat{D}(s^{t+1})$. Since $\hat{D} \leq D$, we have $a(s^{t+1}) \geq -D(s^{t+1})$. At any

⁴That is, the process \tilde{a} supports consumption c such that $U(c|s^t) := V_\ell(0, 0|s^t)$.

successor event $s^{t+1} \succ s^t$, we have

$$\begin{aligned}
y(s^{t+1}) + a(s^{t+1}) &= y(s^{t+1}) + \tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1}) \\
&\geq y(s^{t+1}) + \tilde{a}(s^{t+1}) - \underline{D}(s^{t+1}) \\
&\geq y(s^{t+1}) - \ell(s^{t+1}) + \tilde{a}(s^{t+1}) - \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) \widehat{D}(s^{t+2}) \\
&\geq c(s^{t+2}) + \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) [\tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})] \\
&\geq c(s^{t+2}) + \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) a(s^{t+2})
\end{aligned}$$

where $a(s^{t+2}) := \tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})$.⁵ Observe that $a(s^{t+2}) \geq -D(s^{t+2})$ as $\widehat{D} \leq D$.

Defining $a(s^\tau) := \tilde{a}(s^\tau) - \widehat{D}(s^\tau)$ for any successor $s^\tau \succ s^t$ and iterating the above argument, we can show that (c, a) belongs to the budget set $B(D, -\underline{D}(s^t)|s^t)$. It follows that

$$V(D, -\underline{D}(s^t)|s^t) \geq U(c|s^t) = V_\ell(0, 0|s^t)$$

implying the desired result: $\underline{D}(s^t) \leq D(s^t)$. □

It follows from Claim A.1 that \underline{D} satisfies

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \underline{D}(s^{t+1}), \quad \text{for all } s^t \succeq s^0. \quad (\text{A.5})$$

Applying equation (A.5) recursively, we get

$$\begin{aligned}
p(s^t) \underline{D}(s^t) &= p(s^t) \ell(s^t) + \sum_{s^{t+1} \in S^{t+1}(s^t)} p(s^{t+1}) \ell(s^{t+1}) + \dots \\
&\quad \dots + \sum_{s^T \in S^T(s^t)} p(s^T) \ell(s^T) + \sum_{s^{T+1} \in S^{T+1}(s^t)} p(s^{T+1}) \underline{D}(s^{T+1})
\end{aligned}$$

for any $T > t$. Since \underline{D} is nonnegative, it follows that

$$p(s^t) \underline{D}(s^t) \geq \sum_{\tau=t}^T \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau) \ell(s^\tau).$$

Passing to the limit when T goes to infinity, we get that $\text{PV}(\ell|s^t)$ is finite for any event s^t (in particular for s^0). Recalling that $D \geq \underline{D}$, we also get that $D(s^t) \geq \text{PV}(\ell|s^t)$.

⁵To get the second weak inequality, we use equation (A.4).

A.2.2 Proof of Lemma A.2

Denote by (c, \tilde{a}) the optimal consumption and bond holdings in the budget set $B_\ell(0, 0|s^t)$ for some arbitrary event s^t . We pose $\underline{D} := \text{PV}(\ell)$ and observe that

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \underline{D}(s^{t+1}).$$

It is easy to show that (c, a) is optimal in the budget set $B(\underline{D}, -\underline{D}(s^0)|s^t)$ where $a := \tilde{a} - \underline{D}$. We then deduce that $V^i(\underline{D}, -\underline{D}(s^t)|s^t) = V_\ell(0, 0|s^t)$, so proving the claim.

A.3 Derivation of the Asset Price Equation (3.10)

Fix an event $s^t \succeq s^0$. Market clearing implies that there exists at least one agent $i \in I$ holding a positive amount $\alpha^i(s^t) > 0$ of the tree shares. Fix $\varepsilon \in \mathbb{R}$ such that $\varepsilon \geq -\alpha^i(s^t)$. The following changes in contingent claims and equity's holding are admissible

$$\tilde{\alpha}^i(s^t) := \alpha^i(s^t) + \varepsilon \quad \text{and} \quad \tilde{b}^i(s^{t+1}) := b^i(s^{t+1}) - \varepsilon[P(s^{t+1}) + \delta(s^{t+1})].$$

Since the agent's welfare cannot improve after these changes, we must have

$$P(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1})[P(s^{t+1}) + \delta(s^{t+1})]. \tag{A.6}$$

Given this recursive equation, it follows that $\text{PV}(\delta|s^0)$ is finite. Moreover, for every event s^t , the following limit

$$M(s^t) = \lim_{\tau \rightarrow \infty} \frac{1}{p(s^t)} \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau) P(s^\tau)$$

is well-defined, so we obtain Equation (3.10).

Remark A.1. The proof that the equity's price satisfies the asset-pricing recursive equation (A.6) relies on a standard no-arbitrage argument. This differs from the proof that endowment losses have finite present value (Lemma A.1) where the recursive equation is obtained by means of fixed point of a suitably defined operator on debt limits.

A.4 Proof of Theorem 3.2

Consider first the collateralized debt model where δ denotes the process of dividends of the long-lived tree, and $(\alpha^i(s^{-1}))_{i \in I}$ is the initial allocation of tree holdings. Fix a laissez-

faire equilibrium $(q, P, (c^i, \alpha^i, b^i, 0)_{i \in I})$.⁶ Denote by M the bubble component of the tree's price and choose an arbitrary decomposition $M = \sum_{i \in I} M^i$ where each M^i is a nonnegative exact rollover process. Then, from the decomposition of debt limits in Theorem 3.1 in MPV, it is straightforward to see that the collection $(q, (c^i, a^i, D^i)_{i \in I})$ is a laissez-faire equilibrium in the reputation debt model, where, for every s^t , endowment losses are given by $\ell^i(s^t) := \alpha^i(s^{-1})\delta(s^t)$, debt limits are given by:

$$D^i := \text{PV}(\ell^i) + M^i, \quad (\text{A.7})$$

and contingent claims are given by

$$a^i(s^t) := b^i(s^t) + \alpha^i(\sigma(s^t))[P(s^t) + \delta(s^t)] - D^i(s^t). \quad (\text{A.8})$$

Reciprocally, let $(q, (c^i, a^i, D^i)_{i \in I})$ be a laissez-faire equilibrium in the reputation debt model where endowment losses $(\ell^i)_{i \in I}$ satisfy $\ell^i(s^t) = \alpha^i(s^{-1})\delta(s^t)$ for each event s^t . Recall that $D^i = \text{PV}(\ell^i) + M^i$ where M^i is agent i 's credit bubble. Fix any family $(\alpha^i)_{i \in I}$ of equity shares satisfying market clearing.⁷ Then $(q, P, (c^i, \alpha^i, b^i, 0)_{i \in I})$ constitutes an equilibrium with collateralized debt where δ is the dividend process, equity is given by

$$P := \text{PV}(\delta) - \delta + M, \quad \text{where} \quad M = \sum_{i \in I} M^i, \quad (\text{A.9})$$

and bond holdings are defined by

$$b^i(s^t) := a^i(s^t) + D^i(s^t) - \alpha^i(\sigma(s^t))[P(s^t) + \delta(s^t)]. \quad (\text{A.10})$$

B Omitted Arguments and Proofs of Section 4

B.1 Proof of Theorem 4.1

Let $(q, (c^i, a^i, D^i)_{i \in I})$ be an equilibrium with not-too-tight debt constraints. Since endowment losses are a nonnegligible fraction of aggregate resources, initial endowments have finite present value (see Proposition 2.1). We can then apply [Bloise and Reichlin \(2011\)](#) to

⁶Recall that the debt limits \tilde{D}^i are not too tight if, and only if, they are equal to zero.

⁷In the sense that $\sum_{i \in I} \alpha^i(s^t) = 1$ for all $s^t \succeq s^0$.

deduce that it is not possible to Pareto dominate $(c^i)_{i \in I}$ by another feasible consumption allocation $(\hat{c}^i)_{i \in I}$ that satisfies the participation constraints

$$U^i(\hat{c}^i | s^t) \geq U^i(y^i - \ell^i | s^t)$$

for every agent i and every event $s^t \succeq s^0$. This is sufficient to get the desired result. Indeed, assume by way of contradiction, that there exists another equilibrium $(\hat{q}, (\hat{c}^i, \hat{a}^i, \hat{D}^i)_{i \in I})$ with self-enforcing debt constraints such that the consumption allocation $(\hat{c}^i)_{i \in I}$ Pareto dominates $(c^i)_{i \in I}$. Since (\hat{c}^i, \hat{a}^i) is optimal in the budget set $B^i(\hat{D}^i, a^i(s^0) | s^0)$, it follows from the Principle of Optimality that (\hat{c}^i, \hat{a}^i) is optimal in $B^i(\hat{D}^i, \hat{a}^i(s^t) | s^t)$ for any event s^t . Since $\hat{a}^i(s^t) \geq -\hat{D}^i(s^t)$, we deduce that

$$U^i(\hat{c}^i | s^t) = V^i(\hat{D}^i, \hat{a}^i(s^t) | s^t) \geq V^i(\hat{D}^i, -\hat{D}^i(s^t) | s^t).$$

Since the debt limits \hat{D}^i are self-enforcing, we deduce that $U^i(\hat{c}^i | s^t) \geq U^i(y^i - \ell^i | s^t)$: a contradiction.

B.2 Tight Reputation Debt Limits

In Section 4.3.2 of MPV, we fixed a parameter $d_1 \in [0, y_H)$ representing the debt issued at period $t = 0$, and look for an equilibrium such that $c_0^{i,\varepsilon} = y_0$, $a_1^{i,\varepsilon}(z^i) = -d_1$, $a_1^{i,\varepsilon}(z^j) = d_1$ for $j \neq i$, and $D_1^{i,\varepsilon}(z^i) = d_1$. Since at period $t = 2$, the economy settles in the cyclical steady-state described in Claim 4.2, bond holdings at the end of period $t = 1$ should be $a_2^{i,\varepsilon}(z) = d(\varepsilon)$ if $y_2^i(z) = y_L$, and $a_2^{i,\varepsilon}(z) = -d(\varepsilon)$ if $y_2^i(z) = y_H$. This in turn implies that the corresponding consumption levels at $t = 1$ are given by $c_1^{i,\varepsilon}(z^i) = c_{1,H}(\varepsilon, d_1)$ and $c_1^{i,\varepsilon}(z^j) = c_{1,L}(\varepsilon, d_1)$ for $j \neq i$ where

$$c_{1,H}(\varepsilon, d_1) := y_H - d_1 - q_2^\varepsilon(z)d(\varepsilon) \quad \text{and} \quad c_{1,L}(\varepsilon, d_1) := y_H - d_1 - q_2^\varepsilon(z)d(\varepsilon).$$

The bond prices $q_2^\varepsilon(z)$ at period $t = 1$ are determined by the first-order conditions:

$$q_2^\varepsilon(z) = \beta \frac{u'(c_L(\varepsilon))}{u'(c_{1,H}(\varepsilon, d_1))} =: q_2(\varepsilon, d_1), \quad \text{for } z \in \{z^a, z^b\}.$$

Similarly, the bond prices at period $t = 0$ are determined by the following first-order conditions:

$$q_1^\varepsilon(z) = \beta \pi_L \frac{u'(c_{1,L}(\varepsilon, d_1))}{u'(y_0)} =: q_1(\varepsilon, d_1), \quad \text{for } z \in \{z^a, z^b\}.$$

Optimality requires that:

$$\frac{u'(c_{1,L}(\varepsilon, d_1))}{u'(c_{1,H}(\varepsilon, d_1))} \geq \max \left\{ \frac{\pi_H}{\pi_L}, \frac{u'(c_H(\varepsilon))}{u'(c_L(\varepsilon))} \right\}. \quad (\text{B.1})$$

The inequality obtains from the first-order conditions of the borrowing decisions at $t = 0$ and $t = 1$, respectively, i.e.,

$$q_1(\varepsilon, d_1) \geq \beta \pi_H u'(c_{1,H}(\varepsilon, d_1)) / u'(y_0) \quad \text{and} \quad q_2(\varepsilon, d_1) \geq \beta u'(c_H(\varepsilon)) / u'(c_{1,L}(\varepsilon, d_1)).$$

We denote by $d_1(\varepsilon)$ the level of d_1 that is not too tight given that the debt limits at all successor periods $t \geq 2$ are set to be too tight (see Claim 4.2). The determination of $d_1(\varepsilon)$ requires that we do compute the value functions associated to equilibrium and out-of-equilibrium paths. For this purpose, we introduced the following notations. The continuation value function

$$U_{1,H}(\varepsilon, d_1) := V^i(D^{i,\varepsilon}, -d_1 | (z^i, 1))$$

corresponds to the largest continuation utility when the high-income agent starts at date $t = 1$ with debt d_1 . Observe that

$$U_{1,H}(\varepsilon, d_1) = u(c_{1,H}(\varepsilon, d_1)) + \beta U_L(\varepsilon) \quad \text{where} \quad U_L(\varepsilon) := \frac{u(c_L(\varepsilon)) + \beta u(c_H(\varepsilon))}{1 - \beta^2}.$$

To identify the self-enforcing and not-too-tight level $d_1(\varepsilon)$, we need to compute the value $W_{1,H}(\varepsilon, d_1)$ of the default option of the high-income agent at $t = 1$:

$$W_{1,H}(\varepsilon, d_1) := V_{\ell^*}^i(0, 0 | (z^i, 1)).$$

The value of the default option depends indirectly on the debt level d_1 as it is affected by the bond price $q_2(\varepsilon, d_1)$. An educated guess suggests that

$$W_{1,H}(\varepsilon, d_1) = u(\bar{c}_1(\varepsilon, d_1)) + \beta u(\bar{c}_2(\varepsilon, d_1)) + \beta^2 \left[\frac{u(\bar{c}_H(\varepsilon)) + \beta u(\bar{c}_L(\varepsilon))}{1 - \beta^2} \right]$$

where these consumption levels satisfy the flow budget constraint at $t = 1$

$$\bar{c}_1(\varepsilon, d_1) + q_2(\varepsilon, d_1) \bar{\theta}_2(\varepsilon, d_1) = y_H - \ell^* \quad \text{with} \quad \bar{\theta}_2(\varepsilon, d_1) \geq 0,$$

the flow budget constraint at $t = 2$

$$\bar{c}_2(\varepsilon, d_1) = y_L - \ell^* + \bar{\theta}_2(\varepsilon, d_1),$$

the flow budget constraint at any date $t \geq 3$ when agents are receiving high-income

$$\bar{c}_H(\varepsilon) + q(\varepsilon)\bar{\theta}(\varepsilon) = y_H - \ell^* \quad \text{with} \quad \bar{\theta}(\varepsilon) \geq 0,$$

and the flow budget constraint at any date $t \geq 3$ when agents are receiving low-income

$$\bar{c}_L(\varepsilon) = y_L - \ell^* + \bar{\theta}(\varepsilon).$$

The level $\bar{\theta}_2(\varepsilon, d_1)$ of savings at $t = 1$ is optimal when

$$q_2(\varepsilon, d_1) = \beta \frac{u'(\bar{c}_2(\varepsilon, d_1))}{u'(\bar{c}_1(\varepsilon, d_1))} = \beta \frac{u'(y_L - \ell^* + \bar{\theta}_2(\varepsilon, d_1))}{u'(y_H - \ell^* - q_2(\varepsilon, d_1)\bar{\theta}_2(\varepsilon, d_1))}. \quad (\text{B.2})$$

The decision for not saving at $t = 2$ is optimal when

$$q(\varepsilon) \geq \beta \frac{u'(\bar{c}_H(\varepsilon))}{u'(\bar{c}_2(\varepsilon, d_1))}. \quad (\text{B.3})$$

The level $\bar{\theta}(\varepsilon)$ of savings at any date $t \geq 3$ when receiving high-income is optimal when

$$q(\varepsilon) = \beta \frac{u'(\bar{c}_L(\varepsilon))}{u'(\bar{c}_H(\varepsilon))} = \beta \frac{u'(y_L - \ell^* + \bar{\theta}(\varepsilon))}{u'(y_H - \ell^* - q(\varepsilon)\bar{\theta}(\varepsilon))}. \quad (\text{B.4})$$

The decision for not saving at any date $t \geq 3$ when receiving low-income is optimal when

$$q(\varepsilon) \geq \beta \frac{u'(\bar{c}_H(\varepsilon))}{u'(\bar{c}_L(\varepsilon))}, \quad \text{or, equivalently,} \quad \bar{c}_H(\varepsilon) \geq \bar{c}_L(\varepsilon). \quad (\text{B.5})$$

The level $d_1(\varepsilon)$ is determined by solving the following equation

$$U_{1,H}(\varepsilon, d_1(\varepsilon)) = W_{1,H}(\varepsilon, d_1(\varepsilon)).$$

Equivalently, $d_1(\varepsilon)$ solves

$$\begin{aligned} u(y_H - d_1 - q_2(\varepsilon, d_1)d(\varepsilon)) + \beta U_L(\varepsilon) &= u(y_H - \ell^* - q_2(\varepsilon, d_1)\bar{\theta}_2(\varepsilon, d_1)) \\ &+ \beta u(y_L - \ell^* + \bar{\theta}_2(\varepsilon, d_1)) \\ &+ \beta^2 \left[\frac{u(y_H - \ell^* - q(\varepsilon)\bar{\theta}(\varepsilon)) + \beta u(y_L - \ell^* + \bar{\theta}(\varepsilon))}{1 - \beta^2} \right]. \end{aligned}$$

When $u = \ln$, the values of $\bar{\theta}_2(\varepsilon, d_1)$ and $\bar{\theta}(\varepsilon)$ are given by the following equations

$$\bar{\theta}_2(\varepsilon, d_1) = \frac{\beta(y_H - \ell^*) - q_2(\varepsilon, d_1)(y_L - \ell^*)}{q_2(\varepsilon, d_1)(1 + \beta)}.$$

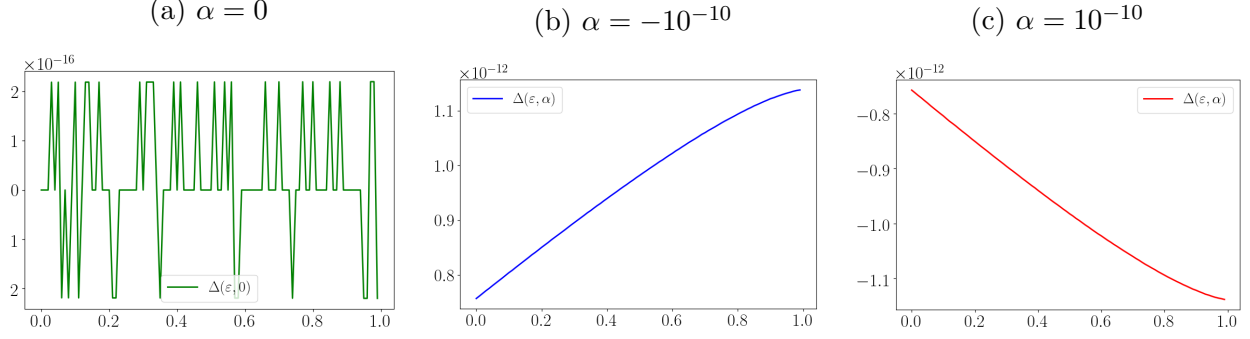


Figure B.1: Approximate Relative Difference $\Delta(\alpha, \varepsilon)$ as a function of the tightening parameter ε .

and

$$\bar{\theta}(\varepsilon) = \frac{\beta(y_H - \ell^*) - q(\varepsilon)(y_L - \ell^*)}{q(\varepsilon)(1 + \beta)}.$$

To compute an approximate value of $d_1(\varepsilon)$, we use the function `fsolve` of the SciPy software in Python. The starting estimate is set to ℓ^* and the parameters are set such that the calculation terminates if the relative error between two consecutive iterates is at most 10^{-11} . To check the accuracy of $d_1(\varepsilon)$, we plot in Figure B.1 the approximate relative difference

$$\Delta(\varepsilon, \alpha) := \frac{U_{1,H}(\varepsilon, (1 + \alpha)d_1(\varepsilon)) - W_{1,H}(\varepsilon, (1 + \alpha)d_1(\varepsilon))}{U_{1,H}(\varepsilon, d_1(\varepsilon))}.$$

We consider the following simpler notations: for variables along the equilibrium path, let

$$c_{1,H}(\varepsilon) := c_{1,H}(\varepsilon, d_1(\varepsilon)), \quad c_{1,L}(\varepsilon) := c_{1,L}(\varepsilon, d_1(\varepsilon)) \quad \text{and} \quad q_t(\varepsilon) := q_t(\varepsilon, d_1(\varepsilon)), \quad \text{for } t \in \{1, 2\};$$

and for variables off the equilibrium path, let

$$\bar{c}_1(\varepsilon) := \bar{c}_1(\varepsilon, d_1(\varepsilon)), \quad \bar{c}_2(\varepsilon) := \bar{c}_2(\varepsilon, d_1(\varepsilon)) \quad \text{and} \quad \bar{\theta}_2(\varepsilon) := \bar{\theta}_2(\varepsilon, d_1(\varepsilon)).$$

Consolidating all the above arguments, we get that Claim 4.3 applies provided that conditions (B.1), (B.3) and (B.5) are satisfied. We show below that this is true for the utility specification and the parameter values we consider. Indeed, when $u = \ln$ Condition (B.1) reads

$$\frac{c_{1,H}(\varepsilon)}{c_{1,L}(\varepsilon)} \geq \max \left\{ \frac{\pi_H}{\pi_L}, \frac{c_L(\varepsilon)}{c_H(\varepsilon)} \right\}.$$

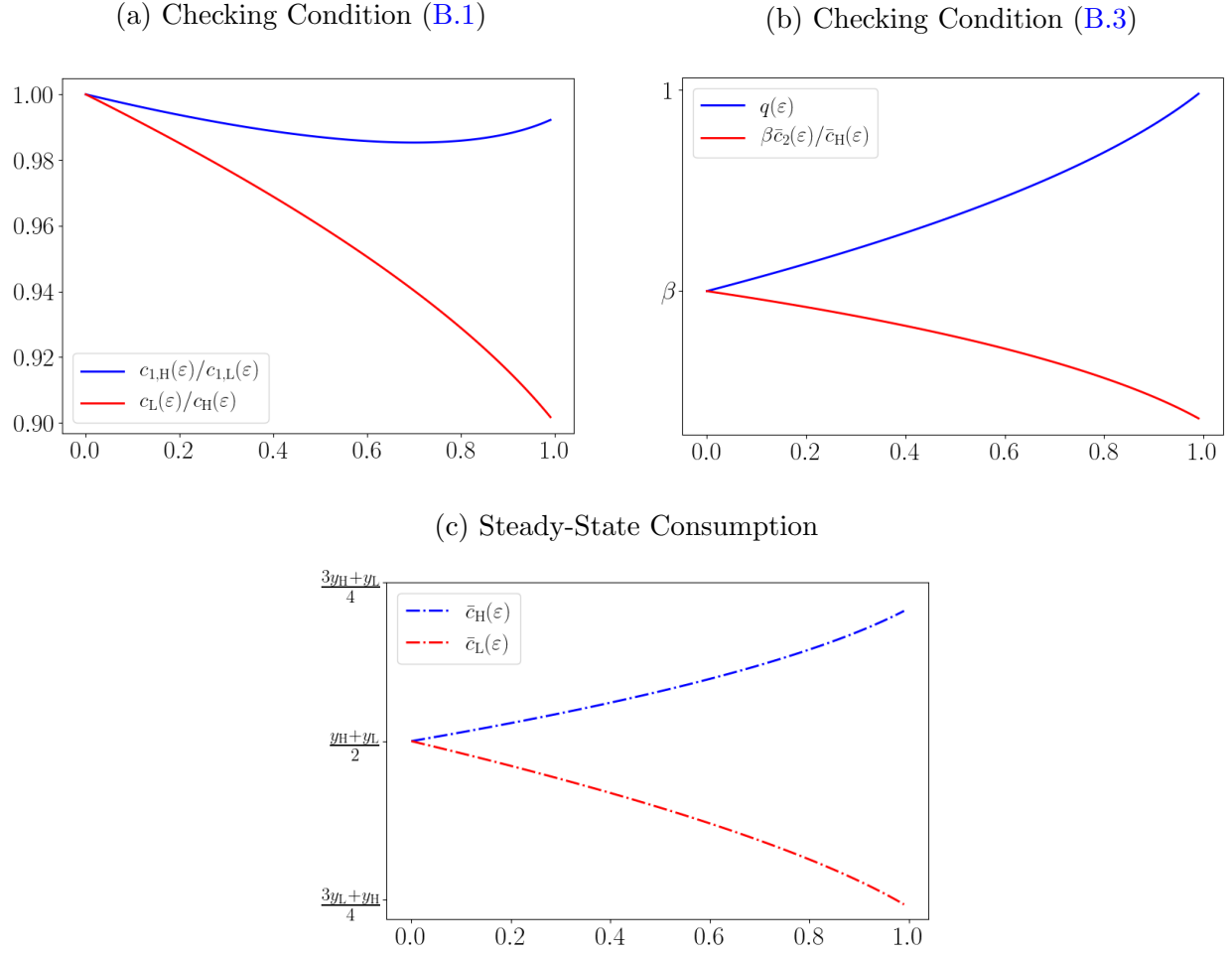


Figure B.2: Checking the FOC for borrowing decisions.

Since $\pi_H/\pi_L < 1/2$, Figure B.2(a) affirms that the above inequality holds true. When $u = \ln$ condition (B.3) reads

$$q(\epsilon) \geq \beta \frac{\bar{c}_2(\epsilon)}{\bar{c}_H(\epsilon)}.$$

Figure B.2(b) shows that the above inequality is also satisfied. Finally, Figure B.2(c) reveals that

$$\bar{c}_H(\epsilon) \geq \bar{c}_L(\epsilon),$$

so condition (B.5) is also satisfied.

B.3 Tight Collateral Constraints

B.3.1 Tightening Debt Limits for $t \geq 1$

Assume that $\eta_1 = 0$ and $\eta_t = \varepsilon > 0$ for every $t \geq 2$. We construct an equilibrium $(q^\varepsilon, (c^{i,\varepsilon}, a^{i,\varepsilon}, D^{i,\varepsilon})_{i \in I})$ where the debt limits satisfy $D_1^{i,\varepsilon}(z) = \text{PV}_1^\varepsilon(\ell^*|z)$ and $D_t^{i,\varepsilon}(z) = (1 - \varepsilon) \text{PV}_t^\varepsilon(\ell^*|z)$ for all $t \geq 2$. The characteristics of the equilibrium are as follows: the economy reaches at period $t = 3$ a cyclical steady-state $(q(\varepsilon), c_H(\varepsilon), c_L(\varepsilon), d(\varepsilon))$ similar to the one obtained in the model with reputation debt (i.e., Claim 4.2 applies for $t \geq 3$). In the transition periods $t \in \{1, 2\}$, consumption, asset holdings and debt limits are symmetric, i.e., for any $z \in \{z^a, z^b\}$,

$$c_t^{i,\varepsilon}(z) = \begin{cases} c_{t,H}(\varepsilon), & \text{if } y_t^i(z) = y_H, \\ c_{t,L}(\varepsilon), & \text{if } y_t^i(z) = y_L; \end{cases} \quad \text{and} \quad a_t^{i,\varepsilon}(z) = \begin{cases} -d_t(\varepsilon), & \text{if } y_t^i(z) = y_H, \\ d_t(\varepsilon), & \text{if } y_t^i(z) = y_L; \end{cases}$$

together with $q_{t+1}^\varepsilon(z) =: q_{t+1}(\varepsilon)$ and $D^{i,\varepsilon}(z) =: D_t(\varepsilon)$ where:

$$D_1(\varepsilon) = \ell^* \left[1 + q_2(\varepsilon) \left(1 + q_3(\varepsilon) \frac{1}{1 - q(\varepsilon)} \right) \right] \quad \text{and} \quad D_2(\varepsilon) = (1 - \varepsilon) \ell^* \left[1 + q_3(\varepsilon) \frac{1}{1 - q(\varepsilon)} \right].$$

At $t = 0$, both agents consume their endowment $c_0^{i,\varepsilon} = y_0$, with asset prices given by:

$$q_1^\varepsilon(z) = \beta \pi_L \frac{u'(c_{1,L}(\varepsilon))}{u'(y_0)}, \quad \text{for each } z \in \{z^a, z^b\}.$$

There are three threshold values $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < 1$ over which equilibrium characteristics differ. We present the main characteristics hereafter. A graphical illustration is given in Figure 4.4 in MPV.

For $\varepsilon \in [0, \varepsilon_1]$, both agents borrow up to the debt limit contingent to the high income at periods $t \in \{1, 2\}$. In this case, the cyclical steady-state is reached at period $t = 2$. Formally, we have $c_{2,L}(\varepsilon) = c_L(\varepsilon)$, $c_{2,H}(\varepsilon) = c_H(\varepsilon)$, $d_2(\varepsilon) = D_2(\varepsilon) = d(\varepsilon)$, $q_3(\varepsilon) = q(\varepsilon)$ with the remaining equilibrium variables be determined by the period-1 first-order condition

$$q_2(\varepsilon) = \beta \frac{u'(c_L(\varepsilon))}{u'(c_{1,H}(\varepsilon))},$$

the binding debt limit

$$d_1(\varepsilon) = D_1(\varepsilon),$$

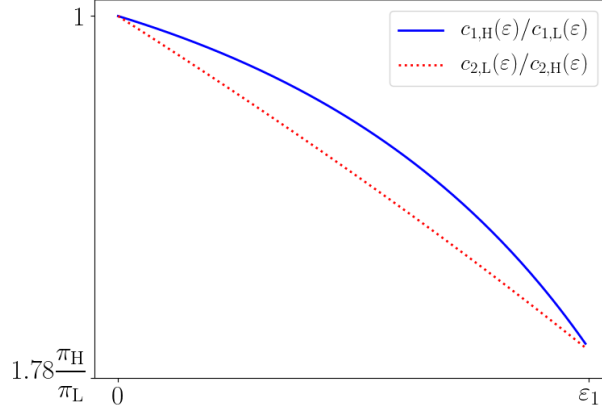


Figure B.3: Phase 1 when $\varepsilon \in [0, \varepsilon_1]$.

and the binding budget constraints:

$$c_{1,H}(\varepsilon) = y_H - d_1(\varepsilon) - q_2(\varepsilon)d(\varepsilon) \quad \text{and} \quad c_{1,L}(\varepsilon) = y_L + d_1(\varepsilon) + q_2(\varepsilon)d(\varepsilon).$$

The variables defined above form an equilibrium if, and only if, the following first-order conditions for the borrowing decisions at $t = 0$ and $t = 1$ are satisfied

$$\frac{u'(c_{1,L}(\varepsilon))}{u'(c_{1,H}(\varepsilon))} \geq \frac{\pi_H}{\pi_L} \quad \text{and} \quad \frac{u'(c_L(\varepsilon))}{u'(c_{1,H}(\varepsilon))} \geq \frac{u'(c_H(\varepsilon))}{u'(c_{1,L}(\varepsilon))}. \quad (\text{B.6})$$

The threshold value ε_1 is determined as the value of ε that equates consumption at periods $t = 1$ and $t = 2$, i.e., it corresponds to the solution of

$$c_{1,H}(\varepsilon) = c_{2,L}(\varepsilon) \quad \text{or} \quad c_{1,L}(\varepsilon) = c_{2,H}(\varepsilon). \quad (\text{B.7})$$

We refer to Figure 4.4(c) for the determination of ε_1 . The validity of (B.6) is presented in Figure B.3

For $\varepsilon \in (\varepsilon_1, \varepsilon_2]$, it is not anymore optimal to borrow up to the debt limit contingent to high income at period $t = 2$, i.e., $d_2(\varepsilon) < D_2(\varepsilon)$. This is because the debt at $t = 1$ is so large that we get perfect consumption smoothing between date $t = 1$ and $t = 2$,

$$c_{1,H}(\varepsilon) = c_{2,L}(\varepsilon) \quad \text{and} \quad c_{1,L}(\varepsilon) = c_{2,H}(\varepsilon).$$

This implies that $q_2(\varepsilon) = \beta$ with the remaining equilibrium variables be determined by the binding debt limit

$$d_1(\varepsilon) = D_1(\varepsilon),$$

the period-2 first-order condition associated to the saving's decision

$$q_3(\varepsilon) = \beta \frac{u'(c_L(\varepsilon))}{u'(c_{2,H}(\varepsilon))},$$

the two equations associated to perfect smoothing

$$\underbrace{y_H - D_1(\varepsilon) - \beta d_2(\varepsilon)}_{c_{1,H}} = \underbrace{y_L + d_2(\varepsilon) + q_3(\varepsilon)d(\varepsilon)}_{c_{2,L}}$$

and

$$\underbrace{y_L + D_1(\varepsilon) + \beta d_2(\varepsilon)}_{c_{1,L}} = \underbrace{y_H - d_2(\varepsilon) - q_3(\varepsilon)d(\varepsilon)}_{c_{2,H}},$$

the period-1 binding flow budget constraints

$$c_{1,H}(\varepsilon) = y_H - D_1(\varepsilon) - q_2(\varepsilon)d_2(\varepsilon) \quad \text{and} \quad c_{1,L}(\varepsilon) = y_L + D_1(\varepsilon) + q_2(\varepsilon)d_2(\varepsilon),$$

and the period-2 binding flow budget constraints

$$c_{2,H}(\varepsilon) = y_H - d_2(\varepsilon) - q_3(\varepsilon)d(\varepsilon) \quad \text{and} \quad c_{2,L}(\varepsilon) = y_L + d_2(\varepsilon) + q_3(\varepsilon)d(\varepsilon).$$

The variables defined above form an equilibrium if, and only if, the first-order conditions for the borrowing decisions at $t = 0$ and $t = 2$ are satisfied:

$$\frac{u'(c_{1,L}(\varepsilon))}{u'(c_{1,H}(\varepsilon))} \geq \frac{\pi_H}{\pi_L} \quad \text{and} \quad \frac{u'(c_L(\varepsilon))}{u'(c_{2,H}(\varepsilon))} \geq \frac{u'(c_H(\varepsilon))}{u'(c_{2,L}(\varepsilon))}, \quad (\text{B.8})$$

and the debt constraint at $t = 1$ is satisfied: $d_2(\varepsilon) \geq -D_2(\varepsilon)$. When ε get close to ε_2 , the debt $d_1(\varepsilon)$ contingent to high income at $t = 1$ is so large that the high-income agent borrows against his low income at $t = 2$, i.e., $d_2(\varepsilon) < 0$. The threshold value ε_2 is determined by the binding constraint $d_2(\varepsilon) = -D_2(\varepsilon)$. Debt limits and bond prices are presented in Figure 4.4(a) and Figure 4.4(b). The validity of (B.8) are presented in Figure B.4.

For $\varepsilon \in (\varepsilon_2, \varepsilon_3]$, agents borrow up to the debt limit contingent to high income at $t = 1$, i.e., $d_1(\varepsilon) = D_1(\varepsilon)$ but they do not anymore perfectly smooth consumption between dates $t = 1$ and $t = 2$. This is because the debt constraint binds at $t = 2$: agents borrow up to the debt limit contingent to low income, i.e., $d_2(\varepsilon) = -D_2(\varepsilon)$. The remaining equilibrium variables are determined by the first-order conditions associated to the saving decisions at $t = 1$ and $t = 2$:

$$q_2(\varepsilon) = \beta \frac{u'(c_{2,H}(\varepsilon))}{u'(c_{1,L}(\varepsilon))} \quad \text{and} \quad q_3(\varepsilon) = \beta \frac{u'(c_L(\varepsilon))}{u'(c_{2,H}(\varepsilon))},$$

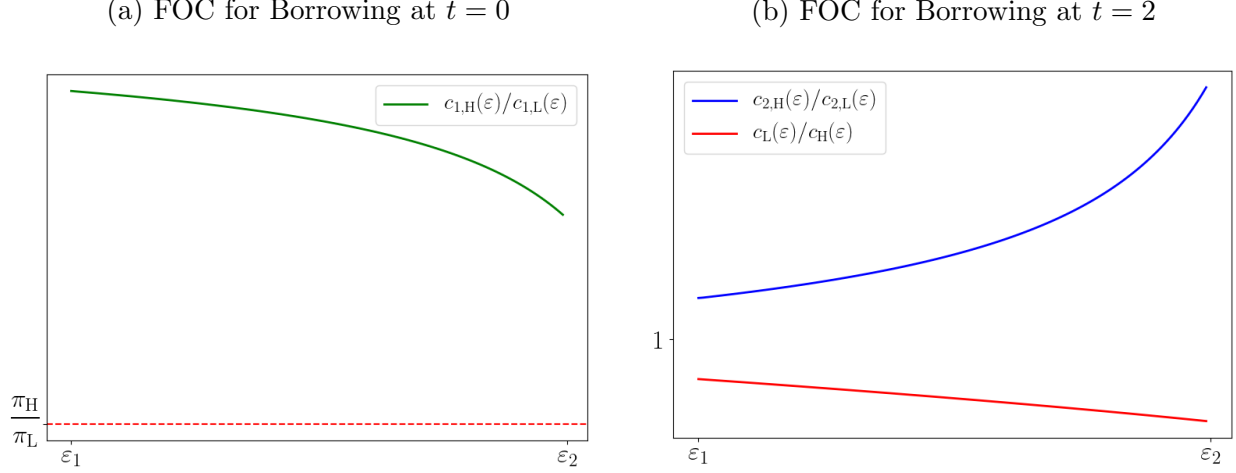


Figure B.4: Phase 2 when $\epsilon \in [\epsilon_1, \epsilon_2]$.

the period-1 binding flow budget constraints:

$$c_{1,H}(\epsilon) = y_H - D_1(\epsilon) + q_2(\epsilon)D_2(\epsilon) \quad \text{and} \quad c_{1,L}(\epsilon) = y_L + D_1(\epsilon) - q_2(\epsilon)D_2(\epsilon),$$

and the period-2 binding flow budget constraints:

$$c_{2,H}(\epsilon) = y_H + D_2(\epsilon) - q_3(\epsilon)d(\epsilon) \quad \text{and} \quad c_{2,L}(\epsilon) = y_L - D_2(\epsilon) + q_3(\epsilon)d(\epsilon).$$

The variables defined above form an equilibrium if, and only if, the first-order conditions for the borrowing decision at $t = 0$, $t = 1$ and $t = 2$ are satisfied:

$$\frac{u'(c_{1,L}(\epsilon))}{u'(c_{1,H}(\epsilon))} \geq \frac{\pi_H}{\pi_L}, \quad \frac{u'(c_{2,H}(\epsilon))}{u'(c_{1,L}(\epsilon))} \geq \frac{u'(c_{2,L}(\epsilon))}{u'(c_{1,H}(\epsilon))} \quad \text{and} \quad \frac{u'(c_L(\epsilon))}{u'(c_{2,H}(\epsilon))} \geq \frac{u'(c_H(\epsilon))}{u'(c_{2,L}(\epsilon))}. \quad (\text{B.9})$$

The threshold level ϵ_3 is attained when the first-order condition for borrowing at $t = 0$ binds, i.e., $\pi_L u'(c_{1,L}(\epsilon)) = \pi_H u'(c_{1,H}(\epsilon))$. Consumption levels and the validity of (B.9) are presented in Figure B.5.

Finally, for $\epsilon \in (\epsilon_3, 1]$, the debt limit level $D_1(\epsilon)$ is so large that we can implement the first best consumption at $t = 1$:

$$c_{1,H}(\epsilon) = \underline{c}^{\text{fb}} \quad \text{and} \quad c_{1,L}(\epsilon) = \bar{c}^{\text{fb}}.$$

The debt level $d_1(\epsilon)$ (which turns out to be strictly lower than the debt limit $D_1(\epsilon)$) and the remaining equilibrium variables are determined by the period-1 binding flow budget

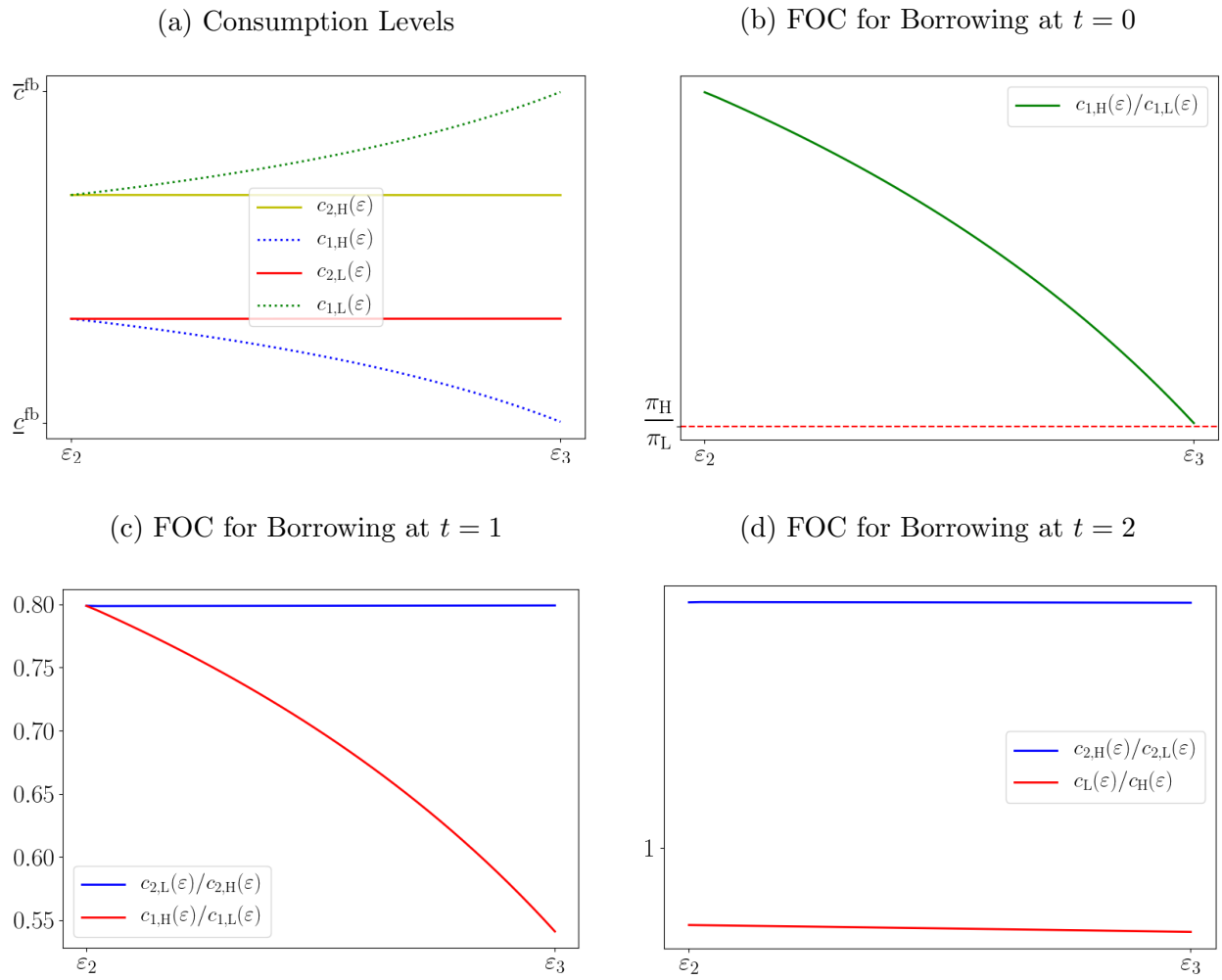


Figure B.5: Phase 3 when $\varepsilon \in [\varepsilon_2, \varepsilon_3]$.

constraints:

$$\underline{c}^{\text{fb}} = y_{\text{H}} - d_1(\varepsilon) + q_2(\varepsilon)D_2(\varepsilon) \quad \text{and} \quad \bar{c}^{\text{fb}} = y_{\text{L}} + d_1(\varepsilon) - q_2(\varepsilon)D_2(\varepsilon),$$

the period-2 binding flow budget constraints:

$$c_{2,\text{H}}(\varepsilon) = y_{\text{H}} + D_2(\varepsilon) - q_3(\varepsilon)d(\varepsilon) \quad \text{and} \quad c_{2,\text{L}}(\varepsilon) = y_{\text{L}} - D_2(\varepsilon) + q_3(\varepsilon)d(\varepsilon),$$

and the first-order conditions associated to the saving decisions at $t = 2$ and $t = 3$:

$$q_2(\varepsilon) = \beta \frac{u'(c_{2,\text{H}}(\varepsilon))}{u'(c_{1,\text{L}}(\varepsilon))} \quad \text{and} \quad q_3(\varepsilon) = \beta \frac{u'(c_{\text{L}}(\varepsilon))}{u'(c_{2,\text{H}}(\varepsilon))}.$$

The variables defined above form an equilibrium if, and only if, the first-order conditions for the borrowing decision at $t = 1$ and $t = 2$ are satisfied:

$$\frac{u'(c_{2,\text{H}}(\varepsilon))}{u'(c_{1,\text{L}}(\varepsilon))} \geq \frac{u'(c_{2,\text{L}}(\varepsilon))}{u'(c_{1,\text{H}}(\varepsilon))} \quad \text{and} \quad \frac{u'(c_{\text{L}}(\varepsilon))}{u'(c_{2,\text{H}}(\varepsilon))} \geq \frac{u'(c_{\text{H}}(\varepsilon))}{u'(c_{2,\text{L}}(\varepsilon))}. \quad (\text{B.10})$$

Debt limits are plotted in Figure 4.4(a) and the validity of (B.6) is presented in Figure B.6.

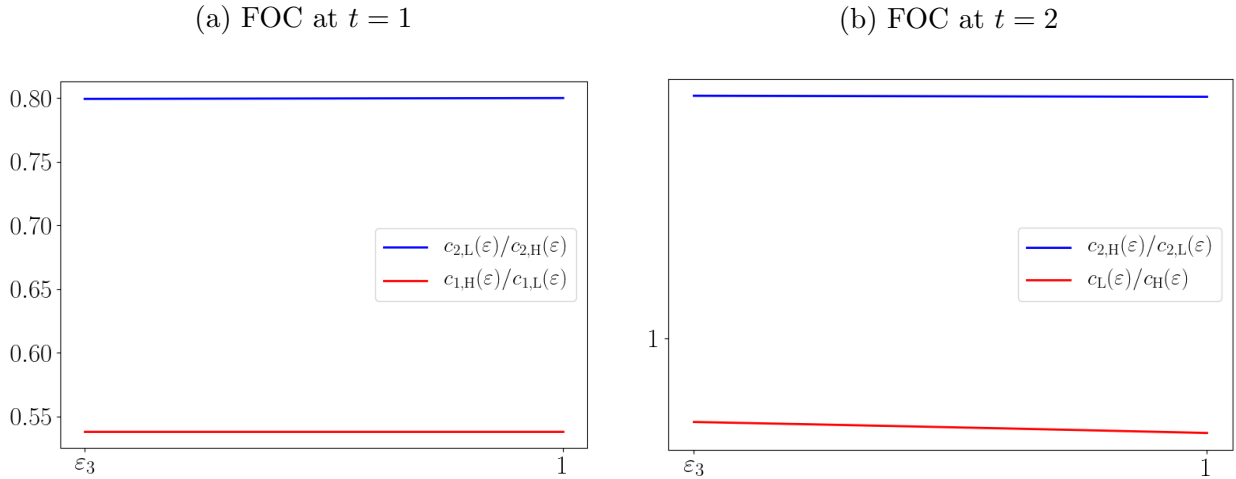


Figure B.6: Phase 4 with $\varepsilon \in [\varepsilon_3, 1]$.

B.3.2 Delaying the Intervention

Consider now the case where the collateral constraints are tightened at all dates $t \geq 2$, but at $t = 0$ and $t = 1$ constraints are not-too-tight. Formally, we assume that $\eta_1 = \eta_2 = 0$ and for every $t \geq 3$, $\eta_t = \varepsilon$ for some $\varepsilon > 0$. As in the previous section, depending on the

value of ε , we exhibit an equilibrium $(q^\varepsilon, (c^{i,\varepsilon}, a^{i,\varepsilon}, D^{i,\varepsilon})_{i \in I})$ that has different characteristics before the economy reaches the cyclical steady-state at $t = 3$. The difference is that we only have a single threshold value $\varepsilon_1 \in (0, 1)$ where the equilibrium variables differ over the transition phase. We hereafter discuss the main equilibrium characteristics.

Agents can borrow at period $t = 1$ up to

$$D_2(\varepsilon) = \ell^* \left[1 + q_3(\varepsilon) \frac{1}{1 - q(\varepsilon)} \right].$$

Figure B.8(a) shows that $D_2(\varepsilon)$ is increasing in ε and explodes to infinite as ε converges to 1 (and $q(\varepsilon) \mapsto 1$), exactly as it happens with $D_1(\varepsilon)$. Figures B.8(b) and B.8(c) plot the equilibrium bond prices and consumption levels for the transition period as functions of the tightening parameter ε .

For $\varepsilon \in [0, \varepsilon_1]$, both agents borrow at $t = 0$ up to the debt limit contingent to period-1 high income state, but they do not exhaust all borrowing opportunities at period $t = 1$ when income is low, that is, the debt constraint is non-binding. In doing so they perfectly smooth consumption between $t = 1$ and $t = 2$ before reaching the cyclical steady-state at $t = 3$. Formally, we have $d_1(\varepsilon) = D_1(\varepsilon)$, $d_2(\varepsilon) \in (-D_2(\varepsilon), D_2(\varepsilon))$ and $d_t(\varepsilon) = d(\varepsilon)$ for every $t \geq 3$. Since debt constraints at $t = 1$ do not bind, we have

$$c_{1,H}(\varepsilon) = c_{2,L}(\varepsilon), \quad c_{1,L}(\varepsilon) = c_{2,H}(\varepsilon) \quad \text{and} \quad q_2(\varepsilon) = \beta.$$

The remaining equilibrium variables are determined by the two equations associated to perfect smoothing

$$\underbrace{y_H - D_1(\varepsilon) - \beta d_2(\varepsilon)}_{c_{1,H}} = \underbrace{y_L + d_2(\varepsilon) + q_3(\varepsilon)d(\varepsilon)}_{c_{2,L}}$$

and

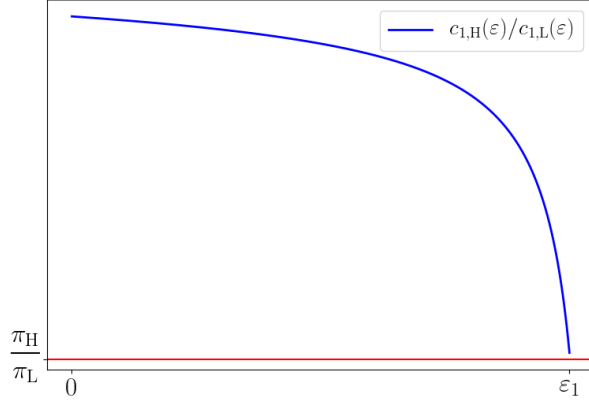
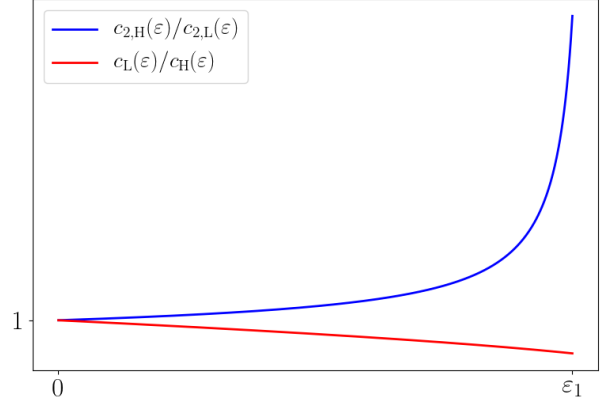
$$\underbrace{y_L + D_1(\varepsilon) + \beta d_2(\varepsilon)}_{c_{1,L}} = \underbrace{y_H - d_2(\varepsilon) - q_3(\varepsilon)d(\varepsilon)}_{c_{2,H}}$$

together with the first-order condition associated to the saving decision at $t = 2$

$$q_3(\varepsilon) = \beta \frac{u'(c_L(\varepsilon))}{u'(c_{2,H}(\varepsilon))}.$$

The variables defined above form an equilibrium if, and only if, the first-order conditions for the borrowing decision at $t = 0$ and $t = 2$ are satisfied:

$$\frac{u'(c_{1,L}(\varepsilon))}{u'(c_{1,H}(\varepsilon))} \geq \frac{\pi_H}{\pi_L} \quad \text{and} \quad \frac{u'(c_L(\varepsilon))}{u'(c_{2,H}(\varepsilon))} \geq \frac{u'(c_H(\varepsilon))}{u'(c_{2,L}(\varepsilon))}. \quad (\text{B.11})$$

(a) FOC for Borrowing at $t = 0$ (b) FOC for Borrowing at $t = 2$ Figure B.7: Validity of the FOCs when $\epsilon \in [0, \epsilon_1]$.

The threshold level ϵ_1 is attained when the first-order condition for borrowing at $t = 0$ binds, i.e., $\pi_L u'(c_{1,L}(\epsilon)) = \pi_H u'(c_{1,H}(\epsilon))$. The validity of (B.11) is presented in Figure B.7.

For $\epsilon \in (\epsilon_1, 1]$, the debt limit $D_1(\epsilon)$ is so large so both agents borrow less than the debt limit contingent to period-1 high income state, i.e., both the period-0 and period-1 debt constraints are non-binding. Not only we support perfect consumption smoothing, but also we implement the first best allocation at $t = 1$ and $t = 2$:

$$c_{1,H}(\epsilon) = c_{2,L}(\epsilon) = \underline{c}^{\text{fb}} \quad \text{and} \quad c_{1,L}(\epsilon) = c_{2,H}(\epsilon) = \bar{c}^{\text{fb}}.$$

This implies that $q_2(\epsilon) = \beta$. The debt levels $d_1(\epsilon)$ and $d_2(\epsilon)$ together with the price $q_3(\epsilon)$ are determined by the two equations associated to perfect smoothing

$$\underbrace{y_H - d_1(\epsilon) - \beta d_2(\epsilon)}_{c_{1,H}} = \underbrace{y_L + d_2(\epsilon) + q_3(\epsilon)d(\epsilon)}_{c_{2,L}}$$

and

$$\underbrace{y_L + d_1(\epsilon) + \beta d_2(\epsilon)}_{c_{1,L}} = \underbrace{y_H - d_2(\epsilon) - q_3(\epsilon)d(\epsilon)}_{c_{2,H}}$$

and the first-order condition associated to the saving decisions at $t = 2$

$$q_3(\epsilon) = \beta \frac{u'(c_{1,L}(\epsilon))}{u'(c_{2,H}(\epsilon))} = \beta \frac{u'(c_{1,L}(\epsilon))}{u'(\bar{c}^{\text{fb}})}.$$

the binding flow budget constraints at $t = 1$

$$\underline{c}^{\text{fb}} = y_H - d_1(\epsilon) + q_2(\epsilon)d_2(\epsilon) \quad \text{and} \quad \bar{c}^{\text{fb}} = y_L + d_1(\epsilon) - q_2(\epsilon)d_2(\epsilon)$$

the binding flow budget constraints at $t = 2$

$$\bar{c}^{\text{fb}} = y_{\text{H}} + d_2(\varepsilon) - q_3(\varepsilon)d(\varepsilon) \quad \text{and} \quad \underline{c}^{\text{fb}} = y_{\text{L}} - d_2(\varepsilon) + q_3(\varepsilon)d(\varepsilon)$$

and the first-order conditions associated to the saving decisions at $t = 2$

$$q_3(\varepsilon) = \beta \frac{u'(c_{\text{L}}(\varepsilon))}{u'(c_{2,\text{H}}(\varepsilon))} = \beta \frac{u'(c_{\text{L}}(\varepsilon))}{u'(\bar{c}^{\text{fb}})}.$$

The variables defined above form an equilibrium if, and only if, the first-order conditions for the borrowing decision at $t = 2$ is satisfied:

$$\frac{u'(c_{\text{L}}(\varepsilon))}{u'(\bar{c}^{\text{fb}})} \geq \frac{u'(c_{\text{H}}(\varepsilon))}{u'(\underline{c}^{\text{fb}})}. \quad (\text{B.12})$$

The validity of (B.12) is presented in Figure B.8(d).

Figure B.9 plots the ex-ante expected utility and shows that the tightening of debt constraints can lead to Pareto improvement of the laissez-faire equilibrium. A comparison with Figure 4.4(f) in MPV also reveals that, the tightening of debt constraints one period ahead, generates higher utility gains. This is because the first-best consumption levels are achieved not only at $t = 1$, but also at $t = 2$.

B.3.3 Tightening Collateral Constraints in the Long Run

The analysis in Section B.3.2 suggests that the later the government decides to intervene in financial markets, by means of tightening the debt constraints, the larger the utility gains. We provide more insight on this issue by showing that late interventions can support equilibria that are as close as possible to the first-best outcome. To formalize this property, given a date $T \geq 2$ and $\varepsilon > 0$, we consider an equilibrium where the tightening of debt constraints is given by

$$\eta_1 = \eta_2 = \dots = \eta_T = 0 \quad \text{and} \quad \eta_t = \varepsilon, \quad \forall t \geq T + 1.$$

When $T \geq 2$ is an even date, choosing ε close enough to one, we can verify (arguing as in the previous section) that there exists a competitive equilibrium where, at any $t \leq T - 1$, the debt constraints are not binding and consumption equals the first-best level, while a

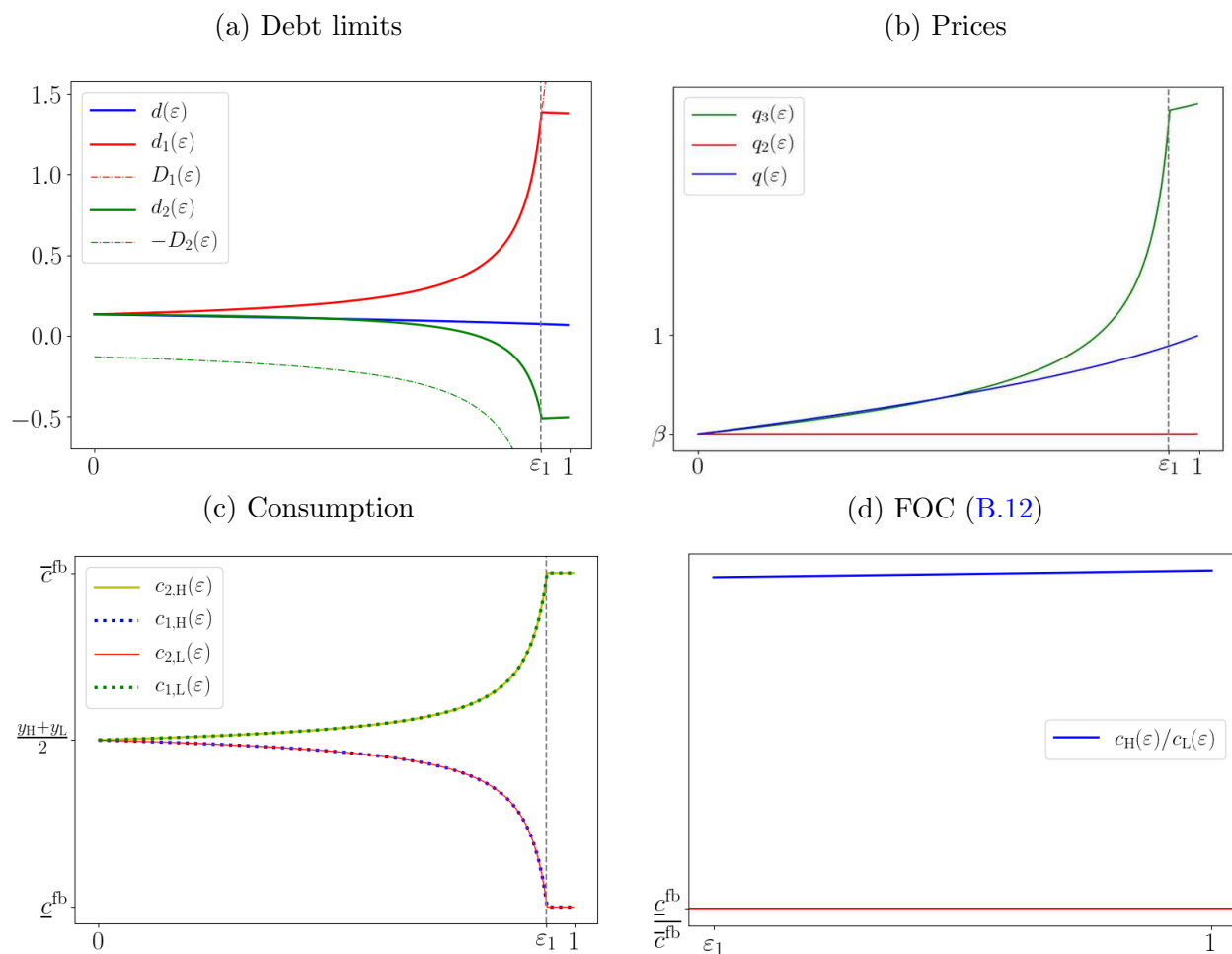


Figure B.8: Equilibrium variables as functions of the tightening coefficient ε .

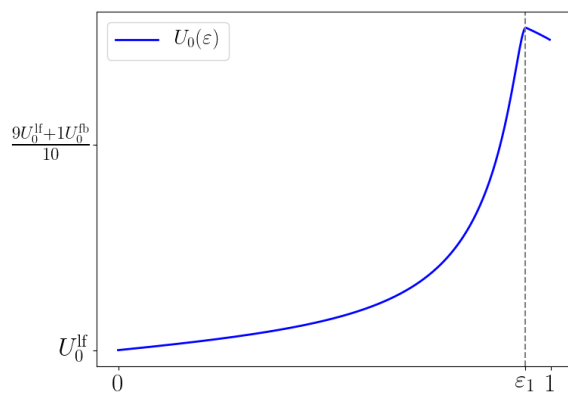


Figure B.9: Expected lifetime utility at $t = 0$.

steady-state is reached at period $T + 1$.⁸ That is, for every $t \leq T$, we have

$$c_t^{i,\varepsilon}(z) = \begin{cases} \bar{c}^{\text{fb}}, & \text{if } y_1^i(z) = y_L, \\ \underline{c}^{\text{fb}}, & \text{if } y_1^i(z) = y_H, \end{cases}$$

and $q_t^\varepsilon(z) = \beta$. For $t \geq T + 1$, we have

$$c_t^{i,\varepsilon}(z) = \begin{cases} c_L(\varepsilon), & \text{if } y_t^i(z) = y_L, \\ c_H(\varepsilon), & \text{if } y_t^i(z) = y_H, \end{cases}$$

and $q_{T+1}^\varepsilon(z) = q(\varepsilon)$.

The bond price $q_{T+1}^\varepsilon(z)$ is determined by the first-order condition of the saving decision of the agent with the high income at date T . Observe that since T is even, the high-income agent at date T had low income at $t = 1$, so his current consumption level is \bar{c}^{fb} . This implies that

$$q_{T+1}^\varepsilon(z) = \beta \frac{u'(c_L(\varepsilon))}{u'(\bar{c}^{\text{fb}})}.$$

Debt limits at every $t \leq T$ satisfy

$$\frac{D_t^\varepsilon}{\ell^*} = 1 + \beta \left[1 + \beta \left[\dots + \left[1 + q_{T+1}^\varepsilon \frac{1}{1 - q(\varepsilon)} \right] \right] \right].$$

We notice that

$$\lim_{\varepsilon \rightarrow 1} c_L(\varepsilon) = y_L + 2d(1),$$

where $d(1)$ is the unique positive value satisfying

$$1 = \beta \frac{u'(y_L + 2d(1))}{u'(y_H - 2d(1))}.$$

Since $\lim_{\varepsilon \rightarrow 1} q(\varepsilon) = 1$, we get that for every $t \leq T$,

$$\lim_{\varepsilon \rightarrow 1} D_t^\varepsilon = \infty.$$

This verifies the claim that, choosing ε close enough to 1, debt constraints do not bind at every $t \leq T - 1$. The debt constraint binds at T , and the variables defined above form a competitive equilibrium if, and only if,

$$\frac{u'(c_L(\varepsilon))}{u'(\bar{c}^{\text{fb}})} \geq \frac{u'(c_H(\varepsilon))}{u'(\underline{c}^{\text{fb}})}. \quad (\text{B.13})$$

⁸Since $\eta_{T+1} = \varepsilon$, the debt constraint $a_{T+1}^{i,\varepsilon}(z) \geq (1 - \varepsilon) \text{PV}(\ell^* | (z, T + 1))$ imposed at date T is too-tight.

The above condition is always satisfied since $c_L(\varepsilon) \leq c_H(\varepsilon)$ and $\underline{c}^{\text{fb}} \leq \bar{c}^{\text{fb}}$. The period-0 expected utility is then given by

$$U(c^{i,\varepsilon}|s^0) = u(y_0) + \pi_H [u(\underline{c}^{\text{fb}}) (\beta + \dots + \beta^T) + \beta^{T+1} U_H(\varepsilon)] + \pi_L [u(\bar{c}^{\text{fb}}) (\beta + \dots + \beta^T) + \beta^{T+1} U_L(\varepsilon)].$$

In particular, we have

$$\sup_{\varepsilon \in [0,1]} U(c^{i,\varepsilon}|s^0) \geq \underline{U}_0(T) := U(c^{i,1}|s^0).$$

Since

$$\lim_{T \rightarrow \infty} \underline{U}_0(T) = u(y_0) + \pi_H \frac{u(\underline{c}^{\text{fb}})}{1 - \beta} + \pi_L \frac{u(\bar{c}^{\text{fb}})}{1 - \beta},$$

we can get as close as desired to the first-best utility level by choosing T large enough and ε close enough to 1. This property is illustrated by Figure B.10 where we plot the function $T \mapsto \underline{U}_0(T)$.

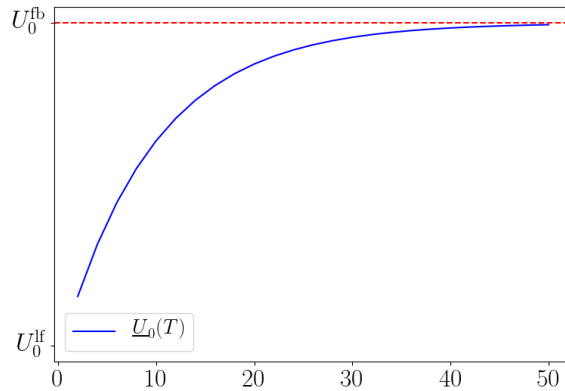


Figure B.10: Expected lifetime utility at $t = 0$ (for tightening parameter ε close enough to 1) as a function of tightening period T .

C Omitted Arguments and Proofs of Section 5

C.1 A Simple Characterization Result

To better understand how corrective subsidies can improve welfare, we here present a characterization of equilibria with limited pledgeability and Pigouvian subsidies on net deliveries. Consider a pair of subsidy rate and lump-sum taxes $(\kappa, (T^i)_{i \in I})$ such that $\kappa \in [0, 1]$

and

$$T^i(s^t) \leq \kappa y^i(s^t) \tag{C.1}$$

for every $i \in I$ and every $s^t \in \Sigma$. Let (c, a) be a post-tax/subsidy plan that satisfies the flow budget constraints (5.1) with equality and the debt constraints (5.2) in MPV. Denote by \tilde{c} the pre-tax/subsidy consumption process defined by:

$$\tilde{c}(s^t) := y^i(s^t) + a(s^t) - \sum_{s^{t+1} \succ s^t} q(s^{t+1})a(s^{t+1}).$$

Observe that:

$$c(s^t) = F^i(\tilde{c}(s^t), s^t) \quad \text{where} \quad F^i(x, s^t) := x - T^i(s^t) + \kappa[y^i(s^t) - x]^+.$$

Equivalently, F^i can also be written as follows:

$$F^i(x, s^t) = \begin{cases} x - T^i(s^t) & \text{if } x > y^i(s^t), \\ (1 - \kappa)x + \kappa y^i(s^t) - T^i(s^t) & \text{elsewhere.} \end{cases}$$

By construction, the function $F^i(\cdot, s^t)$ is well-defined on the whole domain $[0, \infty)$ with non-negative values. Let $\tilde{u}^i(\cdot, s^t)$ be the period utility function defined by:

$$\tilde{u}^i(x, s^t) := u(F^i(x, s^t)).$$

Denote the corresponding continuation utility by:

$$\tilde{U}^i(\tilde{c}|s^t) := \tilde{u}^i(\tilde{c}(s^t), s^t) + \sum_{\tau \geq 1} \beta^\tau \sum_{s^{t+\tau} \succ s^t} \pi(s^{t+\tau}|s^t) \tilde{u}^i(\tilde{c}(s^{t+\tau}), s^{t+\tau}).$$

We can see that a post-tax/subsidy plan (c, a) satisfies the flow budget constraints (5.1) with equality if, and only if, the pre-tax/subsidy plan (\tilde{c}, a) satisfies with equality the standard flow budget constraint

$$\tilde{c}(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})a(s^{t+1}) = y^i(s^t) + a(s^t). \tag{C.2}$$

This implies that (c^i, a^i) maximizes the utility U^i among all plans (c, a) satisfying the post-tax/subsidy flow budget constraints (5.1) and the debt constraints (5.2) if, and only if, (\tilde{c}^i, a^i)

maximizes the utility \tilde{U}^i among all plans (\tilde{c}, a) satisfying the pre-tax/subsidy flow budget constraints (C.2) and the debt constraints (5.2). Moreover, if

$$T^i(s^t) = \kappa \left[-a^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a^i(s^{t+1}) \right]^+,$$

then the pre- and post-tax/subsidy consumption plans coincide, $c^i = \tilde{c}^i$.⁹ It then follows that

$$\tilde{u}'(\tilde{c}^i(s^t), s^t) = \begin{cases} u'(c^i(s^t)) & \text{if } c^i(s^t) > y^i(s^t), \\ (1 - \kappa)u'(c^i(s^t)) & \text{if } c^i(s^t) < y^i(s^t). \end{cases} \quad (\text{C.3})$$

This allows us to establish the following characterization.

Proposition C.1. *Fix a collection $(q, (c^i, a^i, D^i)_{i \in I})$ satisfying market clearing such that*

$$c^i(s^t) \neq y^i(s^t), \quad \text{for all } i \in I \text{ and all } s^t \succ s^0.$$

Let

$$\chi^i(s^t) := \begin{cases} 1 - \kappa & \text{if } c^i(s^t) < y^i(s^t), \\ 1 & \text{if } c^i(s^t) > y^i(s^t). \end{cases}$$

The collection $(q, (c^i, a^i, D^i)_{i \in I})$ is a competitive equilibrium with limited pledgeability and Pigouvian subsidy $\kappa \in [0, 1]$ if, and only if, for each agent i :

- the post-tax/subsidy plan (c^i, a^i) satisfies the pre-tax/subsidy flow budget constraints with equality and the debt constraints;
- debt limits take the following form $D^i = \text{PV}(\ell^i) + M^i$, where M^i is a nonnegative exact rollover process;
- the following Euler equations are satisfied: for every event s^t and each successor event $s^{t+1} \succ s^t$, we have

$$q(s^{t+1}) = \max_{i \in I} \frac{\chi^i(s^{t+1})}{\chi^i(s^t)} \left[\frac{\beta \pi(s^{t+1} | s^t) u'(c^i(s^{t+1}))}{u'(c^i(s^t))} \right]; \quad (\text{C.4})$$

⁹Observe that condition (C.1) is satisfied since $T^i(s^t) = \kappa[y^i(s^t) - \tilde{c}^i(s^t)]^+$.

- the following standard transversality condition holds

$$\liminf_{t \rightarrow \infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u'(c^i(s^t)) = 0;$$

- the lump-sum taxes satisfy

$$T^i(s^t) := \kappa \left[-a^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a^i(s^{t+1}) \right]^+.$$

We remark that the subsidy rate affects only the term $\chi^i(s^{t+1})/\chi^i(s^t)$ of the Euler Equation (C.4). However, this term plays a crucial role in determining the remaining equilibrium variables as we illustrate in subsequent sections. The proof of Proposition C.1 is straightforward once we observe that an equilibrium with Pigouvian subsidies is nothing more than a standard equilibrium with limited pledgeability but with a different period utility function (the function $u(\cdot)$ is replaced by $\tilde{u}^i(\cdot, s^t)$). This also reveals why the microfoundations for limited pledgeability, discussed in Section 3, remain valid.

C.2 Proof of Proposition 5.1

Since the asset price $q_t^\kappa(z)$ is constant and equal to $q(\kappa)$ for every $t \geq 1$, we get that $D_t^{i,\kappa}(z) = \ell^*/(1 - q(\kappa))$ for every $t \geq 1$. This implies that the debt constraints are satisfied by construction. We can also verify that the pre-tax/subsidy flow budget constraints are satisfied with equality. To check the validity of the first-order conditions, we fix an arbitrary period $t \geq 1$. Since the high-income agent saves, the associated FOC should be

$$q(\kappa) = \beta \frac{1}{1 - \kappa} \frac{u'(c_L(\kappa))}{u'(c_H(\kappa))}.$$

The presence of the term $1 - \kappa$ in the denominator follows from Equation (C.3) and the fact that $c_L(\kappa) > y_L(\kappa)$ and $c_H(\kappa) < y_H(\kappa)$. The above equation is satisfied by definition of $q(\kappa)$. The low-income agent borrows up to his debt limit. The associated FOC should be¹⁰

$$q(\kappa) \geq \beta(1 - \kappa) \frac{u'(c_H(\kappa))}{u'(c_L(\kappa))}.$$

¹⁰Again, this follows from Equation (C.3) and the fact that $c_L(\kappa) > y_L(\kappa)$ and $c_H(\kappa) < y_H(\kappa)$.

Equation (5.7) in MPV guarantees that the above inequality is satisfied.¹¹ Finally, the transversality condition is satisfied because, for each agent, the debt limits bind infinitely many often. We have proved that we can apply Proposition C.1 to get the desired result.

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¹¹Indeed, let $m := u'(c_L(\kappa))/u'(c_H(\kappa))$. Equation (5.7) means that $m \geq 1 - \kappa$. This implies that $m^2 \geq (1 - \kappa)^2$, or, equivalently, $m/(1 - \kappa) \geq (1 - \kappa)/m$. Recall that $q(\kappa) = \beta m/(1 - \kappa)$. We then get $q(\kappa) \geq \beta(1 - \kappa)/m$. The term $\beta(1 - \kappa)/m$ is exactly the RHS of the desired inequality.